Title
OPERATOR-VALUED MEASURE THEORY AND THE GROTHENDIECK INEQUALITY

Journal
HOUSTON JOURNAL OF MATHEMATICS, 41(1)

ISSN
0362-1588

Author
Bowers, Adam

Publication Date
2015

Peer reviewed

OPERATOR-VALUED MEASURE THEORY AND THE GROTHENDIECK INEQUALITY

ADAM BOWERS

Abstract. We generalize the operator-valued Grothendieck inequality of Defant and Junge by setting it in the context of operator-valued bimeasures. We then prove a duality theorem in the topological setting.

1. Introduction

The celebrated Grothendieck inequality has generated much interest since it was first published in [11] and then later reformulated as a matrix inequality by Lindenstrauss and Pełczyński [13]. Since then, it has been recast and generalized in many directions [2, 3, 12, 14, 17, 21], and even by the current author in [4]. In this paper, we are interested in the following generalization of Defant and Junge [5]:

Theorem 1.1 (Defant & Junge)**.** *Let X and Y be Banach spaces such that X^* and Y have cotype 2. Then there exists a constant K (depending on X and Y) such that for all $m, n \in \mathbb{N}$, and all $m \times m$ matrices $(a_{ij})_{i,j=1}^m$ with $a_{ij} \in \Gamma_2(X, Y)$,*

$$\Big| \sum_{i,j=1}^{m} \sum_{k=1}^{n} \big\langle a_{ij}(x_i(k)), y_j(k) \big\rangle \Big| \leq K \sup_{t,s} \Big| \sum_{i,j=1}^{m} \big\langle a_{ij}(t(i)), s(j) \big\rangle \Big|,$$

for all $x_1, \ldots, x_m \in B_{\ell_2(X)}$ and $y_1, \ldots, y_m \in B_{\ell_2(Y^*)}$, where the supremum is taken over all $t \in B_{\ell_\infty(X)}$ and $s \in B_{\ell_\infty(Y^*)}$.

(In [5], Defant and Junge proved the converse of this theorem, as well.) In the theorem, we use the standard notation B_E to denote the closed unit ball of the Banach space E. By $\Gamma_2(X, Y)$, we mean the space of all operators $u : X \to Y$ that factor through a Hilbert space [15, p. 21].

In this note, we recast Theorem 1.1 in a measure-theoretic setting:

Theorem 1.2. *Let X and Y be Banach spaces such that X^* and Y^* have cotype 2, and let $(\Omega_1, \mathcal{A})$ and $(\Omega_2, \mathcal{B})$ be two measurable spaces. Then there exists a constant K (depending on X and Y) such that for all operator-valued bimeasures $\mu : \mathcal{A} \times \mathcal{B} \to (X \widehat{\otimes} Y)^*$ with $\mu(A, B) \in \Gamma_2(X, Y^*)$ for $(A, B) \in \mathcal{A} \times \mathcal{B}$,*

$$\Big| \int \langle f, g \rangle \, d\mu \Big| \leq K \, \|\mu\|_{F_2} \, \|f\|_\infty \, \|g\|_\infty,$$

for all bounded measurable functions $f : \Omega_1 \to \ell_2(X)$ and $g : \Omega_2 \to \ell_2(Y)$.

1991 *Mathematics Subject Classification.* 28B05, 28A25, 46A32.

Key words and phrases. vector bimeasures, Grothendieck inequality.

Some explanation of the notation is in order. For Banach spaces X and Y, we let $X\widehat{\otimes}Y$ be the projective tensor product. (See Section 2.1.) The spaces $\ell_2(X)$ and $\ell_2(Y)$ are the spaces of norm-square summable sequences with entries in X and Y, respectively. The inner product in Theorem 1.2 is defined as follows: If $x = (x_j)_{j=1}^\infty \in \ell_2(X)$ and $y = (y_j)_{j=1}^\infty \in \ell_2(Y)$, then $\langle x, y\rangle = \sum_{j=1}^\infty x_j \otimes y_j$. We note that $\langle x, y\rangle \in X\widehat{\otimes}Y$ for all $x \in \ell_2(X)$ and $y \in \ell_2(Y)$, by the Cauchy-Schwarz inequality.

For $f : \Omega_1 \to \ell_2(X)$, we let

$$\|f\|_\infty = \sup_{\omega\in\Omega_1} \|f(\omega)\|_{\ell_2(X)},$$

and similarly for $g : \Omega_2 \to \ell_2(Y)$. When we say $f = (f_k)_{k=1}^\infty$ and $g = (g_k)_{k=1}^\infty$ are *measurable functions*, we mean that the coordinate functions $f_k : \Omega_1 \to X$ and $g_k : \Omega_2 \to Y$ are *strongly measurable* for each $k \in \mathbb{N}$. By strongly measurable, we mean the pointwise limit of simple functions (in norm on X and Y, respectively) [8, Section II.1].

The operator-valued bimeasure $\mu : \mathcal{A} \times \mathcal{B} \to (X\widehat{\otimes}Y)^*$ is a $(X\widehat{\otimes}Y)^*$-valued set function that is countably-additive in each argument separately. The semivariation of μ is denoted by $\|\mu\|_{F_2}$. (See Section 2.2.)

It is well-known that the space $(X\widehat{\otimes}Y)^*$ is isometrically isomorphic to both $\mathcal{B}(X, Y)$, the space of bounded bilinear forms on $X \times Y$, and $\mathcal{L}(X, Y^*)$, the space of bounded linear operators from X into Y^* [8, Corollary VIII.2]. We will take this identification for granted and, by an abuse of notation, we will write

$$\Lambda(x \otimes y) = \Lambda(x, y) = \langle\Lambda(x), y\rangle, \quad (x, y) \in X \times Y,$$

whenever Λ is in any one of these spaces. Consequently, in the statement of Theorem 1.2, when we say $\mu(A, B) \in \Gamma_2(X, Y^*)$ (for $A \in \mathcal{A}$ and $B \in \mathcal{B}$) we mean $\tilde{\mu}(A, B) \in \Gamma_2(X, Y^*)$, where $\langle\tilde{\mu}(A, B)(x), y\rangle = \mu(A, B)(x \otimes y)$ for all $x \in X$ and $y \in Y$. For ease of notation, when μ satisfies the hypotheses of Theorem 1.2, we will say $\mu \in \Gamma_{DJ} = \Gamma_{DJ}(\mathcal{A}, \mathcal{B}; X, Y^*)$.

To recover Theorem 1.1 from Theorem 1.2, take $\Omega_1 = \Omega_2 = \{1, \dots, m\}$. Further, for $1 \le i, j \le m$, let $x_i = f(i)$ and $y_j = g(j)$, and define $a_{ij} : X \to Y^*$ by $\langle a_{ij}(x), y\rangle = \mu(\{i\}, \{j\})(x, y)$ for all $x \in X$ and $y \in Y$. The bound on the right side of the inequality in Theorem 1.1 is precisely $K \|\mu\|_{F_2}$.

Observe that we recover Theorem 1.1 not with X and Y, but with X and Y^*. This may be done without loss of generality [15, Theorem 2.8(b)]. We also take Y to be a subset of Y^{**}. If we wished, we could let g take values not in Y, but in Y^{**}. For notational simplicity, we will not do this.

Throughout this note, X and Y will denote Banach spaces such that X^* and Y^* have cotype 2, and $(\Omega_1, \mathcal{A})$ and $(\Omega_2, \mathcal{B})$ will be measurable spaces. In order to take advantage of tools of operator theory, we will take our scalars to be complex, but the results can be interpreted for the real case.

2. Background

In this section, we will gather some definitions from the literature. Our primary sources are Blei [2]; Diestel, et al [6]; and Villanueva, e.g. [20].

2.1. **Projective tensor products.** Let $X_1, \dots, X_n$ be Banach spaces. The *algebraic tensor product* of these spaces is denoted by $X_1 \otimes \cdots \otimes X_n$ and contains all finite combinations of elementary tensors $x_1 \otimes \cdots \otimes x_n$, where $x_j \in X_j$ for all $j \in \{1, \dots, n\}$. Two elements $\sum_j x_{1,j} \otimes \cdots \otimes x_{n,j}$ and $\sum_k y_{1,k} \otimes \cdots \otimes y_{n,k}$ in $X_1 \otimes \cdots \otimes X_n$ are said to be equal if

$$\sum_j x_1^*(x_{1,j}) \cdots x_n^*(x_{n,j}) = \sum_k x_1^*(y_{1,k}) \cdots x_n^*(y_{n,k}), \tag{1}$$

for all $(x_1^*, \dots, x_n^*) \in X_1^* \times \cdots \times X_n^*$ (e.g., [2, Section IV.6]).

The space $X_1 \widehat{\otimes} \cdots \widehat{\otimes} X_n$ is called the *projective tensor product*, and is the completion of $X_1 \otimes \cdots \otimes X_n$ in the norm

$$\|u\|_\wedge = \inf \Big\{ \sum_j \|x_{1,j}\|_{X_1} \cdots \|x_{n,j}\|_{X_n} : u = \sum_j x_{1,j} \otimes \cdots \otimes x_{n,j} \Big\},$$

where the infimum is taken over representations of u (with finite sums); equality is given by (1). It can be shown that for any $u \in X_1 \widehat{\otimes} \cdots \widehat{\otimes} X_n$,

$$\|u\|_\wedge = \inf \Big\{ \sum_{j=1}^\infty \|x_{1,j}\|_{X_1} \cdots \|x_{n,j}\|_{X_n} : u = \sum_{j=1}^\infty x_{1,j} \otimes \cdots \otimes x_{n,j} \Big\},$$

where again the infimum is taken over representations of u, but this time infinite sums are allowed [6, Proposition 1.1.4].

Let $(\Omega_1, \Sigma_1), \dots, (\Omega_n, \Sigma_n)$ be measurable spaces. For each $1 \le j \le n$, we denote by $L^\infty(\Omega_j, X_j)$ the space of all bounded *strongly measurable* functions on Ω_j taking values in the Banach space X_j. This is a Banach space with the supremum norm, which we denote by $\|\cdot\|_\infty$.

We let $V_n = V_n(\Sigma_1, \dots, \Sigma_n; X_1, \dots, X_n)$ denote the projective tensor product $L^\infty(\Omega_1, X_1) \widehat{\otimes} \cdots \widehat{\otimes} L^\infty(\Omega_n, X_n)$. (The V is for Varopoulos.) It can be shown that two elements $\sum_j f_{1,j} \otimes \cdots \otimes f_{n,j}$ and $\sum_k g_{1,k} \otimes \cdots \otimes g_{n,k}$ of V_n are equal if

$$\sum_j f_{1,j}(\omega_1) \otimes \cdots \otimes f_{n,j}(\omega_n) = \sum_k g_{1,k}(\omega_1) \otimes \cdots \otimes g_{n,k}(\omega_n), \tag{2}$$

for $(\omega_1, \dots, \omega_n) \in \Omega_1 \times \cdots \times \Omega_n$. The equality in (2) is in $X_1 \widehat{\otimes} \cdots \widehat{\otimes} X_n$.

2.2. **Polymeasures.** Let $X_1, \dots, X_n, Z$ be Banach spaces and use the symbol $\mathcal{L}^n(X_1, \dots, X_n; Z)$ to denote the space of bounded n-linear operators from the space $X_1 \times \cdots \times X_n$ into Z. Let $(\Omega_1, \Sigma_1), \dots, (\Omega_n, \Sigma_n)$ be measurable spaces. A set function $\gamma : \Sigma_1 \times \cdots \times \Sigma_n \to \mathcal{L}^n(X_1, \dots, X_n; Z)$ is called an *operator-valued polymeasure* if it is countably additive in each argument separately. The (total) *semivariation* of γ is given by

$$\|\gamma\|_{F_n} = \sup \Big\{ \Big\| \sum_{j_1=1}^{m_1} \cdots \sum_{j_n=1}^{m_n} \gamma(A_{1,j_1}, \dots, A_{n,j_n})(x_{1,j_1}, \dots, x_{n,j_n}) \Big\|_Z \Big\}, \tag{3}$$

where the supremum is taken over measurable partitions $(A_{i,j_i})_{j_i=1}^{m_i}$ in Σ_i, and sequences $(x_{i,j_i})_{j_i=1}^{m_i}$ in the unit ball of X_i $(i = 1, \dots, n)$.

We denote by $F_n = F_n(\Sigma_1, \dots, \Sigma_n; X_1, \dots, X_n; Z)$ the collection of all polymeasures $\gamma : \Sigma_1 \times \cdots \times \Sigma_n \to \mathcal{L}^n(X_1, \dots, X_n; Z)$. We use the notation F_n because, when $Z = \mathbb{C}$ and $X_i = \mathbb{C}$ for each $1 \le i \le n$, the polymeasure γ is called a *Fréchet measure* and the semivariation is called the *Fréchet variation* [2].

2.3. **The topological setting.** Let $\Omega_1, \ldots, \Omega_n$ be compact Hausdorff spaces with respective Borel fields $\Sigma_1, \ldots, \Sigma_n$, and let $X_1, \ldots, X_n$ be Banach spaces. For each $1 \leq j \leq n$, we let $C(\Omega_j, X_j)$ denote the space of X_j-valued continuous functions on the compact Hausdorff space Ω_j. These spaces are Banach spaces when equipped with the supremum norm, $\|\cdot\|_\infty$.

In this setting, we take $F_n = F_n(\Sigma_1, \ldots, \Sigma_n; X_1, \ldots, X_n; \mathbb{C})$ to be the Banach space of weak*-regular polymeasures of finite semivariation from $\Sigma_1 \times \cdots \times \Sigma_n$ into the space of n-linear forms $\mathcal{L}^n(X_1, \ldots, X_n; \mathbb{C})$. That is, F_n is the space of polymeasures $\gamma : \Sigma_1 \times \cdots \times \Sigma_n \to \mathcal{L}^n(X_1, \ldots, X_n; \mathbb{C})$ such that $\gamma(\cdot, \cdots, \cdot)(x_1, \ldots, x_n)$ is a regular Borel measure in each argument separately, for all $(x_1, \ldots, x_n)$ in $X_1 \times \cdots \times X_n$.

We require the following Riesz-type representation theorem, stated here only for the special case we are considering:

Theorem 2.1 (Villanueva)**.** *Let $\Omega_1, \ldots, \Omega_n$ be compact Hausdorff spaces with respective Borel fields $\Sigma_1, \ldots, \Sigma_n$ and let $X_1, \ldots, X_n$ be Banach spaces. Let F_n be the Banach space of weak*-regular polymeasures of finite (total) semivariation from $\Sigma_1 \times \cdots \times \Sigma_n$ into the space of n-linear forms $\mathcal{L}^n(X_1, \ldots, X_n; \mathbb{C})$. Then*

$$\big(C(\Omega_1, X_1) \,\widehat{\otimes} \cdots \widehat{\otimes}\, C(\Omega_n, X_n)\big)^* = F_n.$$

(For the full theorem, see [19, Theorem 3.1] or [20, Theorem 1.1].)

The action of F_n on $C(\Omega_1, X_1) \,\widehat{\otimes} \cdots \widehat{\otimes}\, C(\Omega_n, X_n)$ can be realized as integration with respect to a polymeasure. Let $\mathbb{1}_A$ denote the indicator function for the measurable set A. For each $j \in \{1, \ldots, n\}$, let $s_j = \sum_{k_j} x_{j,k_j} \mathbb{1}_{A_{j,k_j}}$ be a simple measurable function on Ω_j taking values in X_j. (The summation is finite.) Define the integral of $s_1 \otimes \cdots \otimes s_n$ with respect to γ by

$$\int_{\Omega_1 \times \cdots \times \Omega_n} s_1 \otimes \cdots \otimes s_n \, d\gamma = \sum_{k_1, \ldots, k_n} \gamma(A_{1,k_1}, \ldots, A_{n,k_n})(x_{1,k_1}, \ldots, x_{n,k_n}). \tag{4}$$

Now, for each $j \in \{1, \ldots, n\}$, let $f_j \in C(\Omega_j, X_j)$. Then we can define

$$\int_{\Omega_1 \times \cdots \times \Omega_n} f_1 \otimes \cdots \otimes f_n \, d\gamma = \lim_{m \to \infty} \int_{\Omega_1 \times \cdots \times \Omega_n} s_1^{(m)} \otimes \cdots \otimes s_n^{(m)} \, d\gamma, \tag{5}$$

where $(s_j^{(m)})_{m=1}^\infty$ is a sequence of simple functions converging uniformly to f_j for each $j \in \{1, \ldots, n\}$. (See, for example, [20].)

2.4. **Bimeasures.** We are concerned only with the case $n = 2$ and $Z = \mathbb{C}$. In this case, γ is called a *bimeasure* and $\mathcal{L}^2(X_1, X_2; \mathbb{C}) = \mathcal{B}(X_1, X_2) = (X_1 \widehat{\otimes} X_2)^*$.

In the special case that $\Omega_1 = \Omega_2 = \mathbb{N}$ and $X_1 = X_2 = \mathbb{C}$, we write $F_2(\mathbb{N}, \mathbb{N})$ for the space of bimeasures, and we view elements in $F_2(\mathbb{N}, \mathbb{N})$ as scalar-valued functions on $\mathbb{N} \times \mathbb{N}$ [2, Chapter IV].

We will have need of the following useful lemma, due to Blei [2, Corollary IV.7]:

Lemma 2.2. *If $\beta \in F_2(\mathbb{N}, \mathbb{N})$, then $\sum_{i=1}^\infty \sum_{j=1}^\infty \beta(i,j) = \sum_{j=1}^\infty \sum_{i=1}^\infty \beta(i,j)$ and, in particular, the two series converge.*

The key observation in this paper is that, if $a : \mathbb{N} \times \mathbb{N} \to \mathcal{B}(X, Y)$ is given by $a_{ij} = a(i,j)$, then

$$\|a\|_{F_2(\mathbb{N}, \mathbb{N}; X, Y; \mathbb{C})} = \sup_{m,t,s} \Big| \sum_{i,j=1}^m \langle a_{ij}(t(i)), s(j) \rangle \Big|,$$

and so the upper bound in Theorem 1.1 is $K \, \|a\|_{F_2(\mathbb{N}, \mathbb{N}; X, Y; \mathbb{C})}$.

3. Extensions

We begin with an extension of Theorem 1.1:

Theorem 3.1. *Let X and Y be Banach spaces such that X^* and Y^* have cotype 2 and let $(a_{ij})_{i,j=1}^\infty$ be an infinite matrix with $a_{ij} \in \Gamma_2(X, Y^*)$ for all $i, j \in \mathbb{N}$. Then*

$$\sum_{i=1}^\infty \sum_{j=1}^\infty \Big(\sum_{k=1}^n \big\langle a_{ij}(x_i(k)), y_j(k) \big\rangle \Big) = \sum_{j=1}^\infty \sum_{i=1}^\infty \Big(\sum_{k=1}^n \big\langle a_{ij}(x_i(k)), y_j(k) \big\rangle \Big) \tag{6}$$

(and in particular both series converge) and

$$\Big| \sum_{i=1}^\infty \sum_{j=1}^\infty \Big(\sum_{k=1}^n \big\langle a_{ij}(x_i(k)), y_j(k) \big\rangle \Big) \Big| \le K \sup_{m,t,s} \Big| \sum_{i,j=1}^m \big\langle a_{ij}(t(i)), s(j) \big\rangle \Big|, \tag{7}$$

for all $(x_i)_{i=1}^\infty \subset B_{\ell_2(X)}$ and $(y_j)_{j=1}^\infty \subset B_{\ell_2(Y)}$, where the supremum is taken over $m \in \mathbb{N}$, $t \in B_{\ell_\infty(X)}$, and $s \in B_{\ell_\infty(Y)}$. (K is the constant from Theorem 1.1.)

Proof. Define $a : \mathbb{N} \times \mathbb{N} \to \mathcal{B}(X, Y)$ by $a(i,j)(x,y) = \langle a_{ij}(x), y \rangle$ for all $i, j \in \mathbb{N}$ and $(x, y) \in X \times Y$. It follows that

$$\|a\|_{F_2(\mathbb{N},\mathbb{N};X,Y;\mathbb{C})} = \sup_{m,t,s} \Big| \sum_{i,j=1}^m \big\langle a_{ij}(t(i)), s(j) \big\rangle \Big|,$$

where t and s are as given in the statement of the theorem. Consequently, the upper bound in (7) is precisely $K\|a\|_{F_2(\mathbb{N},\mathbb{N};X,Y;\mathbb{C})}$.

The statement in (6) follows from Theorem 1.1 and Lemma 2.2: Define a scalar-valued function $\beta : \mathbb{N} \times \mathbb{N} \to \mathbb{C}$ by

$$\beta(i,j) = \sum_{k=1}^n \big\langle a_{ij}(x_i(k)), y_j(k) \big\rangle, \quad i, j \in \mathbb{N}.$$

It suffices to show $\beta \in F_2(\mathbb{N}, \mathbb{N})$. Let $(\epsilon_i)_{i=1}^\infty$ and $(\delta_j)_{j=1}^\infty$ be sequences of scalars in $B_\mathbb{C}$. Then, for any $m \in \mathbb{N}$,

$$\Big| \sum_{i,j=1}^m \beta(i,j)\epsilon_i\delta_j \Big| = \Big| \sum_{i,j=1}^m \Big(\sum_{k=1}^n \big\langle a_{ij}(\epsilon_i x_i(k)), \delta_j y_j(k) \big\rangle \Big) \Big|.$$

By Theorem 1.1, this is bounded by $K\|a\|_{F_2(\mathbb{N},\mathbb{N};X,Y;\mathbb{C})}$. Taking the supremum over $m \in \mathbb{N}$ and $\epsilon_i, \delta_j \in B_\mathbb{C}$, we conclude $\|\beta\|_{F_2} < \infty$, and so we have (6), by Lemma 2.2. The bound in (7) follows directly. □

We will restate Theorem 3.1 in a measure-theoretic setting. Let $(\Omega_1, \mathcal{A})$ and $(\Omega_2, \mathcal{B})$ be measurable spaces and let $\mu : \mathcal{A} \times \mathcal{B} \to \mathcal{B}(X, Y)$ be a bimeasure such that $\mu(A, B) \in \Gamma_2(X, Y^*)$ for all $(A, B) \in \mathcal{A} \times \mathcal{B}$; that is, let $\mu \in \Gamma_{DJ}$.

Let $(f_k)_{k=1}^n \subset L^\infty(\Omega_1, X)$ and $(g_k)_{k=1}^n \subset L^\infty(\Omega_2, Y)$ be finite sequences of countably valued bounded measurable functions. For each $k \in \{1, \dots, n\}$, let

$$f_k = \sum_{i=1}^\infty x_i(k)\mathbb{1}_{A_i} \quad \text{and} \quad g_k = \sum_{j=1}^\infty y_j(k)\mathbb{1}_{B_j},$$

where $(x_i(k))_{i=1}^\infty$ and $(y_j(k))_{j=1}^\infty$ are bounded sequences in X and Y (respectively) and $(A_i)_{i=1}^\infty$ and $(B_j)_{j=1}^\infty$ are sequences of pairwise-disjoint nonempty measurable sets in $\mathcal{A}$ and $\mathcal{B}$ (respectively). (We may choose the measurable sets independent of k because $n < \infty$; i.e., because there are finitely many functions f_k and g_k.)

Define the integral of $\sum_{k=1}^n f_k \otimes g_k$ with respect to μ by

$$\begin{aligned}\int \Big(\sum_{k=1}^n f_k \otimes g_k\Big) d\mu &= \int_{\Omega_1 \times \Omega_2} \Big(\sum_{k=1}^n f_k(\omega_1) \otimes g_k(\omega_2)\Big) \mu(d\omega_1, d\omega_2) \\ &= \sum_{i=1}^\infty \sum_{j=1}^\infty \Big(\sum_{k=1}^n \mu(A_i, B_j)\big(x_i(k), y_j(k)\big)\Big).\end{aligned} \tag{8}$$

By Theorem 3.1, the series in (8) converges and

$$\begin{aligned}\Big| \int \Big(\sum_{k=1}^n f_k \otimes g_k\Big) d\mu\Big| &\le K\|\mu\|_{F_2} \sup_{i\in\mathbb{N}} \Big(\sum_{k=1}^n \|x_i(k)\|^2\Big)^{1/2} \sup_{j\in\mathbb{N}} \Big(\sum_{k=1}^n \|y_j(k)\|^2\Big)^{1/2} \\ &= K\|\mu\|_{F_2} \sup_{\omega_1\in\Omega_1} \Big(\sum_{k=1}^n \|f_k(\omega_1)\|^2\Big)^{1/2} \sup_{\omega_2\in\Omega_2} \Big(\sum_{k=1}^n \|g_k(\omega_2)\|^2\Big)^{1/2}.\end{aligned} \tag{9}$$

By a simple argument, the integral defined in (8) is independent of representation in $L^\infty(\Omega_1, X) \otimes L^\infty(\Omega_2, Y)$ by functions of the form $\sum_{k=1}^n f_k \otimes g_k$, where $(f_k)_{k=1}^n$ and $(g_k)_{k=1}^n$ are sequences of countably valued bounded measurable functions.

We summarize in the following lemma:

Lemma 3.2. *Let $(f_k)_{k=1}^n \subset L^\infty(\Omega_1, X)$ and $(g_k)_{k=1}^n \subset L^\infty(\Omega_2, Y)$ be finite sequences of countably valued bounded measurable functions, where X^* and Y^* have cotype 2. If $\mu \in \Gamma_{DJ}$, then $\sum_{k=1}^n f_k \otimes g_k$ can be integrated with respect to μ in a well-defined way and $\big|\int \big(\sum_{k=1}^n f_k \otimes g_k\big) d\mu\big|$ is bounded by*

$$K\|\mu\|_{F_2} \sup_{\omega_1\in\Omega_1} \Big(\sum_{k=1}^n \|f_k(\omega_1)\|_X^2\Big)^{1/2} \sup_{\omega_2\in\Omega_2} \Big(\sum_{k=1}^n \|g_k(\omega_2)\|_Y^2\Big)^{1/2}.$$

For convenience, when we are discussing functions of the type described in Lemma 3.2, we will call them *functions of type CV*.

4. The Haagerup tensor product

Let $(\Omega_1, \mathcal{A})$ and $(\Omega_2, \mathcal{B})$ be measurable spaces and let X and Y be Banach spaces such that X^* and Y^* have cotype 2. We will put a norm on the algebraic tensor product $L^\infty(\Omega_1, X) \otimes L^\infty(\Omega_2, Y)$: For each $u \in L^\infty(\Omega_1, X) \otimes L^\infty(\Omega_2, Y)$, let

$$\|u\|_h = \inf\Big\{ \Big\|\sum_{k=1}^n \|f_k\|_X^2\Big\|_{L^\infty(\Omega_1)}^{1/2} \Big\|\sum_{k=1}^n \|g_k\|_Y^2\Big\|_{L^\infty(\Omega_2)}^{1/2} : u = \sum_{k=1}^n f_k \otimes g_k\Big\}, \tag{10}$$

where the infimum is taken over representations of u in $L^\infty(\Omega_1, X) \otimes L^\infty(\Omega_2, Y)$.

The map $u \mapsto \|u\|_h$ determines a norm on $L^\infty(\Omega_1, X) \otimes L^\infty(\Omega_2, Y)$, and an easy calculation shows that $\|u\|_h \le \|u\|_\wedge$ for all $u \in L^\infty(\Omega_1, X) \otimes L^\infty(\Omega_2, Y)$. The completion of $L^\infty(\Omega_1, X) \otimes L^\infty(\Omega_2, Y)$ in the norm $\|\cdot\|_h$ is called the *Haagerup tensor product* and is denoted by $L^\infty(\Omega_1, X) \otimes_h L^\infty(\Omega_2, Y)$. (See, for example, [16].)

Let $\mu \in \Gamma_{DJ}$. We define the integral of $u \in L^\infty(\Omega_1, X) \otimes_h L^\infty(\Omega_2, Y)$ with respect to μ as follows:

$$\int u \, d\mu = \lim_{n\to\infty} \int u_n \, d\mu, \tag{11}$$

where $(u_n)_{n=1}^{\infty}$ is a sequence of functions of type CV (cf. Lemma 3.2) in the algebraic tensor product $L^\infty(\Omega_1, X) \otimes L^\infty(\Omega_2, Y)$ that converges to u in the Haagerup norm, $\|\cdot\|_h$.

Theorem 4.1. *Let $(\Omega_1, \mathcal{A})$ and $(\Omega_2, \mathcal{B})$ be measurable spaces and let X and Y be Banach spaces such that X^* and Y^* have cotype 2. If $u \in L^\infty(\Omega_1, X) \otimes_h L^\infty(\Omega_2, Y)$ and $\mu \in \Gamma_{DJ}$, then $\int u\, d\mu$ is well-defined and*

$$\Big| \int u\, d\mu \Big| \leq K\, \|u\|_h\, \|\mu\|_{F_2},$$

where K is the Defant-Junge constant from Theorem 1.1.

Proof. We begin by showing that functions of the form $\sum_k f_k \otimes g_k$, where $(f_k)_k$ and $(g_k)_k$ are finite sequences of countably valued bounded measurable functions, are dense in $L^\infty(\Omega_1, X) \otimes_h L^\infty(\Omega_2, Y)$. Let $u \in L^\infty(\Omega_1, X) \otimes_h L^\infty(\Omega_2, Y)$ and let $\epsilon > 0$. By definition, there exists a function $\sum_{k=1}^n \phi_k \otimes \psi_k$ in the algebraic tensor product $L^\infty(\Omega_1, X) \otimes L^\infty(\Omega_2, Y)$ such that

$$\Big\| u - \sum_{k=1}^n \phi_k \otimes \psi_k \Big\|_h < \frac{\epsilon}{3}.$$

Let $M = 1 + \max\{\|\phi_k\|_\infty + \|\psi_k\|_\infty : 1 \leq k \leq n\}$.

It is a consequence of the Pettis measurability theorem that countably valued measurable functions are dense in $L^\infty(\Omega_1, X)$ and $L^\infty(\Omega_2, Y)$ [8, Corollary II.1.3]. Thus, for each $k \in \{1, \dots, n\}$, there exist countably valued bounded measurable functions f_k and g_k such that

$$\|\phi_k - f_k\|_\infty < \frac{\epsilon}{3nM} \quad \text{and} \quad \|\psi_k - g_k\|_\infty < \frac{\epsilon}{3nM}. \tag{12}$$

Without loss of generality, we may assume $\frac{\epsilon}{3nM} < 1$, and so $\|g_k\|_\infty < M$ for each $k \in \{1, \dots, n\}$.

By the triangle inequality,

$$\Big\| \sum_{k=1}^n \phi_k \otimes \psi_k - \sum_{k=1}^n f_k \otimes g_k \Big\|_h \leq \Big\| \sum_{k=1}^n \phi_k \otimes (\psi_k - g_k) \Big\|_h + \Big\| \sum_{k=1}^n (\phi_k - f_k) \otimes g_k \Big\|_h.$$

By the definition of the Haagerup norm, this is bounded by

$$\Big\| \sum_{k=1}^n \|\phi_k\|_X^2 \Big\|_{L^\infty(\Omega_1)}^{1/2} \Big\| \sum_{k=1}^n \|\psi_k - g_k\|_Y^2 \Big\|_{L^\infty(\Omega_2)}^{1/2} + \Big\| \sum_{k=1}^n \|\phi_k - f_k\|_X^2 \Big\|_{L^\infty(\Omega_1)}^{1/2} \Big\| \sum_{k=1}^n \|g_k\|_Y^2 \Big\|_{L^\infty(\Omega_2)}^{1/2}.$$

By (12), this is bounded by

$$(\sqrt{n} M) \cdot \Big(\sqrt{n} \frac{\epsilon}{3nM} \Big) + \Big(\sqrt{n} \frac{\epsilon}{3nM} \Big) \cdot (\sqrt{n} M) = \frac{2\epsilon}{3}.$$

Therefore, $\|u - \sum f_k \otimes g_k\|_h < \epsilon$, as required.

Now, let $u \in L^\infty(\Omega_1, X) \otimes_h L^\infty(\Omega_2, Y)$ and let $(u_n)_{n=1}^\infty$ be a sequence of functions of type CV in $L^\infty(\Omega_1, X) \otimes L^\infty(\Omega_2, Y)$ that converges to u in the Haagerup norm. By Lemma 3.2, and arguing as above, we have that $(\int u_n d\mu)_{n=1}^\infty$ is a Cauchy sequence. Therefore,

$$\int u\, d\mu = \lim_{n \to \infty} \int u_n\, d\mu$$

exists. A similar argument shows the limit is independent of the choice of defining sequence $(u_n)_{n=1}^\infty$. Finally, we remark that the bound is obtained from Lemma 3.2, by taking limits. □

We easily obtain the following convergence theorem as a corollary:

Corollary 4.2. *Assume the hypotheses of Theorem 4.1. Then*

$$\int u\, d\mu = \lim_{n\to\infty} \int u_n\, d\mu,$$

whenever $u_n \to u$ *in* $L^\infty(\Omega_1, X) \otimes_h L^\infty(\Omega_2, Y)$.

5. The extended Haagerup tensor product

Again, let $(\Omega_1, \mathcal{A})$ and $(\Omega_2, \mathcal{B})$ be measurable spaces and let X and Y be Banach spaces such that X^* and Y^* have cotype 2. Define the *extended Haagerup tensor product* to be the space:

$$\Big\{\sum_{k=1}^\infty f_k \otimes g_k : \Big\|\sum_{k=1}^\infty \|f_k\|_X^2\Big\|_{L^\infty(\Omega_1)}^{1/2} \Big\|\sum_{k=1}^\infty \|g_k\|_Y^2\Big\|_{L^\infty(\Omega_2)}^{1/2} < \infty\Big\},$$

where $f_k : \Omega_1 \to X$ and $g_k : \Omega_2 \to Y$ are strongly measurable bounded functions for each $k \in \mathbb{N}$. The sum $\sum_{k=1}^\infty f_k \otimes g_k$ converges pointwise, by which we mean that $\sum_{k=1}^\infty f_k(\omega_1) \otimes g_k(\omega_2)$ is in $X \widehat{\otimes} Y$ for $(\omega_1, \omega_2) \in \Omega_1 \times \Omega_2$, so that

$$\Big(\sum_{k=1}^\infty f_k \otimes g_k\Big)(\omega_1, \omega_2)(x^*, y^*) = \sum_{k=1}^\infty f_k(\omega_1)(x^*) g_k(\omega_2)(y^*), \tag{13}$$

for all $(\omega_1, \omega_2) \in \Omega_1 \times \Omega_2$ and $(x^*, y^*) \in X^* \times Y^*$. We denote this space by $L^\infty(\Omega_1, X) \otimes_{eh} L^\infty(\Omega_2, Y)$ and equip it with the norm

$$\|u\|_{eh} = \inf\Big\{\Big\|\sum_{k=1}^\infty \|f_k\|_X^2\Big\|_{L^\infty(\Omega_1)}^{1/2} \Big\|\sum_{k=1}^\infty \|g_k\|_Y^2\Big\|_{L^\infty(\Omega_2)}^{1/2} : u = \sum_{k=1}^\infty f_k \otimes g_k\Big\}.$$

The functions $\sum_{k=1}^\infty f_k \otimes g_k$ and $\sum_{k=1}^\infty f'_k \otimes g'_k$ in $L^\infty(\Omega_1, X) \otimes_{eh} L^\infty(\Omega_2, Y)$ are equal if

$$\sum_{k=1}^\infty f_k(\omega_1)(x^*) g_k(\omega_2)(y^*) = \sum_{k=1}^\infty f'_k(\omega_1)(x^*) g'_k(\omega_2)(y^*),$$

for all $(\omega_1, \omega_2) \in \Omega_1 \times \Omega_2$ and $(x^*, y^*) \in X^* \times Y^*$.

Clearly $\|u\|_{eh} \le \|u\|_h$ whenever $u \in L^\infty(\Omega_1, X) \otimes_h L^\infty(\Omega_2, Y)$. We remark that the sum in (13) converges uniformly only if $\sum_{k=1}^\infty f_k \otimes g_k$ represents an element of $L^\infty(\Omega_1, X) \otimes_h L^\infty(\Omega_2, Y)$.

The extended Haagerup tensor product was introduced in [10] in a general context. It is closely related to the weak*-Haagerup tensor product of [1], and indeed the two notions coincide when the spaces in question are dual operator spaces. In our particular case, we are making use of the characterization given by [1, Theorem 3.1]. See also [18] for a detailed treatment of the extended Haagerup tensor product of spaces of scalar-valued bounded measurable functions.

Let $\mu \in \Gamma_{DJ}$ and let u have representation $u = \sum_{k=1}^\infty f_k \otimes g_k$ in the extended Haagerup tensor product $L^\infty(\Omega_1, X) \otimes_{eh} L^\infty(\Omega_2, Y)$. Define the integral of u with respect to μ by

$$\int u\,d\mu = \lim_{n\to\infty}\sum_{k=1}^{n}\int f_k\otimes g_k\,d\mu. \tag{14}$$

Theorem 5.1. *Let $(\Omega_1,\mathcal{A})$ and $(\Omega_2,\mathcal{B})$ be measurable spaces and let X and Y be Banach spaces such that X^* and Y^* have cotype 2. If both $\mu\in\Gamma_{DJ}$ and $u\in L^\infty(\Omega_1,X)\otimes_{eh}L^\infty(\Omega_2,Y)$, then $\int u\,d\mu$ is well-defined and*

$$\Big|\int u\,d\mu\Big| \le K\,\|u\|_{eh}\,\|\mu\|_{F_2},$$

where K is the Defant-Junge constant from Theorem 1.1.

Proof. This is a consequence of [10, Theorem 5.7]. A bounded linear functional on $L^\infty(\Omega_1,X)\otimes_h L^\infty(\Omega_2,Y)$ determines a bounded linear functional on $L^\infty(\Omega_1,X)\otimes_{eh} L^\infty(\Omega_2,Y)$ of the same norm. Furthermore, the extension is independent of representation of u and can be computed by (14). The rest follows directly. □

The next theorem is the last we need before we can prove the main result of this note, the measure-theoretic operator-valued Grothendieck inequality.

Theorem 5.2. *Let X and Y be Banach spaces and let $(\Omega_1,\mathcal{A})$ and $(\Omega_2,\mathcal{B})$ be two measurable spaces. Let $f:\Omega_1\to\ell_2(X)$ and $g:\Omega_2\to\ell_2(Y)$ be bounded measurable functions such that $f=(f_k)_{k=1}^\infty$ and $g=(g_k)_{k=1}^\infty$. If*

$$\langle f,g\rangle(\omega_1,\omega_2)=\sum_{k=1}^{\infty} f_k(\omega_1)\otimes g_k(\omega_2),\quad (\omega_1,\omega_2)\in\Omega_1\times\Omega_2,$$

then $\langle f,g\rangle\in L^\infty(\Omega_1,X)\otimes_{eh}L^\infty(\Omega_2,Y)$ and $\|\langle f,g\rangle\|_{eh}\le\|f\|_\infty\|g\|_\infty$.

Proof. By assumption,

$$\|f\|_\infty\|g\|_\infty=\sup_{\omega_1\in\Omega_1}\Big(\sum_{k=1}^{\infty}\|f_k(\omega_1)\|_X^2\Big)^{1/2}\sup_{\omega_2\in\Omega_2}\Big(\sum_{k=1}^{\infty}\|g_k(\omega_2)\|_Y^2\Big)^{1/2}<\infty.$$

Consequently, $\langle f,g\rangle\in L^\infty(\Omega_1,X)\otimes_{eh}L^\infty(\Omega_2,Y)$. □

We are now prepared to prove the promised Theorem 1.2, which we now restate as a corollary to Theorems 5.1 and 5.2.

Corollary 5.3 (Measure-theoretic operator-valued Grothendieck inequality)**.** *Let $(\Omega_1,\mathcal{A})$ and $(\Omega_2,\mathcal{B})$ be two measurable spaces and let X and Y be Banach spaces such that X^* and Y^* have cotype 2. If $f:\Omega_1\to\ell_2(X)$ and $g:\Omega_2\to\ell_2(Y)$ are bounded measurable functions, and if $\mu\in\Gamma_{DJ}$, then*

$$\Big|\int\langle f,g\rangle\,d\mu\Big|\le K\,\|\mu\|_{F_2}\sup_{\omega_1\in\Omega_1}\|f(\omega_1)\|_{\ell_2(X)}\sup_{\omega_2\in\Omega_2}\|g(\omega_2)\|_{\ell_2(Y)}.$$

Proof. This follows directly from Theorems 5.1 and 5.2. □

6. The Defant-Junge space of operator-valued bimeasures

In Section 1, we adopted the notation $\mu \in \Gamma_{DJ} = \Gamma_{DJ}(\mathcal{A}, \mathcal{B}; X, Y^*)$ to mean that $\mu : \mathcal{A}\times\mathcal{B} \to \mathcal{B}(X, Y)$ is an operator-valued bimeasure such that $\mu(A, B)$ determines a member of $\Gamma_2(X, Y^*)$ for each $(A, B) \in \mathcal{A} \times \mathcal{B}$. We now define a scalar-valued function on the space Γ_{DJ} as follows:

$$\|\mu\|_{DJ} = \sup\Big\{\Big|\sum_{i,j=1}^{m}\sum_{k=1}^{n}\Big\langle \mu(A_i, B_j)(x_i(k)), y_j(k)\Big\rangle\Big|\Big\}, \tag{15}$$

where the supremum is taken over $m, n \in \mathbb{N}$, finite measurable partitions $(A_i)_{i=1}^m$ and $(B_j)_{j=1}^m$ in $\mathcal{A}$ and $\mathcal{B}$ (respectively), and sequences $x_1, \ldots, x_m \in B_{\ell_2(X)}$ and $y_1, \ldots, y_m \in B_{\ell_2(Y)}$

By Theorem 1.1, $\|\mu\|_{DJ} \le K\|\mu\|_{F_2}$, and so $\|\mu\|_{DJ} < \infty$ for all $\mu \in \Gamma_{DJ}$.

Theorem 6.1. *The function $\|\cdot\|_{DJ}$ determines a complete norm on Γ_{DJ}.*

Proof. We will show only completeness, as the other conditions are straightforward. Suppose $(\mu_n)_{n=1}^\infty$ is a Cauchy sequence in Γ_{DJ}. Let $A \in \mathcal{A}$ and $B \in \mathcal{B}$. Then

$$\sup\Big\{\big|\mu_n(A, B)(x, y) - \mu_m(A, B)(x, y)\big| : x \in B_X, y \in B_Y\Big\} \le \|\mu_n - \mu_m\|_{DJ}.$$

(This follows from the definition, taking $m = 1$ and $n = 1$ in (15).) Consequently,

$$\|\mu_n(A, B) - \mu_m(A, B)\|_{\mathcal{B}(X,Y)} \le \|\mu_n - \mu_m\|_{DJ}, \tag{16}$$

and so $(\mu_n(A, B))_{n=1}^\infty$ is a Cauchy sequence in $\mathcal{B}(X, Y)$. Therefore, we conclude that $(\mu_n(A, B))_{n=1}^\infty$ has a limit in $\mathcal{B}(X, Y)$ for each $A \in \mathcal{A}$ and $B \in \mathcal{B}$.

Define a set function $\mu : \mathcal{A} \times \mathcal{B} \to \mathcal{B}(X, Y)$ by

$$\mu(A, B) = \lim_{n\to\infty} \mu_n(A, B), \quad (A, B) \in \mathcal{A} \times \mathcal{B}.$$

By the Vitali-Hahn-Saks-Nikodým theorem for operator-valued polymeasures (for example, [9, Theorem (VHSN) on p. 490]), the set function μ is an operator-valued bimeasure on $\mathcal{A} \times \mathcal{B}$.

It remains only to show that $\mu(A, B) \in \Gamma_2(X, Y^*)$ for each $A \in \mathcal{A}$ and $B \in \mathcal{B}$, but this follows directly from (16). (See, for example, [7, Theorem 7.5].) ☐

We call Γ_{DJ} the Defant-Junge space of operator-valued bimeasures and refer to elements in Γ_{DJ} as Defant-Junge bimeasures.

7. Duality in the topological setting

In this section, X and Y are Banach spaces such that X^* and Y^* have cotype 2, and Ω_1 and Ω_2 are compact Hausdorff spaces with Borel fields $\mathcal{A}$ and $\mathcal{B}$, respectively. We let F_2 denote the Banach space of weak*-regular bimeasures of finite total semivariation from $\mathcal{A} \times \mathcal{B}$ into $\mathcal{B}(X, Y)$. We recall that, for a Banach space E and a compact Hausdorff space Ω, the symbol $C(\Omega, E)$ denotes the collection of all continuous functions from Ω into E, which is a Banach space when equipped with the supremum norm.

Theorem 7.1. *Let X and Y be Banach spaces such that X^* and Y^* have cotype 2 and let Ω_1 and Ω_2 be compact Hausdorff spaces with Borel fields $\mathcal{A}$ and $\mathcal{B}$, respectively. We have the identification*

$$\big(C(\Omega_1, X) \otimes_h C(\Omega_2, Y)\big)^* = \Gamma_{DJ}(\mathcal{A}, \mathcal{B}; X, Y^*),$$

and if Λ_μ is the bounded linear functional on $C(\Omega_1, X)\otimes_h C(\Omega_2, Y)$ that corresponds to $\mu \in \Gamma_{DJ}$, then

$$\|\mu\|_{DJ} \leq \|\Lambda_\mu\| \leq K\|\mu\|_{DJ}.$$

Proof. Observe that $\|\mu\|_{F_2} \leq \|\mu\|_{DJ}$. By Theorem 4.1, the Defant-Junge bimeasure $\mu \in \Gamma_{DJ}$ determines a bounded linear functional Λ_μ on $C(\Omega_1, X)\otimes_h C(\Omega_2, Y)$ and $\|\Lambda_\mu\| \leq K\,\|\mu\|_{F_2}$. Thus, $\|\Lambda_\mu\| \leq K\,\|\mu\|_{DJ}$.

Now, let Λ be a bounded linear functional on $C(\Omega_1, X)\otimes_h C(\Omega_2, Y)$. We have the inequality $\|\cdot\|_h \leq \|\cdot\|_\wedge$, and so Λ determines a bounded linear functional on $C(\Omega_1, X)\,\widehat{\otimes}\,C(\Omega_2, Y)$. By Theorem 2.1, there exists a bimeasure $\mu \in F_2$ such that $\Lambda(u) = \int u\,d\mu$ for all $u \in C(\Omega_1, X)\,\widehat{\otimes}\,C(\Omega_2, Y)$.

Simple functions are dense in $C(\Omega_1, X)$ and $C(\Omega_2, Y)$. It follows that functions of the form $\sum_k f_k \otimes g_k$, where $(f_k)_k$ and $(g_k)_k$ are finite sequences of simple functions, are dense in $C(\Omega_1, X)\otimes_h C(\Omega_2, Y)$. Consequently, we can conclude that $\Lambda(u) = \int u\,d\mu$ for all $u \in C(\Omega_1, X)\otimes_h C(\Omega_2, Y)$, provided that $\mu \in \Gamma_{DJ}$ (cf. Theorem 4.1).

We need to show that $\mu(A, B) \in \Gamma(X, Y^*)$ for all $A \in \mathcal{A}$ and $B \in \mathcal{B}$. Let $A \in \mathcal{A}$ and $B \in \mathcal{B}$ be fixed. Take finite sequences $(x_j)_{j=1}^n$ and $(y_j)_{j=1}^n$ in X and Y, respectively. Then

$$\sum_{j=1}^n \mu(A, B)(x_j, y_j) = \int \Big(\sum_{j=1}^n x_j \mathbb{1}_A \otimes y_j \mathbb{1}_B\Big) d\mu.$$

Therefore, by Corollary 5.3,

$$\Big|\Big\langle \mu(A, B), \sum_{j=1}^n x_j \otimes y_j \Big\rangle\Big| \leq K\,\|\mu\|_{F_2} \Big(\sum_{j=1}^n \|x_j\|^2\Big)^{1/2} \Big(\sum_{j=1}^n \|y_j\|^2\Big)^{1/2}. \tag{17}$$

This condition precisely identifies elements in $\Gamma(X, Y^*)$ [15, Theorem 2.8(a)].

Finally, we need to show $\|\mu\|_{DJ} \leq \|\Lambda\|$. To that end, let $m, n \in \mathbb{N}$ be arbitrary. Let $x_1, \ldots, x_m \in B_{\ell_2(X)}$ and $y_1, \ldots, y_m \in B_{\ell_2(Y)}$, and let $(A_i)_{i=1}^m$ and $(B_j)_{j=1}^m$ be measurable partitions in $\mathcal{A}$ and $\mathcal{B}$, respectively. Then

$$\sum_{i,j=1}^m \sum_{k=1}^n \Big\langle \mu(A_i, B_j)(x_i(k)), y_j(k)\Big\rangle = \sum_{k=1}^n \int \Big(\sum_{i=1}^m x_i(k)\mathbb{1}_{A_i}\Big) \otimes \Big(\sum_{j=1}^m y_j(k)\mathbb{1}_{B_j}\Big) d\mu.$$

For each $k \in \{1, \ldots, n\}$, let $f_k = \sum_{i=1}^m x_i(k)\mathbb{1}_{A_i}$ and $g_k = \sum_{j=1}^m y_j(k)\mathbb{1}_{B_j}$. Thus,

$$\Big|\sum_{i,j=1}^m \sum_{k=1}^n \Big\langle \mu(A_i, B_j)(x_i(k)), y_j(k)\Big\rangle\Big| = \Big|\sum_{k=1}^n \int f_k \otimes g_k\, d\mu\Big|.$$

Given the relationship between Λ and μ, we have

$$\Big|\sum_{k=1}^n \int f_k \otimes g_k\, d\mu\Big| = \Big|\Lambda\Big(\sum_{k=1}^n f_k \otimes g_k\Big)\Big| \leq \|\Lambda\| \Big\|\sum_{k=1}^n \|f_k\|^2\Big\|_\infty^{1/2} \Big\|\sum_{k=1}^n \|g_k\|^2\Big\|_\infty^{1/2}.$$

Observe that

$$\sup_{\omega_1 \in \Omega_1} \Big(\sum_{k=1}^n \|f_k(\omega_1)\|_X^2\Big)^{1/2} = \sup_i \Big(\sum_{k=1}^n \|x_i(k)\|_X^2\Big)^{1/2} \leq \sup_i \|x_i\|_{\ell_2(X)} \leq 1.$$

Similarly $\big\|\sum_{k=1}^n \|g_k\|^2\big\|_\infty^{1/2} \leq 1$, and so

$$\Big|\sum_{i,j=1}^m \sum_{k=1}^n \Big\langle \mu(A_i, B_j)(x_i(k)), y_j(k)\Big\rangle\Big| \leq \|\Lambda\|.$$

Taking suprema, we have $\|\mu\|_{DJ} \leq \|\Lambda\|$, as required. □

7.1. **A factorization theorem.** Analogous to the classical Grothendieck inequality, the inequality of Corollary 5.3 provides a factorization theorem for bounded linear functionals on $C(\Omega_1, X) \otimes_h C(\Omega_2, Y)$.

Theorem 7.2. *Let Λ be a bounded linear functional on $C(\Omega_1, X) \otimes_h C(\Omega_2, Y)$. There exist (scalar-valued) probability measures ν_1 and ν_2 on Ω_1 and Ω_2, respectively, such that*

$$|\Lambda(f \otimes g)| \leq K\, \|\Lambda\| \Big(\int_{\Omega_1} \|f(\omega_1)\|_X^2 \, \nu_1(d\omega_1) \Big)^{1/2} \Big(\int_{\Omega_2} \|g(\omega_2)\|_Y^2 \, \nu_2(d\omega_2) \Big)^{1/2}.$$

The proof is essentially the same as the proof of the classical Grothendieck factorization theorem in the scalar case, replacing absolute values with norms, using the duality in Theorem 7.1, and applying Corollary 5.3 in place of the classical Grothendieck inequality. Consequently, the proof will be omitted. (The author suggests [2, Theorem V.2] for a treatment of the classical case.)

7.2. **A special case.** Again, let X and Y be Banach spaces such that X^* and Y^* have cotype 2 and let Ω_1 and Ω_2 be compact Hausdorff spaces with Borel fields $\mathcal{A}$ and $\mathcal{B}$, respectively. Suppose X and Y are such that $\mathcal{L}(X, Y^*) = \Gamma_2(X, Y^*)$. In this case, we have that every $\mathcal{B}(X, Y)$-valued bimeasure is in Γ_{DJ}. Consequently, $F_2 = \Gamma_{DJ}$. In addition, $C(\Omega_1, X) \otimes_h C(\Omega_2, Y)$ coincides with the space $C(\Omega_1, X) \widehat{\otimes} C(\Omega_2, Y)$.

It should be noted that the equality $\mathcal{L}(X, Y^*) = \Gamma_2(X, Y^*)$ is known to be true under various circumstances. Such a situation occurs, for example, whenever (in addition to the cotype 2 assumption) either X or Y^* is a GL-space [7, Theorem 17.12]. Defant and Junge showed also that $\mathcal{L}(X, Y^*) = \Gamma_2(X, Y^*)$ whenever (in addition to the cotype 2 assumption) we have that either X or Y^* has the approximation property, is a Banach lattice, or is K-convex. (See Remark (b) in Section 1 of [5].)

7.3. **The weak*-Haagerup tensor product and a characterization of Γ_{DJ}.** Once again, let X and Y be Banach spaces such that X^* and Y^* have cotype 2 and let Ω_1 and Ω_2 be compact Hausdorff spaces with Borel fields $\mathcal{A}$ and $\mathcal{B}$, respectively. The dual space to the Haagerup tensor product was identified by Blecher and Smith [1], where it was dubbed the weak*-Haagerup tensor product. In our case, we have

$$\Gamma_{DJ} = \big(C(\Omega_1, X) \otimes_h C(\Omega_2, Y)\big)^* = M(\Omega_1, X^*) \otimes_{w^*h} M(\Omega_2, Y^*).$$

(For a Banach space E and a compact Hausdorff space Ω, the symbol $M(\Omega, E^*)$ denotes the set of weak*-regular measures on Ω taking values in E^*.)

By Theorem 3.1 in [1], the bimeasure $\mu \in M(\Omega_1, X^*) \otimes_{w^*h} M(\Omega_2, Y^*)$ has a weak*-representation $\mu = \sum_{\alpha \in A} \nu_\alpha \otimes \lambda_\alpha$, where A is an arbitrary (possibly uncountable) index set, $\nu_\alpha \in M(\Omega_1, X^*)$ and $\lambda_\alpha \in M(\Omega_2, Y^*)$ for each $\alpha \in A$, and

$$\|\mu\|_{w^*h} = \Big(\sum_{\alpha \in A} \|\nu_\alpha \star \nu_\alpha^*\|_M^2 \Big)^{1/2} \Big(\sum_{\alpha \in A} \|\lambda_\alpha^* \star \lambda_\alpha\|_M^2 \Big)^{1/2},$$

where $\|\cdot\|_M$ denotes the total variation norm and $*$ and $\star$ denote the involution and multiplication (respectively) in the space of measures viewed as an operator space.

Acknowledgment. The author wishes to express his gratitude to Marius Junge for his encouragement, and to the anonymous referee for several useful suggestions.

References

[1] D. P. Blecher and R. R. Smith, *The dual of the Haagerup tensor product*, J. London Math. Soc. (2), 45 (1992), pp. 126–144.

[2] R. C. Blei, *Analysis in Integer and Fractional Dimensions*, no. 71 in Cambridge studies in advanced mathematics, Cambridge University Press, first ed., 2001.

[3] F. Bombal, D. Pérez-García, and I. Villanueva, *Multilinear extensions of Grothendieck's theorem*, Q. J. Math., 55 (2004), pp. 441–450.

[4] A. Bowers, *A measure-theoretic Grothendieck inequality*, J. Math. Anal. Appl., 369 (2010), pp. 671–677.

[5] A. Defant and M. Junge, *A vector-valued Grothendieck inequality with an application to (p, q)-completely bounded operators*, Indiana Univ. Math. J., 48 (1999), pp. 295–310.

[6] J. Diestel, J. H. Fourie, and J. Swart, *The metric theory of tensor products*, American Mathematical Society, Providence, RI, 2008. Grothendieck's résumé revisited.

[7] J. Diestel, H. Jarchow, and A. Tonge, *Absolutely summing operators*, vol. 43 of Cambridge Studies in Advanced Mathematics, Cambridge University Press, Cambridge, 1995.

[8] J. Diestel and J. Uhl, Jr., *Vector Measures*, no. 15 in Mathematical Surveys, American Mathematical Society, first ed., 1977.

[9] I. Dobrakov, *On integration in Banach spaces. VIII. Polymeasures*, Czechoslovak Math. J., 37(112) (1987), pp. 487–506.

[10] E. G. Effros and Z.-J. Ruan, *Operator space tensor products and Hopf convolution algebras*, J. Operator Theory, 50 (2003), pp. 131–156.

[11] A. Grothendieck, *Résumé de la théorie métrique des produits tensoriels topologiques*, Bol. Soc. Matem. São Paulo, 8 (1956), pp. 1–79.

[12] N. J. Kalton, *Rademacher series and decoupling*, New York J. Math., 11 (2005), pp. 563–595 (electronic).

[13] J. Lindenstrauss and A. Pełczyński, *Absolutely summing operators in $\mathcal{L}_p$-spaces and their applications*, Studia Math., 29 (1968), pp. 275–326.

[14] A. M. Peralta, *New advances on the Grothendieck's inequality problem for bilinear forms on JB*-triples*, Math. Inequal. Appl., 8 (2005), pp. 7–21.

[15] G. Pisier, *Factorization of linear operators and geometry of Banach spaces*, vol. 60 of CBMS Regional Conference Series in Mathematics, Published for the Conference Board of the Mathematical Sciences, Washington, DC, 1986.

[16] ———, *Introduction to operator space theory*, vol. 294 of London Mathematical Society Lecture Note Series, Cambridge University Press, Cambridge, 2003.

[17] G. Pisier and D. Shlyakhtenko, *Grothendieck's theorem for operator spaces*, Invent. Math., 150 (2002), pp. 185–217.

[18] N. Spronk, *Measurable Schur multipliers and completely bounded multipliers of the Fourier algebras*, Proc. London Math. Soc. (3), 89 (2004), pp. 161–192.

[19] I. Villanueva, *Representation of multilinear operators on $C(K, X)$ spaces*, RACSAM Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat., 96 (2002), pp. 35–43.

[20] I. Villanueva, *Multilinear operators on $C(K, X)$ spaces*, Czechoslovak Math. J., 54(129) (2004), pp. 31–54.

[21] Q. Xu, *Operator-space Grothendieck inequalities for noncommutative L_p-spaces*, Duke Math. J., 131 (2006), pp. 525–574.

Department of Mathematics, University of Missouri, Columbia MO 65211, U.S.A

E-mail address: abowers@ucsd.edu

Current address: Department of Mathematics, UCSD, La Jolla, CA 92093, U.S.A.