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TRANSIENT BEHAVIOR OF THE BERKELEY BUBBLE CHAMBER MOTOR GENERATOR
MAGNET SYSTEM DURING EMERGENCY PROCEDURE

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Alper Garren and Warren Heckrotte

June 19, 1958

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ABSTRACT

The transient behavior of the electrical system for the Berkeley bubble-chamber magnets under certain emergency procedures is examined theoretically. Conditions for nonoscillating behavior of the magnet current are derived.

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1. Introduction

The bubble-chamber magnets are powered by one or more motor generator sets. In case of certain kinds of malfunctioning the system will automatically turn off the power in the motor and the power that energizes the generator field and short-circuit the generator field windings, causing the field to decay exponentially. In this note we consider the magnet current as a function of time after the power turnoff. The resulting behavior will also be a function of the time constants of the generator field and the magnet circuit, of the mechanical energy stored initially in the flywheel, and of the electrical energy stored initially in the field of the magnet. Curves of the current vs time are given for two cases of immediate interest, and for another case to illustrate a different type of possible behavior. Finally a relationship between the parameters of the system is derived that determines whether the current will reverse itself.

2. Statement of Problem

The system is shown schematically in Fig. 1. For times after the power shutoff (at $t = 0$) the parameters are

$$t_g = L_g/R_g = \text{time constant of generator field}$$

$$t_m = L_m/R_m = \text{time constant of magnet with leads}$$

$$\phi = \text{flux through generator armature} = \phi_0 \exp(-t/t_g)$$

$$\mathcal{M} = \text{moment of inertia of flywheel}$$

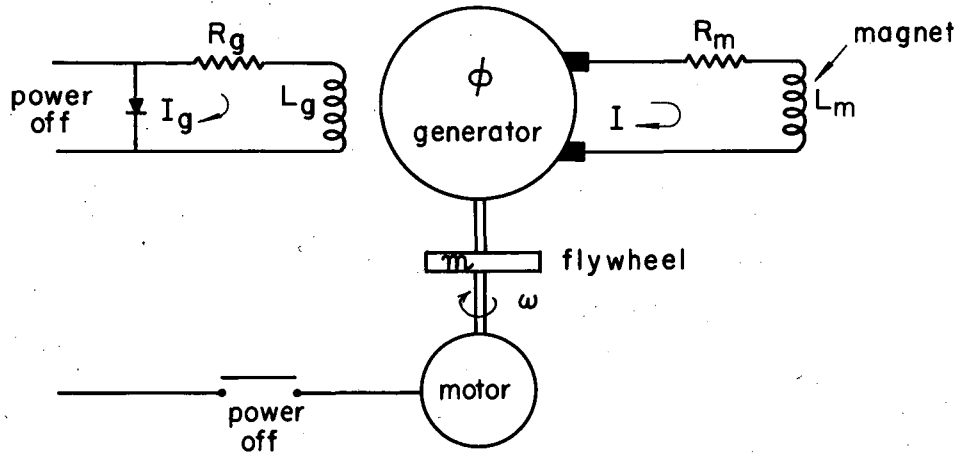
$$V = \text{voltage of generator}$$

$$\omega = \text{angular velocity of armature and flywheel}$$

$$T_g = \text{torque on flywheel from generator}$$

$$T_m = \text{torque on flywheel from motor}$$

$$\tau = t/t_g$$



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Fig. 1. Schematic diagram of the circuit for the bubble chamber motor generator magnet system.

$$\sigma = t_g/t_m, \quad \eta = \sqrt{\frac{1}{2} \mathcal{M} \omega_0^2 / \frac{1}{2} L_m I_0^2}, \quad p = \frac{\sigma - 1}{2}$$

$$\kappa = \sigma/\eta$$

We ignore mechanical friction. The current through the generator field circuit indicated at the left of Fig. 1 is supposed to decay according to

$$L_g \frac{dI_g}{dt} + R_g I_g = 0, \quad I_g = I_{g0} \exp(-t/t_g),$$

where the subscript zero denotes the conditions at $t = 0$. This means that we have assumed that the mutual inductance between the armature and the field circuit is zero on the average, and that the fluctuations in flux are short compared with t_g . The flux through the armature is assumed proportional to I_g , so that we have

$$\phi = \phi_0 \exp(-t/t_g). \quad (1)$$

The voltage of the generator equals the number of lines of flux cut by the armature coils per sec,

$$V = \omega \phi,$$

and this voltage drives the current I through the magnet:

$$\omega \phi = L_m \frac{dI}{dt} + R_m I. \quad (2)$$

The torque from the generator on the flywheel tends to slow down the latter, and is given by $T_g = -I\phi$. (3)

The rate of change of angular momentum of the flywheel is given by

$$\mathcal{M} \frac{d\omega}{dt} = T_g + T_m; \quad T_m = 0 \text{ for } T \geq 0. \quad (4)$$

3. Initial Conditions

Before the power that drives the motor and the generator field is turned off, the system is in a steady state, so that we have $I = \dot{\omega} = 0$ for $t < 0$. From Eqs. (2) and (4) we get

$$\omega_0 \phi_0 = R_m I_0, \quad \phi_0 = \frac{R_m I_0}{\omega_0}, \quad (5)$$

$$0 = T_g + T_m; \quad T_m = -T_g = I_0 \phi_0.$$

At $t = 0$, $T_m = 0$ so that we have

$$\mathcal{M} \dot{\omega}_0 = T_{g0} = -I_0 \phi_0 = -R_m I_0^2 / \omega_0,$$

$$\dot{\omega}_0 = -\frac{R_m I_0^2}{\mathcal{M} \omega_0}$$

Thus the initial conditions are

$$\omega = \omega_0, \quad I = I_0, \quad \dot{I}_0 = 0, \quad \dot{\omega}_0 = -R_m I_0^2 / \mathcal{M} \omega_0. \quad (6)$$

4. Equation of Motion of the System for $t \geq 0$

From Eqs. (1), (3), and (4) we have

$$\begin{aligned} \mathcal{M} \dot{\omega} &= -I \phi_0 \exp(-t/t_g), \\ I &= -(\mathcal{M} / \phi_0) \exp(t/t_g) \dot{\omega}. \end{aligned} \quad (7)$$

Differentiating this latter equation, we obtain

$$\dot{I} = -\frac{\mathcal{M}}{\phi_0} (\ddot{\omega} + \frac{\dot{\omega}}{t_g}) \exp(t/t_g), \quad (8)$$

and substituting Eqs. (1), (7), and (8) in Eq. (2), we get

$$\ddot{\omega} + \left(\frac{1}{t_g} + \frac{1}{t_m} \right) \dot{\omega} + \frac{\phi_0^2}{L_m \mathcal{M}} \left(\exp(-2t/t_g) \right) \omega = 0. \quad (9)$$

This is the equation of motion for ω . When it is solved, the current may be obtained from Eq. (7).

It is convenient to put the equations in dimensionless form. With the help of Eq. (5) and utilizing our definitions given in Section (2), Eq. (9) becomes

$$\frac{d^2 \Omega}{d\tau^2} + (1 + \sigma) \frac{d\Omega}{d\tau} + \kappa^2 \exp(-2\tau) \Omega = 0, \quad (9')$$

where $\Omega = \omega / \omega_0$. The current is given by

$$\bar{\psi} \equiv \frac{I}{I_0} = \exp(\tau) \left(\frac{d\Omega}{d\tau} \right) / \left(\frac{d\Omega}{d\tau} \right)_0 \quad (7')$$

and the initial conditions are

$$\tau = 0: \quad \Omega_0 = 1, \quad \left(\frac{d\Omega}{d\tau} \right)_0 = -\frac{\sigma}{\eta^2} = -\frac{\kappa^2}{\sigma}. \quad (6')$$

We must solve Eq. (9') with initial conditions (6'), and then use (7') to obtain the current.

5. Solution of Equation of Motion

It is convenient to replace the independent variable τ by s , where

$$s = \exp [-(1 + \sigma)\tau], \quad \exp(-\tau) = s^{1/(1 + \sigma)}, \quad \text{and} \quad (10)$$

$$\frac{d}{d\tau} = -(1 + \sigma)s \frac{d}{ds}, \quad \frac{d^2}{d\tau^2} = (1 + \sigma)^2 \left(s^2 \frac{d^2}{ds^2} + s \frac{d}{ds} \right),$$

by which the equation of motion (9') is transformed to

$$\frac{d^2 \Omega}{ds^2} + \left(\frac{\kappa}{1 + \sigma} \right)^2 s^{-2\sigma/(1 + \sigma)} \Omega = 0.$$

This is a Bessel equation, the solution of which is¹

$$\Omega = \sqrt{s} Z_{\frac{1}{2}(1 + \sigma)} \left(\kappa s^{1/(1 + \sigma)} \right) = \exp \left\{ -\frac{1}{2}(1 + \sigma)\tau \right\} Z_{\frac{1}{2}(1 + \sigma)} (\kappa e^{-\tau}) \quad (11)$$

where $Z_\nu(x) = \alpha J_\nu(x) + \beta N_\nu(x)$, and J_ν , N_ν are Bessel functions of the first and second kind respectively, and α, β constants to be determined by the initial conditions. If we differentiate Eq. (11) with respect to τ , and use the formula

$$\frac{dZ_\nu(x)}{dx} = -\frac{\nu}{x} Z_\nu(x) + Z_{\nu-1}(x),$$

we obtain

$$\frac{d\Omega}{d\tau} = -\kappa \exp \left\{ -\frac{1}{2}(1 + \sigma)\tau \right\} e^{-\tau} Z_{\frac{1}{2}(\sigma - 1)} (\kappa e^{-\tau}). \quad (12)$$

We now solve for α and β by using the initial conditions:

$$\Omega_0 = 1 = \alpha J_{p+1}(\kappa) + \beta N_{p+1}(\kappa)$$

$$-\frac{1}{\kappa} \left(\frac{d\Omega}{d\tau} \right)_0 = \frac{\kappa}{\sigma} = \alpha J_p(\kappa) + \beta N_p(\kappa), \quad (13)$$

where

$$p = \frac{\sigma - 1}{2}, \quad \sigma = 2p + 1. \quad (14)$$

¹E. Jahnke and F. Emde, Tables of Functions (Dover, New York, 1945), p. 147

The determinant of these equations is²

$$\begin{vmatrix} J_{p+1}(\kappa) & N_{p+1}(\kappa) \\ J_p(\kappa) & N_p(\kappa) \end{vmatrix} = \frac{2}{\pi\kappa},$$

and we obtain

$$\begin{aligned} \alpha &= \frac{\pi}{2} \kappa \left(N_p(\kappa) - \frac{\kappa}{2p+1} N_{p+1}(\kappa) \right) \\ \beta &= -\frac{\pi}{2} \kappa \left(J_p(\kappa) - \frac{\kappa}{2p+1} J_{p+1}(\kappa) \right) \end{aligned} \tag{15}$$

Inserting these in Eqs. (11) and (12), we obtain from Eqs. (7') and (6'),

$$\begin{aligned} \Omega(\tau) &\equiv \frac{\omega}{\omega_0} = \frac{\pi}{2} \kappa \exp[-(p+1)\tau] \left\{ \left[N_p(\kappa) - \frac{\kappa}{2p+1} N_{p+1}(\kappa) \right] J_{p+1}(\kappa e^{-\tau}) \right. \\ &\quad \left. - \left[J_p(\kappa) - \frac{\kappa}{2p+1} J_{p+1}(\kappa) \right] N_{p+1}(\kappa e^{-\tau}) \right\}, \\ \overline{\Psi}(\tau) &\equiv \frac{I}{I_0} = \frac{\pi}{2} (2p+1) \exp[-(p+1)\tau] \left\{ \left[N_p(\kappa) - \frac{\kappa}{2p+1} N_{p+1}(\kappa) \right] J_p(\kappa e^{-\tau}) \right. \\ &\quad \left. - \left[J_p(\kappa) - \frac{\kappa}{2p+1} J_{p+1}(\kappa) \right] N_p(\kappa e^{-\tau}) \right\}. \end{aligned} \tag{16}$$

Equations (16) express the time dependence of the angular velocity and current after power shutoff. Beside the dimensionless time $\tau = t/t_g$, the formulas involve p and κ , which in turn are determined by the physically more meaningful parameters σ and η .

6. Condition for Positive Current

It may be useful to know whether the magnet current will change sign while it is decaying, and if so, how many times. The answer is given by the following theorem:

Theorem I

The current will change sign n times if and only if there are just n positive, nonzero roots of $F_p(x) = 0$ that are less than κ , i. e. $f_{p,n} < \kappa < f_{p,n+1}$.

²ibid., p. 144.

where $f_{p,n}$ is the n th positive root of $F_p(x) = 0$ and

$$F_p(x) = \begin{cases} F_p^+(x) = [J_p(x) - \frac{x}{2p+1} J_{p+1}(x)], & p \geq 0 \\ F_p^-(x) = [J_{-p}(x) + \frac{x}{2p+1} J_{-p-1}(x)], & -\frac{1}{2} \leq p \leq 0. \end{cases} \quad (18)$$

Hence the current will not change sign at all provided we have $\kappa < f_{p,1}$, where $f_{p,1}$ is the smallest positive root of $F_p(x)=0$. Since we have $\kappa = \sigma/\eta$ the condition may also be written

$$\frac{\sigma}{\eta} < f_{p,1} \quad \text{or} \quad \eta > \frac{\sigma}{f_{p,1}}$$

We have evaluated the $f_{p,1}$ numerically for $\sigma = 0, \frac{1}{2}, 1, 2, 3, 4, 5$, ($p = -\frac{1}{2}, -\frac{1}{4}, 0, \frac{1}{2}, 1, 3/2, 2$). The results are shown in Table I.

Table I

Minimum value of $\eta^2 = W_{\text{mech}}/W_{\text{elec}}$ to assure positive current, as a function of $\sigma = \tau_g/\tau_m$

σ	p	$f_{p,1}$	$\eta_{\text{min}} = \sigma/f_{p,1}$	η_{min}^2
0	$-\frac{1}{2}$	0	$-\infty$	0
$\frac{1}{2}$	$-\frac{1}{4}$	1.420	0.352	0.124
1	0	2.405	0.796	0.634
2	$\frac{1}{2}$	2.028	0.986	0.972
3	1	3.832	1.097	1.203
4	$3/2$	3.407	1.174	1.378
5	2	5.136	1.233	1.520
∞	∞	∞	2.000	4.000

A plot of η_{min}^2 vs σ is shown in Fig. 2. If $\eta^2 = W_{\text{mech}}/W_{\text{elec}}$ is above the curve, for a given $\sigma = \tau_g/\tau_m$, the current will not change sign.

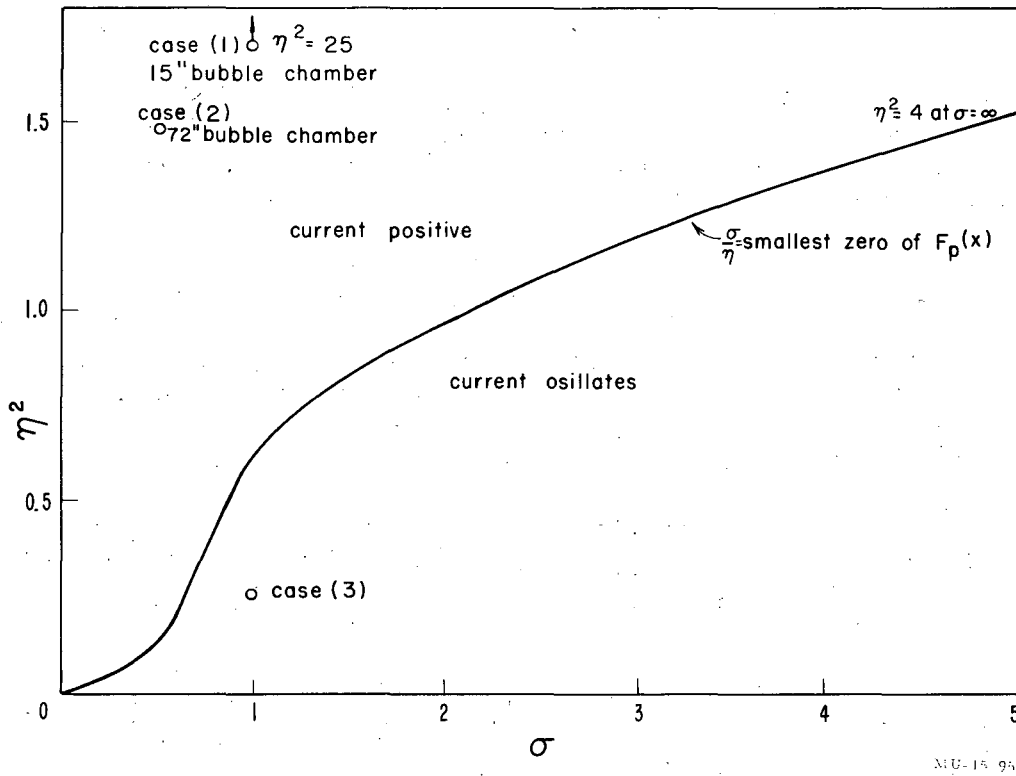


Fig. 2. Condition for positive current.
 $\eta^2 = E_{\text{mech}}/E_{\text{elec}} = \frac{1}{2} \gamma \omega^2 / \frac{1}{2} L_m I^2$;
 $\sigma = t_g/t_m = L_g R_m / R_g L_m$.

7. Behavior of the Flywheel

Referring to Eq. (16), we see that the situation for ω is very similar to that for I. The theorem that applies to ω is the following:

Theorem II

The angular velocity will change sign n times if and only if there are just n positive, nonzero roots of the equation $F_p^+(x) = 0$ that are less than κ .

Note that this condition is identical to that for the current for $p \geq 0$. For $-\frac{1}{2} \leq p < 0$, however, the roots $f_{p,n}^+$ of $F_p^+(x) = 0$ are less than or equal to the roots $f_{p,n}^-$ of $F_p^-(x) = 0$, so that in this case for a given p or σ a larger value of η is required to insure nonreversal of ω than is needed to insure nonreversal of current (since we have $\kappa = \sigma/\eta$).

The condition for completely stopping the flywheel is given by Theorem III, which follows from Theorem II.

Theorem III

The angular velocity goes to zero as $\tau \rightarrow \infty$ if and only if κ equals one of the roots of $F_p^+(x) = 0$.

The proofs of Theorems II and III are obvious, once that of Theorem I is understood.

It is also worth noting, from Eq. (7), that since $I \sim \dot{\omega}$, the zeroes of I occur at the maxima and minima of ω .

The flywheel is critically damped if (a) it never oscillates and (b) it goes to zero as $t \rightarrow \infty$. This occurs only if κ equals the first root of $F_p^+(x) = 0$.

8. Asymptotic Behavior

For very large times the current approaches zero and the angular velocity approaches some constant value (possibly zero). Considering the angular velocity, we have

$$J_{p+1}(y) \rightarrow y^{p+1} \sim \exp [-(p+1)\tau], \quad N_{p+1} \rightarrow y^{-(p+1)} \sim \exp [(p+1)\tau]$$

when $y = \kappa e^{-T} \rightarrow 0$, so that we also have

$$\begin{aligned} \omega &\rightarrow A \exp[-2(p+1)\tau] + B = A \exp[-(\sigma+1)t/t_g] + B = \\ &= A \exp \left\{ -t/[t_g t_m / (t_g + t_m)] \right\} + B. \end{aligned}$$

Thus the angular velocity approaches its constant value with time constant

$$t_{\omega} = t_g / [2(p+1)] = t_g t_m / (t_g + t_m).$$

The current, on the other hand, behaves differently according to $\sigma \gtrless 1$, $p \gtrless 0$. For $\sigma > 1$, $p > 0$ then we have

$$J_p(\kappa e^{-\tau}) \rightarrow e^{-p\tau}, \quad N_p(\kappa e^{-\tau}) \rightarrow e^{p\tau}, \quad I \rightarrow e^{-\tau} = e^{-t/t_g},$$

so that the time constant is the same as that of the generator field, τ_g . But for $\sigma < 1$, $p < 0$ then

$$J_p(\kappa e^{-\tau}) \rightarrow e^{-p\tau}, \quad N_p(\kappa e^{-\tau}) \rightarrow e^{-p\tau}, \quad I \rightarrow e^{-(2p+1)\tau} \\ = e^{-\sigma\tau} = \exp(-t/t_m)$$

so the time constant is the same as that of the magnet circuit, t_m . To sum up, for very large times the angular velocity and the current approach their asymptotic values with time constants

$$t_{\omega} = t_g t_m / (t_g + t_m),$$

and

$$t_I = t_g \text{ or } t_m, \quad \text{whichever is larger.}$$

9. Applications

The time dependence after power shutoff of the current and angular velocity has been computed from Eqs. (16) for the following three cases:

Case I. 15-inch bubble chamber magnet powered by #4 motor generator set

$$\left. \begin{array}{l} t_g = 2.2 \text{ sec} \\ t_m = 1.7 \text{ sec} \end{array} \right\} \text{ Both assumed equal to 2 sec for the calculation:}$$

$$\sigma = t_g / t_m = 1 \\ \eta = \sqrt{\frac{W_{\text{mech}}}{W_{\text{elec}}}} = \sqrt{\frac{\frac{1}{2} M \omega_0^2}{\frac{1}{2} L I_0^2}} = 5; \quad p = \frac{\sigma - 1}{2} = 0,$$

$$\kappa = \frac{\sigma}{\eta} = \frac{1}{5};$$

$$\tau = t/t_g = t_{\text{sec}}/2.$$

The results are plotted in Fig. 3. The angular velocity is hardly reduced at all, therefore the simplified expression that is obtained by integrating Eq. (2) with $\omega = \omega_0$,

$$\frac{I}{I_0} = \frac{t_m}{t_m - t_g} \exp(-t/t_m) - \frac{t_g}{t_m - t_g} \exp(-t/t_g), \quad (20)$$

is fully justified, and in fact for this case the curve obtained from Eq. (20) is about identical with that obtained from (16). Also the lack of oscillation can be predicted from Fig. 2 -- the point $\sigma = 1$, $\eta = 5$ is far above the critical curve. Unfortunately neither curve fits the data too well after about 4 seconds.

Case II. 72-inch bubble chamber powered by two 1.5-megawatt motor generator sets, #8 and #9.

$$t_g = 3 \text{ sec} \quad \sigma = t_g/t_m \approx \frac{1}{2}, \quad p = \frac{\sigma - 1}{2} = -\frac{1}{4}.$$

$$t_m = 5.8 \text{ sec}$$

$$\mathcal{M} = 2 \times 1.054 \times 10^5 \text{ lb-ft}^2 = 8.91 \times 10^3 \text{ kg-meter}^2.$$

$$\omega_0 = 514 \text{ rpm} = 53.8 \text{ radians/sec}$$

$$W_{\text{mech}} = \frac{1}{2} \mathcal{M} \omega_0^2 = 12.8 \times 10^6 \text{ joules.}$$

$$R_m = 0.12 \Omega, \quad L_m = R_m t_m = 0.12 \times 5.8 = 0.696 \text{ henry.}$$

$$I_0 = 5,000 \text{ amp.}$$

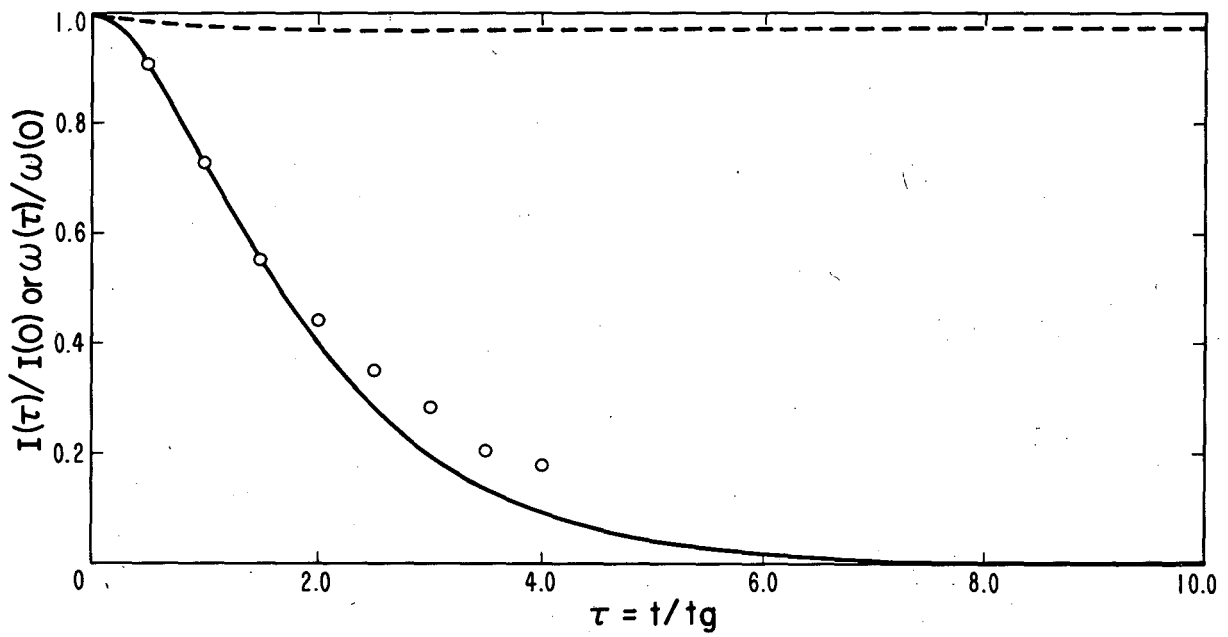
$$W_{\text{elec}} = \frac{1}{2} L_m I_0^2 = 8.7 \times 10^6 \text{ joules.}$$

$$\eta^2 = 12.8/8.7 = 1.47; \quad \eta = 1.213; \quad \kappa = \sigma/\eta = 0.4122.$$

Again we are well above the critical curve of Fig. 2, so that the current should stay positive, as it does (see Fig. 4). Although there is now a significant decrease in angular velocity, it is not enough to significantly affect the magnet current, and again the current calculated from Eq. (16) is almost the same as that calculated from (20).

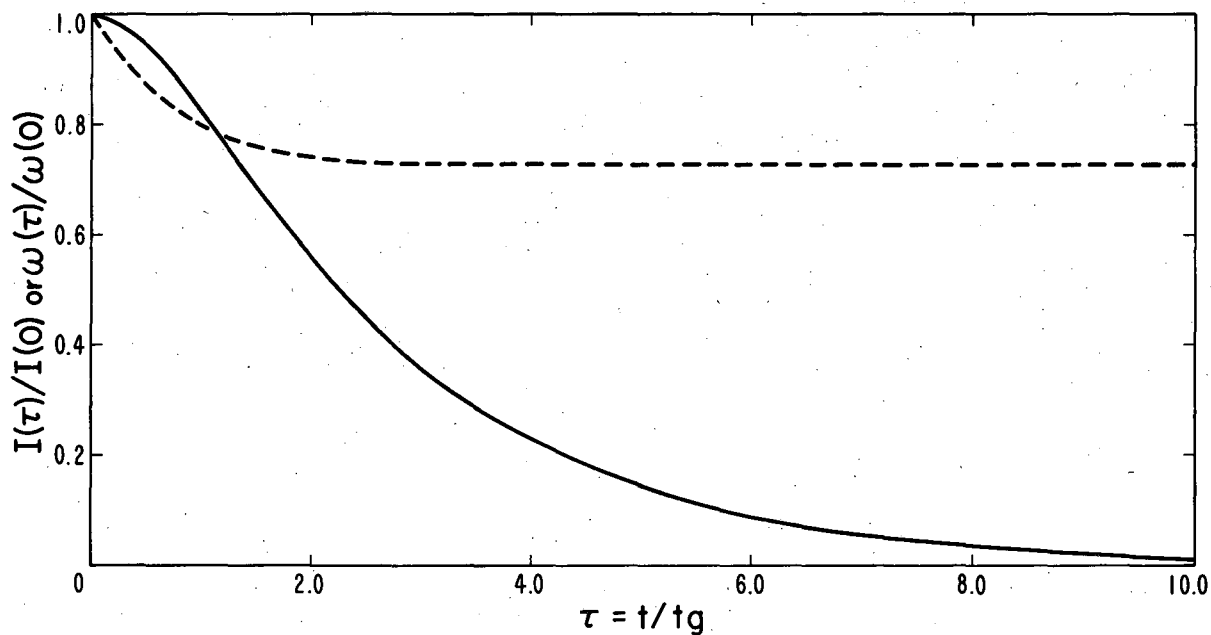
Case III. Hypothetical Example.

We chose the hypothetical example $\sigma = 1$, $\eta^2 = 0.25$ in order to illustrate a case in which the current does change sign. From Fig. 2 we would predict a sign change for this choice of parameters, and in Fig. 5 we see that it does. Note that the zero of the current occurs at the same time as the angular velocity reaches its minimum. This is because we have $I \sim \dot{\omega}$ (See Eq.(7)).



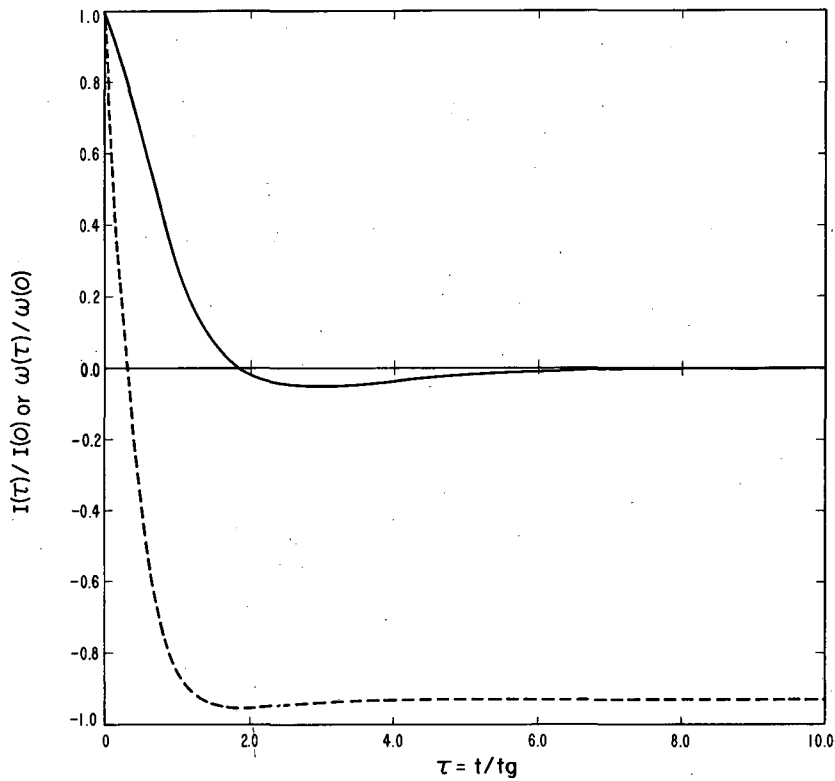
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Fig. 3. Case I. Curves representing predicted current (solid line) and angular velocity (broken line) versus time for $\sigma = 1$, $\eta = 5$. This corresponds approximately to the values for the #4 motor generator set (15-inch bubble chamber). The \times points indicate measured values of the current. $t_g = 2$ sec.



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Fig. 4. Case II. Curves representing predicted current (solid line) and angular velocity (broken line) versus time for $\sigma = 0.5$, $\eta = 1.213$. This corresponds approximately to the values for the #8 and #9 motor generator sets (72-inch bubble chamber). $t_g = 3$ sec.



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Fig. 5. Case III. Curves representing predicted current (solid line) and angular velocity (broken line) versus time for $\sigma = 1$, $\eta = 0.5$. Hypothetical case illustrating oscillatory behavior.

As for the discrepancy mentioned in Case (1) between this theory and the measured data, possibly this can be attributed to a mutual inductance between the generator circuit and the magnet circuit. In this case the equation for the former would be

$$L_g \frac{dI_g}{dt} = -R_g I_g - M \frac{dI}{dt},$$

and since $M \frac{dI}{dt}$ is negative this term tends to make dI_g/dt less negative, so that I_g (and ϕ_g) decays less rapidly. This in turn increases the left side of Eq. (2), which makes dI/dt less negative, so that I decays more slowly.

Acknowledgments

We wish to thank Mr. Clarence A. Harris for drawing our attention to this problem and for discussions relating to the practical aspects of this problem.

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APPENDIX

Proof of Theorem on Changes of Sign of the Current

We will now prove the theorem stated above. Referring to Eq. (16) for the current, we are led to consider the function

$$\psi_p(x, y) = \left[N_p(x) - \frac{x}{2p+1} N_{p+1}(x) \right] J_p(y) - \left[J_p(x) - \frac{x}{2p+1} J_{p+1}(x) \right] N_p(y). \quad (19)$$

If we put $x = \kappa$ and $y = \kappa e^{-\tau}$, then $\psi_p = \frac{2}{\pi(2p+1)} \exp[(p+1)\tau] I(\tau)/I(\tau_0)$, so that the sign of ψ is the same as that of I . In a given case $x = \kappa$ is fixed, and $y = \kappa e^{-\tau}$ varies between $x = \kappa$ and zero. We need to know how many times, if any, $\psi_p(x, y)$ changes sign as the system point (x, y) moves from (κ, κ) to $(\kappa, 0)$ along the $x = \kappa$ line, and this requires that we investigate the qualitative behavior of $\psi_p(x, y)$ in that sector of the x, y plane bonded by the $y=0$ axis and the $x = y$ line, and lying in the positive quadrant (see Fig. 6).

The first feature to notice is that on the 45° line ψ_p has a constant, positive value:

$$\psi_p(x, x) = \frac{x}{2p+1} (J_{p+1}(x) N_p(x) - N_{p+1}(x) J_p(x)) = \frac{x}{2p+1} \cdot \frac{2}{\pi x} = \frac{2}{2p+1}$$

Now we want to find the boundaries between the regions of positive and negative ψ_p . These boundaries are curves on which we have $\psi_p = 0$. Obviously they cannot cross the line $x = y$, where we have $\psi_p = 2/(2p+1)$. However, we can show that they do cross the $y = 0$ axis. To show this we will have to consider the cases $p \geq 0$ and $-\frac{1}{2} \leq p < 0$ separately.

$p \geq 0$.

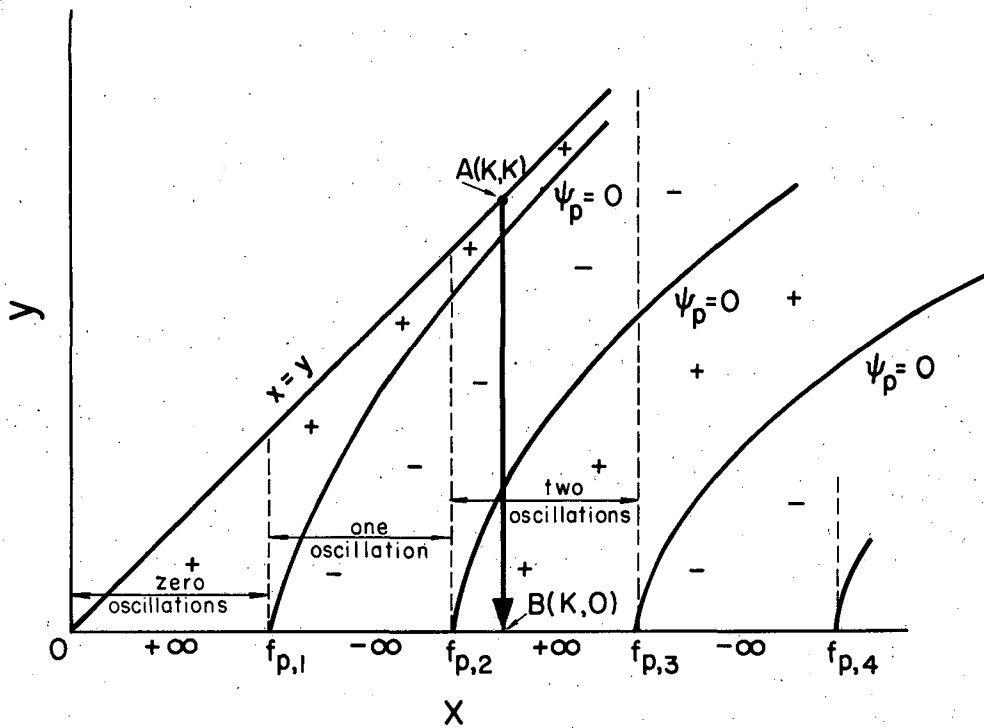
In this case, for $y \rightarrow 0$, $J_p(y)$ becomes finite or zero while $N_p(y)$ becomes negatively infinite. Hence we have

$$\psi_p(x, 0) \approx - \left[J_p(x) - \frac{x}{2p+1} J_{p+1}(x) \right] N_p(0) = \pm \infty$$

according to

$$F_p(x) = \left[J_p(x) - \frac{x}{2p+1} J_{p+1}(x) \right] \begin{cases} \geq 0 \\ \leq 0 \end{cases}$$

Clearly the $\psi_p(x, y) = 0$ curves must cross the x axis where $\psi_p(x, 0)$ changes from $+\infty$ to $-\infty$, that is, where we have $F_p(x) = 0$.



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Fig. 6. Map of the $\psi_p(x, y)$ function. For a particular case the system point moves down the line from A to B as τ goes from 0 to ∞ . The + and - signs indicate the regions where ψ_p is positive and negative.

$-\frac{1}{2} \leq p < 0.$

We use the formulas

$$N_p(y) \sin p\pi = J_p(y) \cos p\pi - J_{-p}(y), \text{ and for}$$

$$y \rightarrow 0: J_p(y) = \frac{(\frac{1}{2}y)^p}{0!p!} + 0(y^{p+2}); J_{-p}(y) = \frac{(\frac{1}{2}y)^p}{0!(-p)!} + 0(y^{-p+2}).$$

Since $p < 0$, $J_p(y) \rightarrow \infty$, $J_{-p}(y) \rightarrow 0$ as $y \rightarrow 0$,

so that we have $N_p(y) \rightarrow J_p(y) \cot p\pi \rightarrow -\infty$ as $y \rightarrow 0$,

and

$$\begin{aligned} \psi_p(x, y) \xrightarrow{y \rightarrow 0} & \left\{ [N_p(x) - \frac{x}{2p+1} N_{p+1}(x)] - [J_p(x) - \frac{x}{2p+1} J_{p+1}(x)] \cot p\pi \right\} J_p(y) \\ & = -\csc(p\pi) [J_{-p}(x) + \frac{x}{2p+1} J_{-p-1}(x)] J_p(y) = -\csc(p\pi) F_p(x) J_p(y) \end{aligned}$$

Hence we have $\psi_p(x, 0) = \pm \infty$ according to $F_p(x) \gtrless 0$, and again the $\psi_p = 0$ curves cross the x axis at the zeroes of $F_p(x)$.

Next we will show that the $\psi_p = 0$ curves tend to the right as y increases, that is, they have positive slope.^p To show this we will first show

(1) The $\psi_p = 0$ curves are never horizontal (except possibly at infinity).

Proof: Suppose a $\psi_p = 0$ curve has zero slope at some point (x, y) . Then at this point

$$\psi_p = \frac{\partial \psi_p}{\partial x} = 0.$$

By using

$$\frac{dZ_\nu}{dx} = -\frac{\nu}{x} Z_\nu + Z_{\nu-1} = \frac{\nu}{x} Z_\nu - Z_{\nu+1}, \tag{a}$$

we find

$$\frac{d}{dx} [Z_p(x) - \frac{x}{2p+1} Z_{p+1}(x)] = -\left(\frac{1}{x} + \frac{x}{2p+1}\right) Z_p + \frac{p+1}{x} (Z_p - \frac{x}{2p+1} Z_{p+1}),$$

therefore

$$\frac{\partial \psi_p}{\partial x} = - \left(\frac{1}{x} + \frac{x}{2p+1} \right) [N_p(x) J_p(y) - J_p(x) N_p(y)] + \frac{p+1}{x} \psi_p = 0.$$

Now, since we have $\psi_p = 0$ and $\left(\frac{1}{x} + \frac{x}{2p+1} \right) \neq 0$, it follows that

$$[N_p(x) J_p(y) - J_p(x) N_p(y)] = 0 \tag{b}$$

Subtracting this from $\psi_p = 0$, with ψ_p given by Eq. (19), we get

$$N_{p+1}(x) J_p(y) - J_{p+1}(x) N_p(y) = 0. \tag{c}$$

Now regarding (b) and (c) as equations for $J_p(y)$ and $N_p(y)$, we have either $J_p(y) = N_p(y) = 0$, which is impossible (except at $y = \infty$), or the determinant $[-N_p(x) J_{p+1}(x) + J_p(x) N_{p+1}(x)]$ equals zero, which is again impossible except for $x = \infty$, since the determinant is $-2/\pi x$. Thus the supposition that a $\psi_p = 0$ curve is horizontal in the finite domain leads to a contradiction, and is therefore false.

(2) The $\psi_p = 0$ curves are never vertical (except possibly at infinity).

Proof: Suppose a $\psi_p = 0$ curve is vertical at some point (x, y) . Then at this point we have

$$\psi_p = \frac{\partial \psi_p}{\partial y} = 0, \text{ and}$$

$$\begin{aligned} \frac{\partial \psi_p}{\partial y} = & [N_p(x) - \frac{x}{2p+1} N_{p+1}(x)] \left(\frac{p}{y} J_p(y) - J_{p+1}(y) \right) \\ & - [J_p(x) - \frac{x}{2p+1} J_{p+1}(x)] \left(\frac{p}{y} N_p(y) - N_{p+1}(y) \right) = 0. \end{aligned}$$

Subtracting this from $\psi_p = 0$, we get

$$[N_p(x) - \frac{x}{2p+1} N_{p+1}(x)] J_{p+1}(y) - [J_p(x) - \frac{x}{2p+1} J_{p+1}(x)] N_{p+1}(y) = 0. \tag{d}$$

Regarding Eq. (19) with $\psi_p = 0$ and (d) taken together as equations for the quantities in square brackets, we have either that the determinant of the equations is zero

$$-J_p(y) N_{p+1}(y) + N_p(y) J_{p+1}(y) = 0,$$

which is impossible (except at $y = \infty$), since this is $2/\pi y$; or that the quantities in square brackets are zero,

$$N_p(x) - \frac{x}{2p+1} N_{p+1}(x) = J_p(x) - \frac{x}{2p+1} J_{p+1}(x) = 0,$$

$$\frac{N_p(x)}{N_{p+1}(x)} = \frac{J_p(x)}{J_{p+1}(x)} = \frac{x}{2p+1}; \quad N_p(x) J_{p+1}(x) - N_{p+1}(x) J_p(x) = 0,$$

which is impossible (except at $x = \infty$), since the last expression is $2/\pi x$. Again we have a contradiction--the $\psi_p = 0$ curves cannot be vertical in the finite domain.

Now we have proven that the $\psi_p = 0$ curves cross the $y = 0$ axis at the roots of $F_p(x) = 0$, that they never cross the line $x = y$, and that they are never horizontal or vertical in the finite domain. Consequently they must tend to the right as y increases from $y = 0$, as shown in Fig. 6. As the system point moves down a vertical line from (κ, κ) to $(\kappa, 0)$, the current will change sign whenever a $\psi_p = 0$ curve is crossed. It is clear from Fig. 6 that the number of such crossings is equal to the number of roots of $F_p(x) = 0$ less than κ , as stated in the theorem.