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Number Theory Type Formulae Appearing in Graphs

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by

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The dissertation of Thomas Anthony Petrillo is approved, and it is acceptable in quality and form for publication on microfilm:

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Chair

University of California, San Diego

2010

DEDICATION

For my mom, Susan Warren Converse.

EPIGRAPH

Still round the corner there may wait
A new road or a secret gate.
— J. R. R. Tolkien

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Thanks to Audrey and my Dad. You are both indispensable.

VITA

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ABSTRACT OF THE DISSERTATION

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by

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Doctor of Philosophy in Mathematics

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In Chapter I, a brief history of expander graphs will be discussed. In Chapter II, I will introduce many elementary terms and concepts related to graphs and graph covers. Subsequently, in Chapter III, I will study logarithmic derivatives of L-functions associated to graph covers. In this chapter I will show how to use the representations associated to a graph covering, to determine the number of paths which split completely in a given cover. In Chapter IV, an explicit formula for graph zeta functions will be presented. Subsequently I will combine elements of the previous chapter to deduce an explicit formula for graph L-functions. In the next chapter, the subject of the universal cover of a graph and its spectrum will be discussed. A result of Angel, Friedman, and Hoory will be discussed. I will prove a theorem allowing one to increase the speed of their result. I will use this to apply the aforementioned result to a new graph; this will allow us to provide new evidence of a conjecture they provide in their paper [3].

Chapter 1

Introduction

The study of graphs is a fundamental component of Discrete Mathematics. Many real world problems may be described through the rigor of graphs. Most notably, graphs are used in Computer Science as models for algorithms, data structures, and networks. Similarly, graphs model social networks for the social sciences. It is natural for much study to focus on 'Expander Graphs.'

Expander Graphs were first introduced by Bassalygo and Pinsker in the early 1970s. Since that time, much effort has been invested in studying this subject. An expander graph is one with high connectivity between points. Fundamentally, in any network, one is interested in making it easy to get from one node to the next, without building too many paths. Unsurprisingly, expander graphs can be made to model and analyze communication networks. Specifically, expansion properties of graphs can be used to investigate hardness results for linear transformations, construction of error correcting codes, and deterministic error amplification for randomized algorithms. They are certainly an important object of study. Parallel to expander graphs, one may study the zeta functions associated to them. Since 1859 when Riemann proposed his conjecture, mathematicians have studied his zeta function. While the Riemann's zeta function has been used to study number theory, the Selberg zeta function is used to study Differential Geometry. The Ruelle zeta function is an object of study in dynamical systems. A different zeta function has been defined for graphs. For a more detailed background on the history of graph theory and expander graphs, the reader should see the survey paper of Hoory et. al. [16].

This paper will investigate some structural problems related to graphs and expanders, using the L-functions associated to a graph. I will establish a new explicit formula for graph coverings. Further, I will establish a method to calculate the number of paths in a graph covering satisfying certain progression type requirements. Specifically, we will establish certain bounds on parameters associated to graphs and associate them with each other.

Chapter 2

Preliminaries

2.1 Graph Theory - Introductory Terms

First, we will need some elementary definitions. All of the information below can be supplemented with Biggs' *Algebraic Graph Theory* [6] or Bollobás *Modern Graph Theory* [7]. Technically, a **Graph** is a pair (V,E) where V is the set of vertexes and E is the set of edges. It is understood that each edge is a pairing of two vertexes, a start and an end. In this paper we will not consider directed graphs; thus if an edge has a given start and end, there is another edge, its inverse, that 'travels' in the opposite direction. Sometimes one will allow loops and multiple edges, thus while each edge is associated to a specific 2-tuple in V^2, more than one edge may be associated to a given 2-tuple; the start and end vertexes may be the same. The number of edges exiting a given vertex is called its **Degree**. A graph is said to be **Regular**, if all its vertexes have the same degree. A graph is **Connected** if any two vertexes may be connected by a path of edges. A graph is **Bipartite** if its vertexes may be divided into disjoint sets U and V, so that all edges have one vertex in U and the other in V. A graph is **Complete** if all vertexes are connected by an edge.

Associated to any graph, G, is its **Adjacency Matrix**, usually denoted A_G or just A when the associated graph is clear. The matrix's rows and columns are indexed by the graph's vertices. $A_G = (a_{ij})$ where $a_{ij} =$ the number of edges from vertex i to vertex j, usually 0 or 1. Assuming our edges are undirected we have $A^{\mathrm{T}} = A$. Assuming G is a regular graph of degree d, we have that

$$A \cdot \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} d \\ \vdots \\ d \end{pmatrix} = d \cdot \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \tag{2.1}$$

Hence, $d \in \mathrm{Spec}(A)$. In fact, it is easy to observe that each $\lambda \in \mathrm{Spec}(A)$ satisfies $|\lambda| \leq d$. One other fact to note concerning the adjacency matrix is that

$$(A^n)_{ij} = \text{number of paths from i to j of length n.} \tag{2.2}$$

This fact is easily proven via induction.

Since A is a real symmetric matrix it must have real eigenvalues. Let

$$d = \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n \tag{2.3}$$

be the eigenvalues of A. Note that n is both the number of eigenvalues of A and the number of vertexes in G. The magnitude of λ_2 reveals information about the expansion ratio of the graph. The graph G is connected if and only if $\lambda_1 > \lambda_2$. The graph G is bipartite if and only if $\lambda_1 = -\lambda_n$. One denotes the spectral radius, λ_1, as $\sigma(G)$ and the second largest eigenvalue in absolute value as $\lambda(G)$. One may question whether it is necessary to attach absolute value signs to λ_1 in the preceding statement. However, this is unnecessary as a result of the Perron-Frobenius Theorem from linear algebra which indicates $\lambda_1 > 0$ for matrices satisfying certain conditions. The interested reader should consult Horn and Johnson's *Matrix Analysis* chapter 8 section 4 [17].

Suppose $S \subseteq V$ is a set of vertexes, then the **Edge Boundary** of S, denoted $\partial S = E(S, S^c)$, is the set of edges from S to its complement, S^c. The **Expansion**

Ratio of G is defined to be

$$h(G) = \min_{\left\{S \subseteq V \middle| |S| \leq \frac{|V|}{2}\right\}} \frac{|\partial S|}{|S|}. \tag{2.4}$$

One defines a sequence of graphs, $\{G_i\}$ to be a **Family of Expanders** if there exists $\epsilon > 0$ such that $h(G_i) \geq \epsilon$ for all i. We have the following theorem due to Dodziuk [2].

Theorem 1. ***(Dodziuk)*** *If a graph, G, is d-regular, then*

$$\frac{d - \lambda_2}{2d} \leq h(G) \tag{2.5}$$

Proof. For a proof one should examine Alon [1] or Fan Chung [9]. □

Importantly we see that $d - \lambda_2 = \lambda_1 - \lambda_2$ bounds the expansion of G. This quantity is known as the **Spectral Gap**.

2.2 Ramanujan Graphs

Alon and Boppana have proven an asymptotic lower bound for the spectral gap.

Theorem 2. ***(Alon and Boppana)*** *Let G_i be a sequence of d-regular connected graphs. Let $n(G_i)$ be the number of vertexes in G_i. If $n(G_i) \to \infty$, then,*

$$\liminf_i \lambda_2(G_i) \geq 2 \cdot \sqrt{d-1}. \tag{2.6}$$

Proof. For a proof see [1] or [19]. □

This suggests the following definition; a d-regular graph is said to be **Ramanujan** if $|\lambda| \leq 2 \cdot \sqrt{d-1}$ for all $\lambda \in \mathrm{Spec}(G)$ such that $|\lambda| \neq d$. As an example, consider K_4 the complete graph on 4 vertexes. K_4 is pictured in Figure 2.1. Notice that

$$A_{K_4} = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}. \tag{2.7}$$

Via inspection, $\text{spec}(A_{K_4}) = \{3, -1, -1, -1\}$ and $2 \cdot \sqrt{3-1} = 2 \cdot \sqrt{2} > 1$. Hence

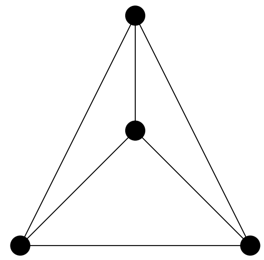

Figure 2.1: A Picture of K_4

K_4 is Ramanujan. In fact, for K_n, $\text{Spec}(A_{K_n}) = \{\underbrace{n-1}_{1 \text{ time}}, \underbrace{-1}_{n-1 \text{ times}}\}$. Hence, K_n is Ramanujan for all n.

2.3 The Vertex Ihara Zeta Function

In studying Ramanujan graphs, it is useful to introduce the Ihara Zeta function. Although we have used the term path earlier, it has not been clearly defined. A **Path**, p, is a sequence of directed edges $p = e_1e_2 \dots e_k$ such that for each i, e_i ends at the beginning of e_{i+1}. A path is said to be **Closed** if e_1 begins at the end of e_k. A path is said to have a **Backtrack** if there is an i such that $e_i = e_{i+1}^{-1}$. A closed path is said to have a **Tail** if $e_1 = e_k^{-1}$. A closed path in a graph is said to be **Prime** if it has no backtracks, no tails, and is not the power of another path. Note, if one has two prime paths starting at the same vertex, $a \neq b$ with $a = a_1a_2 \dots a_k$ and $b = b_1b_2 \dots b_l$ then ab is also a prime path provided that $\left\{a_i^{\pm 1} \middle| 1 \leq i \leq k\right\}$ and $\left\{b_i^{\pm 1} \middle| 1 \leq i \leq l\right\}$ are disjoint. Two prime paths $a = a_1a_2 \dots a_k$ and $b = b_1b_2 \dots b_k$ are said to be **Equivalent** if $b = a_la_{1+1} \dots a_ka_1 \dots a_{l-1}$ for some l. If P is a closed path, we refer to the set of paths equivalent to P as its **Equivalence Class** or just **Class**. By $[P]$ we mean the equivalence class of the closed path P. For a path, $p = e_1e_2 \dots e_k$ we define its **Length** to be $v(p) = k$. Sometimes the use of the word prime will refer to classes, other times it will refer to paths. Care will be taken to avoid confusion.

The **Ihara Vertex Zeta Function** or, when confusion is unlikely, **Zeta Function** is defined by

$$\zeta_G(u) = \prod_{[p] \text{ a prime class}} (1 - u^{v(p)})^{-1}. \tag{2.8}$$

The definition is only used where the product converges. Outside this range, the function will be meromorphically continued to the complex plane. For details about this process the interested reader should consult Conway [8]. In fact the continuation is made easy via Theorem 3. The only trivial example for computing the Zeta function is that of the cycle on n vertexes (hereinafter C_n). For clarity below is a picture of C_6. It is easy to verify that

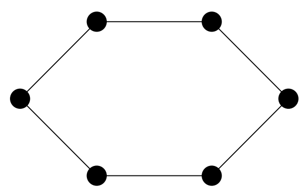

Figure 2.2: A Picture of C_6

$$\zeta_{C_n}(u) = (1 - u^n)^{-2}. \tag{2.9}$$

This is because there are only two prime classes on a cycle graph, the path traveling around the loop and its inverse. Generally, the zeta function of a graph can be seen to be the inverse of a polynomial which is non-zero at the origin. For the graph G, its zeta function's radius of convergence about the origin will be denoted R_G.

The set of paths without backtracks or tails on a graph G, is also denoted as the fundamental group, $\Gamma = \pi_1(G, u)$ associated to the graph, G. Via Seifert's theorem, we know that this is a free group. For reference, see Hatcher's *Algebraic Topology* [14]. For graphs topologically inequivalent to the cycle, one will find infinitely many different primes. Given (2.2), one would suspect that the adjacency matrix can be used to simplify things. Luckily the following theorem allows us to easily calculate the zeta function associated to a graph. For reference examine Stark and Terras [22].

Theorem 3. ***(Ihara)*** *Let G be a graph. Let r be the rank of G's fundamental group. Let Q be a diagonal matrix indexed by G's vertexes with the ith diagonal entry equal to the degree minus 1 of the ith vertex. Then,*

$$\zeta_G(u)^{-1} = (1-u^2)^{r-1} \cdot \det(I - A \cdot u + Q \cdot u^2). \tag{2.10}$$

The quantity r may seem mysterious, but, in fact, $r-1 = |E|-|V|$. Alternatively, the 'r' value may be calculated via finding a spanning tree in a given graph. A **Tree** is a connected graph which does not contain any closed non-backtracking paths aside from the empty path. A **Spanning Tree** of a graph, is a subgraph which contains every vertex and is also a tree. Each spanning tree will contain the same number of edges. The number of edges left off a spanning tree of a given graph is the rank of its fundamental group, r. A few more examples may illuminate the preceding. The tetrahedron graph K_4 (the complete graph on 4 vertexes) is found to have rank 3. A picture is located earlier in the text, labeled as Figure 2.1. Its zeta function is

$$\zeta_{K_4}(u)^{-1} = (1-u^2)^2 \cdot (1-u) \cdot (1-2u) \cdot (1+u+2u^2)^3. \tag{2.11}$$

2.4 The Riemann Hypothesis

A regular graph of degree $d = q+1$ is said to satisfy the **Riemann Hypothesis** if whenever $0 < \Re(s) < 1$ then $\zeta_G(q^{-s})^{-1} = 0$ if and only if $\Re(s) = \frac{1}{2}$. This leads to the following theorem.

Theorem 4. *For a connected regular graph G, G satisfies the Riemann Hypothesis if and only if G is Ramanujan.*

Proof. For a proof consult Stark and Terras [22]. $\square$

One can unravel the hypotheses, noting $\Re(s) = \frac{1}{2}$ is equivalent to $|q^{-s}| = \frac{1}{\sqrt{q}}$. Hence the the theorem equivalently states that $\frac{1}{q} < |u| < 1$ implies $|u| = \frac{1}{\sqrt{q}}$ if u is a zero of the zeta function and $q+1$ is the degree of the graph.

2.5 The Edge Zeta Function

To start, we will need more definitions. Associated to a finite graph G, we attach its **Edge Matrix**, W_G, or, W when no confusion is possible. The rows and columns of W are indexed by the edges of G with entries in $\mathbb{C}$. Henceforth, we will count the number of edges in a graph G as $2m$. Thus there are m undirected edges. Hence, $W \in M_{2m}(\mathbb{C})$ with $W = (w_{ij})$ and

$$w_{ij} = \begin{cases} 0 & \text{if } i = j^{-1}, \\ \text{a complex number} & \text{if } i \neq j^{-1} \text{ and i leads into j}, \\ 0 & \text{otherwise.} \end{cases} \tag{2.12}$$

We obtain the **Non-Backtracking Edge Matrix** by setting each nonzero value of the edge matrix equal to 1. This matrix will be denoted W_1; it will be clear in the text the graph to which this refers. As with the adjacency matrix, the W_1 matrix values reveal much about the paths in a graph.

Proposition 1. *Let G be a graph and W_1 its associated matrix with rows and columns indexed by the $2m$ edges of G. Then if $a \in \mathbb{N}$,*

$$(W_1^a)_{i,j} = \begin{cases} \textit{the number of non-backtracking paths starting on edge } i \\ \textit{and ending on edge } j \textit{ of length } a+1. \end{cases} \tag{2.13}$$

Proof. This is easily verified via induction. □

Finally, for a closed path $P = e_1 e_2 \ldots e_s$ define the **Edge Norm** of P associated to a given Edge Matrix to be

$$N_E(P) = w_{e_1 e_2} w_{e_2 e_3} \ldots w_{e_{s-1} e_s} w_{e_s e_1}. \tag{2.14}$$

The **Edge Zeta Function** is defined as

$$\zeta_E(W) = \prod_{[P] \text{ a prime class}} (1 - N(P))^{-1}. \tag{2.15}$$

By defining each $w_{ij} = u$ when it is not a priori assumed to be 0, the edge zeta function specializes to the vertex zeta function. We have the following generalization of Theorem 3:

Theorem 5. ***(Bass)*** *Let G be a graph with associated non-backtracking edge matrix W. Then,*

$$\zeta_E(W) = \det(I - W)^{-1} \tag{2.16}$$

Proof. Here, $I \in M_{2m}(\mathbb{C})$ is the identity matrix. For a proof, see Stark and Terras II [23]. □

Corollary 1. *Let G be a finite graph with associated W_1 matrix. Then,*

$$\zeta_G(u) = \det(I - uW_1)^{-1}. \tag{2.17}$$

Although not immediately obvious, Theorem 3, can quickly be deduced from Corollary 1. The reader should look at Terras [25]. Combining the two, one realizes that

$$spec(W_1) \subseteq \{\pm 1\} \bigcup \{\lambda \big| Q_\lambda = Q - A\lambda + I\lambda^2 \text{ is not invertible}\}. \tag{2.18}$$

Moreover, via Theorem 4, one can deduce a relationship between the spectral gap of A and the spectrum of W_1 for regular connected graphs. Namely, if a $q+1$ regular graph is Ramanujan, then in terms of A's spectral gap, we know that: $\max\{\lambda : \lambda \in \text{spec}(A), |\lambda| \neq q+1\} \leq 2\sqrt{q}$. Via a combination of the Riemann Hypothesis and the Bass theorem, this is equivalent to knowing that if $\lambda \in \text{spec}(W_1)$ and $1 < |\lambda| < q$, then $|\lambda| = \sqrt{q}$.

2.6 The Path Zeta Function and the Fundamental Group

Recall that the fundamental group has rank $|E| - |V| + 1$. Similarly there are $|E| - |V| + 1$ unoriented edges from a graph left out of its spanning tree. There is a clear correspondence between the two objects. Let $e_1, e_2, ...e_r, e_1^{-1}, e_2^{-1}, ...e_r^{-1}$ be the edges left off of a spanning tree for a given graph G. Notice that any non-backtracking closed tailless path can be determined up to starting point via a listing of the e_i's it passes through. Thus the generators of the fundamental group may be regarded as the edges $e_1, e_2, ...e_r$.

The **Path Matrix** associated to a graph is $Z \in M_{2r}(\mathbb{C})$ with the rows and columns indexed by the e_i and e_i^{-1}. Let $Z = (z_{ij})$. As in the non-backtracking edge matrix,

$$z_{ij} = \begin{cases} 0 & \text{if } i = j^{-1}, \\ \text{a complex number} & \text{otherwise.} \end{cases} \tag{2.19}$$

Given an edge matrix W attached to a graph Z, there is a natural path matrix, $Z(W)$ associated to W. Suppose g and h are non-spanning tree edges of a graph G with $g \neq h^{-1}$; e.g. $g, h \in \left\{e_1^{\pm 1}, e_2^{\pm 1}, \ldots, e_r^{\pm 1}\right\}$. Connect g to h via a path $p = t_1 t_2 \ldots t_k$ with

1. $t_1 = g$
2. $t_l \notin \left\{e_1^{\pm 1}, e_2^{\pm 1}, \ldots, e_r^{\pm 1}\right\}$
3. $t_k = h$

Having done the above we are prepared to define a path matrix $Z(W)$ induced from W by

$$Z(W)_{gh} = \begin{cases} \prod_{l=1}^{k-1} W_{a_l a_{l+1}} & \text{if } g \neq h^{-1}, \\ 0 & \text{otherwise.} \end{cases} \tag{2.20}$$

One defines the path norm for a closed path $C = a_1 a_2 \ldots a_k$ by writing C in terms of the e_i and e_i^{-1}; again we are referring to the edges left off a spanning tree as in the preceding section. If $C \cong b_1 b_2 \ldots b_l$ with each $b_i \in \left\{e_1^{\pm 1}, e_2^{\pm 1}, \ldots, e_r^{\pm 1}\right\}$, then the path norm of C is defined to be

$$N_P(C) = z_{b_1 b_2} z_{b_2 b_3} \ldots z_{b_{l-1} b_l} z_{b_l b_1} \tag{2.21}$$

As with the edge zeta function, the path zeta function is defined as

$$\zeta_P(Z, X) = \prod_{[C] \text{ a non-backtracking tailless path class}} (1 - N_P(C))^{-1} \tag{2.22}$$

Many of the same properties of the edge zeta function hold for the path zeta function. Most notably, a similar determinant formula exists.

Theorem 6.

$$\zeta_P(Z, X)^{-1} = \det(I - Z). \tag{2.23}$$

Proof. For a proof one should consult Terras [25]. □

There is an obvious connection between the edge zeta function and the path zeta function.

Theorem 7.

$$\zeta_E(W) = \zeta_P(Z(W)) \tag{2.24}$$

Proof. This is obvious from the defining formula for each respective function. □

2.7 Graph Coverings

As with field extensions, a notion of graph extensions or coverings will be introduced. This will give rise to a type of Galois theory for graph coverings. The reader looking for original reference should consult [23], or examine [25] for a more straightforward approach. It should be noted that the view of a graph giving rise to covering graphs is directly opposite the view of the founders of the subject, e.g. see [13] or [5]. Given a graph, we associate to any vertex $v \in V(X)$ its **Neighborhood**, the collection of edges protruding from v and the vertex itself. An undirected finite graph Y is said to **Cover** an undirected finite graph X if after an arbitrary assignment of directions to the edges of X there exists an assignment of directions of the edges of Y and a covering map $\pi : V(Y) \to V(X)$ which is bijective on neighborhoods and preserves directions. The reader may be convinced he has spotted a more simplistic definition for covering, the complications in the preceding definition arise from allowing finite graphs to have multiple edges or loops. Generally, having observed that a graph Y covers a graph X, we will fix one map π as the covering map. Without much effort one can observe that a given finite cover Y of a graph X contains an integral number of copies of any spanning tree for X. In fact, this integer is fixed. A graph Y covering a graph X containing d copies of a spanning tree for X is known as a **d-Sheeted Cover** of X. Each of the copies of the spanning tree will be known as a **Sheet**.

Henceforth, unless otherwise stated, it will be assumed that X and Y are both finite graphs with Y covering X via $\pi : V(Y) \to V(X)$. We define the **Galois Group**, denoted Gal(Y/X) by

$$\text{Gal}(Y/X) = \left\{ \sigma : V(Y) \to V(Y) \Big|^{\sigma \text{ is 1-1, onto, direction preserving, and}}_{\pi \circ \sigma = \pi} \right\}. \tag{2.25}$$

Mirroring the Galois theory of field extensions suggests the following definition. A d-sheeted cover Y/X is said to be **Normal** or **Galois** if $|Gal(Y/X)| = d$. Note, if Y is connected, then $|Gal(Y/X)| \leq d$. In fact, a fundamental theorem of graph Galois theory may be formulated, but it escapes the focus of this dissertation. The interested reader should explore Terras [25].

2.8 Graph L-Functions

In this section, Y/X will be a normal d-sheeted covering with Galois group $G = \mathrm{Gal}(Y/X)$. Fix a spanning tree for X and arbitrarily fix one sheet in Y. Notice for each $g \in G$, g sends our fixed sheet to a distinct sheet in Y. Hence, the d-sheets in Y may now be indexed via the elements in G; the reader will have to fix an indexing for a given cover. For a path p in X, one can associate to it a unique path $\tilde{p}$, with $\pi(\tilde{p}) = p$ in Y starting on sheet 1, i.e. the sheet associated to the identity map. The **Normalized Frobenius Automorphism**, $\sigma(p)$, is defined to be the element in G corresponding to the label of the terminal vertex of $\tilde{p}$. The letter σ will be used in other contexts; care will be taken to inform the reader when the Frobenius automorphism is intended. Let $[C]$ be a prime of X which starts and ends at vertex $v \in V(X)$. Let $[D]$ be a prime of Y such that $\pi(D) = C^f$ for some $f \in \mathbb{N}$. The integer f will be referred to as the **Degree** of D over C. Let D start on sheet $g \in G$. Lift C to a path $\tilde{C}$ starting on sheet g and ending on sheet $h \in G$. The **Frobenius Automorphism**, denoted [Y/X, D], is defined to be

$$[Y/X, D] = \left(\frac{Y/X}{D}\right) = hg^{-1} \in G. \tag{2.26}$$

One will notice that for different choices of D, a different automorphism is produced; all such possibilities will be similar as matrices.

In order to introduce L-functions, a knowledge of representation theory will be required. Many results will be assumed and not proven. The interested reader should consult [10]. By $\widehat{G}$, we will denote the set of inequivalent irreducible unitary representations. By d_ρ we will denote the degree of the representation ρ. As with the edge zeta function, we start with the edge matrix W, which is indexed by the edges of our graph G. Let ρ be a representation of G. The **Edge Artin-L-function** associated to ρ is defined by

$$L(W, \rho, Y/X) = L(W, \rho) = \prod_{[C] \text{ a prime class in X}} \det\left(I - \rho\left(\left(\frac{Y/X}{D}\right)\right) N_E(C)\right)^{-1}, \tag{2.27}$$

where [D] is an arbitrarily chosen from primes in Y over C. Notice that regardless of choice of D, via similarity, the same value is produced in the product. The definition is only used when the product converges and the function is meromorphically continued to the complex plane as in (2.8). As with the previous zeta-functions, these L functions admit a determinant formula. Define the **Artin Edge Matrix**, W_ρ, to be the block matrix with rows and columns indexed by the edges of X, with

$$(W_\rho)_{e,f} = \rho(\sigma(e))W_{e,f}. \tag{2.28}$$

As before, σ corresponds to the normalized Frobenius automorphism. Additionally, we set

$$(W_{1,\rho})_{e,f} = \rho(\sigma(e))(W_1)_{e,f}. \tag{2.29}$$

This leads us to a formula for these L-functions.

Theorem 8. *Let Y/X be two finite graphs with Y covering X. Let ρ be a representation of $Gal(Y/X)$. Then,*

$$L(u, \rho, Y/X) = \det(I - uW_{1,\rho})^{-1} \tag{2.30}$$

Proof. The interested reader should consult [25]. □

One may wonder why all this effort has been placed into the preceding developments; the parallel with number theory is not finished. For any graph covering, Y/X, with W_X the edge matrix associated to X, we may form a new edge matrix matrix associated to Y, $\widetilde{W}_{spec}$. If $\tilde{e}$ and $\tilde{f}$ are two edges of Y with $\tilde{e}$ leading into $\tilde{f}$, then set $e = \pi(\tilde{e})$ and $f = \pi(\tilde{f})$. We set $(\widetilde{W}_{spec})_{\tilde{e}\tilde{f}} = (W_X)_{ef}$. If $\tilde{e}$ does not lead into $\tilde{f}$ then $(\widetilde{W}_{spec})_{\tilde{e}\tilde{f}} = 0$ automatically. $\widetilde{W}_{spec}$ is called the **X-Specialized Edge Matrix**.

Theorem 9. *Let Y/X be a normal graph covering, and W_X the edge matrix associated to X. Then,*

$$\zeta_E(\widetilde{W}_{spec}, Y) = \prod_{\rho \in \widehat{Gal(Y/X)}} L(W_X, \rho)^{d_\rho}. \tag{2.31}$$

2.9 The Universal Covering Tree

To this point we have only considered finite graphs. As a graph is a topological space, it naturally comes equipped with a universal cover, which is also denoted as the universal covering tree. The **Universal Covering Tree**, or just **Universal Cover**, of a graph G may be viewed as the unique graph that is a cover of any covering graph of G. Alternatively, the universal covering tree of a graph G is the unique cover of G which does not contain any nontrivial non-backtracking tailless closed paths. In the case of a d-regular graph, the universal cover will be d-regular tree. This notion will be useful in generalizing the Riemann Hypothesis for graphs.

2.10 Irregular Graphs

While the study of regular graphs is well understood, irregular graphs are a far more complicated foe. For an irregular graph, we set $p+1$ to be the minimum degree and $q+1$ to be the maximum degree of the vertexes. We set $\overline{d_G}$ to be the average degree of the vertexes. In an attempt to generalize the Riemann Hypothesis to irregular graphs inspection of the Universal Cover will be useful. We view the adjacency matrix A, as an operator on the set of complex valued functions on the vertexes of a graph. For a graph G, let $f : V(G) \to \mathbb{C}$. For $v \in V(G)$, we define

$$(Af)(v) = \sum_{v \to v'} f(v') \tag{2.32}$$

where the sum runs over all vertexes v' adjacent to v. Similarly, we define the identity operator I, so that $If(v) = f(v)$. A similar definition for eigenvalue may now installed for our operator A, namely $\lambda \in \operatorname{spec}(A)$ if and only if $A - \lambda I$ has no bounded inverse. With this view in mind we can now examine the adjacency operator on the universal cover of a graph. The **Spectral Radius** of an operator A, $\rho(A)$, will be defined by,

$$\rho(A) = \sup \big\{ |\lambda| \big| \lambda \in \operatorname{spec}(A) \big\}. \tag{2.33}$$

Fix the universal cover of a graph G, and denote it T. Define,

$$\sigma_{\tilde{G}} = \rho(A_T), \tag{2.34}$$

the spectral radius of the adjacency operator on the universal cover of G. Additionally, in order to further our goals, it will be helpful to define the following two quantities for a given graph G:

$$\rho_G = \max\left\{|\lambda| \middle| \lambda \in \operatorname{spec}(A_G)\right\}, \tag{2.35}$$

$$\rho'_G = \max\left\{|\lambda| \neq \rho_G \middle| \lambda \in \operatorname{spec}(A_G)\right\}. \tag{2.36}$$

Lubotzky [20] has proposed the following modified Ramanujan inequality, which is applicable to both regular and irregular graphs. A graph G is Ramanujan if

$$\rho'_G \leq \sigma_{\tilde{G}}. \tag{2.37}$$

Although this definition may seem curious, we will see later that when G is a regular graph, the above definition reduces to our previous definition of Ramanujan. In fact, this definition has been made sound by Greenberg in his Ph. D. thesis [12].

Theorem 10. ***(Greenberg)*** *If G_i is a family of graphs with each covered by the same universal cover, then*

$$\liminf \rho'_{(G_i)} \geq \sigma_{\tilde{G}}. \tag{2.38}$$

In the above equation, set $\sigma_{\tilde{G}} = \sigma_{\tilde{G}_i}$ for any i.

This is a powerful generalization of the Alon-Boppana theorem for regular graphs. As with (2.37), the preceding theorem simplifies in the regular case to match with Alon and Boppana's Theorem 2. One may now be convinced that we have spotted the correct Ramanujan inequality. Unfortunately, relating this hypothesis to a suitable 'Riemann Hypothesis' is difficult and remains open. The Ramanujan property of graphs is correlated to values of the spectra of the adjacency matrix, whereas the

Riemann Hypothesis in the regular case correlates to the non-backtracking path matrix. While we have conditions based on the spectrum of the adjacency matrix for an irregular graph G, it would be nice to be able to place different matching conditions on its non-backtracking path matrix; this issue remains open.

Unfortunately, it is frequently, not easy to calculate the spectral radius of the lift of an operator to the universal tree covering one's graph. Angel, Friedman, and Hoory [3] provide a method to calculate the spectra of the lift of operators on a finite graph to the universal cover of the graph. While it allows one to perform such a calculation for a few simple graphs, the associated equations quickly become too complicated for practical application after a few trivial examples are explored.

Any $q+1$ regular graph, has the $q+1$ regular tree as its infinite cover. We denote the adjacency operator A, lifted to the tree, as $\widetilde{A}$. It is relatively easy to deduce that $\rho(\widetilde{A}) = [-2\sqrt{q}, 2\sqrt{q}]$, we will do this in Section 5.2. Hence one sees how Greenberg's theorem generalizes the notion of Ramanujan to irregular graphs. In general, the adjacency operator's spectrum is always real, whereas for regular graphs, the spectrum of the non-backtracking operator $\widetilde{W_1}$ is the union of a circle and two line segments in the complex plane. Of course in discussing the spectrum of the lift of W_1 to the covering tree, we now deal with operators on functions of the edges of the tree. For irregular graphs, the spectrum of $\widetilde{W_1}$ may in fact be 2 dimensional. Subsequently, we will produce experimental data concerning the spectrum of $\widetilde{W_1}$ for a graph.

2.11 A Helpful Theorem from Linear Algebra

It will be helpful to establish a few theorems from linear algebra. Hereinafter, the letter I will denote the identity matrix; we will abuse notation and freely use I for any dimension. The reader should consult Horn and Johnson's classic text [17] for more background and the proof of the following theorem.

Theorem 11. *Let $A, B \in M_n(\mathbb{C})$. Then,*

$$\operatorname{tr}(AB) = \operatorname{tr}(BA) \tag{2.39}$$

Additionally, the reader will need to know that:

Theorem 12. *Let $A(\alpha) \in M_n(\mathbb{C})$ be a collection of matrices indexed by α. Assume that the entries are differentiable with respect to α. If $A(\alpha)$ is invertible when $\alpha = \alpha_0$, then*

$$\left.\frac{\partial \det A(\alpha)}{\partial \alpha}\right|_{\alpha=\alpha_0} = \det A(\alpha_0) \cdot \operatorname{tr}\left(A(\alpha_0)^{-1} \cdot \left(\left.\frac{\partial A(\alpha)}{\partial \alpha}\right|_{\alpha=\alpha_0} \right) \right). \tag{2.40}$$

Proof. For a brute-force proof using Cramer's rule, one should consult Magnus [21] Chapter 8 Section 3. $\square$

In the following theorem, $\rho(A)$ will denote the spectral radius of the matrix A as it did for the operator A. I.e.

$$\rho(A) = \sup \left\{ |\lambda| \middle| \lambda \in \operatorname{spec}(A) \right\}. \tag{2.41}$$

Theorem 13. *Let $A \in M_n(\mathbb{C})$. Then, $\rho(A) < 1$ if and only if $A^n \to 0$.*

Proof. For reference see Horn and Johnson [17] 5.6.12. The reader may be interested in the norm used to determine if $A^n \to 0$. However, since we are operating in a finite dimensional vector space, all norms will be equivalent. $\square$

Chapter 3

Logarithmic Derivatives of L-Functions

3.1 Two Main Sums

In this chapter we will explore the logarithmic derivative of graph theoretic L-functions and determine some results concerning how paths split in covers. To start, it will be helpful to recall a popular lemma. Also, it will be helpful to define a few new quantities. For a graph X, π_n will denote the number of prime paths in X of length n. N_m will denote the number of closed paths of length n. Fix a finite graph covering Y/X. For any path p in X there exists a unique path $\tilde{p}$ starting on sheet 1 with $\pi(\tilde{p}) = p$. For $\sigma \in \mathrm{Gal}(Y/X)$, define $\pi_{\sigma,d}$ to be the number of prime paths in X of length d, which when lifted to start at sheet 1, end on sheet σ. Notice the number of prime classes satisfying the preceding condition is equal to $\frac{\pi_{\sigma,d}}{d}$. $N_{\sigma,d}$ will be the number of closed paths of length d which when lifted to start at sheet 1, end at sheet σ. A path p will be said to **Split Completely** if $\tilde{p}$ starts and ends on the identity sheet.

Lemma 1. *Let χ be a Banach space. If $T \in \mathcal{L}(\chi, \chi)$ and $\|I - T\| < 1$ where I is the* *identity operator, then T is invertible and $\sum_{i=0}^{\infty}(I-T)^i$ converges to T^{-1} in $\mathcal{L}(\chi,\chi)$.*

Proof. The reader should consult Folland [11] page 155. □

Using the above lemma, we have the following equation.

Lemma 2. *Let Y/X be a covering of finite graphs and $\rho \in \mathrm{Gal}(\hat{Y}/X)$. Then for small u,*

$$u \cdot \frac{\partial}{\partial u} \log L(u,\rho,Y/X) = \sum_{i=1}^{\infty} \mathrm{tr}(W_{1,\rho}^i)u^i. \tag{3.1}$$

Proof. Recall via Theorem 8,

$$L(u,\rho,Y/X) = \det(I-uW_{1,\rho})^{-1}. \tag{3.2}$$

Applying Theorem 12, it follows that,

$$\frac{\partial}{\partial u} \log L(u,\rho,Y/X) = \mathrm{tr}((I-uW_{1,\rho})^{-1} \cdot W_{1,\rho}). \tag{3.3}$$

Recalling that $M_n(\mathbb{C})$ is a Banach space under the operator norm we may apply Lemma 1. Thus, applying the above lemma 1 to equation (3.3), it follows that for small enough u

$$\frac{\partial}{\partial u} \log L(u,\rho,Y/X) = \mathrm{tr}((\sum_{i=0}^{\infty}(uW_{1,\rho})^i) \cdot W_{1,\rho}. \tag{3.4}$$

Whence,

$$u \cdot \frac{\partial}{\partial u} \log L(u,\rho,Y/X) = \mathrm{tr}((\sum_{i=1}^{\infty}(uW_{1,\rho})^i = \sum_{i=1}^{\infty} \mathrm{tr}(W_{1,\rho}^i)u^i. \tag{3.5}$$

□

We now establish a second power series for the same function.

Lemma 3. *Assume the same hypotheses as in Lemma 2. Then, for small u,*

$$u \cdot \frac{\partial}{\partial u} \log L(u,\rho,Y/X) = \sum_{\sigma \in \mathrm{Gal}(Y/X)} \sum_{i=1}^{\infty} \sum_{d|i} \pi_{\sigma,d}\, \mathrm{tr}(\rho(\sigma)^{\frac{i}{d}})u^i. \tag{3.6}$$

Proof. Recall via definition,

$$L(u,\rho,Y/X) = \prod_{[C] \text{ a prime class}} \det(I-\rho([Y/X,D])u^{v(c)})^{-1} \tag{3.7}$$

$$= \prod_{n\geq 1}\prod_{[C] \text{ a prime class of length n}} \det(I-\rho([Y/X,D])u^{v(c)})^{-1} \tag{3.8}$$

$$= \prod_{n\geq 1}\prod_{\sigma\in \mathrm{Gal}(Y/X)} \det(I-\rho(\sigma)u^{n})^{-\frac{\pi_{\sigma,n}}{n}} \tag{3.9}$$

Hence, applying the logarithm map, one concludes that

$$\log L(u,\rho,Y/X) = \sum_{n\geq 1}\sum_{\sigma\in \mathrm{Gal}(Y/X)} \log(\det(I-\rho(\sigma)u^{n})^{-\frac{\pi_{\sigma,n}}{n}}) \tag{3.10}$$

$$= \sum_{n\geq 1}\sum_{\sigma\in \mathrm{Gal}(Y/X)} \left(-\frac{\pi_{\sigma,n}}{n}\right) \log(\det(I-\rho(\sigma)u^{n})) \tag{3.11}$$

Now we take the derivative of (3.11) and apply Theorem 12:

$$\frac{\partial}{\partial u}\log L(u,\rho,Y/X) = \sum_{n\geq 1}\sum_{\sigma\in \mathrm{Gal}(Y/X)} \frac{\pi_{\sigma,n}}{n}\,\mathrm{tr}\left((I-\rho(\sigma)u^{n})^{-1}\cdot n\rho(\sigma)u^{n-1}\right) \tag{3.12}$$

Applying Lemma 1, one deduces that

$$\frac{\partial}{\partial u}\log L(u,\rho,Y/X) = \sum_{n\geq 1}\sum_{\sigma\in \mathrm{Gal}(Y/X)} \frac{\pi_{\sigma,n}}{n}\,\mathrm{tr}\left(\sum_{i\geq 0}(\rho(\sigma)u^{n})^{i} n\rho(\sigma)u^{n-1}\right) \tag{3.13}$$

$$= \sum_{n\geq 1}\sum_{\sigma\in \mathrm{Gal}(Y/X)}\sum_{i\geq 0} \pi_{\sigma,n}\,\mathrm{tr}\left(\rho(\sigma)^{i+1}\right)\cdot u^{n(i+1)-1} \tag{3.14}$$

Whence,

$$u\frac{\partial}{\partial u}\log L(u,\rho,Y/X) = \sum_{n\geq 1}\sum_{\sigma\in \mathrm{Gal}(Y/X)}\sum_{i\geq 0} \pi_{\sigma,n}\,\mathrm{tr}\left(\rho(\sigma)^{i+1}\right)\cdot u^{n(i+1)} \tag{3.15}$$

Rearranging the series, one deduces that

$$u\frac{\partial}{\partial u}\log L(u,\rho,Y/X) \;=\; \sum_{i\geq 1}\sum_{\sigma\in \mathrm{Gal}(Y/X)}\sum_{d|i}\pi_{\sigma,d}\,\mathrm{tr}\left(\rho(\sigma)^{\frac{i}{d}}\right)u^i \tag{3.16}$$

□

Lemmas 2 and 3 lead immediately to the following theorem.

Theorem 14. *Let Y/X be a finite covering of graphs and ρ a representation of* $\mathrm{Gal}(Y/X)$, *then*

$$\mathrm{tr}(W_{1,\rho}^n) = \sum_{\sigma\in Gal(Y/X)}\sum_{d|n}\pi_{\sigma,d}\,\mathrm{tr}(\rho(\sigma)^{n/d}) \tag{3.17}$$

Proof. This follows by comparing the coefficients of (3.1) and (3.6). □

Notice, if ρ is the trivial representation, then (3.17) becomes

$$N_n = \sum_{d|n}\pi_d. \tag{3.18}$$

3.2 Simplifying Formula

It will be helpful to reformulate Theorem 14.

Theorem 15. *With assumptions as in the previous paragraph we have*

$$N_{\sigma,n} = \sum_{d|n}\sum_{\alpha\in \mathrm{Gal}(Y/X),\alpha^{\frac{n}{d}}=\sigma}\pi_{\alpha,d} \tag{3.19}$$

Proof. We prove the equality by giving a one to one correspondence between the following sets A and B. Let A be the set of paths in X of length n which when lifted to sheet 1 end on sheet σ. Let B be the set of of prime paths, q, in X of length dividing n with $\hat{q}$ starting on sheet 1 and ending on sheet α, satisfying $\alpha^{\frac{n}{v(q)}} = \sigma$. Suppose p is a closed path of length n in X whose lift $\tilde{p}$ starts on sheet 1 and ends on sheet σ. It follows that $p = q^j$ for some prime path q in X. Notice that $j|n$.

Moreover if $\tilde{q}$ starts at sheet 1 and ends at α, then it follows that $\hat{q^j} = \hat{q}^j$ starts at sheet 1 and ends on sheet α^j. Thus, $\alpha^j = \alpha^{\frac{n}{\frac{n}{j}}} = \sigma$. Similarly if q is a prime path of length $j|n$ in X such that $\hat{q}$ ends on sheet α with $\alpha^{nj} = \sigma$, then $q^{\frac{n}{j}}$ is a closed path of length n whose lift ends on sheet σ. The theorem follows. □

We now rearrange the summation in (3.17) so that it may be combined with the preceding theorem.

Theorem 16. *Let Y/X be a finite Galois cover of graphs and ρ a representation of* $\mathrm{Gal}(Y/X)$*, then*

$$\mathrm{tr}(W_{1,\rho}^n) = \sum_{\sigma \in \mathrm{Gal}(Y/X)} N_{\sigma,n}\,\mathrm{tr}(\rho(\sigma)) \tag{3.20}$$

Proof. Via rearranging our order of summation in (3.17), we have that

$$\mathrm{tr}(W_{1,\rho}^n) = \sum_{\sigma \in Gal(Y/X)} \sum_{d|n} \sum_{\alpha^{\frac{n}{d}} = \sigma} \pi_{\alpha,d}\,\mathrm{tr}(\rho(\sigma)) \tag{3.21}$$

Applying (3.19) now yields the result. □

3.3 Immediate Corollaries

Notice if Y/X is a finite covering of graphs, we may may look at the set of inequivalent irreducible representations of $\mathrm{Gal}(Y/X)$, $\widehat{\mathrm{Gal}(Y/X)} = \hat{G}$. In the following corollaries, e is the identity element of the group $\mathrm{Gal}(Y/X)$. If $\sigma \in G$ then $\{\sigma\} = \left\{\alpha^{-1}\sigma\alpha \middle| \alpha \in G\right\}$. Moreover, we need to recall the Schur Orthogonality relations. Henceforth, for a representation ρ, the character of ρ, $\mathrm{tr} \circ \rho$ will be denoted χ_ρ. Recall,

Theorem 17. *(*The Orthogonality Relation for Group Characters*) Let G be a finite group and let $\chi_1, \chi_2, \ldots, \chi_r$ be the irreducible characters of G over $\mathbb{C}$. Then we have*

$$(\chi_i, \chi_j) = \frac{1}{|G|}\sum_{g \in G} \chi_i(g)\overline{\chi_j(g)} = \delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases} \tag{3.22}$$

Additionally, if $x, y \in G$, then

$$\sum_{i=1}^{r} \chi_i(x)\overline{\chi_i(y)} = \begin{cases} \frac{|G|}{|\{x\}|} & \text{if } x \text{ and } y \text{ are conjugate,} \\ 0 & \text{otherwise.} \end{cases} \tag{3.23}$$

Proof. The interested reader should consult Dummit and Foote [10] Chapter 18. □

Corollary 2. *Let $\sigma \in \mathrm{Gal}(Y/X)$ and $n \in \mathbb{N}$, then*

$$\sum_{\alpha \in \{\sigma\}} N_{\alpha,n} = \frac{|\{\sigma\}|}{|G|} \sum_{\rho \in \widehat{\mathrm{Gal}(Y/X)}} \overline{\chi_\rho(\sigma)}\,\mathrm{tr}(W_{1,\rho}^n) \tag{3.24}$$

Proof. Rewriting (3.20), we have that

$$\mathrm{tr}(W_{1,\rho}^n) = \sum_{\sigma \in \mathrm{Gal}(Y/X)} N_{\sigma,n}\chi_\rho(\sigma). \tag{3.25}$$

Multiplying through by $\overline{\chi_\rho(\alpha)}$ and summing over the set of irreducible representations, while applying (3.23), yields

$$\sum_{\rho \in \widehat{\mathrm{Gal}(Y/X)}} \overline{\chi_\rho(\alpha)}\,\mathrm{tr}(W_{1,\rho}^n) = \sum_{\rho \in \widehat{\mathrm{Gal}(Y/X)}} \sum_{\sigma \in \mathrm{Gal}(Y/X)} \overline{\chi_\rho(\alpha)} N_{\sigma,n}\chi_\rho(\sigma) \tag{3.26}$$

$$= \sum_{\sigma \in \mathrm{Gal}(Y/X)} \sum_{\rho \in \widehat{\mathrm{Gal}(Y/X)}} N_{\sigma,n}\overline{\chi_\rho(\alpha)}\chi_\rho(\sigma) \tag{3.27}$$

$$= \sum_{\alpha \in \{\sigma\}} N_{\alpha,n} \cdot \frac{|G|}{|\{\sigma\}|} \tag{3.28}$$

The result follows. □

Corollary 3. *Making the same assumption as in Corollary 2, then*

$$N_{e,n} = \frac{1}{|G|} \sum_{\rho \in \widehat{\mathrm{Gal}(Y/X)}} d_\rho\,\mathrm{tr}(W_{1,\rho}^n) \tag{3.29}$$

Proof. This is immediate from the preceding corollary. □

Table 3.1: Table of Galois Elements of Y_6 Corresponding to the Pictured Labels

label	1	2	3	4	5	6
S_3 element	e	(1 3)	(1 3 2)	(2 3)	(1 2 3)	(1 2)

Corollary 4. *Making the same assumption as in Corollary 2, then if e is the identity of* Gal(Y/X)

$$\sum_{\rho\in\widehat{\mathrm{Gal}(Y/X)}} \frac{d_\rho}{|G|} u \frac{\partial}{\partial u} \log L(u,\rho,Y/X) = \sum_{\rho\in\widehat{\mathrm{Gal}(Y/X)}} N_{e,n} u^n. \tag{3.30}$$

Proof. This is just the combination of (3.29) and (3.1). □

3.4 A Counter-Example

Based on Corollary 2, the reader may hope that if $\alpha, \beta \in \mathrm{Gal}(Y/X)$ then $N_{n,\alpha} = N_{n,\beta\alpha\beta^{-1}}$ as I was. However the following counterexample demonstrates this is not true. Pictured below is a S_3 cover of K_4 less an edge. Notice that the sheets of Y_6 are labeled with a superscript. Below is a table indicating which sheet corresponds to which element of S_3. Consider N_3. In the graph X, there are clearly 4 closed path classes of length 3; those corresponding to $\left\{c^{\pm 1}, d^{\pm 1}\right\}$. However, both c and c^{-1} lift to sheet (4), corresponding to the element (23) in S_3. Its conjugacy class consists of the three different 2 cycles. Noting that there are only 2 paths left to lift, it is immediate that the lifting of paths does not split evenly over conjugate elements of the Galois group.

3.5 An Example

We use the preceding Corollary 3 to determine the number of paths in a sample graph cover that split completely when lifted to the cover. Pictured in Figure 3.2 is the 'Dumbbell' graph X. Pictured in Figure 3.3 is a 2-sheeted cover of X, Y.

Table 3.2: List of Values of the Normalized Frobenius Automorphism Applied to the Edges of the Dumbbell Graph X, within the Cover, Y/X

e	a	b	c	a^{1-}	b^{-1}	c^{-1}
$\rho(\sigma(e))$	-1	1	1	-1	1	1

Notice how each of the individual edges have been labeled, with a spanning tree in X represented by a red dotted line. The labels of Y correspond to the edge in X that the edges project upon. Since $|\operatorname{Gal}(Y/X)| = 2$, it follows that $\operatorname{Gal}(Y/X) = \{1, \alpha\}$ and $\widehat{\operatorname{Gal}(Y/X)} = \{1, \rho\}$. Notice that $\rho(\alpha) = -1$. In Table 3.2 we list the values of $\rho(\sigma(e))$. Here, $\sigma(e)$ refers to the normalized Frobenius automorphism associated to the edge e. Thus we are able to calculate the $W_{1,\rho}$ and $W_{1,1}$ matrices.

$$\mathbf{W_{1,\rho}} = \begin{pmatrix} -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ -1 & 0 & 0 & -1 & 0 & 0 \end{pmatrix}$$

$$\mathbf{W_{1,1}} = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

Table 3.3: List of Values of $N_{e,n}$ for the Cover Y/X

n	1	2	3	4	5	6
$N_{e,n}$	0	4	0	20	0	76
n	7	8	9	10	11	12
$N_{e,n}$	0	228	0	1084	0	4052

Here the rows and columns of the respective W matrices are ordered $a, b, c, a^{-1}, b^{-1}, c^{-1}$. Thus, one finds that

$$
\begin{aligned}
L(u, \rho, Y/X)^{-1} &= -4u^6 - 8u^5 - 3u^4 + 4u^3 + 6u^2 + 4u + 1 && (3.31)\\
&= -(u+1)^2(u-1)(2u+1)(u+2u^2+1) && (3.32)\\
L(u, 1, Y/X)^{-1} &= \zeta_X(u) && (3.33)\\
&= -4u^6 + 8u^5 - 3u^4 - 4u^3 + 6u^2 + 4u + 1 && (3.34)\\
&= -(2u-1)(u+1)(2u^2-u+1)(u-1)^2 && (3.35)
\end{aligned}
$$

Whence, one calculates that

$$
\begin{aligned}
\frac{1}{2}(u\frac{\partial}{\partial u}\log(L(u, \rho, Y/X)) + u\frac{\partial}{\partial u}\log(L(u, 1, Y/X))) = 4u^2 + 20u^4 \\
+ 76u^6 + 228u^8 + 1084u^{10} + 4052u^{12} + O(u^{13}). \quad (3.36)
\end{aligned}
$$

Hence, via (3.30) one is able to calculate the number of closed paths of given length up to 12. We list them in Table 3.3. It is interesting to note that the 4 smallest paths which split completely when lifted are not prime, namely $a^{\pm 2}, b^{\pm 2}$. However, clearly, the path $cbc^{-1}a$ is of length 4 and prime; one easily observes that it splits completely. Hence the smallest length of a prime which splits completely is 4.

3.6 Growth Controls on the Primes Splitting Completely in a Graph Cover

As seen in the preceding section, the numbers $\pi_{e,n}$ are far more complicated then their counterparts $N_{e,n}$. In this section we will produce a growth control for them. Notice via Corollary 3 and Theorem 15, we have that

$$\sum_{\rho \in \widehat{\mathrm{Gal}(Y/X)}} \frac{1}{|\operatorname{Gal}(Y/X)|} d_\rho \operatorname{tr}(W_{1,\rho}) = \sum_{d|n} \sum_{\sigma^{\frac{n}{d}}=e} \pi_{\sigma,d}. \tag{3.37}$$

In a normal cover of graphs, Y/X, we will let $N_{Y,n}$ denote the number of closed paths of length n in Y; we will let $W_{Y,1}$ denote Y's W_1 matrix. Notice that, via Theorem 5, Theorem 8, and Theorem 9 we have that

$$\sum_{\rho \in \widehat{\mathrm{Gal}(Y/X)}} d_\rho \operatorname{tr}(W_{1,\rho}) = \operatorname{tr}(W_{Y,1}^n) = N_{Y,n}. \tag{3.38}$$

Combining (3.37) and (3.38) yields

$$\frac{N_{Y,n}}{|\operatorname{Gal}(Y/X)|} = \sum_{d|n} \sum_{\sigma^{\frac{n}{d}}=e} \pi_{\sigma,d}. \tag{3.39}$$

In the case of a single graph X, there already exists a prime number theorem. We will use it to make a prime number theorem for graph coverings. Because it will not aid our study further, a simplified version of the theorem will be stated. We will need an additional definition. For a graph X, set

$$\Delta X = \gcd \left\{ v(p) \middle| \text{p a closed path in X} \right\}. \tag{3.40}$$

Theorem 18. *If X is a connected graph, π_n denotes the number of primes paths of length n in X, and R_X is the radius of convergence of the associated zeta function, then*

$$N_n = O(\Delta X \cdot R_X^{-n}). \tag{3.41}$$

and

$$\pi_n = O(\Delta X \cdot R_X^{-n}). \tag{3.42}$$

Proof. For a proof consult [25]. □

This gives rise to the following theorem.

Theorem 19. *Let Y/X be a normal graph covering. Then*

$$\pi_{e,n} = O\Big(\frac{\Delta Y \cdot R_Y^{-n}}{|\operatorname{Gal}(Y/X)|}\Big). \tag{3.43}$$

Proof. Notice, via (3.39),

$$\frac{N_{Y,n}}{|\operatorname{Gal}(Y/X)|} \geq \pi_{e,n}. \tag{3.44}$$

Thus, via Theorem 18, $\pi_{e,n} = O\Big(\frac{\Delta Y R_Y^{-n}}{|\operatorname{Gal}(Y/X)}\Big)$. □

In fact, the bound cannot be improved. For, if $\gcd(n, |\operatorname{Gal}(Y/X)|) = 1$, then

$$\frac{N_{Y,n}}{|\operatorname{Gal}(Y/X)|} = \sum_{d|n} \pi_{e,d}. \tag{3.45}$$

To continue, we will assume the reader has knowledge of the Möbius inversion formula. Here, μ will denote the Möbius μ function. For reference consult [18]. Continuing with the hypothesis $\gcd(n, |\operatorname{Gal}(Y/X)|) = 1$, the Möbius inversion formula implies that

$$\pi_{e,n} = \sum_{d|n} \mu(\frac{n}{d}) \frac{N_{Y,d}}{|\operatorname{Gal}(Y/X)|}. \tag{3.46}$$

Applying the Möbius inversion formula to (3.18) yields

$$\pi_n = \sum_{d|n} \mu(\frac{n}{d}) N_d. \tag{3.47}$$

Combining (3.45) and (3.47) together would imply that

$$\pi_{e,n} = \frac{\pi_n}{|\operatorname{Gal}(Y/X)|}. \tag{3.48}$$

It follows that the growth condition in Theorem 19 cannot be improved.

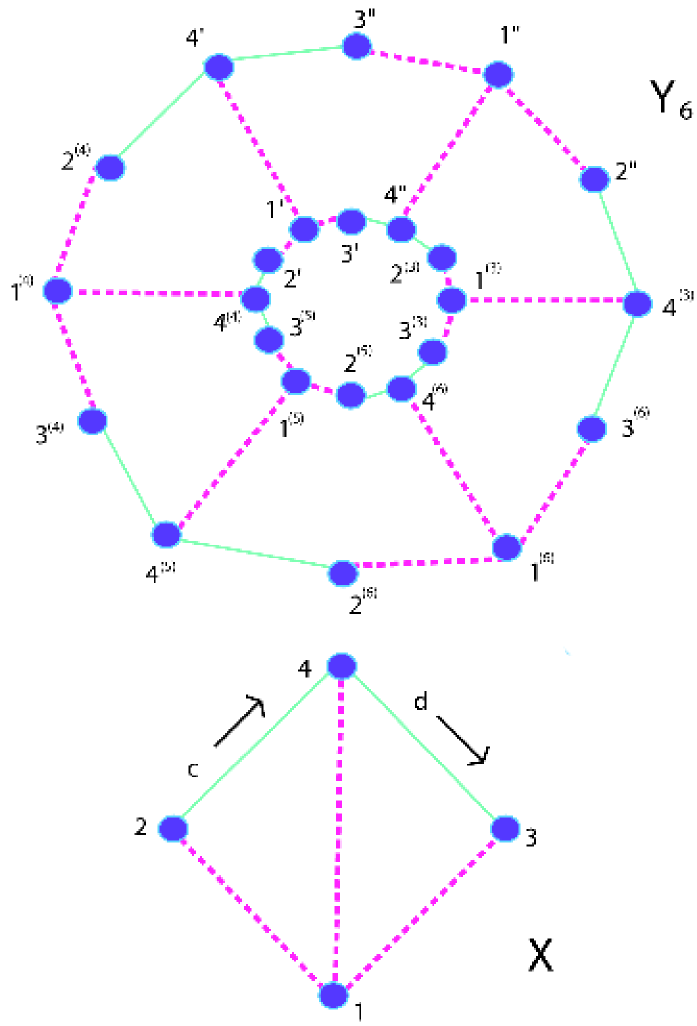

Figure 3.1: A Picture of K_4 Less an Edge, X, Together with a S_3 Cover of X, Y_6

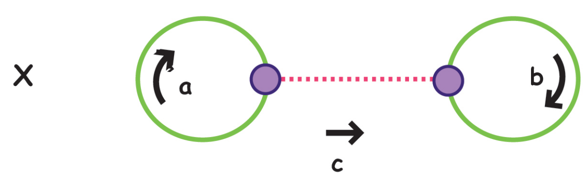

Figure 3.2: A Picture of the Dumbbell Graph, X

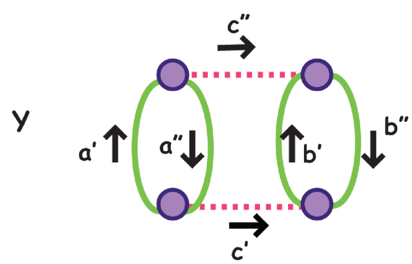

Figure 3.3: A Picture of a 2-Sheeted Cover of the Dumbbell Graph, Y

Chapter 4

Explicit Formulas

We now introduce an explicit formula for graph L-functions motivated by the preceding chapter. First, however we will state an explicit formula for zeta functions of graphs by Terras [26].

4.1 An Explicit Formula for Zeta Functions

Theorem 20. *Let X be a graph and $\epsilon > 0$. Let $0 < a < R_X$ where R_X is the radius of convergence of the zeta function of the graph X about the origin. Let $h(u)$ be a meromorphic function and pole free outside $B(0, a-\epsilon)$. Let $h(u) \in O(|u|^p)$ as $|u| \to \infty$ with $p < -1$. Moreover, let $\hat{h}_a(n)$ satisfy the requirement of absolute convergence on the right hand side of equation (4.1). Then,*

$$\sum_{\rho} \rho h(\rho) = \sum_{n \geq 1} N_n \hat{h}_a(n). \tag{4.1}$$

Here $\hat{h}_a(n)$ is the Fourier transform of h,

$$\hat{h}_a(n) = \frac{1}{2\pi i} \int_{|u|=a} u^n h(u) du. \tag{4.2}$$

Before proving the above we will need a few lemmas.

Lemma 4. *Let G a graph and let N_m be the number of closed paths of length m in G. Then,*

$$u\frac{\partial}{\partial u}\log\zeta_G(u) = \sum_{n\geq 1} N_m u^m \tag{4.3}$$

Proof. This follows from a straightforward manipulation of equation (2.8). □

One should note the similarity of the preceding lemma with Lemma 2.

Lemma 5.

$$u\frac{\partial}{\partial u}\log\zeta_G(u) = -\sum_{\rho \textit{ a pole of } \zeta_G} \frac{u}{u-\rho} \tag{4.4}$$

Proof. Via (2.16), we have

$$\zeta_G(u) = \prod_{\rho \text{ a pole of } \zeta_G} (1-\frac{u}{\rho})^{-1} \tag{4.5}$$

Whence,

$$\log\zeta_G(u) = -\sum_{\rho \text{ a pole of } \zeta_G} \log(1-\frac{u}{\rho}) \tag{4.6}$$

Taking derivatives on both sides,

$$\frac{\partial}{\partial u}\log\zeta_G(u) = -\sum_{\rho \text{ a pole of } \zeta_G} \frac{1}{1-\frac{u}{\rho}}\cdot(-\frac{1}{\rho}) \tag{4.7}$$

Thus,

$$u\frac{\partial}{\partial u}\log\zeta_G(u) = -\sum_{\rho \text{ a pole of } \zeta_G} \frac{u}{u-\rho} \tag{4.8}$$

□

It will be helpful to recall a definition from complex analysis. For a curve $\gamma : [a,b] \to \mathbb{C}$ and a partition of $[a,b]$, $P = \{a = t_0 < t_1 < \ldots < t_z = b\}$, define

$$v(\gamma; P) = \sum_{k=1}^{z} |\gamma(t_k) - \gamma(t_{k-1})|. \tag{4.9}$$

The **Total Variation** of γ, $V(\gamma)$, is defined to be

$$V(\gamma) = \sup_{\text{P a partition of } [a,b]} v(\gamma, P). \tag{4.10}$$

Lemma 6. *Let γ be a curve of finite total variation and further let f be continuous on the image of γ. Let $V(\gamma)$ denote the total variation of γ. Then*

$$|\int_{\gamma} f\big| \leq V(\gamma) \cdot sup\big[|f(z)|\big|z \in \textit{the image of } \gamma\big]. \tag{4.11}$$

Proof. See Conway [8] chapter IV 1.17. □

Lemma 7. *Let $f(x)$ be meromorphic and $f(x) \in O(x^p)$ as $|x| \to \infty$ with $p < -1$. Then,*

$$\lim_{r\to\infty} \int_{|x|=r} f(x)\partial x = 0. \tag{4.12}$$

Proof. Via assumption there exists $C, D \in \mathbb{R}^+$ so that $|f(x)| \leq C \cdot |x|^p$ for all $|x| \geq D$. Thus, letting γ be the circle of radius D, notice via Lemma 4.11

$$\left|\int_{|x|=D} f(x)\partial x\right| \leq \int_{|x|=D} |f(x)||\partial x| \leq \int_{|x|=D} |x|^p|\partial x| \leq V(\gamma)\sup\{|z|^p|z \in \{\gamma\}\} \tag{4.13}$$

Simplifying the right hand side of the above equation, one sees that:

$$\left|\int_{|x|=D} f(x)\partial x\right| \leq D^p \cdot V(\gamma) = D^{p+1} \tag{4.14}$$

Letting $D \to \infty$ one sees that,

$$\lim_{D\to\infty} \left|\int_{|x|=D} f(x)\partial x\right| = 0. \tag{4.15}$$

□

We may now proceed to prove the main theorem of this section.

Proof. Via (4.3) and (4.4) we know that,

$$\sum_{n\geq 1} N_m u^m = -\sum_{\rho \text{ a pole of } \zeta_G} \frac{u}{u-\rho} = u\frac{\partial}{\partial u}\log\zeta_G(u). \tag{4.16}$$

Thus, if $0 < a < R_X < 1 < b$ we have

$$\frac{1}{2\pi i}\int_{|u|=a}\left(u\frac{\partial}{\partial u}\log\zeta_G(u)\right)\cdot h(u)\partial u = \frac{1}{2\pi i}\int_{|u|=a}\left(-\sum_{\rho\text{ a pole of }\zeta_G}\frac{u}{u-\rho}\cdot h(u)\partial u\right). \tag{4.17}$$

Applying the residue theorem, we see that

$$(4.17) = \frac{1}{2\pi i}\int_{|u|=b}\left(-\sum_{\rho\text{ a pole of }\zeta_G}\frac{u}{u-\rho}\cdot h(u)\right)\partial u + \sum_{\rho}\rho h(\rho). \tag{4.18}$$

Notice that ζ is the reciprocal of a polynomial. It follows that $u\frac{\partial}{\partial u}\log\zeta(u) = \frac{u\cdot\zeta'(u)}{\zeta(u)} \in O(1)$ as $|u| \to \infty$. Now we allude to Lemma 7 and let $b \to \infty$, to see that

$$\frac{1}{2\pi i}\int_{|u|=a}\left(u\frac{\partial}{\partial u}\log\zeta_G(u)\right)\cdot h(u)\partial u = \sum_{\rho\text{ a pole of }\zeta_G}\rho h(\rho). \tag{4.19}$$

On the other hand,

$$u\frac{\partial}{\partial u}\log\zeta(u) = \sum N_m u^m \tag{4.20}$$

Whence, taking advantage of our assumption concerning absolute convergence,

$$\frac{1}{2\pi i}\int_{|u|=a}\sum_{m\geq 1}N_m u^m h(u)\partial u = \sum_{m\geq 1}N_m\frac{1}{2\pi i}\int_{|u|=a}u^m h(u)\partial u = \sum_{m\geq 1}N_m\hat{h}_a(m). \tag{4.21}$$

Combining (4.19) and (4.21) yields the result. □

Notice if one sets $h(u) = u^{-l-1}$ for $l \in \mathbb{Z}$ and $l \geq 1$. One calculates that,

$$\hat{h}_a(n) = \begin{cases} 1 & \text{if } l = n, \\ 0 & \text{otherwise.} \end{cases} \tag{4.22}$$

One confirms that,

$$\operatorname{tr}(W_1^l) = \sum_{\rho\text{ a pole of }\zeta_X}\rho^{-l} = N_l. \tag{4.23}$$

4.2 An Explicit Formula for L-Functions

Examining (4.1), one notices that the right hand side must be altered to fit with our definition of L-function. Taking advantage of Theorem 12, one can formulate a statement similar to the preceding section.

Theorem 21. *Let Y/X be a finite graph covering, $\rho \in \widehat{\mathrm{Gal}(Y/X)}$, and $\epsilon > 0$. Let $0 < a < R$ where R is the radius of convergence of the $L(u, \rho, Y/X)$ about the origin. Let $h(u)$ be a meromorphic function and pole free outside $B(0, a-\epsilon)$. Let $h(u) \in O(|u|^p)$ as $|u| \to \infty$ with $p < -1$. Moreover, let $\hat{h}_a(n)$ satisfy the requirement of absolute convergence on the right hand side of equation* (4.24)*. Then,*

$$\sum_{\sigma \text{ a pole of } L(u,Y/X,\rho)} \sigma h(\sigma) = \sum_{n \geq 1} \mathrm{tr}(W_{1,\rho}) \hat{h}_a(n) \tag{4.24}$$

Where $\hat{h}_a(n)$ is the Fourier transform of h,

$$\hat{h}_a(n) = \frac{1}{2\pi i} \int_{|u|=a} u^n h(u) du. \tag{4.25}$$

Before proving the above, we will need a lemma.

Lemma 8.

$$u \frac{\partial}{\partial u} \log L(u, Y/X, \rho) = - \sum_{\sigma \text{ a pole of } L(u,Y/X,\rho)} \frac{u}{u - \sigma} \tag{4.26}$$

Proof. The proof is identical to Lemma 5, the reader now takes advantage of Theorem 8 for L-functions as opposed to our theorem for zeta-functions. □

We now prove Theorem 21 of the chapter.

Proof. Via (3.1) and (4.26) we know that,

$$\sum_{n \geq 1} \mathrm{tr}(W_{1,\rho}^m) u^m = - \sum_{\sigma \text{ a pole of } L(u,Y/X,\rho)} \frac{u}{u - \sigma} = u \frac{\partial}{\partial u} \log \zeta_G(u). \tag{4.27}$$

Using this modified equality, the rest of the proof is now identical to that of Theorem 20. □

Again set $h(u) = u^{-l-1}$ and recall (4.22). Applying Theorem 21, one realizes that

$$\sum_{\sigma \text{ a pole of } L(u,Y/X,\rho)} \sigma^l = \operatorname{tr}(W_{1,\rho}^l). \tag{4.28}$$

The preceding fact is not, entirely, obvious.

Chapter 5

Calculating the Spectrum of Certain Operators on the Universal Cover of a Graph

Unfortunately, it is not easy to calculate the spectral radius of the lift of an operator to the universal covering tree of a given graph. Angel, Friedman, and Hoory [3] provide a method to calculate the spectrum of the lift of operators on a finite graph to the universal cover of the graph. While it allows one to perform such a calculation for a few simple graphs, the associated equations quickly become too complicated for practical application after a few trivial examples are explored. In the next section the method of calculating the spectra of operators on the tree will be briefly explained and not proven. Subsequently, a theorem will be introduced to simplify the calculations, expanding them to a wider range of possible graphs.

5.1 Ratio Systems

Angel, Friedman, and Hoory's [3] method for calculating the spectrum of certain operators on the Universal Cover of a graph revolves around attaching a ratio system to one's tree.

Definition 1. *A ratio system for an operator M is a function* $r : E(G) \to \mathbb{C} \cup \{\infty\}$ *such that:*

1. *If* $r_e \neq 0$ *for some* $e = (u, v)$ *then*
$$0 = M_{vv} + \frac{M_{e^{-1}}}{r_e} + \sum_{e' \ s.t. \ e \to e'} M_{e'} r_{e'} \tag{5.1}$$
with the convention that $e \to e'$ *means* e *leads into* e' *and* $e' \neq e^{-1}$.

2. *If* $r_e = 0$ *then there is an* f *such that* $e \to f$ *and* $r_f = \infty$.

3. *If* $r_f = \infty$ *and* $e \to f$*, then* $r_e = 0$ *and* $r_f \neq 0$.

One now sets:

$$R_{ef} = \begin{cases} \sum_{e' \text{ s.t. } e \to e' \to f} |\frac{M_f}{M_{e'^{-1}}}|^2 & \text{if } r_f = \infty \\ |r_f|^2 & \text{if } r_f \neq \infty \text{ and } e \to f \\ 0 & \text{otherwise} \end{cases}$$

One notes that the matrix R is one particular 'W' matrix. One can verify R satisfies the requirements of the Perron-Frobenius theorem from linear algebra. Set $\alpha(r)$ to be the Perron eigenvalue of R, the matrix associated to our ratio system. A few definitions are needed to state the next result. For a graph G and an operator M on functions on the vertexes of G, set $M_{uv} =< \delta_u, M\delta_v >$ with $u, v \in V(G)$. We say that M is **Symmetric** if $M_{uv} = M_{vu}$ for all $u, v \in V(G)$. M is **Local** if $M_{uv} \neq 0$ only if u and v are adjacent or $u = v$. M is **Pulled-back** if G is a cover and M_{uv} depends only on $\pi(u)$ and $\pi(v)$. Angel, Friedman, and Hoory [3] prove the following theorems.

Theorem 22. *Let* $\pi : T \to G$, *be the universal cover of a finite graph* G. *Let* M *be a pulled-back, symmetric, local operator on* $l^2(V(T))$. *Then* M *has a bounded inverse if and only if* M *has a ratio system* r, *with* $\alpha(r) < 1$.

One can generalize (2.18):

Theorem 23. *Let* G *be a (possibly infinite) graph of bounded degree and,as before, set* $Q_\lambda = I - A\lambda + Q\lambda^2$, *then:*

$$\sigma(W_1) = \{\pm 1\} \cup \{\lambda : Q_\lambda \textit{ is not invertible as a bounded linear operator}\}. \tag{5.2}$$

Though at first one might suspect one has many different ratio's to deal with, we have the following theorem:

Theorem 24. *Let* G *be a finite graph and* r *a ratio system for* G. *Let* $x, y \in E(G)$. *If there exists a graph automorphism* σ *of* G *such that* $\sigma(x) = y$, *then* $r_x = r_y$.

There is one other theorem, which is worthy to note even if not used elsewhere. Let $\widetilde{W_1}$ denote the lifting of the W_1 matrix to a graph's universal covering tree.

Theorem 25. *Let* G *be a connected graph, then*

$$\rho(\widetilde{W_1})^2 = \rho(W_1) \tag{5.3}$$

Notice the contrast with the proposed Riemann Hypothesis. Every graph satisfies our Riemann Hypothesis for the W_1 matrix. Thus, with the preceding results, one is now theoretically capable of calculating or approximating the spectrum of both the adjacency operator and the W_1 operator on the universal covering tree associated to a graph.

5.2 The Spectrum of the Adjacency Operator of a Regular Graph's Universal Cover

In this section we focus on the Universal Cover of a $q+1$ regular graph. For any such graph, the $q+1$ regular tree is its universal cover.

Theorem 26. *Let G be any $q+1$ regular graph and T its universal cover. Let $\tilde{A}$ be the adjacency operator on T. Then,*

$$\mathrm{spec}(\tilde{A}) = [-2\sqrt{q}, 2\sqrt{q}]. \tag{5.4}$$

Proof. For any such graph, its universal cover is the $q+1$ regular tree. Hence it is enough to determine the spectrum when A is lifted from K_{q+2}, the complete graph on $q+2$ vertexes. In K_{q+2} it is obvious that there is only one ratio, call it x. Using (5.1) with $M = \lambda I - A$, one generates:

$$0 = qx^2 - \lambda x + 1. \tag{5.5}$$

Solving,

$$x = \frac{\lambda \pm \sqrt{\lambda^2 - 4q}}{2q} \tag{5.6}$$

One, will note that $R = W_1 \cdot |x^2|$. It is a well known fact that the Perron eigenvalue of W_1 for a $q+1$ regular graph is q. Hence, for any λ, $\alpha(x) \geq 1$ for all ratio systems x associated to λ if and only if $q \cdot |x^2| \geq 1$. It is easy to show that the spectrum of the adjacency operator is real, since it is a bounded self-adjoint operator. Thus, $\lambda \in \mathbb{R}$. Thus, there are three cases to explore from (5.6).

Case 1: x is complex, i.e. $\lambda^2 < 4q$. If x is complex, then there is only one possible value for the absolute value of x:

$$|x|^2 = \frac{1}{4q^2}(\lambda^2 + 4q - \lambda^2) = \frac{1}{q}. \tag{5.7}$$

Hence,

$$q|x|^2 = 1 \geq 1 \tag{5.8}$$

Whence, if $\lambda^2 < 4q$, then $\lambda \in \sigma(\widetilde{A})$.

Case 2: x is a double root, i.e. $\lambda^2 = 4q$. In this case it follows that:

$$|x| = \frac{|\lambda|}{2q} = \frac{1}{\sqrt{q}}. \tag{5.9}$$

Hence,

$$q|x|^2 = 1. \tag{5.10}$$

Hence, if $|\lambda| = 2\sqrt{q}$, then $\lambda \in \sigma(\widetilde{A})$.

Finally, *Case 3*: x is real and not a double root, i.e. $\lambda^2 > 4q$. In this case it follows that for the smaller possible value of x:

$$|x| = \frac{1}{2q}(|\lambda| - \sqrt{\lambda^2 - 4q}). \tag{5.11}$$

Note that via the triangle inequality one can deduce that for positive real numbers $a, b \in \mathbb{R}^+$ with $a > b$ one has $\sqrt{a-b} > \sqrt{a} - \sqrt{b}$. Applying to (5.11),

$$|x| < \frac{1}{2q}(|\lambda| - |\lambda| + 2\sqrt{q}) = \frac{1}{\sqrt{q}}. \tag{5.12}$$

Hence,

$$q|x|^2 < 1. \tag{5.13}$$

It follows that $\sigma(\widetilde{A}) = [-2\sqrt{q}, 2\sqrt{q}]$. □

5.3 Relations Between the Path Matrix and the Edge Matrix

In actually applying the results of the preceding sections to calculate the spectrum of the W_1 operator, I found it to be very cumbersome and computer intensive. In fact, my computer could only calculate the spectra of a graph for a single irregular graph. Inspecting the method of ratio systems, one essentially calculates a determinant and solves an inequality associated to the determinant. Simplifying the determinant to be calculated could increase the speed of calculations.

Theorem 27. ***(P.)*** *Let G be a finite graph with associated edge matrix W containing non-negative entries. Further, let Z(W) be the path matrix induced by W. Then,* $\rho(Z(W)) < 1$ *if and only if* $\rho(W) < 1$. *In the preceding statements,* $\rho(C) = \max\{|\sigma| | \sigma \in \mathrm{spec}(C)\}$ *denotes the spectral radius of the matrix C.*

Proof. We will first state a property of the W and Z(W) matrices.

$$(W^n)_{ij} = \sum_{\substack{\text{non-backtracking paths p, starting at i} \\ \text{ending at j of length n+1}}} N_E(p), \tag{5.14}$$

$$((Z(W))^n)_{ij} = \sum_{\substack{\text{non-backtracking paths p, starting at i ending at j,} \\ \text{represented with n+1 non-spanning tree edges}}} N_E(p). \tag{5.15}$$

The preceding equations are easily observed via induction.

The key to proving the theorem will be to follow the conclusion of Theorem 13. First, suppose $\rho(W) < 1$. Hence, $W^n \to 0$. In fact, $\|W\| < 1$, since otherwise $\|W^n\| \to \infty$. Thus, via Lemma 1, $I - W$ is invertible and

$$\sum_n W^n \to (I - W)^{-1}. \tag{5.16}$$

Fix $\epsilon > 0$. For each pair of non-spanning tree edges i and j, let $\nu(i, j)$ denote the length of a path from i to j which contains only spanning tree edges aside from its start and end. Thus, $\nu(i, j) \geq 2$. Define $C, D \in \mathbb{N}$ by

$$D = \inf\big\{\nu(i, j)\big| i \text{ and } j \text{ are non-spanning tree edges}\big\}, \tag{5.17}$$

$$C = \sup\big\{\nu(i, j)\big| i \text{ and } j \text{ are non-spanning tree edges}\big\}. \tag{5.18}$$

Notice, via our definition of C and D, along with (5.14) and (5.15),

$$((Z(W))^n)_{ij} \leq \sum_{k=n(D-1)}^{nC-1} (W^k)_{ij}. \tag{5.19}$$

In the preceding equation, we subtract 1 from D as to avoid counting an intermediate non-spanning tree edge twice. As a result, we do not need to subtract 1 from the lower index of the sum. Via (5.16), there exists $N \in \mathbb{N}$ so that $m \geq N$ implies

$$\left\| \sum_{l=N}^{m} (W^l)_{ij} \right\| < \epsilon. \tag{5.20}$$

Hence, if $m \geq \frac{N-1}{D}$, then

$$(Z(W)^m)_{ij} \leq \sum_{k=m(D-1)}^{mC-1} (W^k)_{ij} \leq \sum_{k=N}^{mC-1} (W^k)_{ij} < \epsilon. \tag{5.21}$$

Hence $(Z(W))^n \to 0$ and $p(Z(W)) < 1$ via Theorem 13.

Conversely, suppose that $\rho(Z(W)) < 1$. For a given pair of edges i and j find new edges k and l with non-backtracking tailless paths to i and from j respectively so that:

1. The path from k to i is denoted p_i.

2. The path from j to l is denoted p_j.

3. Both k and l are non-spanning tree edges.

4. Both $N_E(p_i) \neq 0$ and $N_E(p_j) \neq 0$.

If one cannot find paths p_i and p_j satisfying the above conditions for any choice of k and l, then for $n \gg 0$,

$$(W^n)_{ij} = 0. \tag{5.22}$$

To observe the last statement, one must note that any non-backtracking path from i to j of large enough length must encounter a non-spanning tree edge. Hence, the norm of such a path must be equal to 0. Assuming such paths do exist, call their lengths l_i and l_j respectively. Then,

$$(W^n)_{ij} \leq \sum_{m=\max(\left\lfloor \frac{n+1+l_i+l_j}{C} \right\rfloor,1)}^{\max(\left\lceil \frac{n+1+l_i+l_j}{D-1} \right\rceil,1)} \frac{(Z(W)^m)_{kl}}{N_E(p_i)N_E(p_j)}. \tag{5.23}$$

The preceding indices may be quite confusing. For the lower bound on the indices, notice that a path of length $n+1$ from i to j gives rise to a path of length $n+1+l_i+l_j$ from k to l. Such a path will encounter at least $\frac{n+1+l_i+l_j}{C}+1$ non-spanning tree edges. Similarly, such a path will encounter at most $\frac{n+1+l_i+l_j}{D-1}+1$ non-spanning tree edges. The maximums are included in the bounds to avoid an exponent less than or equal to 0. These numbers give rise to the bounds of the sum in (5.23). The reader will notice that it will be quite elementary and tedious this time to bound the preceding inequality by ϵ, it is left as an exercise for the reader. The claim now follows as above. □

5.4 The Spectrum of the W_1 Operator on the Covering Tree for the Ice Cream Cone

In Angel, Friedman, and Hoory's paper, they are only able to produce a picture of the spectrum of W_1 lifted for the complete graph on 4 vertexes with one edge removed or K_4 less an edge. I.e. By using Theorem 27, one is able to simplify

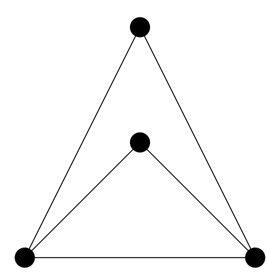

Figure 5.1: A Picture of K_4 Less an Edge

the inequalities one must solve in order to approximate the spectrum of a graph. Using this theorem and Mathematica on my personal computer, I was now able to approximate the spectrum of an additional graph. Adjoining a loop to a double edge produces the graph I nicknamed the 'Ice Cream Cone,' pictured in Figure 5.2. In

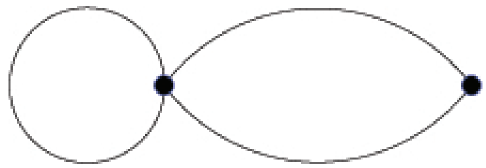

Figure 5.2: A Picture of the 'Ice Cream Cone'

approximately 10 minutes of computation, I was able to produce the following rather crisp picture of the spectrum, Figure 5.4. The x-axis is the real component and the y-axis is the imaginary component. The shaded region corresponds to points in the spectrum. In Appendix A, the Mathematica code for this is produced, along with the Mathematica code for Figure 5.5. Unfortunately, even with this improvement in

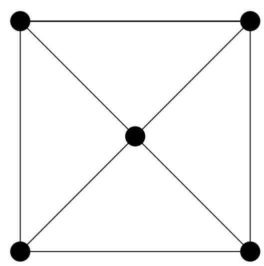

Figure 5.3: A Picture of the 4-Wheel

the computation, it remains difficult to calculate the spectrum of lifts. Attempts at calculating the spectrum for the 4-wheel (Figure 5.3), were unsuccessful due to the complexities of the inequality. Compared to the Mathematica algorithms, it seems likely one could write a more computationally efficient program for approximating the graph of an inequality in the instant case; although, producing precise pictures seems doubtful.

5.5 Approximation of Infinite Cover by Random Lifts

In Angel, Friedman, and Hoory's work [3], they conjecture that in a sequence of random lifts of a graph G, the spectrum of the lifts W_1 matrix will closely approximate the spectrum of the non-backtracking operator on the universal cover. In Figure 5.5, we plot the spectrum of the W_1 matrix of a randomly generated 250-sheeted cover of the ice cream cone graph against its universal cover's spectrum.

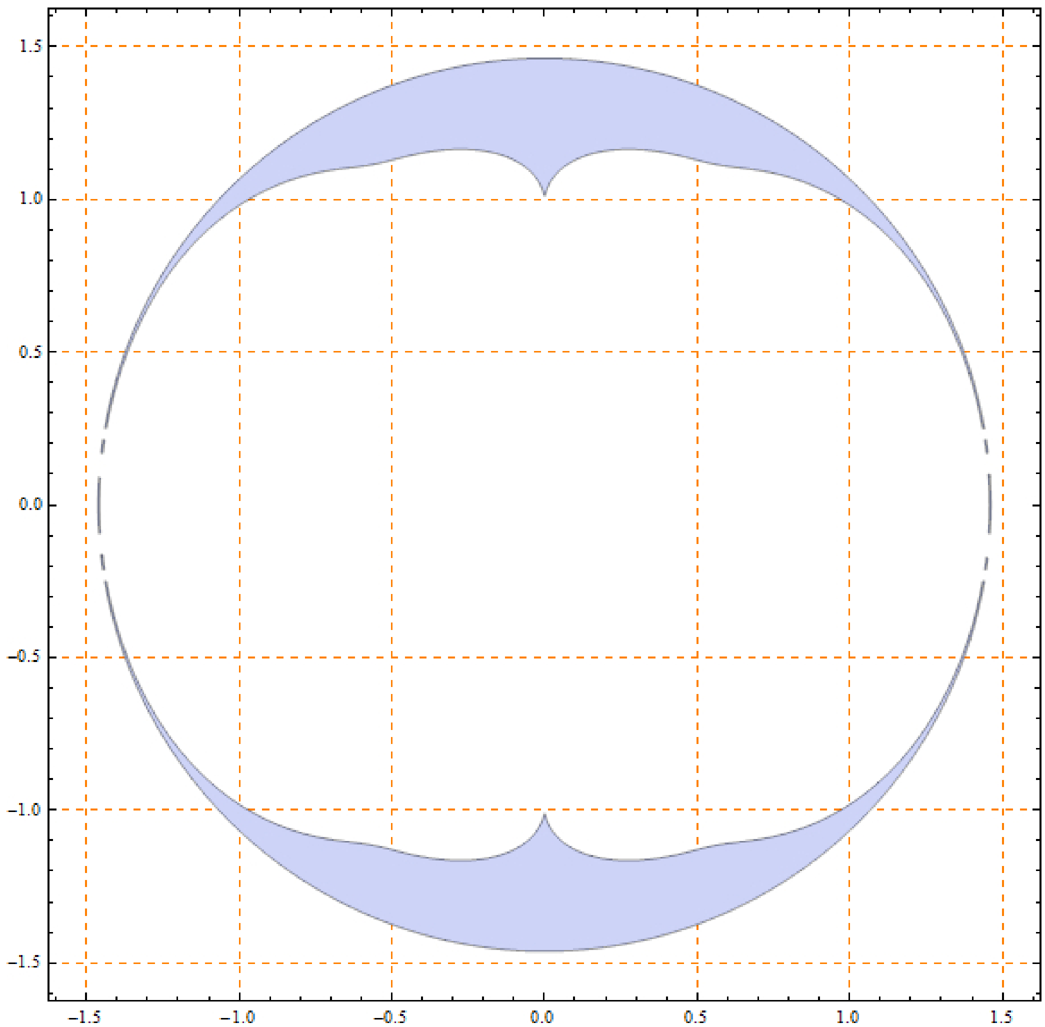

Figure 5.4: A Picture of the Spectrum of the W_1 Operator on the Universal Cover of the 'Ice Cream Cone'

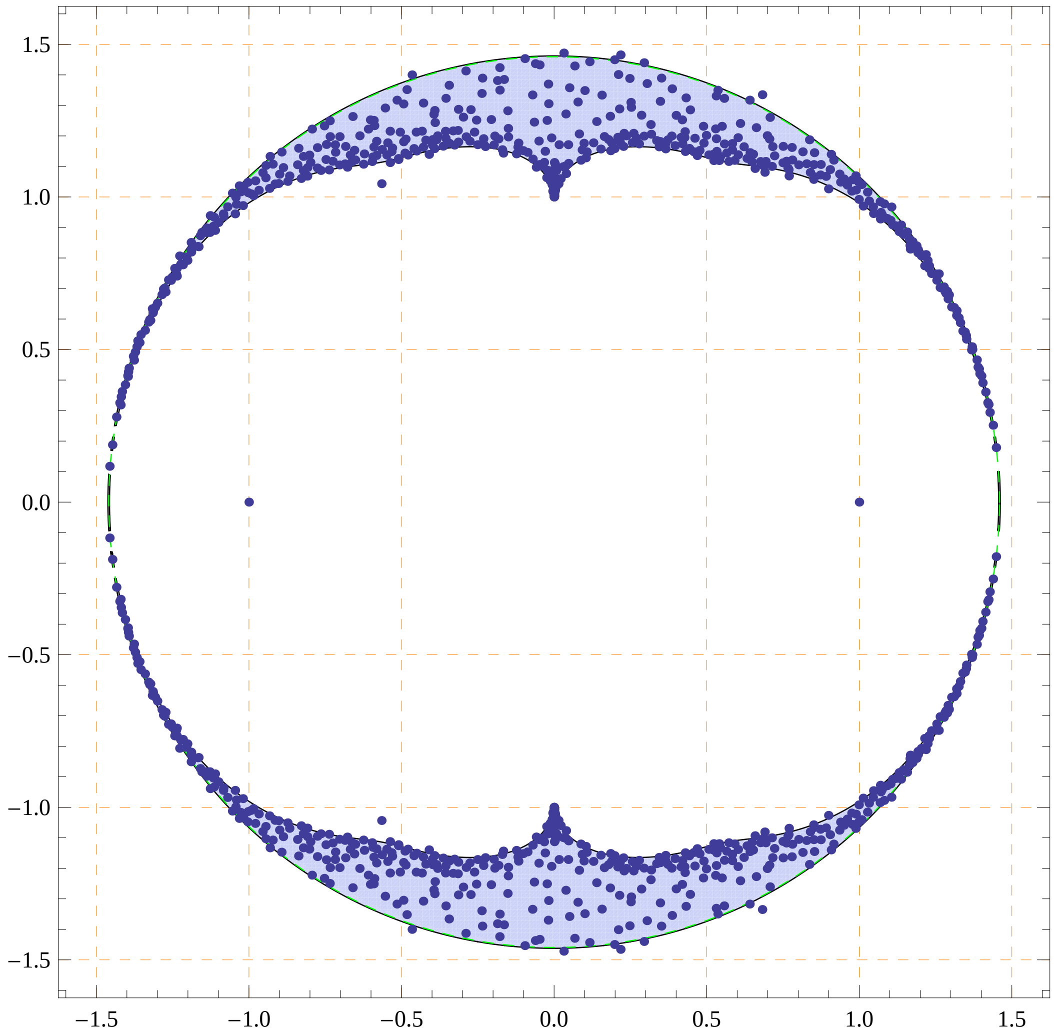

Figure 5.5: A Picture of the Spectrum of the W_1 Matrix for a Random 250-Sheeted Cover of the Ice Cream Cone Against the Universal Cover's Spectrum

Appendix A

Mathematica Code to Plot the Spectrum of the W_1 Operator on the Universal Covering Tree of the 'Ice Cream Cone'

Below is the code used to create Figures 5.4 and 5.5.

```
(*y is the ratio for the self loop, x is for the edges pointing away from
the loop, z is for the edges pointing towards the loop*)
Q={{1,0},{0,3}};
A={{0,2},{2,2}};
Ql=Q-l*A+l^2*IdentityMatrix[2];
e1={(l^2+3)*z==l*(1+z*(x+2*y))};
e2={(l^2+3)*y==l*(1+y*(y+2*x))};
e3={(l^2+1)*x==l*(1+x*z)};
sol=Solve[{e1[[1]],e2[[1]],e3[[1]]},{x,y,z}];
Z=Abs[{{(x*z)^2,0,(y*z)^2,(y*z)^2},{0,(x*z)^2,(y)^2,y^2},
```

```
   {x^2,(z*x)^2,y^2,0},{x^2,(z*x)^2,0,y^2}}];
Z1=(Z//.sol[[1]])//.{l→ s+I*t};
Z2=(Z//.sol[[2]])//.{l→ s+I*t};
Z3=(Z//.sol[[3]])//.{l→ s+I*t};
Z4=(Z//.sol[[4]])//.{l→ s+I*t};
B={{0,1,1,0,0,1},{1,0,0,0,0,0},{0,1,1,1,0,0},{0,0,0,0,1,0},
   {0,0,1,1,0,1},{0,1,0,1,0,1}};
specR=Max[N[Abs[Eigenvalues[B]]]];
range1=(specR^(1/2)+.1);
ErrorT=.005;
Pic1=Show[Show[RegionPlot[Max[Abs[N[Eigenvalues[Z1]]]]+ErrorT≥ 1 &&
   Max[Abs[N[Eigenvalues[Z2]]]]+ErrorT≥ 1 &&
   Max[Abs[N[Eigenvalues[Z3]]]]+ErrorT≥ 1 &&
   Max[Abs[N[Eigenvalues[Z4]]]]+ErrorT≥ 1,
   {s,-range1,range1}, {t,-range1,range1}, PlotPoints→ 250,
   MaxRecursion→ 2, PerformanceGoal → Quality],
   Graphics[{Green, Dashed, Circle[{0,0},specR^(1/2)]}]],
   {GridLines→ {{-1.5, -1, -.5, .5, 1, 1.5},{-1.5, -1, -.5, .5, 1, 1.5}},
   GridLinesStyle→ Directive[Orange,Dashed]}]
RandWCover[n_]:=Module[{J=IdentityMatrix[3*n],
   id=IdentityMatrix[n],
   A=RandPermMat[n],B=RandPermMat[n],
   Z=IdentityMatrix[n]*0,M,N,Y},
   M=ArrayFlatten[{{id,id,Z},{Z,Z,id}}];
   N=ArrayFlatten[{{A,Z,id},{Z,B,Z}}];
   Y=ArrayFlatten[{{Transpose[N].M,
   Transpose[N].N-J},{Transpose[M].M-J,Transpose[M].N}}];Y];
RandPerm[n_]:=Module[{perm={},rem=Range[n],temp,
```

```
   pos},For[i=1,i≤ n,i++,pos=RandomInteger[{1,Length[rem]}];
   temp=rem[[pos]];rem=Drop[rem,{pos}];
   perm=Append[perm,temp]]; perm];
RandPermMat[n_]:=Module[{perm=RandPerm[n],mat={},
   tempMat=IdentityMatrix[n]},
   For[i=1,i≤ n,i++,mat=Append[mat,tempMat[[perm[[i]]]]]]; mat];
PlotEigs[W_]:=Module[{eig=
   Union[N[Round[10^5*Eigenvalues[N[W]]]]]*10^-5,
   tempList={}},For[i=1,i≤ Length[eig],i++,
   tempList=Append[tempList,{N[Re[eig[[i]]]],N[Im[eig[[i]]]]}]];
   ListPlot[tempList,{PlotStyle→ PointSize[0.009],
   GridLines→ {{-1.5,-1,-.5,.5,1,1.5},{-1.5,-1,-.5,.5,1,1.5}},
   GridLinesStyle→ Directive[Orange,Dashed],
   PlotRange→ {{-range1,range1},{-range1,range1}}}]];
Show[Pic1,
   PlotEigs[RandWCover[250]],{GridLines→ {{-1.5,-1,-.5,.5,1,1.5},
   {-1.5,-1,-.5,.5,1,1.5}},
   GridLinesStyle→ Directive[Orange,Dashed]}]
```

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