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Balanced Equilibrium in Pseudo-markets with Endowments[1]

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Abstract

Endowments set a lower bound for agents' utilities in allocation problems. For example, in school choice, one can ensure that low-income families have a shot at high-quality schools by endowing them with a chance of admission in such schools. Common policy objectives, such as walk-zone or sibling placement can be achieved through endowments — and arguably more transparently than via priorities. The policymaker, moreover, could decide to what extent endowment rights should be balanced by an equalizer weight α that shifts individual endowments towards the average endowment.

We introduce a notion of α-balanced individual rationality. The property is compatible with Pareto-efficiency, and asymptotic incentive compatibility. We develop an α-balanced competitive pseudo-market procedure that reaches the desired properties. We also show that envy in such market equilibria is related to a lower contribution of the envying agent's endowment to standard weighted utilitarian welfare than that of the envied agent.

JEL Classification: D47,D50, D60

Keywords: School choice; fairness; justified envy; Walrasian equilibrium; balanced individual rationality.

1 Introduction

We propose new normative criteria for the allocation of discrete resources. Our criteria capture the meaning of fairness among agents who start off from different positions, or who differ in which resources they have property rights over. In particular, we consider a school choice program where property rights are given by explicit endowments, instead of implicitly via priorities, and we propose a notion of fairness among agents who have different endowments.

The policymaker can decide to which extent individual endowment differences shall be taken into account. She can make use of a redistribution parameter, α, that shifts individual endowments towards the average endowment held in the economy. It is in that sense that we propose an idea of α-balanced individual rationality: each individual must prefer her allocation to her α-balanced endowment.

We propose a simple competitive equilibrium procedure that respects such property alongside with Pareto-efficiency and asymptotic incentive compatibility desiderata. Such procedure side-steps the already known equilibrium existence issues arising from a straightaway redistribution of endowments (Hylland and Zeckhauser, 1979).

Moreover, we note that, in such competitive equilibrium allocations, *envy is justified in a utilitarian sense*. If agent i envies agent j, then agent i's endowment's contribution to weighted (by the inverse of marginal utility of income) utilitarian welfare is lower than j's endowment's contribution.

Motivation: school choice, fairness and property rights. School choice is the problem of allocating children to schools when we want to take into account children's (or their parents') preferences. Several large US school districts have in the last 15 years implemented school choice programs that follow economists' recommendation and are based on economic theory.[1] Practical implementation of

[1]Boston (Abdulkadiroğlu and Sönmez, 2003; Abdulkadiroğlu, Pathak, Roth, and Sönmez, 2005), New York (Abdulkadiroğlu, Pathak, and Roth, 2005), and Chicago (Pathak and Sönmez, 2013)

school choice programs presents us with a number of lessons and challenges.

The first lesson is that school choice should be guided by fairness, or *lack of justified envy*. When given the choice of implementing either a fair or an efficient outcome, school districts have consistently chosen fairness (Abdulkadiroğlu, Pathak, Roth, and Sönmez, 2005; Abdulkadiroğlu, Pathak, and Roth, 2005). One reason could be that district administrators are concerned with litigation: if Alice prefers the school that Bob was assigned to, meaning that she envies Bob's allocation, then the district can invoke justified envy to argue as a defense that Bob had a higher priority than Alice at the school in question. It is also likely that district administrators, and society as a whole, have an intrinsic preference for fairness. Such a preference for fairness is important enough to outweigh efficiency.

The second lesson is that school districts want to give children certain rights, like the right to attend a neighborhood school if they wish to, or the right to go to the same school as an older sibling. Rights are achieved by giving children different priorities. For example, Bob might have a high priority for admission in a neighborhood school, or in a school that his brother already attends. While priorities are common in practice, we argue that they are problematic. Priorities do *not* translate immediately into property rights. Alice may have a high priority in one school, but still not get in. Her chances of getting in to a school depends on many things. It depends, for example, on all agents' choices and priorities in the system, not only on her priority at a given school. Given the absence of an immediate translation between priorities and property rights, we propose the use of endowments to ensure property rights.

Endowments provide transparent, and immediate, property rights. A child who is endowed with a seat at her neighborhood school can simply choose to attend that school. Her right to attend a school does not depend on other agents' choices and priorities. Endowments, however, present a new conceptual challenge: What is the meaning of fairness? It is easy to define fairness among agents who start out from identical positions, but how do we understand fairness among unequal agents? One of our contributions will be to formalize the notion of fairness, in the sense of absence of justified envy, for school choice problems with endowments.

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are the leading examples.

Our market notion. Our market equilibrium solution is a hybrid of the standard equal-income market solution, and classical Walrasian equilibrium. Agents' expenses in our market must be debited against a budget constraint that is a weighted average of a fixed income, and an income derived from selling endowments at market prices. Say that the weight on the fixed income is $\alpha \in (0, 1)$ and the weight on the income from endowments is $1 - \alpha$. In the extreme case when α is zero, the market is a textbook Walrasian exchange economy, in which agents derive income purely from selling their endowment at market prices. Unfortunately, when $\alpha = 0$ the Walrasian model may not possess an equilibrium (see our discussion in Section 5.1), and may have Pareto dominated equilibria (see Section 5.2). We show, however, that when $\alpha > 0$ equilibrium always exists, and an equilibrium that is Pareto optimal can be found. Moreover, by choosing $\alpha > 0$ to be arbitrarily small, we can come as close as desired to respecting individual rationality. Finally, as long as $\alpha < 1$, the model allows endowments to matter and play a role in the final allocation. As a consequence, in equilibrium, if Alice envies Bob, her envy must be the reflection of Bob's endowment being more valuable than Alice's, and (under some additional conditions) by a coalition of agents wanting more of Bob's endowments and less of Alice's.

After we first circulated our paper, Garg, Tröbst, and Vazirani (2020) have proposed algorithms that can efficiently compute our notion of balanced equilibrium for special classes of utility functions. Their work, arguably, renders our proposal practically implementable.

2 Related literature.

The papers closest to ours are Hylland and Zeckhauser (1979), Mas-Colell (1992), Le (2017), and McLennan (2018).

Hylland and Zeckhauser (1979) were the first to propose markets over lottery shares to solve centralized allocation problems. They assume a fixed income for each agent, independent of prices. Hylland and Zeckhauser make the point, which we elaborate on in Section 5.1, that a Walrasian market with endowments would not always work because equilibrium may not exist. They also emphasize that equilibrium may not be efficient, and introduce the "cheapest bundle" property that we employ as well in our version of the first welfare theorem. It should be clear that allowing for endowments is a stark departure from the model of Hylland and Zeckhauser (1979), and poses significant challenges.

Many other papers have followed Hylland and Zeckhauser in analyzing competitive equilibria as solutions in market design. For example, Mas-Colell (1992), Budish, Che, Kojima, and Milgrom (2013), Ashlagi and Shi (2015), He, Miralles, Pycia, and Yan (2018), He, Li, and Yan (2015), Le (2017), and McLennan (2018). With the exception of Mas-Colell, Le, and McLennan, three papers that we discuss below, these authors explore markets with exogenously given budgets: $\alpha = 1$ in our model. When all agents have equal budgets, there can be no envy in a competitive equilibrium (an idea stressed by Varian (1974)). But equal budgets of course eliminate any role for the initial endowments in the same blow as they eliminate envy.[2] The textbook model of a Walrasian exchange economy allows for endowments to play a role in justifying envy, but equilibrium, as we have emphasized, may not exist. There may also exist Pareto-ranked Walrasian equilibria (see Section 5.2).

A version of the hybrid model was first introduced by Mas-Colell (1992) and later on by Le (2017). Mas-Colell presents an existence result that is similar to ours, with income that is constituted by an additive fixed price-independent income that is added to the standard Walrasian income that depends on prices and on the endowment. His result requires the first component to be determined as part of the fixed point argument in the equilibrium existence result. Put differently, his result does not give an existence result for a fixed α, but instead determines α endogenously in equilibrium.

We view our result, for fixed α, as having an advantage for market design because in market design we wish to fix the parameters of the market. Moreover, α has some meaning as a policy instrument, capturing the importance of the exogenous equal income relative to the income that is derived from endowments. For example, we can ensure that agents' welfare is as close as desired to what it would be in a purely Walrasian model by choosing α arbitrarily small (see Theorem 2). Finally, our approach allows for a simple connection between equilibrium welfare and property

[2] Eric Budish has pointed out to us that, in the applications to course-bidding in Wharton, agents were awarded different budgets out of fairness considerations. The purpose of our results is different. We seek to understand the meaning of fairness for agents who start out with different endowments. When endowments results from, for example, the presence of neighborhood schools, it is not clear how to relate our model to one with different budgets.

rights, and justified envy (see Theorem 3). Such results are not available for Mas-Colell's notion of equilibrium.

Le's objective was to avoid the non-existence result in Hylland-Zeckhauser, and to be able to talk about justified envy. These themes are common to our paper. There are, however, some important differences between his approach and our results on α-balanced equilibrium. The main difference is that, in his notion of equilibrium, two identical goods may have different prices. As a consequence, there may be envy among identical agents, and it may be necessary for some agents to purchase a more expensive copy of a good when a cheaper one is available. Envy among equals is problematic for normative reasons.[3] Having agents purchase the more expensive copy of an identical good is problematic because it may make it hard to implement Le's equilibria in a decentralized fashion. These issues are illustrated through an example in Section 5.1.

A second, somewhat more technical, issue is that the exact way in which the exogenous and endogenous budgets are combined is different in Mas-Colell's and Le's cases from ours. These authors add them, while we mix them in a convex combination. This may seem like a technicality, yet it matters. For example, Le cannot totally eliminate excess demands unless all agents are endowed with all goods (all endowments are full support). Finally, in Le's result, the efficiency property of equilibrium is weaker: weak Pareto optimality, rather than Pareto optimality.

The third relevant paper is the recent work of McLennan (2018), who presents an existence result for equilibrium with "slack" in a general model. McLennan's general model of an economy allows for production, and encompasses our model as a special case, but his notion of equilibrium with slack differs from ours in important ways. Agents in his (and our) model may be satiated, and his notion of slack controls the distribution of transfers from satiated agents to unsatiated agents. Satiated agents my spend less than their income, and it is important to transfer their unspent income to unsatiated agents. In contrast, our α parameter controls the role of endowments, allowing for α to specify the weight of equal incomes vs. (unequal) endowments. McLennan presents an example to illustrate the difference between the two notions of equilibrium.[4] In his example, no agents are satiated, so the

[3] One could interpret different prices for different copies of the same good as a novel endogenous transfer scheme, but we are unaware of a normative defense of this idea.

[4] The two papers were written independently.

slack in his notion of equilibrium has no role to play; as a consequence, equilibrium allocations are independent of α. In contrast, our equilibrium allocations for his example range from equal division to the autartical consumption of endowments, as α ranges from placing all weight on the exogenous income, to placing all weight on initial endowments.

3 The model

Our model is essentially the textbook model of an exchange economy in general equilibrium theory. The difference with the textbook model is that agents consume lotteries: consumption bundles cannot add up to more than one. This difference is far from minor. For example, it results in the non-existence of Walrasian equilibrium, even for economies that are otherwise well-behaved, and in the presence of Pareto-ranked Walrasian equilibria (see our discussions in Section 5.1 and 5.2).

Notation and preliminary definitions. The simplex $\{x \in \mathbf{R}^n_+ : \sum_{j=1}^n x_j = 1\}$ in $\mathbf{R}^n$ is denoted by $\Delta^n \subseteq \mathbf{R}^n$, while the set $\{x \in \mathbf{R}^n_+ : \sum_{j=1}^n x_j \leq 1\}$ is denoted by $\Delta^n_- \subseteq \mathbf{R}^n$. When n is understood, we simply use the notation Δ and Δ_-.

A function $u : \Delta_- \to \mathbf{R}$ is

- *concave* if, for any $x, z \in \Delta_-$, and $\lambda \in (0,1)$, $\lambda u(z) + (1-\lambda)u(x) \leq u(\lambda z + (1-\lambda)x)$;

- *quasi-concave* if, for any $x, z \in \Delta_-$, and $\lambda \in (0,1)$, $\min\{u(z), u(x)\} \leq u(\lambda z + (1-\lambda)x)$.

- *semi-strictly quasi-concave* if it is quasi-concave, and for any $x, z \in \Delta_-$, $u(z) \neq u(x)$ and $\lambda \in (0,1)$ imply that $\min\{u(z), u(x)\} < u(\lambda z + (1-\lambda)x)$.

- *strictly quasi-concave* if it is quasi-concave and, for any $x, z \in \Delta_-$ with $z \neq x$, and $\lambda \in (0,1)$, $\min\{u(z), u(x)\} < u(\lambda z + (1-\lambda)x)$.

- *strictly increasing* if, for any $x, z \in \Delta_-$ with $x > z$, $u(x) > u(z)$.

- C^1 if it can be extended to a continuously differentiable function defined on an open set that contains Δ_-.

Model. A *discrete allocation problem* is a tuple $\Gamma = \{O, I, Q, (u^i, \omega^i)_{i\in I}\}$, where:

- $O = \{1, \dots, L\}$ represents a finite set of indivisible objects, or goods.
- $I = \{1, \dots, N\}$ represents a finite set of agents, each of whom demands exactly one copy of an object.
- $Q = (q_l)_{l\in O}$ is a capacity vector, and $q_l \in \mathbf{N}$ is the quantity of object l. For simplicity, we assume that $\sum_{l\in O} q_l = N$, i.e., the number of copies of objects is equal to the number of agents.
- For each agent i, $u^i : \Delta^L_- \to \mathbf{R}$ is a continuous and strictly increasing utility function defined on Δ^L_-.
- For each agent i, $\omega^i \in \Delta^L$ is i's endowment vector such that ω^i_l is the fraction of object l owned by i. We assume that all objects are owned by agents. So $\sum_{i=1}^N \omega^i = Q$. We denote by $\bar{\omega} = Q/N$ the *average endowment* of the economy.

Allocations, Pareto optimality and individual rationality. An *allocation* in Γ is a vector $x \in \mathbf{R}^{LN}_+$, which we write as $x = (x^i)_{i=1}^N$, with $x^i \in \Delta^L_-$, such that

$$\sum_{i\in I} x^i_l = q_l$$

for all $i \in I$ and all $l \in O$. When $x^i_l \in \{0, 1\}$ for all i and all l, x is a deterministic allocation. The Birkhoff-von Neumann theorem (Birkhoff, 1946; Von Neumann, 1953) implies that every allocation is a convex combination of deterministic allocations.

An allocation x is *Pareto optimal* (PO) if there is no allocation y such that $u^i(y^i) \geq u^i(x^i)$ for all i and $u^j(y^j) > u^j(x^j)$ for some j.

An allocation x is *acceptable* to agent i if $u^i(x^i) \geq u^i(\omega^i)$; x is *individually rational* (IR) if it is acceptable to all agents. We also define a notion of approximate individual rationality: for any $\varepsilon > 0$, x is *ε-individually rational* (ε-IR) if $u^i(x^i) > u^i(\omega^i) - \varepsilon$ for all i.

Allocations and α-balanced individual rationality. We introduce a novel notion of individual rationality in models with endowments: for any $\alpha \in (0, 1)$, x is *α-balanced individually rational* if $u^i(x^i) \geq u^i(\alpha\bar{\omega} + (1-\alpha)\omega^i)$ for all i. The idea in balanced individual rationality is that a central planner "moderates" individual agents' claims to be as well off as in their endowments. It introduces a degree of resource egalitarianism, in the sense that it pulls all agents partially to a claim on the economy's average endowment. Of course, the degree of such resource egalitarianism is controlled by the parameter α. One may imagine a planner using real taxes and subsidies to "moderate" individual agents endowments and make the starting point of the economy more egalitarian.

Allocations and equity. We define equity in our model. An agent i *envies* another agent j in an allocation x if $u^i(x^j) > u^i(x^i)$. An allocation is *envy-free* if no agent envies any other agent. We also define a notion of approximate equity: for any $\varepsilon > 0$, x is *ε-envy-free* (ε-EF) if, for every distinct i and j, $u^i(x^i) > u^i(x^j) - \varepsilon$.

Balanced Walrasian equilibrium. Given any $\alpha \in (0,1)$, an *α-balanced Walrasian equilibrium* is a pair (x,p) such that $x \in \Delta_-^N$, and $p = (p_l)_{l \in O} \in \mathbf{R}_+^L$ is a price vector such that

1. the market clears: $\sum_{i=1}^N x^i = \sum_{i=1}^N \omega^i$; and
2. x^i maximizes i's utility within his α-balanced budget:

$$x^i \in \operatorname{argmax}\{u^i(z^i) : z^i \in \Delta_- \text{ and } p \cdot z^i \leq \alpha + (1-\alpha) p \cdot \omega^i\};$$

A *Walrasian equilibrium* is an extreme case of α-balanced Walrasian equilibrium in which we set $\alpha = 0$. It is well-known that Walrasian equilibria may not exist in our model, even for very well behaved utility functions (Hylland and Zeckhauser, 1979). Section 5.1 elaborates further.

On the other extreme, when $\alpha = 1$, we obtain the competitive equilibrium from equal incomes (Hylland and Zeckhauser, 1979; Varian, 1974)).

4 Main Results

Let $\Gamma = \{O, I, Q, (u^i, \omega^i)_{i \in I}\}$ be a discrete allocation problem. We first prove that for any $\alpha > 0$, α-balanced Walrasian equilibria exist.

Theorem 1. *Suppose that agents' utility functions in Γ are quasi-concave. For any $\alpha \in (0,1]$, there exists an α-balanced Walrasian equilibrium (x,p). The equilibrium* *allocation x is α-balanced individually rational. Moreover, if agents' utility functions are semi-strictly quasi-concave, then x can be chosen to be Pareto optimal.*

The next result proves that by choosing α arbitrarily close to 0, the market designer can obtain an equilibrium allocation arbitrarily close to individual rationality, while by choosing α arbitrarily close to 1, the market designer can obtain an equilibrium allocation arbitrarily close to equity.[5]

Theorem 2. *Suppose that agents' utility functions in Γ are semi-strictly quasi-concave. For any $\varepsilon > 0$:*

1. *There is $\alpha \in (0, 1)$ and an α-balanced Walrasian equilibrium (x, p) such that x is Pareto optimal and for every i,*

$$\max\{u^i(y) : y \in \Delta_- \textit{ and } p \cdot y \leq p \cdot \omega^i\} - u^i(x) < \varepsilon.$$

   In particular, x is ε-individually rational.

2. *There is $\alpha \in (0, 1)$ and an α-balanced Walrasian equilibrium (x, p) such that x is Pareto optimal and for every distinct i and j,*

$$\max\{u^i(y) : y \in \Delta_- \textit{ and } p \cdot y \leq \alpha + (1 - \alpha)p \cdot \omega^j\} - u^i(x) < \varepsilon.$$

   In particular, x is ε-envy-free.

The second part of Theorem 2 means that in the equilibrium allocation x, i does not envy j's assignment as well as any consumption in j's budget set in an approximate sense. Without approximation, Varian (1976) calls this property *opportunity fairness.*

Suppose that an agent i envies another agent j in an α-balanced Walrasian equilibrium with some $\alpha \in (0, 1)$. Then it must be that $p \cdot \omega^j > p \cdot \omega^i$. In other words, i's envy is not justified, because j's endowment is *more valuable* at market prices than i's. This means that the society values j's endowment more than i's, in a sense made precise in Theorem 3.

Theorem 3. *Suppose that agents' utility functions in Γ are concave and C^1. Let (x, p) be an α-balanced Walrasian equilibrium. Denote by $S = \{i : u^i(x^i) =$*

[5] Of course, we can obtain exact equity by choosing $\alpha = 1$. The purpose of this result is to show that to obtain approximate equity, α needs not to be one.

$\max_{z^i \in \Delta_-} u^i(z^i)\}$ *the set of satiated agents, and by* $U = I \setminus S$ *the set of unsatiated agents. Suppose that* $\sum_{i \in U} x^i \gg 0$. *If* i *envies* j *in* x, *then* $p \cdot \omega^j > p \cdot \omega^i$, *and there exists welfare weights* $\theta \in \mathbf{R}^U_{++}$ *such that if*

$$v(t) = \sup\{\sum_{i \in U} \theta^i u^i(\tilde{x}^i) : (\tilde{x}^i) \in \Delta^U_- \text{ and } \sum_{i \in U} \tilde{x}^i \leq Q + t(\omega^i - \omega^j) - \sum_{i \in S} x^i\},$$

then $(x^i)_{i \in U}$ *solves the problem for* $v(0)$, *and* $v(t) < v(0)$ *for all* t *small enough.*

The meaning of Theorem 3 is that if i envies j then j's endowment is more valuable than i's in two senses. First, it is more valuable at equilibrium prices. Second, the higer price valuation translates into a statement about how much agents value the endowment. In particular, j's endowment is more valuable than i's to a coalition of players U (a coalition that includes i!) in the sense that there are welfare weights for the members of U such that a change in agents' endowment towards having more of i's endowment and less of j's leads to a worse weighted utilitarian outcome. The results requires $\sum_{i \in U} x^i \gg 0$ simply to ensure that when we subtract ω^j we do not force some agent to consume negative quantities of some good.[6]

An important take-away from these results is that the market designer can flexibly balance equity and individual rationality by choosing the value of α. The following example further illustrates this point.

Example 1. *Given is an economy with two agents and two objects. Each object has one copy. Agents have expected utilities given by the following vNM indexes:*

i	u^i_l	$u^i_{l'}$
1	*100*	*1*
2	*100*	*1*

Endowments are $\omega^1 = (2/3, 1/3)$ *and* $\omega^2 = (1/3, 2/3)$. *In words, the two agents both prefer* l *to* l', *and agent 1 owns more amount of* l *than agent 2 does.*

For any $\alpha \in (0, 1]$, *there is an* α*-balanced Walrasian equilibrium where the price vector is* $p = (2, 0)$ *and the allocation is* $x_\alpha = (x^i_\alpha)_{i=1,2}$ *where*

$$x^1_\alpha = (\frac{4-\alpha}{6}, \frac{2+\alpha}{6}), \quad x^2_\alpha = (\frac{2+\alpha}{6}, \frac{4-\alpha}{6}).$$

[6] The $\sum_{i \in U} x^i \gg 0$ hypothesis in Theorem 3 is stronger than what we need. It suffices that if $\omega^j_l > 0$ then $\sum_{i \in U} x^i_l > 0$.

When α converges to 1, x_α converges to the only envy-free allocation where $x^1 = x^2 = (1/2, 1/2)$. When α converges to 0, x_α converges to the only individually rational allocation, the initial endowment ω. As α varies from 0 to 1, x_α becomes more equitable but more away from individual rationality.

We proceed to discuss other more nuanced aspects of our results.

5 Discussion

5.1 The Hylland and Zeckhauser example

A Walrasian equilibrium (a 0-balanced equilibrium) may not exist in our model. We present a non-existence example originally due to Hylland and Zeckhauser (1979). We show that the symmetric Pareto optimal allocation in this example can be sustained as an α-balanced Walrasian equilibrium with any $\alpha \in (0, 1]$.

Example 2 (Hylland-Zeckhauser example)**.** *Given is an economy with three agents and two objects. Object l has one copy, while object l' has two copies. Agents have expected utilities given by the following vNM indexes:*

i	u^i_l	$u^i_{l'}$
1	*100*	*1*
2	*100*	*1*
3	*1*	*100*

Endowments are $\omega^i = (1/3, 2/3)$ for $i = 1, 2, 3$.

Proposition 1. *There is no Walrasian equilibrium in Example 2.*

Proof. Suppose (towards a contradiction) that (x, p) is a Walrasian equilibrium. Suppose first that $p_{l'} > 0$. Normalize $P_{l'}$ to one. Then all agents have the same positive budget. If $p_l = 0$, then 1 and 2 would each buy one copy of l, which is a contradiction. So p_l must be positive. The preferences of agents imply that 1 and 2 must each obtain a half of l. Therefore, $1/3p_l + 2/3 \geq 1/2p_l$, and we obtain $p_l \leq 4$. However, if $p_l < 4$, 1 and 2 would spend all of their budgets on l, and each obtain more than a half of l, which is a contradiction. So it must be that $1/3p_l + 2/3 = 1/2p_l$ and $p_l = 4$. But then 1 and 2 would still spend all of their budgets on l, and l' must have excess supply, which is a contradiction.

Now suppose $p_{l'} = 0$ and $p_l > 0$. Then 3 must obtain one copy of l'. Since p_l is positive, 1 and 2 must each obtain a half of l. However, their budget $1/3p_l$ cannot afford such an allocation. $\square$

Consider the symmetric Pareto optimal allocation x defined by:

i	x^i_l	$x^i_{l'}$
1	1/2	1/2
2	1/2	1/2
3	0	1

Proposition 2. *For any $\alpha \in (0, 1]$, there is an α-balanced Walrasian equilibrium that supports the allocation x in Example 2.*

Proof. For any $\alpha \in (0, 1]$, let

$$p = (\frac{6\alpha}{1 + 2\alpha}, 0).$$

Then for $i = 1, 2$, $p \cdot \omega^i = \frac{2\alpha}{1+2\alpha}$ and

$$\alpha + (1 - \alpha)p \cdot \omega^i = \frac{\alpha + 2\alpha^2 + (2\alpha - 2\alpha^2)}{1 + 2\alpha} = \frac{3\alpha}{1 + 2\alpha} = p \cdot x^i.$$

Agents 1 and 2 can improve by purchasing more l, but they cannot afford any more. They can only afford 1/2 share of l and buy 1/2 share of l' for free. They can improve by purchasing more l' at the zero price, but that would not be feasible in Δ_-. Agent 3 is optimizing by choosing 1 share of l' for free. $\square$

Note that in the above α-balanced Walrasian equilibrium supporting x, the value of 1 and 2's endowments ($p \cdot \omega^i$) in equilibrium is $2\alpha/(1 + 2\alpha)$. So the value of the exogenous part of the budget relative to the endogenous value of endowments $p \cdot \omega^i$ is

$$\frac{\alpha \cdot 1}{(1 - \alpha) \cdot p \cdot \omega^i} = \frac{1 + 2\alpha}{2(1 - \alpha)} \to \frac{1}{2} \quad \text{as } \alpha \to 0.$$

So as α shrinks to zero, the value of the exogenous income is not negligible. In the same spirit, the following proposition shows that the average endogenous budget will always be below the exogenous budget of one.

Proposition 3. *If (x, p) is an α-balanced Walrasian equilibrium with $\alpha \in (0, 1]$ then*

$$p \cdot \bar{\omega} \leq 1$$

Proof. Note that $p \cdot (x^i - \omega^i) \leq \alpha(1 - p \cdot \omega^i)$ for all i. Sum over i to obtain:

$$0 = p \cdot \left(\sum_i x^i - Q \right) \leq \alpha(N - p \cdot Q).$$

□

Proposition 3 puts an upper bound on the average endogenous income. It cannot exceed the exogenous income of one. In particular, this means that the economy needs outside "money."

Proposition 3 reveals more than the proof of Theorem 1, which bound prices by the inequality:

$$\frac{p_l(\min_{l \in O} q_l - \varepsilon)}{N} \leq 1.$$

Finally, we consider the resolution presented in Le (2017) of Example 2, namely that the allocation x can be obtained in a market equilibrium with different prices for the two copies of l'. Specifically, let

$$p = (100, 1, \frac{101}{2}).$$

That is, the price of one copy of l' is 1, and the price of the other copy is 101/2. Then all agents have an endogenous income of 101/2. The unique optimal bundle for agents 1 and 2 is $x^i = (1/2, 1/2, 0)$. Agent 3 is willing to spend all his income on buying the more expensive copy of good l', so $x^3 = (0, 0, 1)$ is in his demand correspondence.

Consider a variation of the Hylland-Zeckhauser example, with three agents and the same utility functions, but where endowments are $\omega^1 = (1/3, 1/2, 1/6)$, $\omega^2 = (1/3, 1/6, 1/2)$, and $\omega^3 = (1/3, 1/3, 1/3)$. Agents essentially have the same endowments as above. Then $p = (100, 1, \frac{101}{2})$ is still an equilibrium price, with $x^1 = (\frac{5}{12}, \frac{7}{12}, 0)$, $x^2 = (\frac{7}{12}, \frac{5}{12}, 0)$, and $x^3 = (0, 0, 1)$. But observe that agent 1 envies 2, despite having the same utility and the same endowment: 1/3 of l and 2/3 of l', which happened to be split unequally over the two copies of l'. More generally, in Le's resolution two objects may be perfect substitutes but end up having different prices. This leads to envy among equals, which is undesirable.

5.2 Efficiency in balanced Walrasian equilibrium

The first welfare theorem is not true in our model. Walrasian equilibria, and even α-balanced Walrasian equilibria with any $\alpha > 0$, may fail to be Pareto optimal.

Example 3 below illustrates the point by exhibiting Pareto-ranked Walrasian equilibria.

Example 3. *Given is an economy with two agents and two objects. Each object has one copy. Agents have expected utilities given by the following vNM indexes:*

i	u^i_l	$u^i_{l'}$
1	*1*	*1*
2	*1*	*100*

Endowments are $\omega^i = (1/2, 1/2)$ *for* $i = 1, 2$.

Consider the allocations $x = ((1,0),(0,1))$ *and* $y = ((1/2,1/2),(1/2,1/2))$. *Note that* x *Pareto dominates* y.

The following table summarizes how both x *and* y *may be supported as Walrasian equilibria, both with* $\alpha > 0$ *and* $\alpha = 0$. *The first welfare theorem fails because agents have satiated preferences, not because we focus on* α*-balanced Walrasian equilibria.*

α	*allocation*	p	$\alpha + (1-\alpha)p \cdot \omega^i$
0	x	$(1,1)$	1
	y	$(0,1)$	$1/2$
$1/2$	x	$(1,1)$	1
	y	$(0,2)$	1

The table is hopefully obvious, but it may be useful to detail why y *is an equilibrium allocation with* $\alpha = 0$. *Note that the income with prices* $(0,1)$ *is* $1/2$ *for each agent. Agent 1 is happy to spend his income purchasing* $x^1 = (1/2, 1/2)$ *for a (global) utility maximum. Agent 2 spends all his income on* l' *and purchases* $1/2$ *share of* l'*, and obtains* $1/2$ *share of* l *for free.*

Theorem 1 asserts the existence of Pareto optimal α-balanced equilibria. This finding relies on the the following property: an α-balanced Walrasian equilibrium (x, p) satisfies the *cheapest-bundle property* if, for each i, x^i minimizes expenditure $p \cdot z^i$ among all the $z^i \in \Delta_-$ for which $u^i(z^i) = u^i(x^i)$. The notion of a cheapest bundle, and its role in the first welfare theorem, was already established by Hylland and Zeckhauser (1979). In Theorem 1 we impose semi-strictly quasiconcave utilities in order to ensure the existence of an α-balanced Walrasian equilibrium with the cheapest-bundle property. The first welfare theorem holds for such equilibria.

Proposition 4. *Any α-balanced Walrasian equilibrium allocation is weakly Pareto optimal, and any α-balanced Walrasian equilibrium with the cheapest-bundle property is Pareto optimal.*

5.3 Incentive Compatibility

The market procedure described in this paper is subject to well known positive results on incentive compatibility in large markets. In order to tackle the incentive compatibility problem, one has to give mechanism structure to our competitive procedure. Let $(x, p)[(\tilde{u}^i)_i]$ denote the implemented α-balanced Walrasian equilibrium when declared preferences correspond to the array $(\tilde{u}^i)_i$.[7] We can separate allocation and prices so that $p[(\tilde{u}^i)_i]$ denotes the implemented α-balanced Walrasian equilibrium price vector.

We study large markets in the replica sense. We denote with Γ^k the k-replica of the allocation problem Γ.[8] Let $(x, p)^k$ denote the competitive equilibrium mechanism that applies to the k-replica allocation problem. Let p^k denote the corresponding equilibrium prices in the k-replica.

Borrowing an idea from Jackson (1992), we define a replicated mechanism array $[(x, p)^k]_{k\in\mathbf{N}}$ as *regular* (at an original allocation problem $\Gamma = \{S, I, Q, (u^i, \omega^i)_{i\in I}\}$) if, for all $i \in I$ and all $\tilde{u}^i$, for every $\epsilon > 0$ there is k_ϵ such that for all $k > k_\epsilon$ we obtain $||p^k[\tilde{u}^i, (u^j)_{j\neq i}] - p^k[(u^j)_{j\in I}]|| < \epsilon$.

Let $V^i(p)$ denote agent i's indirect utility when prices are p. We next define large-market incentive compatibility and asymptotic incentive compatibility.

Given an original allocation problem $\Gamma = \{S, I, Q, (u^i, \omega^i)_{i\in I}\}$, a replicated mechanism array $[(x, p)^k]_{k\in\mathbf{N}}$ is *large-market incentive compatible* if for all $i \in I$ and all $\tilde{u}^i$, there is k^* such that for all $k > k^*$ we have $V^i(p^k[\tilde{u}^i, (u^j)_{j\neq i}]) \leq V^i(p^k[(u^j)_{j\in I}])$.

Given an original allocation problem $\Gamma = \{S, I, Q, (u^i, \omega^i)_{i\in I}\}$, a replicated mechanism array $[(x, p)^k]_{k\in\mathbf{N}}$ is *asymptotically incentive compatible* if for all $i \in I$ and all $\tilde{u}^i$, and for all $\varepsilon > 0$, there is k_ε such that for all $k > k_\varepsilon$ we have $V^i(p^k[\tilde{u}^i, (u^j)_{j\neq i}]) - V^i(p^k[(u^j)_{j\in I}]) < \varepsilon$.

[7] Throughout this section, we only consider utility functions allowing for the existence of α-balanced Walrasian equilibria.

[8] This is the standard replica in the sense of multiplying capacities by k and at the same time replicating agents with their endowments and preferences k times.

The former concept borrows again from Jackson (1992) while the later borrows from the seminal paper by Roberts and Postlewaite (1976). We state the main result in this subsection without proof. Such a proof would just mimic those provided in the above cited papers.

Theorem 4. *Given an allocation problem $\Gamma = \{S, I, Q, (u^i, \omega^i)_{i \in I}\}$ and a regular replica mechanism array $[(x,p)^k]_{k \in \mathbf{N}}$ constituted by α-balanced Walrasian equilibria, the mechanism array is asymptotically incentive compatible. Morevore, if all utility functions are strictly quasi-concave, such a mechanism array is large-market incentive compatible.*

6 Proof of Theorem 1.

Existence. Given any $\alpha \in (0,1]$, we prove the existence of an α-balanced Walrasian equilibrium with the cheapest bundle property under semi-strict quasi-concavity. See Remark 1 below for a more general result.

Let

$$\begin{aligned} v^i &= \max\{u^i(x) : x \in \Delta_-\} \\ B^i(p) &= \{x \in \Delta_- : p \cdot x \leq \alpha + (1-\alpha) p \cdot \omega^i\} \\ d^i(p) &= \operatorname{argmax}\{u^i(x) : x \in B^i(p)\} \\ \underline{d}^i(p) &= \operatorname{argmin}\{p \cdot x : x \in d^i(p)\} \\ V^i(p) &= \max\{u^i(x) : x \in B^i(p)\} \\ z^i(p) &= \underline{d}^i(p) - \omega^i \text{ and } z(p) = \sum_{i=1}^{N} z^i(p). \end{aligned}$$

Note that v^i is the largest utility that i can attain. B^i is the budget set, d^i is demand, $\underline{d}^i$ is cheapest-demand, V^i is i's indirect utility function. z^i is i's excess demand correspondence given the cheapest-bundle selection, and Z the aggregate excess demand.

Lemma 1. *If $V^i(p) < v^i$ then $d^i(p) = \underline{d}^i(p)$.*

Proof. Let $x \in d^i(p)$. We shall prove that $p \cdot x = \alpha + (1-\alpha) p \cdot \omega^i$, which means we are done because it implies that all bundles in $d^i(p)$ cost the same at prices

p. Let $z \in \Delta_-$ be such that $u^i(z) = v^i > u^i(x)$, and note that for any $\varepsilon \in (0,1)$, $u^i(\varepsilon z + (1-\varepsilon)x) > u^i(x)$ by the semi-strict quasi-concavity of u^i. Since $\varepsilon z + (1-\varepsilon)x \in \Delta_-$, this means that $p \cdot (\varepsilon z + (1-\varepsilon)x) > \alpha + (1-\alpha)p \cdot \omega^i$ for any $\varepsilon \in (0,1)$. But this is only possible, for arbitrarily small ε, if $p \cdot x \geq \alpha + (1-\alpha)p \cdot \omega^i$. Since $x \in B^i(p)$ we have established that $p \cdot x = \alpha + (1-\alpha)p \cdot \omega^i$. □

Lemma 2. *If $V^i(p) = v^i$ then*

$$\underline{d}^i(p) = \arg\min\{p \cdot x : u^i(x) = v^i \text{ and } x \in \Delta_-\}.$$

Proof. Let $x \in \underline{d}^i(p)$. Then for any $z \in \Delta_-$ with $p \cdot z < p \cdot x$, $z \in B^i(p)$. So $u^i(z) < v^i$. Therefore, if $z \in \text{argmin}\{p \cdot x : u^i(x) = v^i \text{ and } x \in \Delta_-\}$, then $p \cdot z = p \cdot x \leq \alpha + (1-\alpha)p \cdot \omega^i$, and therefore

$$\underline{d}^i(p) \supseteq \text{argmin}\{p \cdot x : u^i(x) = v^i \text{ and } x \in \Delta_-\}.$$

The converse set inclusion follows similarly because if x is not in the righ-hand set, there would exist a $z \in \Delta_-$ with $p \cdot z < p \cdot x$ and $u^i(z) = v^i$, which is not possible as such a z^i would be in $B^i(p)$. □

Lemma 3. *If $\alpha > 0$ then d^i is uppper hemicontinuous.*

Proof. Let $(x_n, p_n) \to (x, p)$, with $x_n \in d^i(p_n)$. Suppose that there is $x' \in B^i(p)$ with $u^i(x') > u^i(x)$. If $p \cdot x' < \alpha + (1-\alpha)p \cdot \omega^i$, then this strict inequality will be true for p_n with n large enough; a contradiction, as u^i is continuous. If $p \cdot x' = \alpha + (1-\alpha)p \cdot \omega^i$, then $\alpha > 0$ implies that $p \cdot x' > 0$. Then there is $\lambda \in (0,1)$ large enough that $u^i(\lambda x') > u^i(x)$, $p \cdot (\lambda x') < p \cdot x'$, and $\lambda x' \in \Delta_-$. The argument for the case of a strict inequality then applies. □

Let $\varepsilon \in (0, \min_{l \in O} q_l)$ and

$$\bar{p} = \frac{N}{\min_{l \in O} q_l - \varepsilon} > 0.$$

Lemma 4. *$\underline{d}^i$ is upper hemi-continuous on $[0, \bar{p}]^L$.*

Proof. We shall prove that $\underline{d}^i$ has a closed graph. Let $(x_n, p_n) \to (x, p)$ with $x_n \in \underline{d}^i(p_n)$ for all n.

First, consider the case where $V^i(p) < v^i$. By the maximum theorem, V^i is continuous, so $V^i(p_n) < v^i$ for all n large enough. Then Lemma 1 implies that $x \in \underline{d}^i(p)$ as d^i is upper hemi-continuous.

Second, consider the case where $V^i(p) = v^i$. We know that $x \in d^i(p)$ as d^i is upper hemi continuous. Suppose (towards a contradiction) that $x \notin \underline{d}^i(p)$. Then there is $y \in d^i(p)$ with

$$p \cdot y < p \cdot x \leq \alpha + (1-\alpha)p \cdot \omega^i.$$

Then $p_n \cdot y < \alpha + (1-\alpha)p_n \cdot \omega^i$ for all n large enough. Since $y \in d^i(p)$ and $V^i(p) = v^i$, $u^i(y) = v^i$. This means that $V^i(p_n) = v^i$ for all n large enough, as $y \in B^i(p_n)$.

By Lemma 2, $x_n \in \operatorname{argmin}\{p \cdot x : u^i(x) = v^i \text{ and } x \in \Delta_-\}$ for all n large enough. But the correspondence

$$p \mapsto \operatorname{argmin}\{p \cdot x : u^i(x) = v^i \text{ and } x \in \Delta_-\}.$$

is upper hemicontinous (by the maximum theorem), so $x \in \operatorname{argmin}\{p \cdot x : u^i(x) = v^i \text{ and } x \in \Delta_-\}$; a contradiction. □

Consider the correspondence $\phi : [0, \bar{p}]^L \to [0, \bar{p}]^L$ defined by

$$\phi_l(p) = \{\min\{\max\{0, \zeta_l + p_l\}, \bar{p}\} : \zeta \in z(p)\}.$$

Lemma 5. *ϕ is upper hemi-continuous, convex- and compact- valued.*

Proof. The aggregate excess demand, z, is upper hemi-continuous by Lemma 4. It is easy to see that this implies the upper hemi-continuity of ϕ. Similarly, convex and compact values are immediate. □

By Kakutani's fixed point theorem there is $p^* \in [0, \bar{p}]^L$ with $p^* \in \phi(p^*)$. We shall prove that p^* is an equilibrium price. Note that there exists $\zeta \in z(p^*)$ such that

$$p^*_l = \min\{\max\{0, \zeta_l + p^*_l\}, \bar{p}\}. \tag{1}$$

Lemma 6. $p^* \cdot \zeta \geq 0$.

Proof. If $p^* \cdot \zeta < 0$ then there is some object l with $p^*_l > 0$ and $\zeta_l < 0$. By Equation 1, then, $p^*_l = p^*_l + \zeta_l$, which is not possible as $\zeta_l < 0$. □

Lemma 7. *$p^*_l < \bar{p}$ for all $l \in O$.*

Proof. Suppose towards a contradiction that there is l for which $p_l^* = \bar{p}$. Then $p_l^* > 0$, so Equation 1 means that $\bar{p} \leq \zeta_l + p_l^* = \zeta_l + \bar{p}$. Let $\zeta = \sum_i x^i - Q$, with $x^i \in d^i(p^*)$. The definition of $B^i(p)$ means that

$$p^* \cdot (x^i - \omega^i) \leq \alpha(1 - p^* \cdot \omega^i),$$

for all $i \in I$. Thus, summing over i we obtain that $p^* \cdot \zeta \leq \alpha(N - p^* \cdot Q)$.

Now, by definition of $\bar{p}$, we have that

$$p^* \cdot Q \geq \bar{p} q_l > \bar{p}(\min_{l \in O} q_l - \varepsilon) = N.$$

Thus, $p^* \cdot \zeta \leq \alpha(N - p^* \cdot Q)$ implies that $p^* \cdot \zeta < 0$, in contradiction to Lemma 6. □

Lemma 8. $\zeta = 0$

Proof. By Lemma 7 and Equation (1),

$$p_l^* = \max\{0, \zeta_l + p_l^*\} \tag{2}$$

for all $l \in O$.

Equation 2 implies two things. First, that $\zeta_l > 0$ is not possible for any l. Hence $\zeta \leq 0$. Second, that if $\zeta_l < 0$ then $p_l^* = 0$.

Suppose then, towards a contradiction, that $\zeta_l < 0$ for some good l, and correspondingly that $p_l^* = 0$. Now, $\zeta_l < 0$ and $\zeta \leq 0$ means that

$$0 > \sum_{l \in O} \zeta_l = \sum_{l \in O} \sum_{i \in I} x_l^i - \sum_{l \in O} q_l = \sum_{i \in I} \sum_{l \in O} x_l^i - N.$$

So there is some agent i for which $\sum_l x_l^i < 1$. Agent i can then increase his consumption of good l without violating the constraint that consumption lie in Δ_-. Given that $p_l^* = 0$, the increase in consumption of good l would also not violate the budget constraint. So there exist a bundle in $B^i(p)$ with strictly more of good l, and the same amount of every other good, than x^i. This contradicts the strict monotonicity of u^i, and the fact that $x^i \in d^i(p^*)$. □

Proof of α-balanced individual rationality.

Proof. By Proposition 3, we know that $p \cdot \bar{\omega} \leq 1$. This means that with income $\alpha + (1 - \alpha) p \cdot \omega^i$ the bundle $\alpha \bar{\omega} + (1 - \alpha) w^i$ is affordable at p. □

Remark 1. *The proof uses semi-strict quasiconcavity only in the proof of upper hemicontinuity of* $\underline{d}^i$*. To prove existence of an* α*-balanced equilibrium without imposing the cheapest-bundle property, observe that continuity and quasiconcavity of* u^i *is enough to ensure that* d^i *is upper hemicontinuous, and takes convex and compact valued. If* z *is defined from* d^i *in place of* $\underline{d}^i$*, the proof as written above shows the existence of an* α*-balanced Walrasian equilibrium.*

7 Proof of Theorem 2

Let d_H denote the Hausdorff distance between any two sets A, B in $\mathbf{R}^L$. So,

$$d_H(A,B) = \max\big\{\sup\{\inf\{\|x-y\| : y \in B\} : x \in A\}, \sup\{\inf\{\|x-y\| : x \in A\} : y \in B\}\big\}.$$

Let $B^i(p,\alpha) = \{x \in \Delta_- : p \cdot x \leq \alpha + (1-\alpha)p \cdot \omega^i\}$ denote agent i's budget set given a price vector p and $\alpha \in (0,1]$.

To prove the **first part** of the theorem, we prove the following lemma.

Lemma 9. *For any* $\delta > 0$*, there is* $\alpha \in (0,1)$ *such that if* p *is the price vector in an* α*-balanced Walrasian equilibrium found in Theorem 1, then for any* $i \in I$*, either* $p \cdot \omega^i < 1$ *or* $d_H(B^i(p,\alpha), B^i(p,0)) < \delta$*.*

Proof. Consider the price $\bar{p}$ defined in the proof of Theorem 1. Note that if p is a price obtained by application of the theorem, then $p \in [0,\bar{p}]^L$. Note also that $\bar{p}$ is independent of α.

Let $K = \sup\{\|x\| : x \in \Delta_-\}$. For any $\alpha \in (0,1]$ and any $p \in [0,\bar{p}]^L$ such that $p \cdot \omega^i \geq 1$, define

$$\gamma(\alpha,p) = \frac{\alpha + (1-\alpha)p \cdot \omega^i}{p \cdot \omega^i}.$$

Note that $\gamma(\alpha,p) \in (0,1]$.

Now choose $\alpha \in (0,1)$ such that

$$\sup\{\frac{1-\gamma(\alpha,p)}{\gamma(\alpha,p)}K : p \in [0,\bar{p}]^L \text{ and } p \cdot \omega^i \geq 1\} < \delta.$$

Let p be the price vector in an α-balanced Walrasian equilibrium found in Theorem 1. If $p \cdot \omega^i < 1$, we are done.

If $p \cdot \omega^i \geq 1$, then $B^i(p,0) \supseteq B^i(p,\alpha)$. Let $x \in B^i(p,0)$, then $\gamma(\alpha,p)x \in B^i(p,\alpha)$. Note that

$$\|x - \gamma(\alpha,p)x\| = (1-\gamma(\alpha,p))\|x\| < \delta.$$

Thus $\inf\{\|x-y\| : y \in B^i(p,\alpha)\} < \delta$, and therefore

$$\sup\{\inf\{\|x-y\| : y \in B^i(p,\alpha)\} : x \in B^i(p,0)\} < \delta.$$

In a similar vein, let $x \in B^i(p,\alpha)$, then $\dfrac{x}{\gamma(\alpha,p)} \in B^i(p,0)$. Note that

$$\|x - \frac{x}{\gamma(\alpha,p)}\| = \frac{1-\gamma(\alpha,p)}{\gamma(\alpha,p)}\|x\| < \delta.$$

Thus $\inf\{\|x-y\| : y \in B^i(p,0)\} < \delta$, and therefore

$$\sup\{\inf\{\|x-y\| : y \in B^i(p,0)\} : x \in B^i(p,\alpha)\} < \delta.$$

Thus $d_H(B^i(p,0), B^i(p,\alpha)) < \delta$. □

Let $\delta > 0$ be such that, for any $p \in [0,\bar{p}]^L$, if $d_H(B^i(p,0), B^i(p,\alpha)) < \delta$ then

$$\left|\max\{u^i(x) : x \in B^i(p,\alpha)\} - \max\{u^i(x) : x \in B^i(p,0)\}\right| < \varepsilon.$$

For such δ, let α be as in Lemma 9.

Then for any i, if $p \cdot \omega^i < 1$ then $B^i(p,0) \subseteq B^i(p,\alpha)$, so

$$\max\{u^i(y) : y \in \Delta_- \text{ and } p \cdot y \leq p \cdot \omega^i\} - u^i(x) < 0 < \varepsilon.$$

If, on the contrary, $p \cdot \omega^i \geq 1$, then Lemma 9 implies that $d_H(B^i(p,0), B^i(p,\alpha)) < \delta$, and the result follows from the definition of δ.

To prove the **second part** of the theorem, we prove the following lemma.

Lemma 10. *For any $\delta > 0$, there is $\alpha \in (0,1)$ such that if p is the price vector in an α-balanced Walrasian equilibrium found in Theorem 1, then for any distinct i and j, either $p \cdot \omega^j < p \cdot \omega^i$ or $d_H(B^i(p,\alpha), B^j(p,\alpha)) < \delta$.*

Proof. Consider the price $\bar{p}$ defined in the proof of Theorem 1. Let $K = \sup\{\|x\| : x \in \Delta_-\}$. For any $\alpha \in (0,1]$ and any $p \in [0,\bar{p}]^L$ such that $p \cdot \omega^j \geq p \cdot \omega^i$, define

$$\beta(\alpha,p) = \frac{\alpha + (1-\alpha)p \cdot \omega^i}{\alpha + (1-\alpha)p \cdot \omega^j}.$$

Note that $\beta(\alpha,p) \in (0,1]$.

Now choose $\alpha \in (0,1)$ such that

$$\sup\{\frac{1-\beta(\alpha,p)}{\beta(\alpha,p)}K : p \in [0,\bar{p}]^L \text{ and } p \cdot \omega^j \geq p \cdot \omega^i\} < \delta.$$

Let p be the price vector in an α-balanced Walrasian equilibrium found in Theorem 1. If $p \cdot \omega^j < p \cdot \omega^i$, we are done.

If $p \cdot \omega^j \geq p \cdot \omega^i$, then $B^j(p, \alpha) \supseteq B^i(p, \alpha)$. Let $x \in B^j(p, \alpha)$, then $\beta(\alpha, p)x \in B^i(p, \alpha)$. Note that

$$\|x - \beta(\alpha, p)x\| = (1 - \beta(\alpha, p))\|x\| < \delta.$$

Thus $\inf\{\|x - y\| : y \in B^i(p, \alpha)\} < \delta$, and therefore

$$\sup\{\inf\{\|x - y\| : y \in B^i(p, \alpha)\} : x \in B^j(p, \alpha)\} < \delta.$$

In a similar vein, let $x \in B^i(p, \alpha)$, then $\frac{x}{\beta(\alpha, p)} \in B^j(p, \alpha)$. Note that

$$\|x - \frac{x}{\beta(\alpha, p)}\| = \frac{1 - \beta(\alpha, p)}{\beta(\alpha, p)}\|x\| < \delta.$$

Thus $\inf\{\|x - y\| : y \in B^j(p, \alpha)\} < \delta$, and therefore

$$\sup\{\inf\{\|x - y\| : y \in B^j(p, \alpha)\} : x \in B^i(p, \alpha)\} < \delta.$$

Thus $d_H(B^i(p, \alpha), B^j(p, \alpha)) < \delta$. □

Let $\delta > 0$ be such that, for any $p \in [0, \bar{p}]^L$, if $d_H(B^i(p, \alpha), B^j(p, \alpha)) < \delta$ then

$$\left|\max\{u^i(x) : x \in B^i(p, \alpha)\} - \max\{u^i(x) : x \in B^j(p, \alpha)\}\right| < \varepsilon.$$

For such δ, let α be as in Lemma 10.

Then for any distinct i and j, if $p \cdot \omega^j < p \cdot \omega^i$ then $B^j(p, \alpha) \subseteq B^i(p, \alpha)$. So

$$\max\{u^i(y) : y \in \Delta_- \text{ and } p \cdot y \leq \alpha + (1 - \alpha)p \cdot \omega^j\} - u^i(x) \leq 0 < \varepsilon.$$

If, on the contrary, $p \cdot \omega^j > p \cdot \omega^i$, then Lemma 10 implies that $d_H(B^i(p, \alpha), B^j(p, \alpha)) < \delta$, and the result follows from the definition of δ.

8 Proof of Theorem 3

Our first observation establishes the relation between envy and the value of endowments at equilibrium prices.

Lemma 11. *Let (x, p) be an α-balanced Walrasian equilibrium for any $\alpha \in (0, 1]$. If i envies j, then $p \cdot (x^j - x^i) > 0$ and $p \cdot (\omega^j - \omega^i) > 0$.*

Proof. Let i envy j, so $u^i(x^j) > u^i(x^i)$. Then utility maximization implies that

$$\alpha + (1-\alpha)p \cdot \omega^j \geq p \cdot x^j > \alpha + (1-\alpha)p \cdot \omega^i \geq p \cdot x^i,$$

where the strict inequality follows because $x^j \in \Delta_-$. So $p \cdot (x^j - x^i) > 0$ and $p \cdot (\omega^j - \omega^i) > 0$. $\square$

Now consider an α-balanced Walrasian equilibrium (x, p). Agent i's maximization problem is:

$$\max_{x \in \mathbf{R}^L_+} u^i(x) + \lambda^i(I^i - p \cdot x) + \gamma^i(1 - \mathbf{1} \cdot x)$$

Where $I^i = \alpha + (1-\alpha)p \cdot \omega^i$, λ^i is a multiplier for the budget constraint, and γ^i for the $\sum_l x^i_l \leq 1$ constraint.

Utility functions are C^1. The first-order conditions for the maximization problems are then:

$$\partial_l u^i(x^i) - \lambda^i p_l - g^i \begin{cases} = 0 & \text{if } x^i_l > 0 \\ \leq 0 & \text{if } x^i_l = 0, \end{cases}$$

where $\partial_l u^i(x^i)$ denotes the partial derivative of u^i with respect to x^i_l.

Observe that if $p \cdot x^i < \alpha + (1-\alpha)p \cdot \omega^i$, then the budget constraint is not binding and $\lambda^i = 0$. As a consequence, $u^i(x^i) = \max\{u^i(z^i) : z^i \in \Delta_-\}$. Let $S = \{i \in I : p \cdot x^i < \alpha + (1-\alpha)p \cdot \omega^i\}$ be the set of *satiated* agents. Let $U = \{i \in I : p \cdot x^i = \alpha + (1-\alpha)p \cdot \omega^i\}$ be the set of *unsatiated* agents, and observe that we can let $\lambda^i > 0$ for all $i \in U$. Consider the two stage social program:

Stage 1:

$$\max_{\tilde{y} \in (\Delta_-)^S} \sum_{i \in S} u^i(\tilde{y}^i)$$

Stage 2:

$$\begin{array}{ll} \max_{\tilde{y} \in (\Delta_-)^U} \sum_{i \in U} \frac{1}{\lambda^i} u^i(\tilde{y}^i) \\ \text{subject to} \quad \sum_{i \in U} \tilde{y}^i \leq Q - \sum_{i \in S} x^i \end{array}$$

Note that $(x^i)_{i \in S}$ solves Stage 1, while satisfying $\sum_{i \in S} x^i \leq Q$, and that given $(x^i)_{i \in S}$, $(x^i)_{i \in U}$ solves Stage 2. That this is so follows from the fact that $(x^i)_{i \in U}$ solves the first-order conditions for the Stage 2 problem with Lagrange multiplier p for the constraint that $\sum_{i \in U} \tilde{y}^i \leq Q - \sum_{i \in S} x^i$.

Now use the assumption that $\sum_{i \in U} x^i \gg 0$. This means that there exists $\bar{t} > 0$ such that if $t \in (0, \bar{t}]$ then the set of $\tilde{y} \in (\Delta_-)^U$ such that $\sum_{i \in U} \tilde{y}^i \leq Q + t(\omega^i -$

$\omega^j) - \sum_{i \in S} x^i$ is nonempty (and, for constraint qualification, contains an element that satisfies all constraints with slack).

Consider the problem

$$\begin{array}{ll} \max_{\tilde{y} \in (\Delta_-^U)} \sum_{i \in U} \frac{1}{\lambda^i} u^i(\tilde{y}^i) & \\ \text{subject to} \quad \sum_{i \in U} \tilde{y}^i \leq Q + t(\omega^i - \omega^j) - \sum_{i \in S} x^i & \end{array}$$

Note that for each $t \in (0, \bar{t}]$ there exists $(\nu(t), \gamma(t), \alpha(t))$ such that

$$v(t) = \sup\{\sum_{i \in U} \frac{1}{\lambda^i} u^i \cdot \tilde{y}^i + \nu(t) \cdot (\bar{\omega} - \sum_{i \in S} \tilde{y}^i + t(\omega^i - \omega^j)) - \sum_{i \in U} \tilde{y}^i) + \sum_{i \in U} \gamma_i(t)(1 - \sum_{l \in O} \tilde{y}^i_l) + \sum_{i \in U} \alpha_i(t) \tilde{y}^i_l.\}$$

Here $\nu(t)$ is the Lagrange multiplier for the constraint that $\sum_{i \in U} \tilde{y}^i \leq Q - \sum_{i \in S} x^i + t(\omega^i - \omega^j)$, while $\gamma(t)$ and $\alpha(t)$ are the Lagrange multipliers for the constraint that $(\tilde{y}^i) \in (\Delta_-)^N$. Choose a selection $(\nu(t), \gamma(t), \alpha(t))$ such that $\nu(0) = p$.

Let $\tilde{\omega} = Q - \sum_{i \in S} x^i$. The saddle point inequalities imply that

$$\begin{aligned} (t' - t)\nu(t) \cdot (\omega^i - \omega^j) = & \sum_{i \in U} \frac{1}{\lambda^i} u^i(x^i(t')) + \nu(t) \cdot (\tilde{\omega} + t'(\omega^i - \omega^j) - \sum_{i \in U} x^i(t')) \\ & + \sum_{i \in U} \gamma_i(t)(1 - \sum_{l \in O} x^i_l(t')) + \sum_{i \in U} \alpha_i(t) x^i_l(t') \\ & - \Bigg(\sum_{i \in U} \frac{1}{\lambda^i} u^i(x^i(t')) + \nu(t) \cdot (\tilde{\omega} + t(\omega^i - \omega^j) - \sum_{i \in U} x^i(t')) \\ & + \sum_{i \in U} \gamma_i(t)(1 - \sum_{l \in O} x^i_l(t')) + \sum_{i \in U} \alpha_i(t) x^i_l(t') \Bigg) \\ & \geq v(t') - v(t) \end{aligned}$$

Now recall that $\nu(0) = p$. Then Lemma 11, together with the above inequality, imply that

$$0 > p \cdot (\omega^i - \omega^j)t' \geq v(t') - v(0)$$

for all $t' > 0$ with $t' \leq \bar{t}$.

9 Proof of Proposition 4

Let (x, p) be an α-balanced Walrasian equilibrium for any $\alpha \in (0, 1]$. Suppose that y is an allocation such that $u^i(y^i) > u^i(x^i)$ for all i. Then

$$p \cdot (y^i - \omega^i) > \alpha(1 - p \cdot \omega^i) \geq p \cdot (x^i - \omega^i).$$

Sum over i to obtain:

$$p \cdot \left(\sum_{i \in I} y^i - Q \right) > \alpha(N - p \cdot Q) \geq p \cdot \left(\sum_{i \in I} x^i - Q \right) = 0.$$

Thus y cannot be an allocation. So x is weakly Pareto optimal.

In second place, suppose that (x, p) is an α-balanced Walrasian equilibrium in which each x^i satisfies the cheapest-bundle property. Then, for any $y^i \in \Delta_-$, $u^i(y^i) \geq u^i(x^i)$ implies that $p \cdot y^i \geq p \cdot x^i$, while $u^i(y^i) > u^i(x^i)$ implies that $p \cdot y^i > p \cdot x^i$. Thus, if $(y^i)_{i \in I}$ Pareto dominates x, adding up gives $p \cdot \sum_{i \in I} y^i > p \cdot \sum_{i \in I} x^i = p \cdot Q$, as x is an allocation. Then $(y^i)_{i \in I}$ cannot be an allocation. So x is Pareto optimal.

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