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ON A CLASS OF CAPILLARY SURFACES

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On a Class of Capillary Surfaces

by

Paul Concus and Robert Finn

1. In this paper we solve approximately the problem of determining the liquid-vapor interface that results when a wedge formed by two intersecting vertical planes is dipped into a liquid reservoir. We assume the gravitational field directed vertically[1] and the free surface of the reservoir to be a horizontal plane. We have already given a number of estimates for such surfaces in our earlier announcement [1]; here we present the explicit asymptotic form for the most general such surface, correct to within an additive constant.

Our considerations will be limited to surfaces whose height u above the (x,y) plane of the reservoir can be represented by a single valued function $u(x,y)$. The function is to be characterized as a stationary point for the variational problem

$$(1) \qquad \delta\{\iint\sqrt{1+u_x^2+u_y^2}\,dxdy + \tfrac{1}{2}\,\kappa\iint u^2\,dxdy\} = 0$$

among all surfaces which meet the bounding wedge walls in a prescribed angle γ. Here, $\kappa = \rho g \sigma^{-1}$ is the "capillarity constant", σ being the surface tensiom, ρ the difference in densities of the liquid and gas, and g the gravitational constant (positive when the field is directed downward). Thus, if we set $Au \equiv \frac{u_x}{W}$, $Bu \equiv \frac{u_y}{W}$, $W = \sqrt{1+u_x^2+u_y^2}$, then $u(x,y)$ must satisfy the equation

(2a) $$Nu \equiv (Au)_x + (Bu)_y = \kappa u$$

interior to the wedge, and setting $Tu \equiv \frac{1}{W}\nabla u$, there holds

(2b) $$Tu \cdot n]_\Gamma = \cos\gamma$$

with respect to the exterior directed normal n on the boundaries Γ of the wedge domain. The angle γ is to be measured within the fluid. It clearly suffices to suppose $0 \le \gamma \le \frac{\pi}{2}$, as the complementary case can be reduced to this one. We suppose that $u(x,y)$ possesses all smoothness properties indicated by the operations; however, we do not assume $u(x,y)$ to be defined at the vertex V, nor do we impose growth conditions as V is approached.

2. The case $\kappa > 0$: Let 2α be the interior wedge angle, and set $k = \sin\alpha\sec\gamma$. Introducing polar coordinates ρ,θ centered at V, we consider the function

(3) $$v(x,y;\gamma) = \frac{\cos\theta - \sqrt{k^2 - \sin^2\theta}}{k\kappa\rho}$$

which is defined and positive in the range $-\alpha \le \theta \le \alpha$, $\rho > 0$, provided $\alpha + \gamma < \frac{\pi}{2}$.

The following two properties of $v(x,y;\gamma)$ are basic for our result. Both can be verified by formal calculation: if $\kappa > 0$, then

(4a) $$Nv = \kappa v + \eta(x,y), \qquad |\eta(x,y)| < C\rho^3$$

(4b) $$Tv \cdot n]_\Gamma = \cos\gamma + \mu(\rho), \qquad -C\rho^4 < \mu(\rho) \le 0$$

for some positive constant C, as $\rho \to 0$.

In what follows, C will denote a generic constant, the value of which may change with the context.

Theorem 1. Let $u(x,y)$ be a solution of (2a,b) in the circular sector $\mathcal{D}_r$ determined by the wedge angle and by an arc Γ_r of radius r centered at V. Then, if $\kappa > 0$ and $\alpha + \gamma < \frac{\pi}{2}$, there holds throughout $\mathcal{D}_r$

(5) $$v(x,y) - C < u(x,y) < v(x,y) + C$$

while if $\alpha + \gamma \geq \frac{\pi}{2}$ there holds simply

(6) $$-C < u(x,y) < C .$$

Note that if $\alpha + \gamma < \frac{\pi}{2}$ all solutions are unbounded, while if $\alpha + \gamma \geq \frac{\pi}{2}$ only bounded solutions can exist. We emphasize again that no hypothesis is introduced on the behavior of $u(x,y)$ as V is approached, either from within $\mathcal{D}_r$ or on Γ.

We consider in this paper only the case $\alpha + \gamma < \frac{\pi}{2}$, as the other situation is covered in our earlier report [1].

To prove the result, we may first choose C such that $v < u + C$ on Γ_r, and such that $|\eta(x,y)| < \kappa C$. The function $\bar{v} \equiv v - C$ then satisfies

(7a) $$N\bar{v} = \kappa\bar{v} + \kappa C + \eta(x,y)$$

in $\mathcal{D}_r$,

(7b) $$T\bar{v}\cdot n]_{\Gamma} = \cos\gamma + \mu(r)$$

on Γ, and the function $w(x,y) = \bar{v} - u$ satisfies

(8) $$w]_{\Gamma_r} < 0 .$$

We intend to show $w < 0$ throughout $\mathcal{D}_r$. If this were not so, then for some $\varepsilon > 0$ the component $\mathcal{D}_r^{\varepsilon} \subset \mathcal{D}_r$ of points (x,y) whose distances from V exceed ε, and such that $w(x,y) > 0$, would be non-empty (cf. Fig. 1). We consider then the formal identity

(9) $$\iint_{\mathcal{D}_r^{\varepsilon}} w[N\bar{v} - Nu]dxdy = \kappa \iint_{\mathcal{D}_r^{\varepsilon}} w^2\, dxdy + \kappa C \iint_{\mathcal{D}_r^{\varepsilon}} (1 + \frac{\eta}{\kappa C})w\, dxdy$$

$$= - \iint_{\mathcal{D}_r^{\varepsilon}} \{w_x(A\bar{v} - Au) + w_y(B\bar{v} - Bu)\}dxdy$$

$$+ \int_{\Gamma_+} w(T\bar{v} - Tu)\cdot n\, ds + \int_{\Gamma_{\varepsilon}} w(T\bar{v} - Tu)\cdot n\, ds .$$

The integral over Γ_o vanishes, since $w = 0$ there[2]. Further, the convexity of the integrand in the variational expression (1) implies that the first term on the right is non-positive, see, e.g., the proof of Lemma II.2 in [2].

On Γ_+, $w \geq 0$ while, by (7b, 4b), $T\bar{v}\cdot n < \cos\gamma = Tu\cdot n$. Hence, $\int_{\Gamma_+} w(T\bar{v} - Tu)\cdot n\, ds \leq 0$. We conclude

$$\Omega(\varepsilon) \equiv \kappa \iint_{\mathcal{D}_r^{\varepsilon}} w^2\, dxdy \leq \int_{\Gamma_{\varepsilon}} w(T\bar{v} - Tu)\cdot n\, ds \leq 2(2\alpha\varepsilon)^{\frac{1}{2}} \left\{ \int_{\Gamma_{\varepsilon}} w^2\, ds \right\}^{\frac{1}{2}}$$

since $|T\varphi \cdot n| < 1$ for any function $\varphi(x,y)$. Interpreting the integral on the right in terms of $\Omega(\varepsilon)$, we obtain[3]

$$(10) \qquad \kappa\,\Omega^2(\varepsilon) \leq -8\alpha\varepsilon \frac{d\Omega}{d\varepsilon}$$

the integration of which leads to the result

$$\Omega(0) = \lim_{\varepsilon \to 0} \kappa \iint_{\mathcal{D}_r^\varepsilon} w^2\, dxdy = 0 .$$

Hence $\mathcal{D}_r^\varepsilon$ is empty for every $\varepsilon > 0$. This establishes the left side of (5).

The right side of (5) would follow similarly, except that in order to ensure the proper sign for the integral over Γ_+, the boundary condition (4b) must be changed.[4] We replace γ in (4b) by $\bar{\gamma} < \gamma$, chosen so that $\cos\bar{\gamma} + \mu(r) > \cos\gamma$. Clearly it suffices to set $\bar{\gamma} = \gamma - \bar{C}\,r^4$ for a suitable constant $\bar{C}$. Thus,

$$(11) \qquad u(x,y;\gamma) < v(x,y;\bar{\gamma}) + C < v(x,y;\gamma) + C\,\frac{r^4}{\rho} + C$$

where the constant C does not depend on r or ρ, or on $\bar{\gamma}$ in the range considered. Taking (11) to hold for $r = r_o < 1$ with ρ in the interval $r_o^2 \leq \rho < r_o$, and then for $r = r_o^2$ with ρ in the interval $r_o^4 \leq \rho < r_o^2$, we find

$$u(x,y;\gamma) < v(x,y;\gamma) + C + C\,r_o^2 + C\,r_o^4 .$$

Iteration of this procedure yields

$$u(x,y;\gamma) < v(x,y;\gamma) + \frac{C}{1-r^2}$$

in $\mathcal{D}_r$. This establishes the stated result.

3. The case $\kappa < 0$: This case corresponds to a situation in which the gravitational field is reversed. There is experimental evidence to indicate that in some circumstances a capillary surface will then continue to exist and to be stable with respect to small disturbances [4].

The previous discussion needs only minor modification. We introduce the function

$$\hat{v}(x,y) = \frac{\cos\theta + \sqrt{k^2 - \sin^2\theta}}{\kappa k \rho} \quad (12)$$

The estimates (4a,b) still hold, and we obtain

Theorem 2. *Under the hypotheses of Theorem 1, if* $\kappa < 0$, *then*

$$\hat{v}(x,y) - C < u(x,y) < \hat{v}(x,y) + C \quad (13)$$

if $\alpha + \gamma < \frac{\pi}{2}$, *while if* $\alpha + \gamma \geq \frac{\pi}{2}$ *there holds*

$$-C < u(x,y) < C . \quad (14)$$

It seems unlikely that the configuration (13) could be realized physically, cf. the discussion in [1, § 12].

Footnotes

p. 1 [1]The case of negative gravitational field, which we consider in § 3, can of course not be realized physically by this model. However, the physical situations in which capillary free surfaces remain stable under reversal of gravity forces lead to an equation that is easily transformed to the one considered here.

p. 4 [2]The difficulties arising from possible irregularities of Γ_0 can be overcome by a simple approximation procedure, based on the fact that Γ_0 is a level set of $w(x,y)$; thus, $\nabla w = 0$ at possible irregular points, so that w vanishes at least to second order at such points. From this follows that the irregular set of Γ_0 can be replaced, e.g., by segments of a successive refinement of rectilinear lattices.

p. 5 [3]The existence of $\frac{d\Omega}{d\varepsilon}$ for all ε is of course not evident. This differentiation can however be avoided by an averaging procedure. See, e.g., Shiffman [3, pp. 280-281] where the same differential inequality occurs in another context.

p. 5 [4]In the case $\gamma = 0$ there holds $\mu(r) \equiv 0$. The integral over Γ_+ then vanishes, and no change is necessary.

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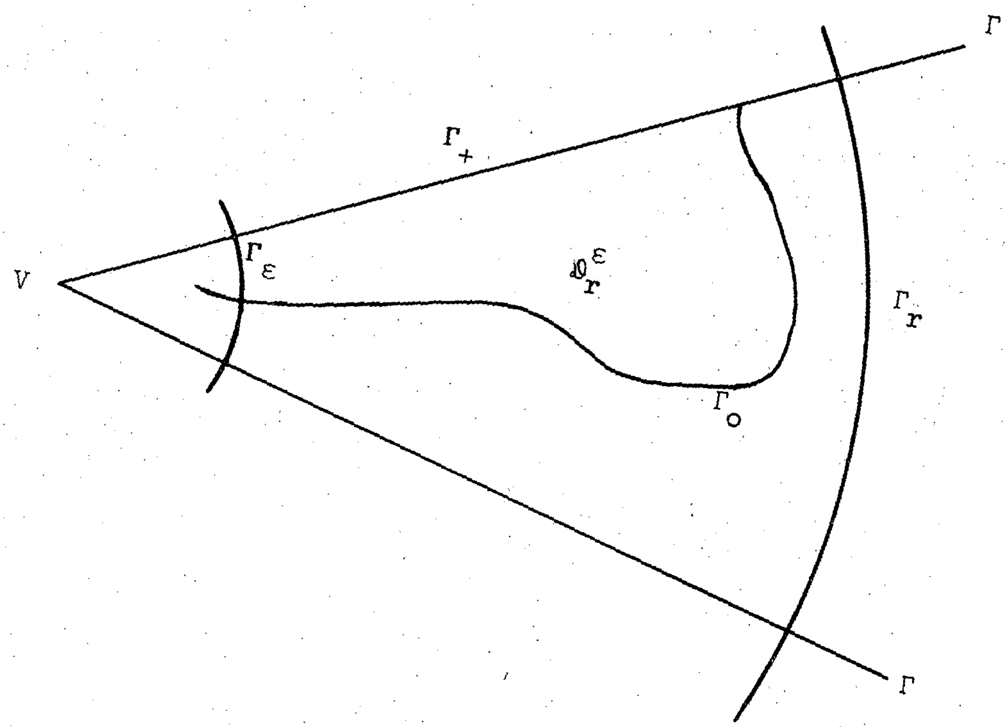

Γ
Γ_+
V
Γ_ε
$\mathcal{D}_r^\varepsilon$
Γ_r
Γ_0
Γ

Figure 1

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