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Linear SCY Groups With Virtual First Betti Number Equal to Four

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Linear SCY Groups With Virtual First Betti Number Equal to Four

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&

Lois Anne Sperka Trent

two women who could not be more different,

but who both knew the value of an education

ABSTRACT OF THE DISSERTATION

Linear SCY Groups With Virtual First Betti Number Equal to Four

by

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Doctor of Philosophy, Graduate Program in Mathematics
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Symplectic 4-manifolds are coarsely classified by Kodaira dimension; those of Kodaira dimension 0 are characterized by torsion canonical class. A symplectic Calabi-Yau 4-manifold (SCY) is a symplectic 4-manifold with trivial canonical class. SCYs satisfy strict homological constraints: the (virtual) first Betti number is 0, 2, 3, or 4. The fundamental group of an SCY satisfies the same constraints and is called an SCY group. A symplectic manifold is almost complex and so admits a canonical spin^c structure, permitting access to Seiberg-Witten theory by which it is shown that linear SCY groups with virtual first Betti number 4 are virtually solvable and hence elementary amenable; Hirsch length calculations force such a group to be virtually $\mathbb{Z}^4$.

Contents

Chapter 1

Introduction

One of the main questions in geometric topology is the classification of compact connected orientable manifolds of a given dimension. In dimension 1 the only such space is the circle S^1. In dimension 2 the only examples are the sphere S^2 and the surfaces of genus $g \geq 1$. In dimension 3 the possibilities are far greater but can be classified according to the eight Thurston geometries.

In each of the three dimensions above, the fundamental group encodes a great deal of information. In dimensions 1 and 2 a manifold is uniquely determined by its fundamental group: this goes without saying for S^1 and is a classical result for surfaces. The situation in dimension 3 is more subtle.

A compact connected orientable 3-manifold M decomposes into a connected sum $P_1 \# \cdots \# P_k$ of *prime* 3*-manifolds*, manifolds that admit only the trivial connected sum decomposition $P = P \# S^3$. This prime decomposition is unique up to permuting the factors.

The prime 3-manifold P_i can be decomposed along tori as $N_{i,1} \#_{T^2} \cdots \#_{T^2} N_{i,k}$ so that the fundamental group of $N_{i,j}$ is determined by the Thurston geometry of $N_{i,j}$. The fundamental group of P_i is recovered from the fundamental groups of this decomposition by Seifert-van Kampen. The fundamental group of M is recovered in the same way as the free product $\pi_1(P_1) * \cdots * \pi_1(P_k)$.

Satisfied with the perceivable dimensions, one's attention then advances to dimension 4. How might 4-manifolds be classified, and what groups occur as their fundamental groups?

The answer—regardless of whether one considers topological manifolds or smooth manifolds, and due to A. A. Markov in [21]—is that every finitely presentable group occurs as the fundamental group of a 4-manifold. A classification in the style of those attained in lower dimensions is thus out of reach: such a classification would provide a solution to the isomorphism problem for finitely presentable groups, which is a classical example of an undecidable problem.

In [9], R. Gompf proved that the situation is unchanged by restricting one's attention to symplectic 4-manifolds, for every finitely presentable group occurs as the fundamental group of a symplectic 4-manifold. Hence, from the point of view of the fundamental group, there is no difference between studying topological, smooth, or symplectic 4-manifolds.

One could continue to place additional structures on the 4-manifolds in question, subject to compatibility conditions with the other structures, and might, in this way, arrive at the study of *Kähler surfaces*, complex surfaces that admit a symplectic form and metric such that the symplectic form, metric, and complex structure are compatible in the appropriate sense. This confluence of structures proves sufficient to limit the possible fundamental groups to a tractable class.

Alternatively, one might remain in the symplectic realm and seek to, at first, classify the symplectic 4-manifolds according to a coarser scheme in the hopes that the class of fundamental groups associated with each class is sufficiently tame.

1.1 Kodaira Dimension 0

The notion of Kodaira dimension, originally defined on algebraic varieties but adapted by T. J. Li in [18] to symplectic 4-manifolds, classifies symplectic 4-manifolds into four classes: Kodaira dimensions $-\infty$, 0, 1, and 2.

Given a symplectic 4-manifold (M, ω), its *first Chern class* is a characteristic class $c_1(M, \omega) \in H^2(M; \mathbb{Z})$ associated with the symplectic form ω (see §2.1). The *canonical class* of (M, ω) is defined to be $K_\omega = -c_1(M, \omega)$.

If (M, ω) is a minimal symplectic 4-manifold (see §2.1), then its *Kodaira dimension* is defined to be

$$\kappa(M, \omega) = \begin{cases} -\infty & \langle K_\omega \cup [\omega], [M] \rangle < 0 \text{ or } \langle K_\omega^2, [M] \rangle < 0 \\ 0 & \langle K_\omega \cup [\omega], [M] \rangle = 0 \text{ and } \langle K_\omega^2, [M] \rangle = 0 \\ 1 & \langle K_\omega \cup [\omega], [M] \rangle > 0 \text{ and } \langle K_\omega^2, [M] \rangle = 0 \\ 2 & \langle K_\omega \cup [\omega], [M] \rangle > 0 \text{ and } \langle K_\omega^2, [M] \rangle > 0 \end{cases},$$

where $\cup\colon H^k(M; \mathbb{Z}) \times H^\ell(M; \mathbb{Z}) \to H^{k+\ell}(M; \mathbb{Z})$ is the cup product and $\langle [\sigma], [\alpha] \rangle$ denotes the evaluation of $[\sigma] \in H^k(M;\mathbb{Z})$ on $[\alpha] \in H_k(M;\mathbb{Z})$. If (M,ω) is not minimal, then it has the Kodaira dimension of any of its minimal models.

While a classification analogous to those for low dimensional manifolds is impossible for aforementioned reasons, there is hope for understanding 4-manifolds of certain Kodaira dimensions. For example, every member of Kodaira dimension $-\infty$ is a Kähler surface [20], although the converse does not hold: T^4 is Kähler but its Kodaira dimension is 0 [18].

T. J. Li proved in [18] that Kodaira dimension 0 is characterized by K_ω being torsion. The same paper of Li's, and [26] by J. Morgan and Z. Szabó, provides a classification of symplectic 4-manifolds (M,ω) of Kodaira dimension 0 with finite fundamental group. If M is simply connected, then M is homeomorphic to the K3 surface and if $\pi_1(M)$ is finite but nontrivial, then M is homeomorphic to the Enriques surface.

The K3 surface is a distinguished object in algebraic geometry, but for the purpose of this dissertation it is a simply-connected symplectic 4-manifold defined by the equation

$$x^4 + y^4 + z^4 + w^4 = 0$$

in $\mathbb{C}\mathrm{P}^3$. The Enriques surface is a quotient of the K3 surface by a free $\mathbb{Z}_2$-action, hence its fundamental group is isomorphic to $\mathbb{Z}_2$. In particular, this entails, by the above result of Li, Morgan, and Szabó, that, in Kodaira dimension 0, the only nontrivial finite fundamental group is $\mathbb{Z}_2$.

Symplectic 4-manifolds with torsion K_ω can be divided into two classes: one where K_ω is trivial and one where K_ω is nontrivial. The latter class contains only manifolds with the same integral homology as the Enriques surface [18, Prop. 6.3]. This dissertation thus ignores the Enriques surface and its brethren, opting instead to focus on the case when $K_\omega = 0$.

A symplectic 4-manifold (M, ω) of Kodaira dimension 0 with $K_\omega = 0$ is called a *symplectic Calabi-Yau manifold*, or an *SCY manifold* for short. In addition to the K3 surface, which is the unique simply connected SCY manifold, and 4-manifolds with isomorphic integral homology, there are two known classes of examples of SCY manifolds. The first of which is produced from the work of H. Geiges in [7] and T. J. Li in [18] and [19]: T^2-bundles over T^2. The second class of examples is due to M. Fernández *et alii* in [5] and D. McDuff and D. Salamon in [22]. It consists of spaces X that are constructed as the total spaces of certain S^1-bundles over 3-manifold torus bundles.

Both T^2-bundles over T^2 and the members of the second class are so-called cohomologically symplectic infrasolvmanifolds (see §2.3). Furthermore, a member of either class or a cohomologically symplectic infrasolvmanifold always admits a T^2-bundle over T^2 as a finite-sheeted cover.

1.2 Homological Constraints

Work by J. W. Morgan and Z. Szabó in [26] revealed that an SCY manifold (M, ω) with $b_1(M) = 0$ must be an integral homology K3 surface. Moreover, any simply-connected SCY manifold is homeomorphic to the K3 surface.

Constraints on SCY manifolds with positive first Betti number were proved by the work of T. J. Li in [17] and S. Bauer in [1]. An SCY manifold (M, ω) with $b_1(M) \geq 1$ satisfies $2 \leq b_1(M) \leq 4$ and has vanishing Euler characteristic and signature. Alongside

Morgan-Szabó, this pair of results puts strict homological constraints on SCY manifolds and their fundamental groups.

The class of spaces referred to as *solvmanifolds* are those spaces arising as the compact quotients of simply-connected solvable Lie groups by closed subgroups. As it would happen, K. Hasegawa proved in [10] that a 4-dimensional solvmanifold M satisfies $2 \leq b_1(M) \leq 4$ if and only if M is a T^2-bundle over T^2. As every known example of an SCY manifold with positive first Betti number is a T^2-bundle over T^2, one might wonder if it is any easier to show that this class consists solely of solvmanifolds.

As suggested by their names, solvmanifolds and infrasolvmanifolds are related: every solvmanifold is an infrasolvmanifold and every infrasolvmanifold admits a finite-sheeted solvmanifold cover. One might weaken the above question and ask if every SCY manifold with positive first Betti number is an infrasolvmanifold. This possibility is further suggested by the fact that infrasolvmanifolds have vanishing Euler characteristic and made desirable by the fact that infrasolvmanifolds are determined, within the class of infrasovlmanifolds, up to diffeomorphism by their fundamental groups [2].

Thus the big question that this dissertation approaches is: are SCY manifolds with $b_1 \geq 2$ infrasolvmanifolds? In terms of the fundamental group, this question becomes: are SCY groups with $b_1 \geq 2$ virtually poly-$\mathbb{Z}$? One can then weaken this question to asking: are SCY groups with $b_1 \geq 2$ virtually solvable?

Under the assumptions that the SCY group is linear and $b_1 = 4$, Theorem 5.1.3 answers this last question in the affirmative. Alongside Theorem 5.2.2 one obtains a proof of the main result of this dissertation: that every SCY group with $vb_1 = 4$ is virtually

$\mathbb{Z}^4$, which implies by [6] that every SCY manifold with $vb_1 = 4$ is finitely covered by T^4. As T^4 is a solvmanifold, this result is in accord with the conjecture that SCY manifolds with $b_1 \geq 2$ are infrasolvmanifolds as it is shows that every SCY manifold with linear fundamental group is finitely covered by a solvmanifold.

Chapter 2

Symplectic Calabi-Yau 4-Manifolds & Groups

The notion of a Calabi-Yau manifold is an algebro-geometric one, originating in the study of projective algebraic varieties. In complex dimension 2, the Calabi-Yau surfaces are completely classified by the K3 surfaces and elliptic fibrations. In complex dimension 3, the pursuit of a similar classification led to the so-called "mirror symmetry", which relates complex and symplectic structures of Calabi-Yaus in this complex dimension. Impressed with the need to understand the symplectic structures of Calabi-Yaus, one might return to complex dimension 2 where the Calabi-Yau surfaces are better understood.

This dissertation views symplectic Calabi-Yau surfaces from a topological viewpoint. The situation above thus changes when these objects are instead viewed as (real) dimension 4 manifolds. In particular, the K3 surfaces belong to the same homeomorphism class, so one might speak of *the* K3 surface. Additionally, there exist symplectic Calabi-

Yau 4-manifolds that fail to be Calabi-Yau surfaces: the Kodaira-Thurston manifold is a well-known example of such, as it fails to be Kähler.

2.1 Introduction

Manifolds are assumed to be smooth, connected, and closed. A manifold is *closed* if it is compact with empty boundary.

If V is an even-dimensional real vector space, then a *symplectic form* on V is a nondegenerate skew-symmetric bilinear form $\omega\colon V \times V \to \mathbb{R}$. A *symplectomorphism* is a linear isomorphism $\Phi\colon V \to V$ such that $\Phi^*\omega = \omega$, that is, $\omega(\Phi\mathbf{u}, \Phi\mathbf{v}) = \omega(\mathbf{u}, \mathbf{v})$ for all $\mathbf{u}, \mathbf{v} \in V$. The collection of all symplectomorphisms $\mathrm{Sp}(V, \omega)$ is a subgroup of $\mathrm{GL}(V)$ called the *symplectic group* of (V, ω).

If $V = \mathbb{R}^{2n}$, then the standard symplectic form ω_0 on $\mathbb{R}^{2n}$ is defined by

$$\omega_0(\mathbf{u}, \mathbf{v}) = \sum_{i=1}^{n} (u_{2i-1}v_{2i} - u_{2i}v_{2i-1}).$$

The symplectic group on $(\mathbb{R}^{2n}, \omega_0)$ is denoted by $\mathrm{Sp}(2n; \mathbb{R})$ or, in brief, just $\mathrm{Sp}(2n)$.

A *symplectic vector bundle* over a smooth $2n$-manifold M is a pair (E, ω) of a vector bundle E and a family of symplectic forms that smoothly vary over the fibers. That is, for each $p \in M$ there exists a symplectic form $\omega_p\colon E_p \times E_p \to \mathbb{R}$ such that the transition maps for E are smooth maps into $\mathrm{Sp}(2n; \mathbb{R})$.

Definition 2.1.1. A *symplectic manifold* is a pair (M, ω) of a smooth manifold M and a 2-form ω such that $(\mathrm{T}M, \omega)$ is a symplectic vector bundle; then ω is said to be a *symplectic form.*

Equivalently, a symplectic form on M can be characterized as a choice of closed nondegenerate 2-form ω. Hence ω is a representative for some nonzero class in $H^2(M;\mathbb{R})$; thus $b_2(M) \geq 1$.

Almost Complex Structures & Chern Classes

An *almost complex structure* on a smooth manifold M is an automorphism J of $\mathrm{T}M$ such that $J^2 = -\mathrm{id}_{\mathrm{T}M}$. Equivalently, $\mathrm{T}M$ admits a complex structure—although this need not descend to M itself.

Given a symplectic manifold (M, ω), an almost complex structure J on M is said to be ω-*compatible* if for all $p \in M$ and $\mathbf{u}, \mathbf{v} \in \mathrm{T}_pM$ the equation

$$\omega_p(J_p\mathbf{u}, J_p\mathbf{v}) = \omega_p(\mathbf{u}, \mathbf{v})$$

is satisfied. The space of ω-compatible almost complex structures is nonempty and contractible [23, §2.6], so up to homotopy there is a unique almost complex structure on (M, ω).

As a consequence of this, complex vector bundle invariants are well-defined on symplectic manifolds. In particular one has access to the Chern classes.

Definition 2.1.2. Let (M, ω) be a symplectic manifold and J any ω-compatible almost complex structure on M. Let $c_n(\mathrm{T}M, J) \in H^{2n}(M;\mathbb{Z})$ be the nth Chern class associated with the complex vector bundle $(\mathrm{T}M, J)$. The *nth Chern class of* (M, ω) is the class $c_n(M, \omega) = c_n(\mathrm{T}M, J)$.

When the symplectic form on M is known from context, one might write $c_n(M)$ instead. Likewise, if M is fixed and the symplectic form is the focus, then $c_n(\omega)$ is employed instead.

Of particular interest is the *canonical class* of (M, ω), the class $K_\omega = -c_1(M, \omega)$. Alongside $[\omega] \in H^2(M; \mathbb{R})$, the canonical class is one of two fundamental cohomological invariants in the study of symplectic manifolds.

If M is a $4n$-dimensional manifold, then its cup product induces a non-degenerate symmetric quadratic form q on $H^{2n}(M; \mathbb{R})$. This quadratic form splits $H^{2n}(M; \mathbb{R})$ into its positive and negative eigenspaces, denoted $H^{2n}_+(M; \mathbb{R})$ and $H^{2n}_-(M; \mathbb{R})$ respectively. The dimension of the positive and negative eigenspace of q is written as $b^+(M)$ and $b^-(M)$ respectively. Note that $b_n(M) = b^+(M) + b^-(M)$.

If M is symplectic, then $b^+(M)$ is necessarily positive as $[\omega]^n$ provides a nonzero element of $H^{2n}_+(M; \mathbb{R})$.

The *Euler characteristic* $\chi(M)$ and the *signature* $\sigma(M)$ of a $4n$-dimensional manifold are

$$\chi(M) = \sum_{k \in \mathbb{N}} (-1)^k b_k(M) \qquad \text{and} \qquad \sigma(M) = b^+(M) - b^-(M).$$

(The signature is restricted to $4n$-dimensional manifolds, but the Euler characteristic is defined in all dimensions. However, by Poincaré duality, the Euler characteristic vanishes for odd-dimensional manifolds.)

For symplectic manifolds in dimension 4, the Euler characteristic and the signature are connected through the following proposition, known as the Hirzebruch signature theorem.

Proposition 2.1.3. *If (M, ω) is a symplectic 4-manifold, then*

$$\langle K_\omega^2, [M] \rangle = 2\chi(M) + 3\sigma(M),$$

where $[M]$ is the fundamental class of M.

Kodaira Dimension

Here on out the discussion specializes to dimension 4, although symplectic Calabi-Yau manifolds can be defined in any even dimension with the proper generalization of a minimal symplectic manifold and Kodaira dimension (see [19, §7.1]).

By change of coefficients, K_ω can be considered as an element of $H^2(M;\mathbb{R})$ alongside $[\omega]$. Although $\langle[\omega]^2,[M]\rangle > 0$ for all symplectic manifolds, the signs of the products and powers $[\omega]$ and K_ω evaluated on the fundamental class of M can vary. The Kodaira dimension—introduced for symplectic manifolds in [18], although originating in the study of projective varieties—classifies symplectic manifolds according to these signs.

A symplectic 4-manifold (M,ω) is said to be *minimal* if M cannot be written as the connect sum $N \# \overline{\mathbb{CP}}^2$ for any symplectic manifold N; the notation $\overline{\mathbb{CP}}^2$ denotes $\mathbb{CP}^2$ with the opposite orientation. If (M,ω) fails to be minimal, then there are a finite number of complex projective lines of self-intersection -1 preventing M from being minimal; by replacing a tubular neighborhood of each with a copy of D^4, one blows down to a *minimal model* for (M,ω).

Topologically, this presents as writing M as $N \# \overline{\mathbb{CP}}^2 \# \cdots \# \overline{\mathbb{CP}}^2$, where N cannot be written as $P \# \overline{\mathbb{CP}}^2$, and then calling N a minimal model for M. The previous description is necessary, however, for unlike $\mathbb{CP}^2$, the reversal $\overline{\mathbb{CP}}^2$ fails to be symplectic. This follows from the observation that the non-degeneracy of the symplectic form forces b^+ to be nontrivial. It is known that $b_2(\mathbb{CP}^2) = 1$, hence $b^+(\mathbb{CP}^2) = 1$ and $b^-(\mathbb{CP}^2) = 0$. Reversing the orientation on the manifold exchanges the values of b^+ and b^-, hence $b^+(\overline{\mathbb{CP}}^2) = 0$, precluding the existence of a symplectic form.

Definition 2.1.4. Let (M, ω) be a 4-dimensional symplectic manifold with canonical class K_ω. If (M, ω) is minimal, then its *Kodaira dimension* is

$$\kappa(M,\omega) = \begin{cases} -\infty & \langle K_\omega \cup [\omega], [M]\rangle < 0 \text{ or } \langle K_\omega^2, [M]\rangle < 0 \\ 0 & \langle K_\omega \cup [\omega], [M]\rangle = 0 \text{ and } \langle K_\omega^2, [M]\rangle = 0 \\ 1 & \langle K_\omega \cup [\omega], [M]\rangle > 0 \text{ and } \langle K_\omega^2, [M]\rangle = 0 \\ 2 & \langle K_\omega \cup [\omega], [M]\rangle > 0 \text{ and } \langle K_\omega^2, [M]\rangle > 0 \end{cases}.$$

If (M, ω) is not minimal, then $\kappa(M, \omega)$ is defined to be the Kodaira dimension of any minimal model for (M, ω).

An astute eye might notice a missing combination from the above pairs: what about the case when $\langle K_\omega \cup [\omega], [M]\rangle = 0$ and $\langle K_\omega^2, [M]\rangle > 0$? This case is rendered vacuous by the following lemma in [18].

Lemma 2.1.5. *If (M, ω) is a minimal symplectic 4-manifold with $\langle K_\omega \cup [\omega], [M]\rangle = 0$ and $\langle K_\omega^2, [M]\rangle \geq 0$, then K_ω is torsion and hence $\langle K_\omega^2, [M]\rangle = 0$.*

Corollary 2.1.6. *A symplectic manifold (M, ω) has Kodaira dimension 0 if and only if K_ω is torsion.*

Thus there are two cases in Kodaira dimension 0: when K_ω is a nontrivial torsion class; and when $K_\omega = 0$. The former consists solely of the integral homology Enriques surfaces, so $b_1 = 0$ and $b_2 = 10$ (see [18, Table 1]); the latter forms the class of spaces that is the focus of this dissertation. Note that when K_ω is nontrivial it is specifically a 2-torsion class.

Definition 2.1.7. A symplectic 4-manifold (M, ω) is said to be a *symplectic Calabi-Yau (SCY) manifold* if its canonical class is trivial. A group G is said to be a *symplectic Calabi-Yau (SCY) group* if it is the fundamental group of an SCY manifold.

Of note are the homological properties of SCY manifolds, and thus of SCY groups. Such results are discussed in §2.2 below.

Among non-homological results, a useful property of SCY manifolds is that they are closed under passing to finite-sheeted covers.

Proposition 2.1.8. *Let M be an SCY manifold. If $\widetilde{M}$ is a finite-sheeted cover of M, then $\widetilde{M}$ is an SCY manifold.*

Proof. Assume (M, ω) is an SCY manifold and $\widetilde{M}$ a finite-sheeted cover of M with covering map π. Since π is a local diffeomorphism, $(\widetilde{M}, \pi^*\omega)$ is a symplectic manifold. If K_ω is trivial, then $\pi^*(K_\omega) = 0$ as well. By [24, Lem. 3.1], $\mathrm{T}\widetilde{M}$ is isomorphic to the pullback of $\mathrm{T}M$ along π, hence, as K_ω is a characteristic class, $\pi^*(K_\omega) = K_{\pi^*\omega}$. □

Corollary 2.1.9. *Let G be an SCY group. If $H \leq G$ is a finite-index subgroup, then H is an SCY group.*

2.2 Homological Constraints

A homological result for SCY manifolds that is immediate from Definition 2.1.7 is the following corollary to the Hirzebruch signature formula of Proposition 2.1.3.

Corollary 2.2.1. *If M is an SCY manifold, then $2\chi(M) + 3\sigma(M) = 0$.*

On a grander scale, two theorems divide the realm of SCY groups according to their homology. The first was proved by J. Morgan and Z. Szabó in [26].

Theorem 2.2.2 (Morgan-Szabó). *Let G be an SCY group with $b_1(G) = 0$. If M is an SCY manifold with fundamental group G, then M is an integral homology K3 surface. Moreover, if G is trivial, then M is homeomorphic to the K3 surface.*

This accounts, up to homeomorphism, for the simply-connected SCY manifolds. The following theorem, due to S. Bauer [1] and T. J. Li [18], describes the basic homological invariants of SCY manifolds with positive first Betti number.

Theorem 2.2.3 (Bauer & Li). *If G is an SCY group with $b_1(G) \geq 1$, then $2 \leq b_1(G) \leq 4$. Moreover, if M is an SCY manifold with fundamental group G, then $\chi(M) = 0$ and $\sigma(M) = 0$.*

Let G be a group and R a ring. The *virtual nth Betti number* of G with coefficients in R is

$$vb_n(G; R) = \sup\{b_n(H; R) \mid H \leq G \text{ is finite-index}\}.$$

The virtual nth Betti number of a space is defined analogously, with finite-sheeted covers in place of finite-index subgroups.

Corollary 2.2.4. *If G is an SCY group with $b_1(G) \geq 1$, then $2 \leq b_1(G) \leq vb_1(G) \leq 4$.*

Proof. By Theorem 2.2.3 one has $2 \leq b_1(G) \leq 4$. If $H \leq G$ is a finite-index subgroup then it is also an SCY group by Corollary 2.1.9. Since $b_1(H) \geq b_1(G)$ it satisfies the hypotheses of Theorem 2.2.3 as well, hence H satisfies $2 \leq b_1(G) \leq b_1(H) \leq 4$. As H was an arbitrary finite-index subgroup of G, this implies that $2 \leq b_1(G) \leq vb_1(G) \leq 4$. □

An immediate consequence of Theorems 2.2.2 and 2.2.3 is that the rational homology of an SCY group falls into one of four classes.

Corollary 2.2.5. *If M is an SCY manifold, then the pair $(b_1(M), b_2(M))$ must be one of $(0, 22)$, $(2, 2)$, $(3, 4)$, or $(4, 6)$.*

Proof. The simply-connected case is the integral homology of the K3 surface which is well known (see, for example, [18, Table 1]). The remaining three cases can be computed from Theorem 2.2.3. $\square$

2.3 Examples of SCY 4-Manifolds with $b_1 \geq 1$

Recall from Theorem 2.2.3 that an SCY 4-manifold with positive b_1 in fact satisfies the stronger condition that $2 \leq b_1 \leq 4$.

The known examples of SCY 4-manifolds with $b_1 \geq 1$ fall into two overlapping classes. The first class are the T^2-bundles over T^2, which were proven symplectic by H. Geiges in [7] and shown to have trivial canonical class by T. J. Li in [18] and [19]. The second class are the total spaces X of a bundle $S^1 \hookrightarrow X \to N^3$ such that N^3 is the total space of a bundle $T^2 \hookrightarrow N^3 \to S^1$ and the S^1-action on N^3 is trivial on the T^2-fibers; that these are SCYs was proven by M. Fernández *et alii* in [5] and D. McDuff and D. Salamon in [22].

Both classes belong to the *a priori* larger class of so-called cohomologically symplectic infrasolvmanifolds. However, as Proposition 2.3.4 below describes, the class of cohomologically symplectic infrasolvmanifolds is no larger than the class of SCY 4-manifolds.

Cohomologically Symplectic Infrasolvmanifolds

A $2n$-manifold M is *cohomologically symplectic* if there exists $[\alpha] \in H^2(M;\mathbb{R})$ such that $\langle [\alpha]^n, [M] \rangle > 0$. Every symplectic manifold is seen to be cohomologically symplectic by choosing $[\alpha]$ to be the class of the symplectic form. The converse does not hold in general, but the additional hypothesis that M is an infrasolvmanifold was proved sufficient by H. Kasuya in [13].

The definition of an infrasolvmanifold is more involved than that of cohomologically symplectic.

Definition 2.3.1. Let S be a simply-connected solvable Lie group. Let Γ be a closed torsion-free subgroup of $S \rtimes \mathrm{Aut}(S)$ satisfying the following conditions:

- The identity component of Γ is contained in the maximal connected nilpotent normal subgroup of S;
- The closure of $\Gamma/(\Gamma \cap S)$ is compact in $\mathrm{Aut}(S)$;
- S/Γ is compact.

The manifold $M = S/\Gamma$ is called an *infrasolvmanifold.*

Infrasolvmanifolds generalize the class of solvmanifolds. A *solvmanifold* is the compact quotient of a simply-connected solvable Lie group by a closed subgroup. Hence every solvmanifold is an infrasolvmanifold as well.

Infrasolvmanifolds satisfy several properties that are quite useful and of significant interest, especially when it comes to the relationship between infrasolvmanifolds and SCY manifolds with $b_1 \geq 1$.

Proposition 2.3.2. *If M is an infrasolvmanifold, then it satisfies the following properties:*

- *There exists a finite-sheeted cover $\widetilde{M}$ of M such that $\widetilde{M}$ is a solvmanifold;*
- $\chi(M) = 0$*;*
- *If G is the fundamental group of M, then G is torsion-free and virtually poly-$\mathbb{Z}$;*
- *If N is an infrasolvmanifold, then M and N are diffeomorphic if and only if $\pi_1(M)$ and $\pi_1(N)$ are isomorphic.*

In addition to enjoying the above properties *a fortiori*, the class of solvmanifolds in dimension 4 satisfies the following proposition, proved by K. Hasegawa in [10].

Proposition 2.3.3. *If M is a 4-dimensional solvmanifold, then either M is a T^2-bundle over T^2 or $b_1(M) = 1$.*

Thus, as far as the study of SCY manifolds is concerned, solvmanifolds and T^2-bundles over T^2 are equivalent.

Proposition 2.3.4. *If M is a 4-dimensional cohomologically symplectic infrasolvmanifold, then M is SCY.*

Proof. Let M be a cohomologically symplectic infrasolvmanifold in dimension 4. That M is truly symplectic is known from H. Kasuya's work in [13].

By Proposition 2.3.2, there exists a finite-sheeted cover $\widetilde{M}$ of M such that $\widetilde{M}$ is a solvmanifold. As M is 4-dimensional, so too is $\widetilde{M}$; assume $\widetilde{M}$ is not a T^2-bundle over T^2, then $b_1(\widetilde{M}) = 1$ by Proposition 2.3.3.

Combining $b_1(\widetilde{M}) = 1$ with the result of Proposition 2.3.2 that $\chi(\widetilde{M}) = 0$ one can perform the following calculations:

$$0 = 2b_0(\widetilde{M}) - 2b_1(\widetilde{M}) + b_2(\widetilde{M})$$

$$0 = 2 - 2 + b_2(\widetilde{M})$$

to obtain that $b_2(\widetilde{M}) = 0$. Since $\widetilde{M}$ is a finte-sheeted cover of M, it satisfies $b_2(\widetilde{M}) \geq b_2(M)$; that is to say, $b_2(M) = 0$ as well. However, this contradicts that M is symplectic. Thus $\widetilde{M}$ must be a T^2-bundle over T^2.

As $\widetilde{M}$ is now known to be an SCY, its canonical, $K_{\widetilde{\omega}}$, vanishes. Since π is a local diffeomorphism, $\pi^* K_\omega = K_{\widetilde{\omega}}$. Over rational cohomology, π^* is injective, so K_ω is trivial rationally, hence over $\mathbb{Z}$ one concludes that K_ω is torsion.

By Corollary 2.1.6, either M is an SCY or M has the integral homology of the Enriques surface. If M is an integral homology Enriques surface, then $b_1(M) = 0$ and $b_2(M) = 10$. As $\widetilde{M}$ is an SCY with $b_1 \geq 1$ its second Betti number is 2, 4, or 6; all three cases contradict that $b_2(\widetilde{M}) \geq b_2(M)$. Hence M must be an SCY. $\square$

Symplectic Calabi-Yau Manifolds & Infrasolvmanifolds

The two overlapping classes of known examples of SCY manifolds with $b_1 \geq 1$ are, as mentioned above: T^2-bundles over T^2; total spaces of certain types of T^2-bundles. Furthermore, these two classes are contained within the class of cohomologically symplectic infrasolvmanifolds which are, themselves, SCY. Moreover, as observed in [6], every known example of an SCY 4-manifold with $b_1 \geq 1$ is either a T^2-bundle over T^2 or admits a T^2-bundle over T^2 as a finite-sheeted cover.

If every SCY manifold was a solvmanifold, then Proposition 2.3.3 would suffice to show that the classes of certain total spaces of T^2-bundles and cohomologically symplectic infrasolvmanifolds reduce to T^2-bundles over T^2. A weaker result would be to determine if every SCY manifold was an infrasolvmanifold, although it is at present unknown whether solvmanifolds exhaust the class of infrasolvmanifolds in dimension 4.

The idea that SCY manifolds and 4-dimensional cohomologically symplectic infrasolvmanifolds might coincide has merits and the final property of Proposition 2.3.2 makes the identification desirable. The first two results of Proposition 2.3.2 hold in similar fashion for SCY manifolds: see Proposition 2.1.8 and Theorem 2.2.3.

2.4 Non-SCY Powers of Thompson's Group T

One can compute, from Theorem 2.2.2 and by referencing the table in Corollary 2.2.5, that any SCY 4-manifold with $b_1 = 0$ has $\chi = 24$. Since the Euler characteristic is multiplicative with respect to taking finite-sheeted covers, one can make the following observation.

Proposition 2.4.1. *If M is an SCY 4-manifold with $b_1(M) = 0$, then M admits no nontrivial finite-sheeted covers.*

Proof. Suppose $\widetilde{M}$ is a k-sheeted cover of M, then $\widetilde{M}$ is an SCY manifold by Proposition 2.1.8. By Theorem 2.2.2 and Theorem 2.2.3, the Euler characteristic of $\widetilde{M}$ is either 24 or 0. However, as a k-sheeted cover of M, it must satisfy $\chi(\widetilde{M}) = k\chi(M)$.

By Theorem 2.2.2 it is known that $\chi(M) = 24$, hence the above cases are $24 = 24k$ and $0 = 24k$. The only possibilities for k are then 1 and 0. In the former, $\widetilde{M}$ is a trivial cover, being 1-sheeted, and the latter is not a solution as $\widetilde{M}$ cannot be 0-sheeted. Thus the only finite-sheeted cover of M is the trivial cover. $\square$

Corollary 2.4.2. *If G is an SCY group with $b_1(G) = 0$, then G admits no nontrivial finite-index subgroups.*

Corollary 2.4.2 provides a method by which to prove certain groups fail to be SCY. Namely, if $b_1(G) = 0$ but G possesses a nontrivial finite-index subgroup, then G cannot be SCY.

An SCY 4-manifolds with $b_1 = 0$ must, by virtue of Theorem 2.2.2, be integral homology K3 surfaces. In the converse direction, one might hope that the only integral homology K3 surface to be SCY is the K3 surface itself. By Corollary 2.4.2, any counterexamples must originate in the class of finitely generated groups without finite-index subgroups.

This class includes the finitely generated infinite simple groups, for if $H < G$ is a finite-index proper subgroup, then the normal core $H_G = \bigcap_{g\in G} gHg^{-1}$ of H in G is a finite-index normal subgroup of G that satisfies $H_G \leq H < G$. Clearly $H_G \neq G$ and the trivial subgroup of an infinite group is not finite-index, hence H_G is a nontrivial proper normal subgroup, preventing G from being simple. When one mentions finitely generated infinite simple groups, Thompson's groups are seldom far behind.

Consider Thompson's group T, which is an infinite simple with finite presentation

$$T = \langle c, d \mid c^3, d^4, (cd)^5, [d^2cdcd^2, cd^2cdcd^2c^{-1}]\rangle.$$

Recall that the bracket $[g, h]$ is the commutator of g and h, that is, $[g, h] = g^{-1}h^{-1}gh$.

In [1], S. Bauer asks if T—or another of Thompson's simple groups—is SCY. This section does not provide an answer to Bauer's question, but employs an inequality that here so far appears to have gone unexploited in the process of determining if a group without finite-index subgroups is SCY.

To wit, observe that if M is some topological space with fundamental group G, then $b_1(G) = b_1(M)$ but $b_2(G) \leq b_2(M)$. Since the b_2 of an SCY manifold is constrained to be one of 22, 2, 4, or 6 depending on the value of b_1 (see Corollary 2.2.5), this induces a bound on the b_2 of an SCY group that can then be used to exclude certain groups from being SCY.

Of course, making use of these constraints for Thompson's group T requires knowledge of $b_1(T)$ and $b_2(T)$. Computing $b_1(T)$ can be done in a straightforward manner by computing the abelianization of T, that is, $H_1(T;\mathbb{Z})$, directly. The finite presentation for T given above becomes a finite presentation for $H_1(T;\mathbb{Z})$ by the addition of the commutator $[c,d]$ as a relation:

$$H_1(T;\mathbb{Z}) = \langle c, d \mid c^3, d^4, (cd)^5, [d^2cdcd^2, cd^2cdcd^2c^{-1}], [c,d]\rangle.$$

Of course, if c and d commute, then the relation $[d^2cdcd^2, cd^2cdcd^2c^{-1}]$ becomes redundant in the presentation and $(cd)^5$ becomes c^5d^5, reducing the presentation to

$$H_1(T;\mathbb{Z}) = \langle c, d \mid c^3, d^4, c^5d^5, [c,d]\rangle.$$

Applying the relations c^3 and d^4 to c^5d^5 reduces the latter to c^2d, hence $c^2 = d^{-1}$. On the other hand, the relation c^3 implies that $c^2 = c^{-1}$, hence $c = d$, reducing the presentation to $H_1(T;\mathbb{Z}) = \langle c \mid c^3, c^4\rangle$. In the presence of the relation c^3, the relation c^4 reduces to c, hence $H_1(T;\mathbb{Z}) = 0$ and so $b_1(T) = 0$.

A calculation of $b_2(T)$, by E. Ghys and V. Sergiescu, can be found in [8]. The argument is far from elementary, but the end result is that $H^2(T;\mathbb{Z}) \cong \mathbb{Z}^2$, hence $b_2(T) = 2$.

Since T satisfies both $b_1(T) = 0$ and $b_2(T) \leq 22$ it is impossible for the constraints of Theorem 2.2.2 to preclude T from being SCY. However, any construction that increases b_2 while leaving b_1 unchanged can be used to produce groups derived from T which are prevented from being SCY by the constraints. In particular, the groups $T^{\oplus n}$ and T^{*n}, the direct and free products of n copies of T respectively, can be shown to satisfy the first but violate the second for sufficiently large n. (Note that $T^{\oplus n}$ and T^{*n} are not simple when $n \geq 2$, but they do lack finite-index subgroups and hence it is still desirable to exclude them from the list of SCY groups with $b_1 = 0$.)

Proposition 2.4.3. *If $n \geq 12$, then $T^{\oplus n}$ and T^{*n} are not SCY.*

Proof. Beginning with the first Betti number, observe that, for an arbitrary group G, one has that $b_1(G \oplus G) = b_1(G) + b_1(G)$. Likewise, using the fact that $K(G * G, 1)$ is homotopy equivalent to $K(G, 1) \vee K(G, 1)$, one obtains $b_1(G * G) = b_1(G) + b_1(G)$. Thus in general $b_1(T^{\oplus n}) = b_1(T^{*n}) = 0$.

For the second Betti number, observe that $b_2(T \oplus T) = 2b_2(T) + b_1(T)^2$ by the Künneth formula and so by $b_1(T) = 0$, one obtains $b_2(T \oplus T) = 2b_2(T)$. In general, $b_2(T^{\oplus n}) = nb_2(T)$, hence $b_2(T^{\oplus n}) = 2n$. Likewise, $b_2(T * T) = b_2(T) + b_2(T)$ and so $b_2(T^{*n}) = 2n$.

Let G_n denote $T^{\oplus n}$ or T^{*n}, then $b_1(G_n) = 0$ and $b_2(G_n) = 2n$. When $n \geq 12$ we have that $b_1(G_n) = 0$ but $b_2(G_n) > 22$, hence by Theorem 2.2.2, it is impossible for G_n to be an SCY group. $\square$

Chapter 3

Seiberg-Witten Theory

The purpose of this chapter is to state Theorem 3.4.3 and Theorem 3.4.6. These results involve the machinery of Seiberg-Witten theory, so much of this chapter is devoted to developing the necessary infrastructure, especially that of spin^c groups and spin^c structures.

3.1 Clifford Algebras & Spin Groups

Definition 3.1.1. Let V be a real vector space of dimension n equipped with an inner product $\langle \cdot, \cdot \rangle$. If $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ is an orthonormal basis for V, then the *Clifford algebra* $\mathcal{C}\ell(V)$ of V is the unital algebra generated by $\mathbf{e}_1, \ldots, \mathbf{e}_n$ according to vector addition in V and the formal multiplication rules

$$\mathbf{e}_i^2 = -1 \qquad \text{and} \qquad \mathbf{e}_i \mathbf{e}_j = -\mathbf{e}_j \mathbf{e}_i$$

for all $1 \leq i, j \leq n$ such that $i \neq j$.

If $\mathbf{v}, \mathbf{w} \in V$, then it is a straightforward exercise in summation manipulation to show that they satisfy the relation $\mathbf{v}\mathbf{w} + \mathbf{w}\mathbf{v} = -2\langle \mathbf{v}, \mathbf{w} \rangle$.

An element of $\mathcal{C}\ell(V)$ is *simple* if can be written as a product of basis elements of V. The simple elements generate $\mathcal{C}\ell(V)$ as a vector space, but do *not* form a basis.

For fixed $n \in \mathbb{N}$, a *multi-index* is a set $I \subseteq \{1, \ldots, n\}$; for $n = 0$ the only multi-index is the empty set. Choosing n to be the dimension of V as above and given a multi-index $I = \{i_1 < \cdots < i_k\}$, define $\mathbf{e}_I = \mathbf{e}_{i_1} \cdots \mathbf{e}_{i_k}$. When $I = \emptyset$, then $\mathbf{e}_I = 1$. The *degree* of $\mathbf{e}_I$ is k, the cardinality of I.

Observe that any simple element can be written as $\pm\mathbf{e}_I$ for some multi-index I by a finite sequence of transpositions and cancellations given by the formal multiplication rules of Definition 3.1.1. Hence $\{\mathbf{e}_I\}_{I \subseteq \{1,\ldots,n\}}$ is a 2^n-dimensional basis for $\mathcal{C}\ell(V)$ as a vector space.

An arbitrary element $\sum_I \alpha_I \mathbf{e}_I \in \mathcal{C}\ell(V)$ has *degree* k if $\alpha_I = 0$ whenever $|I| \neq k$. Note that 0 is degree k for all $k \in \mathbb{N}$. The vector subspace of all degree k elements is denoted by $\mathcal{C}\ell^k(V)$ and thus $\mathcal{C}\ell(V)$ is a graded algebra of the form

$$\mathcal{C}\ell(V) = \bigoplus_{k \in \mathbb{N}} \mathcal{C}\ell^k(V).$$

Observe that $\mathcal{C}\ell^0(V) = \mathbb{R}$ and $\mathcal{C}\ell^1(V) = V$. Likewise, $\mathcal{C}\ell^n(V) \cong V$ and if $k > n$, then $\mathcal{C}\ell^k(V)$ is trivial.

In general, $\mathcal{C}\ell^k(V)$ fails to be a subalgebra of $\mathcal{C}\ell(V)$, but

$$\mathcal{C}\ell^{\mathrm{ev}}(V) = \bigoplus_{k \text{ even}} \mathcal{C}\ell^k(V)$$

is a subalgebra and is called the *even subalgebra* of $\mathcal{C}\ell(V)$. Its odd counterpart, $\mathcal{C}\ell^{\mathrm{odd}}(V)$, however, is not.

The purpose behind this introductory exposition on Clifford algebras is to set the foundation upon which one can define the spin group and, ultimately, the spin^c group.

Definition 3.1.2. Let V be a real vector space of dimension n and equipped with an inner product $\langle \cdot, \cdot \rangle$. The *pin group* of V is the group $\mathrm{Pin}(V)$ of all elements of the form $\mathbf{v}_1 \cdots \mathbf{v}_k \in \mathcal{C}\ell(V)$ where $\mathbf{v}_i \in V$ and $\langle \mathbf{v}_i, \mathbf{v}_i \rangle = 1$. The *spin group* of V is the group $\mathrm{Spin}(V) = \mathrm{Pin}(V) \cap \mathcal{C}\ell^{\mathrm{ev}}(V)$.

In the case that V is n-dimensional Euclidean space equipped with the standard inner product structure, one writes $\mathrm{Spin}(n)$ instead of $\mathrm{Spin}(\mathbb{R}^n)$ and refers to $\mathrm{Spin}(n)$ as the *nth spin group*. An equivalent characterization of $\mathrm{Spin}(n)$ is that it is the unique nontrivial double cover of $\mathrm{SO}(n)$ such that, when $n \geq 2$, there exists a short exact sequence

$$1 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathrm{Spin}(n) \longrightarrow \mathrm{SO}(n) \longrightarrow 1$$

of Lie groups. To see this, note that $\mathrm{Pin}(n)$ acts orthogonally by conjugation on $\mathbb{R}^n$, yielding a map $\mathrm{Pin}(n) \to \mathrm{O}(n)$. This induces the map $\mathrm{Spin}(n) \to \mathrm{SO}(n)$ above.

To show that this map has kernel $\{\pm 1\}$, observe that the only elements of $\mathrm{Pin}(n)$ which act trivially are those that commute with everything in $\mathbb{R}^n$, which is to say, the scalars. Note that the inner product on $\mathbb{R}^n$ induces a natural multiplicative inner product structure on $\mathcal{C}\ell(n)$ [25]. Since every generator $\mathbf{v}$ of $\mathrm{Pin}(n)$ satisfies $\langle \mathbf{v}, \mathbf{v} \rangle = 1$, so too does every element of $\mathrm{Pin}(n)$. Hence the only scalars in $\mathrm{Pin}(n)$ are ± 1, likewise for $\mathrm{Spin}(n)$.

In low dimensions, $\mathrm{Spin}(n)$ admits so-called "exceptional isomorphisms" that relate it to other Lie groups but do not persist into higher values of n. Selected exceptional isomorphisms of particular use or interest are collected below.

Proposition 3.1.3. *The nth spin group satisfies the following exceptional isomorphisms:*

$$\begin{array}{ll} \mathrm{Spin}(1) \cong \mathbb{Z}_2 & \mathrm{Spin}(2) \cong \mathrm{U}(1) \\ \mathrm{Spin}(3) \cong \mathrm{SU}(2) & \mathrm{Spin}(4) \cong \mathrm{SU}(2) \times \mathrm{SU}(2) \end{array} .$$

As $\mathcal{C}\ell(V)$ is even-dimensional, one can consider its complexification $\mathcal{C}\ell(V) \otimes \mathbb{C}$, likewise with $\mathcal{C}\ell^{\mathrm{ev}}(V)$. One could define $\mathrm{Pin}^c(V)$ as the analogue to $\mathrm{Pin}(V)$ and hence define $\mathrm{Spin}^c(V)$ this way. It is however more convenient to define the *spin^c group* $\mathrm{Spin}^c(V)$ to be a subgroup of $\mathcal{C}\ell(V) \otimes \mathbb{C}$ generated by $\mathrm{Spin}(V)$ and $\mathrm{U}(1)$, with $\mathrm{U}(1)$ identified with the scalars of norm 1 in $\mathbb{C}$ as is done in [25]. The *nth spin^c group* is the group $\mathrm{Spin}^c(\mathbb{R}^n)$ and is abbreviated to $\mathrm{Spin}^c(n)$.

Since scalars commute with everything in $\mathcal{C}\ell(V)$, any element of $\mathrm{Spin}^c(V)$ can be written as $e^{i\theta}s$, where $\theta \in \mathbb{R}$ and $s \in \mathrm{Spin}(V)$. This yields a map $\mathrm{Spin}(V) \times \mathrm{U}(1) \to \mathrm{Spin}^c(V)$ that multiplies elements of a pair together. Any element (s, z) of the kernel must satisfy $zs = 1$. Since $s \in \mathrm{Spin}(V)$ it is either ± 1 or else is not a scalar, hence $s = \pm 1$. Thus $\pm z = 1$, so $z = \pm 1$ as well and so the kernel of this map is $\{(1, 1), (-1, -1)\}$. Hence there is an isomorphism $\mathrm{Spin}^c(V) \cong \mathrm{Spin}(V) \times_{\mathbb{Z}_2} \mathrm{U}(1)$, taking the $\mathbb{Z}_2$-action to be the diagonal action.

This twisted product corresponds to a short exact sequence for $\mathrm{Spin}^c(n)$ analogous to the one for $\mathrm{Spin}(n)$ above:

$$1 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathrm{Spin}^c(n) \longrightarrow \mathrm{SO}(n) \times \mathrm{U}(1) \longrightarrow 1 \ .$$

Hence $\mathrm{Spin}^c(n)$ is a double cover of $\mathrm{SO}(n) \times \mathrm{U}(1)$; its $\mathrm{Spin}(n)$ subgroup covering the $\mathrm{SO}(n)$ factor and its $\mathrm{U}(1)$ subgroup double covering the $\mathrm{U}(1)$ factor.

Like $\mathrm{Spin}(n)$, there are exceptional isomorphisms for $\mathrm{Spin}^c(n)$ in low dimensions.

Proposition 3.1.4. *The nth spin^c group satisfies the following exceptional isomorphisms:*

$$\begin{array}{ll} \mathrm{Spin}^c(1) \cong \mathrm{U}(1) & \mathrm{Spin}^c(2) \cong \mathrm{U}(1) \times \mathrm{U}(1) \\ \mathrm{Spin}^c(3) \cong \mathrm{U}(2) & \mathrm{Spin}^c(4) \cong \mathrm{U}(2) \times_{\mathrm{U}(1)} \mathrm{U}(2) \end{array} .$$

Note that the isomorphisms $\mathrm{Spin}^c(1) \cong \mathrm{U}(1)$ and $\mathrm{Spin}^c(2) \cong \mathrm{U}(1) \times \mathrm{U}(1)$ should be thought as identifying the spin^c groups with double covers of $\mathrm{U}(1)$ and $\mathrm{U}(1) \times \mathrm{U}(1)$ respectively, both of which are isomorphic to their base space. The argument for the $\mathrm{Spin}^c(4)$ exceptional isomorphism is sketched in §3.2 below.

3.2 Spin^c Structures

The treatment of spin^c structures in this section follows that of [14]. Recall that if G is a Lie group, then a *principal G-bundle* is a principal bundle with structure group G.

Definition 3.2.1. Let (X, g) be an orientable Riemannian 4-manifold with orthonormal frame bundle P. For a given principal $\mathrm{U}(1)$-bundle Q over X that satisfies $c_1(Q) \equiv w_2(X)$ modulo 2, the *spin^c structure* on (X, g) associated with Q is a double covering of $P \times Q$ by a principal $\mathrm{Spin}^c(4)$-bundle $\widetilde{P}$ such that the covering map is equivariant for the principal bundle actions with respect to the double covering $\mathrm{Spin}^c(4) \to \mathrm{SO}(4) \times \mathrm{U}(1)$.

A priori, the spin^c structure $\widetilde{P}$ depends on Q and the Riemannian metric g. However, the space of Riemannian metrics is path-connected and contractible, hence any two metrics g_1 and g_2 on X are homotopy equivalent. As a result, the spin^c structures $\widetilde{P}_1$ and $\widetilde{P}_2$ associated with (g_1, Q) and (g_2, Q) respectively can be canonically identified, leaving the spin^c structure dependent only on the choice of associated bundle Q.

Given a spin^c structure $\widetilde{P}$, the associated bundle Q can be recovered as $\det(\widetilde{P})$, that is, as the bundle whose transition functions are the determinants of the transition functions of $\widetilde{P}$.

Using the exceptional isomorphism for $\mathrm{Spin}(4)$ in Proposition 3.1.3 and the isomorphism $\mathrm{Spin}^c(n) \cong \mathrm{Spin}(n) \times_{\mathbb{Z}_2} \mathrm{U}(1)$, one can see that $\mathrm{Spin}^c(4)$ is the quotient of $\mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{U}(1)$ by a free $\mathbb{Z}_2$-action. Each $\mathrm{SU}(2)$ factor induces a projection map $\mathrm{Spin}^c(4) \to \mathrm{U}(2)$, hence given a spin^c structure $\widetilde{P}$ (and a metric g) each factor yields a $\mathrm{U}(2)$-bundle over X. These bundles are called the *positive and negative spinor bundles* of the spin^c bundle and are denoted by V^+ and V^- respectively. The *spinor bundle* is the bundle $W = V^+ \oplus V^-$.

These projections can be used to obtain another exceptional isomorphism for $\mathrm{Spin}^c(4)$. By combining projections, one obtains a map $\mathrm{Spin}^c(4) \to \mathrm{U}(2) \times \mathrm{U}(2)$. The kernel of this map is $\mathrm{U}(1)$ as it is forgotten by the maps induced by the projections. Hence $\mathrm{Spin}^c(4) \cong \mathrm{U}(2) \times_{\mathrm{U}(1)} \mathrm{U}(2)$.

This exceptional isomorphism reveals that V^+ and V^- are not so terribly different. In particular, it reveals that if an element of $\mathrm{Spin}^c(4)$ is thought of as a matrix, then it can be written in the form $\left[\begin{smallmatrix} B & 0 \\ 0 & C \end{smallmatrix}\right]$ where $B, C \in \mathrm{U}(2)$ satisfy $\det(B) = \det(C)$, recalling that $\det\colon \mathrm{U}(2) \to \mathrm{U}(1)$. Hence the determinants of the transition functions of V^+ and V^- are equal. Collectively V^+ and V^- are denoted by $V^{\pm}$.

An equivalent characterization of spin^c structures, such as in [28, Ch. 5], defines a spin^c structure on X as a homomorphism $\Gamma\colon \mathrm{T}X \to \mathrm{End}(W)$ subject to certain conditions. The bundle $\widetilde{P}$ can then be recovered as a bundle of spin^c isomorphisms from a model spinor bundle W_0 to W.

Regardless of characterization, a key object to the study of spin^c structures is the characteristic line bundle and its first Chern class in $H^2(X;\mathbb{Z})$. The characteristic line bundle is constructed from $V^{\pm}$.

Definition 3.2.2. Let $\widetilde{P}$ be a spin^c structure on (X, g). The *characteristic line bundle* of $\widetilde{P}$ is the complex line bundle $L = \det(V^{\pm})$.

That is to say, the transition functions of L are multiplication by the (complex) determinant of the associated transition functions of V^+ or, equivalently, V^-.

A complex line bundle E acts on a spin^c structure $\widetilde{P}$, producing a *twisted* spin^c structure denoted by $\widetilde{P}_E$. The description of this construction can be found, for example, in [28, Proof of Thm. 5.8iii], but the interest of this dissertation lies more on the effects of twisting on the characteristic line bundle. The effect of "twisting" by E on the spinor bundle is straightforward: $W_E = W \otimes E$ and $V_E^{\pm} = V^{\pm} \otimes E$. The proposition below [28, Thm. 5.8] describes the twisted characteristic line bundle and characterizes the complex line bundles that fix (the isomorphism class of) $\widetilde{P}$.

Proposition 3.2.3. *Let (X, g) be an orientable Riemannian manifold of even dimension. If $\widetilde{P}$ is a spin^c structure on X with characteristic line bundle L and E a complex line bundle over X, then the characteristic line bundle of $\widetilde{P}_E$ is*

$$L_E = L \otimes (E \otimes E).$$

Furthermore, if $\widetilde{P}_1$ and $\widetilde{P}_2$ are spin^c structures on X, then there exists a complex line bundle E such that $\widetilde{P}_{1,E} \cong \widetilde{P}_2$. Moreover, $\widetilde{P}_1 \cong \widetilde{P}_2$ if and only if $c_1(E) = 0$.

Recall that if A and B are complex vector bundles, then the first Chern class of the tensor product $A \otimes B$ is given by $c_1(A \otimes B) = \operatorname{rank}(A) c_1(B) + c_1(A) \operatorname{rank}(B)$. Hence the effects of twisting by E on the first Chern class can be quickly computed.

Corollary 3.2.4. $c_1(L_E) = c_1(L) + 2c_1(E)$.

As the first Chern classes of Q, $\widetilde{P}$, and L coincide, Corollary 3.2.4 is equivalent to $c_1(\widetilde{P}_E) = c_1(\widetilde{P}) + 2c_1(E)$.

3.3 The Seiberg-Witten Invariant

Given a spin^c manifold $(X, g, \widetilde{P})$ with associated bundle $Q = \det(\widetilde{P})$ and spinor bundles $V^{\pm}$, the *Seiberg-Witten equations* on $(X, g, \widetilde{P})$ are a pair of differential equations

$$D_A\phi = 0$$
$$\rho(F_A^+) = \sigma(\phi, \phi)$$

whose solutions are pairs (A, ϕ) of a connection A on Q and a section ϕ of V^+. The operator D_A maps sections of V^+ to sections of V^- and F_A^+ is the self-dual part of the curvature of A; both of which depend on choice of metric, but this dependence is not of practical concern for this dissertation. The maps ρ and σ are bundle maps

$$\rho\colon \Omega^2_+(X; \mathbb{C}) \to \mathrm{End}_{\mathbb{C}}(V^+)$$
$$\sigma\colon V^+ \otimes V^+ \to \mathrm{End}_{\mathbb{C}}(V^+)$$

whose precise definition will be left unstated, as the interest of this dissertation is in the moduli space of solutions rather than the equations themselves, which shall remain in a black box. For a more detailed survey of the Seiberg-Witten equations, see [14].

If ϕ is the zero section, then a pair (A, ϕ) is *reducible* and if not, then the pair is *irreducible.*

The *gauge group* $\mathcal{G}$ for Q is the automorphism group of the principal U(1)-bundle. This acts on pairs (A, ϕ) by the action $u \cdot (A, \phi) = ((u^2)^*A, u^{-1}\phi)$; if (A, ϕ) is a solution to the Seiberg-Witten equations, so too will $u \cdot (A, \phi)$ be. Denote by $\widehat{\mathcal{B}}$ the space of all irreducible pairs (A, ϕ) modulo this action.

The moduli space of solutions to the Seiberg-Witten equations is denoted by $\mathcal{M}(X, g, \widetilde{P})$ or, if $(X, g, \widetilde{P})$ is clear from context, by $\mathcal{M}$ for convenience. When $b^+(X) \geq 1$ and for a generic choice of metric, $\mathcal{M}$ is a smooth compact orientable submanifold of $\widehat{\mathcal{B}}$ (see [14, Fact 1.4]). If given an orientation, $\mathcal{M}$ then has a fundamental class $[\mathcal{M}] \in H_*(\widehat{\mathcal{B}}; \mathbb{Z})$. In order to remove the *a priori* dependence of $[\mathcal{M}]$ on the choice of metric, one requires that $b^+(X) \geq 2$.

An orientation is given by choosing a *homology orientation* of X, that is an orientation α of the vector space

$$H^0(X; \mathbb{R}) \otimes (\Lambda^{b_1(X)} H^1(X; \mathbb{R}))^* \otimes \Lambda^{b^+(X)} H^2_+(X; \mathbb{R}),$$

where $H^2_+(X; \mathbb{R})$ is a maximal subspace of $H^2(X; \mathbb{R})$ on which the cup product is a positive definite quadratic form and Λ^k denotes the kth exterior power. Observe that this vector space is 1-dimensional as each factor is 1-dimensional (recall that if V is n-dimensional, then $\Lambda^k V$ is $\binom{n}{k}$-dimensional).

Proposition 3.3.1. *Let the* formal dimension *of $\mathcal{M}$ be the quantity*

$$\frac{1}{4}[\langle c_1(Q)^2, [X]\rangle - (2\chi(X) + 3\sigma(X))].$$

If $\mathcal{M}$ is nonempty, then it has dimension equal to its formal dimension.

That the formal dimension is an integer reveals a necessary condition on X. It can be shown [28, §7.4] that in order for $\langle c_1(Q)^2, [X]\rangle - (2\chi(X) + 3\sigma(X))$ to be divisible by 4, one must have that $b^+(X) - b_1(X)$ is odd. Verifying this is straightforward in the symplectic Calabi-Yau case that concerns this dissertation (see §3.4 below for proof that symplectic manifolds are spin^c) and the proof somewhat demystifies the condition.

Lemma 3.3.2. *If X is an SCY 4-manifold with $b_1(X) \geq 2$, then $b^+(X) - b_1(X)$ is odd.*

Proof. It is equivalent to show that $1 - b_1(X) + b^+(X)$ is even. By Corollary 2.2.1, one has that $2\chi(X) + 3\sigma(X) = 0$. Expanding the Euler characteristic and signature, one obtains

$$\begin{aligned} 2\chi(X) + 3\sigma(X) &= 4 - 4b_1(X) + 2b_2(X) + 3b^+(X) - 3b^-(X) \\ &= 4 - 4b_1(X) + 5b^+(X) - b^-(X) \\ &= 4(1 - b_1(X) + b^+(X)) + \sigma(X), \end{aligned}$$

hence $1 - b_1(X) + b^+(X)$ is even if and only if $\sigma(X)$ is divisible by 8. From Theorem 2.2.3, it is known that $\sigma(X) = 0$. $\square$

All that is left is to define the Seiberg-Witten invariant.

Definition 3.3.3. Let $(X, \widetilde{P})$ be a spin^c manifold with $b^+(X) \geq 2$ and α a homology orientation of X. The *Seiberg-Witten invariant* of $\widetilde{P}$ is defined to be $\mathrm{sw}(\widetilde{P}) = [\mathcal{M}]$, where $[\mathcal{M}] \in H_d(\widehat{\mathcal{B}}; \mathbb{Z})$ is the fundamental class of $\mathcal{M}$ and d is the dimension of $\mathcal{M}$.

In the case that $\mathcal{M}$ is 0-dimensional, the Seiberg-Witten invariant is a number.

3.4 Applications of the Seiberg-Witten Invariant

The introduction of the Seiberg-Witten invariant to this dissertation is motivated by the desire to make use of two theorems. The first is due to C. H. Taubes in [29] and [30], although the notation here is closer to that of [14], and the second is due to D. Ruberman and S. Strle in [27].

As the Taubes constraints apply to symplectic 4-manifolds, this section begins with a confirmation of the existence of a spin^c structures on symplectic manifolds by showing that every almost complex 4-manifold is spin^c (see [14] for more detail).

If X is an almost complex 4-manifold with almost complex structure J, and equipped with a J-compatible metric, then J corresponds to a reduction of the structure group for $\mathrm{T}X$ to $\mathrm{U}(2)$. Consider the map $\iota \times \det\colon \mathrm{U}(2) \to \mathrm{SO}(4) \times \mathrm{U}(1)$, where $\iota\colon \mathrm{U}(2) \to \mathrm{SO}(4)$ is the map

$$\begin{bmatrix} a_{11}+ib_{11} & a_{21}+ib_{21} \\ a_{12}+ib_{12} & a_{22}+ib_{22} \end{bmatrix} \mapsto \begin{bmatrix} a_{11} & b_{11} & a_{21} & b_{21} \\ -b_{11} & a_{11} & -b_{21} & a_{21} \\ a_{12} & b_{12} & a_{22} & b_{22} \\ -b_{12} & a_{12} & -b_{22} & a_{22} \end{bmatrix}$$

and det is the standard complex determinant. The map $\iota \times \det$ lifts to $\mathrm{Spin}^c(4)$ through the double cover $\mathrm{Spin}^c(4) \to \mathrm{SO}(4) \times \mathrm{U}(1)$, inducing a canonical spin^c structure on X whose characteristic line bundle has $c_1(\mathrm{T}X, J)$ for its first Chern class [14].

As discussed in §2.1, every symplectic manifold (X, ω) admits, up to homotopy equivalence, a unique almost complex structure compatible with the symplectic form. Any two homotopic almost complex structures have equal first Chern class, hence the first Chern class depends only on the symplectic form. As a result, every symplectic manifold admits a canonical spin^c structure $\widetilde{P}_{\mathrm{can}}$ whose first Chern class is $-K_\omega = c_1(\mathrm{T}X, J)$, where K_ω is the canonical class of (X, ω) (see Chapter 2).

Corollary 3.4.1. *Let (X, ω) be a symplectic 4-manifold and $\mathcal{M}$ the moduli space of solutions to the Seiberg-Witten equations for the canonical spin^c structure on X. If $\mathcal{M}$ is nonempty, then $\mathcal{M}$ is 0-dimensional.*

Proof. Since $\mathcal{M}$ is nonempty, Proposition 3.3.1 gives its dimension as

$$\frac{1}{4}[\langle K_\omega^2, [X]\rangle - (2\chi(X) + 3\sigma(X))].$$

By the Hirzebruch signature formula of Proposition 2.1.3, this quantity is 0. $\square$

Hence the canonical spin^c structure on X has a Seiberg-Witten invariant that lives in $H_0(\widehat{\mathcal{B}}; \mathbb{Z})$, the value of which can be found in Theorem 3.4.3 below. One might wonder about the dimension of the Seiberg-Witten invariants associated with other spin^c structures on X. The result stated in Theorem 3.4.2 below is a consequence of Theorem 3.4.3, but is stated now in order to resolve this question. A proof can be found in [14] and originates from [31] and [32].

Theorem 3.4.2. *Let X be a symplectic 4-manifold with $b^+(X) \geq 2$. If $\widetilde{P}$ is a spin^c structure on X with nontrivial $\mathrm{sw}(\widetilde{P}) \in H_*(\widehat{\mathcal{B}}; \mathbb{Z})$, then $\mathcal{M}$ is 0-dimensional.*

A spin^c structure $\widetilde{P}$ can be identified with its first Chern class, which coincides with $c_1(L)$, where L is the characteristic line bundle for $\widetilde{P}$. As a result, $H^2(X; \mathbb{Z})$ classifies spin^c structures on X. Identifying $H^2(X; \mathbb{Z})$ with the space of spin^c structures on X requires a preferred spin^c structure to act as an origin.

If (X, ω) is symplectic, then its canonical spin^c structure $\widetilde{P}_{\text{can}}$ is a preferred spin^c structure with $c_1(\widetilde{P}_{\text{can}}) = -K_\omega$. The classification is presented in the following way. Given a class $[\alpha] \in H^2(X; \mathbb{Z})$, the *spin^c structure associated with the twist* $[\alpha]$ is the spin^c structure

$\widetilde{P}_{[\alpha]}$ obtained by twisting the canonical spin^c structure by the complex line bundle $E_{[\alpha]}$ that satisfies $c_1(E_{[\alpha]}) = [\alpha]$.

It is important to note that $c_1(\widetilde{P}_{[\alpha]}) \neq [\alpha]$, but instead $c_1(\widetilde{P}_{[\alpha]}) = -K_\omega + 2[\alpha]$ by Corollary 3.2.4. In this formulation, the canonical spin^c structure on X is $\widetilde{P}_0$, the spin^c structure associated with the trivial twist.

Taubes Constraints

Theorem 3.4.3 (Taubes)**.** *Let (X, ω) be a closed symplectic 4-manifold with $b^+(X) \geq 2$ and canonical class K_ω. If $\widetilde{P}_0$ and $\widetilde{P}_{K_\omega}$ are the spin^c structures on M associated with the trivial twist and K_ω respectively, then $\mathrm{sw}(\widetilde{P}_0) = 1$ and $\mathrm{sw}(\widetilde{P}_{K_\omega}) = (-1)^{(\chi(X)+\sigma(X))/4}$. Moreover, if $[\alpha] \in H^2(X; \mathbb{Z})$ is such that $\mathrm{sw}(\widetilde{P}_{[\alpha]}) \neq 0$ then*

$$0 \leq \langle [\alpha] \cup [\omega], [X] \rangle \leq \langle K_\omega \cup [\omega], [X] \rangle.$$

Furthermore, $[\alpha] = 0$ if and only if $0 = \langle [\alpha] \cup [\omega], [X] \rangle$ and $[\alpha] = K_\omega$ if and only if $\langle [\alpha] \cup [\omega], [X] \rangle = \langle K_\omega \cup [\omega], [X] \rangle$.

That the expression $\frac{1}{4}(\chi(X) + \sigma(X))$ is always an integer is equivalent to requiring that $\frac{1}{2}(1 - b_1(X) + b^+(X))$ be an integer, that is, $1 - b_1(X) + b^+(X)$ must be even. For SCY 4-manifolds this is guaranteed by Lemma 3.3.2; for general symplectic 4-manifolds a more general argument is required (see, for example, [28, Remark 13.3]).

When (X, ω) is an SCY, the inequality in Theorem 3.4.3 collapses, forcing the evaluation $\langle [\alpha] \cup [\omega], [X] \rangle$ to vanish. This highly constrains the spin^c structures on SCY 4-manifolds that can have nontrivial Seiberg-Witten invariant.

Corollary 3.4.4. *If (X, ω) is an SCY 4-manifold with $b^+(X) \geq 2$, then the only class $[\alpha] \in H^2(X; \mathbb{Z})$ such that $\mathrm{sw}(\widetilde{P}_{[\alpha]}) \neq 0$ is the trivial class.*

Proof. Let $[\alpha] \in H^2(X; \mathbb{Z})$ be such that $\mathrm{sw}(\widetilde{P}_{[\alpha]}) \neq 0$, then by Theorem 3.4.3

$$0 \leq \langle [\alpha] \cup [\omega], [X] \rangle \leq \langle K_\omega \cup [\omega], [X] \rangle.$$

However, as X is an SCY, $K_\omega = 0$, hence Taubes's inequality becomes

$$0 \leq \langle [\alpha] \cup [\omega], [X] \rangle \leq 0,$$

which collapses to $\langle [\alpha] \cup [\omega], [X] \rangle = 0$, so by Theorem 3.4.3 it must be that $[\alpha] = 0$. $\square$

That is, for an SCY 4-manifold, the canonical spin^c structure is the only spin^c structure with nontrivial Seiberg-Witten invariant.

A Result of Ruberman & Strle

Recall that, if X and Y are manifolds, then X is an *(integral) homology* Y if $H_p(X; \mathbb{Z}) \cong H_p(Y; \mathbb{Z})$ for all $p \geq 0$. A *rational homology* Y is defined analogously. Note that this does *not* imply that the ring structures of the cohomologies of X and Y are isomorphic, merely that the individual grades are isomorphic as groups.

Definition 3.4.5. Let X be a rational homology 4-torus. Pick a basis $[\alpha_1], \ldots, [\alpha_4]$ for $H^1(X; \mathbb{Z})$. The *determinant* of X is

$$\det(X) = |\langle [\alpha_1] \cup [\alpha_2] \cup [\alpha_3] \cup [\alpha_4], [X] \rangle|,$$

where $[X]$ is the fundamental class of X.

Note that $H^1(X;\mathbb{Z})$ is torsion-free by the universal coefficient theorem. To elaborate, it splits as the direct sum of the free part of $H_1(X;\mathbb{Z})$ and the torsion part of $H_0(X;\mathbb{Z})$. The latter is trivial, hence one is guaranteed that $H^1(X;\mathbb{Z}) \cong \mathbb{Z}^4$ even if $H_1(X;\mathbb{Z})$ has torsion.

The determinant does not depend on choice of basis. Let $[\beta_1], \dots, [\beta_4]$ be another basis for $H^1(X;\mathbb{Z})$ and $A \in \mathrm{GL}(4;\mathbb{Z})$ the change of basis matrix that takes $[\alpha_1], \dots, [\alpha_4]$ to $[\beta_1], \dots, [\beta_4]$; thus

$$|\langle [\beta_1] \cup [\beta_2] \cup [\beta_3] \cup [\beta_4], [X] \rangle| = |\det(A)||\langle [\alpha_1] \cup [\alpha_2] \cup [\alpha_3] \cup [\alpha_4], [X] \rangle|.$$

Matrices in $\mathrm{GL}(4;\mathbb{Z})$ have determinant ± 1, hence the two bases determine the same determinant for X.

Theorem 3.4.6 (Ruberman-Strle)**.** *If X is a symplectic rational homology 4-torus with a spin structure that lifts to a spin^c structure $\widetilde{P}$ whose characteristic line bundle is trivial, then* $\mathrm{sw}(\widetilde{P}) \equiv \det(X)$ *modulo* 2.

A *spin structure* on an orientable Riemannian n-manifold (X, g) is a double cover of its orthonormal frame bundle by a principal $\mathrm{Spin}(n)$-bundle such that the covering map is equivariant with respect to the double cover $\mathrm{Spin}(n) \to \mathrm{SO}(n)$. Spin structures on (X, g) are classified by $H^1(X;\mathbb{Z}_2)$, so there is a twisting $H^1(X;\mathbb{Z}_2)$-action for spin structures analogous to the $H^2(X;\mathbb{Z})$-action that twists spin^c structures.

Every spin manifold is a spin^c manifold as $\mathrm{Spin}(n)$ is a subgroup of $\mathrm{Spin}^c(n)$. Let $\mathcal{S}(X)$ and $\mathcal{S}^c(X)$ denote the collections of all spin and spin^c structures on (X, g) respectively, then there is a map $\Psi\colon \mathcal{S}(X) \to \mathcal{S}^c(X)$ that turns spin structures into their associated spin^c structures.

Given a spin structure S on X, the associated spin^c structure $\Psi(S)$ always has trivial characteristic line bundle by the following argument.

From the group extension $\mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}_2$ one obtains a long exact sequence in cohomology for the space X:

$$\cdots \longrightarrow H^n(X;\mathbb{Z}) \xrightarrow{\times 2} H^n(X;\mathbb{Z}) \longrightarrow H^n(X;\mathbb{Z}_2) \xrightarrow{\beta} H^{n+1}(X;\mathbb{Z}) \longrightarrow \cdots .$$

The map $\beta\colon H^n(X;\mathbb{Z}_2) \to H^{n+1}(X;\mathbb{Z})$ is the *Bockstein map* and the map Ψ above satisfies

$$\Psi([\alpha]\cdot S) = \beta([\alpha])\cdot \Psi(S)$$

where the action on the left twists a spin structure by a class in $H^1(X;\mathbb{Z}_2)$ and the action on the right twists a spin^c structure by a class in $H^2(X;\mathbb{Z})$. As noted in [33, §3], the image of Ψ is thus exactly those spin^c structures with first Chern class equal to 0 and hence trivial characteristic line bundle.

Theorem 3.4.6 above differs from the original theorem in [27], which assumes X to be an *integral* homology 4-torus. The result still holds in the rational case, [19, §7.3] for example, applies the original result of [27] to such a case.

To elaborate a bit on why the result passes to rational homology tori: a key step in Ruberman and Strle's proof is finding a preferred choice of U(1)-connection on $L^{1/2}$, the square root of the characteristic line bundle L of $\widetilde{P}$. As $H^2(X;\mathbb{Z})$ has no torsion in the integral homology case, $c_1(L^{1/2}) = 0$ and is thus trivializable; the preferred U(1)-connection is the product connection.

In the rational homology case, $H^2(X;\mathbb{Z})$ may very well have 2-torsion—and in fact the argument in Chapter 5 depends on it—so now all one can conclude is that $c_1(L^{1/2})$ is

2-torsion. A complex line bundle admits a flat connection if and only if its first Chern class is torsion, so $L^{1/2}$ is at least a flat bundle and by personal correspondence with S. Strle a flat connection is sufficient as a preferred choice and the rest of the proof proceeds as in the integral homology case.

Chapter 4

Hirsch Length

In the theory of groups, the notion of Hirsch length is classically defined for *polycyclic groups*—groups G that admit a subnormal series

$$1 = G_0 \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G_{n-1} \trianglelefteq G_n = G$$

such that the quotients G_i/G_{i-1} are cyclic groups for all $1 \leq i \leq n$. There, the Hirsch length of G is the number of infinite quotients that occur in a given subnormal series. This is independent of the choice of subnormal series and hence is a group invariant.

4.1 Elementary Amenable Groups

Work of J. Hillman in [11] and in collaboration with P. Linnell in [12] generalizes the notion of Hirsch length to the so-called elementary amenable groups. The construction therein references certain metaproperties of groups, which are collected here for later reference.

Definition 4.1.1. Let $\mathscr{P}$ and $\mathscr{Q}$ be collections of groups. A group G is said to be

- *$\mathscr{P}$-by-$\mathscr{Q}$* if G is an extension of $Q \in \mathscr{Q}$ by $K \in \mathscr{P}$, that is to say, if there exists $K \in \mathscr{P}$ and $Q \in \mathscr{Q}$ such that

$$1 \longrightarrow K \longrightarrow G \longrightarrow Q \longrightarrow 1$$

  is a short exact sequence of groups;

- *locally $\mathscr{P}$* if every finitely generated subgroup $H \leq G$ is in $\mathscr{P}$;

- *locally $\mathscr{P}$-insulated* if for every finitely generated subgroup $H \leq G$ there exists $K \in \mathscr{P}$ such that $H \leq K \leq G$;

- *poly-$\mathscr{P}$* if there exists a subnormal series for G whose quotients are in $\mathscr{P}$;

- *virtually $\mathscr{P}$* if there exists a finite-index subgroup $H \leq G$ such that $H \in \mathscr{P}$.

The notation $L\mathscr{P}$ is employed in [11] and [12] to represent what is here called "locally $\mathscr{P}$-insulated" and the subgroup H from that definition would be referred to by Hillman as a "$\mathscr{P}$-subgroup". The choice of "locally $\mathscr{P}$-insulated" here is *ad hoc*, chosen with the intent to resemble that of its fellow metaproperties, to avoid any suggestions that $L\mathscr{P}$ might make to mean "locally $\mathscr{P}$" instead, and to avoid the ungrammatical sounding "G is locally $\mathscr{P}$-subgroup". Any similarity to nomenclature found elsewhere is purely coincidental.

When metaproperties intermix, hyphenation indicates "order of operation" as it were. That is, a locally-$\mathscr{P}$ by $\mathscr{Q}$ group is an extension of a $\mathscr{Q}$ group by a locally $\mathscr{P}$ group, whereas a locally $\mathscr{P}$-by-$\mathscr{Q}$ group is a group for which every finitely generated subgroup is an extension of a $\mathscr{Q}$ group by a $\mathscr{P}$ group.

Hillman defines the class of elementary amenable groups via transfinite induction. A collection of groups $\mathscr{X}_\alpha$ is constructed for each ordinal α in the following way. Recall that a nonzero ordinal β is called a *successor ordinal* if there exists an ordinal α such that $\beta = \alpha + 1$, otherwise β is said to be a *limit ordinal.* Note that, as per this recollection, every ordinal is either 0, a successor ordinal, or a limit ordinal.

The only $\mathscr{X}_0$ group is the trivial group. The class $\mathscr{X}_1$ consists of all finitely generated virtually abelian groups, that is, all finitely generated groups that admit a finite-index abelian subgroup. If $\mathscr{X}_\alpha$ has been defined, then a group G is $\mathscr{X}_{\alpha+1}$ if it is locally-$\mathscr{X}_\alpha$-insulated by $\mathscr{X}_1$. If β is a limit ordinal such that for all $\alpha < \beta$ the class $\mathscr{X}_\alpha$ has been defined, then $\mathscr{X}_\beta = \bigcup_{\alpha<\beta} \mathscr{X}_\alpha$.

Definition 4.1.2. A group G is *elementary amenable* if there exists an ordinal α such that $G \in \mathscr{X}_\alpha$. Denote by $\alpha(G)$ the least ordinal such that $G \in \mathscr{X}_{\alpha(G)}$.

An arbitrary elementary amenable group cannot be assumed finitely generated or even of countable cardinality and the results below should be taken to apply to groups that do not satisfy these requirements unless explicitly stated or without a stronger assumption (e.g., being the fundamental group of a compact manifold).

The class of elementary amenable groups is very broad and very difficult to escape from by the usual group theoretic constructions. The following proposition states a variety of closure properties for the class and characterizes it according to them (cf. [11, §1] and [12]).

Proposition 4.1.3. *The class of elementary amenable groups satisfies the following properties.*

- *Every finite group and $\mathbb{Z}$ are elementary amenable.*
- *If G is elementary amenable and $H \leq G$, then H is elementary amenable.*
- *If G is elementary amenable and $N \trianglelefteq G$, then G/N is elementary amenable.*
- *If A and B are elementary amenable and G forms a short exact sequence*

$$1 \longrightarrow A \longrightarrow G \longrightarrow B \longrightarrow 1 \ ,$$

  then G is elementary amenable.

- *If G is a group and $\{H_j\}_{j\in J}$ an ascending family of elementary amenable subgroups of G such that $G = \bigcup_{j\in J} H_j$, then G is elementary amenable.*

Furthermore, if a class of groups $\mathscr{A}$ satisfies the above properties, then every elementary amenable group is an $\mathscr{A}$ group.

Note that the statement in [11] that corresponds to Proposition 4.1.3 is missing the statement that $\mathbb{Z}$ is elementary amenable, without which the above properties no longer characterize the class of elementary amenable groups; the correct characterization can be found in [12]. A common equivalent characterization replaces $\mathbb{Z}$ with all abelian groups, finitely generated or not. Proposition 4.1.4 below provides one direction of the proof of this equivalence and is of use in proving Corollary 4.1.5.

Proposition 4.1.4. *Abelian groups are elementary amenable.*

Proof. Let $\mathcal{S}$ be a generating set for G indexed by some ordinal σ. For each $\lambda \leq \sigma$ define $G_\lambda = \langle g_\alpha \rangle_{\alpha<\lambda}$. If λ is finite, then G_λ is a finitely generated abelian group and hence $\mathscr{X}_1$.

Let $\lambda \leq \sigma$ be such that G_λ is elementary amenable. Observe that $G_{\lambda+1}$ is an extension of $\langle g_\lambda \rangle$ by G_λ. The former is a finitely generated abelian group and the latter was assumed elementary amenable, hence $G_{\lambda+1}$ is elementary amenable.

Let $\lambda \leq \sigma$ be such that if $\beta < \lambda$, then G_β is elementary amenable. Observe that $\{G_\beta\}_{\beta<\lambda}$ is an ascending family of elementary amenable subgroups and $G_\lambda = \bigcup_{\beta<\lambda} G_\beta$; hence G_λ is elementary amenable.

Thus by transfinite induction, G_σ, which is to say G, is elementary amenable. $\square$

For the purposes of this dissertation, the elementary amenable groups are of interest because they contain the virtually solvable groups. As solvable is the classical name for polyabelian, Proposition 4.1.4 renders considerable aid in verifying this.

Corollary 4.1.5. *Virtually solvable groups are elementary amenable.*

Proof. Observe that virtually solvable is equivalent to solvable-by-finite, so it suffices to show that solvable groups are elementary amenable. If G is solvable, then it admits a subnormal series

$$1 = G_0 \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G_{n-1} \trianglelefteq G_n = G$$

whose quotients are abelian.

By Proposition 4.1.4, the group G_1 is elementary amenable. Observe that G_2 is abelian-by-abelian, making it elementary amenable in turn. In general, if G_{i-1} is elementary amenable, then G_i is an extension of an abelian group by G_{i-1} and thus elementary amenable. $\square$

4.2 Hirsch Length

The Hirsch length of a group G takes values in $\mathbb{N} \cup \{\infty\}$ and is denoted by $h(G)$. In [11], the Hirsch length is defined on the classes $\mathscr{X}_\alpha$ defined in §4.1 above via transfinite induction.

If G is $\mathscr{X}_0$, that is, if G is the trivial group, define $h(G) = 0$. If G is $\mathscr{X}_1$, then there exists a finite-index abelian subgroup $A \leq G$; define $h(G) = b_1(A)$, that is, the torsion-free rank of A. To see that this is a well-defined choice for $h(G)$, consider the following.

Let A be a finite-index abelian subgroup of G, then $Z(G)$, the center of G, necessarily has finite-index and A is necessarily finite-index in $Z(G)$. A subgroup of an abelian group has finite-index if and only if its first Betti number agrees with that of its supergroup. Hence $b_1(A) = b_1(Z(G))$ and the same is true of any other finite-index abelian subgroup of G.

Suppose the Hirsch length has been defined on $\mathscr{X}_\alpha$ and assume G is $\mathscr{X}_{\alpha+1}$. By construction, G is an extension of an $\mathscr{X}_1$ group by a locally $\mathscr{X}_\alpha$-insulated group. If G is locally $\mathscr{X}_\alpha$-insulated, define $h(G) = \sup\{h(H) \mid H \leq G \text{ and } H \in \mathscr{X}_\alpha\}$. If G is a general $\mathscr{X}_{\alpha+1}$ group then it possesses a locally $\mathscr{X}_\alpha$-insulated normal subgroup K such that G/K is $\mathscr{X}_1$; define $h(G) = h(K) + h(G/K)$.

Suppose β is a limit ordinal such that for all $\alpha < \beta$ the Hirsch length has been defined on $\mathscr{X}_\alpha$. If G is $\mathscr{X}_\beta$, then there exists $\alpha < \beta$ such that G is $\mathscr{X}_\alpha$ and so $h(G)$ is defined.

As one can see, this is less a concise definition and more a collection of inductively defined computational tools. Theorem 4.2.1 below [11, Thm. 1] collects the computational results that are corollaries of the above definition.

Theorem 4.2.1. *If G is a (not necessarily finitely generated) elementary amenable group then $h(G)$ is well-defined. Furthermore, the Hirsch length satisfies the following properties.*

- *If G is finitely generated and virtually abelian, then $h(G) = b_1(A)$ for any finite-index abelian subgroup $A \leq G$.*
- *If $H \leq G$, then $h(H) \leq h(G)$.*
- $h(G) = \sup\{h(H) \mid H \leq G \text{ and } H \text{ is finitely generated}\}$.
- *If $N \trianglelefteq G$, then $h(G) = h(N) + h(G/N)$.*

The Hirsch length, despite being defined on finite groups—by virtue of their being finitely generated and virtually abelian—is truly an invariant of infinite groups.

Corollary 4.2.2. *If G is a finite group, then $h(G) = 0$.*

Proof. Finite groups are finitely generated and virtually abelian, hence $h(G) = b_1(G)$. However, finite groups are torsion, so $b_1(G) = 0$. □

In fact, trivial Hirsch length does not even characterize the class of finite groups, but instead that of locally finite elementary amenable groups.

Corollary 4.2.3. *An elementary amenable group G is locally finite if and only if $h(G) = 0$.*

Proof. Assume $h(G) = 0$ and let $H \leq G$ be finitely generated. If $a \in H$ is not a torsion element, then $\langle a \rangle$ is a free abelian subgroup of G. Hence, on the one hand, $h(\langle a \rangle) = 1$ by virtue of being free abelian, but on the other hand, $h(\langle a \rangle) = 0$ since $h(\langle a \rangle) \leq h(G)$. Thus one can conclude that every element of H is torsion. As a finitely generated torsion group, H is finite, hence G is locally finite.

Conversely, assume G is locally finite. By Theorem 4.2.1, the Hirsch length of G is given by

$$h(G) = \sup\{h(H) \mid H \leq G \text{ and } H \text{ is finitely generated}\}.$$

Let H be a finitely generated subgroup of G, then by assumption H is finite. By Corollary 4.2.2, one has that $h(H) = 0$ and thus $h(G) = 0$. □

When dealing with virtual properties, it is often desirable to pass to finite-index subgroups. Doing so does not affect the computation of the Hirsch length of the ambient group.

Corollary 4.2.4. *Let G be an elementary amenable group. If $H \leq G$ is finite-index, then $h(H) = h(G)$.*

Proof. Recall that the *normal core* of a subgroup H of G is the subgroup $N = \bigcap_{g \in G} gHg^{-1}$ which is normal in G, a subgroup of H, and if H has finite-index in G, then so too does N.

Since the three groups satisfy $N \leq H \leq G$ their Hirsch lengths satisfy

$$h(N) \leq h(H) \leq h(G).$$

By Theorem 4.2.1, $h(G) = h(N) + h(G/N)$, which reduces to $h(G) = h(N)$ by Corollary 4.2.3. Hence the above inequality collapses to $h(H) = h(G)$ □

Like the first Betti number, the Hirsch length of a group measures its "size" in some sense. The Hirsch length can be viewed as a variant of the first Betti number that is better behaved with respect to group extensions.

All that can be said about the first Betti number of G from a short exact sequence

$$1 \longrightarrow K \longrightarrow G \longrightarrow Q \longrightarrow 1$$

of groups, is that $b_1(G) \leq b_1(K) + b_1(Q)$, with equality attained in the case that the extension is trivial. However, if the groups involved are elementary amenable—such as when G is elementary amenable (see Proposition 4.1.3)—then $h(G) = h(K) + h(Q)$.

The two invariants coincide for finitely generated abelian groups and then diverge as one takes group extensions and advances along the classes $\mathscr{X}_\alpha$. The inequality and equation above predict that the first Betti number will lag behind the Hirsch length on this journey. This is exactly what happens, with the first Betti number providing a lower bound for the Hirsch length. The proof is made cleaner by proving Lemma 4.2.5 below as a separate result.

An infinitely generated abelian group has arbitrary Hirsch length. Consider, for example the groups $\bigoplus_{k\in\mathbb{N}} \mathbb{Z}$ and $\mathbb{Z}^n \oplus \bigoplus_{k\in\mathbb{N}} \mathbb{Z}_2$ for any $n \in \mathbb{N}$. The former has infinite Hirsch length as it has a finitely generated subgroup isomorphic to $\mathbb{Z}^n$ for all $n \in \mathbb{N}$ and the latter has Hirsch length n. Although infinitely generated abelian groups need not resemble one of the forms above, the situation suggested by them, that an infinitely generated torsion subgroup is responsible for this behavior, holds in general.

Lemma 4.2.5. *If G is an infinitely generated torsion-free abelian group, then $h(G) = \infty$.*

Proof. Let $\mathcal{S}$ be an infinite generating set for G. Given $g_1 \in \mathcal{S}$ the subgroup $S_1 = \langle g_1 \rangle$ is a finitely generated torsion-free cyclic group, hence $S_1 \cong \mathbb{Z}$ and thus $h(S_1) = 1$.

Assume for some $k \geq 1$ and $g_1, \ldots, g_k \in \mathcal{S}$ that $S_k = \langle g_1, \ldots, g_k \rangle$ has Hirsch length k. Now pick $g_{k+1} \in \mathcal{S}$ such $g_{k+1} \notin S_k$ and define $S_{k+1} = \langle g_1, \ldots, g_k, g_{k+1} \rangle$. If no such g_{k+1}

exists, then $S_k = G$ which contradicts that G is infinitely generated.

Observe that $S_k \leq S_{k+1}$ and, as G is abelian, in fact one has that $S_k \trianglelefteq S_{k+1}$. Theorem 4.2.1 then obtains $h(S_{k+1}) = h(S_k) + h(S_{k+1}/S_k)$. By assumption, $h(S_k) = k$ and the quotient $S_{k+1}/S_k \cong \langle g_{k+1} \rangle$ which, as a finitely generated torsion-free cyclic group, has Hirsch length 1. Hence $h(S_{k+1}) = k + 1$ as desired.

Through this process one constructs a sequence of finitely generated subgroups of G whose Hirsch lengths are unbounded. By Theorem 4.2.1 this forces $h(G) = \infty$. $\square$

One now proceeds to bound the Hirsch length of a group from below by its first Betti number.

Proposition 4.2.6. *Let G be an elementary amenable group, then $h(G) \geq b_1(G)$.*

Proof. Let T denote the torsion subgroup of $H_1(G;\mathbb{Z})$, then there exists $N \trianglelefteq G$ such that

$$1 \longrightarrow N \longrightarrow G \longrightarrow H_1(G;\mathbb{Z})/T \longrightarrow 1$$

is a short exact sequence.

Observe that $H_1(G;\mathbb{Z})/T$ is a torsion-free abelian group, hence if it is infinitely generated, then $b_1(G) = \infty$ by definition and $h(H_1(G;\mathbb{Z})/T) = \infty$ by Lemma 4.2.5.

On the other hand, if $H_1(G;\mathbb{Z})/T$ is finitely generated, then $H_1(G;\mathbb{Z})/T \cong \mathbb{Z}^{b_1(G)}$, so by definition, $h(H_1(G;\mathbb{Z})/T) = b_1(G)$. By Theorem 4.2.1, the short exact sequence above yields $h(G) = h(N) + h(H_1(G;\mathbb{Z})/T)$. Thus, $h(G) = h(N) + b_1(G)$ and so $h(G) \geq b_1(G)$. $\square$

4.3 Poincaré Duality Groups & Manifolds

As the Hirsch length of an elementary amenable group G is a purely group theoretic invariant, one might wonder if it encodes any topological information about manifolds that have G as their fundamental group. A bridge of sorts is provided by [11, Thm. 6] which sorts a particular quotient of the fundamental group into one of three classes according to its Hirsch length.

In addition to Hillman's result itself, which is presented below, there is a lemma used in proving the theorem that is of independent use in this dissertation.

Lemma 4.3.1. *If G is elementary amenable, then $H^s(G; \mathbb{Z}[G]) = 0$ for all $s < h(G)$.*

Recall that $H^s(G; \mathbb{Z}[G])$ is the sth cohomology group of G with coefficients in the ring $\mathbb{Z}[G]$. The cohomology of a group can be taken as the cohomology of any $K(G, 1)$.

The main use of Lemma 4.3.1 in this dissertation is to satisfy $H^2(G; \mathbb{Z}[G]) = 0$ which is necessary in order to invoke the following theorem of B. Eckmann in [4].

Theorem 4.3.2. *Let M be a connected orientable closed 4-manifold whose fundamental group G is elementary amenable. If G is infinite, finitely presentable,* not *virtually infinite cyclic, $H^2(G; \mathbb{Z}[G]) = 0$, and $\chi(M) = 0$, then M is a $K(G, 1)$ and G is a* PD_4*-group.*

One should recall that a group G is a *Poincaré duality group of dimension n*, or briefly PD_n*-group*, if for any $\mathbb{Z}[G]$-module A the isomorphism $H^i(G; A) \cong H_{n-i}(G; A)$ holds. Note that G is a PD_n-group if there exists a $K(G, 1)$ that is a connected orientable closed n-manifold. In that case the homology of $K(G, 1)$ possesses Poincaré duality in dimension n, hence the name.

Lemma 4.3.1 is used by Hillman to prove the following result. Note that the use case in this dissertation for Theorem 4.3.3 below has T trivial, which simplifies the result to Corollary 4.3.4.

Theorem 4.3.3. *Let M be a closed 4-manifold with $\chi(M) = 0$ and elementary amenable fundamental group G. Let T be the maximal locally-finite normal subgroup of G such that the finite subgroups of G/T have bounded order, then the following hold:*

- *If $h(G/T) = 1$, then $G/T \cong \mathbb{Z}$ or $G/T \cong \mathbb{Z}_2 * \mathbb{Z}_2$ and T is finite;*
- *If $h(G/T) = 2$, then G/T is solvable and virtually an extension of $\mathbb{Z}$ by a subgroup of $\mathbb{Q}$;*
- *If $h(G/T) \geq 3$, then $h(G/T) = 4$ and G/T is virtually poly-$\mathbb{Z}$, and if T is finite, then M is aspherical.*

Theorem 4.3.3 is stated in [11] without characterizing the results by Hirsch length, but these arise explicitly from Hillman's proof.

When G is torsion-free, as happens when M is a finite-dimensional $K(G, 1)$, Theorem 4.3.3 becomes more succinct.

Corollary 4.3.4. *Let M be a closed 4-manifold with $\chi(M) = 0$ and elementary amenable fundamental group G. If G is torsion-free, then the following hold:*

- *If $h(G) = 1$, then $G \cong \mathbb{Z}$;*
- *If $h(G) = 2$, then G is solvable and virtually an extension of $\mathbb{Z}$ by a subgroup of $\mathbb{Q}$;*
- *If $h(G) \geq 3$, then G is virtually poly-$\mathbb{Z}$ and $h(G) = 4$ and M is aspherical.*

Chapter 5

Linear SCY Groups with $vb_1 = 4$

Recall that a group G is said to be *linear* if it is isomorphic to some matrix group and that $vb_1(G)$ denotes the *virtual first Betti number* of G (see §2.2).

This chapter proves the following conjecture.

Conjecture 5.0.1. *If G is a linear SCY group with $vb_1(G) = 4$, then G is virtually $\mathbb{Z}^4$.*

This is accomplished in two pieces. First, the hypotheses are shown to imply that G is virtually solvable. Second, under this condition, G admits a finite-index subgroup that is poly-$\mathbb{Z}$; the Hirsch length of G forces this subgroup to then be isomorphic to $\mathbb{Z}^4$.

5.1 Linear SCY Groups with $vb_1 = 4$ are Virtually Solvable

The proof of the first step is accomplished by contradiction. In assuming that G is a linear SCY group with $vb_1(G) = 4$ but is not virtually solvable, one is able to produce two non-isomorphic spin^c structures with matching first Chern classes. The constraints of Taubes in Theorem 3.4.3 allow one to compute the Seiberg-Witten invariants of these spin^c structures to be 1 and 0 respectively. However, a result of Ruberman and Strle (Theorem 3.4.6) forces the Seiberg-Witten invariants to be congruent modulo 2.

The ability to produce such spin^c structures, however, depends on Lemma 5.1.2 below, which in turn depends on the following theorem from [15].

Theorem 5.1.1. *Let G be a finitely generated linear group, then either G is virtually solvable or for all primes p it is the case that $vb_1(G;\mathbb{Z}_p) = \infty$.*

The hypothesis of finite generation does not restrict the utility of Theorem 5.1.1 in proving Conjecture 5.0.1 *a fortiori*: G is finitely presentable by virtue of being the fundamental group of a compact manifold.

Lemma 5.1.2. *Suppose M is an SCY 4-manifold with $vb_1(M;\mathbb{Z}) = 4$ and $\pi_1(M)$ linear. If $\pi_1(M)$ is not virtually solvable, then there exists a finite-sheeted cover $\widetilde{M}$ of M such that $H_1(\widetilde{M};\mathbb{Z})$ has nontrivial 2-torsion.*

Proof. Assume that $\pi_1(M)$ is not virtually solvable, then as it is a linear group, Theorem 5.1.1 reveals that $vb_1(M;\mathbb{Z}_p) = \infty$ for any prime p; consider the case when $p = 2$. Given any $k \in \mathbb{Z}$ there then exists a finite-sheeted cover $\widetilde{M}_k$ of M such that $b_1(\widetilde{M}_k;\mathbb{Z}_2) > k$; in particular, choose $\widetilde{M}$ to be a cover that satisfies $b_1(\widetilde{M};\mathbb{Z}_2) > 4$.

Returning to the realm of integral coefficients, observe that, as $\widetilde{M}$ is a finite-sheeted cover, $b_1(M;\mathbb{Z}) \leq b_1(\widetilde{M};\mathbb{Z})$, that is, $b_1(\widetilde{M};\mathbb{Z}) \geq 4$. On the other hand, $\widetilde{M}$ is an SCY manifold itself by virtue of Proposition 2.1.8 and so $b_1(\widetilde{M};\mathbb{Z}) \leq vb_1(M;\mathbb{Z})$, hence $b_1(\widetilde{M};\mathbb{Z}) = 4$.

Thus in the realm of integral coefficients, $b_1(\widetilde{M};\mathbb{Z}) = 4$, yet over $\mathbb{Z}_2$ it happens that $b_1(\widetilde{M};\mathbb{Z}_2) > 4$. For this to occur necessitates that $H_1(\widetilde{M};\mathbb{Z})$ has nontrivial 2-torsion. □

With Lemma 5.1.2 in hand, the nontrivial 2-torsion can be employed to produce the desired spin^c structure.

Theorem 5.1.3. *Let M be an SCY 4-manifold with $vb_1(M) = 4$. If $\pi_1(M)$ is linear, then $\pi_1(M)$ is virtually solvable.*

Proof. Assume that $\pi_1(M)$ is not virtually solvable, then by Lemma 5.1.2 there exists a finite-sheeted cover $\widetilde{M}$ of M such that $H_1(\widetilde{M};\mathbb{Z})$ has nontrivial 2-torsion. Observe that $\widetilde{M}$ is an SCY manifold and $vb_1(\widetilde{M};\mathbb{Z}) = 4$. Assume, for simplicity of notation, that $H_1(M;\mathbb{Z})$ itself has nontrivial 2-torsion.

Let $\widetilde{P}_0$ be the canonical spin^c structure on M and $\widetilde{P}_E$ an arbitrary spin^c structure that differs from $\widetilde{P}_0$ by a complex line bundle E; denote by L_0 and L_E the respective characteristic line bundles. By Corollary 3.2.4, one has that $c_1(L_0) = c_1(L_E)$ if and only if $c_1(E)$ is a 2-torsion class in $H^2(M;\mathbb{Z})$.

By the universal coefficient theorem, there is a short exact sequence

$$0 \longrightarrow \mathrm{Ext}(H_1(M;\mathbb{Z}),\mathbb{Z}) \longrightarrow H^2(M;\mathbb{Z}) \longrightarrow \mathrm{Hom}(H_2(M;\mathbb{Z}),\mathbb{Z}) \longrightarrow 0 \ .$$

Observe that $\mathrm{Hom}(H_2(M;\mathbb{Z}),\mathbb{Z})$ is torsion-free, hence all torsion in $H^2(M;\mathbb{Z})$ comes from the torsion in $\mathrm{Ext}(H_1(M;\mathbb{Z}),\mathbb{Z})$. Moreover, $\mathrm{Ext}(H_1(M;\mathbb{Z}),\mathbb{Z})$ is isomorphic to the torsion part of $H_1(M;\mathbb{Z})$, hence the 2-torsion part of $H^2(M;\mathbb{Z})$ is the 2-torsion part of $H_1(M;\mathbb{Z})$.

Let $[\alpha] \in H^2(M;\mathbb{Z})$ be a nontrivial 2-torsion class and let E be its associated complex line bundle. Then $c_1(\widetilde{P}_0) = c_1(\widetilde{P}_E)$ but $\widetilde{P}_0 \ncong \widetilde{P}_E$ by Proposition 3.2.3.

The constraints in Theorem 3.4.3 determine the Seiberg-Witten invariants of $\widetilde{P}_0$ and $\widetilde{P}_E$. As the canonical spin^c structure on M, it is immediate that $\mathrm{sw}(\widetilde{P}_0) = 1$. For

$\widetilde{P}_E$, observe that by Corollary 3.4.4 it is known that $\mathrm{sw}(\widetilde{P}_E) = 0$, as $[\alpha]$ was chosen to be nontrivial.

Furthermore, observe that both L_0 and L_E are trivial complex line bundles, as the first Chern class vanishes for each. Invoking Theorem 3.4.6, one obtains a congruence between $\mathrm{sw}(\widetilde{P}_0)$ and $\mathrm{sw}(\widetilde{P}_E)$ modulo 2; a contradiction, hence $\pi_1(M)$ must be virtually solvable. □

5.2 Virtually Solvable SCY Groups with $vb_1 = 4$ are Virtually $\mathbb{Z}^4$

The proof that every linear SCY group G with $vb_1(G) = 4$ is virtually solvable allows use of the machinery in Chapter 4. The relationship between the Hirsch length of a group and its first Betti number, expressed in Proposition 4.2.6, enable the use of Theorem 4.3.3 whose result implies the desired conclusion: that G is virtually $\mathbb{Z}^4$.

Lemma 5.2.1. *Let M be an SCY 4-manifold with fundamental group G. If G is virtually solvable and $vb_1(M;\mathbb{Z}) = 4$, then G is a torsion-free* PD_4*-group, M is a $K(G,1)$, and $h(G) = 4$.*

Proof. Begin by proving the result assuming $b_1(M;\mathbb{Z}) = 4$. The Hirsch length of G is defined, as G is virtually solvable and hence by Corollary 4.1.5 is elementary amenable. From Proposition 4.2.6 one obtains $h(G) \geq 4$. By Lemma 4.3.1 this implies, in particular, that $H^2(G;\mathbb{Z}[G]) = 0$, which is one of the hypotheses required for Theorem 4.3.2. One now proves that Theorem 4.3.2 applies.

First, G is assumed virtually solvable, hence by Corollary 4.1.5, it is elementary amenable. Second, G is infinite, for if it were finite, then $b_1(G)$ would vanish. Third, as the fundamental group of a compact manifold, G must be finitely presentable. Fourth, if G were virtually infinite cyclic, then M would possess a finite-sheeted cover $\widetilde{M}$ with infinite cyclic fundamental group, hence $b_1(\widetilde{M}) = 1$. However, as a finite-sheeted cover, $\widetilde{M}$ must satisfy $b_1(M) \leq b_1(\widetilde{M})$, but by assumption, $b_1(M) = 4$; hence G cannot be virtually infinite cyclic. Finally, since M is an SCY manifold with $vb_1(M) = 4$, the constraints of Bauer and Li conclude that $\chi(M) = 0$.

Thus Theorem 4.3.2 implies that G is a PD_4-group; moreover, it implies that M is a $K(G, 1)$. Furthermore, it is a standard result that if a finite-dimensional CW complex is a $K(G, 1)$, then its fundamental group is torsion-free, hence G is torsion-free. As G is a PD_4-group, it satisfies $H^4(G; \mathbb{Z}[G]) \cong \mathbb{Z}$ and hence the contrapositive of Lemma 4.3.1 yields $h(G) \leq 4$.

Alongside the above bound of $h(G) \geq 4$, one now has that $h(G) = 4$.

When $vb_1(M; \mathbb{Z}) = 4$, there exists a finite-sheeted cover $\widetilde{M}$ of M such that $b_1(\widetilde{M}; \mathbb{Z}) = 4$; let $H = \pi_1(\widetilde{M})$. By Proposition 2.1.8, $\widetilde{M}$ is an SCY 4-manifold. Since H is a finite-index subgroup of G it is virtually solvable, thus the above proves that $h(H) = 4$. Moreover, $h(H) = h(G)$ by Corollary 4.2.4, proving the result in full. $\square$

Theorem 5.2.2. *Let M be a closed SCY 4-manifold with fundamental group G. If G is virtually solvable and $vb_1(M; \mathbb{Z}) = 4$, then G is virtually $\mathbb{Z}^4$.*

Proof. Theorem 2.2.3 and Lemma 5.2.1 imply that M and G satisfy the hypotheses of Corollary 4.3.4, hence G is virtually poly-$\mathbb{Z}$. As such, G possess a finite-index subgroup H

that admits a subnormal series

$$1 = H_0 \trianglelefteq H_1 \trianglelefteq H_2 \trianglelefteq H_3 \trianglelefteq H_4 = H$$

such that $H_i/H_{i-1} \cong \mathbb{Z}$ for all $1 \leq i \leq 4$. As $h(H) = 4$ and $b_1(H) = 4$, the subnormal series is of length 4.

From the subnormal series one obtains the short exact sequence

$$1 \longrightarrow H_3 \longrightarrow H \longrightarrow \mathbb{Z} \longrightarrow 1 \ .$$

In order to satisfy $h(H) = h(H_3) + h(\mathbb{Z})$ one must have $h(H_3) = 3$. In terms of first Betti number, the sequence implies that $b_1(H) \leq b_1(H_3)+1$, that is, $b_1(H_3) \geq 3$. Proposition 4.2.6 yields the additional bound $h(H_3) \geq b_1(H_3)$ and so one concludes that $b_1(H_3) = 3$. An identical argument, proceeding from this one, yields $b_1(H_2) = 2$.

Observe that $H_1 \cong \mathbb{Z}$, so H_2 sits inside the following short exact sequence:

$$1 \longrightarrow \mathbb{Z} \longrightarrow H_2 \longrightarrow \mathbb{Z} \longrightarrow 1 \ .$$

The result that $b_1(H_2) = 2$ forces this to be a trivial group extension, hence $H_2 \cong \mathbb{Z}^2$. This produces the following short exact sequence for H_3:

$$1 \longrightarrow \mathbb{Z}^2 \longrightarrow H_3 \longrightarrow \mathbb{Z} \longrightarrow 1 \ .$$

Since $b_1(H_3) = 3$, the extension must be trivial, which forces $H_3 \cong \mathbb{Z}^3$. Finally one arrives at the following short exact sequence for H_4:

$$1 \longrightarrow \mathbb{Z}^3 \longrightarrow H_4 \longrightarrow \mathbb{Z} \longrightarrow 1 \ .$$

Which, alongside $b_1(H_4) = 4$, forces the extension to be trivial and hence H_4, which is to say, H, is isomorphic to $\mathbb{Z}^4$. Thus G is virtually $\mathbb{Z}^4$. $\square$

5.3 Main Result

Theorem 5.1.3 and Theorem 5.2.2 each provide half of the proof for Conjecture 5.0.1. One starts with a linear SCY group G with $vb_1(G) = 4$ whence Theorem 5.1.3 yields that G is virtually solvable. This allows the invocation of Theorem 5.2.2 to obtain that G is virtually $\mathbb{Z}^4$. In light of this, Conjecture 5.0.1 can be restated as a simple corollary of this pair of theorems.

Corollary 5.3.1. *If G is a linear SCY group with $vb_1(G) = 4$, then G is virtually $\mathbb{Z}^4$.*

Proof. See the above discussion. □

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