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LINEAR-QUADRATIC STOCHASTIC DIFFERENTIAL GAMES ON DIRECTED CHAIN NETWORKS

Jean-Pierre Fouque* Yichen Feng† Tomoyuki Ichiba‡

ABSTRACT

We study linear-quadratic stochastic differential games on directed chains inspired by the directed chain stochastic differential equations introduced by Detering, Fouque & Ichiba [**?**]. We solve explicitly for Nash equilibria with a finite number of players and we study more general finite-player games with a mixture of both directed chain interaction and mean field interaction. We investigate and compare the corresponding games in the limit when the number of players tends to infinity. The limit is characterized by Catalan functions and the dynamics under equilibrium is an infinite-dimensional Gaussian process described by a Catalan Markov chain, with or without the presence of mean field interaction.

Key Words and Phrases: Linear-quadratic stochastic games, directed chain network, Nash equilibrium, Catalan functions, Catalan Markov chain, mean field games.

AMS 2010 Subject Classifications: 91A15, 60H30

1 Introduction

Stochastic differential games on networks is a broad area. There are two extreme situations. On one hand, we can consider a fully connected network with interaction of mean-field type. When the number of players goes to infinity, this kind of game can be approximated by a mean field game. The mean field convergence problem has been discussed widely, for instance in Lacker [**?**]. Other networks and games have been proposed and studied. For example, Delarue [**?**] investigates an example of a game with a large number of players in mean-field interaction when the graph connection between them is of Erdos-Rényi type, and Lacker & Ramanan [**?**] studies the limit of an interacting diffusive particle system on a large sparse interaction graph with finite average degree. On the other hand, we can consider a very structured network such as a one-dimensional directed chain which has been studied in Detering, Fouque & Ichiba [**?**] without the game aspect. It is a complete opposite to mean field games since, on a directed chain network, each player interacts with its neighbor in a given direction. In this paper, we introduce a game aspect of the directed chain and identify Nash equilibria. We also consider the limit when the number of players goes to infinity.

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Interestingly, the equilibrium dynamics on the network discussed in this paper turns out to be different from the dynamics suggested in [**?**], in particular, with long time variance behavior. The equilibrium dynamics for the infinite-player game is described by a Catalan Markov chain introduced in this paper.

The goal of this paper is to consider a game on a directed chain network and to find its Nash equilibrium. We focus on open-loop Nash equilibria. We want to understand how the structure of the network affects this Nash equilibrium. We propose three directed chain networks shown in Figures 1 and 2. Starting from a finite directed chain, we also discuss a periodic directed chain in a ring structure and we compare with the game on a infinite directed chain network.

The paper is organized as follows. In section 2, we propose a finite-player game model on a directed chain and construct an open-loop Nash equilibrium. We discuss general boundary conditions as well as two special cases to illustrate that the boundary condition does not actually affect the Nash equilibrium. Section 3 is devoted to the analysis of an infinite-player stochastic differential game on a directed chain. We try to find an open-loop Nash equilibrium and get a similar Riccati system to that of the finite-player game. The solutions are called Catalan functions and we use them to build a Catalan Markov chain, discussed in section 4. We find that its long-time asymptotic variance and covariance are finite. In sections 5 and 6, we discuss the finite-player and infinite-player games for a mixed system including both a directed chain interaction and a mean-field interaction. We can adjust the model to be a purely mean field game or a purely directed chain game or a mixed one by introducing a tuning parameter $u \in [0,1]$. We repeat the same steps as sections 2, 3, and 4 to find the Nash equilibria and we construct a generalized Catalan Markov chain describing the two effects. We find that the long-time asymptotic variance of the process with the purely directed chain interaction is finite, which is different from the case with mean-field interaction as shown in Table 1 in [**?**]. In section 7, we propose a finite-player periodic directed chain game and we construct an open-loop Nash equilibrium. We conjecture that its infinite-player limit is the same as the one found for other boundary condition. this conjecture is supported by numerical results. Section 8 gives a conclusion and open problems. Appendix A includes some technical proofs and discussions.

Figure 1: Finite Directed Chain and Infinite Directed Chain

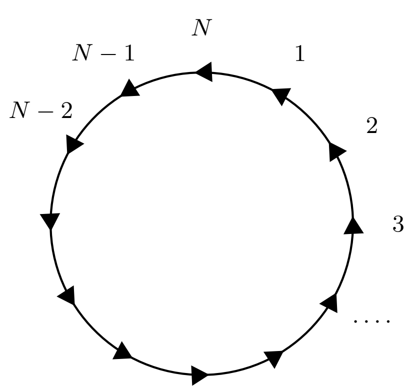


Figure 2: Periodic Directed Chain

2 N-Player Directed Chain Game

2.1 Setup and Assumptions

We consider a stochastic game in continuous time, involving N players indexed from 1 to N. Each player is controlling its own, real-valued private state X_t^i by taking a real-valued action α_t^i at time $t \in [0,T]$. The dynamics of the states of the N individual players are given by N stochastic differential equations of the form:

$$dX_t^i = \alpha_t^i \mathrm{d}t + \sigma \mathrm{d}W_t^i, \quad i = 1, \cdots, N, \tag{1}$$

where $0 \leq t \leq T$ and $(W_t^i)_{0 \leq t \leq T}$, $i = 1, \cdots, N$ are independent standard Brownian motions. For simplicity, we assume that the diffusion is one-dimensional and the diffusion coefficients are constant and identical denoted by $\sigma > 0$. The drift coefficients α^i's are adapted to the filtration of the Brownian motions and satisfy $\mathbb{E}[\int_0^T |\alpha_t^i|^2 dt] < \infty$ for $i = 1, \ldots, N$. The system starts at time $t = 0$ from $i.i.d.$ square-integrable random variables $X_0^i = \xi_i$ independent of the Brownian motions and, without loss of generality, we assume $\mathbb{E}(\xi_i) = 0$ for $i = 1, \ldots, N$.

In this model, among the first $N-1$ players, each player i chooses its own strategy α^i, in order to minimize its objective function given by:

$$1 \leq i \leq N-1: \qquad J^i(\alpha^1, \cdots, \alpha^N) = \mathbb{E}\left\{ \int_0^T \left(\frac{1}{2}(\alpha_t^i)^2 + \frac{\epsilon}{2}(X_t^{i+1} - X_t^i)^2 \right) \mathrm{d}t + \frac{c}{2}(X_T^{i+1} - X_T^i)^2 \right\}, \tag{2}$$

for some constants $\epsilon > 0$ and $c \geq 0$. The running cost and the terminal cost functions are defined by $f^i(x, \alpha^i) = \frac{1}{2}(\alpha^i)^2 + \frac{\epsilon}{2}(x^{i+1} - x^i)^2$ and $g^i(x) = \frac{c}{2}(x^{i+1} - x^i)^2$, respectively. This is a *Linear-Quadratic* differential game on a directed chain network, since X^i interacts only with X^{i+1} through the cost functions for $i = 1, \ldots, N-1$. The system is completed by describing the behavior of player N which will be done in the following section, when we discuss the boundary condition of the system.

2.2 Open-Loop Nash Equilibrium

In this section, we search for an open-loop Nash equilibrium of the system among strategies $\{\alpha_t^i, i = 1, \cdots, N\}$ and we study the effect of boundary conditions. We will discuss a general boundary condition for the game and then show two particular choices in Section 2.2.2 and 2.2.3. We construct the equilibrium by the Pontryagin stochastic maximum principle.

2.2.1 General Boundary Condition

We consider a setup with general boundary condition for the directed chain where the last player N does not depend on the other players. The cost functional for player N is defined by:

$$J^N(\alpha^N) = \mathbb{E}\left\{ \int_0^T \left(\frac{1}{2}(\alpha_t^N)^2 + q_2(X_t^N) \right) \mathrm{d}t + Q_2(X_T^N) \right\}.$$

Here, $q_2(x) = \frac{a_1}{2}(x - m)^2 + a_2$ and $Q_2(x) = \frac{c_1}{2}(x - m)^2 + c_2$ are non-degenerate convex quadratic functions in x, where a_1, a_2, m, c_1, c_2 are some constants with $a_1 > 0$ and $c_1 > 0$. The running cost function is defined by $f^N(x, \alpha^N) = \frac{1}{2}(\alpha^N)^2 + q_2(x)$ and the terminal cost function is defined by $g^N(x) = Q_2(x)$. This can be seen as a control problem for the player N and we assume its state is attracted to some constant level m.

The Hamiltonian for player $i \leq N-1$ is given by:

$$H^i(x^1, \cdots, x^N, y^{i,1}, \cdots, y^{i,N}, \alpha^1, \cdots, \alpha^N) = \sum_{k=1}^N \alpha^k y^{i,k} + \frac{1}{2}(\alpha^i)^2 + \frac{\epsilon}{2}(x^{i+1} - x^i)^2,$$

and the Hamiltonian for player N is given by:

$$H^N(x^1,\cdots,x^N,y^{i,1},\cdots,y^{i,N},\alpha^1,\cdots,\alpha^N)=\sum_{k=1}^{N}\alpha^k y^{i,k}+\frac{1}{2}(\alpha^N)^2+\frac{a_1}{2}(x^N-m)^2+a_2.$$

For $i=1,\dots,N$ the value of α^i minimizing the Hamiltonian $H^i(\cdot)$ with respect to α^i, when all the other variables including α^j for $j\neq i$ are fixed, is given by the first order condition

$$\partial_{\alpha^i}H^i=y^{i,i}+\alpha^i=0 \quad \text{leading to the choice:} \quad \hat{\alpha}^i=-y^{i,i}.$$

The adjoint processes $Y_t^i=(Y_t^{i,j};j=1,\cdots,N)$ and $Z_t^i=(Z_t^{i,j,k};j=1,\cdots,N,k=0,\cdots,N)$ for $i=1,\cdots,N$ are defined as the solutions of the system of backward stochastic differential equations (BSDEs): for $j=1,\dots,N$

$$\begin{aligned} i\le N-1: &\quad \begin{cases} \mathrm{d}Y_t^{i,j} &= -\partial_{x^j}H^i(X_t,Y_t^i,\alpha_t)\mathrm{d}t+\sum_{k=0}^{N}Z_t^{i,j,k}\mathrm{d}W_t^k \\ &= -\epsilon(X_t^{i+1}-X_t^i)(\delta_{i+1,j}-\delta_{i,j})\mathrm{d}t+\sum_{k=0}^{N}Z_t^{i,j,k}\mathrm{d}W_t^k, \\ Y_T^{i,j} &= \partial_{x^j}g_i(X_T)=c(X_T^{i+1}-X_T^i)(\delta_{i+1,j}-\delta_{i,j}); \end{cases} \\ i=N: &\quad \begin{cases} \mathrm{d}Y_t^{N,j} &= -a_1(X_t^N-m)\delta_{N,j}\mathrm{d}t+\sum_{k=0}^{N}Z_t^{N,j,k}\mathrm{d}W_t^k, \\ Y_T^{N,j} &= c_1(X_T^N-m)\delta_{N,j}. \end{cases} \end{aligned} \tag{3}$$

Particularly, for $j=i$, it becomes:

$$\begin{cases} \mathrm{d}Y_t^{i,i}=\epsilon(X_t^{i+1}-X_t^i)\mathrm{d}t+\sum_{k=0}^{N}Z_t^{i,i,k}\mathrm{d}W_t^k, & Y_T^{i,i}=-c(X_T^{i+1}-X_T^i), \quad i\le N-1 \\ \mathrm{d}Y_t^{N,N}=-a_1(X_t^N-m)\mathrm{d}t+\sum_{k=0}^{N}Z_t^{N,N,k}\mathrm{d}W_t^k, & Y_T^{N,N}=c_1(X_T^N-m). \end{cases} \tag{4}$$

Considering the BSDE system and its initial condition (4), we make the ansatz:

$$Y_t^{i,i}=\sum_{j=i}^{N-1}\phi_t^{N,i,j}X_t^j+\underbrace{(\phi_t^{N,i,N}X_t^N+\psi_t^{N,i})}_{\text{affine in } X^N\text{, depending on B.C.}}=\sum_{j=i}^{N}\phi_t^{N,i,j}X_t^j+\psi_t^{N,i}, \tag{5}$$

for some deterministic scalar functions ϕ_t (depending on N) satisfying the terminal conditions: for $1\le i\le N-1$, $\phi_T^{N,i,i}=c,\phi_T^{N,i,i+1}=-c,\phi_T^{N,i,j}=0$ for $j\ge i+2$, $\psi_T^{N,i}=0$; and $\phi_T^{N,N,N}=c_1$, $\psi_T^{N,N}=-c_1m$. With this ansatz, the optimal strategy and the forward equation become

$$\begin{cases} \hat{\alpha}_t^i=-Y_t^{i,i}=-\big(\sum_{j=i}^{N}\phi_t^{N,i,j}X_t^j+\psi_t^{N,i}\big), \\ \mathrm{d}X_t^j=-\big(\sum_{k=j}^{N}\phi_t^{N,j,k}X_t^k+\psi_t^{N,j}\big)\mathrm{d}t+\sigma\mathrm{d}W_t^j. \end{cases}$$

Differentiating the ansatz leads to:

$$
\begin{aligned}
\mathrm{d}Y_t^{i,i} &= \sum_{j=i}^{N}[X_t^j\dot{\phi}_t^{N,i,j}\mathrm{d}t + \phi_t^{N,i,j}dX_t^j] + \dot{\psi}_t^{N,i}\mathrm{d}t \\
&= \sum_{j=i}^{N} X_t^j\dot{\phi}_t^{N,i,j}\mathrm{d}t + \dot{\psi}_t^{N,i}\mathrm{d}t \\
&\quad + \sum_{j=i}^{N}\phi_t^{N,i,j}\big(-\sum_{k=j}^{N}\phi_t^{N,j,k}X_t^k\mathrm{d}t - \psi_t^{N,j}\mathrm{d}t + \sigma\mathrm{d}W_t^j\big) \\
&= \sum_{k=i}^{N}\left(\dot{\phi}_t^{N,i,k} - \sum_{j=i}^{k}\phi_t^{N,i,j}\phi_t^{N,j,k}\right)X_t^k\mathrm{d}t + \sum_{k=i}^{N}\sigma\phi_t^{N,i,k}\mathrm{d}W_t^k \\
&\quad + \dot{\psi}_t^{N,i}\mathrm{d}t - \sum_{j=i}^{N}\psi_t^{N,j}\phi_t^{N,i,j}\mathrm{d}t \\
&= \big\{\sum_{k=i}^{N}\big(\dot{\phi}_t^{N,i,k} - \sum_{j=i}^{k}\phi_t^{N,i,j}\phi_t^{N,j,k}\big)X_t^k + \big[\dot{\psi}_t^{N,i} - \sum_{j=i}^{N}\psi_t^{N,j}\phi_t^{N,i,j}\big]\big\}\mathrm{d}t \\
&\quad + \sigma\sum_{k=i}^{N}\phi_t^{N,i,k}\mathrm{d}W_t^k.
\end{aligned}
\tag{6}
$$

Here $\dot{\phi}_t$ represents the time derivative of ϕ_t. Comparing the martingale parts and drifts of two Itô's decompositions (4) and (6) of $Y_t^{i,i}$, the martingale terms give the deterministic (and therefore adapted) processes $Z_t^{i,i,k}$:

$$
Z_t^{i,i,0} = 0;\ Z_t^{i,i,k} = 0 \text{ for } k < i \text{ and } \ Z_t^{i,i,k} = \sigma\phi_t^{N,i,k} \text{ for } k \geq i. \tag{7}
$$

The drift terms show that the functions $\phi_t^{N,\cdot,\cdot}$ and $\psi_t^{N,\cdot}$ must satisfy the system of Riccati equations :
for $i \leq N-1$,

$$
\begin{aligned}
\dot{\phi}_t^{N,i,i} &= \phi_t^{N,i,i}\cdot\phi_t^{N,i,i} - \epsilon, & \phi_T^{N,i,i} &= c, \\
\dot{\phi}_t^{N,i,i+1} &= \phi_t^{N,i,i}\cdot\phi_t^{N,i,i+1} + \phi_t^{N,i,i+1}\cdot\phi_t^{N,i+1,i+1} + \epsilon, & \phi_T^{N,i,i+1} &= -c, \\
&\vdots \\
\dot{\phi}_t^{N,i,\ell} &= \phi_t^{N,i,i}\cdot\phi_t^{N,i,\ell} + \phi_t^{N,i,i+1}\cdot\phi_t^{N,i+1,\ell} \\
&\quad + \cdots + \phi_t^{N,i,\ell-1}\cdot\phi_t^{N,\ell-1,\ell} + \phi_t^{N,i,\ell}\cdot\phi_t^{N,\ell,\ell}, & \phi_T^{N,i,\ell} &= 0, \\
&\vdots \\
\dot{\phi}_t^{N,i,N-1} &= \phi_t^{N,i,i}\cdot\phi_t^{N,i,N-1} + \cdots + \phi_t^{N,i,N-1}\cdot\phi_t^{N,N-1,N-1}, & \phi_T^{N,i,N-1} &= 0, \\
\dot{\phi}_t^{N,i,N} &= \phi_t^{N,i,i}\phi_t^{N,i,N} + \cdots + \phi_t^{N,i,N-1}\phi_t^{N,N-1,N} + \phi_t^{N,i,N}\phi_t^{N,N,N}, & \phi_T^{N,i,N} &= 0;
\end{aligned}
\tag{8}
$$

for $i = N$,

$$
\dot{\phi}_t^{N,N,N} \quad = \phi_t^{N,N,N}\cdot\phi_t^{N,N,N} - a_1 \quad , \phi_T^{N,N,N} = c_1.
$$

And

$$
\begin{cases}
\dot{\psi}_t^{N,i} &= \sum_{j=i}^{N}\psi_t^{N,j}\phi_t^{N,i,j}, & \psi_T^{N,i} = 0, \\
&\vdots \\
\dot{\psi}_t^{N,N-1} &= \psi_t^{N,N-1}\phi_t^{N,N-1,N-1} + \psi_t^{N,N}\phi_t^{N,N-1,N}, & \psi_T^{N,N-1} = 0, \\
\dot{\psi}_t^{N,N} &= \psi_t^{N,N}\phi_t^{N,N,N} + a_1 m, & \psi_T^{N,N} = -c_1 m.
\end{cases}
$$

From the equations above, the functions $\phi_t^{N,i,i}$ for all $i = 1,\cdots,N-1$ are identical; the functions $\phi_t^{N,i,i+1}$ for all $i = 1,\cdots,N-2$ are identical ;$\cdots$; and the functions $\phi_t^{N,i,N-2} = \phi_t^{N,i+1,N-1}$. The functions $\phi_t^{N,i,N}$ for all i depend on $\phi_t^{N,N,N}$ of the last player which is determined by the boundary condition. However, the functions $\phi_t^{N,i,i},\cdots,\phi_t^{N,i,N-1}$ are independent of $\phi_t^{N,i,N}$ and the boundary condition. The functions $\psi^{N,\cdot}$ depend on the ϕ functions and have no effect on $\phi^{N,i,j}$ $(j < N)$ as well. In conclusion, these $\phi^{N,i,j}$ $(j < N)$ functions are solvable, identical and independent of the boundary condition as long as the boundary condition defines the last player as a self-controlled problem.

As the number of players goes to infinity, we can get rid of the boundary condition and get a sequence of functions $\{\phi_t^j, j = 1, 2, \cdots\}$, defined by $\phi_t^0 = \phi_t^{N,i,i}$, $\phi_t^1 = \phi_t^{N,i,i+1}$, $\cdots$, $\phi_t^j = \phi_t^{N,i,i+j}$ for large N and so on. It indicates that the Nash equilibrium converges to a limit independent of the boundary condition. Therefore, it is natural to study a similar game with infinite players and we conjecture that the limit of the Nash equilibrium of the finite-player game gives us the Nash equilibrium of the infinite-player game. And the sequence of functions $\{\phi_t^j, i \in \mathbb{N}\}$ is the solution to the Riccati equation system of the infinite-player game. This will be discussed in Section 3. Next, two particular examples are discussed to better illustrate the effect of the special boundary condition.

2.2.2 Boundary Condition 1: X^N is attracted to 0

Here, we discuss the case when X^N is attracted to 0 which is also the mean of the initial condition. It is equivalent to the general boundary condition when $m = 0$. Without loss of generality, we can take constants: $a_1 = \epsilon$, $c_1 = c$ and $a_2 = c_2 = 0$. Then the cost functional for player N is given by:

$$J^N(\alpha^N) = \mathbb{E}\left\{\int_0^T \left(\frac{1}{2}(\alpha_t^N)^2 + \frac{\epsilon}{2}(X_t^N)^2\right) \mathrm{d}t + \frac{c}{2}(X_T^N)^2\right\}.$$

The running cost function is defined by $f^N(x, \alpha^N) = \frac{1}{2}(\alpha^N)^2 + \frac{\epsilon}{2}x^2$ and the terminal cost function is defined by $g^N(x) = \frac{c}{2}c^2$. Then, X^N is independent of the other players and is the solution of a self-controlled problem. We then make the same ansatz as (5) with $\psi_t^{N,i} = 0$ for all i. As a result, the martingale terms give the same processes $Z_t^{i,i,k}$ as (7). And from the drift terms, we get the system of Riccati equations:
for $i \leq N-1$,

$$\begin{aligned}
\dot{\phi}_t^{N,i,i} &= \phi_t^{N,i,i} \cdot \phi_t^{N,i,i} - \epsilon, & \phi_T^{N,i,i} &= c,\\
\dot{\phi}_t^{N,i,i+1} &= \phi_t^{N,i,i} \cdot \phi_t^{N,i,i+1} + \phi_t^{N,i,i+1} \cdot \phi_t^{N,i+1,i+1} + \epsilon, & \phi_T^{N,i,i+1} &= -c,\\
&\vdots\\
\dot{\phi}_t^{N,i,l} &= \phi_t^{N,i,i} \cdot \phi_t^{N,i,l} + \phi_t^{N,i,i+1} \cdot \phi_t^{N,i+1,l} + \cdots + \phi_t^{N,i,l-1} \cdot \phi_t^{N,l-1,l} + \phi_t^{N,i,l} \cdot \phi_t^{N,l,l}, & \phi_T^{N,i,l} &= 0,\\
&\vdots\\
\dot{\phi}_t^{N,i,N} &= \phi_t^{N,i,i}\phi_t^{N,i,N} + \phi_t^{N,i,i+1}\phi_t^{N,i+1,N} + \cdots + \phi_t^{N,i,N-1}\phi_t^{N,N-1,N} + \phi_t^{N,i,N}\phi_t^{N,N,N}, & \phi_T^{N,i,N} &= 0;
\end{aligned}$$

for $i = N$,

$$\dot{\phi}_t^{N,N,N} = \phi_t^{N,N,N} \cdot \phi_t^{N,N,N} - \epsilon, \quad \phi_T^{N,N,N} = c,$$

From above, we have the same conclusion: the functions $\phi_t^{N,i,i+k} = \phi_t^{N,j,j+k}$ for all $i, j \geq 1, k \geq 0$ and $i + k < N, j + k < N$; and functions $\phi_t^{N,i,j} (j < N)$ are independent of the boundary condition. Notice that in this case $\phi_t^{N,N,N}$ has the same solution as $\phi_t^{N,i,i}$ $(i < N)$. Thus, in the ansatz, we can actually assume $\phi^{N,j-i}$ instead of $\phi^{N,i,j}$.

2.2.3 Boundary Condition 2: $\alpha^N = 0$

We study the case when there is no control for the last player X^N, i.e. the dynamics of the state is given by:

$$\mathrm{d}X_t^N = \sigma \mathrm{d}W_t^N; \qquad X_0^N = \xi_N, \quad \mathbb{E}(\xi_N) = 0.$$

Player i chooses the strategy α_t^i $(i < N)$ to minimize J^i given above and $\alpha_t^N = 0$. We make the same ansatz as in (5) with $\psi_t^{N,i} = 0$ for all i. The martingale terms give the same processes $Z_t^{i,i,k}$ as in (7).
From the drift terms, we get the system of Riccati equations :

for $i \leq N-1$,

$$\begin{aligned}
\dot{\phi}_t^{N,i,i} &= \phi_t^{N,i,i} \cdot \phi_t^{N,i,i} - \epsilon, & \phi_T^{N,i,i} = c,\\
\dot{\phi}_t^{N,i,i+1} &= \phi_t^{N,i,i} \cdot \phi_t^{N,i,i+1} + \phi_t^{N,i,i+1} \cdot \phi_t^{N,i+1,i+1} + \epsilon, & \phi_T^{N,i,i+1} = -c,\\
&\vdots\\
\dot{\phi}_t^{N,i,l} &= \phi_t^{N,i,i} \cdot \phi_t^{N,i,l} + \phi_t^{N,i,i+1} \cdot \phi_t^{N,i+1,l} + \cdots + \phi_t^{N,i,l-1} \cdot \phi_t^{N,l-1,l} + \phi_t^{N,i,l} \cdot \phi_t^{N,l,l}, & \phi_T^{N,i,l} = 0,\\
&\vdots\\
\dot{\phi}_t^{N,i,N-1} &= \phi_t^{N,i,i}\phi_t^{N,i,N-1} + \phi_t^{N,i,i+1}\phi_t^{N,i+1,N-1} + \cdots + \phi_t^{N,i,N-1}\phi_t^{N,N-1,N-1}, & \phi_T^{N,i,N-1} = 0,\\
\dot{\phi}_t^{N,i,N} &= \sum_{j=i}^{N-1} \phi_t^{N,i,j}\phi_t^{N,j,N}\\
&= \phi_t^{N,i,i}\phi_t^{N,i,N} + \phi_t^{N,i,i+1}\phi_t^{N,i+1,N} + \cdots + \phi_t^{N,i,N-1}\phi_t^{N,N-1,N}, & \phi_T^{N,i,N} = 0;
\end{aligned}$$

for $i = N$,

$$\dot{\phi}_t^{N,N,N} = -\epsilon, \quad \phi_T^{N,N,N} = c,$$

From above, it is demonstrated again that the boundary condition does not affect the solutions $\phi_t^{N,i,j}$ $(j < N)$, however, the functions $\phi_t^{N,i,N}$ for all i are different from those in Section 2.2.2, which are dependent on the boundary condition.

3 Infinite-Player Game Model

Motivated by the limit of the finite-player game discussed in Section 2, we define the game with infinite players on a directed chain structure. In Remark 1 in section 3.1, we will see that the Hamiltonian only depends on finite players, which will make it well-defined. We assume that the state dynamics of all players are given by the stochastic differential equations of the form:

$$\mathrm{d}X_t^i = \alpha_t^i \mathrm{d}t + \sigma \mathrm{d}W_t^i, \quad 0 \leq t \leq T,$$

where $(W_t^i)_{0\leq t\leq T}$, $i \geq 1$ are one-dimensional, independent Brownian motions. Similar to the setup for the finite-player games in Section 2, we assume that the drift coefficients α^i are adapted to the filtration of the Brownian motions and satisfy $\mathbb{E}[\int_0^T |\alpha_t^i|^2 \mathrm{d}t] < \infty$. We also assume that the diffusion coefficients are constant and identically denoted by $\sigma > 0$. The system starts at time $t = 0$ from $i.i.d.$ square-integrable random variables $X_0^i = \xi_i$ independent of the Brownian motions and such that $\mathbb{E}(\xi_i) = 0$. In this model, player i chooses its own strategy α^i in order to minimize its cost function of the form: $J^i(\boldsymbol{\alpha}) = \mathbb{E}\big[\int_0^T f^i(s, X_s, \alpha_s)\mathrm{d}s + g^i(X_T)\big]$, where $f^i(x, \alpha^i) = \frac{1}{2}(\alpha^i)^2 + \frac{\epsilon}{2}(x^{i+1} - x^i)^2$ and $g^i(x) = \frac{c}{2}(x^{i+1} - x^i)^2$, $x^i \in \mathbb{R}$ for $i \geq 1$.

3.1 Open-Loop Nash Equilibrium

We search for an open-loop Nash equilibrium of the infinite system among strategies $\{\alpha_t^i, i = 1, 2, \cdots\}$. First, we have the Hamiltonian of the form:

$$H^i(x^1, x^2, \cdots, y^{i,1}, \cdots, y^{i,n_i}, \alpha^1, \alpha^2, \cdots) = \sum_{k=1}^{n_i} \alpha^k y^{i,k} + \frac{1}{2}(\alpha^i)^2 + \frac{\epsilon}{2}(x^{i+1} - x^i)^2,$$

assuming it is defined on Y_t^i's where only finitely many $Y_t^{i,k}$'s are non-zero for every given i. Here, n_i is a finite number depending on i with $n_i > i$. This assumption is checked in Remark 1 below. Thus, H^i is well defined for $i \geq 1$.

The adjoint processes $Y_t^i = (Y_t^{i,j}; j = 1, \cdots, n_i)$ and $Z_t^i = (Z_t^{i,j,k}; 1 \le j \le n_i, k \ge 0)$ for $i = 1, 2, \cdots$ are the solutions of the system of backward stochastic differential equations (BSDEs):

$$\begin{cases} dY_t^{i,j} & = -\partial_{x^j} H^i(X_t, Y_t^i, \alpha_t)\mathrm{d}t + \sum\limits_{k=0}^{\infty} Z_t^{i,j,k} \mathrm{d}W_t^k \\ & = -\epsilon(X_t^{i+1} - X_t^i)(\delta_{i+1,j} - \delta_{i,j})\mathrm{d}t + \sum\limits_{k=0}^{\infty} Z_t^{i,j,k} \mathrm{d}W_t^k, \\ Y_T^{i,j} & = \partial_{x^j} g_i(X_T) = c(X_T^{i+1} - X_T^i)(\delta_{i+1,j} - \delta_{i,j}). \end{cases} \tag{9}$$

Remark 1. *For every $j \neq i$ or $i+1$, $dY_t^{i,j} = \sum\limits_{k=0}^{\infty} Z_t^{i,j,k} dW_t^k$ and $Y_T^{i,j} = 0$ implies $Z_t^{i,j,k} = 0$ for all k. Thus, there must be finitely many non-zero $Y^{i,j}$'s for every i. Hence, the Hamiltonian can be rewritten as*

$$H^i(x^1, x^2, \cdots, y^{i,i}, y^{i,i+1}, \alpha^1, \alpha^2, \cdots) = \alpha^i y^{i,i} + \alpha^{i+1} y^{i,i+1} + \frac{1}{2}(\alpha^i)^2 + \frac{\epsilon}{2}(x^{i+1} - x^i)^2.$$

By minimizing the Hamiltonian with respect to α^i, we can get the open-loop Nash equilibrium: $\hat{\alpha}^i = -y^{i,i}$ for all i. Inspired by the conclusion from the finite-player game, we then make the ansatz of the form:

$$Y_t^{i,i} = \sum_{j=i}^{\infty} \phi_t^{j-i} X_t^j, \tag{10}$$

for some deterministic scalar functions ϕ_t satisfying the terminal conditions: $\phi_T^0 = c, \phi_T^1 = -c, \phi_T^k = 0$ for $k \ge 2$. Using the ansatz, the optimal strategy $\hat{\alpha}^i$ and the forward equation for $X_\cdot^i$ in (1) become:

$$\begin{cases} \hat{\alpha}_t^i = -Y_t^{i,i} = -\sum\limits_{j=i}^{\infty} \phi_t^{j-i} X_t^j, \\ \mathrm{d}X_t^i = -\sum\limits_{j=i}^{\infty} \phi_t^{j-i} X_t^j \mathrm{d}t + \sigma \mathrm{d}W_t^i. \end{cases} \tag{11}$$

Differentiating the ansatz (10), we obtain

$$\begin{aligned} \mathrm{d}Y_t^{i,i} & = \sum_{j=i}^{\infty} [X_t^j \dot{\phi}_t^{j-i} \mathrm{d}t + \phi_t^{j-i} \mathrm{d}X_t^j] \\ & = \sum_{k=0}^{\infty} \dot{\phi}_t^k X_t^{i+k} \mathrm{d}t - \sum_{k=0}^{\infty} \left(\sum_{j=0}^{k} \phi_t^j \phi_t^{k-j} \right) X_t^{i+k} \mathrm{d}t + \sigma \sum_{k=i}^{\infty} \phi_t^{k-i} \mathrm{d}W_t^k. \end{aligned} \tag{12}$$

Now we compare the two Itô's decompositions (12) and (9) of $Y_t^{i,i}$. The martingale terms give the processes $Z_t^{i,j,k}$:

$$Z_t^{i,i,0} = 0;\ Z_t^{i,i,k} = 0 \text{ for } k < i \text{ and } \ Z_t^{i,i,k} = \sigma \phi_t^{k-i} \text{ for } k \ge i.$$

And from the drift terms, we get the system of Riccati equations:

$$\begin{array}{lll} \text{for } k = 0: & \dot{\phi}_t^0 = \phi_t^0 \cdot \phi_t^0 - \epsilon, & \phi_T^0 = c, \\ \text{for } k = 1: & \dot{\phi}_t^1 = 2\phi_t^0 \cdot \phi_t^1 + \epsilon, & \phi_T^1 = -c, \\ \text{for } k \ge 2: & \dot{\phi}_t^k = \phi_t^0 \cdot \phi_t^k + \phi_t^1 \cdot \phi_t^{k-1} + \cdots + \phi_t^{k-1} \cdot \phi_t^1 + \phi_t^k \cdot \phi_t^0, & \phi_T^k = 0. \end{array} \tag{13}$$

The solutions to this Riccati system coincide with the limit of the solutions to the ODE system (8) of the N-player directed chain game in Section 2, i.e., $\phi_\cdot^i = \lim\limits_{N \to \infty} \phi_\cdot^{N,i,i+j}$ in the supremum norm.

Lemma 1. *We have*

$$\sum_{k=0}^{\infty} \phi_t^k = 0, \qquad \phi_t^0 = \frac{(-\epsilon - c\sqrt{\epsilon})e^{2\sqrt{\epsilon}(T-t)} + \epsilon - c\sqrt{\epsilon}}{(-\sqrt{\epsilon} - c)e^{2\sqrt{\epsilon}(T-t)} - \sqrt{\epsilon} + c},$$

and the functions ϕ^k's are obtained by a series expansion of $S_t(z)$ given by (66).

Proof. Given in Appendix A.1. □

Remark 2. *It follows from Lemma 1 that the forward dynamics (11) can be written as:*

$$\begin{aligned} \mathrm{d}X_t^i &= -\sum_{j=0}^{\infty}\phi_t^j X_t^{i+j}\mathrm{d}t + \sigma \mathrm{d}W_t^i \\ &= -\phi_t^0 \cdot \sum_{j=1}^{\infty}\tfrac{\phi_t^j}{\phi_t^0}X_t^{i+j}\mathrm{d}t - \phi_t^0 X_t^i \mathrm{d}t + \sigma \mathrm{d}W_t^i \\ &= \phi_t^0 \cdot \big(\sum_{j=1}^{\infty}\tfrac{-\phi_t^j}{\phi_t^0}X_t^{i+j} - X_t^i\big)\mathrm{d}t + \sigma \mathrm{d}W_t^i, \end{aligned}$$

which shows that this is a mean-reverting type process, since $\phi_t^0 > 0$. We also see that this system is invariant under the shift of indices of individuals. In particular, the law of X^i is the same as the law of X^1 for every i and also X^i is independent of $(W^1, \cdots, W^{i-1})$.

4 Catalan Markov Chain

In order to simplify our analysis, we look at the stationary solution $\{\phi^k, k \geq 0\}$ of the Riccati system (13) in Section 3. Without loss of generality, we assume $\epsilon = 1$. By taking $T \to \infty$, we obtain the stationary long-time behavior satisfying $\dot{\phi}^k_{\cdot} = 0$ for all k. Then, (13) gives the recurrence relation: $\phi^0 = 1$ and $\sum_{k=0}^{n}\phi^k\phi^{n-k} = 0$ for every $n \geq 0$. This is closely related to the recurrence relation of Catalan numbers. By using a moment generating function method as in Appendix A.1, we get the stationary solutions (that we call Catalan functions): $\phi^0 = 1, \phi^1 = -\frac{1}{2}$, $\phi^k = -\dfrac{(2k-3)!}{(k-2)!k!2^{2k-2}}$ for $k \geq 2$.

Let $p_0 = -\phi^0 = 1, p_1 = -\phi^1 = \frac{1}{2}$, and $p_k = -\phi^k = \frac{(2k-3)!}{(k-2)!k!}\,\frac{1}{2^{2k-2}}$ for $k \geq 2$. We consider the continuous-time Markov chain $M(\cdot)$ with state space $\mathbb{N}_0$ and generator matrix

$$\mathbf{Q} = \begin{pmatrix} -1 & p_1 & p_2 & p_3 & \cdots \\ 0 & -1 & p_1 & p_2 & \ddots \\ 0 & 0 & -1 & p_1 & \ddots \\ & \ddots & \ddots & \ddots & \ddots \end{pmatrix}.$$

The infinite particle system (11) can be represented as a stochastic evolution equation:

$$\mathrm{d}\mathbf{X}_t = \mathbf{Q}\,\mathbf{X}_t\mathrm{d}t + \mathrm{d}\mathbf{W}_t, \tag{14}$$

where $\mathbf{X}_{\cdot} = (X_{\cdot,k}, k \in \mathbb{N}_0)$ with $\mathbf{X}_0 = \mathbf{x}_0$ and $\mathbf{W}_{\cdot} = (W_{\cdot,k}.k \in \mathbb{N}_0)$. By Itô's formula we have

$$\mathrm{d}\big(\int_0^t e^{(t-s)\mathbf{Q}}\mathrm{d}\mathbf{W}_s\big) \;=\; \big(\mathbf{Q}\int_0^t e^{(t-s)\mathbf{Q}}\mathrm{d}\mathbf{W}_s\big)\mathrm{d}t + \mathrm{d}\mathbf{W}_s; \quad t \geq 0, \tag{15}$$

and thus, the solution to (14) is:

$$\mathbf{X}_t = \mathbf{x_0} + \int_0^t e^{(t-s)\mathbf{Q}}\mathrm{d}\mathbf{W}_s; \quad t \geq 0. \tag{16}$$

Note that the transition probabilities of the continuous-time Markov chain $M(\cdot)$ are: $p_{i,k}(t) = \mathbb{P}(M(t) = k|M(0) = i) = (e^{t\mathbf{Q}})_{i,k}$, $i, k \in \mathbb{N}_0$, $t \geq 0$. Without loss of generality, let us assume $\mathbf{X}_0 = \mathbf{0}$. Then,

$$\begin{aligned} \mathbf{X}_t &= \int_0^t \sum_{k=0}^{\infty} p_{0,k}(t-s)\mathrm{d}W_{s,k} \\ &= \int_0^t \sum_{k=0}^{\infty} \mathbb{P}(M(t-s) = k|M(0) = 0)\mathrm{d}W_{s,k} \\ &= \mathbb{E}^M\big[\int_0^t \sum_{k=0}^{\infty} \mathbf{1}_{(M(t-s)=k)}\mathrm{d}W_{s,k}|M(0) = 0\big]; \quad t \geq 0, \end{aligned} \tag{17}$$

where the expectation is taken with respect to the probability induced by the Markov chain $M(\cdot)$, independent of the Brownian motions $(W_{\cdot,k},\ k \in \mathbb{N}_0)$. Therefore, we obtained a Feynman–Kac representation formula for the generator $\mathbf{Q}$.

Proposition 1. *The Gaussian process* $X_j(t)\,,\ j \in \mathbb{N}_0\,,\ t \geq 0$*, corresponding to the Catalan Markov chain, is*

$$X_j(t) \ := \sum_{k=0}^{\infty} \int_0^t (\exp(Q(t-s)))_{j,k} \mathrm{d}W_k(s) \ = \sum_{k=j}^{\infty} \int_0^t \frac{(t-s)^{2(k-j)}}{(k-j)!} \cdot F^{(k-j)}(-(t-s)^2) \mathrm{d}W_k(s)$$

$$= \sum_{k=j}^{\infty} \int_0^t \frac{(t-s)^{2(k-j)}}{(k-j)!} \cdot \rho_{k-j}(-(t-s)^2)\, e^{-(t-s)} \cdot \mathrm{d}W_k(s), \tag{18}$$

where $W_k(\cdot)\,,\, k \in \mathbb{N}_0$ *are independent standard Brownian motions and* $\rho_k(x) = \frac{1}{2^k} \sum_{j=k}^{2k-1} \frac{(j-1)!}{(2j-2k)!!(2k-j-1)!} (-x)^{-\frac{j}{2}}$, *for* $k \geq 1$.

Proof. Given in **Appendix** A.2. ☐

4.1 Asymptotic Behavior of the Variances as $t \to \infty$

For $t \geq 0$, we have:

$$\mathrm{Var}(X_0(t)) \ = \ \mathrm{Var}\Big(\sum_{k=0}^{\infty} \int_0^t \frac{(t-s)^{2k}}{k!} F^{(k)}(-(t-s)^2) \mathrm{d}W_k(s)\Big)$$

$$= \sum_{k=0}^{\infty} \int_0^t \frac{(t-s)^{4k}}{(k!)^2} |\rho_k(-(t-s)^2)|^2 e^{-2(t-s)} \mathrm{d}s.$$

Remark 3. *To evaluate the variance, we need some estimates of* $\rho_k(\cdot)\,,\ k \in \mathbb{N}_0$. *It can be shown that*

$$\rho_k(-\nu^2) = \frac{1}{2^k \nu^k} \cdot \sqrt{\frac{2\nu}{\pi}} \cdot e^{\nu} \cdot K_{k-(1/2)}(\nu)\,; \quad k \geq 1\,,$$

where $K_n(x)$ *(*$= \int_0^{\infty} e^{-x \cosh t} \cosh(nt) \mathrm{d}t\,; \quad n > -1, \ \ x > 0$*) is the modified Bessel function of the second kind. Then* $\mathrm{Var}(X_0(t)) \ = \ \sum_{k=1}^{\infty} \int_0^t \frac{2}{\pi} \frac{\nu^{2k+1}}{(k!)^2\, 4^k} \big(K_{k-(1/2)}(\nu)\big)^2 \mathrm{d}\nu + \frac{1-e^{-2t}}{2}$ *for* $t \geq 0$. *Details are given in the Appendix A.3.*

Proposition 2. *The asymptotic variance is finite, i.e.,* $\lim_{t\to\infty} \mathrm{Var}(X_0(t)) \ = \ \frac{1}{\sqrt{2}} < \infty$.

Proof. Given in Appendix A.4. ☐

4.2 Asymptotic Independence

With $X_0 = 0$, it follows from Proposition 1 and Remark 3 that:

$$X_j(t) = \sum_{i=0}^{\infty} \int_0^t \frac{1}{\sqrt{\pi} i!} \frac{(t-s)^{i+1/2}}{2^{i-1/2}} K_{i-1/2}(t-s) \mathrm{d}W_{j+i}(s). \tag{19}$$

Then the auto-covariance and cross-covariance are given by:

$$\mathbb{E}[X_0(s)X_0(t)] = \sum_{k=0}^{\infty} \int_0^s \frac{1}{\pi (k!)^2 2^{2k-1}} ((t-s+\alpha)\alpha)^{k+1/2} K_{k-1/2}(t-s+\alpha) K_{k-1/2}(\alpha) \mathrm{d}\alpha, \quad 0 \leq s \leq t \tag{20}$$

$$\mathbb{E}[X_0(t)X_k(t)] = \sum_{j=0}^{\infty} \frac{1}{\pi (k+j)! j!} \frac{1}{2^{k+2j-1}} \int_0^t s^{k+2j+1} K_{k+j-1/2}(s) K_{j-1/2}(s) \mathrm{d}s, \quad t \geq 0. \tag{21}$$

The following propositions give two results about these covariances and the details of the proofs are given in Appendix A.5.

Proposition 3 (Asymptotic behavior of the auto-covariance)**.** *According to equation (20), the auto-covariance* $\mathbb{E}[X_0(s)X_0(t)]$ *is positive since* $K_n(x) > 0$*. Fixing* $s > 0$*, when* $t - s \to \infty$*, the auto-covariance does not converge to* 0*, i.e. the process has no stationary distribution.*

Proposition 4 (Asymptotic behavior of the cross-covariance)**.** *Similarly, the cross-covariance* $\mathbb{E}[X_0(t)X_k(t)]$ *is positive for any* $t > 0$ *and*

$$0 < \lim_{t\to\infty} \mathbb{E}[X_0(t)X_k(t)] = \sum_{j=0}^{\infty} \frac{1}{\pi(k+j)!j!} \frac{1}{2^{k+2j-1}} \int_0^{\infty} s^{k+2j+1} K_{k+j-1/2}(s) K_{j-1/2}(s) \mathrm{d}s < \frac{1}{\sqrt{2}}.$$

The asymptotic cross-covariance is positive and bounded above, which means the states are asymptoticaly dependent in the directed chain game.

5 Mixture of Directed Chain and Mean Field Interaction on a Finite-player System

In the spirit of the paper, we want to look at the game on a mixed system, including the directed chain interaction and the mean field interaction for finite players. This section repeats the same steps as before to analyse the mixed system game. We assume the state dynamics of all payers are of the form:

$$\mathrm{d}X_t^i = \alpha_t^i \mathrm{d}t + \sigma \mathrm{d}W_t^i,$$

as in the previous sections. In this model, player i chooses its own strategy α^i in order to minimize its objective function of the mixed form:

$$i \le N-1: \qquad J^i(\alpha^1, \cdots, \alpha^N) = \mathbb{E}\Big\{ \int_0^T \big(\frac{1}{2}(\alpha_t^i)^2 \quad + u\tfrac{\epsilon}{2}(X_t^{i+1} - X_t^i)^2 + (1-u)\tfrac{\epsilon}{2}(\bar{X}_t - X_t^i)^2\big)\mathrm{d}t \\ + u\tfrac{c}{2}(X_T^{i+1} - X_T^i)^2 + (1-u)\tfrac{c}{2}(\bar{X}_T - X_T^i)^2 \Big\}, \tag{22}$$

for some positive constants ϵ, c and $u \in [0,1]$. The notation $\bar{X}_t$ is defined as the empirical mean, i.e., $\bar{X}_t = \frac{1}{N}\sum_{i=1}^{N} X_t^i$. The running cost function is defined by $f^i(x, \alpha^i) = \frac{1}{2}(\alpha^i)^2 + u\frac{\epsilon}{2}(x^{i+1} - x^i)^2 + (1-u)\frac{\epsilon}{2}(\bar{x} - x^i)^2$ and the terminal cost function is defined by $g^i(x) = u\frac{c}{2}(x^{i+1} - x^i)^2 + (1-u)\frac{c}{2}(\bar{x} - x^i)^2$. The system is completed by describing the behavior of player N. For simplicity, we consider the boundary condition of the system where X^N is attracted to 0. Then we can compare the result with that of Section 2.2.2. The cost functional for player N is given by:

$$J^N(\alpha^N) = \mathbb{E}\Big\{ \int_0^T \big(\frac{1}{2}(\alpha_t^N)^2 \quad + u\tfrac{\epsilon}{2}(X_t^N)^2 + (1-u)\tfrac{\epsilon}{2}(\bar{X}_t - X_t^N)^2\big)\mathrm{d}t \\ + u\tfrac{c}{2}(X_T^N)^2 + (1-u)\tfrac{c}{2}(\bar{X}_T - X_T^N)^2 \Big\}. \tag{23}$$

The running cost function is defined by $f^N(x, \alpha^N) = \frac{1}{2}(\alpha^N)^2 + u\frac{\epsilon}{2}(x^N)^2 + +(1-u)\frac{\epsilon}{2}(\bar{x} - x^N)^2$ and the terminal cost function is defined by $g^N(x) = u\frac{c}{2}(x^N)^2 + (1-u)\frac{c}{2}(\bar{x} - x^N)^2$. If $u = 1$, the system becomes the directed chain system discussed before. If $u = 0$, it becomes a mean-field system where each player is attracted towards the mean of the system.

5.1 Open-Loop Nash Equilibrium

We search for an open-loop Nash equilibrium of the system among strategies $\{\alpha_t^i, i = 1, \cdots, N\}$. The Hamiltonian for player i is given by:

$$H^i(x^1, \cdots, x^N, y^{i,1}, \cdots, y^{i,N}, \alpha^1, \cdots, \alpha^N) = \sum_{k=1}^{N} \alpha^k y^{i,k} + \frac{1}{2}(\alpha^i)^2 + u\frac{\epsilon}{2}(x^{i+1} - x^i)^2 + (1-u)\frac{\epsilon}{2}(\bar{x} - x^i)^2,$$

and the Hamiltonian for player N is given by:

$$H^N(x^1,\cdots,x^N,y^{i,1},\cdots,y^{i,N},\alpha^1,\cdots,\alpha^N)=\sum_{k=1}^{N}\alpha^k y^{i,k}+\frac{1}{2}(\alpha^i)^2+u\frac{\epsilon}{2}(x^N)^2+(1-u)\frac{\epsilon}{2}(\bar{x}-x^i)^2.$$

The value of α^i minimizing the Hamiltonian with respect to α^i is given by:

$$\partial_{\alpha^i}H^i=y^{i,i}+\alpha^i=0 \quad \text{leading to the choice:} \quad \hat{\alpha}^i=-y^{i,i}.$$

The adjoint processes $Y_t^i=(Y_t^{i,j};j=1,\cdots,N)$ and $Z_t^i=(Z_t^{i,j,k};j=1,\cdots,N,k=0,\cdots,N)$ for $i=1,\cdots,N$ are defined as the solutions of the backward stochastic differential equations (BSDEs):

$$\begin{aligned}
i<N:&\quad \begin{cases}
\mathrm{d}Y_t^{i,j} &= -\partial_{x^j}H^i(X_t,Y_t^i,\alpha_t)\mathrm{d}t+\sum\limits_{k=0}^{N}Z_t^{i,j,k}\mathrm{d}W_t^k\\
&= -\big\{u\epsilon(X_t^{i+1}-X_t^i)(\delta_{i+1,j}-\delta_{i,j})+(1-u)\epsilon(\bar{X}_t-X_t^i)(\frac{1}{N}-\delta_{i,j})\big\}\mathrm{d}t+\sum\limits_{k=0}^{N}Z_t^{i,j,k}\mathrm{d}W_t^k,\\
Y_T^{i,j} &= \partial_{x^j}g_i(X_T)=uc(X_T^{i+1}-X_T^i)(\delta_{i+1,j}-\delta_{i,j})+(1-u)c(\bar{X}_T-X_T^i)(\frac{1}{N}-\delta_{i,j}).
\end{cases}\\
i=N:&\quad \begin{cases}
\mathrm{d}Y_t^{N,j} &= -\big\{u\epsilon X_t^N\delta_{N,j}+(1-u)\epsilon(\bar{X}_t-X_t^N)(\frac{1}{N}-\delta_{N,j})\big\}\mathrm{d}t+\sum\limits_{k=0}^{N}Z_t^{N,j,k}\mathrm{d}W_t^k,\\
Y_T^{N,j} &= ucX_T^N\delta_{N,j}+(1-u)c(\bar{X}_T-X_T^N)(\frac{1}{N}-\delta_{N,j}).
\end{cases}
\end{aligned} \tag{24}$$

When $j=i$, it becomes:

$$\begin{cases}
\mathrm{d}Y_t^{i,i} &= \big\{u\epsilon(X_t^{i+1}-X_t^i)+(1-u)\epsilon(\bar{X}_t-X_t^i)(1-\frac{1}{N})\big\}\mathrm{d}t+\sum\limits_{k=0}^{N}Z_t^{i,i,k}\mathrm{d}W_t^k,\\
Y_T^{i,i} &= -uc(X_T^{i+1}-X_T^i)-(1-u)c(\bar{X}_T-X_T^i)(1-\frac{1}{N}), \quad i<N\\
\mathrm{d}Y_t^{N,N} &= \big\{-u\epsilon X_t^N+(1-u)\epsilon(\bar{X}_t-X_t^N)(1-\frac{1}{N})\big\}\mathrm{d}t+\sum\limits_{k=0}^{N}Z_t^{N,N,k}\mathrm{d}W_t^k,\\
Y_T^{N,N} &= ucX_T^N-(1-u)c(\bar{X}_T-X_T^N)(1-\frac{1}{N}).
\end{cases} \tag{25}$$

Considering the BSDE system and the initial condition, we then make the following ansatz with function parameters depending on N:

$$Y_t^{i,i}=u\sum_{j=i}^{N}\phi_t^{N,i,j}X_t^j-(1-u)(\bar{X}_t-X_t^i)\theta_t^N, \tag{26}$$

for some deterministic scalar functions ϕ_t,θ_t satisfying the terminal condition: when $i<N$, $\phi_T^{N,i,i}=c,\phi_T^{N,i,i+1}=-c,\phi_T^{N,i,j}=0$ for $N\geq j\geq i+2$; $\phi_T^{N,N,N}=c$ and $\theta_T^N=c(1-\frac{1}{N})$. For simplicity of notation, we denote $\theta_t=\theta_t^N$. Using the ansatz, the optimal strategy and forward equation become:

$$\begin{cases}
\hat{\alpha}^i=-Y_t^{i,i}=-u\sum\limits_{j=i}^{N}\phi_t^{N,i,j}X_t^j+(1-u)(\bar{X}_t-X_t^i)\theta_t,\\
\mathrm{d}X_t^j=\big[-u\sum\limits_{k=j}^{N}\phi_t^{N,j,k}X_t^k+(1-u)(\bar{X}_t-X_t^j)\theta_t\mathrm{d}t\big]+\sigma\mathrm{d}W_t^j.
\end{cases}$$

By summation, we can get:

$$\begin{aligned}
\mathrm{d}\bar{X}_t &= -u(\tfrac{1}{N}\sum_{j=1}^{N}\sum_{k=j}^{N}\phi_t^{N,j,k}X_t^k)\mathrm{d}t+\sigma(\tfrac{1}{N}\sum_{j=1}^{N}\mathrm{d}W_t^j)\\
&= -u\tfrac{1}{N}\sum_{k=1}^{N}(\sum_{j=1}^{k}\phi_t^{N,j,k})X_t^k\mathrm{d}t+\sigma(\tfrac{1}{N}\sum_{k=1}^{N}\mathrm{d}W_t^k).
\end{aligned}$$

Consequently, one obtains:

$$
\begin{aligned}
\mathrm{d}(\bar{X}_t - X_t^i) &= -u\tfrac{1}{N}\sum_{k=1}^{N}(\sum_{j=1}^{k}\phi_t^{N,j,k})X_t^k\mathrm{d}t + \sigma(\tfrac{1}{N}\sum_{k=1}^{N}\mathrm{d}W_t^k) \\
&\quad + u\sum_{k=i}^{N}\phi_t^{N,i,k}X_t^k - (1-u)(\bar{X}_t - X_t^i)\theta_t\mathrm{d}t - \sigma\mathrm{d}W_t^i \\
&= -u\tfrac{1}{N}\sum_{k=1}^{i-1}(\sum_{j=1}^{k}\phi_t^{N,j,k})X_t^k\mathrm{d}t + u\sum_{k=i+1}^{N}(\phi_t^{N,i,k} - \tfrac{1}{N}\sum_{j=1}^{k}\phi_t^{N,j,k})X_t^k\mathrm{d}t \\
&\quad + \big(u\phi_t^{N,i,i} - u\tfrac{1}{N}\sum_{j=1}^{i}\phi_t^{N,j,i} + (1-u)\theta_t\big)X_t^i\mathrm{d}t - (1-u)\bar{X}_t\theta_t\mathrm{d}t \\
&\quad + \sigma(\tfrac{1}{N}\sum_{k=1}^{N}\mathrm{d}W_t^k - ud W_t^i).
\end{aligned}
\tag{27}
$$

Differentiating the ansatz (26) and using (27), we get:

$$
\begin{aligned}
\mathrm{d}Y_t^{i,i} &= u\cdot\sum_{j=i}^{N}[X_t^j\dot{\phi}_t^{N,i,j}\mathrm{d}t + \phi_t^{N,i,j}dX_t^j] - (1-u)\cdot\Big(\dot{\theta}_t(\bar{X}_t - X_t^i)\mathrm{d}t + \theta_t\mathrm{d}(\bar{X}_t - X_t^i)\Big) \\
&\overset{\text{def}}{=} u\cdot I - (1-u)\cdot II.
\end{aligned}
\tag{28}
$$

First,

$$
\begin{aligned}
I &= \sum_{j=i}^{N}[X_t^j\dot{\phi}_t^{N,i,j}\mathrm{d}t + \phi_t^{N,i,j}\mathrm{d}X_t^j] \\
&= \sum_{j=i}^{N}X_t^j\dot{\phi}_t^{N,i,j}\mathrm{d}t + \sum_{j=i}^{N}\phi_t^{N,i,j}\cdot\big\{\big[-u\sum_{k=j}^{N}\phi_t^{N,j,k}X_t^k + (1-u)(\bar{X}_t - X_t^j)\theta_t\mathrm{d}t\big] + \sigma\mathrm{d}W_t^j\big\} \\
&= \sum_{k=i}^{N}X_t^k\dot{\phi}_t^{N,i,k}\mathrm{d}t - u\sum_{j=i}^{N}\sum_{k=j}^{N}\phi_t^{N,i,j}\phi_t^{N,j,k}X_t^k\mathrm{d}t + (1-u)\theta_t\sum_{k=i}^{N}\phi_t^{N,i,k}(\bar{X}_t - X_t^k)\mathrm{d}t + \sigma\sum_{k=i}^{N}\phi_t^{N,i,k}\mathrm{d}W_t^k \\
&= \sum_{k=i}^{N}\big(\dot{\phi}_t^{N,i,k} - u\sum_{j=i}^{k}\phi_t^{N,i,j}\phi_t^{N,j,k} - (1-u)\theta_t\phi_t^{N,i,k}\big)X_t^k\mathrm{d}t \\
&\quad + (1-u)\theta_t\sum_{k=i}^{N}\phi_t^{N,i,k}\cdot\bar{X}_tdt + \sigma\sum_{k=i}^{N}\phi_t^{N,i,k}dW_t^k.
\end{aligned}
\tag{29}
$$

Then,

$$
\begin{aligned}
II &= \dot{\theta}_t(\bar{X}_t - X_t^i)\mathrm{d}t + \theta_t\mathrm{d}(\bar{X}_t - X_t^i) \\
&= \dot{\theta}_t(\bar{X}_t - X_t^i)\mathrm{d}t \\
&\quad + \theta_t\cdot\big\{-u\tfrac{1}{N}\sum_{k=1}^{i-1}(\sum_{j=1}^{k}\phi_t^{N,j,k})X_t^k\mathrm{d}t + u\sum_{k=i+1}^{N}(\phi_t^{N,i,k} - \tfrac{1}{N}\sum_{j=1}^{k}\phi_t^{N,j,k})X_t^k\mathrm{d}t \\
&\quad + \big(u\phi_t^{N,i,i} - u\tfrac{1}{N}\sum_{j=1}^{i}\phi_t^{N,j,i} + (1-u)\theta_t\big)X_t^i\mathrm{d}t - (1-u)\bar{X}_t\theta_t\mathrm{d}t \\
&\quad + \sigma(\tfrac{1}{N}\sum_{k=1}^{N}\mathrm{d}W_t^k - \mathrm{d}W_t^i)\big\} \\
&= -u\theta_t\tfrac{1}{N}\sum_{k=1}^{i-1}(\sum_{j=1}^{k}\phi_t^{N,j,k})X_t^kdt + u\theta_t\sum_{k=i+1}^{N}(\phi_t^{N,i,k} - \tfrac{1}{N}\sum_{j=1}^{k}\phi_t^{N,j,k})X_t^kdt \\
&\quad - [\dot{\theta}_t - u\theta_t(\phi_t^{N,i,i} - \tfrac{1}{N}\sum_{j=1}^{i}\phi_t^{N,j,i}) - (1-u)\theta_t^2]X_t^idt \\
&\quad + (\dot{\theta}_t - (1-u)\theta_t^2)\bar{X}_tdt + \sigma(\tfrac{1}{N}\sum_{k=1}^{N}dW_t^k - dW_t^i).
\end{aligned}
\tag{30}
$$

So the equation (28) can be written as:

$$
\begin{aligned}
\mathrm{d}Y_t^{i,i} &= u\cdot I-(1-u)\cdot II\\
&= u\cdot\Big\{\sum_{k=i}^{N}\big(\dot\phi_t^{N,i,k}-u\sum_{j=i}^{k}\phi_t^{N,i,j}\phi_t^{N,j,k}-(1-u)\theta_t\phi_t^{N,i,k}\big)X_t^k\mathrm{d}t\\
&\quad+(1-u)\theta_t\sum_{k=i}^{N}\phi_t^{N,i,k}\cdot\bar X_t\mathrm{d}t+\sigma\sum_{k=i}^{N}\phi_t^{N,i,k}\mathrm{d}W_t^k\Big\}\\
&\quad-(1-u)\cdot\Big\{-u\theta_t\tfrac{1}{N}\sum_{k=1}^{i-1}(\sum_{j=1}^{k}\phi_t^{N,j,k})X_t^k\mathrm{d}t+u\theta_t\sum_{k=i+1}^{N}(\phi_t^{N,i,k}-\tfrac{1}{N}\sum_{j=1}^{k}\phi_t^{N,j,k})X_t^k\mathrm{d}t\\
&\quad-[\dot\theta_t-u\theta_t(\phi_t^{N,i,i}-\tfrac{1}{N}\sum_{j=1}^{i}\phi_t^{N,j,i})-(1-u)\theta_t^2]X_t^i\mathrm{d}t\\
&\quad+(\dot\theta_t-(1-u)\theta_t^2)\bar X_t\mathrm{d}t+\sigma(\tfrac{1}{N}\sum_{k=1}^{N}\mathrm{d}W_t^k-\mathrm{d}W_t^i)\Big\}\\
&=\sum_{k=1}^{i-1}(u(1-u)\theta_t\tfrac{1}{N}\sum_{j=1}^{k}\phi_t^{N,j,k})X_t^k\mathrm{d}t\\
&\quad+\sum_{k=i+1}^{N}[u\dot\phi_t^{N,i,k}-u^2\sum_{j=i}^{k}\phi_t^{N,i,j}\phi_t^{N,j,k}-u(1-u)\theta_t\phi_t^{N,i,k}-u(1-u)\theta_t(\phi_t^{N,i,k}-\tfrac{1}{N}\sum_{j=1}^{k}\phi_t^{N,j,k})]X_t^k\mathrm{d}t\\
&\quad+[u\dot\phi_t^{N,i,i}-u^2(\phi_t^{N,i,i})^2-2u(1-u)\theta_t\phi_t^{N,i,i}+(1-u)\dot\theta_t+u(1-u)\theta_t\tfrac{1}{N}\sum_{j=1}^{i}\phi_t^{N,j,i}-(1-u)^2\theta_t^2]X_t^i\mathrm{d}t\\
&\quad+[u(1-u)\theta_t\sum_{k=i}^{N}\phi_t^{N,i,k}-(1-u)\dot\theta_t+(1-u)^2\theta_t^2]\bar X_t\mathrm{d}t\\
&\quad+u\sigma\sum_{k=i}^{N}\phi_t^{N,i,k}\mathrm{d}W_t^k-(1-u)\sigma\theta_t(\tfrac{1}{N}\sum_{k=1}^{N}\mathrm{d}W_t^k-dW_t^i)
\end{aligned}
\tag{31}
$$

Now we compare the two Itô's decompositions (25) and (31). The martingale terms give the processes $Z_t^{i,j,k}$:

$$
\begin{aligned}
&Z_t^{i,i,0}=0;\ Z_t^{i,i,k}=-(1-u)\sigma\theta_t\tfrac{1}{N}\quad \text{for } k<i,\\
&Z_t^{i,i,i}=u\sigma\phi_t^{N,i,i}+(1-u)\sigma\theta_t(1-\tfrac{1}{N})\ \text{and}\ Z_t^{i,i,k}=u\sigma\phi_t^{N,i,k}\quad \text{for } k>i.
\end{aligned}
$$

And from the drift terms, we get:
when $i<N$,

$$
\begin{aligned}
\text{for } i:\quad & u\dot\phi_t^{N,i,i}-u^2(\phi_t^{N,i,i})^2-2u(1-u)\theta_t\phi_t^{N,i,i}+(1-u)\dot\theta_t(1-\tfrac{1}{N})-(1-u)^2\theta_t^2(1-\tfrac{1}{N})\\
&\quad+u(1-u)\theta_t\tfrac{1}{N}(\sum_{j=1}^{i}\phi_t^{N,j,i}+\sum_{k=i}^{N}\phi_t^{N,i,k})=-u\epsilon-(1-u)\epsilon(1-\tfrac{1}{N})^2,\quad \phi_T^{N,i,i}=c,\\
\text{for } i+1:\quad & u\dot\phi_t^{N,i,i+1}-u^2(\phi_t^{N,i,i}\phi_t^{N,i,i+1}+\phi_t^{N,i,i+1}\phi_t^{N,i+1,i+1})\\
&\quad-2u(1-u)\theta_t\phi_t^{N,i,i+1}-(1-u)\dot\theta_t\tfrac{1}{N}+(1-u)^2\theta_t^2\tfrac{1}{N}\\
&\quad+u(1-u)\theta_t\tfrac{1}{N}(\sum_{j=1}^{i+1}\phi_t^{N,j,i+1}+\sum_{k=i}^{N}\phi_t^{N,i,k})\\
&=u\epsilon+(1-u)\epsilon(1-\tfrac{1}{N})\tfrac{1}{N},\quad \phi_T^{N,i,i+1}=-c,\\
\text{for } \ell\ge i+2:\quad & u\dot\phi_t^{N,i,\ell}-u^2\sum_{j=i}^{l}\phi_t^{N,i,j}\phi_t^{N,j,\ell}-2u(1-u)\theta_t\phi_t^{N,i,\ell}-(1-u)\dot\theta_t\tfrac{1}{N}+(1-u)^2\theta_t^2\tfrac{1}{N}\\
&\quad+u(1-u)\theta_t\tfrac{1}{N}(\sum_{j=1}^{l}\phi_t^{N,j,\ell}+\sum_{k=i}^{N}\phi_t^{N,i,k})=(1-u)\epsilon(1-\tfrac{1}{N})\tfrac{1}{N},\quad \phi_T^{N,i,\ell}=0,\\
\text{and}\quad & u(1-u)\theta_t\sum_{k=i}^{N}\phi_t^{N,i,k}-(1-u)\dot\theta_t+(1-u)^2\theta_t^2=(1-u)\epsilon(1-\tfrac{1}{N}),\quad \theta_T=c(1-\tfrac{1}{N});
\end{aligned}
\tag{32}
$$

When $i=N$,

$$
\begin{aligned}
& u\dot\phi_t^{N,N,N}-u^2(\phi_t^{N,N,N})^2-2u(1-u)\theta_t\phi_t^{N,N,N}+(1-u)\dot\theta_t(1-\tfrac{1}{N})-(1-u)^2\theta_t^2(1-\tfrac{1}{N})\\
&\quad+u(1-u)\theta_t\tfrac{1}{N}(\sum_{j=1}^{N}\phi_t^{N,j,N}+\phi_t^{N,N,N})=-u\epsilon-(1-u)\epsilon(1-\tfrac{1}{N})^2,\quad \phi_T^{N,i,i}=c,\\
\textit{and}\quad & u(1-u)\theta_t\phi_t^{N,N,N}-(1-u)\dot\theta_t+(1-u)^2\theta_t^2=(1-u)\epsilon(1-\tfrac{1}{N}),\quad \theta_T=c(1-\tfrac{1}{N});
\end{aligned}
\tag{33}
$$

When $u = 1$, the systems (32) and (33) are exactly what we got for finite-player directed chain game in section 2. We have the similar conclusion that the boundary condition does not affect the functions $\phi_t^{N,i,j}$ $(j < N)$ for all $i < N$. We can also compare thesystem (32) with the system (45) - (48). Under suitable assumptions, the system (32) converges as the number of players N goes to infinity.

6 Infinite-Player Game Model with Mean-Field Interaction

Motivated by Section 5 and following Section 3, we can define a game with infinite players on a mixed system, including the directed chain interaction and the mean field interaction. This section searches for an open-loop Nash equilibrium and repeats the same steps as before to analyse the infinite mixed system game. We have a more general Catalan Markov chain and Table 1 below shows the asymptotic behaviors of the variances and covariances as $t \to \infty$ for the process with different types of interactions. Comparing it with Table 1 in Detering, Fouque & Ichiba [**?**], we have similar conclusions except that our asymptotic variance of purely directed chain does not explode.

The game model is given by:

$$\mathrm{d}X_t^i = \alpha_t^i \mathrm{d}t + \sigma \mathrm{d}W_t^i; \quad i = 1, 2, \cdots, \quad 0 \le t \le T, \tag{34}$$

where $(W_t^i)_{0 \le t \le T}$, $i \in \mathbb{N}$ are independent standard Brownian motions. We assume the same drift and diffusion coefficients and the initial conditions as the finite-player game. By choosing α_t^i, player i tries to minimize:

$$J^i(\alpha^1, \alpha^2, \cdots) = \mathbb{E}\Big\{ \int_0^T \big(\frac{1}{2}(\alpha_t^i)^2 \quad + u\tfrac{\epsilon}{2}(X_t^{i+1} - X_t^i)^2 + (1-u)\tfrac{\epsilon}{2}(m_t - X_t^i)^2\big)\mathrm{d}t \\ + u\tfrac{c}{2}(X_T^{i+1} - X_T^i)^2 + (1-u)\tfrac{c}{2}(m_T - X_T^i)^2 \Big\},$$

for some positive constants ϵ, c and $u \in [0, 1]$. Here, there is a real issue on the choice of m_t. Intuitively, it should come from the finite-player mixed game described in Section 5 as the limit of $\bar{X}_t = \frac{1}{N}\sum_{i=1}^{N} X_t^i$ as $N \to \infty$. Combined with the fact that we had $\mathbb{E}\{X_t^i\}$ independent of i, it is natural to set $m_t = \mathbb{E}\{X_t^i\}$ and check afterwards that this mean value does not depend on i de facto after solving the fixed point step. Note that the case $u = 0$ is very particular, and consists in solving the same mean field game problem for every i. The case $u = 1$ has already been studied in Section 3, and therefore, in what follows, we concentrate on the case $u \in (0, 1)$.

6.1 Open-Loop Nash Equilibrium

We search for Nash equilibria of the system among strategies $\{\alpha_t^i, i = 1, 2, \cdots\}$. The Hamiltonian for individual i is given by:

$$H^i(t, x^1, x^2, \cdots, y^{i,1}, y^{i,2}, \cdots, \alpha^1, \alpha^2, \cdots) = \sum_{k=1}^{\infty} \alpha^k y^{i,k} + \frac{1}{2}(\alpha^i)^2 + u\frac{\epsilon}{2}(x^{i+1} - x^i)^2 + (1-u)\frac{\epsilon}{2}(m_t - x^i)^2. \tag{35}$$

The adjoint processes $Y_t^i = (Y_t^{i,j}; j \ge 1)$ and $Z_t^i = (Z_t^{i,j,k}; j \ge 1, k \ge 0)$ for $i = 1, 2, \cdots$ are defined as the solutions of the backward stochastic differential equations (BSDEs):

$$\begin{cases} \mathrm{d}Y_t^{i,j} &= -\big\{u\epsilon(X_t^{i+1} - X_t^i)(\delta_{i+1,j} - \delta_{i,j}) + (1-u)\epsilon(m_t - X_t^i)(-\delta_{i,j})\big\}\mathrm{d}t + \sum_{k=1}^{\infty} Z_t^{i,j,k}\mathrm{d}W_t^k, \\ Y_T^{i,j} &= \partial_{x^j} g_i(X_T) = uc(X_T^{i+1} - X_T^i)(\delta_{i+1,j} - \delta_{i,j}) + (1-u)c(m_T - X_T^i)(-\delta_{i,j}). \end{cases} \tag{36}$$

When $j = i$, it becomes:

$$\begin{cases} \mathrm{d}Y_t^{i,i} &= \big\{u\epsilon(X_t^{i+1} - X_t^i) + (1-u)\epsilon(m_t - X_t^i)\big\}\mathrm{d}t + \sum_{k=1}^{\infty} Z_t^{i,i,k}\mathrm{d}W_t^k, \\ Y_T^{i,i} &= -uc(X_T^{i+1} - X_T^i) - (1-u)c(m_T - X_T^i). \end{cases} \tag{37}$$

According to the Pontryagin stochastic maximum principle, by minimizing the Hamiltonian H^i with respect to α^i, we can get the optimal strategy: $\hat{\alpha}^i = -y^{i,i}$. Then the forward equation becomes:

$$\mathrm{d}X_t^i = -Y_t^{i,i}\mathrm{d}t + \sigma \mathrm{d}W_t^i. \tag{38}$$

Similar as Carmona, Fouque, and Sun [?], we define $m_t^X = \mathbb{E}(X_t)$ and $m_t^Y = \mathbb{E}(Y_t)$. In equilibrium, we have: $m_t^X = m_t$ for $t \leq T$.
Taking expectation in (37), we have: $dm_t^Y = 0$ and $m_T^Y = 0 \Longrightarrow m_t^Y = 0$ for $t \leq T$.
Taking expectation in (38) we get: $\mathrm{d}m_t^X = -m_t^Y\mathrm{d}t = 0$ and $m_0^X = \mathbb{E}(\xi) = 0 \Longrightarrow m_t^X = 0$.
Now we make the ansatz:

$$Y_t^{i,i} = u\sum_{j=i}^{\infty}\phi_t^{j-i}X_t^j - (1-u)(m_t - X_t^i)\psi_t, \tag{39}$$

for some deterministic scalar functions ϕ_t, ψ_t satisfying the terminal condition $\phi_T^0 = c, \phi_T^1 = -c, \phi_T^k = 0$ for $k \geq 2$ and $\psi_T = c$. Using this ansatz, the forward equation (34) becomes

$$\left\{\begin{array}{l} \hat{\alpha}^i = -Y_t^{i,i} = -u\sum_{j=i}^{\infty}\phi_t^{j-i}X_t^j + (1-u)(m_t - X_t^i)\psi_t, \\ \mathrm{d}X_t^i = \big(-u\sum_{j=i}^{\infty}\phi_t^{j-i}X_t^j + (1-u)(m_t - X_t^i)\psi_t\big)\mathrm{d}t + \sigma\mathrm{d}W_t^i. \end{array}\right. \tag{40}$$

Using (40) and $\mathrm{d}m_t = \mathrm{d}m_t^X = 0$, we can differentiate the ansatz (39) to obtain

$$\begin{aligned} \mathrm{d}Y_t^{i,i} &= u\cdot\sum_{j=i}^{\infty}[X_t^j\dot{\phi}_t^{j-i}\mathrm{d}t + \phi_t^{j-i}\mathrm{d}X_t^j] - (1-u)\cdot\Big(\dot{\psi}_t(m_t - X_t^i)\mathrm{d}t + \psi_t\mathrm{d}(m_t - X_t^i)\Big) \\ &\stackrel{\text{def}}{=} u\cdot I - (1-u)\cdot II. \end{aligned} \tag{41}$$

First,

$$\begin{aligned} I &= \sum_{j=i}^{\infty}[X_t^j\dot{\phi}_t^{j-i}\mathrm{d}t + \phi_t^{j-i}\mathrm{d}X_t^j] \\ &= \sum_{j=i}^{\infty}[X_t^j\dot{\phi}_t^{j-i}\mathrm{d}t + \phi_t^{j-i}(-u\sum_{k=j}^{\infty}\phi_t^{k-j}X_t^k + (1-u)(m_t - X_t^j)\psi_t\mathrm{d}t + \sigma\mathrm{d}W_t^j)] \\ &= \sum_{j=i}^{\infty}X_t^j\dot{\phi}_t^{j-i}\mathrm{d}t - u\sum_{j=i}^{\infty}\phi_t^{j-i}\sum_{k=j}^{\infty}\phi_t^{k-j}X_t^k\mathrm{d}t + (1-u)\psi_t\sum_{j=i}^{\infty}\phi_t^{j-i}(m_t - X_t^j)\mathrm{d}t + \sum_{j=i}^{\infty}\sigma\phi_t^{j-i}\mathrm{d}W_t^j \\ &= \sum_{k=0}^{\infty}\dot{\phi}_t^kX_t^{i+k}\mathrm{d}t - u\sum_{k=0}^{\infty}\left(\sum_{j=0}^{k}\phi_t^j\phi_t^{k-j}\right)X_t^{i+k}\mathrm{d}t + (1-u)\psi_t\sum_{k=0}^{\infty}\phi_t^k(m_t - X_t^{i+k})\mathrm{d}t + \sigma\sum_{k=i}^{\infty}\phi_t^{k-i}\mathrm{d}W_t^k \\ &= \sum_{k=0}^{\infty}\left(\dot{\phi}_t^k - u\sum_{j=0}^{k}\phi_t^j\phi_t^{k-j}\right)X_t^{i+k}\mathrm{d}t + (1-u)\psi_t\sum_{k=0}^{\infty}\phi_t^k(m_t - X_t^{i+k})\mathrm{d}t + \sigma\sum_{k=i}^{\infty}\phi_t^{k-i}\mathrm{d}W_t^k. \end{aligned} \tag{42}$$

Then,

$$\begin{aligned} II &= \dot{\psi}_t(m_t - X_t^i)\mathrm{d}t + \psi_t\mathrm{d}(m_t - X_t^i) \\ &= \dot{\psi}_t(m_t - X_t^i)\mathrm{d}t + \psi_t\left(u\sum_{j=i}^{\infty}\phi_t^{j-i}X_t^j - (1-u)(m_t - X_t^i)\psi_t\mathrm{d}t - \sigma\mathrm{d}W_t^i\right) \\ &= u\psi_t\sum_{k=0}^{\infty}\phi_t^kX_t^{i+k}\mathrm{d}t + (\dot{\psi}_t - (1-u)\psi_t^2)(m_t - X_t^i)\mathrm{d}t - \psi_t\sigma\mathrm{d}W_t^i. \end{aligned} \tag{43}$$

Thus, equation (41) can be written as:

$$\begin{aligned}
\mathrm{d}Y_t^{i,i} &= u\cdot I-(1-u)\cdot II\\
&= u\cdot\left\{\sum_{k=0}^{\infty}\left(\dot\phi_t^k-u\sum_{j=0}^{k}\phi_t^j\phi_t^{k-j}\right)X_t^{i+k}\mathrm{d}t+(1-u)\psi_t\sum_{k=0}^{\infty}\phi^k(m_t-X_t^{i+k})\mathrm{d}t+\sigma\sum_{k=i}^{\infty}\phi_t^{k-i}\mathrm{d}W_t^k\right\}\\
&\quad-(1-u)\cdot\left\{u\psi_t\sum_{k=0}^{\infty}\phi_t^kX_t^{i+k}\mathrm{d}t+(\dot\psi_t-(1-u)\psi_t^2)(m_t-X_t^i)\mathrm{d}t-\psi_t\sigma\mathrm{d}W_t^i\right\}\\
&=\sum_{k=0}^{\infty}\left\{u\dot\phi_t^k-u^2\sum_{j=0}^{k}\phi_t^j\phi_t^{k-j}-u(1-u)\psi_t\phi_t^k\right\}X_t^{i+k}\mathrm{d}t\\
&\quad+u(1-u)\psi_t\sum_{k=0}^{\infty}\phi_t^k(m_t-X_t^{i+k})\mathrm{d}t-(1-u)(\dot\psi_t-(1-u)\psi_t^2)(m_t-X_t^i)\mathrm{d}t\\
&\quad+(u\sigma\phi_t^0+(1-u)\sigma\psi_t)\mathrm{d}W_t^i+u\sigma\sum_{k=i+1}^{\infty}\phi_t^{k-i}\mathrm{d}W_t^k\\
&=[u\dot\phi_t^0-u^2(\phi_t^0)^2-2u(1-u)\psi_t\phi_t^0+(1-u)\dot\psi_t-(1-u)^2\psi_t^2]X_t^idt\\
&\quad+\sum_{k=1}^{\infty}[u\dot\phi_t^k-u^2\sum_{j=0}^{k}\phi_t^j\phi_t^{k-j}-2u(1-u)\psi_t\phi_t^k]X_t^{i+k}\mathrm{d}t\\
&\quad+[u(1-u)\psi_t\sum_{k=0}^{\infty}\phi_t^k-(1-u)\dot\psi_t+(1-u)^2\psi_t^2]m_t\mathrm{d}t\\
&\quad+(u\sigma\phi_t^0+(1-u)\sigma\psi_t)\mathrm{d}W_t^i+u\sigma\sum_{k=i+1}^{\infty}\phi_t^{k-i}\mathrm{d}W_t^k.
\end{aligned}\tag{44}$$

Now we compare the two Itô's decompositions (44) and (37). First, the martingale terms give the processes $Z_t^{i,j,k}$:

$$Z_t^{i,i,0}=0;\ Z_t^{i,i,k}=0 \text{ for } k<i,\ Z_t^{i,i,i}=u\sigma\phi_t^0+(1-u)\sigma\psi_t \text{ and } Z_t^{i,i,k}=u\sigma\phi_t^{k-i} \text{ for } k>i.$$

And from the drift terms:

$$\text{for } k=0: u\dot\phi_t^0-u^2(\phi_t^0)^2-2u(1-u)\psi_t\phi_t^0+(1-u)\dot\psi_t-(1-u)^2\psi_t^2=-\epsilon,\qquad \psi_T=c,\ \phi_T^0=c \tag{45}$$

$$\text{for } k=1: u\dot\phi_t^1-2u^2\phi_t^0\phi_t^1-2u(1-u)\psi_t\phi_t^1=u\epsilon,\qquad \phi_T^1=-c \tag{46}$$

$$\text{for } k\geq 2: u\dot\phi_t^k-u^2\sum_{j=0}^{k}\phi_t^j\phi_t^{k-j}-2u(1-u)\psi_t\phi_t^k=0,\qquad \phi_T^k=0 \tag{47}$$

$$\text{and}\quad u(1-u)\psi_t\sum_{k=0}^{\infty}\phi_t^k-(1-u)\dot\psi_t+(1-u)^2\psi_t^2=(1-u)\epsilon,\qquad \psi_T=c. \tag{48}$$

In Appendix A.6 we show the following result which simplifies equation (48) considerably.

Proposition 5. $\sum_{k=0}^{\infty}\phi_t^k=0.$

Using Proposition 5 and $0<u<1$, we can simplify the equations (45) to (48):

$$\begin{array}{lll}
 & \dot\psi_t=(1-u)\psi_t^2-\epsilon, & \psi_T=c \textit{ (Riccati)},\\
\text{for } k=0: & \dot\phi_t^0=u\phi_t^0\cdot\phi_t^0+2(1-u)\psi_t\phi_t^0-\epsilon, & \phi_T^0=c \textit{ (Riccati)},\\
\text{for } k=1: & \dot\phi_t^1=2u\phi_t^0\cdot\phi_t^1+2(1-u)\psi_t\phi_t^1+\epsilon, & \phi_T^1=-c,\\
\text{for } k\geq 2: & \dot\phi_t^k=u(\phi_t^0\cdot\phi_t^k+\phi_t^1\cdot\phi_t^{k-1}+\cdots+\phi_t^{k-1}\cdot\phi_t^1+\phi_t^k\cdot\phi_t^0)+2(1-u)\psi_t\phi_t^k, & \phi_T^k=0.
\end{array}\tag{49}$$

Looking at the stationary solution (in the limit ($T\to\infty$), and without loss of generality assuming $\epsilon=1$, the recurrence relation can be solved by the method of moment generating function to obtain:

$$\left\{\begin{array}{l}
\psi=\sqrt{\frac{1}{1-u}},\\
\phi^0=\frac{1-\sqrt{1-u}}{u},\\
\phi^1=-\frac{1}{2},\\
\phi^k=-\dfrac{(2k-3)!}{(k-2)!k!2^{2k-2}}u^{k-1},\quad for\, k\geq 2.
\end{array}\right.$$

6.2 Catalan Markov Chain for the Mixed Model

As in Section 4, we consider a continuous-time Markov chain $M^{(u)}(\cdot)$ in the state space $\mathbb{N}_0$ with generator matrix

$$\mathbf{Q}^{(u)} = \begin{pmatrix} -1 & q_1 & q_2 & q_3 & \cdots \\ 0 & -1 & q_1 & q_2 & \ddots \\ 0 & 0 & -1 & q_1 & \ddots \\ & \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$ where $q_1 = \frac{u}{2}$, $q_k = \frac{(2k-3)!}{(k-2)!k!}\frac{1}{2^{2k-2}}u^k (= -u\phi^k = p_k u^k)$ for $k \geq 2$.

For $0 < u < 1$, $\mathbf{Q}^{(u)}$ is the generator of the Markov chain with jump rate $1 - \sqrt{1-u}$ from i and killed with probability $1 - \sum_{k=1}^{\infty} q_k = \sqrt{1-u}$. The infinite particle system (40) can be represented as the infinite-dimensional stochastic evolution equation:

$$\mathrm{d}\mathbf{X}^{(\mathbf{u})}_{\mathbf{t}} = \mathbf{Q}^{(\mathbf{u})}\,\mathbf{X}^{(\mathbf{u})}_{\mathbf{t}}\mathrm{d}t + \mathrm{d}\mathbf{W}_t, \tag{50}$$

where $\mathbf{X}^{(\mathbf{u})}_{\cdot} = (X^{(u)}_{\cdot,k}, k \in \mathbb{N}_0)$ with $\mathbf{X}^{(\mathbf{u})}{}_0 = \mathbf{x}^{(\mathbf{u})}_{\mathbf{0}}$ and $\mathbf{W}_{\cdot} = (W_{\cdot,k}.k \in \mathbb{N}_0)$. The solution is:

$$\mathbf{X}^{(\mathbf{u})}_{\mathbf{t}} = \mathbf{x}^{(\mathbf{u})}_{\mathbf{0}} + \int_0^t e^{(t-s)\mathbf{Q}^{(\mathbf{u})}}\mathrm{d}\mathbf{W}_s; \quad t \geq 0. \tag{51}$$

Note that the transition probabilities of the continuous-time Markov chain $M^{(u)}(\cdot)$ is : $p_{i,k}(t) = \mathbb{P}(M^{(u)}(t) = k | M^{(u)}(0) = i) = (e^{t\mathbf{Q}^{(\mathbf{u})}})_{i,k}$, $i, k \in \mathbb{N}_0$, $t \geq 0$. Without loss of generality, assume $\mathbf{x}^{(\mathbf{u})}_{\mathbf{0}} = \mathbf{0}$. Then,

$$\begin{aligned} \mathbf{X}^{(u)}_t &= \int_0^t \sum_{k=0}^{\infty} p_{0,k}(t-s)\mathrm{d}W_{s,k} \\ &= \int_0^t \sum_{k=0}^{\infty} \mathbb{P}(M(t-s) = k | M(0) = 0)\mathrm{d}W_{s,k} \\ &= \mathbb{E}^M\Big[\int_0^t \sum_{k=0}^{\infty} \mathbf{1}_{(M(t-s)=k)}\mathrm{d}W_{s,k} | M(0) = 0\Big]; \quad t \geq 0, \end{aligned} \tag{52}$$

where the expectation is taken with respect to the probability induced by the Markov chain $M^{(u)}(\cdot)$, independent of the Brownian motions $(W_{\cdot,k},\, k \in \mathbb{N}_0)$. Therefore, we have a Feynman–Kac representation formula for the generator $\mathbf{Q}^{(\mathbf{u})}$.

Since $\sum_{i=1}^{k-1} q_i q_{k-i} = u^2 \sum_{i=1}^{k-1} \phi_t^{(i)}\phi_t^{(k-i)} = -2u\phi_t^{(k)} = 2q_k$ we have $(\mathbf{Q}^{(\mathbf{u})})^2 = I - uB$ with B having 1's on the upper second diagonal and 0's elsewhere, i.e.,

$$(\mathbf{Q}^{(\mathbf{u})})^2 = \begin{pmatrix} 1 & -u & 0 & \cdots & \\ 0 & 1 & -u & \ddots & \\ & \ddots & \ddots & \ddots & \end{pmatrix}$$

The matrix exponential of $\mathbf{Q}^{(\mathbf{u})}t$, $t \geq 0$ is written formally as

$$\exp(\mathbf{Q}^{(\mathbf{u})}t) = F(-(\mathbf{Q}^{(\mathbf{u})})^2 t^2), \quad t \geq 0, \quad F(x) := \exp(-\sqrt{-x}),\ x \in \mathbb{C}.$$

Since F is smooth, one can write

$$\exp(\mathbf{Q}^{(\mathbf{u})}t) = F((-I + uB)t^2) = \sum_{k=0}^{\infty} \frac{F^{(k)}(-t^2)}{k!}(uBt^2)^k = \sum_{k=0}^{\infty} \frac{u^k t^{2k} F^{(k)}(-t^2)}{k!} B^k.$$

The (j,k)-element of $\exp(\mathbf{Q}t)$ is formally given by

$$(\exp(\mathbf{Q}^{(\mathbf{u})}t))_{j,k} = \frac{u^{k-j}t^{2(k-j)} \cdot F^{(k-j)}(-t^2)}{(k-j)!}, \quad j \leq k, \quad \text{where } F^{(k)}(x) := \frac{\mathrm{d}^k F}{\mathrm{d}x^k}(x); \quad x > 0,\ k \in \mathbb{N},$$

and $(\exp(\mathbf{Q}^{(\mathbf{u})}t))_{j,k} = 0\,,\ j > k$ for $t \geq 0\,.$

As in Section 4, we have the same solution for $F^{(k)}(x)$: $F^{(k)}(x) = \rho_k(x)e^{-\sqrt{-x}}$, where $\rho_0(x) = 1,\ \rho_k(x) = \frac{1}{2^k}\sum_{j=k}^{2k-1} \frac{(j-1)!}{(2j-2k)!!(2k-j-1)!}\,(-x)^{-\frac{j}{2}}$.

Proposition 6. *The Gaussian process $X_j^{(u)}(t),\ i \in \mathbb{N}_0,\ t \geq 0$, corresponding to the (Catalan) general Markov chain, is*

$$X_j^{(u)}(t) \,:=\, \sum_{k=0}^{\infty}\int_0^t (\exp(\mathbf{Q}^{(\mathbf{u})}(t-s)))_{j,k}\mathrm{d}W_k(s) \,=\, \sum_{k=j}^{\infty}\int_0^t \frac{u^{k-j}(t-s)^{2(k-j)}}{(k-j)!}\cdot F^{(k-j)}(-(t-s)^2)\mathrm{d}W_k(s)$$

$$= \sum_{k=j}^{\infty}\int_0^t \frac{u^{k-j}(t-s)^{2(k-j)}}{(k-j)!}\cdot \rho_{k-j}(-(t-s)^2)\,e^{-(t-s)}\cdot \mathrm{d}W_k(s), \tag{53}$$

where $W_k(\cdot),\ k \in \mathbb{N}_0$ are independent standard Brownian motions.

6.3 Asymptotic Behavior

Table 1 exhibits the asymptotic behaviors of their variances and covariances as $t \to \infty$. The calculation is given in Appendix A.7. We find that only when $u = 0$ (i.e. pure mean field game), the asymptotic cross-covariance is zero, which means the states are asymptotically independent. Otherwise, they are dependent and their covariance is finite. Note that in the purely nearest neighbor interaction studied in Detering, Fouque, and Ichiba [**?**], in the case $u = 0$, the variance does not stabilized as in our "Catalan" interaction equilibrium dynamics.

u	Interaction Type	Asymptotic Variance	Asymptotic Independence
$u = 0$	Purely mean-field	Stabilized	Independent
$u \in (0, 1)$	Mixed interaction	Stabilized	Dependent
$u = 1$	Purely directed chain	Stabilized	Dependent

Table 1: Asymptotic behaviors as $t \to \infty$

7 Periodic Directed Chain Game

We consider a stochastic game with finite players on a periodic ring structure. We assume the dynamics of the states of the individual players are given by N stochastic differential equations of the form:

$$\mathrm{d}X_t^i = \alpha_t^i\mathrm{d}t + \sigma\mathrm{d}W_t^i, \quad i = 1, \cdots, N, \quad 0 \leq t \leq T, \tag{54}$$

where $(W_t^i)_{0\leq t\leq T}$, $i = 1, \cdots, N$ are one-dimensional independent standard Brownian motions. The drift coefficient function, the diffusion coefficient and the initial conditions are assumed to be the same as those in Section 2. In this model, player i chooses its own strategy α^i in order to minimize its objective function of the form:

$$J^i(\alpha^1, \cdots, \alpha^N) = \mathbb{E}\left\{\int_0^T \left(\frac{1}{2}(\alpha_t^i)^2 + \frac{\epsilon}{2}(X_t^{i+1} - X_t^i)^2\right)\mathrm{d}t + \frac{c}{2}(X_T^{i+1} - X_T^i)^2\right\}, \tag{55}$$

for constants $\epsilon > 0$, and $c \geq 0$, and we define $X_t^{N+1} = X_t^1$.

7.1 Construction of an Open-Loop Nash Equilibrium

We search for Nash equilibria of the system among strategies $\{\alpha_t^i, i = 1, \cdots, N\}$. We construct an open-loop Nash equilibrium by the Pontryagin stochastic maximum principle. The Hamiltonian for player i is given by:

$$H^i(x^1, \cdots, x^N, y^{i,1}, \cdots, y^{i,N}, \alpha^1, \cdots, \alpha^N) = \sum_{k=1}^{N} \alpha^k y^{i,k} + \frac{1}{2}(\alpha^i)^2 + \frac{\epsilon}{2}(x^{i+1} - x^i)^2. \tag{56}$$

The adjoint processes $Y_t^i = (Y_t^{i,j}; j = 1, \cdots, N)$ and $Z_t^i = (Z_t^{i,j,k}; j, k = 1, \cdots, N)$ for $i = 1, \cdots, N$ are defined as the solutions of the system of the backward stochastic differential equations (BSDEs):

$$\begin{cases} dY_t^{i,j} &= -\epsilon(X_t^{i+1} - X_t^i)(\delta_{i+1,j} - \delta_{i,j})\mathrm{d}t + \sum_{k=1}^{N} Z_t^{i,j,k}\mathrm{d}W_t^k, \\ Y_T^{i,j} &= \partial_{x^j} g_i(X_T) = c(X_T^{i+1} - X_T^i)(\delta_{i+1,j} - \delta_{i,j}). \end{cases} \tag{57}$$

Based on the sufficiency part of the Pontryagin stochastic maximum principle, we can get an open-loop Nash equilibrium by minimizing the Hamiltonian H^i with respect to α^i:

$$\partial_{\alpha^i} H^i = y^{i,i} + \alpha^i = 0 \quad \textit{leading to the choice:} \quad \hat{\alpha}^i = -y^{i,i}. \tag{58}$$

With this choice for the controls α^i's, the forward equation (54) becomes coupled with the backward equation (57). We make the ansatz:

$$Y_t^{i,i} = \sum_{j=0}^{N-1} \phi_t^{N,j} X_t^{i+j}, \tag{59}$$

for some deterministic scalar functions ϕ_t satisfying the terminal conditions: $\phi_T^{N,0} = c, \phi_T^{N,1} = -c, \phi_T^{N,k} = 0$ for $k \geq 2$ and $X_t^{i+j} \overset{def}{=} X_t^{(i+j) \mod N}$. Using the ansatz, the optimal strategy (58) and the forward equation (54) become:

$$\begin{cases} \hat{\alpha}^i = -Y_t^{i,i} = -\sum_{j=0}^{N-1} \phi_t^{N,j} X_t^{i+j}, \\ \mathrm{d}X_t^i = -\sum_{j=0}^{N-1} \phi_t^{N,j} X_t^{i+j}\mathrm{d}t + \sigma \mathrm{d}W_t^i. \end{cases} \tag{60}$$

Using the equations (60), we can differentiate the ansatz (59):

$$\begin{aligned} \mathrm{d}Y_t^{i,i} &= \sum_{j=0}^{N-1} [X_t^{i+j} \dot{\phi}_t^{N,j}\mathrm{d}t + \phi_t^{N,j}\mathrm{d}X_t^{i+j}] \\ &= \sum_{j=0}^{N-1} X_t^{i+j} \dot{\phi}_t^{N,j}\mathrm{d}t - \sum_{j=0}^{N-1} \phi_t^{N,j} \sum_{k=0}^{N-1} \phi_t^{N,k} X_t^{i+j+k}\mathrm{d}t + \sum_{j=0}^{N-1} \sigma \phi_t^{N,j}\mathrm{d}W_t^{i+j} \end{aligned} \tag{61}$$

Now we compare the two Itô's decompositions (61) and (57) of $Y_t^{i,i}$. The martingale terms give the processes $Z_t^{i,j,k}$:

$$Z_t^{i,i,0} = 0; \; Z_t^{i,i,k} = \sigma \phi_t^{N,N+k-i} \text{ for } 1 \leq k < i \text{ and } \; Z_t^{i,i,k} = \sigma \phi_t^{N,k-i} \text{ for } i \leq k \leq N.$$

And from the drift terms, we get:

$$\begin{aligned} &\text{for } k = 0: && \dot{\phi}_t^{N,0} = \phi_t^{N,0} \cdot \phi_t^{N,0} + \sum_{i=1}^{N-1} \phi_t^{N,i}\phi_t^{N,N-i} - \epsilon, && \phi_T^{N,0} = c, \\ &\text{for } k = 1: && \dot{\phi}_t^{N,1} = \phi_t^{N,0} \cdot \phi_t^{N,1} + \phi_t^{N,1} \cdot \phi_t^{N,0} + \sum_{i=2}^{N-1} \phi_t^{N,i}\phi_t^{N,N+1-i} + \epsilon, && \phi_T^{N,1} = -c, \\ &\text{for } N-1 > k \geq 2: && \dot{\phi}_t^{N,k} = \sum_{j=0}^{k} \phi_t^{N,j}\phi_t^{N,k-j} + \sum_{i=k+1}^{N-1} \phi_t^{N,i}\phi_t^{N,N+k-i}, && \phi_T^{N,k} = 0, \\ &\text{for } k = N-1: && \dot{\phi}_t^{N,N-1} = \sum_{j=0}^{N-1} \phi_t^{N,j}\phi_t^{N,N-1-j}, && \phi_T^{N,N-1} = 0. \end{aligned} \tag{62}$$

It can be written as a matrix Ricatti equation:

$$\dot{\Phi}^N(t) = \Phi^N(t)\Phi^N(t) - \mathbb{E}\,, \quad \Phi^N(T) := \mathbf{C}\,, \tag{63}$$

where $\Phi^N(\cdot)$ is the $N \times N$ matrix-valued function given by

$$\Phi^N(t) := \begin{pmatrix} \phi_t^{N,0} & \phi_t^{N,N-1} & \cdots & & \phi_t^{N,1} \\ \phi_t^{N,1} & \phi_t^{N,0} & \ddots & & \phi_t^{N,2} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \phi_t^{N,N-1} \\ \phi_t^{N,N-1} & \cdots & & \phi_t^{N,1} & \phi_t^{N,0} \end{pmatrix},$$

and

$$\mathbf{E} := \begin{pmatrix} \epsilon & 0 & \cdots & 0 & -\epsilon \\ -\epsilon & \epsilon & \ddots & \ddots & 0 \\ 0 & -\epsilon & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & -\epsilon & \epsilon \end{pmatrix}, \quad \mathbf{C} := \begin{pmatrix} c & 0 & \cdots & 0 & -c \\ -c & c & \ddots & \ddots & 0 \\ 0 & -c & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & -c & c \end{pmatrix}.$$

Proposition 7. *We have the relation:* $\sum_{k=0}^{N-1} \phi_t^{N,k} = 0$.

Proof. Given in Appendix A.8. □

With finite N, these equations are not easy to solve explicitely. If we take $N = \infty$, we expect that the system converges to the Riccati system of the infinite-player game studied in Section 3.

Conjecture 1. *$\Phi^N(t) \to \Phi^\infty(t)$, i.e. $\phi_t^{N,i}$ converges when $N \to \infty$ for each $i \leq N$, where $\Phi^\infty(t)$ is an infinite dimensional matrix-valued function given by*

$$\Phi^\infty(t) := \begin{pmatrix} \phi_t^0 & 0 & 0 & \cdots & \cdots \\ \phi_t^1 & \phi_t^0 & 0 & \ddots & \vdots \\ \phi_t^2 & \phi_t^1 & \phi_t^0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \cdots & \cdots & & \cdots & \cdots \end{pmatrix},$$

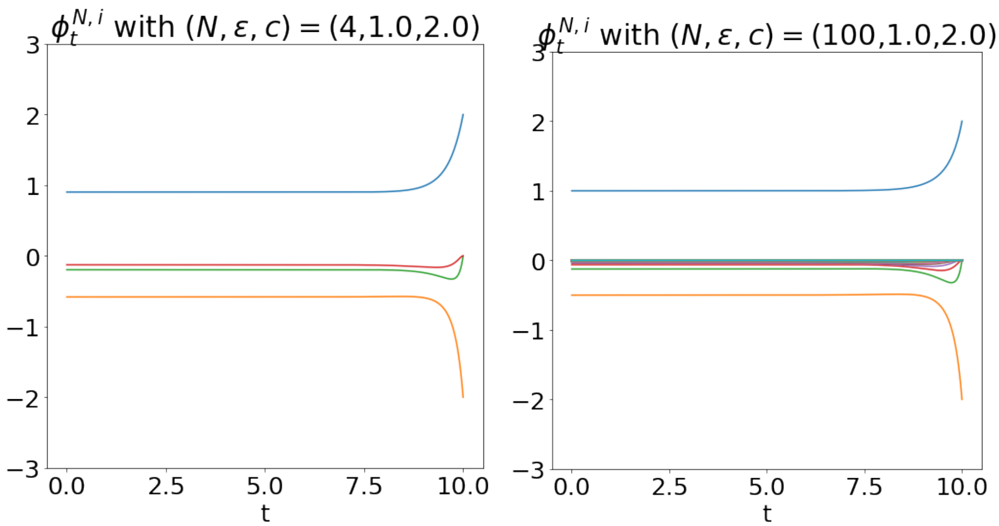
where the functions ϕ^k's are given by the system of differential equations (13).

Remark 4. *Proving this conjecture is equivalent to show that* $\sum_{k=j+1}^{N-1} \phi_t^{N,k}\phi_t^{N,N+j-k} \to 0$ *as* $N \to \infty$*. For instance, for* $j = 0$*, one needs to show that* $\sum_{k=1}^{N-1} \phi_t^{N,k}\phi_t^{N,N-k} \to 0$*. As of now, this remains an open problem.*

Our conjecture is substantiated by numerical evidences presented below.

7.2 Numerical Results

Using the methods given in [**?**], we can get the numerical solution of the matrix Riccati equation (63). Taking $\epsilon = 2$, $c = 1$, $T = 10$ (large terminal time), Figure 3 shows the behaviors of the ϕ functions defined by the system of differential equations (13) for $N = 4$ and $N = 100$. They converge to the constant solutions of the infinite game given in Section 4, except in the tail close to maturity as T is large but not infinite. This result confirms our conjecture stated in the previous section. Figure 4 shows the behavior of the function $\sum_{k=1}^{N-1} \phi_t^{N,k} \phi_t^{N,N-k}$ for different values of N. As we can see, the sum converges to 0 when N becomes larger, which supports the statement in remark 4. Though these numerical results give us strong evidence and confidence that the conjecture is true, a mathematical proof is still needed and it is part of our ongoing research.



(a) N=4

(b) N=100

Figure 3: As N increases, the blue line $\phi_t^{N,0} \to 1$, the orange line $\phi_t^{N,1} \to -\frac{1}{2}$, and $\phi_t^{N,k} \to 0$ for ≥ 2.

8 Conclusion

We studied a linear-quadratic stochastic differential game on a directed chain network. We were able to identify Nash equilibria in the case of finite chain with various boundary conditions and in the case of an infinite chain. This last case allows for more explicit computation in terms of Catalan functions and Catalan Markov chain. The Catalan

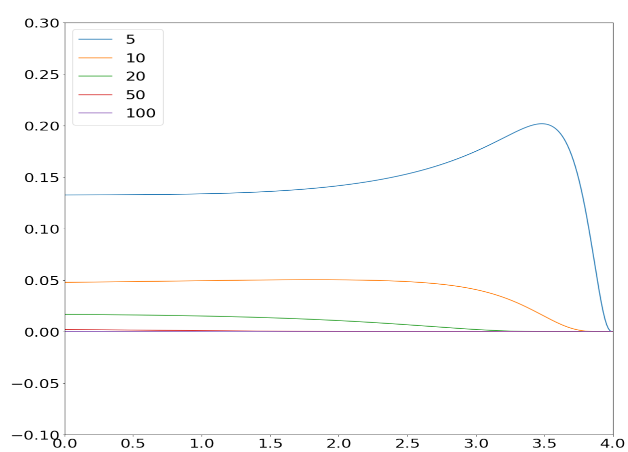


Figure 4: $\sum_{k=1}^{N-1} \phi_t^{N,k} \phi_t^{N,N-k}$ for different values of N

open-loop Nash equilibrium that we obtained is characterized by interactions with all the neighbors in one direction of the chain weighted by Catalan functions, event though the interaction in the objective functions is only with the nearest neighbor. Under equilibrium the variance of a state converges in the infinite time limit as opposed to the diverging behavior observed in the nearest neighbor dynamics studied in Detering, Fouque & Ichiba [**?**]. Our analysis is extended to mixed games with directed chain and mean field interaction so that our game model includes the two extreme network interactions, fully connected and only one neighbor connection. Our ongoing and future research concerns games with interactions on directed tree-like networks and stochastic networks.

A Appendix

A.1 Proof of Lemma 1

Define $S_t(z) = \sum_{k=0}^{\infty} z^k \, \phi_t^{(k)}$ where $0 \le z < 1$ and $\phi_t^{(k)} = \phi_t^k$ in equation (13) to avoid confusion. Then

$$
\begin{aligned}
\dot{S}_t(z) \quad &= \sum_{k=0}^{\infty} z^k \dot{\phi}_t^{(k)} \\
&= \big(\phi_t^{(0)}\phi_t^{(0)} - \epsilon\big) + z\big(\phi_t^{(0)}\phi_t^{(1)} + \phi_t^{(1)}\phi_t^{(0)} + \epsilon\big) + \cdots \\
&\quad + z^k \left(\phi_t^{(0)}\phi_t^{(k)} + \phi_t^{(1)}\phi_t^{(k-1)} + \cdots + \phi_t^{(k-1)}\phi_t^{(1)} + \phi_t^{(k)}\phi_t^{(0)}\right) + \cdots \\
&= \left(\phi_t^{(0)} S_t(z) + z\phi_t^{(1)} S_t(z) + \cdots + z^k \phi_t^{(k)} S_t(z) + \cdots\right) - \epsilon + z\epsilon \\
&= (S_t(z))^2 - \epsilon(1-z), \\
S_T(z) \quad &= c(1-z).
\end{aligned}
\tag{64}
$$

- For $z = 1$, we get the ODE:

$$\dot{S}_t(1) = (S_t(1))^2 \,, \qquad S_T(1) = 0. \tag{65}$$

  The solution is $S_t(1) = 0$, and we deduce:

$$\sum_{k=0}^{\infty} \phi_t^{(k)} = 0, \quad i.e., \quad \phi_t^{(0)} = -\sum_{k=1}^{\infty} \phi_t^{(k)}.$$

  One needs to be careful when taking $z = 1$ because the series defining $S_t(1)$ may not converge. Instead, we take a sequence $\{z_n\} \to 1$, the limit of $S_t(z_n)$ converges to the ODE (65), and we get the conclusion.

- For $z \ne 1$, the solution to the Riccati equation (64) is:

$$
\begin{aligned}
S_t(z) \quad &= \frac{-\epsilon(1-z)\big(e^{2\sqrt{\epsilon(1-z)}(T-t)} - 1\big) - c(1-z)\big(\sqrt{\epsilon(1-z)}e^{2\sqrt{\epsilon(1-z)}(T-t)} + \sqrt{\epsilon(1-z)}\big)}{\big(-\sqrt{\epsilon(1-z)}e^{2\sqrt{\epsilon(1-z)}(T-t)} - \sqrt{\epsilon(1-z)}\big) - c(1-z)\big(e^{2\sqrt{\epsilon(1-z)}(T-t)} - 1\big)} \\
&= \frac{\big(-\epsilon(1-z) - c\sqrt{\epsilon(1-z)}(1-z)\big)e^{2\sqrt{\epsilon(1-z)}(T-t)} + \epsilon(1-z) - c\sqrt{\epsilon(1-z)}(1-z)}{\big(-\sqrt{\epsilon(1-z)} - c(1-z)\big)e^{2\sqrt{\epsilon(1-z)}(T-t)} - \sqrt{\epsilon(1-z)} + c(1-z)} \\
&\overset{T\to\infty}{\longrightarrow} \sqrt{\epsilon(1-z)}.
\end{aligned}
\tag{66}
$$

A.2 Catalan Markov Chain

We have the Catalan probabilities: $\sum_{k=1}^{\infty} p_k = 1$ and $p_k = \frac{1}{2}\sum_{i=1}^{k-1} p_i p_{k-i}$. Then, it is easily seen that $\mathbf{Q}^2 = I - B$ with B having 1 's on the upper second diagonal and 0 's elsewhere, i.e.,

$$\mathbf{Q}^2 = \begin{pmatrix} 1 & -1 & 0 & \cdots & \\ 0 & 1 & -1 & \ddots & \\ & \ddots & \ddots & \ddots & \end{pmatrix} = -J_\infty(-1)\,, \quad J_\infty(\lambda) := \begin{pmatrix} \lambda & 1 & 0 & \cdots & \\ 0 & \lambda & 1 & \ddots & \\ & \ddots & \ddots & \ddots & \end{pmatrix}.$$

Here, $J_\infty(\lambda)$ is the infinite Jordan block matrix with diagonal components λ.

The matrix exponential of $\mathbf{Q}t$, $t \ge 0$, is written formally as

$$\exp(\mathbf{Q}t) = F(-\mathbf{Q}^2 t^2) = F(J_\infty(-1)\cdot t^2)\,, \ t \ge 0\,, \quad F(x) := \exp(-\sqrt{-x})\,, \ x \in \mathbb{C}\,.$$

Since a smooth function of a Jordan block matrix can be expressed as

$$F(J_\infty(\lambda)) = F(\lambda I + B) = \sum_{k=0}^{\infty} \frac{F^{(k)}(\lambda)}{k!} B^k = \begin{pmatrix} F(\lambda) & F^{(1)}(\lambda) & \frac{F^{(2)}(\lambda)}{2!} & \cdots & \frac{F^{(k)}(\lambda)}{k!} & \cdots \\ & \ddots & \ddots & \ddots & & \ddots \\ & & \ddots & \ddots & \ddots & \end{pmatrix},$$

we get

$$\exp(\mathbf{Q}t) = F(J(-\infty)\cdot t^2) = F((-I+B)t^2) = \sum_{k=0}^{\infty} \frac{F^{(k)}(-t^2)}{k!}(Bt^2)^k = \sum_{k=0}^{\infty} \frac{t^{2k}F^{(k)}(-t^2)}{k!} B^k.$$

The (j,k)-element of $\exp(\mathbf{Q}t)$ is formally given by

$$(\exp(\mathbf{Q}t))_{j,k} = \frac{t^{2(k-j)}\cdot F^{(k-j)}(-t^2)}{(k-j)!}, \quad j \le k, \text{ where } F^{(k)}(x) := \frac{\mathrm{d}^k F}{\mathrm{d}x^k}(x); \quad x > 0, \ k \in \mathbb{N},$$

and $(\exp(\mathbf{Q}t))_{j,k} = 0$, $j > k$ for $t \ge 0$. Here the k-th derivative $F^{(k)}(x)$ of $F(\cdot)$ can be written as $F^{(k)}(x) = \rho_k(x)e^{-\sqrt{-x}}$, where $\rho_k(x)$ satisfies the recursive equation

$$\rho_{k+1}(x) = \rho_k'(x) + \frac{\rho_k(x)}{2\sqrt{-x}}; \quad k \ge 0,$$

with $\rho_0(x) = 1$, $x \in \mathbb{C}$. For example,

$$\rho_0(x) = 1, \quad \rho_1(x) := \frac{+1}{2}(-x)^{-\frac{1}{2}}, \quad \rho_2(x) := \frac{1}{4}(-x)^{-\frac{2}{2}} + \frac{+1}{4}(-x)^{-\frac{3}{2}},$$

$$\rho_3(x) := \frac{1}{8}(-x)^{-\frac{3}{2}} + \frac{3}{8}(-x)^{-\frac{4}{2}} + \frac{3}{8}(-x)^{-\frac{5}{2}},$$

$$\rho_4(x) := \frac{1}{16}(-x)^{-\frac{4}{2}} + \frac{6}{16}(-x)^{-\frac{5}{2}} + \frac{15}{16}(-x)^{-\frac{6}{2}} + \frac{15}{16}(-x)^{-\frac{7}{2}},$$

$$\rho_5(x) := \frac{1}{32}(-x)^{-\frac{5}{2}} + \frac{10}{32}(-x)^{-\frac{6}{2}} + \frac{45}{32}(-x)^{-\frac{7}{2}} + \frac{105}{32}(-x)^{-\frac{8}{2}} + \frac{105}{32}(-x)^{-\frac{9}{2}}.$$

More generally we have

$$\begin{aligned}
\rho_k(x) &= \sum_{j=k}^{2k-1} P_{k,j}\,(-x)^{-\frac{j}{2}} \\
&\quad \Big(\text{where } P_{k,j} = \frac{1}{2^k}\frac{(j-1)!}{(2j-2k)!!(2k-j-1)!} \textit{ for } k \le j \le 2k-1\Big) \\
&= \sum_{j=k}^{2k-1} \frac{1}{2^k}\frac{(j-1)!}{(2j-2k)!!(2k-j-1)!}(-x)^{-\frac{j}{2}} \\
&= \frac{1}{2^k}\sum_{j=k}^{2k-1} \frac{(j-1)!}{(2j-2k)!!(2k-j-1)!}(-x)^{-\frac{j}{2}}, \quad \textit{for} \quad k \ge 1., \qquad (67)
\end{aligned}$$

This formula is justified by induction in the prroof below.

Proof. First, $\rho_1(x) = \frac{1}{2}(-x)^{-\frac{1}{2}}$. Assume $\rho_k(x) = \frac{1}{2^k}\sum_{j=k}^{2k-1} \frac{(j-1)!}{(2j-2k)!!(2k-j-1)!}(-x)^{-\frac{j}{2}}$.

Then,

$$\begin{aligned}
\rho_k'(x) &= \tfrac{1}{2^k}\sum_{j=k}^{2k-1} \tfrac{(j-1)!}{(2j-2k)!!(2k-j-1)!}\tfrac{j}{2}(-x)^{-\frac{j+2}{2}} \\
(i = j+1) \quad &= \tfrac{1}{2^{k+1}}\sum_{i=k+1}^{2k} \tfrac{(i-1)!}{(2i-2k-2)!!(2k-i)!}(-x)^{-\frac{i+1}{2}} \\
&= \tfrac{1}{2^{k+1}}\Big(\sum_{i=k+1}^{2k-1} \tfrac{(i-1)!}{(2i-2k-2)!!(2k-i)!}(-x)^{-\frac{i+1}{2}} + \tfrac{(2k-1)!}{(2k-2)!!}(-x)^{-\frac{2k+1}{2}}\Big), \\
\tfrac{\rho_k(x)}{2\sqrt{-x}} &= \tfrac{1}{2^{k+1}}\sum_{j=k}^{2k-1} \tfrac{(j-1)!}{(2j-2k)!!(2k-j-1)!}(-x)^{-\frac{j+1}{2}} \\
&= \tfrac{1}{2^{k+1}}\Big((-x)^{-\frac{k+1}{2}} + \sum_{j=k+1}^{2k-1} \tfrac{(j-1)!}{(2j-2k)!!(2k-j-1)!}(-x)^{-\frac{j+1}{2}}\Big).
\end{aligned}$$

Therefore,

$$\begin{aligned}\rho_k'(x)+\tfrac{\rho_k(x)}{2\sqrt{-x}} &= \tfrac{1}{2^{k+1}}\Big\{\tfrac{(2k-1)!}{(2k-2)!!}\,(-x)^{-\frac{2k+1}{2}}+(-x)^{-\frac{k+1}{2}}\\ &\quad+\sum_{i=k+1}^{2k-1}(i-1)!\,\tfrac{(2i-2k)+(2k-i)}{(2i-2k)!!(2k-i)!}\,(-x)^{-\frac{i+1}{2}}\Big\}\\ &= \tfrac{1}{2^{k+1}}\sum_{i=k}^{2k}\tfrac{i!}{(2i-2k)!!(2k-i)!}\,(-x)^{-\frac{i+1}{2}}\\ (j=i+1)\quad &= \tfrac{1}{2^{k+1}}\sum_{j=k+1}^{2k+1}\tfrac{(j-1)!}{(2j-2(k+1))!!(2(k+1)-j-1)!}\,(-x)^{-\frac{j}{2}}\\ &= \rho_{k+1}(x).\end{aligned}$$

□

Thus, the Gaussian process $X_j(t)$, $j\in\mathbb{N}_0$, $t\geq 0$, corresponding to the Catalan Markov chain, is

$$X_j(t)\;:=\;\sum_{k=0}^{\infty}\int_0^t(\exp(Q(t-s)))_{j,k}\mathrm{d}W_k(s)\;=\;\sum_{k=j}^{\infty}\int_0^t\frac{(t-s)^{2(k-j)}}{(k-j)!}\cdot F^{(k-j)}(-(t-s)^2)\mathrm{d}W_k(s)$$

$$=\sum_{k=j}^{\infty}\int_0^t\frac{(t-s)^{2(k-j)}}{(k-j)!}\cdot\rho_{k-j}(-(t-s)^2)\,e^{-(t-s)}\cdot\mathrm{d}W_k(s),\tag{68}$$

where $W_k(\cdot)$, $k\in\mathbb{N}_0$ are independent standard Brownian motions.

A.3 Proof of Remark 3

By ρ_k's formula (67), we have:

$$\begin{aligned}\rho_k(-\nu^2) = &\;\tfrac{1}{2^k}\sum_{j=k}^{2k-1}\tfrac{(j-1)!}{(2j-2k)!!(2k-j-1)!}\,\nu^{-j}\text{ for }\nu\geq 0\\ &=\tfrac{1}{2^k}\sum_{i=0}^{k-1}\tfrac{(i+k-1)!}{(2i)!!(k-i-1)!}\,\nu^{-(i+k)}\\ &=(2\nu)^{-k}\sum_{i=0}^{k-1}\tfrac{(i+k-1)!}{(2i)!!(k-i-1)!}\,\nu^{-i}\\ &=(2\nu)^{-k}\sum_{i=0}^{k-1}\tfrac{(i+k-1)!}{i!(k-i-1)!}\,(2\nu)^{-i}\\ &=\;\frac{1}{2^k\nu^k}\cdot\sqrt{\frac{2\nu}{\pi}}\cdot e^{\nu}\cdot K_{k-(1/2)}(\nu)\,;\quad k\geq 1\,,\end{aligned}$$

where $K_n(x)$ is the modified Bessel function of the second kind, i.e.,

$$K_n(x)\;=\;\int_0^{\infty}e^{-x\cosh t}\cosh(nt)\mathrm{d}t\,;\quad n>-1,\;\;x>0\,.$$

Then,

$$\mathrm{Var}(X_0(t))\;=\;\sum_{k=0}^{\infty}\int_0^t\frac{(t-s)^{4k}}{(k!)^2}|\rho_k(-(t-s)^2)|^2e^{-2(t-s)}\mathrm{d}s$$

$$(\nu=t-s\geq 0)\;=\;\sum_{k=0}^{\infty}\int_0^t\frac{\nu^{4k}}{(k!)^2}|\rho_k(-\nu^2)|^2e^{-2\nu}\mathrm{d}\nu$$

$$=\sum_{k=1}^{\infty}\int_0^t\frac{2}{\pi}\,\frac{\nu^{2k+1}}{(k!)^2\,4^k}\big(K_{k-(1/2)}(\nu)\big)^2\mathrm{d}\nu+\frac{1-e^{-2t}}{2}\,;\quad t\geq 0\,.$$

A.4 Proof of Proposition 2

First, we observe that:

$$\begin{aligned}
&\int_0^\infty t^{\alpha-1}(K_\nu(t))^2dt = \tfrac{\sqrt{\pi}}{4\Gamma((\alpha+1)/2)}\Gamma(\tfrac{\alpha}{2})\Gamma(\tfrac{\alpha}{2}-\nu)\Gamma(\tfrac{\alpha}{2}+\nu),\\
&(1-4x)^{-\frac{1}{2}} = \sum_{k=0}^{\infty}\tbinom{-\frac{1}{2}}{k}(-4x)^k = \sum_{k=0}^{\infty}\tfrac{(-1)^k(2k-1)!!}{2^k k!}(-4x)^k = \sum_{k=0}^{\infty}\tfrac{(2k-1)!!2^k}{k!}x^k = \sum_{k=0}^{\infty}\tbinom{2k}{k}x^k,\\
&\Gamma(n+\tfrac{1}{2}) = \tfrac{(2n-1)!!}{2^n}\sqrt{\pi}, \quad (2n-1)!! = \tfrac{(2n)!}{n!2^n}.
\end{aligned}$$

As $t\to\infty$, based on the Remark 3, we obtain

$$\begin{aligned}
\lim_{t\to\infty}\mathrm{Var}(X_0(t)) &= \frac{1}{2} + \sum_{k=1}^{\infty}\int_0^\infty \frac{2\,s^{2k+1}}{\pi(k!)^2 4^k}\cdot[K_{k-(1/2)}(s)]^2\mathrm{d}s\\
&= \frac{1}{2} + \sum_{k=1}^{\infty}\frac{2}{\pi\,(k!)^2 4^k}\int_0^\infty s^{2k+1}[K_{k-(1/2)}(s)]^2\mathrm{d}s\\
&= \frac{1}{2} + \sum_{k=1}^{\infty}\frac{2}{\pi(k!)^2 4^k}\cdot\frac{\pi\,\Gamma(k+1)\,\Gamma(2k+(1/2))}{8\,\Gamma(k+(3/2))}\\
&= \frac{1}{2} + \sum_{k=1}^{\infty}\frac{\Gamma(2k+(1/2))}{4^{k+1}\,k!\,\Gamma(k+(3/2))}\\
&= \frac{1}{2} + \sum_{k=1}^{\infty}\frac{1}{4^{k+1}k!}\,\frac{(4k-1)!!\sqrt{\pi}}{2^{2k}}\Big/\left(\frac{(2k+1)!!\sqrt{\pi}}{2^{k+1}}\right)\\
&= \frac{1}{2} + \sum_{k=1}^{\infty}\frac{1}{4^{k+1}k!}\,\frac{(4k-1)!!}{(2k+1)!!}\,\frac{1}{2^{k-1}}\\
&= \frac{1}{2} + \frac{1}{2}\sum_{k=1}^{\infty}\frac{(4k-1)!!}{(2k+1)!!}\,\frac{1}{k!8^k}
\end{aligned}$$

$$\left(\text{since}\quad \frac{(4k-1)!!}{(2k+1)!!} = \frac{(4k)!}{2^{2k}(2k)!}\Big/\frac{(2k+2)!}{2^{k+1}(k+1)!} = \frac{1}{2^{k-1}}\frac{(4k)!}{(2k)!}\frac{(k+1)!}{(2k+1)!} = \frac{1}{2^{k-1}}2^k\frac{1}{2^{k+1}} = \frac{1}{2^k}\right)$$

$$\begin{aligned}
&= \frac{1}{2} + \frac{1}{2}\sum_{k=1}^{\infty}\frac{1}{2^k k!}\frac{1}{8^k} = \frac{1}{2} + \frac{1}{2}\sum_{k=1}^{\infty}\binom{2k}{k}\frac{1}{8^k} = \frac{1}{2} + \frac{1}{2}(-1+(1-\frac{1}{2})^{-\frac{1}{2}})\\
&= \frac{1}{2} + \frac{-1+\sqrt{2}}{2} = \frac{1}{\sqrt{2}}.
\end{aligned}$$

A.5 Proofs of Proposition 3 and Proposition 4

From the expression (19) for $X_j(t)$, the auto-covariance is:

$$\begin{aligned}
\mathbb{E}[X_0(s)X_0(t)] &= \mathbb{E}\bigg(\sum_{k=0}^{\infty}\int_0^t \frac{1}{\sqrt{\pi}k!}\frac{(t-\nu)^{k+1/2}}{2^{k-1/2}}K_{k-1/2}(t-\nu)\mathrm{d}W_k(\nu)\\
&\quad\cdot\sum_{k=0}^{\infty}\int_0^s \frac{1}{\sqrt{\pi}k!}\frac{(s-\gamma)^{k+1/2}}{2^{k-1/2}}K_{k-1/2}(s-\gamma)\mathrm{d}W_k(\gamma)\bigg)\\
&= \sum_{k=0}^{\infty}\int_0^s \frac{1}{\pi(k!)^2 2^{2k-1}}(t-\nu)^{k+1/2}(s-\nu)^{k+1/2}K_{k-1/2}(t-\nu)K_{k-1/2}(s-\nu)\mathrm{d}\nu\\
&= \sum_{k=0}^{\infty}\int_0^s \frac{1}{\pi(k!)^2 2^{2k-1}}((t-s+\alpha)\alpha)^{k+1/2}K_{k-1/2}(t-s+\alpha)K_{k-1/2}(\alpha)\mathrm{d}\alpha\\
&> 0;
\end{aligned}$$

the cross-covariance is:

$$\begin{aligned}\mathbb{E}[X_0(t)X_k(t)] &= \mathbb{E}\Big(\sum_{i=0}^{\infty}\int_0^t \frac{1}{\sqrt{\pi}i!}\frac{(t-\nu)^{i+1/2}}{2^{i-1/2}}K_{i-1/2}(t-\nu)\mathrm{d}W_i(\nu)\\ &\quad\cdot\sum_{j=0}^{\infty}\int_0^t \frac{1}{\sqrt{\pi}j!}\frac{(t-s)^{j+1/2}}{2^{j-1/2}}K_{j-1/2}(t-s)\mathrm{d}W_{k+j}(\gamma)\Big)\\ &= \mathbb{E}\Big(\sum_{i=k}^{\infty}\int_0^t \frac{1}{\sqrt{\pi}i!}\frac{(t-\nu)^{i+1/2}}{2^{i-1/2}}K_{i-1/2}(t-\nu)\mathrm{d}W_i(\nu)\\ &\quad\cdot\sum_{i=k}^{\infty}\int_0^t \frac{1}{\sqrt{\pi}(i-k)!}\frac{(t-s)^{i-k+1/2}}{2^{i-k-1/2}}K_{i-k-1/2}(t-s)\mathrm{d}W_i(\gamma)\Big)\\ &= \sum_{i=k}^{\infty}\int_0^t \frac{1}{\pi i!(i-k)!)}\frac{(t-\nu)^{2i-k+1}}{2^{2i-k-1}}K_{i-1/2}(t-\nu)K_{i-k-1/2}(t-\nu)\mathrm{d}\nu\\ &= \sum_{j=0}^{\infty}\int_0^t \frac{1}{\pi(k+j)!j!}\frac{(t-\nu)^{k+2j+1}}{2^{k+2j-1}}K_{k+j-1/2}(t-\nu)K_{j-1/2}(t-\nu)\mathrm{d}\nu\\ &= \sum_{j=0}^{\infty}\int_0^t \frac{1}{\pi(k+j)!j!}\frac{s^{k+2j+1}}{2^{k+2j-1}}K_{k+j-1/2}(s)K_{j-1/2}(s)\mathrm{d}s\\ &= \sum_{j=0}^{\infty}\frac{1}{\pi(k+j)!j!}\frac{1}{2^{k+2j-1}}\int_0^t s^{k+2j+1}K_{k+j-1/2}(s)K_{j-1/2}(s)\mathrm{d}s,\end{aligned}$$

and as $t\to\infty$, it converges to

$$\sum_{j=0}^{\infty}\frac{1}{\pi(k+j)!j!}\frac{1}{2^{k+2j-1}}\int_0^\infty s^{k+2j+1}K_{k+j-1/2}(s)K_{j-1/2}(s)\mathrm{d}s \quad (>\mathbf{0}). \tag{69}$$

We have the bound

$$\begin{aligned}\int_0^\infty s^{k+2j+1}K_{k+j-1/2}(s)K_{j-1/2}(s)\mathrm{d}s &\le \Big(\int_0^\infty \big(s^{k+j+1/2}K_{k+j-1/2}(s)\big)^2\mathrm{d}s\cdot\int_0^\infty\big(s^{j+1/2}K_{j-1/2}(s)\big)^2\mathrm{d}s\Big)^{\frac{1}{2}}\\ &= \Big(\int_0^\infty s^{2k+2j+1}(K_{k+j-1/2}(s))^2\mathrm{d}s\cdot\int_0^\infty s^{2j+1}(K_{j-1/2}(s))^2\mathrm{d}s\Big)^{\frac{1}{2}}\\ &= \Big(\frac{\pi\Gamma(k+j+1)\Gamma(2k+2j+(1/2))}{8\Gamma(k+j+(3/2))}\cdot\frac{\pi\Gamma(j+1)\Gamma(2j+(1/2))}{8\Gamma(j+(3/2))}\Big)^{\frac{1}{2}}.\end{aligned}$$

We deduce a bound for the cross-covariance given by (69):

$$\begin{aligned}&\lim_{t\to\infty}\mathbb{E}[X_0(t)X_k(t)]\\ &\le \sum_{j=0}^{\infty}\frac{1}{\pi(k+j)!j!}\frac{1}{2^{k+2j-1}}\Big(\frac{\pi\Gamma(k+j+1)\Gamma(2k+2j+(1/2))}{8\Gamma(k+j+(3/2))}\cdot\frac{\pi\Gamma(j+1)\Gamma(2j+(1/2))}{8\Gamma(j+(3/2))}\Big)^{\frac{1}{2}}\\ &= \sum_{j=0}^{\infty}\Big[\frac{1}{(k+j)!4^{k+j+1}}\frac{\Gamma(2k+2j+(1/2))}{\Gamma(k+j+(3/2))}\cdot\frac{1}{j!4^{j+1}}\frac{\Gamma(2j+(1/2))}{\Gamma(j+(3/2))}\Big]^{\frac{1}{2}}\\ &\quad\text{(since } \sum_{j=0}^{\infty}\frac{1}{j!4^{j+1}}\frac{\Gamma(2j+(1/2))}{\Gamma(j+(3/2))} \text{ is convergent, we can apply Cauchy–Schwarz inequality)}\\ &\le \Big[\sum_{j=0}^{\infty}\frac{1}{(k+j)!4^{k+j+1}}\frac{\Gamma(2k+2j+(1/2))}{\Gamma(k+j+(3/2))}\cdot\sum_{j=0}^{\infty}\frac{1}{j!4^{j+1}}\frac{\Gamma(2j+(1/2))}{\Gamma(j+(3/2))}\Big]^{\frac{1}{2}}\\ &< \big(\frac{1}{\sqrt{2}}\cdot\frac{1}{\sqrt{2}}\big)^{\frac{1}{2}} = \frac{1}{\sqrt{2}}\end{aligned}$$

A.6 Proof of Proposition 5

Adding equations (45) and (48), for $0 < u < 1$, we get:

$$
\begin{aligned}
-u\epsilon &= u\dot{\phi}_t^0 - u^2(\phi_t^0)^2 - 2u(1-u)\psi_t\phi_t^0 + u(1-u)\psi_t \sum_{k=0}^{\infty}\phi_t^k \\
\implies \quad u\dot{\phi}_t^0 &= u^2(\phi_t^0)^2 + 2u(1-u)\psi_t\phi_t^0 - u(1-u)\psi_t \sum_{k=0}^{\infty}\phi_t^k - u\epsilon.
\end{aligned}
$$

Then (47) and (48) can be written as:

$$
\begin{aligned}
u\dot{\phi}_t^1 &= 2u^2\phi_t^0\phi_t^1 + 2u(1-u)\psi_t\phi_t^1 + u\epsilon, \\
u\dot{\phi}_t^k &= u^2\sum_{j=0}^{k}\phi_t^j\phi_t^{k-j} + 2u(1-u)\psi_t\phi_t^k, \quad \text{for } k \geq 2.
\end{aligned}
$$

Define $S_t(z) = \sum_{k=0}^{\infty} z^k\phi_t^{(k)}$ where $0 \leq z \leq 1$ and $\phi_t^{(k)} = \phi_t^k$ in equations above to avoid confusion. Then

$$
\begin{aligned}
u\dot{S}_t(z) \quad &= \sum_{k=0}^{\infty} z^k u\dot{\phi}_t^{(k)} \\
&= u^2(\phi_t^{(0)})^2 + 2u(1-u)\psi_t\phi_t^{(0)} - u(1-u)\psi_t \sum_{k=0}^{\infty}\phi_t^{(k)} - u\epsilon \\
&+ z\big(u^2(\phi_t^{(0)}\phi_t^{(1)} + \phi_t^{(1)}\phi_t^{(0)}) + 2u(1-u)\psi_t\phi_t^{(1)} + u\epsilon\big) + \cdots \\
&+ z^k\big(u^2\sum_{j=0}^{k}\phi_t^{(j)}\phi_t^{(k-j)} + 2u(1-u)\psi_t\phi_t^{(k)}\big) + \cdots \\
&= u^2(S_t(z))^2 + u(1-u)\psi_t S_t(z) - u(1-z)\epsilon, \\
uS_T(z) \quad &= u(1-z)c.
\end{aligned}
\tag{70}
$$

For $z = 1$, we obtain the ODE:

$$
u\dot{S}_t(1) = u^2(S_t(1))^2 + u(1-u)\psi_t S_t(1)\,, \quad uS_T(1) = 0. \tag{71}
$$

The solution is given by $S_t(1) = 0$ and we deduce $\sum_{k=0}^{\infty}\phi_t^{(k)} = 0$.

A.7 About Table 1

According to proposition 6, for $t \geq 0$, we have:

$$
\begin{aligned}
\text{Var}(X_0^{(u)}(t)) \;&=\; \text{Var}\Big(\sum_{k=0}^{\infty}\int_0^t \frac{u^k(t-s)^{2k}}{k!}F^{(k)}(-(t-s)^2)\mathrm{d}W_k(s)\Big) \\
&= \sum_{k=0}^{\infty}\int_0^t \frac{u^{2k}(t-s)^{4k}}{(k!)^2}|\rho_k(-(t-s)^2)|^2 e^{-2(t-s)}\mathrm{d}s \\
&= \sum_{k=1}^{\infty}\int_0^t \frac{2u^{2k}}{\pi(k!)^2 4^k}\nu^{2k+1}(K_{k-\frac{1}{2}}(\nu))^2\mathrm{d}\nu + \frac{1-e^{-2t}}{2}.
\end{aligned}
\tag{72}
$$

As $t \to \infty$, for $u < 1$, we obtain

$$
\begin{aligned}
\lim_{t\to\infty} \mathrm{Var}(X_0^{(u)}(t)) &= \frac{1}{2} + \sum_{k=1}^{\infty} \int_0^{\infty} \frac{2u^{2k}\, s^{2k+1}}{\pi(k!)^2 4^k} \cdot [K_{k-(1/2)}(s)]^2 \mathrm{d}s \\
&= \frac{1}{2} + \sum_{k=1}^{\infty} \frac{2u^{2k}}{\pi\,(k!)^2 4^k} \int_0^{\infty} s^{2k+1} [K_{k-(1/2)}(s)]^2 \mathrm{d}s \\
&= \frac{1}{2} + \sum_{k=1}^{\infty} \frac{2u^{2k}}{\pi(k!)^2 4^k} \cdot \frac{\pi\,\Gamma(k+1)\,\Gamma(2k+(1/2))}{8\,\Gamma(k+(3/2))} \\
&= \frac{1}{2} + \sum_{k=1}^{\infty} u^{2k} \cdot \frac{\Gamma(2k+(1/2))}{4^{k+1}\, k!\,\Gamma(k+(3/2))} \\
&= \frac{1}{2} + \frac{1}{2} \sum_{k=1}^{\infty} \binom{2k}{k} \frac{u^{2k}}{8^k} \\
&= \frac{1}{2} + \frac{1}{2}((1-4\frac{u^2}{8})^{-\frac{1}{2}} - 1) \\
&= \frac{1}{2}(1-\frac{u^2}{2})^{-\frac{1}{2}} < \infty.
\end{aligned}
$$

Since

$$
\begin{aligned}
X_j^{(u)}(t) &= \sum_{k=j}^{\infty} \int_0^t \frac{u^{k-j}(t-s)^{2(k-j)}}{(k-j)!} \rho_{k-j}(-(t-s)^2)\, e^{-(t-s)} \mathrm{d}W_k(s) \\
&= \sum_{i=0}^{\infty} \int_0^t \frac{u^i (t-s)^{2i}}{i!} \rho_i(-(t-s)^2)\, e^{-(t-s)} \mathrm{d}W_{j+i}(s) \\
&= \sum_{i=0}^{\infty} \int_0^t \frac{u^i}{\sqrt{\pi} i!} \frac{(t-s)^{i+1/2}}{2^{i-1/2}} K_{i-1/2}(t-s) \mathrm{d}W_{j+i}(s),
\end{aligned}
$$

the (auto)covariance is:

$$
\begin{aligned}
\mathbb{E}[X_0^{(u)}(s) X_0^{(u)}(t)] &= \mathbb{E}\Big(\sum_{k=0}^{\infty} \int_0^t \frac{u^k}{\sqrt{\pi} k!} \frac{(t-\nu)^{k+1/2}}{2^{k-1/2}} K_{k-1/2}(t-\nu) \mathrm{d}W_k(\nu) \\
&\quad \cdot \sum_{k=0}^{\infty} \int_0^s \frac{u^k}{\sqrt{\pi} k!} \frac{(s-\gamma)^{k+1/2}}{2^{k-1/2}} K_{k-1/2}(s-\gamma) \mathrm{d}W_k(\gamma) \Big) \\
&= \sum_{k=0}^{\infty} \int_0^s \frac{u^{2k}}{\pi (k!)^2 2^{2k-1}} (t-\nu)^{k+1/2} (s-\nu)^{k+1/2} K_{k-1/2}(t-\nu) K_{k-1/2}(s-\nu) \mathrm{d}\nu \\
&= \sum_{k=0}^{\infty} \int_0^s \frac{u^{2k}}{\pi (k!)^2 2^{2k-1}} ((t-s+\alpha)\alpha)^{k+1/2} K_{k-1/2}(t-s+\alpha) K_{k-1/2}(\alpha) \mathrm{d}\alpha \\
&\neq 0\,.
\end{aligned}
$$

The cross-covariance is:

$$
\begin{aligned}
\mathbb{E}[X_0^{(u)}(t)X_k^{(u)}(t)] &= \mathbb{E}\Big(\sum_{i=0}^{\infty}\int_0^t \frac{u^i}{\sqrt{\pi}i!}\frac{(t-\nu)^{i+1/2}}{2^{i-1/2}}K_{i-1/2}(t-\nu)\mathrm{d}W_i(\nu) \\
&\qquad \cdot\sum_{j=0}^{\infty}\int_0^t \frac{u^j}{\sqrt{\pi}j!}\frac{(t-s)^{j+1/2}}{2^{j-1/2}}K_{j-1/2}(t-s)\mathrm{d}W_{k+j}(\gamma)\Big) \\
&= \mathbb{E}\Big(\sum_{i=k}^{\infty}\int_0^t \frac{u^i}{\sqrt{\pi}i!}\frac{(t-\nu)^{i+1/2}}{2^{i-1/2}}K_{i-1/2}(t-\nu)\mathrm{d}W_i(\nu) \\
&\qquad \cdot\sum_{i=k}^{\infty}\int_0^t \frac{u^{i-k}}{\sqrt{\pi}(i-k)!}\frac{(t-s)^{i-k+1/2}}{2^{i-k-1/2}}K_{i-k-1/2}(t-s)\mathrm{d}W_i(\gamma)\Big) \\
&= \sum_{i=k}^{\infty}\int_0^t \frac{u^{2i-k}}{\pi i!(i-k)!)}\frac{(t-\nu)^{2i-k+1}}{2^{2i-k-1}}K_{i-1/2}(t-\nu)K_{i-k-1/2}(t-\nu)\mathrm{d}\nu \\
&= \sum_{j=0}^{\infty}\int_0^t \frac{u^{k+2j}}{\pi(k+j)!j!}\frac{(t-\nu)^{k+2j+1}}{2^{k+2j-1}}K_{k+j-1/2}(t-\nu)K_{j-1/2}(t-\nu)\mathrm{d}\nu \\
&= \sum_{j=0}^{\infty}\int_0^t \frac{u^{k+2j}}{\pi(k+j)!j!}\frac{s^{k+2j+1}}{2^{k+2j-1}}K_{k+j-1/2}(s)K_{j-1/2}(s)\mathrm{d}s \\
&= \sum_{j=0}^{\infty}\frac{u^{k+2j}}{\pi(k+j)!j!}\frac{1}{2^{k+2j-1}}\int_0^t s^{k+2j+1}K_{k+j-1/2}(s)K_{j-1/2}(s)\mathrm{d}s\,,
\end{aligned}
$$

and as $t\to\infty$ it converges to

$$
\sum_{j=0}^{\infty}\frac{u^{k+2j}}{\pi(k+j)!j!}\frac{1}{2^{k+2j-1}}\int_0^{\infty} s^{k+2j+1}K_{k+j-1/2}(s)K_{j-1/2}(s)\mathrm{d}s \quad (\neq \mathbf{0},\ \textbf{if}\ \mathbf{u}\neq\mathbf{0}), \tag{73}
$$

and we deduce the bound:

$$
\begin{aligned}
&\lim_{t\to\infty}\mathbb{E}[X_0^{(u)}(t)X_k^{(u)}(t)] \\
&\le \sum_{j=0}^{\infty}\frac{u^{k+2j}}{\pi(k+j)!j!}\frac{1}{2^{k+2j-1}}\Big(\frac{\pi\Gamma(k+j+1)\Gamma(2k+2j+(1/2))}{8\Gamma(k+j+(3/2))}\cdot\frac{\pi\Gamma(j+1)\Gamma(2j+(1/2))}{8\Gamma(j+(3/2))}\Big)^{\frac{1}{2}} \\
&= \sum_{j=0}^{\infty}\Big[\frac{u^{2(k+j)}}{(k+j)!4^{k+j+1}}\frac{\Gamma(2k+2j+(1/2))}{\Gamma(k+j+(3/2))}\cdot\frac{u^{2j}}{j!4^{j+1}}\frac{\Gamma(2j+(1/2))}{\Gamma(j+(3/2))}\Big]^{\frac{1}{2}} \\
&\le \Big[\sum_{j=0}^{\infty}\frac{u^{2(k+j)}}{(k+j)!4^{k+j+1}}\frac{\Gamma(2k+2j+(1/2))}{\Gamma(k+j+(3/2))}\cdot\sum_{j=0}^{\infty}\frac{u^{2j}}{j!4^{j+1}}\frac{\Gamma(2j+(1/2))}{\Gamma(j+(3/2))}\Big]^{\frac{1}{2}} \\
&< \frac{1}{2}(1-\frac{u^2}{2})^{-1/2}
\end{aligned}
$$

A.8 Proof of Proposition 7

Define $S_t^N(z) = \sum_{k=0}^{N-1} z^k \phi_t^{N,k}$, then, by (62), we have:

$$
\begin{aligned}
\dot{S}_t^N(z) &= \sum_{k=0}^{N-1} z^k \dot{\phi}_t^{N,k} \\
&= \phi_t^{N,0}\cdot\phi_t^{N,0} + \phi_t^{N,1}\cdot\phi_t^{N,N-1} + \cdots + \phi_t^{N,N-1}\cdot\phi_t^{N,1} - \epsilon \\
&\quad + z\Big(\phi_t^{N,0}\cdot\phi_t^{N,1} + \phi_t^{N,1}\cdot\phi_t^{N,0} + \phi_t^{N,2}\cdot\phi_t^{N,N-1} + \cdots + \phi_t^{N,N-1}\cdot\phi_t^{N,2} + \epsilon\Big) + \cdots \\
&\quad + z^{N-2}\Big(\phi_t^{N,0}\cdot\phi_t^{N,N-2} + \phi_t^{N,1}\cdot\phi_t^{N,N-3} + \phi_t^{N,2}\cdot\phi_t^{N,N-4} + \cdots + \phi_t^{N,N-2}\cdot\phi_t^{N,0} + \phi_t^{N,N-1}\cdot\phi_t^{N,N-1}\Big) \\
&\quad + z^{N-1}\Big(\phi_t^{N,0}\cdot\phi_t^{N,N-1} + \phi_t^{N,1}\cdot\phi_t^{N,N-2} + \cdots + \phi_t^{N,N-2}\cdot\phi_t^{N,1} + \phi_t^{N,N-1}\cdot\phi_t^{N,0}\Big) \\
&= \phi_t^{N,0} S_t^N(z) - (1-z)\epsilon \\
&\quad + z\phi_t^{N,1}\big(S_t^N(z) - z^{N-1}\phi_t^{N,N-1}\big) + \phi_t^{N,1}\phi_t^{N,N-1} \\
&\quad + z^2\phi_t^{N,2}\big(S_t^N(z) - z^{N-1}\phi_t^{N,N-1} - z^{N-2}\phi_t^{N,N-2}\big) + \phi_t^{N,2}\phi_t^{N,N-2} + z\phi_t^{N,2}\phi_t^{N,N-1} + \cdots \\
&\quad + z^{N-2}\phi_t^{N,N-2}\big(S_t^N(z) - z^{N-1}\phi_t^{N,N-1} - \cdots - z^2\phi_t^{N,2}\big) \\
&\quad + \phi_t^{N,N-2}\phi_t^{N,2} + z\phi_t^{N,N-2}\phi_t^{N,3} + \cdots + z^{N-3}\phi_t^{N,N-2}\phi_t^{N,N-1} \\
&\quad + z^{N-1}\phi_t^{N,N-1}\big(S_t^N(z) - z^{N-1}\phi_t^{N,N-1} - \cdots - z\phi_t^{N,1}\big) \\
&\quad + \phi_t^{N,N-1}\phi_t^{N,1} + z\phi_t^{N,N-1}\phi_t^{N,2} + \cdots + z^{N-2}\phi_t^{N,N-1}\phi_t^{N,N-1} \\
&= (S_t^N(z))^2 + (1-z^N)\phi_t^{N,1}\phi_t^{N,N-1} - (1-z)\epsilon \\
&\quad + (1-z^N)[\phi_t^{N,2}\phi_t^{N,N-2} + z\phi_t^{N,2}\phi_t^{N,N-1}] + \cdots \\
&\quad + (1-z^N)[\phi_t^{N,N-2}\phi_t^{N,2} + z\phi_t^{N,N-2}\phi_t^{N,3} + \cdots + z^{N-3}\phi_t^{N,N-2}\phi_t^{N,N-1}] \\
&\quad + (1-z^N)[\phi_t^{N,N-1}\phi_t^{N,1} + z\phi_t^{N,N-1}\phi_t^{N,2} + \cdots + z^{N-2}\phi_t^{N,N-1}\phi_t^{N,N-1}] \\
&= (S_t^N(z))^2 + (1-z^N)\big[\sum_{k=1}^{N-1}\phi_t^{N,k}\phi_t^{N,N-k} + z\sum_{k=2}^{N-1}\phi_t^{N,k}\phi_t^{N,N+1-k} \\
&\qquad + \cdots + z^{N-2}\sum_{k=N-1}^{N-1}\phi_t^{N,k}\phi_t^{N,N+(N-2)-k}\big] - (1-z)\epsilon \\
&= (S_t^N(z))^2 + (1-z^N)\big[\sum_{j=0}^{N-2} z^j \cdot \sum_{k=j+1}^{N-1}\phi_t^{N,k}\phi_t^{N,N+j-k}\big] - (1-z)\epsilon, \\
&\text{with } \; S_T^N(z) = (1-z)c.
\end{aligned} \tag{74}
$$

For $z=1$, $\dot{S}_t^N(1) = (S_t^N(1))^2,\ S_T^N(1) = 0$, we can get:

$$
S_t^N(1) = \sum_{k=0}^{N-1}\phi_t^{N,k} = 0,\ i.e., \quad \phi_t^{N,0} = -\sum_{k=1}^{N-1}\phi_t^{N,k}.
$$