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A BIRATIONAL NEVANLINNA CONSTANT AND ITS CONSEQUENCES

MIN RU AND PAUL VOJTA

Abstract. The purpose of this paper is to modify the notion of the Nevanlinna constant $\operatorname{Nev}(D)$ introduced by the first author (see [Ru15] and [Ru17]) for an effective Cartier divisor on a projective variety X. The modified notion is called the *birational Nevanlinna constant* and is denoted by $\operatorname{Nev}_{\mathrm{bir}}(D)$. The goal of $\operatorname{Nev}(D)$ and $\operatorname{Nev}_{\mathrm{bir}}(D)$ is to measure what is possible using the filtration method developed by Corvaja and Zannier and, independently, by Evertse and Ferretti. By computing $\operatorname{Nev}_{\mathrm{bir}}(D)$ using subsequent work of Autissier [Aut11], we establish a general result (see the General Theorem in Section 1), in both the arithmetic and complex cases, which extends the results of Evertse–Ferretti [EF08] and of Ru [Ru09] to general divisors. The notion $\operatorname{Nev}_{\mathrm{bir}}(D)$ originally came from applications involving Weil functions, but it also can be defined in terms of local effectivity of Cartier divisors after lifting by a proper birational map.

1. Introduction

We consider the following questions: *For a given effective Cartier divisor D on a given projective variety X, find the conditions (for D and X) such that every holomorphic mapping $f\colon \mathbb{C} \to X\backslash D$ must be degenerate (i.e. its image is contained some proper subvariety of X); If both D and X are defined over a number field k, then one also asks when every set of integral points of $X\backslash D$ must be degenerate.*

For the latter question, Schmidt's Subspace Theorem has been a key tool since its appearance in the early 1970s. In 1994 Faltings and Wüstholz [FW94] introduced a new geometric method of applying the Subspace Theorem, called the filtration method, which involved working with "many" sections of a line bundle and producing many linear combinations of them vanishing along appropriate divisors. This was further developed by Evertse and Ferretti [F00], [EF08] who reframed it using Mumford's theory of the degree of contact and the classical Schmidt subspace theorem (instead of

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the more general subspace theorem of [FW94] involving probability distributions). Independently, Corvaja and Zannier also also worked with filtrations of the same kind. They did this first on curves, leading to a new proof of Siegel's theorem [CZ02], and subsequently in higher dimension [CZ04b].

Of course these advances also apply to the theory of holomorphic curves, using a theorem of Cartan in place of Schmidt's subspace theorem.

With the goal of unifying the proofs of many known results proved using the filtration method, the first author introduced ([Ru15] and [Ru17]) the notion of the *Nevanlinna constant*, denoted by $\mathrm{Nev}(D)$, and proved that *if* $\mathrm{Nev}(D) < 1$*, then every holomorphic mapping* $f\colon \mathbb{C} \to X\setminus D$ *must be degenerate, and every set of integral points of* $X\setminus D$ *must also be degenerate if both* D *and* X *are defined over a number field* k. Moreover, quantitative versions of the above results, in the spirit of Nevanlinna–Roth–Cartan–Schmidt, were also obtained.

We now recall the definition of the Nevanlinna constant. For notations see Section 2.

Definition 1.1. *Let* X *be a complete variety, let* D *be an effective Cartier divisor on* X*, and let* $\mathscr{L}$ *be a line sheaf on* X*. If* X *is normal, then we define*

$$\mathrm{Nev}(\mathscr{L}, D) = \inf_{N,V,\mu} \frac{\dim V}{\mu}\ .$$

Here the inf is taken over all triples (N, V, μ) *such that* $N \in \mathbb{Z}_{>0}$*,* V *is a linear subspace of* $H^0(X, \mathscr{L}^N)$ *with* $\dim V > 1$*, and* $\mu > 0$ *is a rational number, that have the following property. For all* $P \in X$ *there is a basis* $\mathcal{B}$ *of* V *such that*

$$\sum_{s\in\mathcal{B}} (s) \ge \mu N D \tag{1}$$

in a Zariski-open neighborhood U *of* P*, relative to the cone of effective* $\mathbb{Q}$*-divisors on* U*. If there are no such triples* (N, V, μ)*, then* $\mathrm{Nev}(\mathscr{L}, D)$ *is defined to be* $+\infty$*. For a general complete variety* X*,* $\mathrm{Nev}(\mathscr{L}, D)$ *is defined by pulling back to the normalization of* X*.*

Remark 1.2. (a). In [Ru15], the Nevanlinna constant was only defined for $\mathscr{L} = \mathscr{O}(D)$, which is denoted by $\mathrm{Nev}(D) := \mathrm{Nev}(\mathscr{O}(D), D)$. As was done already by Autissier, it is more general to separate the roles of $\mathscr{L}$ and D in the above definition and elsewhere (for example, Theorem 1.4, Theorem 1.5 and the General Theorem below), so that results can be obtained for a general line bundle $\mathscr{L}$, in addition to $\mathscr{O}(D)$. This is used, for example, in Corollaries 1.12 and 1.13. (b). In [Ru15], the condition (1) above was stated as, *for all* $P \in \mathrm{Supp}\, D$*, there exists a basis* $\mathcal{B}$ *of* V *with* $\sum_{s\in\mathcal{B}} \mathrm{ord}_E(s) \ge \mu\, \mathrm{ord}_E(ND)$ *for all irreducible components* E *of* D *passing through* P. Here we impose the condition on all divisor components passing through P (not just those occurring in D). Also, since we are taking the infimum in Definition 1.1, we can require μ to be rational. With these two changes, the condition on the triple is equivalent to requiring that the $\mathbb{Q}$-divisor $\sum_{s\in\mathcal{B}}(s) - \mu ND$ be effective near P.

Theorem A (See [Ru15]). *Let X be a complex projective variety, and let D be an effective Cartier divisor on X. Then, for every $\epsilon > 0$,*

$$m_f(r, D) \leq_{\text{exc}} (\text{Nev}(D) + \epsilon)\, T_{f,D}(r)$$

holds for any holomorphic mapping $f\colon \mathbb{C} \to X$ with Zariski-dense image. Here the notation $\leq_{\text{exc}}$ means that the inequality holds for all $r \in (0, \infty)$ outside of a set of finite Lebesgue measure.

Theorem B (See [Ru17]). *Let k be a number field, and let S be a finite set of places of k containing all of the archimedean places. Let X be a projective variety over k, and let D be an effective Cartier divisor on X. Then, for every $\epsilon > 0$, the inequality*

$$m_S(x, D) \leq (\text{Nev}(D) + \epsilon)\, h_D(x)$$

holds for all $x \in X(k)$ outside a proper Zariski closed subset of X.

As was shown in [Ru15] and [Ru17], by computing $\text{Nev}(D)$, the above results recover the previous known results, such as [CZ04], [EF08], [Ru04] and of [Ru09], as well as derive new results for divisors which are not necessarily linear equivalent on X. More importantly, it led to a unified proof (for the known results) by simply computing $\text{Nev}(D)$.

In attempting to use the filtration constructed by Autissier in [Aut11] to derive a more general result, the authors realized that the notion of $\text{Nev}(\mathscr{L}, D)$ is not general enough for the purpose. More specifically, as we shall see, the (pointwise) maximum of finitely many Weil functions occurs in the proofs, and this is not in general a Weil function. However, as was shown in [Voj96, § 7], it becomes a Weil function after pulling back by a suitable proper birational morphism. In light of more recent developments in the Mori program, it is natural to think of maxima of Weil functions as Weil functions for b-divisors as introduced by Shokurov [Cor07].

These facts motivate the following modified definition.

Let $\mathscr{L}$ be a line sheaf on a variety X and let $\mathcal{B}$ be a finite set of global sections of $\mathscr{L}$. Then the notation $(\mathcal{B})$ will denote the sum of the divisors (s) for all $s \in \mathcal{B}$:

$$(\mathcal{B}) = \sum_{s\in\mathcal{B}}(s)\ . \tag{2}$$

Definition 1.3. *Let X be a normal complete variety, let D be an effective Cartier divisor on X, and let $\mathscr{L}$ be a line sheaf on X. Then*

$$\operatorname{Nev}_{\text{bir}}(\mathscr{L}, D) = \inf_{N,V,\mu} \frac{\dim V}{\mu} ,$$

where the infimum passes over all triples (N, V, μ) such that $N \in \mathbb{Z}_{>0}$, V is a linear subspace of $H^0(X, \mathscr{L}^N)$ with $\dim V > 1$, and $\mu \in \mathbb{Q}_{>0}$, with the following property. There are finitely many bases $\mathcal{B}_1, \dots, \mathcal{B}_\ell$ of V; Weil functions $\lambda_{\mathcal{B}_1}, \dots, \lambda_{\mathcal{B}_\ell}$ for the divisors $(\mathcal{B}_1), \dots, (\mathcal{B}_\ell)$, respectively; a Weil function λ_D for D; and an M_k-constant c such that

$$\max_{1 \le i \le \ell} \lambda_{\mathcal{B}_i} \ge \mu N \lambda_D - c \tag{3}$$

(as functions $\coprod_{v \in M_k} X(\bar{k}_v) \to \mathbb{R} \cup \{+\infty\}$). (Here we use the same convention as in Definition 1.1 when there are no triples (N, V, μ) that satisfy the condition.)

If L is a Cartier divisor or Cartier divisor class on X, then we define $\operatorname{Nev}_{\text{bir}}(L, D) = \operatorname{Nev}_{\text{bir}}(\mathscr{O}(L), D)$. We also define $\operatorname{Nev}_{\text{bir}}(D) = \operatorname{Nev}_{\text{bir}}(D, D)$.

With the notation $\operatorname{Nev}_{\text{bir}}(\mathscr{L}, D)$, we modify Theorems A and B as follows:

Theorem 1.4. *Let X be a complex projective variety, let D be an effective Cartier divisor and $\mathscr{L}$ be a line sheaf on X with $\dim H^0(X, \mathscr{L}^N) \ge 1$ for some $N > 0$. Let $f \colon \mathbb{C} \to X$ be a holomorphic mapping with Zariski-dense image. Then, for every $\epsilon > 0$,*

$$m_f(r, D) \le_{\text{exc}} (\operatorname{Nev}_{\text{bir}}(\mathscr{L}, D) + \epsilon)\, T_{f,\mathscr{L}}(r). \tag{4}$$

Theorem 1.5. *Let k be a number field, and let S be a finite set of places of k containing all archimedean places. Let X be a projective variety over k, and let D be an effective Cartier divisor on X. Let $\mathscr{L}$ be a line sheaf on X with $\dim H^0(X, \mathscr{L}^N) \ge 1$ for some $N > 0$. Then, for every $\epsilon > 0$, there is a proper Zariski-closed subset Z of X such that the inequality*

$$m_S(x, D) \le (\operatorname{Nev}_{\text{bir}}(\mathscr{L}, D) + \epsilon)\, h_{\mathscr{L}}(x) \tag{5}$$

holds for all $x \in X(k)$ outside of Z.

Corollary 1.6. *Let X be a projective variety over a number field k, and let D be an effective Cartier divisor on X. If $\operatorname{Nev}_{\text{bir}}(D) < 1$ then there is a proper Zariski-closed subset Z of X such that any set of D-integral points on X has only finitely many points outside of Z. A similar statement holds for holomorphic curves.*

We note that while the above definition of $\operatorname{Nev}_{\text{bir}}(\mathscr{L}, D)$ is convenient for use in applications, it involves Weil functions in its definition. As we shall see later, one can actually replace the Weil functions in the definition of

$\operatorname{Nev}_{\text{bir}}(\mathscr{L}, D)$ with divisors on a proper birational lifting. So we propose the second (equivalent) definition $\operatorname{Nev}_{\text{bir}}(\mathscr{L}, D)$.

Definition 1.7. *Let X be a complete variety, let D be an effective Cartier divisor on X, and let $\mathscr{L}$ be a line sheaf on X. If X is normal, then we define*

$$\operatorname{Nev}_{\text{bir}}(\mathscr{L}, D) = \inf_{N,V,\mu} \frac{\dim V}{\mu} ,$$

where the infimum passes over all triples (N, V, μ) such that $N \in \mathbb{Z}_{>0}$, V is a linear subspace of $H^0(X, \mathscr{L}^N)$ with $\dim V > 1$, and $\mu \in \mathbb{Q}_{>0}$, with the following property. There exist a variety Y and a proper birational morphism $\phi\colon Y \to X$ such that the following condition holds. For all $Q \in Y$ there is a basis $\mathcal{B}$ of V such that

$$\phi^*(\mathcal{B}) \ge \mu N \phi^* D \tag{6}$$

in a Zariski-open neighborhood U of Q, relative to the cone of effective $\mathbb{Q}$-divisors on U. If there are no such triples (N, V, μ), then $\operatorname{Nev}_{\text{bir}}(\mathscr{L}, D)$ is defined to be $+\infty$. For a general complete variety X, $\operatorname{Nev}_{\text{bir}}(\mathscr{L}, D)$ is defined by pulling back to the normalization of X.

Note that a *birational morphism* from X to Y is a morphism $X \to Y$ that has an inverse as a rational map; in other words, it is a birational map $X \dashrightarrow Y$ that is regular everywhere on X.

Remark 1.8. It is easy to see from the definitions that if n is a positive integer then

$$\operatorname{Nev}(\mathscr{L}, nD) = n \operatorname{Nev}(\mathscr{L}, D) \qquad \text{and} \qquad \operatorname{Nev}_{\text{bir}}(\mathscr{L}, nD) = n \operatorname{Nev}_{\text{bir}}(\mathscr{L}, D) .$$

One of the goals of this paper is to prove that these two definitions (Definitions 1.3 and 1.7) are equivalent. This is done in Corollary 4.16. See also Corollary 4.17, which shows that $\operatorname{Nev}_{\text{bir}}(\mathscr{L}, D)$ can also be defined using b-divisors.

Another goal of this paper is to use Theorems 1.4 and 1.5, together with additional work of Autissier [Aut11] on the filtration method, to prove the following two General Theorems (in the arithmetic and analytic cases).

Throughout this section, we use $h^0(\mathscr{L})$ to denote $\dim H^0(X, \mathscr{L})$ for a line sheaf $\mathscr{L}$ on X, $h^0(D)$ to denote $\dim H^0(X, \mathscr{O}(D))$ for an effective divisor D on X, and $\mathscr{L}(-D)$ to denote $\mathscr{L} \otimes \mathscr{O}(-D)$.

Definition 1.9. *Let $\mathscr{L}$ be a big line sheaf and let D be a nonzero effective Cartier divisor on a complete variety X. We define*

$$\beta(\mathscr{L}, D) = \liminf_{N\to\infty} \frac{\sum_{m\ge1} h^0(\mathscr{L}^N(-mD))}{N h^0(\mathscr{L}^N)} . \tag{7}$$

(Note that $|\mathscr{L}^N|$ does not have to be base point free.)

This is a limit of Autissier's $\alpha(\mathscr{L}, D)$ [Aut11, Def. 2.4].

Using the fact that the volume $\operatorname{vol}(\mathscr{L})$ of $\mathscr{L}$ can be defined as a limit [Laz04, Example 11.4.7], it is possible (with some care) to show that the above lim sup is actually a limit.

General Theorem (Arithmetic Part)**.** *Let X be a projective variety over a number field k, and let $D_1, \dots, D_q$ be nonzero effective Cartier divisors intersecting properly on X. Let $\mathscr{L}$ be a big line sheaf on X. Let $S \subset M_k$ be a finite set of places. Then, for every $\epsilon > 0$, the inequality*

$$\sum_{i=1}^{q} \beta(\mathscr{L}, D_i) m_S(x, D_i) \le (1+\epsilon) h_{\mathscr{L}}(x) \tag{8}$$

holds for all k-rational points outside a proper Zariski-closed subset of X.

General Theorem (Analytic Part)**.** *Let X be a complex projective variety and let $D_1, \dots, D_q$ be nonzero effective Cartier divisors intersecting properly on X. Let $\mathscr{L}$ be a big line sheaf on X. Let $f : \mathbb{C} \to X$ be a holomorphic mapping with Zariski-dense image. Then, for every $\epsilon > 0$,*

$$\sum_{i=1}^{q} \beta(\mathscr{L}, D_i) m_f(r, D_i) \le (1+\epsilon) T_{f,\mathscr{L}}(r)\ . \tag{9}$$

When restricted to integral points, the arithmetic part of the General Theorem is equivalent to [Aut11, Thm. 2.11], and thus is a quantitative extension of the latter.

We also note that Heier and Levin [HL] recently obtained the following result in the case when $\mathcal{L}$ is ample (we only state their Arithmetic Part result, the complex part also holds).

Theorem 1.10 (Heier–Levin)**.** *Let X be a projective variety over a number field k, and let $D_1, \dots, D_q$ be nonzero effective Cartier divisors in general position on X. Let A be an ample divisor on X. Let $S \subset M_k$ be a finite set of places. Then, for every $\epsilon > 0$, the inequality*

$$\sum_{i=1}^{q} \epsilon_{A,D_i} m_S(x, D_i) \le (\dim X + \epsilon) h_A(x)$$

holds for all k-rational points outside a proper Zariski-closed subset of X. Here ϵ_{A,D_i} is the (generalized) Seshadri constant *which is defined as $\epsilon_{A,D_i} = \sup\{\gamma \mid A - \gamma D_i \text{ is nef}\}$.*

Indeed, by Lemma 5.4 of [Aut11], when A is ample, we have $\beta(A, D) \ge \frac{\epsilon_{A,D}}{n+1}$. So the above result of Heier and Levin is a Corollary of our General Theorem.

One case in which $\beta(\mathscr{L}, D_j)$ can be computed is when $D_1, \ldots, D_q$ are effective Cartier divisors on X in general position, such that each D_j is linearly equivalent to a fixed ample divisor A. Write $\beta(D_j) := \beta(\mathscr{O}(D), D_j)$ with $D := D_1 + \cdots + D_q$. By the Riemann–Roch theorem, with $n = \dim X$,

$$h^0(ND) = h^0(qNA) = \frac{(qN)^n A^n}{n!} + o(N^n)$$

and

$$h^0(ND - mD_j) = h^0((qN - m)A) = \frac{(qN - m)^n A^n}{n!} + o(N^n).$$

Thus

$$\sum_{m \geq 1} h^0(ND - mD_j) = \frac{A^n}{n!} \sum_{l=0}^{qN-1} l^n + o(N^{n+1}) = \frac{A^n (qN - 1)^{n+1}}{(n+1)!} + o(N^{n+1}).$$

Hence

$$\beta(D_j) = \lim_{N \to \infty} \frac{\frac{A^n (qN-1)^{n+1}}{(n+1)!} + o(N^{n+1})}{N \frac{(qN)^n A^n}{n!} + o(N^{n+1})} = \frac{q}{n+1} \ .$$

Therefore, the General Theorems imply the following results of Evertse–Ferretti in the case when X is Cohen–Macaulay (for example X is smooth) as well as the result of Ru.

Theorem C (Evertse–Ferretti [EF08])**.** *Let X be a projective variety over a number field k, and let $D_1, \ldots, D_q$ be effective divisors on X in general position. Let $S \subset M_k$ be a finite set of places. Assume that there exist an ample divisor A on X and positive integers d_i such that D_i is linearly equivalent to $d_i A$ for $i = 1, \ldots, q$. Then, for every $\epsilon > 0$,*

$$\sum_{i=1}^{q} \frac{1}{d_i} m_S(x, D_i) \leq (\dim X + 1 + \epsilon) h_A(x)$$

holds for all k-rational points outside a proper Zariski closed subset of X.

Theorem D ([Ru09])**.** *Let X be a smooth complex projective variety and $D_1, \ldots, D_q$ be effective divisors on X, located in general position. Suppose that there exists an ample divisor A on X and positive integers d_i such that D_i is linearly equivalent to $d_i A$ on X for $i = 1, \ldots, q$. Let $f : \mathbb{C} \to X$ be a holomorphic mapping with Zariski-dense image. Then, for every $\epsilon > 0$,*

$$\sum_{i=1}^{q} \frac{1}{d_i} m_f(r, D_i) \leq_{\text{exc}} (\dim X + 1 + \epsilon) T_{f,A}(r).$$

We note that our General Theorem can be proved without using the notion of $\operatorname{Nev}_{\text{bir}}(\mathscr{L}, D)$ (and thus not using Theorem 1.4 and Theorem 1.5) by, instead, applying Schmidt's subspace theorem and H. Cartan's theorem directly (see the proof in Section 6). More importantly our proof greatly simplifies the original proofs of Evertse–Ferretti and of Ru, which involved Chow and Hilbert weights.

Another case we can compute $\beta(D_j)$ is when the divisor $D = D_1 + D_2 + \cdots + D_q$ is of equi-degree, i.e. $D_i \,.\, D^{n-1} = \frac{1}{q} D^n$ for all $i = 1, \ldots, q$. In this case, according to the calculations in [Ru15] (see pages 18–20), one can show that $\beta(D_i) > \frac{q}{2n}$ when $D_j, 1 \le j \le q$, are big and nef and $n = \dim X \ge 2$. Also according to Lemma 9.7 in [Lev09], if D_j, $1 \le j \le q$, are big and nef Cartier divisors, then there exist positive real numbers r_j such that $D = \sum_{j=1}^q r_j D_j$ has equi-degree. Therefore the General Theorem gives

Theorem 1.11 (Arithmetic Part)**.** *Let k be a number field and let $S \subseteq M_k$ be a finite set containing all archimedean places. Let X be a projective variety of dimension ≥ 2 over k, and let $D_1, \ldots, D_q$ be big and nef Cartier divisors on X that intersect properly. Then, for $\epsilon_0 > 0$ small enough (which only depends on the given divisors), the inequality*

$$m_S(x, D) \le \left(\frac{2 \dim X}{q} - \epsilon_0 \right) h_D(x) + O(1)$$

holds for all k-rational points $x \in X(k)$ outside of a proper Zariski-closed subset of X.

The analytic counterpart to the above theorem also holds.

In the context of integral points, this theorem is [CZ04b, Thm. 1] when X is a nonsingular surface and [Aut11, Thm. 1.3] in general.

Separating the roles of $\mathscr{L}$ and D in the General Theorem appears to be useful. We state the following two corollaries, in which $\mathscr{L}$ is the anticanonical line bundle $-K_X$.

Corollary 1.12. *Let k be a field, let $\pi\colon X \to \mathbb{P}^1_k \times \mathbb{P}^1_k$ be the blowing-up at the point $(1,1)$, and let $D = D_1 + D_2 + D_3 + D_4$, where $D_1 = \pi^*(\{0\} \times \mathbb{P}^1)$, $D_2 = \pi^*(\{\infty\} \times \mathbb{P}^1)$, $D_3 = \pi^*(\mathbb{P}^1 \times \{0\})$, and $D_4 = \pi^*(\mathbb{P}^1 \times \{\infty\})$.*

(a). *Assume that k is a number field, and let $S \subset M_k$ be a finite set of places. Then, for any $\epsilon > 0$, the inequality*

$$m_S(x, D) \le \left(\frac{8}{7} + \epsilon \right) h_{-K_X}(x) \tag{10}$$

holds for all points $x \in X(k)$ outside a proper Zariski-closed subset.

(b). *Assume that $k = \mathbb{C}$, and let $f\colon \mathbb{C} \to X$ be a holomorphic map with Zariski-dense image. Then*

$$T_f(r, D) - \left(\frac{8}{7} + \epsilon \right) T_{f,-K_X}(r) \le_{\text{exc}} N_f(r, D) \,. \tag{11}$$

Proof. From [GW17], we know that $\beta(-K_X, \pi^*D_i) \geq 7/8$ for $i = 1, 2, 3, 4$. Hence this corollary follows from the general theorem above. □

Note that, if conjectures of Griffiths and the second author hold, then $8/7 + \epsilon$ could be replaced by $1 + \epsilon$ in (10) and (11). The inequality (11) played a crucial role in [GW17] on the GCD analogue for entire functions.

To see how the above setup is related to greatest common divisors, let $y, z \in \mathbb{Z}$. If $(y, z) \neq (1, 1)$, then the height of the point $\pi^{-1}([1 : y : z]) \in X(\mathbb{Q})$ relative to the exceptional divisor of the blowing-up is equal to $\log \gcd(y - 1, z - 1) + O(1)$. Given a finite set S of places of $\mathbb{Q}$, if y and z are divisible only by primes in S then (10) reduces to $2h(y) + 2h(z) \leq (8/7 + \epsilon)(2h(y) + 2h(z) - \log \gcd(y, z))$. Bugeaud, Corvaja, and Zannier [BCZ03] showed that if a and b are multiplicatively prime integers ≥ 2 then $\log \gcd(a^n - 1, b^n - 1) = o(n)$ for $n \in \mathbb{N}$. This is sharper than (10), but covers a smaller family of points.

The following corollary provides one way in which it may be possible to find examples (other than curves) in which the second author's conjecture could be proved to hold. It boils down to Schmidt's subspace theorem in the case $X = \mathbb{P}^n$.

Corollary 1.13. *Let k be a number field and let $S \subset M_k$ be a finite set of places. Let X be a smooth projective variety over k and let $D_1, \dots, D_q$ be effective divisors on X that intersect properly. Assume that $-K_X$ is ample (i.e., X is a Fano variety), and that $\beta(-K_X, D_j) \geq 1$ for $j = 1, \dots, q$. Then the second author's conjectured inequality holds in this case; i.e.,*

$$\sum_{j=1}^{q} m_S(x, D_j) + h_{K_X}(x) \leq \epsilon h_A(x)$$

holds for all k-rational points outside a proper Zariski-closed subset of X, where A is an ample divisor.

Proof. This is immediate from the General Theorem and the assumption on the $\beta(-K_X, D_j)$. □

In the remainder of this paper we will primarily discuss the number field cases of our results. The corresponding results in Nevanlinna theory can be proved by similar methods, as can the diophantine results over function fields of characteristic zero.

2. Notation and Preliminaries

In this paper, $\mathbb{N} = \{0, 1, 2, \dots\}$ and $\mathbb{R}^+$ is the interval $[0, \infty)$.

2.1. **Notation and Conventions in Number Theory.** For a number field k, recall that M_k denotes the set of places of k, and that k_v denotes the completion of k at a place $v \in M_k$. Norms $\|\cdot\|_v$ on k are normalized so that

$$\|x\|_v = |\sigma(x)|^{[k_v:\mathbb{R}]} \qquad \text{or} \qquad \|p\|_v = p^{-[k_v:\mathbb{Q}_p]}$$

if $v \in M_k$ is an archimedean place corresponding to an embedding $\sigma\colon k \hookrightarrow \mathbb{C}$ or a non-archimedean place lying over a rational prime p, respectively.

An M_k-constant is a collection $(c_v)_{v\in M_k}$ of real constants such that $c_v = 0$ for all but finitely many v. Heights are logarithmic and relative to the number field used as a base field, which is always denoted by k. For example, if P is a point on $\mathbb{P}^n_k$ with homogeneous coordinates $[x_0 : \dots : x_n]$ in k, then

$$h(P) = h_{\mathscr{O}(1)}(P) = \sum_{v\in M_k} \log\max\{\|x_0\|_v, \dots, \|x_n\|_v\} \ .$$

We use the standard notations of Nevanlinna theory and Diophantine approximation (see, for example, [Vojcm], [Voj87] [Ru15] and [Ru17]).

2.2. **Notation in Algebraic Geometry.** A *variety* over a field k is an integral scheme, separated and of finite type over $\operatorname{Spec} k$. A morphism of varieties is a morphism of schemes over k. A *line sheaf* is a locally free sheaf of rank 1 (an invertible sheaf).

For a variety X over a number field k, we let $X(M_k)$ denote the disjoint union $\coprod_{v\in M_k} X(\overline{k}_v)$, where $\overline{k}_v$ is an algebraic closure of the completion k_v of k at v. If $\mathscr{L}$ is a line sheaf on a variety X, then $\mathscr{L}^n$ denotes the n^{th} tensor power $\mathscr{L}^{\otimes n}$, and if D is a Cartier divisor on X, then $\mathscr{L}(D)$ denotes $\mathscr{L} \otimes \mathscr{O}(D)$.

Definition 2.1. *Let $D_1, \dots, D_q$ be effective Cartier divisors on a variety X of dimension n.*

(a). *We say that $D_1, \dots, D_q$ lie in* ***general position*** *if for any $I \subseteq \{1, \dots, q\}$, we have $\dim(\bigcap_{i\in I} \operatorname{Supp} D_i) = n - \#I$ if $\#I \le n$, and $\bigcap_{i\in I} \operatorname{Supp} D_i = \emptyset$ if $\#I > n$.*

(b). *We say that $D_1, \dots, D_q$* ***intersect properly*** *if for any subset $I \subseteq \{1, \dots, q\}$ and any $x \in \bigcap_{i\in I} \operatorname{Supp} D_i$, the sequence $(\phi_i)_{i\in I}$ is a regular sequence in the local ring $\mathscr{O}_{X,x}$, where ϕ_i are the local defining functions of D_i, $1 \le i \le q$.*

Remark 2.2. By [Mat86, Thm. 17.4], if $D_1, \dots, D_q$ intersect properly, then they lie in general position. The converse holds if X is Cohen–Macaulay (this is true if X is nonsingular).

2.3. **Weil functions.** Let X be a variety over $\mathbb{C}$, let D be an effective Cartier divisor on X, and let $s = 1_D$ be a canonical section of $\mathscr{O}(D)$ (i.e., a global section for which $(s) = D$). Choose a smooth metric $|\cdot|$ on $\mathscr{O}(D)$. In Nevanlinna theory, one often encounters the function

$$\lambda_D(x) := -\log |s(x)| \; ; \tag{12}$$

this is a real-valued function on $X(\mathbb{C}) \setminus \operatorname{Supp} D$. It is linear in D (over a suitable domain), so by linearity and continuity it can be extended to a definition of λ_D for a general Cartier divisor D on X.

Weil functions are counterparts to such functions in number theory. For its definition and detailed properties, see [Lan83, Ch. 10] or [Vojcm, Sect. 8]. For example, let k be a number field, and let H be a hyperplane in $\mathbb{P}^n_k$. Let $a_0x_0 + \cdots + a_nx_n$ be a linear form with $a_0, \dots, a_n \in k$ whose vanishing determines the hyperplane H. Then, for all $v \in M_k$ and all rational points $P \in \mathbb{P}^n_k(k) \setminus H$ with homogeneous coordinates $[x_0 : \cdots : x_n]$, we let

$$\lambda_{H,v}(P) = -\log \frac{\|a_0x_0 + \cdots + a_nx_n\|_v}{\max\{\|a_0\|_v, \dots, \|a_n\|_v\} \cdot \max\{\|x_0\|_v, \dots, \|x_n\|_v\}} \; .$$

This quantity is independent of the choice of homogeneous coordinates for P, and also does not depend on the linear form $a_0x_0 + \cdots + a_nx_n$ chosen above. In the complex case, if H is a hyperplane in $\mathbb{P}^n_{\mathbb{C}}$ and $P \in \mathbb{P}^n(\mathbb{C}) \setminus H$, then

$$\lambda_H(P) = -\frac{1}{2} \log \frac{|a_0x_0 + \cdots + a_nx_n|^2}{(|a_0|^2 + \cdots + |a_n|^2)(|x_0|^2 + \cdots + |x_n|^2)}.$$

For this paper, the main properties of Weil functions are additivity, functoriality, and the fact that Weil function depend only on D up to addition of a function whose absolute value is bounded by an M_k-constant—see [Vojcm, Thm. 8.8]. We also need to show which Weil functions on normal complete varieties correspond to effective Cartier divisors.

Remark 2.3. On a normal variety, the Cartier divisors are exactly the locally principal Weil divisors [Har77, II 6.11.2], and a Cartier divisor is effective as a Cartier divisor if and only if it is effective as a Weil divisor [Har77, II 6.3A].

Proposition 2.4. *Let X be a normal complete variety, let D be a Cartier divisor on X, and let λ_D be a Weil function for D. Then the following conditions are equivalent.*

(i) *D is effective.*
(ii) *λ_D is bounded from below by an M_k-constant.*
(iii) *for all $v \in M_k$, $\lambda_{D,v}$ is bounded from below.*
(iv) *there exists a $v \in M_k$ such that $\lambda_{D,v}$ is bounded from below.*

Proof. For the implication (i) $\Longrightarrow$ (ii), see [Lan83, Ch. 10, Prop. 3.1]. The implications (ii) $\Longrightarrow$ (iii) $\Longrightarrow$ (iv) are trivial.

To show that (iv) $\Longrightarrow$ (i), assume that D is not effective. By Remark 2.3, D (as a Weil divisor) has at least one component with negative multiplicity. Fix a closed point $x \in X$ that lies on that prime divisor, but not on any other irreducible component of $\operatorname{Supp} D$, and let U be a Zariski-open neighborhood of x. Take a sequence (x_n) of points in $U(\overline{k}_v) \setminus \operatorname{Supp} D$ that converges to x in the v-topology. Then the sequence $\lambda_{D,v}(x_n)$ goes to $-\infty$; thus $\lambda_{D,v}$ is not bounded from below. ☐

Lemma 2.5. *Let X be a projective variety, and let $U_1, \dots, U_n$ be Zariski-open subsets of X that cover X. Let $D_1, \dots, D_n$ be Cartier divisors on X such that $D_i\big|_{U_i}$ is effective for all i, and let λ_{D_i} be Weil functions for D_i for all i. Then there is an M_k-constant $\gamma = (\gamma_v)$ such that, for all v and all $x \in X(\overline{k}_v)$ there is an i such that $x \in U_i$ and $\lambda_{D_i,v}(x) \ge \gamma_v$.*

Proof. By taking a finite refinement of $\{U_i\}$, we may assume that each U_i is affine, and that $D_i\big|_{U_i} = (f_i)$ for some nonzero $f_i \in \mathscr{O}(U_i)$ for all i.

Then λ_{D_i} is locally M_k-bounded from below on $U_i(M_k)$ for all i. Indeed, by [Lan83, Ch. 10, Prop. 1.3], the function $-\log \|f_i\|$ is locally M_k-bounded from below on $U_i(M_k)$; therefore so is λ_{D_i} since by definition of Weil function and Néron function the difference between the two functions is locally M_k-bounded.

By [Lan83, Ch. 10, Prop. 1.2], there are affine M_k-bounded sets $E_1, \dots, E_n$ such that $\bigcup E_i = X(M_k)$ and such that $E_i \subseteq U_i(M_k)$ for all i.

By definition of locally M_k-bounded function (see [Lan83, Ch. 10, Sect. 1]), for each i there is an M_k-constant γ_i such that $\lambda_{D_i} \ge \gamma_i$ on E_i. This concludes the proof, with $\gamma = \min\{\gamma_1, \dots, \gamma_n\}$. ☐

2.4. **Theorems of Schmidt and Cartan.** Schmidt's Subspace Theorem and the corresponding theorem of Cartan play a central role in this paper.

Theorem 2.6 (Schmidt's subspace theorem)**.** *Let k be a number field, let S be a finite set of places of k containing all archimedean places, let n be a positive integer, let $H_1, \dots, H_q$ be hyperplanes in $\mathbb{P}^n_k$, let $\epsilon > 0$, and let $c \in \mathbb{R}$. Then there is a finite union Z of proper linear subspace of $\mathbb{P}^n_k$, depending only on k, S, n, $H_1, \dots, H_q$, ϵ, and c, such that the inequality*

$$\sum_{v \in S} \max_J \sum_{j \in J} \lambda_{H_j,v}(x) \le (n+1+\epsilon)h(x) + c$$

holds for all $x \in (\mathbb{P}^n_k \setminus Z)(k)$. Here the set J ranges over all subsets of $\{1, \dots, q\}$ such that the hyperplanes $(H_j)_{j \in J}$ lie in general position.

Theorem 2.7 (Cartan's Second Main Theorem)**.** *Let $H_1, \dots, H_q$ be hyperplanes in $\mathbb{P}^n_{\mathbb{C}}$ with $n \ge 1$, and let $f\colon \mathbb{C} \to \mathbb{P}^n_{\mathbb{C}}$ be a holomorphic curve whose image is not contained in a hyperplane. Then, for any $\epsilon > 0$,*

$$\int_0^{2\pi} \max_J \sum_{j\in J} \lambda_{H_j}(f(re^{i\theta}))\,\frac{d\theta}{2\pi} \le_{\text{exc}} (n+1+\epsilon)T_f(r),$$

where J varies over the same collection of sets as in Theorem 2.6, and where the notation $\le_{\text{exc}}$ means that the inequality holds for all $r \in (0,\infty)$ outside of a set of finite Lebesgue measure.

Remark 2.8. In the above two theorems, there is a finite union Z_1 of proper linear subspaces of $\mathbb{P}^n_k$ or $\mathbb{P}^n_{\mathbb{C}}$ (in Theorems 2.6 and 2.7, respectively), depending only on the hyperplanes $H_1, \dots, H_q$, with the following properties. In Theorem 2.6, the subset Z may be taken to be the union of Z_1 and a finite union of points, and in Theorem 2.7, the condition on the holomorphic curve f may be relaxed to allow any nonconstant holomorphic map $f\colon \mathbb{C} \to \mathbb{P}^n_{\mathbb{C}}$ whose image is not contained in Z_1. See [Voj89] and [Voj97], respectively.

Remark 2.9. Since the functions $\lambda_{H,v}$ are bounded from below, we may assume in Theorems 2.6 and 2.7 that $\bigcap H_i = \emptyset$ (include more hyperplanes). Then, in each of these theorems, we may also require that all of the sets J have $n+1$ elements.

We phrase Theorems 2.6 and 2.7 in terms of divisors in a linear system as below (see also [Aut11, Prop. 4.2]).

Theorem 2.10. *Let k be a number field, let S be a finite set of places of k containing all archimedean places, let X be a complete variety over k, let D be a Cartier divisor on X, let V be a nonzero linear subspace of $H^0(X, \mathscr{O}(D))$, let $s_1, \dots, s_q$ be nonzero elements of V, let $\epsilon > 0$, and let $c \in \mathbb{R}$. For each $i = 1, \dots, q$, let D_j be the Cartier divisor (s_j), and let λ_{D_j} be a Weil function for D_j. Then there is a proper Zariski-closed subset Z of X, depending only on k, S, X, L, V, $s_1, \dots, s_q$, ϵ, c, and the choices of Weil and height functions, such that the inequality*

$$\sum_{v\in S} \max_J \sum_{j\in J} \lambda_{D_j,v}(x) \le (\dim V + \epsilon) h_D(x) + c \tag{13}$$

holds for all $x \in (X \setminus Z)(k)$. Here the set J ranges over all subsets of $\{1, \dots, q\}$ such that the sections $(s_j)_{j\in J}$ are linearly independent.

Proof. Let $d = \dim V$. We may assume that $d > 1$ (otherwise, all D_j are the same divisor, and the sets J have at most one element each, so (13) follows immediately from (the number theory version of) the First Main Theorem).

Let $\Phi\colon X \dashrightarrow \mathbb{P}_k^{d-1}$ be the rational map associated to the linear system V. Let X' be the closure of the graph of Φ, and let $p\colon X' \to X$ and $\phi\colon X' \to \mathbb{P}_k^{d-1}$ be the projection morphisms.

Note that, even though Φ extends to the morphism $\phi\colon X' \to \mathbb{P}_k^{d-1}$, the linear system of $H^0(X', p^*\mathscr{O}(D))$ corresponding to V may still have base points. What is true, however, is that there is an effective Cartier divisor B on X' such that, for each nonzero $s \in V$, there is a hyperplane H in $\mathbb{P}_k^{d-1}$ such that $p^*(s) - B = \phi^*H$. (More precisely, $\phi^*\mathscr{O}(1) \cong \mathscr{O}(p^*D - B)$. The map

$$\alpha\colon H^0(X', \mathscr{O}(p^*D - B)) \to H^0(X, \mathscr{O}(p^*D))$$

defined by tensoring with the canonical global section 1_B of $\mathscr{O}(B)$ is injective, and its image contains $p^*(V)$. The preimage $\alpha^{-1}(p^*(V))$ corresponds to a base-point-free linear system for the divisor $p^*D - B$.)

For each $j = 1, \dots, q$ let H_j be the hyperplane in $\mathbb{P}_k^{d-1}$ for which $p^*(s_j) - B = \phi^*H_j$. Choose a Weil function λ_B for B. Then, for all $v \in S$ and all $j = 1, \dots, q$, we have

$$p^*\lambda_{D_j,v} = \phi^*\lambda_{H_j,v} + \lambda_{B,v} + O(1) .$$

Therefore it will suffice to prove that for any $c' \in \mathbb{R}$ the inequality

$$\sum_{v\in S} \max_J \sum_{j\in J} (\lambda_{H_j,v}(\phi(x)) + \lambda_{B,v}(x)) \le (\dim V + \epsilon) h_D(p(x)) + c' \tag{14}$$

holds for all $x \in X'(k)$ outside of some proper Zariski-closed subset Z' of X'. Indeed, for suitable c' this will imply (13) outside of $Z := p(Z' \cup \operatorname{Supp} B)$. The set Z is Zariski-closed in X because $p\colon X' \to X$ is a proper morphism, and $Z \ne X$ because p is birational.

For any subset J of $\{1, \dots, q\}$, the sections s_j, $j \in J$ are linearly independent elements of V if and only if the hyperplanes H_j, $j \in J$ lie in general position in $\mathbb{P}_k^{d-1}$. Therefore we may apply Theorem 2.6, to obtain that for any c'' the inequality

$$\sum_{v\in S} \max_J \sum_{j\in J} \lambda_{H_j,v}(\phi(x)) \le (\dim V + \epsilon) h(\phi(x)) + c'' \tag{15}$$

holds for all $x \in X'(k)$ for which $\phi(x)$ does not lie in a finite union Z'' of proper linear subspaces of $\mathbb{P}_k^{d-1}$. Here Z'' depends on k, S, d, $H_1, \dots, H_q$, ϵ, and c'', but not on x.

Since each set J as above has at most $\dim V$ elements and B is effective, we have

$$(\#J)\lambda_{B,v}(x) \le (\dim V)\lambda_{B,v}(x) + O(1)$$

for all $x \in X'(k)$. Therefore, (15) implies (14) for all $x \in X'(k)$ outside of $Z' := \phi^{-1}(Z'')$. Since the coordinates of ϕ are associated to linearly independent elements of $p^*(V)$, the (closed) set Z' is not all of X'. □

Theorem 2.11. *Let X be a complex projective variety, let D be a Cartier divisor on X, let V be a nonzero linear subspace of $H^0(X, \mathscr{O}(D))$, and let $s_1, \dots, s_q$ be nonzero elements of V. For each $i = 1, \dots, q$, let D_j be the Cartier divisor (s_j), and let λ_{D_j} be a Weil function for D_j. Let $f \colon \mathbb{C} \to X$ be a holomorphic map with Zariski-dense image. Then, for any $\epsilon > 0$,*

$$\int_0^{2\pi} \max_J \sum_{j \in J} \lambda_{D_j}(f(re^{i\theta})) \frac{d\theta}{2\pi} \le_{\text{exc}} (\dim V) T_{f,D}(r) + O(\log^+ T_{f,D}(r)) + o(\log r) .$$

Here the set J ranges over all subsets of $\{1, \dots, q\}$ such that the sections $(s_j)_{j \in J}$ are linearly independent.

The proof of this theorem is very similar to the proof of Theorem 2.10, and is omitted.

Remark 2.12. In Theorems 2.10 and 2.11, if the rational map $X \dashrightarrow \mathbb{P}^{d-1}$ is generically finite, then there is a proper Zariski-closed subset Z_1 of X, depending only on D, V, and $s_1, \dots, s_q$, with the following properties. In Theorem 2.10, the subset Z may be taken to be the union of Z_1 and a finite union of points, and in Theorem 2.11, the condition on the holomorphic curve f may be relaxed to allow any nonconstant holomorphic map $f \colon \mathbb{C} \to \mathbb{P}^n_{\mathbb{C}}$ whose image is not contained in Z_1. Indeed, in the notation of the proof of Theorem 2.10, we may take $Z_1 = p(\phi^{-1}(Z_1') \cup Z_2 \cup \operatorname{Supp} B)$, where $Z_1' \subseteq \mathbb{P}^{d-1}$ is the closed subset of Remark 2.8 and Z_2 is the subset of X' where ϕ is not finite.

Remark 2.13. In Theorems 2.10 and 2.11, we may assume (by shrinking V or using more sections) that $s_1, \dots, s_q$ span V. Under this assumption, we may instead take the maximum over all sets J such that $(s_j)_{j \in J}$ is a basis of V.

3. The birational Nevanlinna constant

Our goal is to prove the equivalence of Definition 1.3 and Definition 1.7. We start by proving the following easy comparison between $\operatorname{Nev}(\mathscr{L}, D)$ and $\operatorname{Nev}_{\text{bir}}(\mathscr{L}, D)$.

Lemma 3.1. *Let X, D, and $\mathscr{L}$ be as in Definitions 1.1 and 1.7. Then*

$$\operatorname{Nev}_{\text{bir}}(\mathscr{L}, D) \le \operatorname{Nev}(\mathscr{L}, D) .$$

Proof. We may assume that X is normal, because in both Definition 1.1 and Definition 1.7 the general case is handled by pulling back to the normalization.

If a triple (N, V, μ) satisfies the condition of Definition 1.1, then it also satisfies the condition of Definition 1.7, because in the latter condition we can take $Y = X$ and let ϕ be the identity map. Thus, the infimum in Definition 1.7 is being taken over a larger set. □

The only difference between Definition 1.1 and Definition 1.7 is that in $\mathrm{Nev}_{\mathrm{bir}}(\mathscr{L}, D)$, the basis $\mathcal{B}$ is allowed to be taken locally on a blowing-up of X, rather than on X itself.

We now show that $\mathrm{Nev}_{\mathrm{bir}}$ can be viewed as a *birationalization* of Nev.

Proposition 3.2. *Let X be a complete variety over a number field k, let D be an effective Cartier divisor on X, and let $\mathscr{L}$ be a line sheaf on X. Then:*

(a). $\mathrm{Nev}_{\mathrm{bir}}(\mathscr{L}, D)$ *is a birational invariant, in the sense that if $\phi\colon Y \to X$ is a model of X (i.e., a proper birational morphism—see Definition 4.1), then*
$$\mathrm{Nev}_{\mathrm{bir}}(\phi^*\mathscr{L}, \phi^*D) = \mathrm{Nev}_{\mathrm{bir}}(\mathscr{L}, D)\ .$$

(b). *For all $\epsilon > 0$ there is a model $\phi\colon Y \to X$ of X such that*
$$\mathrm{Nev}(\phi^*\mathscr{L}, \phi^*D) < \mathrm{Nev}_{\mathrm{bir}}(\mathscr{L}, D) + \epsilon\ .$$

(c). *In particular,*
$$\mathrm{Nev}_{\mathrm{bir}}(\mathscr{L}, D) = \inf_{\phi\colon Y\to X} \mathrm{Nev}(\phi^*\mathscr{L}, \phi^*D)\ ,$$
where the infimum is over all models of X.

Proof. (a). This is clear from Definition 1.7, since the condition on (N, V, μ) remains true if Y is replaced by another variety Y' that dominates Y (i.e., there is a proper birational morphism $Y' \to X$ that factors through Y).

(b). Let (N, V, μ) be a triple satisfying the condition of Definition 1.7, for which
$$\frac{\dim V}{\mu} < \mathrm{Nev}_{\mathrm{bir}}(\mathscr{L}, D) + \epsilon\ ,$$
and let $\phi\colon Y \to X$ be as in Definition 1.7. Then this part follows directly from the fact that (N, ϕ^*V, μ) is a triple that satisfies the condition of Definition 1.1 for $\mathrm{Nev}(\phi^*\mathscr{L}, \phi^*D)$.

(c). This part is immediate from part (b) and from Lemma 3.1. □

To conclude this section, we prove the following Proposition which would lead to the equivalence of the two definitions of $\mathrm{Nev}_{\mathrm{bir}}(D)$.

Proposition 3.3. *Let X be a normal complete variety over a number field, let D be an effective Cartier divisor on X, let $\mathscr{L}$ be a line sheaf on X, let V be a linear subspace of $H^0(X,\mathscr{L})$ with $\dim V > 1$, and let $\mu > 0$ be a rational number.*

Consider the following conditions.

(i) *There exist a variety Y and a proper birational morphism $\phi\colon Y \to X$ such that for all $Q \in Y$ there is a basis $\mathcal{B}$ of V such that*

$$\phi^*(\mathcal{B}) \ge \mu\phi^* D$$

in a Zariski-open neighborhood U of Q, relative to the cone of effective $\mathbb{Q}$-divisors on U.

(ii) *There are finitely many bases $\mathcal{B}_1,\dots,\mathcal{B}_\ell$ of V; Weil functions $\lambda_{\mathcal{B}_1},\dots,\lambda_{\mathcal{B}_\ell}$ for the divisors $(\mathcal{B}_1),\dots,(\mathcal{B}_\ell)$, respectively; a Weil function λ_D for D; and an M_k-constant c such that*

$$\max_{1\le i\le \ell} \lambda_{\mathcal{B}_i} \ge \mu\lambda_D - c \tag{16}$$

(as functions $X(M_k) \to \mathbb{R}\cup\{+\infty\}$).

If (i) is true, then so is (ii).

Proof. Assume that (i) holds.

By quasi-compactness, we may assume that only finitely many open subsets U occur in (i). Let $U_1,\dots,U_\ell$ be a collection of such subsets. Since the condition on $\mathcal{B}$ only depends on U, we may fix a basis $\mathcal{B}_i$ for each open subset U_i. Also, for each i let $\lambda_{\mathcal{B}_i}$ be a Weil function for the divisor $(\mathcal{B}_i)$, and let λ_D be a Weil function for D.

Fix a positive integer n such that $n\mu \in \mathbb{Z}$, and such that the divisor $n\phi^*(\mathcal{B}_i) - n\mu\phi^* D$ is effective on U_i for all i. The above Weil functions can be pulled back to give Weil functions for $\phi^*(\mathcal{B}_i)$ and $\phi^* D$, respectively, on Y.

By Lemma 2.5 applied to the divisors $D_i := n\phi^*(\mathcal{B}_i) - n\mu\phi^* D$ for all i and to the open sets $U_1,\dots,U_\ell$, there is an M_k-constant γ such that

$$\max_{i=1,\dots,\ell} \bigl(n\phi^*\lambda_{\mathcal{B}_i} - n\mu\phi^*\lambda_D\bigr) \ge \gamma\ .$$

Therefore (ii) holds, with $c = -\gamma/n$. □

The converse will be proved in the next section (Proposition 4.13).

4. Models of varieties, b-divisors, and b-Weil functions

In light of the birational nature of $\operatorname{Nev}_{\text{bir}}$, it is useful to consider birationalizations of the definitions of Cartier divisor and Weil function. (Birational variants of Cartier divisors have already been developed as part of the minimal model program.) These allow one to finish the proof of Proposition 3.3, and therefore to show that $\operatorname{Nev}_{\text{bir}}$ can be defined using Weil functions. (This was the original definition of $\operatorname{Nev}_{\text{bir}}$.)

In this section, we define the notion of b-Cartier divisor, and show that the group of these objects (on a fixed variety X over some field), when partially ordered by the condition that $\mathbf{D}_1 \ge \mathbf{D}_2$ if $\mathbf{D}_1 - \mathbf{D}_2$ is effective, forms a lattice (i.e., a partially ordered set in which every nonempty finite set has a least upper bound and a greatest lower bound). We then define the related concept of b-Weil function on X, and show that a corresponding group (obtained by modding out by the subgroup of M_k-bounded functions) is also a lattice, and is naturally isomorphic to the partially ordered group of b-Cartier divisors on X.

Once these b-divisors and b-Weil functions have been defined and their elementary properties discussed, the main result of this section (Proposition 4.13) is stated and proved. This result gives alternative descriptions of the main condition of Definition 1.7 using b-Cartier divisors and using b-Weil functions.

We begin by recalling some definitions from the minimal model program (the Mori program). The notion of b-divisor is originally due to Shokurov; see [Cor07, Def. 1.7.4 and § 2.3] for details. The prefix 'b' stands for *birational.*

Definition 4.1. *Let X be a complete variety over a field k.*

(a). *A **model** of X is a proper birational morphism $Y \to X$ over k, where Y is a variety over k. We often use Y to denote the model.*
(b). *The category of models of X is the category whose objects are models of X and whose morphisms are morphisms over X. We say that a model Y_1 of X **dominates** a model Y_2 of X if there is a morphism $Y_1 \to Y_2$ (necessarily unique) in this category.*
(c). *A **b-Cartier divisor** (resp. **$\mathbb{Q}$-b-Cartier divisor**) on X is an equivalence class of pairs (Y, D), where Y is a model of X and D is a Cartier divisor (resp. $\mathbb{Q}$-Cartier divisor) on Y; here equivalence classes are those for the equivalence relation generated by the relation $(Y_1, D_1) \sim (Y_2, D_2)$ if Y_1 dominates Y_2 via $\phi\colon Y_1 \to Y_2$, and $D_1 = \phi^* D_2$.*
(d). *A b-Cartier divisor or $\mathbb{Q}$-b-Cartier divisor $\mathbf{D}$ on X is **effective** if it is represented by a pair (Y, D) such that D is effective.*

Remark 4.2. Definition 4.1(c) is different from the definition given in [Cor07], but it is equivalent. In *op. cit.*, X is required to be normal, and one works in the category of normal models of X. A b-divisor on X is an element

$$\mathbf{D} = (\mathbf{D}_Y)_Y \in \varprojlim_Y \operatorname{Div}(Y) \;,$$

where $\operatorname{Div}(Y)$ is the group of Weil divisors on a normal model Y of X, and the projective limit is relative to push-forwards $\phi_*\colon \operatorname{Div} Y_1 \to \operatorname{Div} Y_2$ via morphisms $\phi\colon Y_1 \to Y_2$, if Y_1 dominates Y_2 via ϕ. A b-divisor $\mathbf{D}$ on X is b-Cartier if there is a normal model Y of X and a Cartier divisor D on Y such that $\mathbf{D}_{Y_1} = \phi^*D$ for all normal models Y_1 of Y with X-morphisms $\phi\colon Y_1 \to Y$.

To see that this definition is equivalent to Definition 4.1c, we first note that restricting models in Definition 4.1 to normal models does not change the definition. Then, since the normalization of X is a final object in the category of normal models of X, we may assume that X is normal. It is then straightforward to see that this definition agrees with the definition in *op. cit.*

Lemma 4.3. *Let X be a variety, let $\mathbf{D}$ be a b-Cartier divisor on X, and let (Y, D) be a pair that represents $\mathbf{D}$. Assume that Y is normal. Then $\mathbf{D}$ is effective if and only if D is effective.*

Proof. The reverse implication is immediate from the definitions.

To prove the forward implication, assume that $\mathbf{D}$ is effective. By definition, there is a pair (Y', D') representing $\mathbf{D}$ such that D' is effective. By pulling back to a possibly larger model, we may assume that Y' dominates Y, say by $f\colon Y' \to Y$, and that Y' is normal. Then $D = f_*D'$ (here f_* refers to Weil divisors; note that $f_*f^*D = D$). In particular, by Remark 2.3, D is effective. □

The following definition generalizes the definition of Weil function to b-Cartier divisors. This definition comes from [Voj96, § 7]. In *loc. cit.* they were called *generalized Weil functions*, but it is now apparent that it is more natural to call them *b-Weil functions*.

Definition 4.4. *Let X be a complete variety over a number field k. Then a **b-Weil function** on X (resp. a $\mathbb{Q}$-**b-Weil function** on X) is an equivalence class of pairs (U, λ), where U is a nonempty Zariski-open subset of X and $\lambda\colon U(M_k) \to \mathbb{R}$ is a function such that there exist a model $\phi\colon Y \to X$ of X and a Cartier divisor (resp. $\mathbb{Q}$-Cartier divisor) D on Y such that $\lambda \circ \phi$ extends to a Weil function for D (resp. such that $n\lambda \circ \phi$ extends to a Weil function for nD for some (and hence all) nonzero integers n for which nD*

*is a Cartier divisor). Pairs (U,λ) and (U',λ') are **equivalent** if $\lambda=\lambda'$ on $(U\cap U')(M_k)$. Local b-Weil functions and local $\mathbb{Q}$-b-Weil functions on X are defined similarly.*

It is clear that every Weil function on a variety X over k is also a b-Weil function on X, and that b-Weil functions on X form an abelian group under addition. Also, if $\phi\colon X \dashrightarrow Y$ is a dominant rational map and if λ is a b-Weil function on Y, then $\phi^*\lambda$ (defined in the obvious way) is a b-Weil function on X. The same facts are true for local b-Weil functions at a given place v.

Definition 4.5. *Let X be a complete variety over a number field k, let λ be a b-Weil function on X, and let $\mathbf{D}$ be a b-Cartier divisor on X. We say that λ is a **b-Weil function for** $\mathbf{D}$ if $\mathbf{D}$ is represented by a pair (Y,D) as in Definition 4.1, such that if $\phi\colon Y\to X$ is the structural morphism of Y, then $\lambda\circ\phi$ extends to a Weil function for D on Y.*

Proposition 4.6. *Let X be a complete variety over a number field k.*

- (a). *For $i=1,2$ let $\mathbf{D}_i$ be a b-Cartier divisor on X and let λ_i be a b-Weil function for $\mathbf{D}_i$. Then $-\lambda_1$ and $\lambda_1+\lambda_2$ are b-Weil functions for $-\mathbf{D}_1$ and $\mathbf{D}_1+\mathbf{D}_2$, respectively.*
- (b). *Let $\mathbf{D}$ be a b-Cartier divisor and let λ be a b-Weil function on X for $\mathbf{D}$. Then λ is M_k-bounded from below if and only if $\mathbf{D}$ is effective, and λ is M_k-bounded if and only if $\mathbf{D}=0$.*
- (c). *Let λ be a b-Weil function on X. Then there is a unique b-Cartier divisor $\mathbf{D}$ such that λ is a b-Weil function for $\mathbf{D}$.*
- (d). *Let $\mathbf{D}$ be a b-Cartier divisor on X. Then there is a b-Weil function λ for $\mathbf{D}$.*
- (e). *The map $\lambda\mapsto\mathbf{D}$ in part (c) gives a group isomorphism from the group of b-Weil functions on X, modulo addition of M_k-bounded functions, to the group of b-Cartier divisors on X.*

Analogous statements hold for local b-Weil functions at a fixed place v of k.

Proof. (a). For each i let $\phi_i\colon Y_i\to X$ be a model of X and let D_i be a Cartier divisor on Y_i such that $\phi_i\circ\lambda_i$ extends to a Weil function on Y_i for D_i, and such that (Y_i,D_i) represents $\mathbf{D}_i$. Then the first assertion is immediate, since $-\phi_1\circ\lambda_1$ extends to a Weil function for $-D_1$ on Y_1. For the second assertion, we may replace Y_1 and Y_2 with a model Y for X that dominates both of them, and pull back D_1 and D_2 to Y. Then the assertion follows from additivity of Weil functions on Y.

(b). Let $\phi\colon Y\to X$ be a model of X and let D be a Cartier divisor on Y such that $\phi\circ\lambda$ extends to a Weil function for D on Y, and such that (Y,D) represents $\mathbf{D}$. By replacing Y with its normalization, we may assume that Y

is normal. Then λ is M_k-bounded from below if and only if D is effective, by Proposition 2.4. If D is effective, then so is $\mathbf{D}$. Conversely, if $\mathbf{D}$ is effective, then it is represented by a pair (Y', D') with D' effective. Let Y'' be a normal model of X that dominates both Y and Y', and let $\psi\colon Y'' \to Y$ and $\psi'\colon Y'' \to Y'$ be the implied morphisms. Then $D = \psi_*(\psi')^*D'$ is effective, so λ is M_k-bounded from below.

The second assertion follows formally by applying the first assertion also to $-\lambda$ and $-\mathbf{D}$.

(c). Let λ be a b-Weil function on X. By Definition 4.4 there exist a model $\phi\colon Y \to X$ for X and a Cartier divisor D on Y such that $\lambda \circ \phi$ extends to a Weil function for D on Y. Then λ is a b-Weil function for the b-Cartier divisor $\mathbf{D}$ represented by the pair (Y, D). To show uniqueness, suppose that λ is a b-Weil function for b-Cartier divisors $\mathbf{D}_1$ and $\mathbf{D}_2$. Then $\lambda - \lambda = 0$ is a b-Weil function for $\mathbf{D}_1 - \mathbf{D}_2$. This divisor must be zero, by part (b) applied to $\pm(\mathbf{D}_1 - \mathbf{D}_2)$.

(d). Let $\mathbf{D}$ be a b-Cartier divisor, and let (Y, D) be a pair representing it. By Chow's lemma, we may assume that Y is projective. By [Lan83, Ch. 10, Thm. 3.5], there is a Weil function for D on Y. This defines a b-Weil function for $\mathbf{D}$.

(e). Part (c) determines a well-defined function from the group of b-Weil functions on X to the group of b-Cartier divisors on X. Part (a) implies that it is a group homomorphism, part (b) implies that its kernel is the subgroup of M_k-bounded b-Weil functions, and part (d) implies that it is surjective.

The proofs of corresponding statements for local b-Weil functions are left to the reader. □

The main reason for defining b-Weil functions in [Voj96] was the fact that the (pointwise) maximum of two Weil functions may not be a Weil function, but the maximum of two b-Weil functions is another b-Weil function. The main result of this section shows that the group of b-Cartier divisors on a variety also has a least upper bound, relative to the cone of effective b-Cartier divisors, and that this lub corresponds to the maximum of b-Weil functions.

We start with a lemma on the underlying geometry of a least upper bound of b-Cartier divisors. It will not be used until later (Proposition 4.13), but it is stated here because it provides valuable intuition.

Lemma 4.7. *Let X be a variety over a field, and let $\mathbf{D}$ and $\mathbf{D}_1, \dots, \mathbf{D}_\ell$ be b-Cartier divisors on X. Let $\phi\colon Y \to X$ be a normal model of X such that $\mathbf{D}$ and $\mathbf{D}_1, \dots, \mathbf{D}_\ell$ are represented by Cartier divisors D and $D_1, \dots, D_\ell$ on*

Y, respectively. Then $\mathbf{D}$ is a least upper bound of $\mathbf{D}_1,\dots,\mathbf{D}_\ell$ if and only if $D-D_i$ is effective for all i and

$$\bigcap_{i=1}^{\ell}\operatorname{Supp}(D-D_i)=\emptyset\,.$$

Proof. By Lemma 4.3, $\mathbf{D}$ is an upper bound of $\mathbf{D}_1,\dots,\mathbf{D}_\ell$ if and only if $D-D_i$ is effective for all i.

Let $Z=\bigcap\operatorname{Supp}(D-D_i)$ and suppose that $Z\ne\emptyset$. Let $f\colon Y'\to Y$ be the blowing-up of Y along Z, and let E be the exceptional divisor. Then E is a nonzero effective Cartier divisor, and $f^*(D-D_i)-E$ is effective for all i. This shows that (Y',f^*D-E) represents another upper bound of $\mathbf{D}_1,\dots,\mathbf{D}_\ell$, and therefore $\mathbf{D}$ is not a *least* upper bound. Thus $Z=\emptyset$. ☐

This next lemma shows that the above situation is not uncommon. It is taken from the proof of [Voj96, Prop. 7.3].

Lemma 4.8. *Let D be a Cartier divisor on a variety Y over a field k. Then there exist a proper model $\phi\colon Z\to Y$ and effective Cartier divisors D' and D'' on Z such that $\phi^*D=D'-D''$, and such that the supports of D' and D'' are disjoint.*

Proof. If U is an open subset of Y on which D is equal to a principal divisor (f), then define Z_U to be the closure of the graph of the rational function $U\dashrightarrow\mathbb{P}^1$ given by f, and let D' and D'' be the pull-backs of the divisors $[0]$ and $[\infty]$ on $\mathbb{P}^1$, respectively. This construction is compatible with restricting to an open subset U, and multiplying f by an element of $\mathscr{O}_U^*$ induces an automorphism of Z_U that fixes D' and D''. Therefore the schemes Z_U and divisors D' and D'' on Z_U glue together to give a scheme Z, proper over Y, and Cartier divisors D' and D'' on Z, that satisfy the conditions of the lemma. ☐

For the next step, we recall that a **lattice** is a partially ordered set in which every pair of elements a,b has a least upper bound and a greatest lower bound. These are called the **join** and **meet**, respectively, and are denoted $a\vee b$ and $a\wedge b$, respectively.

The following definition comes from [St10, Ch. 2].

Definition 4.9. *A **lattice-ordered group** is a group G, together with a partial ordering on G that respects the group operation (i.e., $x\le y\iff xz\le yz\iff zx\le zy$ for all $x,y,z\in G$), such that the partial ordering forms a lattice.*

In this paper, all lattice-ordered groups are abelian, and are written additively.

Proposition 4.10. *Let X be a complete variety over a field k.*

(a). *Let the set of b-Cartier divisors on X be partially ordered by the relation $\mathbf{D}_1 \le \mathbf{D}_2$ if $D_2 - D_1$ is effective. Then the group of b-Cartier divisors on X is a lattice-ordered group.*

(b). *Assume that k is a number field. Let G be the group of b-Weil functions on X, modulo the set of M_k-bounded functions. Let G be partially ordered by the condition that $\lambda_1 \le \lambda_2$ if $\lambda_2 - \lambda_1$ is M_k-bounded from below. Then G is isomorphic to the partially ordered group of b-Cartier divisors on X under the isomorphism of Proposition 4.6. In particular, it is a lattice-ordered group.*

(c). *Assume that k is a number field, and let G be the group of part (b). Let λ_1 and λ_2 be b-Weil functions for b-Cartier divisors $\mathbf{D}_1$ and $\mathbf{D}_2$, respectively, on X. Then the function $\max\{\lambda_1, \lambda_2\}$ is a b-Weil function for the b-Cartier divisor $\mathbf{D}_1 \vee \mathbf{D}_2$, and its image in G is the join of the images of λ_1 and λ_2 in G.*

Proof. (a). That the group is a partially ordered group is clear from the definition of the ordering.

To check that it is a lattice, by the group property it suffices to check that for any b-Cartier divisor $\mathbf{D}$ on X, the pair $\mathbf{D}, \mathbf{0}$ has a least upper bound. To do this, let (Y, D) be a representative for $\mathbf{D}$. By replacing Y with the model constructed in Lemma 4.8, we may assume that $D = D' - D''$, where D' and D'' are effective Cartier divisors with disjoint supports. We may also assume that Y is normal. Then (Y, D') represents a least upper bound for the pair $\mathbf{D}, \mathbf{0}$. Indeed, this is true by Lemma 4.7, because the divisors $D' - D = D''$ and $D' - 0$ are effective divisors with disjoint supports.

(b). Part (b) of Proposition 4.6 implies that the group isomorphism preserves the ordering, so G is a lattice-ordered group.

(c). Again, we may assume that $\lambda_2 = 0$. Then $\mathbf{D}_2 = \mathbf{0}$. As in the proof of (a), we may let (Y, D) be a representative for $\mathbf{D}_1$, and may assume that Y is normal and that $D = D' - D''$, where D' and D'' are effective with disjoint supports. Then (Y, D') represents $\mathbf{D}_1 \vee \mathbf{0}$.

Let $\phi \colon Y \to X$ be the structural morphism of Y. Then $\phi^*\lambda_1$ is a Weil function for D. By [Lan83, Ch. 10, Prop. 3.2], $\max\{\phi^*\lambda_1, 0\}$ is a Weil function for D', and therefore $\max\{\lambda_1, 0\}$ is a b-Weil function for the b-Cartier divisor represented by the pair (Y, D'). Thus, it is a b-Weil function for $\mathbf{D}_1 \vee \mathbf{0}$. This proves the first assertion. The other assertion then follows from (b). □

Remark 4.11. Many definitions and results so far in this section have been stated only for b-Cartier divisors and b-Weil functions, but they all extend in an obvious way to $\mathbb{Q}$-b-Cartier divisors and $\mathbb{Q}$-b-Weil functions.

We now can give some equivalent formulations of Definition 1.7 using b-divisors and b-Weil functions. We start with a definition that focuses on the part of the definition of $\mathrm{Nev}_{\mathrm{bir}}$ that varies.

Definition 4.12. *Let X be a normal complete variety, let D be an effective Cartier divisor on X, let $\mathscr{L}$ be a line sheaf on X, let V be a linear subspace of $H^0(X,\mathscr{L})$ with $\dim V > 1$, and let $\mu > 0$ be a rational number. We say that D has **μ-b-growth with respect to V and $\mathscr{L}$** if there is a model $\phi\colon Y \to X$ of X such that for all $Q \in Y$ there is a basis $\mathcal{B}$ of V such that*

$$\phi^*(\mathcal{B}) \ge \mu\phi^* D \tag{17}$$

*in a Zariski-open neighborhood U of Q, relative to the cone of effective $\mathbb{Q}$-divisors on U. Also, we say that D has **μ-b-growth with respect to V** if it satisfies the above condition with $\mathscr{L} = \mathscr{O}(D)$.*

Then Definition 1.7 basically says that $\mathrm{Nev}_{\mathrm{bir}}(D)$ is the infimum of $(\dim V)/\mu$ over all triples (N, V, μ) such that ND has μ-b-growth with respect to V and $\mathscr{L}^D$. (The corresponding condition for Definition 1.1 is called μ-growth; see [Ru17]. The proof of Lemma 3.1 then amounts to saying that if D has μ-growth with respect to V, then it also has μ-b-growth with respect to V.)

The following proposition completes Proposition 3.3 (and adds more equivalent conditions).

Proposition 4.13. *Let X be a normal complete variety, let D be an effective Cartier divisor on X, let $\mathscr{L}$ be a line sheaf on X, let V be a linear subspace of $H^0(X,\mathscr{L})$ with $\dim V > 1$, and let $\mu > 0$ be a rational number. Then the following are equivalent.*

(i) *D has μ-b-growth with respect to V and $\mathscr{L}$.*
(ii) *There are bases $\mathcal{B}_1, \dots, \mathcal{B}_\ell$ of V such that*

$$\bigvee_{i=1}^{\ell} (\mathcal{B}_i) \ge \mu D \tag{18}$$

(relative to the cone of effective $\mathbb{Q}$-b-Cartier divisors).
(iii) *There are bases $\mathcal{B}_1, \dots, \mathcal{B}_\ell$ of V; Weil functions $\lambda_{\mathcal{B}_1}, \dots, \lambda_{\mathcal{B}_\ell}$ for the divisors $(\mathcal{B}_1), \dots, (\mathcal{B}_\ell)$, respectively; a Weil function λ_D for D; and an M_k-constant c such that*

$$\max_{1\le i\le \ell} \lambda_{\mathcal{B}_i} \ge \mu\lambda_D - c \tag{19}$$

(as functions $X(M_k) \to \mathbb{R} \cup \{+\infty\}$).

(iv) *For each place $v \in M_k$ there are finitely many bases $\mathcal{B}_1, \dots, \mathcal{B}_\ell$ of V; local Weil functions $\lambda_{\mathcal{B}_1,v}, \dots, \lambda_{\mathcal{B}_\ell,v}$ for the divisors $(\mathcal{B}_1), \dots, (\mathcal{B}_\ell)$, respectively, at v; a local Weil function $\lambda_{D,v}$ for D at v; and a constant c such that*

$$\max_{1\le i\le \ell} \lambda_{\mathcal{B}_i,v} \ge \mu\lambda_{D,v} - c \tag{20}$$

(as functions $X(\overline{k}_v) \to \mathbb{R} \cup \{+\infty\}$).

(v) *The condition of (iv) holds for at least one place v.*

Proof. Conditions (ii) and (iii) are equivalent by Proposition 4.10 and Remark 4.11. Conditions (iii)–(v) are equivalent by Proposition 2.4. The implication (i) $\Longrightarrow$ (iii) is Proposition 3.3. Finally, (ii) $\Longrightarrow$ (i) follows from Lemma 4.14 (below), with $D_i = (\mathcal{B}_i)$ for all i. □

Lemma 4.14. *Let X be a normal complete variety, and let $\mu > 0$ be a rational number. Let D and $D_1, \dots, D_\ell$ be $\mathbb{Q}$-Cartier divisors on X. Assume that*

$$\bigvee_{i=1}^{\ell} D_i \ge \mu D \tag{21}$$

relative to the cone of effective $\mathbb{Q}$-b-Cartier divisors. Then there is a model $\phi\colon Y \to X$ of X such that for all $Q \in Y$ there is an index i such that $\phi^*D_i \ge \mu\phi^*D$ in a Zariski-open neighborhood U of Q, relative to the cone of effective $\mathbb{Q}$-divisors on U.

Proof. Assume that (21) is true. After multiplying D and all D_i by some positive integer n, we may assume that they are all (integral) Cartier divisors. Let $\mathbf{E} = \bigvee D_i$, and let $\phi\colon Y \to X$ be a normal model of X such that $\mathbf{E}$ is represented by a Cartier divisor E on Y. By Lemma 4.7, $E - \phi^*D_i$ is effective for all i, and $\bigcap \operatorname{Supp}(E - \phi^*D_i) = \emptyset$. Therefore, for any given $Q \in Y$ there is an index i such that $Q \notin \operatorname{Supp}(E - \phi^*D_i)$. Fix such an i, and let $U_i = Y \setminus \operatorname{Supp}(E - \phi^*D_i)$. Then $Q \in U_i$. Moreover, by (21),

$$\phi^*D_i\big|_{U_i} = E\big|_{U_i} \ge \mu\phi^*D\big|_{U_i}$$

relative to the cone of effective divisors on U_i. Therefore $\phi^*D_i \ge \mu\phi^*D$ on U_i relative to the cone of effective $\mathbb{Q}$-divisors on U_i, as was to be shown. □

Remark 4.15. In light of Proposition 3.2, one may regard (6) or (18) as being conditions that are *local on the Zariski–Riemann space*, as opposed to (1), which is local in the Zariski topology. (The Zariski–Riemann space of a complete variety X is the inverse limit of all models of X [ZS60, Ch. VI, § 17].)

Proposition 4.13 leads to the following corollaries on equivalence of definitions of $\operatorname{Nev}_{\text{bir}}(\mathscr{L}, D)$.

Corollary 4.16. *Definitions 1.3 and 1.7 are equivalent.*

Corollary 4.17. *Let X be a normal complete variety, let D be an effective Cartier divisor on X, and let $\mathscr{L}$ be a line sheaf on X. Then*

$$\operatorname{Nev}_{\text{bir}}(\mathscr{L}, D) = \inf_{N,V,\mu} \frac{\dim V}{\mu} ,$$

where the infimum passes over all triples (N, V, μ) such that $N \in \mathbb{Z}_{>0}$, V is a linear subspace of $H^0(X, \mathscr{L}^N)$ with $\dim V > 1$, and $\mu \in \mathbb{Q}_{>0}$, with the following property. There are finitely many bases $\mathcal{B}_1, \dots, \mathcal{B}_\ell$ of V such that

$$\bigvee_{i=1}^{\ell} (\mathcal{B}_i) \ge \mu N D . \tag{22}$$

(Here we use the same convention as in Definition 1.7 when there are no triples (N, V, μ) that satisfy the condition.)

Similar corollaries also hold for conditions (iv) and (v) of Proposition 4.13.

The following proposition will be used in the proof of the General Theorem in Section 6.

Proposition 4.18. *Let X be a variety, let $\mathscr{L}$ be a line sheaf on X, and let $E_1, \dots, E_m$ be effective Cartier divisors on X. Let s be a nonzero global section of $\mathscr{L}$ lying in the (coherent) subsheaf of $\mathscr{L}$ generated by $\{\mathscr{L}(-E_j) : j = 1, \dots, m\}$. Then*

$$(s) \ge \bigwedge_{j=1}^{m} E_j . \tag{23}$$

Proof. Let $\mathbf{E} = \bigwedge E_j$, and let $\phi \colon Y \to X$ be a model of X on which $\mathbf{E}$ is represented by a Cartier divisor E. Since $\phi^* E_j - E$ is effective for all j, the sheaf $\phi^*\mathscr{L}(-E)$ contains the sheaves $\phi^*(\mathscr{L}(-E_j))$ for all j, and therefore $\phi^* s$ is a global section of $\phi^*\mathscr{L}(-E)$.

This implies that the divisor $\phi^*(s) - E$ is effective, which implies (23). □

5. The Proof of Theorems 1.4 and 1.5 for $\operatorname{Nev}_{\text{bir}}(\mathscr{L}, D)$

In this section, we prove Theorems 1.4 and 1.5, which are the variations of Theorems A and B with $\operatorname{Nev}(D)$ replaced by $\operatorname{Nev}_{\text{bir}}(\mathscr{L}, D)$. We will only prove Theorem 1.5 (the number field case), since the proof of Theorem 1.4 is very similar. However, for this theorem we will give both a complete proof, based on Proposition 5.6 and Theorem B with $\operatorname{Nev}(\mathscr{L}, D)$ (note that Theorem B still holds, with the same proof, if $\operatorname{Nev}(D)$ is replaced by $\operatorname{Nev}(\mathscr{L}, D)$), and a sketch of how to prove the theorem directly, based on the proof of Theorem B in [Ru17].

Proof of Theorem 1.5. Let k, S, X, and D be as in the statement of the theorem, and let $\epsilon > 0$ be given. By Proposition 3.2b, there is a model $\phi\colon Y \to X$ of X such that

$$\operatorname{Nev}(\phi^*\mathscr{L}, \phi^*D) < \operatorname{Nev}_{\mathrm{bir}}(\mathscr{L}, D) + \epsilon\ . \tag{24}$$

Let $Z_0 \subseteq Y$ be the ramification locus of ϕ.

By Theorem B with $\operatorname{Nev}(\mathscr{L}, D)$, there is a proper Zariski-closed subset Z_1 of Y such that the inequality

$$m_S(y, \phi^*D) \le (\operatorname{Nev}(\phi^*\mathscr{L}, \phi^*D) + \epsilon)\, h_{\phi^*\mathscr{L}}(y) \tag{25}$$

holds for all $y \in Y(k)$ outside of Z_1.

By functoriality of proximity functions, (25), (24), and functoriality of heights, we then have

$$\begin{aligned} m_S(x, D) &= m_S(\phi^{-1}(x), \phi^*D) + O(1) \\ &\le (\operatorname{Nev}(\phi^*\mathscr{L}, \phi^*D) + \epsilon)\, h_{\phi^*\mathscr{L}}(\phi^{-1}(x)) + O(1) \\ &\le (\operatorname{Nev}_{\mathrm{bir}}(\mathscr{L}, D) + 2\epsilon)\, h_{\phi^*\mathscr{L}}(\phi^{-1}(x)) + O(1) \\ &= (\operatorname{Nev}_{\mathrm{bir}}(\mathscr{L}, D) + 2\epsilon)\, h_{\mathscr{L}}(x) + O(1) \end{aligned}$$

for all $x \in X(k)$ outside of $Z := \phi(Z_0 \cup Z_1)$. (Note that this set is closed since ϕ is proper, and that ϕ induces an isomorphism over $X \setminus Z$ since ϕ is unramified over that set.) □

We now indicate how Theorem 1.5 can be proved using the methods of [Ru17, Sect. 2]. According to the argument as appears in the end of Sect. 2 of [Ru17], we only need to show that, assuming that ND has μ-b-growth with respect to $V \subseteq H^0(X, \mathscr{L}^N)$ and $\mathscr{L}$, for each $\epsilon > 0$, there is a proper Zariski-closed subset Z of X such that the inequality

$$m_S(x, D) \le \left(\frac{\dim V}{\mu} + \epsilon\right) h_{\mathscr{L}}(x) \tag{26}$$

holds for all $x \in X(k)$ outside of Z. Indeed, by Corollary 4.16, there are bases $\mathcal{B}_1, \dots, \mathcal{B}_\ell$ of V; Weil functions $\lambda_{\mathcal{B}_1}, \dots, \lambda_{\mathcal{B}_\ell}$ for the divisors $(\mathcal{B}_1), \dots, (\mathcal{B}_\ell)$, respectively; a Weil function λ_D for D; and an M_k-constant c such that

$$\max_{1 \le i \le \ell} \lambda_{\mathcal{B}_i} \ge \mu N \lambda_D - c \tag{27}$$

(as functions $X(M_k) \to \mathbb{R} \cup \{+\infty\}$).

Write

$$\bigcup_{i=1}^{\ell} \mathcal{B}_i = \{s_1, \dots, s_q\} .$$

and for each $j = 1, \dots, q$ choose a Weil function λ_{s_j} for the divisor (s_j). For each $i = 1, \dots, \ell$, let $J_i \subseteq \{1, \dots, q\}$ be the subset such that $\mathcal{B}_i = \{s_j : j \in J_i\}$. Then, by (27), for each $v \in S$ there are constants c_v and c'_v such that

$$\mu N \lambda_{D,v} \le \max_{1\le i\le \ell} \lambda_{\mathcal{B}_i,v} + c'_v \le \max_{1\le i\le \ell} \sum_{j\in J_i} \lambda_{s_j,v} + c_v . \tag{28}$$

By Schmidt's Subspace Theorem in the form of Theorem 2.10, there is a proper Zariski-closed subset Z of X such that the inequality

$$\sum_{v\in S} \max_J \sum_{j\in J} \lambda_{s_j,v}(x) \le (\dim V + \epsilon) h_{\mathscr{L}^N}(x) \tag{29}$$

holds for all $x \in X(k)$ outside of Z; here the maximum is taken over all subsets J of $\{1, \dots, q\}$ for which the sections s_j, $j \in J$, are linearly independent.

Combining (28) and (29) gives

$$\begin{aligned} \mu N m_S(D, x) &= \mu N \sum_{v\in S} \lambda_{D,v}(x) + O(1) \le \sum_{v\in S} \max_{1\le i\le \ell} \sum_{j\in J_i} \lambda_{s_j,v}(x) + O(1) \\ &\le \sum_{v\in S} \max_J \sum_{j\in J} \lambda_{s_j,v}(x) + O(1) \le (\dim V + \epsilon) h_{\mathscr{L}^N}(x) + O(1) \\ &= N(\dim V + \epsilon) h_{\mathscr{L}}(x) + O(1) \end{aligned}$$

for all $x \in X(k)$ outside of Z. Here we used the fact that all of the J_i occur among the J in (29). Hence (26) holds.

6. The General Theorem

In this section, we prove the General Theorem (in both the arithmetic and analytic cases) by using the filtration method of Corvaja and Zannier [CZ04b], as further developed by Autissier [Aut11]. We first review Autissier's results.

Let $D_1, \dots, D_r$ be effective Cartier divisors on a projective variety X. Assume that they intersect properly on X, and that $\bigcap_{i=1}^r D_i$ is non-empty. Let $\mathscr{L}$ be a line sheaf over X with $l := h^0(\mathscr{L}) \ge 1$.

Definition 6.1. *A subset $N \subset \mathbb{N}^r$ is said to be* ***saturated*** *if $\mathbf{a} + \mathbf{b} \in N$ for any $\mathbf{a} \in \mathbb{N}^r$ and $\mathbf{b} \in N$.*

Lemma 6.2 (Lemma 3.2, [Aut11]). *Let A be a local ring and $(\phi_1, \dots, \phi_r)$ be a regular sequence of A. Let M and N be two saturated subsets of $\mathbb{N}^r$. Then*

$$\mathcal{I}(M) \cap \mathcal{I}(N) = \mathcal{I}(M \cap N),$$

where, for $N \subset \mathbb{N}^r$, $\mathcal{I}(N)$ is the ideal of A generated by $\{\phi_1^{b_1} \cdots \phi_q^{b_r} \mid \mathbf{b} \in N\}$.

Remark 6.3. We use Lemma 6.2 in the following particular situation: Let $\square = (\mathbb{R}^+)^r \setminus \{\mathbf{0}\}$. For each $\mathbf{t} \in \square$ and $x \in \mathbb{R}^+$, let

$$N(\mathbf{t}, x) = \{\mathbf{b} \in \mathbb{N}^r \mid t_1 b_1 + \dots + t_r b_r \geq x\}.$$

Notice that $N(\mathbf{t}, x) \cap N(\mathbf{u}, y) \subset N(\lambda\mathbf{t} + (1-\lambda)\mathbf{u}, \lambda x + (1-\lambda) y)$ for all $\lambda \in [0, 1]$. So, from Lemma 6.2, we have

$$\mathcal{I}(N(\mathbf{t}, x)) \cap \mathcal{I}(N(\mathbf{u}, y)) \subset \mathcal{I}(N(\lambda\mathbf{t} + (1-\lambda)\mathbf{u}, \lambda x + (1-\lambda)y)) \tag{30}$$

for any $\mathbf{t}, \mathbf{u} \in \square$; $x, y \in \mathbb{R}^+$; and $\lambda \in [0, 1]$.

Definition 6.4. *Let W be a vector space of finite dimension. A* ***filtration*** *of W is a family of subspaces $\mathcal{F} = (\mathcal{F}_x)_{x \in \mathbb{R}^+}$ of subspaces of W such that $\mathcal{F}_x \supseteq \mathcal{F}_y$ whenever $x \leq y$, and such that $\mathcal{F}_x = \{0\}$ for x big enough. A basis $\mathcal{B}$ of W is said to be* ***adapted to*** *$\mathcal{F}$ if $\mathcal{B} \cap \mathcal{F}_x$ is a basis of $\mathcal{F}_x$ for every real number $x \geq 0$.*

Lemma 6.5 (Corvaja–Zannier [CZ04b, Lemma 3.2], Levin [Lev09], Autissier [Aut11]). *Let $\mathcal{F}$ and $\mathcal{G}$ be two filtrations of W. Then there exists a basis of W which is adapted to both $\mathcal{F}$ and $\mathcal{G}$.*

For any fixed $\mathbf{t} \in \square$, we construct a filtration of $H^0(X, \mathscr{L})$ as follows: for $x \in \mathbb{R}^+$, one defines the ideal $\mathcal{I}(\mathbf{t}, x)$ of $\mathscr{O}_X$ by

$$\mathcal{I}(\mathbf{t}, x) = \sum_{\mathbf{b} \in N(\mathbf{t}, x)} \mathscr{O}_X\Big(-\sum_{i=1}^r b_i D_i\Big) \ , \tag{31}$$

and let

$$\mathcal{F}(\mathbf{t})_x = H^0(X, \mathscr{L} \otimes \mathcal{I}(\mathbf{t}, x)) \ . \tag{32}$$

Then $(\mathcal{F}(\mathbf{t})_x)_{x \in \mathbb{R}^+}$ is a filtration of $H^0(X, \mathscr{L})$.

For $s \in H^0(X, \mathscr{L}) - \{0\}$, let $\mu_{\mathbf{t}}(s) = \sup\{y \in \mathbb{R}^+ \mid s \in \mathcal{F}(\mathbf{t})_y\}$. Also let

$$F(\mathbf{t}) = \frac{1}{h^0(\mathscr{L})} \int_0^{+\infty} (\dim \mathcal{F}(\mathbf{t})_x) \, dx \ . \tag{33}$$

Note that, for all $u > 0$ and all $\mathbf{t} \in \square$, we have $N(u\mathbf{t}, x) = N(\mathbf{t}, u^{-1}x)$, which implies $\mathcal{F}(u\mathbf{t})_x = \mathcal{F}(\mathbf{t})_{u^{-1}x}$, and therefore

$$F(u\mathbf{t}) = \int_0^\infty \frac{\dim \mathscr{F}(\mathbf{t})_{u^{-1}x}}{h^0(\mathscr{L})} \, dx = u \int_0^\infty \frac{\dim \mathscr{F}(\mathbf{t})_y}{h^0(\mathscr{L})} \, dy = uF(\mathbf{t}) \ . \tag{34}$$

Remark 6.6. Let $\mathcal{B} = \{s_1, \ldots, s_l\}$ be a basis of $H^0(X, \mathscr{L})$ with $l = h^0(\mathscr{L})$. Then we have

$$F(\mathbf{t}) \geq \frac{1}{l} \int_0^\infty \#(\mathcal{F}(\mathbf{t})_x \cap \mathcal{B}) dx = \frac{1}{l} \sum_{k=1}^{l} \mu_{\mathbf{t}}(s_k),$$

where equality holds if $\mathcal{B}$ is adapted to the filtration $(\mathcal{F}(\mathbf{t})_x)_{x \in \mathbb{R}^+}$.

The key result we will use about this filtration is the following Proposition.

Proposition 6.7 (Théorème 3.6 in [Aut11])**.** *With the notations and assumptions above, let $F : \square \to \mathbb{R}^+$ be the map defined in (33). Then F is concave. In particular, for all $\beta_1, \ldots, \beta_r \in (0, \infty)$ and all $\mathbf{t} \in \square$ satisfying $\sum \beta_i t_i = 1$,*

$$F(\mathbf{t}) \geq \min_i \left(\frac{1}{\beta_i} \sum_{m \geq 1} \frac{h^0(\mathscr{L}(-mD_i))}{h^0(\mathscr{L})} \right) . \tag{35}$$

We include a proof here for the sake of completeness.

Proof. For any $\mathbf{t}, \mathbf{u} \in \square$ and $\lambda \in [0, 1]$, we need to prove that

$$F(\lambda \mathbf{t} + (1 - \lambda)\mathbf{u}) \geq \lambda F(\mathbf{t}) + (1 - \lambda) F(\mathbf{u}). \tag{36}$$

By Lemma 6.5, there exists a basis $\mathcal{B} = \{s_1, \ldots, s_l\}$ of $H^0(X, \mathscr{L})$ with $l = h^0(\mathscr{L})$, which is adapted both to $(\mathcal{F}(\mathbf{t})_x)_{x \in \mathbb{R}^+}$ and to $(\mathcal{F}(\mathbf{u})_y)_{y \in \mathbb{R}^+}$. For $x, y \in \mathbb{R}^+$, by Lemma 6.2 (or Remark 6.3), since $D_1, \ldots, D_r$ intersect properly on X,

$$\mathcal{F}(\mathbf{t})_x \cap \mathcal{F}(\mathbf{u})_y \subset \mathcal{F}(\lambda \mathbf{t} + (1 - \lambda)\mathbf{u})_{\lambda x + (1 - \lambda) y}.$$

For $s \in H^0(X, \mathscr{L}) - \{0\}$, we have, from the definition of $\mu_{\mathbf{t}}(s)$ and $\mu_{\mathbf{u}}(s)$, $s \in \mathcal{F}(\lambda \mathbf{t} + (1 - \lambda)\mathbf{u})_{\lambda x + (1 - \lambda) y}$ for $x < \mu_{\mathbf{t}}(s)$ and $y < \mu_{\mathbf{u}}(s)$, and thus

$$\mu_{\lambda \mathbf{t} + (1 - \lambda)\mathbf{u}}(s) \geq \lambda \mu_{\mathbf{t}}(s) + (1 - \lambda) \mu_{\mathbf{u}}(s).$$

Taking $s = s_j$ and summing it over $j = 1, \ldots, l$, we get, by Remark 6.6,

$$F(\lambda \mathbf{t} + (1 - \lambda)\mathbf{u}) \geq \lambda \frac{1}{l} \sum_{j=1}^{l} \mu_{\mathbf{t}}(s_j) + (1 - \lambda) \frac{1}{l} \sum_{j=1}^{l} \mu_{\mathbf{u}}(s_j).$$

On the other hand, since $\mathcal{B} = \{s_1, \ldots, s_l\}$ is a basis adapted to both $\mathcal{F}(\mathbf{t})$ and $\mathcal{F}(\mathbf{u})$, from Remark 6.6, $F(\mathbf{t}) = \frac{1}{l} \sum_{j=1}^{l} \mu_{\mathbf{t}}(s_j)$ and $F(\mathbf{u}) = \frac{1}{l} \sum_{j=1}^{l} \mu_{\mathbf{u}}(s_j)$. Thus

$$F(\lambda \mathbf{t} + (1 - \lambda)\mathbf{u}) \geq \lambda F(\mathbf{t}) + (1 - \lambda) F(\mathbf{u}),$$

which proves that F is a convex function.

To prove (35), let $\mathbf{e}_1 = (1,0,\dots,0)$, $\cdots$, $\mathbf{e}_r = (0,0,\dots,1)$ be the standard basis of $\mathbb{R}^r$, and let $\mathbf{t}$ be as in (35). Then, by convexity of F and by (34), we get

$$F(\mathbf{t}) \ge \min_i F(\beta_i^{-1}\mathbf{e}_i) = \min_i \beta_i^{-1} F(\mathbf{e}_i)$$

and, obviously, $F(\mathbf{e}_i) = \frac{1}{h^0(\mathscr{L})}\sum_{m\ge 1} h^0(\mathscr{L}(-mD_i))$ for $i = 1,\dots,r$. □

We are now ready to prove the General Theorems. This proof will be done using b-divisors, so it will simultaneously handle the arithmetic case and the complex case.

Let $D_1,\dots,D_q$ be nonzero effective Cartier divisors intersecting properly on X, and let $\mathscr{L}$ be a big line sheaf on X. Recall that $n = \dim X$.

Let $\epsilon > 0$ be given. Since the quantities $m_S(x,D_i)/h_{\mathscr{L}}(x)$ and $m_f(r,D_i)/T_{f,\mathscr{L}}(r)$ are bounded when their respective denominators are sufficiently large and (in the number field case) when x lies outside of a proper Zariski-closed subset, it suffices to prove (8) or (9) with a slightly smaller $\epsilon > 0$ and with $\beta(\mathscr{L}, D_i)$ replaced by slightly smaller $\beta_i \in \mathbb{Q}$ for all i. (Alternatively, this step could be avoided by defining $\mathrm{Nev}_{\mathrm{bir}}$ using R-divisors.)

Choose positive integers N and b such that

$$\left(1+\frac{n}{b}\right)\max_{1\le i\le q}\frac{\beta_i N h^0(X,\mathscr{L}^N)}{\sum_{m\ge 1} h^0(X,\mathscr{L}^N(-mD_i))} < 1+\epsilon\ . \tag{37}$$

Let

$$\Sigma = \left\{\sigma \subseteq \{1,\dots,q\} \;\Big|\; \bigcap_{j\in\sigma}\operatorname{Supp} D_j \ne \emptyset\right\}.$$

For $\sigma\in\Sigma$, let

$$\triangle_\sigma = \left\{\mathbf{a} = (a_i) \in \prod_{i\in\sigma}\beta_i^{-1}\mathbb{N} \;\Bigg|\; \sum_{i\in\sigma}\beta_i a_i = b\right\}\ .$$

For $\mathbf{a}\in\triangle_\sigma$ as above, one defines (see (31), (32), and (33)) the ideal $\mathcal{I}_{\mathbf{a}}(x)$ of $\mathscr{O}_X$ by

$$\mathcal{I}_{\mathbf{a}}(x) = \sum_{\mathbf{b}} \mathscr{O}_X\left(-\sum_{i\in\sigma} b_i D_i\right) \tag{38}$$

where the sum is taken for all $\mathbf{b}\in\mathbb{N}^{\#\sigma}$ with $\sum_{i\in\sigma} a_i b_i \ge bx$. Let

$$\mathcal{F}(\sigma;\mathbf{a})_x = H^0(X,\mathscr{L}^N\otimes\mathcal{I}_{\mathbf{a}}(x))\ ,$$

which we regard as a subspace of $H^0(X,\mathscr{L}^N)$, and let

$$F(\sigma;\mathbf{a}) = \frac{1}{h^0(\mathscr{L}^N)}\int_0^{+\infty}(\dim\mathcal{F}(\sigma;\mathbf{a})_x)\,dx\ .$$

Applying Proposition 6.7 with the line sheaf being taken as $\mathscr{L}^N$, we have

$$F(\sigma;\mathbf{a}) \geq \min_{1\leq i\leq q}\left(\frac{b}{\beta_i h^0(\mathscr{L}^N)}\sum_{m\geq 1} h^0(\mathscr{L}^N(-mD_i))\right) .$$

As before, for any nonzero $s \in H^0(X, \mathscr{L}^N)$, we also define

(39) $$\mu_{\mathbf{a}}(s) = \sup\{x \in \mathbb{R}^+ : s \in \mathcal{F}(\sigma;\mathbf{a})_x\} .$$

Let $\mathcal{B}_{\sigma;\mathbf{a}}$ be a basis of $H^0(X, \mathscr{L}^N)$ adapted to the above filtration $\{\mathcal{F}(\sigma;\mathbf{a})_x\}_{x\in\mathbb{R}^+}$. By Remark 6.6, $F(\sigma,\mathbf{a}) = \frac{1}{h^0(\mathscr{L}^N)}\sum_{s\in\mathcal{B}_{\sigma;\mathbf{a}}}\mu_{\mathbf{a}}(s)$. Hence

(40) $$\sum_{s\in\mathcal{B}_{\sigma;\mathbf{a}}}\mu_{\mathbf{a}}(s) \geq \min_{1\leq i\leq q}\frac{b}{\beta_i}\sum_{m\geq 1} h^0(\mathscr{L}^N(-mD_i)) .$$

It is important to note that there are only finitely many ordered pairs $(\sigma,\mathbf{a})$ with $\sigma \in \Sigma$ and $\mathbf{a} \in \triangle_\sigma$.

Let $\sigma \in \Sigma$, $\mathbf{a} \in \triangle_\sigma$, and $s \in H^0(X, \mathscr{L}^N)$ with $s \neq 0$. Since the divisors D_i are all effective, it suffices to use only the leading terms in (38). The union of the sets of leading terms as x ranges over the interval $[0, \mu_{\mathbf{a}}(s)]$ is finite, and each such $\mathbf{b}$ occurs in the sum (38) for a closed set of x. Therefore the supremum (39) is actually a maximum.

Similarly, we have

$$\mathscr{L}^N \otimes \mathcal{I}_{\mathbf{a}}(\mu_{\mathbf{a}}(s)) = \sum_{\mathbf{b}\in K}\mathscr{L}^N\left(-\sum_{i\in\sigma} b_i D_i\right) ,$$

where $K = K_{\sigma,\mathbf{a},s}$ is the set of minimal elements of $\{\mathbf{b} \in \mathbb{N}^{\#\sigma} \mid \sum_{i\in\sigma} a_i b_i \geq \mu_{\mathbf{a}}(s)\}$ relative to the product partial ordering on $\mathbb{N}^{\#\sigma}$. This set is finite, so Proposition 4.18 applies, and we have

(41) $$(s) \geq \bigwedge_{\mathbf{b}\in K}\sum_{i\in\sigma} b_i D_i .$$

For a basis $\mathcal{B}$ of $H^0(X, \mathscr{L}^N)$, recall from (2) that $(\mathcal{B})$ denotes the sum of the divisors (s) for all $s \in \mathcal{B}$.

Lemma 6.8. *With the above notation, we have*

(42) $$\bigvee_{\substack{\sigma\in\Sigma\\ \mathbf{a}\in\Delta_\sigma}}(\mathcal{B}_{\sigma;\mathbf{a}}) \geq \frac{b}{b+n}\left(\min_{1\leq i\leq q}\sum_{m=1}^{\infty}\frac{h^0(X,\mathscr{L}^N(-mD_i))}{\beta_i}\right)\sum_{i=1}^{q}\beta_i D_i .$$

Proof. Let $\mathbf{D}' = \bigvee_{\sigma,\mathbf{a}}(\mathcal{B}_{\sigma;\mathbf{a}})$, let $\phi\colon Y \to X$ be a normal model of X on which $\mathbf{D}'$ is represented by a Cartier divisor D', and let E be a prime divisor on Y. For some point $P \in \phi(\operatorname{Supp} E)$, let

$$\sigma = \{i \in \{1,\dots,q\} : P \in \operatorname{Supp} D_i\} .$$

Let ν', $\nu_{\sigma,\mathbf{a}}$ (for $\mathbf{a} \in \triangle_\sigma$), and ν_i ($i = 1,\dots,q$) be the multiplicities of E in D', $\phi^*(\mathcal{B}_{\sigma;\mathbf{a}})$, and D_i, respectively, and let $\nu = \sum_{i=1}^q \beta_i\nu_i$. Since $\nu' \ge \nu_{\sigma,\mathbf{a}}$ for all $\mathbf{a} \in \triangle_\sigma$, the proof is a matter of finding some $\mathbf{a}$ such that

$$\nu_{\sigma,\mathbf{a}} \ge \frac{b}{b+n}\left(\min_{1\le i\le q}\sum_{m=1}^{\infty}\frac{h^0(X,\mathscr{L}^N(-mD_i))}{\beta_i}\right)\nu . \tag{43}$$

If $\nu = 0$ then there is nothing to prove, so we assume that $\nu > 0$.

For $i \in \sigma$, let

$$t_i = \frac{\nu_i}{\nu} . \tag{44}$$

Note that $\nu_i = 0$ for all $i \notin \sigma$, so

$$\sum_{i\in\sigma}\beta_i\nu_i = \sum_{i=1}^q \beta_i\nu_i = \nu ;$$

hence $\sum_{i\in\sigma}\beta_i t_i = 1$.

From the assumption that $D_1,\dots,D_q$ intersect properly (and hence lie in general position), we have $\#\sigma \le n$. Therefore $b \le \sum_{i\in\sigma}\lfloor(b+n)\beta_i t_i\rfloor \le b+n$, and we may choose $\mathbf{a} = (a_i) \in \triangle_\sigma$ such that

$$a_i \le (b+n)t_i \qquad \text{for all } i \in \sigma . \tag{45}$$

For any $s \in \mathcal{B}_{\sigma;\mathbf{a}}$ let ν_s be the multiplicity of E in the divisor $\phi^*(s)$. Using (41), (44), (45), and $\sum_{i\in\sigma} a_ib_i \ge \mu_{\mathbf{a}}(s)$, we get

$$\nu_s \ge \min_{\mathbf{b}\in K}\sum_{i\in\sigma} b_i\nu_i = \left(\min_{\mathbf{b}\in K}\sum_{i\in\sigma} b_it_i\right)\nu \ge \left(\min_{\mathbf{b}\in K}\sum_{i\in\sigma}\frac{a_ib_i}{b+n}\right)\nu \ge \frac{\mu_{\mathbf{a}}(s)\nu}{b+n} , \tag{46}$$

where the set $K = K_{\sigma,\mathbf{a},s}$ is as in (41). Combining (46) and (40) then gives

$$\frac{\nu_{\sigma,\mathbf{a}}}{\nu} = \frac{1}{\nu}\sum_{s\in\mathcal{B}_{\sigma;\mathbf{a}}}\nu_s \ge \frac{1}{b+n}\sum_{s\in\mathcal{B}_{\sigma;\mathbf{a}}}\mu_{\mathbf{a}}(s) \ge \frac{b}{b+n}\min_{1\le i\le q}\sum_{m\ge1}\frac{h^0(\mathscr{L}^N(-mD_i))}{\beta_i} , \tag{47}$$

which gives (43). □

By Lemma 6.8, the triple (N,V,μ) with $V = H^0(X,\mathscr{L}^N)$ satisfies the condition of Corollary 4.17, where

$$\frac{\dim V}{\mu} = \left(1+\frac{n}{b}\right)\max_{1\le i\le q}\frac{\beta_iNh^0(X,\mathscr{L}^N)}{\sum_{m\ge1}h^0(X,\mathscr{L}^N(-mD_i))} < 1+\epsilon \tag{48}$$

by (37). Since $\epsilon > 0$ was arbitrary, we get

$$\mathrm{Nev}_{\mathrm{bir}}\left(\mathscr{L}, \sum_{i=1}^{q} \beta_i D_i\right) \le 1$$

and thus the General Theorem follows from Theorems 1.4 and 1.5. This finishes the proof.

Note that we can continue the proof without using the notion of $\mathrm{Nev}_{\mathrm{bir}}(\mathscr{L}, D_j)$ and Theorems 1.4 and 1.5.

We start with the arithmetic case. Again, we replace $\beta(\mathscr{L}, D_i)$ with a slightly smaller $\beta_i \in \mathbb{Q}$ for all i. Let $\epsilon > 0$ be as in the statement of the theorem. Instead of (37), choose $\epsilon_1 > 0$, and positive integers N and b such that

$$\left(1 + \frac{n}{b}\right) \max_{1 \le i \le q} \frac{\beta_i N(h^0(X, \mathscr{L}^N) + \epsilon_1)}{\sum_{m \ge 1} h^0(X, \mathscr{L}^N(-mD_i))} < 1 + \epsilon\ . \tag{49}$$

Write

$$\bigcup_{\sigma;\mathbf{a}} \mathcal{B}_{\sigma;\mathbf{a}} = \mathcal{B}_1 \cup \cdots \cup \mathcal{B}_{T_1} = \{s_1, \dots, s_{T_2}\}.$$

For each $i = 1, \dots, T_1$, let $J_i \subseteq \{1, \dots, T_2\}$ be the subset such that $\mathcal{B}_i = \{s_j : j \in J_i\}$. Choose Weil functions λ_D, $\lambda_{\mathcal{B}_i}$ $(i = 1, \dots, T_1)$, and λ_{s_j} $(j = 1, \dots, T_2)$ for the divisors D, $(\mathcal{B}_i)$, and (s_j), respectively. Then, by (42) and Proposition 4.10, for each $v \in S$,

$$\begin{aligned} &\frac{b}{b+n}\left(\min_{1 \le i \le q} \sum_{m \ge 1} \frac{h^0(\mathscr{L}^N(-mD_i))}{\beta_i}\right) \sum_{i=1}^{q} \beta_i \lambda_{D_i,v} \\ &\qquad \le \max_{1 \le i \le T_1} \lambda_{\mathcal{B}_i,v} + O_v(1) = \max_{1 \le i \le T_1} \sum_{j \in J_i} \lambda_{s_j,v} + O_v(1). \end{aligned} \tag{50}$$

By Theorem 2.10 with ϵ_1 in place of ϵ, there is a proper Zariski-closed subset Z of X such that the inequality

$$\sum_{v \in S} \max_J \sum_{j \in J} \lambda_{s_j,v}(x) \le \left(h^0(\mathscr{L}^N) + \epsilon_1\right) h_{\mathscr{L}^N}(x) + O(1) \tag{51}$$

holds for all $x \in X(k)$ outside of Z; here the maximum is taken over all subsets J of $\{1, \dots, T_2\}$ for which the sections s_j, $j \in J$, are linearly independent.

Combining (50) and (51) gives

$$\sum_{i=1}^{q} \sum_{v \in S} \beta_i \lambda_{D_i,v}(x) \le \left(1 + \frac{n}{b}\right) \max_{1 \le i \le q} \frac{\beta_i(h^0(\mathscr{L}^N) + \epsilon_1)}{\sum_{m \ge 1} h^0(\mathscr{L}^N(-mD_i))} h_{\mathscr{L}^N}(x) + O(1)$$

for all $x \in X(k)$ outside of Z. Here we used the fact that all of the J_i occur among the J in (51). Using (49) and the fact that $h_{\mathscr{L}^N}(x) = N h_{\mathscr{L}}(x)$, we have

$$\sum_{i=1}^{q} \sum_{v \in S} \beta_i \lambda_{D_i,v}(x) \leq (1+\epsilon)\, h_{\mathscr{L}}(x) + O(1)$$

for all $x \in X(k)$ outside of Z. By the choices of β_i, this implies that

$$\sum_{i=1}^{q} \beta(\mathscr{L}, D_i) m_S(x, D_i) \leq (1+\epsilon) h_{\mathscr{L}}(x) + O(1)$$

holds for all k-rational points outside a proper Zariski-closed set. This proves the General Theorem for the arithmetic case.

The proof in the analytic case is similar, by replacing Theorem 2.10 (Schmidt's Subspace Theorem) with Theorem 2.11 (H. Cartan's theorem).

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