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Zeta Function of Hypersurfaces with ADE Singularities in $\mathbb{P}^3$

DISSERTATION

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by

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DEDICATION

This paper dedicated to my dad who supported me all these years, my mom who passed away early in my life, and my friends who supported me all this way.

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SOFTWARE

Zeta Function Code `https://zenodo.org/badge/latestdoi/422038282`
A Sage algorithm for computing zeta functions and basis for rigid cohomology

ABSTRACT OF THE DISSERTATION

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Doctor of Philosophy in Mathematics

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Professor Vladimir Baranovsky, Chair

Stetson and Baranovsky provided an algorithm with Mathematica to compute the zeta function of projective hypersurface over $\mathbb{F}_p$ with isolated ordinary double points. In this thesis, I extend this algorithm to hypersurfaces with ADE singularities over $\mathbb{P}^3$. In the process of doing so, I characterize the Jacobian ideal as a zero set of differential operators.

Chapter 1

Introduction

1.1 Zeta Function

In the past, the zeta functions for smooth surfaces have been computed using different methods. One method to computer zeta function of smooth surfaces is the deformation method given in Lauder [12] and [13]. A second method involves computing the action of Frobenius operator and reducing the image in cohomology–see Costa, Harvey, and Kedlaya [4] and Rybakov [16]. A Sage code for computing zeta function of smooth surfaces is given by Sperber and Voight [20]. Stetson and Baranovsky [19] extended zeta function computation to hypersurfaces with ordinary double points, and Scott Stetson created a code in Mathematica computing the zeta functions of hypersurfaces with ordinary double points. I will generalize the theory from Stetson and Baranovsky [19] and give an algorithm for computing the zeta functions of hypersurfaces with ADE singularities in the end of Chapter 3. Furthermore, I will provide a Sage code for computing such zeta functions.

Definition 1.1. . *Given a projective hypersurface X defined by equation $f(w:x:y:z)=0$ in $\mathbb{P}^3$ with coordinates over a PID $\mathbb{K}$, the affine cone over X is zero set of $f(w:x:y:z)$*

viewed as a function over $\mathbb{K}^4$*.*

Definition 1.2. . *Let* X *be a hypersurface in* $\mathbb{P}^3$ *given by the zero set of* $f(w:x:y:z)$*. Let* $\tilde{f}$ *be* f *viewed as a function on the affine chart* $V=\{w=1\}$*. Let* s *be a* $\mathbb{Z}_p$ *valued point on* $X\cap V$*. Then by linear change of coordinates,* s *corresponds to the origin given by a map* $\mathbb{Z}_p[x,y,z]\to\mathbb{Z}_p$*. The formal completion along the kernel of the map which is given by the power series* $\mathbb{Z}_p[[x,y,z]]$*. Let* g *be a function given by one of the following equations:*

$$A_n : x_0^2 + x_1^2 + x_2^{n+1} = 0$$
$$D_n : x_0^2 + x_2(x_1^2 + x_2^{n-2}) = 0$$
$$E_6 : x_0^2 + x_1^3 + x_2^4 = 0$$
$$E_7 : x_0^2 + x_1(x_1^2 + x_2^3) = 0$$
$$E_8 : x_0^2 + x_1^3 + x_2^5 = 0$$

Then s *is an equisingular ADE singularity over* $\mathbb{Z}_p$ *if there is an isomorphism* $\phi:\mathbb{Z}_p[[x,y,z]]\to\mathbb{Z}_p[[x_0,x_1,x_2]]$ *such that* $\phi(\tilde{f})=\tilde{g}$ *and* ϕ *is compatible with the surjections onto* $\mathbb{Z}_p$*.*

There is a definition for ADE singularities over finite fields is given in Greuel [10]. We will stick with the 5 given for now. The procedure we will create will cover the other forms given in Greuel [10].

Let p be a prime. Let $N_r = |X(\mathbb{F}_{p^r})|$. Let $\tilde{X}$ be the reduction of X in $\mathbb{P}^3$ to a variety over $\mathbb{F}_p$. Then the zeta function of a variety $\tilde{X}$ over $\mathbb{F}_p$ is a generating function for N_r given by

$$Z(\tilde{X},t) = Z(t) = \exp\sum_{r=1}^{\infty}\frac{N_r t^r}{r}.$$

Let $\tilde{U}=\mathbb{P}^3-\tilde{X}$. Then $\tilde{U}$ is affine and smooth. From formula of Gerkmann [9] applied to

$\mathbb{P}^3$,

$$Z(\tilde{X}, t) = \frac{1}{(1-t)(1-pt)(1-p^2t)v(t)}$$

where $v(t) = \det(1 - tq^3\mathrm{Frob}_q^{-1}|H^2_{\mathrm{rig}}(\tilde{U}))$ and $H^2_{\mathrm{rig}}(\tilde{U})$ is a rigid cohomology group of $\tilde{U}$. For information on rigid cohomology, I refer the reader to Stum [21]. For affine schemes such as $\tilde{U}$, rigid cohomology is equivalent to Monsky-Washnitzer cohomology which I will discuss in the beginning of Chapter 2. Furthermore, we simplify even more. Suppose $\tilde{U} = \hat{P} - \hat{X}$ where $\hat{P}$ is smooth over $\mathbb{Z}_p$ and $\hat{X}$ is a smooth strict relative divisor with normal crossings. This will be defined in Chapter 4. If we know $\hat{P}$ and $\hat{X}$ exists, then theorem from Baldassarri and Chiarellotto [2] holds, i.e.

$$H^i_{\mathrm{rig}}(\tilde{U}) \cong H^i_{\mathrm{dR}}(U_{\mathbb{Q}_p}) \qquad 0 \leq i \leq 2\mathrm{dim}(U)$$

where the right hand side is the de Rham cohomology on $U_{\mathbb{Q}_p}$ and $U_{\mathbb{Q}_p}$ is the complement to the zero set of f over $\mathbb{Q}_p$. We will show the theorem applies with $\hat{P}$ being appropriate blow up of $\mathbb{P}^3$.

1.1.1 De Rham Cohomology on the Complement of a Hypersurface

A differential n-form in $\mathbb{Q}_p^4$ is a form $\omega = \sum_I c_I dx_{i_1} \wedge ... \wedge dx_{i_n}$ where $I = (i_1, ..., i_n)$ and $c_I \in \mathbb{Q}_p[x_0, ..., x_3]$. Let Ω^n_m be the space of n-forms of weight m where if $\omega = x_0^{a_0}...x_s^{a_s}dx_{i_1}\wedge...\wedge dx_{i_s}$,

$$|\omega| = a_0 + ... + a_s + s.$$

A differential form on $U_{\mathbb{Q}_p}$ in $\mathbb{P}^3$ pulls back to a differential form on an open subset $\tilde{U}_{\mathbb{Q}_p}$ in $\mathbb{Q}_p^4$. From Dimca [5], the pullback n-form ω for $n > 0$ on $\tilde{U}_{\mathbb{Q}_p}$ is written as $\omega = \frac{\Delta(\gamma)}{f^s}$, where Δ is the contraction of the Euler vector field $\sum x_i \frac{\partial}{\partial x_i}$ and $\gamma \in \Omega_{sN}^{v+1}$ where $N = \deg(f)$ and $|\Delta(\gamma)| = sN$. One can calculate that $d\omega = -\frac{\Delta(fd\gamma - sdf\wedge\gamma)}{f^{s+1}}$.

Definition 1.3. *For $n \geq 0$, we define our differential operator $d_f \colon \Omega^n \longrightarrow \Omega^{n+1}$ to be*

$$d_f(\omega) = fd\omega - \frac{|\omega|}{N} df \wedge \omega$$

for homogeneous differential form ω.

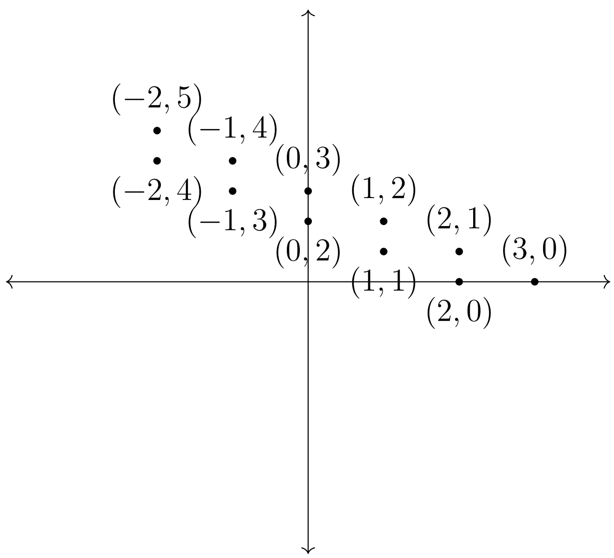
In other words, the homogeneity condition allows us to forget denominators and work with polynomials. However, our differential is no longer the usual one since the differential is now in the form of the Koszul differential plus the de Rham differential.

Definition 1.4. *Let (B, d', d'') be the double complex given by $B^{s,t} = \Omega_{tN}^{s+t+1}$ where $d' = d$ and $d''(\omega) = -|\omega|N^{-1}df \wedge \omega$ for a homogeneous differential form ω.*

Definition 1.5. *Let $(Tot(B)^*, D_f)$ be the total complex given by $Tot(B)^m = \bigoplus_{s+t=m} B^{s,t}$ with filtration $F^s Tot(B)^m = \bigoplus_{k\geq s} B^{k,m-k}$ where $D_f = d' + d''$.*

The goal is to compute the cohomology of the double complex which gives the cohomology on $U_{\mathbb{Q}_p}$. For this, we use the theory of spectral sequences. For purposes of results regarding the spectral sequence, we use $\mathbb{Q}_p$ and $\mathbb{C}$ interchangeably since result holds for both fields due to the embedding of $\mathbb{Q}_p$ into $\mathbb{C}$. Let $\dim_{\mathbb{Q}_p} f^{-1}(0)$ is the dimension of the singular locus. Let $H^k(K_f^*)$ is the cohomology with respect to the d'' differential. Saito [17] shows if $m = \dim_{\mathbb{Q}_p} f^{-1}(0)$, then $H^k(K_f^*) = 0$ for $k \leq 3 - m$. In the smooth case, $m = 0$ so only the top cohomology group $H^4(K_f^*)_{tN}$ is nonzero. As only one diagonal remains on the E_1 page, the de Rham differential is trivial; hence, in the smooth case, the spectral sequence degenerates at the E_1 page and converges to the cohomology of the total complex.

In the singular case, $m = 1$ so $H^3(K_f^*)_{tN}$ and $H^4(K_f^*)_{tN}$ are nonzero. Since there are two diagonals on the E_1 page, the de Rham differential need not be trivial. Below is a piece of the E_1 page which displays the top diagonal and subdiagonal below it.



For the purposes of this paper, we define the local Milnor number as the index of the type A_n, D_n, E_n singularities. For example, a type A_3 singularity has Milnor number 3. The global Milnor number, $\mu(X)$, is defined to be the sum of all local Milnor numbers of singular points. By Corollary 1.5 of Dimca and Sticlaru [8], $H^n(K_f)_m = \mu(X)$ for $m \geq 3(N-2)$ where $N = \deg(f)$. (The general formula is $n(N-2)$ if we are working in $\mathbb{P}^n$ instead of $\mathbb{P}^3$.) Hence, the dimensions of the vector spaces of the diagonals eventually stabilize to the global Milnor number. Furthermore, Theorem 2 of Saito [18] proved that for hypersurfaces with weighted homogeneous singularities, the spectral sequence degenerates on the E_2 page. From equation 2.10 of Dimca and Sticlaru [7], for weighted homogeneous equations, all nonzero terms on the E_2 page lie inside the first quadrant, not including the x-axis and y-axis. Using this, I constructed a Sage code computing the basis elements on the E_2 page. The code mainly involves constructing the matrix for the two differentials and using linear algebra to compute the quotient groups.

Chapter 2

Frobenius and Cohomology

2.1 Monsky-Washnitzer Cohomology

Let U be a smooth affine variety over $\mathbb{Z}_p$ that is the complement to a hypesurface X given by the zero set of a polynomial f. Let $f_1, ..., f_m$ denote the partial derivatives. Then the rigid cohomology groups coincide with the Monsky-Washnitzer cohomology groups. We define the Monsky Washnitzer cohomology groups which is much simpler than the general rigid cohomology groups. Let

$$\mathbb{Z}_p < w, x, y, z >^{\dagger} = \{\sum a_\alpha X^\alpha \in \mathbb{Z}_p[[w, x, y, z]] : \exists C > 0, \rho \in (0, 1) \ with |a_\alpha|_p \leq C\rho^{|\alpha|} \forall \alpha\}$$

where X denotes any monomial in w, x, y, z and α is a multi-index and $|\alpha|$ is the sum of the exponents of w, x, y, z. Let $A^\dagger = \mathbb{Z}_p < w, x, y, z >^\dagger /(f_1, ..., f_m)$. Let $\tilde{\Omega}_{A^\dagger/\mathbb{Z}_p} = \bigoplus_i^n A^\dagger x_i / <$ $\sum \frac{\partial f_k}{\partial f_j} dx_j | k \in \{1...m\} >$. Now let $\Omega^i = \bigwedge^i \tilde{\Omega}_{A^\dagger/\mathbb{Z}_p}$. Let $d\bar{x} = dx_{j_i} \wedge ... \wedge dx_{j_i}$. Define

$$d^i : \Omega^i \to \Omega^{i+1}, f d\bar{x} \to \left(\sum_{j=1}^{4} \frac{\partial f}{\partial x_j} dx_j \right) \wedge d\bar{x}$$

Then the Monsky-Washnitzer cohomology groups are the cohomology groups of the complex given by

$$0 \to A^{\dagger} \xrightarrow{d^0} \Omega^1 \xrightarrow{d^1} \Omega^2 \to ...$$

The theorem from Baldassarri and Chiarellotto [2] given in Chapter 1 allows us to work with de Rham cohomology groups on U instead of the Monsky-Washnitzer cohomology groups on U. The Monsky-Washnitzer cohomology groups come with an action of the Frobenius operator which we define as follows.

2.2 Action of Frobenius Operator

The basis elements on the E_2 page of the spectral sequence given before gives a basis on the de Rham cohomology of U which gives a basis on the Monsky-Washnitzer cohomology groups.For the remainder of the text, let $\Omega = dw \wedge dx \wedge dy \wedge dz$. A basis element h on the E_2 page corresponds to the basis element $\frac{h\Omega}{f^{\ell}}$ on the Monsky-Washnitzer cohomology groups. From equation 4.1 of Gerkmann [9], the action of the lifted Frobenius operator, $\hat{F}$, is

$$\hat{F}\left(\frac{h\Omega}{f^{\ell}}\right) = p^3 \frac{h(w^p, x^p, y^p, z^p) \prod_{i=0}^{3} x_i^{p-1} \Omega}{f^{p\ell}} \left(\sum_{k=0}^{\infty} p^k \frac{\alpha_k g^k}{f^{pk}}\right),$$

where α_k is the k-th coefficient of the power series expansion $(1-t)^{-\ell} = \alpha_0 + \alpha_1 t + \alpha_2 t^2 + ...$, $x_0 = w, x_1 = x, x_2 = y, x_3 = z$, $pg = f(w,x,y,z)^p - f(w^p, x^p, y^p, z^p)$, and the element on the right hand side of the equation is an element on the Monsky-Washnitzer complex. Given each term in the sum, the goal is to express the image of Frobenius that is cohomologically equivalent to a linear combination of the basis elements on the E_2 page.

Since the action of the Frobenius operator is an infinite sum, we first truncate the sum into a finite sum. We compute the truncated action and express the image into something that is cohomologically equivalent. This turns out to be a linear combination of the basis elements of the Monsky-Washnitzer cohomology groups. The truncated action of Frobenius, applied to each basis element, gives a square matrix for the action. Corollary 4.2 of Gerkmann [9] goes over how far one needs to truncate, but for the purposes of Sage calculation, each additional term in the sum generally gives us accuracy up to the next power of p. To be more accurate, the p-adic expansion of the numbers in the truncated Frobenius matrix converge to the p-adic expansion of the actual value of the Frobenius matrix. If $k = 0$ gives accuracy up to p^2, then going up to $k = 1$ gives accuracy up to p^3. If one of the entries is -5, then one might see

$$\begin{aligned} k = 0 &\to 5 + 5^2 + 2 \cdot 5^3 + 3 * 5^4 + ... \\ k = 0,1 &\to 5 + 4 \cdot 5^2 + 3 \cdot 5^3 + 2 \cdot 5^4 + ... \\ k = 0,1,2 &\to 5 + 4 \cdot 5^2 + 4 \cdot 5^3 + \cdot 2 \cdot 5^4 + ... \end{aligned}$$

Continuing on, one will see the values converge to the $5 + 4 \cdot 5^2 + 4 \cdot 5^3 + 4 \cdot 5^4 + ...$, the p-adic expansion of -5.

While the goal is clear, there are several issues that come into play. The first issue is that reduction in cohomology is computationally infeasible if we go the direct route which is explained in the following example. The second issue is that reduction involves using a Gröbner basis algorithm which may get computationally large. The third issue is that the image of Frobenius has high powers of f in the denominator. Although my code does not fix the Gröbner basis issue, my code does fix the other two problems.

Example 2.1. *Let f be degree 3 with $p = 5$ and h_1, h_2, h_3 be our basis elements on the E_2 page of degree 2. Then by homogeneity, $\ell = 2$ in the Frobenius equation above. Now to make* *things simple, let us consider $k = 0$. The sum goes away and on the denominator we have $f^{p\ell} = f^{10}$. Since f is cubic, the denominator is degree 30. In order to express the image as a linear combination of h_1, h_2, h_3, one will need to compute the cokernel of the $df\wedge$ map. Computing the quotient is computationally long, and we only reduce the degree of the image by 3. Furthermore, this is just the $k = 0$ term of the summation and only for one of our basis elements.*

We now explain the action of inverse of the Frobenius operator which solves one of the issues listed above.

2.3 Action of the Inverse of Frobenius Operator

To work with lower powers in general, we decide to use the left inverse of Frobenius. Remke [15] showed that on the level of varieties, Frobenius has a left inverse. As Frobenius is invertible after passing to the level of cohomology, we have an action for the left inverse. Let us denote the left inverse by $\hat{F}^{-1}$. Let $\psi : A^\dagger \to A^\dagger$ be the Q_p linear operator given by

$$\psi(\prod x_i^{a_i}) = \begin{cases} \prod x_i^{a_i/p} & \text{if } a_i \equiv 0 (\text{mod } p) \quad \forall i \\ 0 & \text{otherwise} \end{cases}$$

Note that since the action of Frobenius operator is taking p-th powers, the inverse should involve taking p-th roots. Taking $x_0 = w, x_1 = x, x_2 = y, x_3 = z$ and $\Delta = f(w,x,y,z)^p - f(w^p, x^p, y^p, z^p)$, the action of the inverse is given by

$$\hat{F}^{-1}\left(\frac{h\Omega}{f^\ell}\right) = \left(\sum_k \frac{\psi(f^{p-\ell} h \prod_{i=0}^3 x_i \Delta^i)}{f^{k+1}}\right) \frac{\Omega}{p^3 \prod_{i=0}^3 x_i}.$$

Note this fixes one of the issues given in Example 1. In Example 1, for $k = 0$, the summation gives a degree 26 image for the coefficient of the 4 form on the numerator. The image coefficient of the 4 form on the numerator for $k = 1$ for the inverse is only degree 2. The image coefficient of the 4-form on the numerator for $k = 9$ of the inverse is of degree 26 which is the degree in the original Frobenius image for $k = 0$. Working with low degrees for high values of k makes computation slightly easier. The issue about computing the matrices and quotients still remain.

Chapter 3

Differential Operators and Subdiagonal

Recall from Dimca and Sticlaru [7], $H^n(K_f)_m = \mu(X)$ for $m \geq 3(N-2)$. We call the values of m such that $m \geq 3(N-2)$ the *stable range*. Now, since the top diagonal for a smooth hypersurface on the E_1 page lies only in the first quadrant and the Euler characteristic is independent of whether the hypersurface is smooth or singular, we can conclude that $H^{n+1}(K_f)_m = \mu(X)$. Moreover, the E_2 page is 0 in this stable range.

We now dive into where the assumption of our hypersurface X having ADE singularities is used aside from showing the theorem from Baldassarri and Chiarellotto [2] holds which is explained in Chapter 4.

Suppose $h_1, ..., h_m$ are basis elements of the Monsky-Washnitzer cohomology groups. Let h be the element corresponding to the deepest pole of the truncated image of the Frobenius operator applied to one of the h_i. Then suppose $h \in P_k\Omega_4$. Then find a basis for $P_{k+1}\Omega_3$. As we are in the stable range, there are $\mu(X)$ basis elements which we name as $\beta_1, ..., \beta_{\mu(x)}$.

How we find these basis elements in high degree will be explained later. Applying the de Rham differential, $d\beta_1, ..., d\beta_{\mu(X)} \in P_k\Omega_4$. As the E_2 page is 0 on the stable range, lifting h back to E_0 gives $h = a_1 d\beta_1 + ... + a_{\mu(X)} d\beta_{\mu(X)} + f_w h_1 + f_x h_2 + f_y h_3 + f_z h_4$. Choose a singular point s of our hypersurface X. The partials evaluated at s is 0. Stetson and Baranovsky [19] showed that if all singularities are type A_1, we can evaluate at the singular points and can solve for the variables given.

To see this, suppose there exists one A_1 singularity. Then $g - a_1 d\beta_1 = b_1 f_w + b_2 f_x + b_3 f_y + b_4 f_z$. To find a_1, evaluate both sides at the singular point. Then the right hand side is 0 by definition of a singular point. Note, plugging in any other point will give an equation but the issue is that b_1, b_2, b_3, b_4 are unknown. Similarly, suppose there are k A_1 singularities. Then $g - a_1 d\beta_1 - ... - a_k d\beta_k = b_1 f_w + b_2 f_x + b_3 f_y + b_4 f_z$. We need to find $a_1, ..., a_k$; so k equations are needed. Evaluation at each of the singular points will give k equations. The equations will be linearly independent. In fact, I will show linear independence for the general ADE case later in the paper. Hence, for A_1 singularities, finding the de Rham component of g is simple.

Suppose our hypersurface has one A_2 singularity. Then $g - a_1 d\beta_1 - a_2 d\beta_2 = b_1 f_w + b_2 f_x + b_3 f_y + b_4 f_z$. We need to find a_1 and a_2 but evaluating at the singular point only gives 1 equation. Where will the second equation come from? In this case, the normal form of an A_2 singularity is $uv = t^3$. The partials are given by $v, u, 3t^2$. Along with evaluation at the origin, the operator given by $\frac{\partial}{\partial t}|_{(0,0,0)}$ annihilates the Jacobian ideal. The idea is to transfer this operator to the original coordinates to obtain the second operator for the second equation. In the general case, I establish an equality between the space annihilated by specific operators depending on our ADE singularities and the Jacobian ideal for polynomials with degree in the stable range–See Theorem 3.1.

3.1 Operators on ADE Singularities

Before we continue, in the case that there are two singularities in the same affine open set, we need an algebraic way of working locally around the singularity.

Definition 3.1. *Let M be a finite dimensional module over a polynomial ring R in several variables over $\mathbb{C}$. Let $\tilde{R}$ be the power series ring in the variables of R. We define the formal completion of M as $M \otimes_R \tilde{R}$.*

Definition 3.2. *A module over a polynomial ring in variables (x, y, z) is supported at (α, β, γ) if $\exists N$ such that $\forall\ k \geq N$, $(x-\alpha)^k M = (y-\beta)^k M = (z-\gamma)^k M = 0$.*

Proposition 3.1. *Suppose M is a finite dimensional module over a polynomial ring supported at the origin. Let $\tilde{M}$ be the formal completion of M. Then $M \cong \tilde{M}$ as R-modules.*

Proposition 3.2. *Suppose M is a finite dimensional module over a polynomial ring supported at $(\alpha, \beta, \gamma) \neq (0, 0, 0)$. Let $\tilde{M}$ be the formal completion of M. Then $\tilde{M} = 0$.*

Assuming these two claims, formal completion is a way to study a singularity locally. Proposition 3.1 justifies working with the polynomial ring as opposed to the power series ring.

Proof. I will prove Proposition 3.1 first. Let $R = \mathbb{C}[x, y, z]$. Suppose M is generated by $s_1, ..., s_j$. Then an arbitrary element of $\tilde{M}$ is of the form $\sum_i^j s_i \otimes f_i$ where $f_i \in \tilde{R}$. As M is supported at the origin, there exists N such that for $k \geq N$, $x^k M = 0, y^k M = 0, z^k M = 0$. Now given $h \in \tilde{R}$, we can express h as

$$h = x^N h_1 + y^N h_2 + z^N h_3 + h_4$$

where h_4 is a polynomial with powers of x, y, z smaller than N and $h_1, h_2, h_3 \in \tilde{R}$. Let

$h^{<N} = h_4$. Define ϕ by

$$\phi : \tilde{M} \longrightarrow M$$
$$\phi(\sum_{i=1}^{j} s_i \otimes f_i) = \sum_{i=1}^{j} f_i^{<N} s_i$$

Then to show linearity,

$$\begin{aligned}
\phi(\sum_i s_i \otimes f_i + \sum_i s_i \otimes h_i) &= \phi(\sum_i s_i \otimes (h_i + f_i)) \\
&= \sum_i (h_i + f_i)^{<N} s_i = \sum_i h_i^{<N} s_i + \sum_i f_i^{<N} s_i \\
&= \phi(\sum_i s_i \otimes f_i) + \phi(\sum_i s_i \otimes h_i)
\end{aligned}$$

and $\forall r \in R$,

$$\begin{aligned}
\phi(r(\sum_i s_i \otimes f_i)) &= \phi(\sum_i s_i \otimes r f_i) = \sum_i r f_i^{<N} s_i \\
&= \sum_i r^{<N} f_i^{<N} s_i = r^{<N} \sum_i f_i^{<N} s_i \\
&= r \sum_i f_i^{<N} s_i = r\phi(\sum_i s_i \otimes f_i)
\end{aligned}$$

where the first and second to last equality on the previous line is because any degree N piece or higher acts by 0 since M is supported by the origin. Hence, ϕ is an $R-$module homomorphism. The map is surjective as any element of M is given by $\sum_i r_i s_i$ for $r_i \in R$ and ϕ maps the element $\sum s_i \otimes r_i$ to $\sum r_i s_i$. Now suppose $\phi(\sum_i s_i \otimes f_i) = \sum f_i^{<N} s_i = 0$. Then we can write $f_i^{\geq N} = f_i - f_i^{<N}$. Then

$$\sum_i s_i \otimes f_i = \sum_i s_i \otimes f_i^{<N} + \sum_i s_i \otimes f_i^{\geq N}.$$

For the first sum, as we tensor over R,

$$\sum_i s_i \otimes f_i^{<N} = \sum_i s_i f_i^{<N} \otimes 1 = (\sum_i s_i f_i^{<N}) \otimes 1 = 0 \otimes 1 = 0.$$

For the second sum,

$$\begin{aligned}\sum_i s_i \otimes f_i^{\geq N} &= \sum_i s_i \otimes x^N f_{i,1} + \sum_i s_i \otimes y^N f_{i,2} + \sum_i s_i \otimes z^N f_{i,3} \\ &= \sum_i x^N s_i \otimes f_{i,1} + \sum_i y^N s_i \otimes f_{i,2} + \sum_i z^N s_i \otimes f_{i,3} = 0.\end{aligned}$$

Hence, the map is injective. So we have an isomorphism. □

Proof. For Proposition 3.2, proceed the same way. For $(\alpha, \beta, \gamma) \neq (0,0,0)$, there exists N such that for $k \geq N$, $(x-\alpha)^k M = (y-\beta)^k M = (z-\gamma)^k M = 0$. Without loss of generality, let us just assume $\alpha \neq 0$. Now consider $\tilde{M}$. First note that $(x-\alpha)^N \tilde{M} = 0$ as its action on M is zero. But then

$$1 \cdot M \otimes \tilde{R} = M \otimes 1 \cdot \tilde{R} = M \otimes (x-\alpha)^N (x-\alpha)^{-N} \tilde{R} = (x-\alpha)^N M \otimes (x-\alpha)^{-N} \tilde{R} = 0.$$

Hence, $\tilde{M}$ is the 0 module. Note, we are using the fact that in the power series ring, a power series with constant term is invertible; therefore, since $\alpha \neq 0$, $(x-\alpha)^N$ is an invertible element in $\tilde{R}$. □

Before continuing to the main theorem, I will show that the quotient by the Jacobian is invariant under analytic change of coordinates.

Let $\tilde{f}$ be the defining equation of our hypersurface X over $\mathbb{P}^3$ with coordinates w, x, y, z and suppose X has a type A_n, D_n, or E_n singularity. Let g be the standard equation of the given singularity. Without loss of generality, assume the singularity lies in the affine set $w = 1$. Let $f = \tilde{f}(1, x, y, z)$. Then there exists $g_1, g_2, g_3 \in \mathbb{C}[[u, v, t]]$ and an analytic change of coordinates map $\phi : \mathbb{C}[[x, y, z]] \to \mathbb{C}[[u, v, t]]$ given by

$$\phi(x) = g_1(u, v, t), \phi(y) = g_2(u, v, t), \phi(z) = g_3(u, v, t)$$

that maps $f(x, y, z)$ to $g(u, v, t)$. Furthermore, there exists $f_1, f_2, f_3 \in \mathbb{C}[[x, y, z]]$ and an analytic change of coordinates map, $\psi : \mathbb{C}[[u, v, t]] \to \mathbb{C}[[x, y, z]]$, given by

$$\psi(u) = f_1(x, y, z), \psi(v) = f_2(x, y, z), \psi(t) = f_3(x, y, z)$$

such that $\psi \circ \phi$ is the identity map.

Lemma 3.1. *The quotient by the Jacobian is invariant under analytic change of coordinates if the maps ϕ and ψ defined above descend to an isomorphism on quotients rings $\mathbb{C}[[x, y, z]]/(f)$ and $\mathbb{C}[[u, v, t]]/(g)$.*

Proof. To show isomorphism on the quotient rings, we have to show the map ϕ is well defined when passing to the quotient. First consider the ideal $(f_x, f_y, f_z)\mathbb{C}[[x, y, z]]$.

$$\phi(\frac{\partial f}{\partial x}) = \phi(\frac{\partial t}{\partial x})(\frac{\partial g}{\partial t}) + \phi(\frac{\partial u}{\partial x})(\frac{\partial g}{\partial u}) + \phi(\frac{\partial v}{\partial x})(\frac{\partial g}{\partial v}) = \phi(\frac{\partial t}{\partial x})g_t + \phi(\frac{\partial u}{\partial x})g_u + \phi(\frac{\partial v}{\partial x})g_v.$$

Now $\frac{\partial t}{\partial x}, \frac{\partial u}{\partial x}, \frac{\partial v}{\partial x} \in \mathbb{C}[[x, y, z]]$ as these are just the derivative with respect to x of f_1, f_2, f_3. ϕ maps these elements to a power series in u, v, t. Hence we have that $\phi(\frac{\partial f}{\partial x}) \in \mathbb{C}[[u, v, t]]$. By same argument, $\phi(\frac{\partial f}{\partial y}), \phi(\frac{\partial f}{\partial z}) \in \mathbb{C}[[u, v, t]]$ since

$$\phi(\frac{\partial f}{\partial y}) = \phi(\frac{\partial t}{\partial y})(\frac{\partial g}{\partial t}) + \phi(\frac{\partial u}{\partial y})(\frac{\partial g}{\partial u}) + \phi(\frac{\partial v}{\partial y})(\frac{\partial g}{\partial v}) = \phi(\frac{\partial t}{\partial y})g_t + \phi(\frac{\partial u}{\partial y})g_u + \phi(\frac{\partial v}{\partial y})g_v$$

and

$$\phi(\frac{\partial f}{\partial z}) = \phi(\frac{\partial t}{\partial z})(\frac{\partial g}{\partial t}) + \phi(\frac{\partial u}{\partial z})(\frac{\partial g}{\partial u}) + \phi(\frac{\partial v}{\partial z})(\frac{\partial g}{\partial v}) = \phi(\frac{\partial t}{\partial z})g_t + \phi(\frac{\partial u}{\partial z})g_u + \phi(\frac{\partial v}{\partial z})g_v.$$

Hence, $\phi((f_x, f_y, f_z)\mathbb{C}[[x,y,z]]) \subset (g_u.g_v, g_t)\mathbb{C}[[u,v,t]]$. Hence ϕ induces a map

$$\tilde{\phi} : \mathbb{C}[[x,y,z]]/(f_x, f_y, f_z) \to \mathbb{C}[[u,v,t]]/(g_u, g_v, g_t)$$
$$\tilde{\phi}([h]) = [\phi(h)]$$

where h is any lift of $[h]$.

We prove the same holds for ψ.

$$\psi(\frac{\partial g}{\partial u}) = \psi(\frac{\partial x}{\partial u})(\frac{\partial f}{\partial x}) + \psi(\frac{\partial y}{\partial u})(\frac{\partial f}{\partial y}) + \psi(\frac{\partial z}{\partial u})(\frac{\partial f}{\partial z})$$

$$\psi(\frac{\partial g}{\partial v}) = \psi(\frac{\partial x}{\partial v})(\frac{\partial f}{\partial x}) + \psi(\frac{\partial y}{\partial v})(\frac{\partial f}{\partial y}) + \psi(\frac{\partial z}{\partial v})(\frac{\partial f}{\partial z})$$

and

$$\psi(\frac{\partial g}{\partial t}) = \psi(\frac{\partial x}{\partial t})(\frac{\partial f}{\partial x}) + \psi(\frac{\partial y}{\partial t})(\frac{\partial f}{\partial y}) + \psi(\frac{\partial z}{\partial t})(\frac{\partial f}{\partial z}).$$

Hence, ψ induces a map

$$\tilde{\psi} : \mathbb{C}[[u,v,t]]/(g_u, g_v, g_t) \longrightarrow \mathbb{C}[[x,y,z]]/(f_x, f_y, f_z)$$
$$\tilde{\psi}([h]) = [\psi(h)]$$

where h is any lift of $[h]$. Since the composition of ψ and ϕ is the identity, we have an isomorphism of quotient rings. □

We now prove one of the main theorems and a following Proposition involving operators on ADE singularities.

Theorem 3.1. *Let s be the origin. Let m denote a number in the stable range. Let $k \in \mathbb{Z}$*

and $\mathbb{C}[u,v,t,r]_k$ *be the polynomials of degree* k. *Let* J_k *denote the polynomials in the Jacobian of degree* k. *For type* A_n *singularities, the space of polynomials in* $\mathbb{C}[u,v,t,r]_{\geq m}$ *annihilated by the differential operators*

$$ev|_s, \frac{\partial}{\partial t}|_s, \frac{\partial^2}{\partial^2 t}|_s, \ldots \frac{\partial^{n-1}}{\partial^{n-1} t}|_s$$

is equal to J_m. *For* D_n *singularities, the space of polynomials in* $\mathbb{C}[u,v,t,r]_m$ *annihilated by the differential operators*

$$\frac{\partial}{\partial v}|_s, \frac{\partial}{\partial t}|_s, \frac{\partial^2}{\partial^2 t}|_s, \ldots, \frac{\partial^{n-3}}{\partial^{n-3} t}|_s$$

is equal to the J_m. *For* E_6 *singularities, the operators are*

$$\frac{\partial}{\partial v}|_s, \frac{\partial}{\partial t}|_s, \frac{\partial}{\partial v}\frac{\partial}{\partial t}|_s, \frac{\partial^2}{\partial^2 t}|_s, \frac{\partial^2}{\partial^2 t}\frac{\partial}{\partial v}|_s.$$

For E_7 *singularities, the operators are*

$$\frac{\partial}{\partial v}|_s, \frac{\partial}{\partial t}|_s, \frac{\partial}{\partial v}\frac{\partial}{\partial t}|_s, \frac{\partial^2}{\partial^2 t}|_s, \frac{\partial^3}{\partial^3 t} - \frac{\partial^2}{\partial^2 v}|_s, \frac{\partial^4}{\partial^4 t} - 3\frac{\partial^2}{\partial^2 v}\frac{\partial}{\partial t}|_s.$$

For E_8 *singularities, the operators are*

$$\frac{\partial}{\partial v}|_s, \frac{\partial}{\partial t}|_s, \frac{\partial}{\partial v}\frac{\partial}{\partial t}|_s, \frac{\partial^2}{\partial^2 t}|_s, \frac{\partial^2}{\partial^2 t}\frac{\partial}{\partial v}|_s, \frac{\partial^3}{\partial^3 t}|_s, \frac{\partial^3}{\partial^3 t}\frac{\partial}{\partial v}|_s.$$

Proposition 3.3. *The differential operator* $\frac{\partial^k}{\partial^k t}|_{(0,0,0)}$ *is mapped by inverse change of coordinates to a combination of* k *order and lower differential operators in* x, y, z *evaluated at the origin with polynomial coefficients through the analytic change of coordinates. The same holds true if we replace* t *with* u *or* v.

Proof. To prove Theorem 3.1, we first consider the A_n case. Let S be the space of polynomials annihilated by corresponding differential operators given in Theorem 3.1. Let J be the Jacobian ideal. The partials of the standard A_n equation are given by u, v, t^n. Note that $J_m \subset S_m$ since the differential operators given in Theorem 3.1 annihilate the Jacobian ideal. To prove this, we proceed by induction. First, assume the singular point lies in the affine set $r = 1$. It is clear that evaluation at the origin annihilates the partials since this is the singular point. Let $h = h_1 u + h_2 v + h_3 t^n$ where $h_1, h_2, h_3 \in \mathbb{C}[u, v, t]$ such that $h \in \mathbb{C}[u, v, t, r]_m$. Note r is irrelevant as the operators involve only u, v, t so evaluation at $r = 1$ annihilates any r term. For simplicity, let s be the point $t = 0, u = 0, v = 0, r = 1$. The product rule shows that

$$\frac{\partial}{\partial t} h|_s = \frac{\partial}{\partial t} h_1|_s \cdot u|_s + \frac{\partial}{\partial t} h_2|_s \cdot v|_s + \frac{\partial}{\partial t} h_3|_s \cdot t^n|_s + n h_3 t^{n-1}|_s = 0$$

Suppose that

$$\frac{\partial^i}{\partial^i t} h_1 u|_s, \frac{\partial^i}{\partial^i t} h_2 v|_s, \frac{\partial^i}{\partial^i t} h_3 t^n|_s = 0$$

for $i = 0, ..., k$ where $i = 0$ is the evaluation operator. Then for simplicity of notation, let $D(i, j) f_1 f_2 = \frac{\partial^i}{\partial^i t} f_1|_s \frac{\partial^j}{\partial^j t} f_2|_s$. Then for $k + 1 \leq n - 1$

$$\frac{\partial^{k+1}}{\partial^{k+1} t} h_1 u|_s = D(k+1, 0) h_1 u + D(k, 1) h_1 u + ... + D(1, k) h_1 u + D(0, k+1) h_1 u$$

$D(k+1, 0) h_1 u = 0$ because u evaluates to 0, and $D(0, k+1) h_1 u = 0$ since we are differentiating u with respect to t. The other terms are 0 by the induction hypothesis. For

$$\frac{\partial^{k+1}}{\partial^{k+1} t} h_2 v|_s = D(k+1, 0) h_2 v + D(k, 1) h_2 v + ... + D(1, k) h_2 v + D(0, k+1) h_2 v,$$

$D(k+1, 0) h_1 v = 0$ because v evaluates to 0, and $D(0, k+1) h_2 v = 0$ as we are differentiating

v with respect to t. The other terms are 0 by induction hypothesis. For

$$\frac{\partial^{k+1}}{\partial^{k+1}t}h_3t^n|_s = D(k+1,0)h_3t^n + D(k,1)h_3t^n + ... + D(1,k)h_3t^n + D(0,k+1)h_3t^n,$$

$D(k+1,0)h_1t^n = 0$ because t^n evaluates to 0 and $D(0,k+1)h_3t^n = 0$ as we are differentiating t^n with respect to t $k+1$ times. For $k+1 \leq n-1$, this leads to $C \cdot t^j$ for some $j > 0$ and C a constant so evaluation at the origin gives 0 .The other terms are 0 by induction hypothesis. Therefore, $J_m \subset S_m$.

Define

$$\phi : \mathbb{C}[u,v,t,r]_m \longrightarrow \mathbb{C}^n$$
$$\phi(h) = (\mathrm{ev}(h)|_s, \frac{\partial}{\partial t}h|_s, ..., \frac{\partial^{n-1}}{\partial^{n-1}t}h|_s).$$

The kernel of ϕ is S. To show surjectivity, let e_i be the vector that is 1 on the ith component and 0 elsewhere. Then 1 is mapped to e_1, t is mapped to e_2, t^2 is mapped to $2e_3$, and continuing on, t^n is mapped to $n!e_n$. Hence, the map is surjective. So we have

$$\mathbb{C}[u,v,t,r]_m/S_m \cong \mathbb{C}^n.$$

However, if the quotient by the standard equation is u, v, t^n, the quotient $\mathbb{C}[u,v,t,r]/J_m \cong \mathbb{C}^n$ is the space generated by $r^m, r^{m-1}t, ..., t^{n-1}r^{m-n+1}$. Since from above, $J_m \subset S_m$, we have that $J_m = S_m$.

Note the same proof works for type D_n and type E_n singularities as well. The standard equation for D_n is given by $u^2 + tv^2 + t^{n-1} = 0$. The Jacobian ideal is $J = (u, vt, t^{n-2} + v^2)$. The n operators that annihilate any element of the Jacobian are evaluation at the origin,

$$\frac{\partial}{\partial v}|_s, \frac{\partial}{\partial t}|_s, \frac{\partial^2}{\partial^2 t}|_s, ..., \frac{\partial^{n-3}}{\partial^{n-3}t}|_s.$$

Let S_m be the space of degree m polynomials annihilated by all the differential operators. As in the proof of Theorem 3.1, the space S_m contains J_m. Define

$$\phi : \mathbb{C}[u,v,t,r]_m \longrightarrow \mathbb{C}^n$$
$$\phi(h) = (\mathrm{ev}(h)|_s, \frac{\partial}{\partial v}h|_s, \frac{\partial}{\partial t}h|_s, ..., \frac{\partial^{n-3}}{\partial^{n-3}t}h|_s).$$

The kernel is S_m and the map is surjective as the polynomials $r^m, r^{m-1}v, r^{m-1}t, ..., r^{m-n+3}t^{n-3}$ give a constant times vectors $e_1, ..., e_n$ respectively as in the proof of Theorem 3.1. Hence, by rank nullity, we have $\mathbb{C}[u,v,t,r]_m/S \cong \mathbb{C}^n$. In the stable range, we have that the quotient by Jacobian in degree m is a space of dimension n, hence $S_m = J_m$.

Now the standard E_6 equation is given by $u^2 + v^3 + t^4 = 0$. The Jacobian ideal is $J = (u, v^2, t^3)$. Along with evaluation at the origin, the operators that annihilate any element of the Jacobian ideal is

$$\frac{\partial}{\partial v}|_s, \frac{\partial}{\partial t}|_s, \frac{\partial}{\partial v}\frac{\partial}{\partial t}|_s, \frac{\partial^2}{\partial^2 t}|_s, \frac{\partial^2}{\partial^2 t}\frac{\partial}{\partial v}|_s.$$

Again, let S_m be the space of polynomials of degree m annihilated by our operators and we have $J_m \subset S_m$. Define

$$\phi : \mathbb{C}[u,v,t,r]_m \longrightarrow \mathbb{C}^6$$
$$\phi(h) = (\mathrm{ev}(h)|_s, \frac{\partial}{\partial v}h|_s, \frac{\partial}{\partial t}h|_s, \frac{\partial}{\partial v}\frac{\partial}{\partial t}h|_s, \frac{\partial^2}{\partial^2 t}h|_s, \frac{\partial^2}{\partial^2 t}\frac{\partial}{\partial v}h|_s).$$

The kernel is S and the map is surjective as the polynomials $r^m, r^{m-1}v, r^{m-1}t, r^{m-2}vt, r^{m-2}t^2, r^{m-3}vt^2$ map to a constant times vectors $e_1, ..., e_6$ respectively. Hence, $S_m = J_m$

The standard E_7 equation is given by $u^2 + v^3 + vt^3 = 0$. The Jacobian ideal is given by $J = (u, 3v^2 + t^3, 3vt^2)$ Along with evaluation at the origin, the operators that annihilate any element of the Jacobian are

$$\frac{\partial}{\partial v}|_s, \frac{\partial}{\partial t}|_s, \frac{\partial}{\partial v}\frac{\partial}{\partial t}|_s, \frac{\partial^2}{\partial^2 t}|_s, \frac{\partial^3}{\partial^3 t} - \frac{\partial^2}{\partial^2 v}|_s, \frac{\partial^4}{\partial^4 t} - 3\frac{\partial^2}{\partial^2 v}\frac{\partial}{\partial t}|_s.$$

Let S_m be the space of polynomials of degree m annihilated by all the differential operators. Again, we have $J_m \subset S_m$. Define

$$\begin{aligned} &\phi : \mathbb{C}[u, v, t, r]_m \longrightarrow \mathbb{C}^7 \\ &\phi(h) = (\mathrm{ev}(h)|_s, \frac{\partial}{\partial v}h|_s, \frac{\partial}{\partial t}h|_s, ..., \frac{\partial^4}{\partial^4 t}h - 3\frac{\partial^2}{\partial^2 v}\frac{\partial}{\partial t}h|_s). \end{aligned}$$

The kernel is S_m and the map is surjective as the polynomials $r^m, r^{m-1}v, r^{m-1}t, r^{m-2}vt, r^{m-2}t^2, r^{m-3}t^3 - r^{m-2}v^2, r^{m-4}t^4 - r^{m-3}v^2t$ map to a constant times vectors $e_1, ..., e_7$ respectively. Hence, $S_m = J_m$.

The standard E_8 equation is given by $u^2 + v^3 + t^5$. The Jacobian ideal is given by $J = (u, v^2, t^4)$. Along with evaluation at the origin, the operators that annihilate any element of the Jacobian area

$$\frac{\partial}{\partial v}|_s, \frac{\partial}{\partial t}|_s, \frac{\partial}{\partial v}\frac{\partial}{\partial t}|_s, \frac{\partial^2}{\partial^2 t}|_s, \frac{\partial^2}{\partial^2 t}\frac{\partial}{\partial v}|_s, \frac{\partial^3}{\partial^3 t}|_s, \frac{\partial^3}{\partial^3 t}\frac{\partial}{\partial v}|_s.$$

Let S_m be the space of polynomials of degree m annihilated by all the differential operators. We have $J_m \subset S_m$. Define

$$\begin{aligned} &\phi : \mathbb{C}[u, v, t, r]_m \longrightarrow \mathbb{C}^8 \\ &\phi(h) = (\mathrm{ev}(h)|_s, \frac{\partial}{\partial v}h|_s, \frac{\partial}{\partial t}h|_s, ..., \frac{\partial^3}{\partial^3 t}\frac{\partial}{\partial v}h|_s). \end{aligned}$$

The kernel is S and the map is surjective as the polynomials $r^m, r^{m-1}v, r^{m-1}t, r^{m-2}vt, r^{m-2}t^2, r^{m-3}t^2v, r^{m-3}t$ map to a constant times vectors $e_1, ..., e_8$ respectively. Hence, $S_m = J_m$. $\square$

As of now, Theorem 3.1 applies to the standard equations of ADE singularities. Our goal is to apply Theorem 3.1 to operators on the original equation before the analytic change of coordinates to the standard equations. We prove Proposition 3.3 which will give a set of differential operators in x, y, z coordinates with which we can apply Theorem 3.1.

Proof. To prove Proposition 3.3, recall the multivariable chain rule.

$$\frac{\partial}{\partial t}h = \frac{\partial x}{\partial t}|_s \cdot \frac{\partial}{\partial x}h|_s + \frac{\partial y}{\partial t}|_s \cdot \frac{\partial}{\partial y}h|_s + \frac{\partial z}{\partial t}|_s \cdot \frac{\partial}{\partial z}h|_s.$$

Let $x = f_1(u, v, t), y = f_2(u, v, t), z = f_3(u, v, t) \in \mathbb{C}[[u, v, t]]$ be the analytic change of coordinates. Then $\frac{\partial x}{\partial t}|_s$ is the coefficient of t in the power series of x. $\frac{\partial w}{\partial t}|_s$ is the coefficient of t in the power series of y, and $\frac{\partial z}{\partial t}|_s$ is the coefficient t in the power series of z. Hence, the operator $\frac{\partial}{\partial t}$ is a linear combination of first order operators in x, y, z.

What about $(\frac{\partial}{\partial t})^2$? This is

$$\begin{aligned}
\frac{\partial}{\partial t}\frac{\partial}{\partial t} &= \frac{\partial}{\partial t}\left(\frac{\partial x}{\partial t} \cdot \frac{\partial}{\partial x} + \frac{\partial y}{\partial t} \cdot \frac{\partial}{\partial y} + \frac{\partial z}{\partial t} \cdot \frac{\partial}{\partial z}\right) \\
&= \frac{\partial}{\partial t}(\frac{\partial x}{\partial t} \cdot \frac{\partial}{\partial x}) + \frac{\partial}{\partial t}(\frac{\partial y}{\partial t} \cdot \frac{\partial}{\partial y}) + (\frac{\partial}{\partial t}\frac{\partial z}{\partial t} \cdot \frac{\partial}{\partial z}).
\end{aligned}$$

I will compute the first term and the rest follow in the exact same way. In the first half of the product rule, what I want to do is take the derivative of x with respect to t twice and evaluate at 0. This is equivalent to 2 times the coefficient of t^2 in the power series expansion of x. In the second half of the product rule, we have

$$(\frac{\partial}{\partial t}\frac{\partial}{\partial x})\frac{\partial x}{\partial t}|_{(0,0,0)} = C \cdot \frac{\partial}{\partial t}\frac{\partial}{\partial x}|_{(0,0,0)},$$

where C is the coefficient of t in the power series of x. Now

$$\begin{aligned} C\frac{\partial}{\partial t}\frac{\partial}{\partial x}|_s &= C(\frac{\partial x}{\partial t}\cdot\frac{\partial}{\partial x}\frac{\partial}{\partial x}+\frac{\partial y}{\partial t}\cdot\frac{\partial}{\partial y}\frac{\partial}{\partial x}+\frac{\partial z}{\partial t}\cdot\frac{\partial}{\partial z}\frac{\partial}{\partial x})|_s \\ &= C(C\frac{\partial}{\partial x}\frac{\partial}{\partial x}+C_1\frac{\partial}{\partial y}\frac{\partial}{\partial x}+C_2\frac{\partial}{\partial z}\frac{\partial}{\partial x})|_s \end{aligned}$$

where C_1, C_2 are the coefficients of t in the power series of y and z respectively. Hence, we have a linear combination of second order partials.

Suppose $\frac{\partial^i}{\partial^i t}|_s$ is a linear combination of k order and lower differential operators in x, y, z for i up to k. So $\frac{\partial^k}{\partial^k t} = C_0D_0 + C_1D_1 + C_2D_2 + ... + C_kD_k = D$ where C_i are constants and D_i are differential operators of order i evaluated at the origin. Then for $k \leq n$,

$$\begin{aligned} \frac{\partial^{k+1}}{\partial^{k+1}t}h|_s &= \frac{\partial}{\partial t}\frac{\partial^k}{\partial^k t}h|_s = \frac{\partial}{\partial t}|_s Dh \\ &= \left(\frac{\partial x}{\partial t}|_s\cdot\frac{\partial}{\partial x}h|_s+\frac{\partial y}{\partial t}|_s\cdot\frac{\partial}{\partial y}h|_s+\frac{\partial z}{\partial t}|_s\cdot\frac{\partial}{\partial z}h|_s\right)Dh. \end{aligned}$$

So $(\frac{\partial x}{\partial t}|_s \cdot \frac{\partial}{\partial x}h|_s)$ applied to C_iD_i gives an order $i+1$ operator given by $\frac{\partial}{\partial x}h|_sD_i$. So as the highest order operator is D_k, our operator is at most order $k+1$. The same applies for the other terms. Note this actually holds in the u, v variables as we just repeat the proof replacing u with t or v with t. This proves Proposition 3.3.

$\square$

We have shown for the polynomial ring, $S_m = J_m$. Originally, the space in consideration is the power series ring. From Proposition 3.1, we show $\mathbb{C}[[x, y, z]]/(f_x, f_y, f_z)$ is supported at the origin. The analytic change of coordinates maps the origin to the origin. So the analytic change of coordinates has no constant term. Since $\mathbb{C}[[u, v, t]]/(g_u, g_v, g_t)$ is supported at the origin, there exists N such that

$$u^N\mathbb{C}[[u,v,t]]/(g_u,g_v,g_t), v^N\mathbb{C}[[u,v,t]]/(g_u,g_v,g_t), t^N\mathbb{C}[[u,v,t]]/(g_u,g_v,g_t) = 0.$$

Let $x = h_1(u,v,t), y = h_2(u,v,t), z = h_3(u,v,t)$. Then by the analytic change of coordinates

$$x^{3N}\mathbb{C}[[x,y,z]]/(f_x,f_y,f_z), y^{3N}\mathbb{C}[[x,y,z]]/(f_x,f_y,f_z), z^{3N}\mathbb{C}[[x,y,z]]/(f_x,f_y,f_z)$$

map to zero since the change of coordinates gives an isomorphism on the level of quotients as shown in the proof of invariance of the Jacobian ideal. Hence, $\mathbb{C}[[x,y,z]]/(f_x,f_y,f_z)$ is supported at the origin as well so Proposition 3.1 applies. Hence, we can restrict to work with the polynomial ring instead of the power series ring.

Lemma 3.2. *Given a polynomial in the non standard coordinates w,x,y,z with ADE singularities, there exists homogeneous operators in w,x,y,z such that being in the Jacobian ideal is equivalent to being annihilated by these operators in the stable range.*

Proof. This comes immediately from Theorem 3.1 and Proposition 3.3 The fact the operators are homogeneous is because the standard equations are in an affine set. Without loss of generality, the affine set is $w = 1$. Then we can homogenize the operators by multiplication by w. $\square$

Here, we will go over examples on finding the differential operators given in Lemma 3.2.

Example 3.1. *Let $f(w,x,y,z) = zwx + w^2y + x^3 - y^2x$. The partials are given by*

$$f_w = zx + 2wy$$

$$f_x = zw + 3x^2 - y^2$$

$$f_y = w^2 - 2yx$$

$$f_z = wx.$$

The singular point $s = [0:0:0:1]$ is of type A_4. One can check that $\frac{\partial}{\partial y}|_s$ annihilates the partials.

$$(\frac{\partial}{\partial y})^2(f_w h)|_s = ((\frac{\partial}{\partial y})^2 f_w)h|_s = 0$$
$$(\frac{\partial}{\partial y})^2(f_x h)|_s = ((\frac{\partial}{\partial y})^2 f_x)h|_s = -2h(s).$$

To fix this, we add

$$2\frac{\partial}{\partial w}|_s.$$

This will annihilate $f_x h$. Since this operator annihilates f_x, we have that

$$(\frac{\partial}{\partial y})^2 + 2\frac{\partial}{\partial w}|_s$$

annihilates $f_w h$ and $f_x h$ for all h. Similarly, this operator annihilates $f_y h$ and $f_z h$.

The third order operator is

$$(\frac{\partial}{\partial y})^3 + 2\frac{\partial}{\partial w}\frac{\partial}{\partial y} - 2\frac{\partial}{\partial x}|_s.$$

Instead of showing all calculations, let me summarize what is getting fixed. Applying $(\frac{\partial}{\partial y})^3|_s$ to $f_x h$ does not annihilate $f_x h$. To fix this, we add in $2\frac{\partial}{\partial w}\frac{\partial}{\partial y}|_s$. This now annihilates $f_x h$ but *does not annihilate $f_w h$. To fix this, we add in $-2\frac{\partial}{\partial x}|_s$.*

Example 3.2. *Let $f(w,x,y,z) = wzx + w^3 + x^3 - y^2x$. The partials are given by*

$$f_w = zx + 3w^2$$

$$f_x = wz + 3x^2 - y^2$$

$$f_y = -2yx$$

$$f_z = wx.$$

The singular point is $s = [0:0:0:1]$ is of type A_5. Instead of showing all the calculations, it is more helpful to explain what doesn't get annihilated and what the fix is. For first order operator, we have that $\frac{\partial}{\partial y}|_s$ annihilates all partials.

For second order, we have

$$(\frac{\partial}{\partial y})^2 + 2\frac{\partial}{\partial w}|_s.$$

$(\frac{\partial}{\partial y})^2|_s$ does not annihilate $f_x h$, so we add in $2\frac{\partial}{\partial w}|_s$.

For third order, we have

$$(\frac{\partial}{\partial y})^3 + 6\frac{\partial}{\partial y}\frac{\partial}{\partial w}|_s.$$

$(\frac{\partial}{\partial y})^3|_s$ does not annihilate $f_x h$ so we add in $6\frac{\partial}{\partial y}\frac{\partial}{\partial w}|_s$.

For fourth order, we have

$$(\frac{\partial}{\partial y})^4 + 2\binom{4}{2}(\frac{\partial}{\partial y})^2\frac{\partial}{\partial w} + 4\binom{4}{2}(\frac{\partial}{\partial w})^2 - 24\binom{4}{2}\frac{\partial}{\partial x}|_s.$$

So $(\frac{\partial}{\partial y})^4|_s$ *applied to* $f_x h$ *is not zero. Let us call this the error term. To fix this, applying* $2\binom{4}{2}(\frac{\partial}{\partial y})^2\frac{\partial}{\partial w}|_s$ *gives us negative the error term + another term. So adding these two operators gets rid of the error term but we are left with another term. Now to get rid of this other term, we add* $4\binom{4}{2}(\frac{\partial}{\partial w})^2|_s$*. This operator now annihilates* $f_x h$ *but in doing so, this operator does not annihilate* $f_w h$*. To fix this, we add in* $-24\binom{4}{2}\frac{\partial}{\partial x}|_s$*. Now, this operator annihilates any linear combination of the partials.*

Example 3.3. *Let* $f(w,x,y,z) = zx^2 - zwy + w^2x - wx^2$*. This has one A1 singularity at* $[0:0:0:1]$ *and one A3 singularity at* $[0:0:1:0]$*. We work locally around the A3 singularity by letting* $y = 1$*. Then let*

$$g(w,x,z) = f(w,x,1,z) = zx^2 - zw + w^2x - wx^2,$$

where g *has a singularity at the origin. The partials are given by*

$$g_x = 2zx - 2wx$$
$$g_w = -z + 2wx - x^2$$
$$g_z = x^2 - w.$$

Consider the change of coordinates given by

$$u = -z + wx - x^2 + x^3$$
$$v = w - x^2$$
$$t = x \cdot \sqrt[4]{1-x}.$$

Let us reinterpret the derivative with respect to t in terms of our original coordinates. We have

$$\frac{\partial}{\partial t} = \frac{\partial x}{\partial t} \cdot \frac{\partial}{\partial x} + \frac{\partial w}{\partial t} \cdot \frac{\partial}{\partial w} + \frac{\partial z}{\partial t} \cdot \frac{\partial}{\partial z}.$$

Note that since $t^4 = x^4 - x^5$, $4t^3 dt = (4x^3 - 5x^4)dx$. Therefore, we have

$$\frac{\partial x}{\partial t} = \frac{4t^3}{4x^3 - 5x^4} = \frac{4x^3(1-x)^{3/4}}{4x^3 - 5x^4} = \frac{4(1-x)^{3/4}}{4-5x}.$$

Thus our expression above is

$$\frac{\partial}{\partial t} = \frac{4(1-x)^{3/4}}{4-5x} \cdot \frac{\partial}{\partial x} + \frac{\partial w}{\partial t} \cdot \frac{\partial}{\partial w} + \frac{\partial z}{\partial t} \cdot \frac{\partial}{\partial z}.$$

We have

$$\begin{aligned}
\frac{\partial w}{\partial t} &= 2x\frac{\partial x}{\partial t} \\
\frac{\partial z}{\partial t} &= x\frac{\partial w}{\partial t} + w\frac{\partial x}{\partial t} - 2x\frac{\partial x}{\partial t} + 3x^2\frac{\partial x}{\partial t}.
\end{aligned}$$

What about $(\frac{\partial}{\partial t})^2$? This is

$$\begin{aligned}
\frac{\partial}{\partial t}\frac{\partial}{\partial t} &= \frac{\partial}{\partial t}\left(\frac{\partial x}{\partial t} \cdot \frac{\partial}{\partial x} + \frac{\partial w}{\partial t} \cdot \frac{\partial}{\partial w} + \frac{\partial z}{\partial t} \cdot \frac{\partial}{\partial z}\right) \\
&= \frac{\partial}{\partial t}(\frac{\partial x}{\partial t} \cdot \frac{\partial}{\partial x}) + \frac{\partial}{\partial t}(\frac{\partial w}{\partial t} \cdot \frac{\partial}{\partial w}) + (\frac{\partial}{\partial t}\frac{\partial z}{\partial t} \cdot \frac{\partial}{\partial z}).
\end{aligned}$$

Let us calculate each of the 3 terms separately.

1st term *In the first half of the product rule, we want to take the derivative of* x *with respect to* t *twice and evaluate at* 0. *This is equivalent to 2 times the coefficient of* t^2 *in the power series expansion of* x. *Let* s *denote the origin. From* $\frac{\partial x}{\partial t}|_s = 1$ *and evaluation at the origin being* 0, *the expansion of* x *is given as*

$$x = (0 + t + a_2 t^2 + ...).$$

We have that

$$t^4 = x^4 - x^5 = (t + a_2 t^2 + ...)^4 - (t + a_2 t^2 + ...)^5.$$

The t^5 *coefficient in* x^4 *is* $4a_2$ *and the* t^5 *coefficient in* x^5 *is 1. Thus* $a_2 = \frac{1}{4}$, *and so evaluation at* 0 *gives* $\frac{1}{2}$. *In the second half of the product rule, we have*

$$\left(\frac{\partial}{\partial t}\frac{\partial}{\partial x}\right)\frac{\partial x}{\partial t}|_s = \frac{\partial}{\partial t}\frac{\partial}{\partial x}|_s$$
$$\frac{\partial}{\partial t}\frac{\partial}{\partial x}|_s = (\frac{\partial x}{\partial t}\cdot\frac{\partial}{\partial x} + \frac{\partial w}{\partial t}\cdot\frac{\partial}{\partial w} + \frac{\partial z}{\partial t}\cdot\frac{\partial}{\partial z})|_s = (\frac{\partial}{\partial x})^2|_s.$$

So first term gives $(\frac{\partial}{\partial x})^2 + \frac{1}{2}\frac{\partial}{\partial x}$.

2nd term *Using the fact* $\frac{\partial w}{\partial t} = 2x\frac{\partial x}{\partial t}$,

$$\begin{aligned}\frac{\partial}{\partial t}\left(\frac{\partial w}{\partial t}\frac{\partial}{\partial w}\right)|_s &= \left(\frac{\partial}{\partial t}\frac{\partial w}{\partial t}\right)\frac{\partial}{\partial w}|_s + \frac{\partial w}{\partial t}\left(\frac{\partial}{\partial t}\frac{\partial}{\partial w}\right)|_s\\ &= \left(2\frac{\partial}{\partial t}x\right)\frac{\partial x}{\partial t}\frac{\partial}{\partial w}|_s + 2x\left(\frac{\partial}{\partial t}\frac{\partial x}{\partial t}\right)\frac{\partial}{\partial w}|_s + \frac{\partial w}{\partial t}\left(\frac{\partial}{\partial t}\frac{\partial}{\partial w}\right)|_s\\ &= \left(2\frac{\partial}{\partial t}x\right)\frac{\partial x}{\partial t}\frac{\partial}{\partial w}|_s + \frac{\partial w}{\partial t}\left(\frac{\partial}{\partial t}\frac{\partial}{\partial w}\right)|_s\\ &= 2\frac{\partial}{\partial w} + \frac{\partial w}{\partial t}\left(\frac{\partial}{\partial t}\frac{\partial}{\partial w}\right)|_s = 2\frac{\partial}{\partial w}.\end{aligned}$$

3rd term

$$\begin{aligned}
\frac{\partial}{\partial t}\left(\frac{\partial z}{\partial t}\frac{\partial}{\partial z}\right)|_s &= \left(\frac{\partial}{\partial t}\frac{\partial z}{\partial t}\right)\frac{\partial}{\partial z}|_s + \frac{\partial z}{\partial t}\left(\frac{\partial}{\partial t}\frac{\partial}{\partial z}\right)|_s \\
&= \left(\frac{\partial}{\partial t}\frac{\partial z}{\partial t}\right)\frac{\partial}{\partial z}|_s = \frac{\partial}{\partial t}\frac{\partial z}{\partial t}|_s = \frac{\partial}{\partial t}\left(x\frac{\partial w}{\partial t} + w\frac{\partial x}{\partial t} - 2x\frac{\partial x}{\partial t} + 3x^2\frac{\partial x}{\partial t}\right)|_s \\
&= \frac{\partial}{\partial t}\left(x\frac{\partial w}{\partial t}\right)|_s + \frac{\partial}{\partial t}\left(w\frac{\partial x}{\partial t}\right)|_s - \frac{\partial}{\partial t}\left(2x\frac{\partial x}{\partial t}\right)|_s + \frac{\partial}{\partial t}\left(3x^2\frac{\partial x}{\partial t}\right)|_s \\
&= -2
\end{aligned}$$

So

$$\left(\frac{\partial}{\partial t}\frac{\partial z}{\partial t}\right)\frac{\partial}{\partial z}|_s = -2\frac{\partial}{\partial z}$$

Therefore, our second degree operator is $(\frac{\partial}{\partial x})^2 + \frac{1}{2}\frac{\partial}{\partial x} + 2\frac{\partial}{\partial w} - 2\frac{\partial}{\partial z}$. *Indeed, applying this operator and evaluating at the origin annihilates all the partial derivatives of* f.

3.2 Algorithm

Before giving algorithm for computing the zeta function of a hypersurface with ADE singularities over $\mathbb{P}^3$, we define a method for finding a basis of a vector space of the subdiagonal given a basis of a vector space of the subdiagonal of lower degree.

Definition 3.3. *Let* D^k *be a differential operator of degree* k. *Let* f *and* g *be two polynomials. Let* D *be a differential operator. Then one has*

$$D(fg) = \sum D^i f D^j g$$

such that $i + j$ *is the order of the operator* D. *We define the non-zero order operators of* D *to be the* D^i *such that* $i > 0$.

Theorem 3.2. *Suppose we have a basis $\beta_1, \beta_2, ..., \beta_M$ in degree $4N$ on the subdiagonal. Suppose there exists a degree $(d-4)N$ polynomial L satisfying the following properties. Assume L is not annihilated by evaluation at any of the singular points. Let $D_1, ..., D_M$ be the operators as given in Lemma 3.2. Furthermore, assume that non-zero order operators as defined in Definition 3.3 annihilate L. Define χ to be the multiplication by L map. This map is well-defined. Furthermore, $L\beta_1, L\beta_2, ...L\beta_M$ is a basis on the higher degree of our subdiagonal.*

Proof. We will first show χ is well-defined. Since we can extend by linearity, consider the 3-form $hdx \wedge dy \wedge dz$. Let us call the lower degree on subdiagonal B_V and the upper degree on the subdiagonal B_U. Let L be our multiplying factor. Then we have a map χ given by multiplication by the factor L.

$$\chi : B_V \longrightarrow B_U$$

$$\chi(\omega) = L\omega$$

for a 3-form ω. By linearity, suppose $hdx \wedge dy \wedge dz = (f_x h_1 + f_y h_2 + f_z h_3) dx \wedge dy \wedge dz$. Then

$$\chi(hdx \wedge dy \wedge dz) = h \cdot Ldx \wedge dy \wedge dz = (f_x h_1 L + f_y h_2 L + f_z h_3 L) dx \wedge dy \wedge dz,$$

which remains in the Jacobian.

For example, if the higher order operator is $(\frac{\partial}{\partial z})^2 + \frac{\partial}{\partial x}\frac{\partial}{\partial y} + \frac{\partial}{\partial w}$, then the assumption is that each term in the sum annihilates L and $\frac{\partial}{\partial z}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial w}$ also annihilate L. Then suppose that h does not lie in the Jacobian ideal. We wish to show that hL also does not lie in the Jacobian ideal. Suppose that

$$hL = f_x h_1 + f_y h_2 + f_z h_3$$

Since h does not lie in the Jacobian, there exists D_i that does not annihilate h. Applying D_i

to the right hand side gives 0. Applying D_i to the left hand side, by the assumption on L, we get $D_i(hL) = (D_ih)ev(L) \neq 0$. Hence, we have a contradiction. Thus, we can conclude the image of an element not in the image of Koszul differential will not be in the image of Koszul differential.

Using the fact that elements not in the Jacobian ideal are mapped to elements not in the Jacobian ideal, we can now show that the image is a basis. Suppose we have linear independence. Then from the result that the dimension of the space is the global Milnor number, we immediately get that the M elements span the whole space. Suppose there is a nontrivial linear combination

$$c_1L\beta_1 + c_2L\beta_2 + ... + c_ML\beta_M = 0$$

where 0 is a representative of an element in the Jacobian ideal as we are on the E_1 page. Then this means the term above lies in the Jacobian ideal so the term is annihilated by the corresponding differential operators. We evaluate at a singular point s and get a contradiction because the coefficients $\tilde{c_1} = c_1L|_s, ..., \tilde{c_M}L|_s$ give a nontrivial linear combination on the lower level of the subdiagonal. Hence, we must have linear independence of the new basis elements and from the argument above, these M terms form a basis for the subdiagonal of degree dN. This proves Theorem 3.2.

□

Example 3.4. *In the case we have a single singularity at say* $[1:0:0:0]$*, the operators are in the variables* x, y, z *since we work in the affine open set. For an example of a hypersurface, one can take the equation* $f = x^2 + y^2 + z^2 = 0$*. We can take* L *to be* w^k *for the appropriate power of* k*. Evaluation at* $[1:0:0:0]$ *does not annihilate* L *while all the other operators annihilate* L *since the other operators are in the variables* x, y, z*.*

Example 3.5. *Suppose the singularities are the standard coordinates in the affine open set. In other words, the singularities are* $[1:0:0:0], [0:1:0:0], [0:0:1:0]$*, and*

$[0 : 0 : 0 : 1]$. *For an example of an equation, one can take the Cayley cubic given by $f = xyz + wyz + wxz + wxy = 0$. In this case, suppose our corresponding operators have at most degree k. Then $L = w^j + x^j + y^j + z^j$ for $j > k$ will be a valid choice for Theorem 5. Since all operators are of degree at most k, applying the operators to L and evaluating at the origin will annihilate L, and evaluating at the singular points will not annihilate L by construction. For degrees lower, one will have to construct the matrix.*

Before giving an algorithm, we state the following theorem.

Theorem 3.3. *The subdiagonal on the E_2 page vanishes in the case the hypersurface in $\mathbb{P}^3$ has only ADE singularities.*

Proof. Following the notation of Theorem 5.3 of Dimca and Saito [6] except for the fact we replace p with m, let $z_1, ..., z_r$ be the singularities of f. Let η_j be the 3-forms generated by the generators of $C[x, y, z]/(dh_k)$,where h_k is the local equation of f around z_k. Let $\alpha_{h_k,j}$ be the weight of η_j. Then from Theorem 5.3 of Dimca and Saito [6],

$$\dim(N^2_m) \leq \# \left\{(k, j)| \quad \alpha_{h_k,j} = \frac{m}{d}\right\},$$

where N^2 is the subdiagonal on the E_2 page. We only care about powers of f, and this is when m is a multiple of d. In this case, we only care when $\alpha_{h_k,j} = \frac{m}{d} \in \mathbb{Z}$. Second, the inequality runs through all singularities. If we can show that on each singularity the inequality shows that the dimension is 0, we are done since

$$\# \left\{(k, j)| \quad \alpha_{h_k,j} = \frac{m}{d}\right\} = \sum_i \# \left\{j| \quad \alpha_{h_i,j} = \frac{m}{d}\right\}$$

Let $\mathrm{wt}(h\Omega)$ denote the weight of the form $h\Omega$. Let us first assume that our hypersurface has a type A_n singularity. Then using notation from Theorem 5.3 of Dimca and Saito [6], in a local analytic coordinate system around our singularity, the function of the hypersurface can be written in the form $xy = z^{n+1}$. The weights of x, y, z are $\frac{1}{2}, \frac{1}{2}, \frac{1}{n+1}$ respectively. The partials with respect to x, y, z are $y, x, (n+1)z^n$; so the quotient $\mathbb{C}[x, y, z]/(y, x, z^n)$ is generated by $1, z, z^2, ..., z^{n-1}$ over $\mathbb{C}$. Hence the monomial basis of the quotient is given by

$$dx \wedge dy \wedge dz, zdx \wedge dy \wedge dz, ..., z^{n-1}dx \wedge dy \wedge dz.$$

The weight of $dx \wedge dy \wedge dz$ is

$$\frac{1}{2} + \frac{1}{2} + \frac{1}{n+1} = \frac{n+2}{n+1}.$$

Hence the weight of our forms are

$$\frac{n+2}{n+1}, \frac{n+3}{n+1}, ..., \frac{2n}{n+1}.$$

Let us label these values by α_i respectively. For example, $\alpha_1 = \frac{n+2}{n+1}$ and $\alpha_2 = \frac{n+3}{n+1}$. By Dimca-Saito([6],Theorem 5.3),

$$\dim(N^2_{p+d}) \leq \# \left\{ k | \ \ \alpha_k = \frac{p}{d} \right\},$$

where N^2_j is the dimension of the subdiagonal on the E_2 page of degree j. From above, since the value of α_k ranges between 1 and 2 for all k, there is no way that $\alpha_k = \frac{p}{d}$. Hence, $\dim(N^2_{p+d}) = 0$, and so the subdiagonal vanishes on the E_2 page. This extends to hypersurfaces with multiple A_n singularities as it was noted that we can focus on one singularity at a time.

Now suppose our hypersurface has a type D_n singularity. Then in a local analytic system, our function can be written in the form $z^2 + yx^2 + y^{n-1}$. The weights of x, y, z are $\frac{n-2}{2(n-1)}, \frac{1}{n-1}, \frac{1}{2}$ respectively. The Jacobian ideal is given by $(2z, x^2 + xy, y^{n-1})$. The quotient $\mathbb{C}[x, y, z]/(2z, x^2 + xy, y^{n-1})$ is generated by $1, xy^k, y^j$, where k and j run from 0 to $n-2$.

The weight of $dx \wedge dy \wedge dz$ is $\frac{2n-1}{2n-2}$.

Let us consider the basis given by $y^j dx \wedge dy \wedge dz$. This has weight

$$\frac{2j}{2(n-1)} + \frac{2n-1}{2n-2} = 1 + \frac{2j+1}{2n-2},$$

which is never an integer since the numerator is odd and denominator is even.

Let us now consider the basis given by $xy^j dx \wedge dy \wedge dz$. This has weight

$$\frac{2j}{2(n-1)} + \frac{2n-1}{2n-2} + \frac{n-2}{2(n-1)} = 1 + \frac{2j+n-1}{2n-2}.$$

Now j runs from 0 to $n-2$. At 0, the value is between 1,and 2, and at $n-2$, the value is between 2 and 3. So the only case we need to consider is whether the value can be 2. However, the value 2 means $p = 2d$ so we are calculating the dimension of N_2^{2d} which is not part of the first quadrant. Hence, the subdiagonal vanishes in the case our hypersurface has type D_n singularity.

Suppose the hypersurface has an E_6 singularity. Then there exists a local analytic system where the function of the hypersurface can be written in the form $x^2 + y^3 + z^4$. The weights of x, y, z are $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}$ respectively. The Jacobian ideal is given by $J = (2x, 3y^2, 4z^3)$. The quotient $\mathbb{C}[x, y, z]/(2x, 3y^2, 4z^3)$ is generated by $1, y, z, z^2.yz, yz^2$. The weight of $dx \wedge dy \wedge dz$ is given by $\frac{13}{12}$. We have

$$\begin{aligned}
&\mathrm{wt}(1dx \wedge dy \wedge dz) = \frac{13}{12}, \mathrm{wt}(ydx \wedge dy \wedge dz) = \frac{17}{12}, \mathrm{wt}(zdx \wedge dy \wedge dz) = \frac{16}{12}, \\
&\mathrm{wt}(z^2dx \wedge dy \wedge dz) = \frac{19}{12}, \mathrm{wt}(yzdx \wedge dy \wedge dz) = \frac{20}{12}, \mathrm{wt}(yz^2dx \wedge dy \wedge dz) = \frac{23}{12}.
\end{aligned}$$

None are integers, so the subdiagonal vanishes.

Suppose the hypersurface has an E_7 singularity. Then there exists a local analytic system where the function of the hypersurface can be written as $x^2 + y^3 + yz^3 = 0$. The weights of x, y, z are $\frac{1}{2}, \frac{1}{3}, \frac{2}{9}$ respectively. The Jacobian ideal is given by $J = (2x, 3y^2 + z^3, 3z^2)$.

The quotient $\mathbb{C}[x,y,z]/(2x, 3y^2+z^3, 3z^2y)$ is generated by $1, y, z, y^2, yz, z^2, y^2z$. We have $\mathrm{wt}(dx \wedge dy \wedge dz) = \frac{19}{18}$. Then

$$\mathrm{wt}(ydx \wedge dy \wedge dz) = \frac{25}{18}, \mathrm{wt}(zdx \wedge dy \wedge dz) = \frac{23}{18}, \mathrm{wt}(y^2dx \wedge dy \wedge dz) = \frac{31}{18},$$
$$\mathrm{wt}(yzdx \wedge dy \wedge dz) = \frac{29}{18}, \mathrm{wt}(z^2dx \wedge dy \wedge dz) = \frac{23}{18}, \mathrm{wt}(y^2z) = \frac{35}{18}.$$

Hence, since none are integers, the subdiagonal vanishes. Suppose the hypersurface has an E_8 singularity. Then there exists a local analytic system where the function of the hypersurface can be written as $x^2+y^3+z^5 = 0$. Then the weights of x, y, z are $\frac{1}{2}, \frac{1}{3}, \frac{1}{5}$ respectively. The Jacobian ideal is given by $J = (2x, 3y^2, 5z^4)$. The quotient $\mathbb{C}[x,y,z]/(2x, 3y^2, 5z^4)$ is generated by $1, y, z, yz, z^2, z^2y, z^3, z^3y$. We have $\mathrm{wt}(dx \wedge dy \wedge dz) = \frac{31}{30}$. Then

$$\mathrm{wt}(ydx \wedge dy \wedge dz) = \frac{41}{30}, \mathrm{wt}(zdx \wedge dy \wedge dz) = \frac{37}{30}, \mathrm{wt}(yzdx \wedge dy \wedge dz) = \frac{47}{30},$$
$$\mathrm{wt}(z^2dx \wedge dy \wedge dz) = \frac{43}{30}, \mathrm{wt}(z^2ydx \wedge dy \wedge dz) = \frac{53}{20}, \mathrm{wt}(z^3dx \wedge dy \wedge dz) = \frac{49}{30},$$
$$\mathrm{wt}(z^3ydx \wedge dy \wedge dz) = \frac{59}{30}.$$

None are integers so the subdiagonal vanishes. This concludes the proof. □

We now go over the algorithm.

1. Calculate the basis on the E_2 page. Along with this, calculate the basis on the subdiagonal of the E_1 page for the smallest degree in the stable range. Using Theorem 3.2, find an L to obtain a basis on higher levels of the subdiagonal.

2. For each basis element on rigid cohomology, compute the truncated image of the inverse Frobenius and rewrite the truncated image into an element that is cohomologically equivalent and expressed as a linear combination of the basis elements.

3. Compute the characteristic polynomial and use the Weil conjectures to obtain the zeta

function.

Step 1 is done using the Sage code given and finding a multiplying factor L. We explain how to perform step 2.

Let h be a polynomial of the truncated image of inverse Frobenius operator of degree dN for some $d \geq 4$. By Theorem 3.2, we can apply the de Rham differential on the basis for the vector space of the subdiagonal of degree dN and call the images $\alpha_1, ..., \alpha_M$. Since all terms on the E_2 page are 0 past the first quadrant, there exist constants $a_1, ..., a_M$ such that

$$h - a_1\alpha_1 - ... - a_M\alpha_M = f_w h_1 + f_x h_2 + f_y h_3 + f_z h_4.$$

There are M variables $a_1, ..., a_M$ that need to be solved for. From Theorem 3.1 and Proposition 3.3, there exist M differential operators that eliminate the Jacobian ideal. Applying these M differential operators to the equation above gives a system of M equations with M variables. The system of equations have solution because $h - a_1\alpha_1 - ... - a_M\alpha_M$ is in the Jacobian ideal, and we established above that being in the Jacobian ideal is equivalent to being annihilated by the M differential operators. From here, we find a preimage of $h - a_1\alpha_1 - ... - a_M\alpha_M$ under the Koszul differential using a Gröbner basis. We continue this process to reach the vector spaces on the top diagonal–which contain the basis elements of the E_2 page.

For step 3, after expressing the truncated image as a linear combination of the basis elements of E_2 page, we represent this action as a matrix and compute the characteristic polynomial. Because we used truncation, the eigenvalues are approximations of the true eigenvalues. We use the Weil conjectures to recover the actual eigenvalues.

Chapter 4

Results on Blow up and Zeta Functions

Next, given a hypersurface X in $\mathbb{P}^3$ with ADE singularities over $\mathbb{Z}_p$, we show the theorem by Theorem 2.4 of Baldassarri and Chiarellotto [2] holds after a sequence of iterated blow ups.

4.1 De Rham vs Rigid Cohomology

Theorem 4.1. *Let X be a hypersurface in $\mathbb{P}^3$ with equisingular ADE singularities over $\mathbb{Z}_p$ given by vanishing of f (See Definition 1.2). Let U be the complement to X, $U_{\mathbb{F}_p}$ be the special fiber over U, and $U_{\mathbb{Q}_p}$ be the generic fiber over U. Then*

$$H^i_{rig}(U_{\mathbb{F}_p}) \cong H^i_{dR}(U_{\mathbb{Q}_p}) \qquad 0 \leq i \leq 2dim(U)$$

Proof. Since our singularities are isolated, by Theorem 2.4 of Baldassarri and Chiarellotto [2], it suffices to show that iterated blow ups of a single singularity of type ADE give a smooth strict relative divisor with normal crossings. We consider this in cases.

Definition 4.1. *Let X be a hypersurface in $\mathbb{P}^3$ with defining equation f on an affine subset with coordinates x, y, z over a field or principal ideal domain $\mathbb{K}$. Let Ω be the free rank 3 module over $\mathbb{K}[x, y, z]/(f)$ generated by dx, dy, dz. Let $df = f_x dx + f_y dy + f_z dz$. X is smooth when Ω/df is a projective module over $\mathbb{K}[x, y, z]/(f)$.*

For checking smoothness, the module is not projective when df is 0.

To understand future definitions, we define the completion of a ring.

Definition 4.2. *Let A be a commutative ring. Let I be an ideal of A and let I^n be the nth power of I. We have natural homomorphisms $A/I^{k+1} \to A/I^k$ which makes A/I^n into an inverse system of rings. We say the formal completion of A with respect to I is given by the following inverse limit, $\tilde{A} = \varprojlim A/I^n$. A ring is complete if $\tilde{A} \cong A$.*

An example is the formal completion at the ideal of the origin of a polynomial ring is the power series ring.

We plan to blow up $\mathbb{P}^3$ at centers that are smooth over $Spec(\mathbb{Z}_p)$ but closed subsets of X. After a sequence of iterated blow ups, the strict transform of X is given by E_0 and the exceptional divisor is given by a union of irreducible components $E_1 \cup ... \cup E_i$. Take a point in the union $E_0 \cup ... \cup E_i$. Then in a neighborhood of this point called V, we will be doing the blow up such that there will always be at most 3 components that intersect the neighborhood. Let $\phi : V \to \mathbb{A}^3$ such that $\phi(s) = (f_1(s), f_2(s), f_3(s))$ If the intersection is 3 components, the 3 components are the vanishing of f_0, f_1, f_2. If intersection is 2 components, the 2 components are the vanishing of f_0, f_1 and f_2 may be chosen to be transversal to f_0, f_1. If we are considering 1 component, the component is given by vanishing of f_0 and f_1, f_2 are chosen to be transversal to f_0.

Definition 4.3. *Let Y be locally Noetherian and $\phi : X \to Y$ is locally of finite type. Let*

$x \in X$ and $y = \phi(x)$. Then let $\psi : \tilde{\mathcal{O}}_{Y,y} \to \tilde{\mathcal{O}}_{X,x}$ be the induced map on completed local rings. The map ϕ is etale at x if the following holds:

1. *$\tilde{\mathcal{O}}_{X,x}/m_Y\tilde{\mathcal{O}}_{X,x}$ is a finite separable field extension of residue field $k(y)$. Note m_Y is the maximal ideal of $\tilde{\mathcal{O}}_{Y,y}$.*

2. *$\mathcal{O}_{X,x}$ is a free $\mathcal{O}_{Y,y}$ module.*

Definition 4.4. *The divisor given by the 2 dimension stratification above is a strict relative normal crossings divisor if the map $\phi : V \to \mathbb{A}^3$ is etale.*

We now show we obtain a smooth strict normal crossings divisor on the standard equations through iterated blow ups. We first consider the case X is an affine hypersurface with standard ADE equations.

4.1.1 Standard A_n case

Let X be an affine hypersurface with the A_n standard equation given by $xy = z^{n+1}$. The blow up at the origin is a subset of $\mathbb{A}^3 \times \mathbb{P}^2$ given by vanishes of all 2x2 minors of the matrix with entries given by coordinates of $\mathbb{A}^3$ and $\mathbb{P}^2$. Let x, y, z be the coordinates of $\mathbb{A}^3$ and x_1, y_1, z_1 be the coordinates of $\mathbb{P}^2$. Then the matrix with coordinates of $\mathbb{A}^3$ and $\mathbb{P}^2$ is given by

$$\begin{bmatrix} x & y & z \\ x_1 & y_1 & z_1 \end{bmatrix}$$

Consider the affine patch on $\mathbb{P}^2$ where $x_1 = 1$. Then the vanishing of all 2 by 2 minors of the matrix

$$\begin{bmatrix} x & y & z \\ 1 & y_1 & z_1 \end{bmatrix}$$

is given by $y = xy_1$ and $z = xz_1$. Then our equation $xy = z^{n+1}$ becomes $x^2(y_1 - x^{n-1}z_1^{n+1}) = 0$. The surface $x = 0$ is our exceptional divisor of the blow up of $\mathbb{A}^3$. The second component is the strict transform of our surface given by equation $y_1 - x^{n-1}z_1^{n+1}$ which is smooth. Let c be any value in $\mathbb{Z}_p$. The intersection of the strict transform with the exceptional divisor is given by $x = 0, y_1 = 0$. Take a point s in the intersection given by $x = 0, y_1 = 0, z_1 = c$ for some $c \in \mathbb{Z}_p$. This corresponds to ideals $(x, y_1, z_1 - c)$ and $(x, y_1, z_1 - c, p)$. The blow up defines a map

$$\begin{aligned}
&\phi : V(x, y_1, z_1) \to \mathbb{A}^3(u, v, w) \\
&\phi(x, y_1, z_1) = (x, y_1 - x^{n-1}(z_1 + c)^{n+1}, z) = (u, v, w)
\end{aligned}$$

where V is a neighborhood of the point. The image of $x = 0, y_1 = 0, z_1 = c$ is $u = 0, v = 0, w = c$ which corresponds to ideals $(u, v, w - c)$ and $(u, v, w - c, p)$. For ideal $(x, y_1, z_1 - c)$, we have $\tilde{\mathcal{O}}_{X,s} = \mathbb{Q}_p[[x, y_1, z_1 - c]]$ and $\tilde{\mathcal{O}}_{Y,\phi(s)} = \mathbb{Q}_p[[u, v, w - c]] = \mathbb{Q}_p[[x, y_1 - x^{n-1}z_1^{n+1}, z_1 - c]]$. For ideal $(x, y_1, z_1 - c, p)$, we have $\tilde{\mathcal{O}}_{X,s} = \mathbb{F}_p[[x, y_1, z_1 - c]]$ and $\tilde{\mathcal{O}}_{Y,f(s)} = \mathbb{F}_p[[u, v, w - c]] = \mathbb{F}_p[[x, y_1 - x^{n-1}z_1^{n+1}, z_1 - c]]$. In either case, condition 2 of 4.1 is satisfied because the quotient field is either $\mathbb{F}_p$ or $\mathbb{Q}_p$ which is a finite separable extension of itself. For Condition 1, there is a natural action of u, v, w given by the action of $x, y_1 - x^{n-1}z_1^{n+1}, z_1$ which makes $\tilde{\mathcal{O}}_{X,s}$ into an $\tilde{\mathcal{O}}_{Y,\phi(s)}$ module. This module is free of rank 1 is because $\mathbb{Q}_p[[x, y_1, z_1 - c]] \cong \mathbb{Q}_p[[x, y_1 - x^{n-1}z_1^{n+1}, z - c]]$ and $\mathbb{F}_p[[x, y_1, z_1 - c]] \cong \mathbb{F}_p[[x, y_1 - x^{n-1}z_1^{n+1}, z_1 - c]]$.

Consider the affine patch of $\mathbb{P}^2$ where $y_1 = 1$. The vanishing of all 2 by 2 minors of the matrix

$$\begin{bmatrix} x & y & z \\ x_1 & 1 & z_1 \end{bmatrix}$$

is given by $x = yx_1$ and $z = yz_1$. Then our equation $xy = z^{n+1}$ becomes $y^2(x_1 - y^{n-1}z_1^{n+1}) = 0$. The surface $y = 0$ is our exceptional divisor of the blow up of $\mathbb{A}^3$. The second component is the strict transform of our surface given by equation $x_1 - y^{n-1}z_1^{n+1}$, which is smooth. The reason we have a strict normal crossing divisor here is the same reason as the first patch by symmetry.

Consider the affine patch on $\mathbb{P}^2$ where $z_1 = 1$. Then the vanishing of all 2 by 2 minors of the matrix

$$\begin{bmatrix} x & y & z \\ x_1 & y_1 & 1 \end{bmatrix}$$

is given by $y = zby_1$ and $x = zx_1$. Then our equation $xy = z^{n+1}$ becomes $z^2(x_1y_1 - z^{n-1}) = 0$. This is a type A_{n-2} singularity. Hence, it suffices to show that the result holds for type A_1 and type A_2 singularity and the rest follows by induction.

For type A_1, $n+1 = 2$ so the equation on the third patch of the blow up is given by $z^2(x_1y_1 - 1) = 0$. Let $a \neq 0 \in \mathbb{Z}_p$. Consider a point in the intersection of $z = 0$ and $x_1y_1 - 1 = 0$ given by $x = a, y = \frac{1}{a}, z = 0$. We shift to the origin to get equation $z^2((x_1 - a)(y_1 - \frac{1}{a}) - 1) = 0$. Let s denote the origin. The blow up defines a map

$$\begin{aligned} &\phi : V(x_1, y_1, z) \to \mathbb{A}^3(u, v, w) \\ &\phi(x_1, y_1, z) = (x, (x_1 - a)(y_1 - \frac{1}{a}) - 1, z) = (u, v, w) \end{aligned}$$

where V is a neighborhood of the origin. The image of $(0, 0, 0)$ under ϕ is $(0, 0, 0)$ which corresponds to ideals (u,v,z) and (u,v,z,p). For ideal (x_1,y_1,z), we have $\tilde{\mathcal{O}}_{X,s} = \mathbb{Q}_p[[x_1,y_1,z]]$ and $\tilde{\mathcal{O}}_{Y,\phi(s)} = \mathbb{Q}_p[[u,v,w]] = \mathbb{Q}_p[[x,(x_1-a)(y_1-\frac{1}{a})-1,z]]$. For ideal (x_1,y_1,z,p), we have $\tilde{\mathcal{O}}_{X,s} = \mathbb{F}_p[[x,(x_1-a)(y_1-\frac{1}{a})-1,z]]$ and $\tilde{\mathcal{O}}_{Y,\phi(s)} = \mathbb{F}_p[[u,v,w]] = \mathbb{F}_p[[x,(x_1-a)(y_1-\frac{1}{a})-1,z]]$. In either case, Condition 2 of Definition 4.1 is satisfied because the quotient field is either $\mathbb{F}_p$ or $\mathbb{Q}_p$ which is a finite separable extension of itself. For Condition 1, we check the determinant of the 3x3 matrix formed from the coefficients of the linear terms of the map ϕ is nonzero. The matrix is given by

$$\begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{a} & -a & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The determinant is nonzero so $\tilde{\mathcal{O}}_{X,s}$ is free over $\tilde{\mathcal{O}}_{Y,\phi(s)}$.

For type A_2, $n+1=3$ so the equation on the third patch of the blow up is given by $z^2(x_1y_1-z)=0$. We blow up the y_1-axis given by the ideal (x,z). Let us label the coordinates of $\mathbb{P}^1$ as $[x_2:z_2]$.

On the first patch, we have $z = x_1z_2$. The equation from the first blow up $z^2(x_1y_1-z)=0$ is then given by $x_1^3z_2^2(y_1-z_2)=0$. There are three surfaces given by $x_1=0$, $z_2=0$, $y_1-z_2=0$. Let $a \in \mathbb{Z}_p$. Let $a \in \mathbb{Z}_p$. This covers the case for a point in the double and triple intersections because for triple intersections, we take $a=0, b=0$. Take a point s in the double or triple intersection given by at $x_1=0$, $z_2=a$, and $y_1=a$. We do the case where $a=0$ as the rest is the same.

$$\phi : V(x,y,z) \to \mathbb{A}^3(u,v,w)$$

$$\phi(x,y,z) = (x, y-z, z) = (u,v,w)$$

The map ϕ is an isomorphism so it is etale.

On the second patch, we have $x = z_1 x_2$. The equation from the first blow up $z^2(x_1 y_1 - z) = 0$ is then given by $z_1^3(x_2 y_1 - 1) = 0$. The two surfaces are given by $z_1 = 0$ and $x_2 y_1 - 1 = 0$. This is a normal crossing divisor as this is the exact same case as the A_1 case.

4.1.2 Standard D_n case

Now let X be an affine hypersurface with standard D_n equation given by $x^2 + yz^2 + y^{n-1} = 0$. Similar to before, the blowup is a subset of $\mathbb{A}^3 \times \mathbb{P}^2$ with coordinates x, y, z, x_1, y_1, z_1.

Consider the affine patch on $\mathbb{P}^2$ where $x_1 = 1$. Then the vanishing of all 2 by 2 minors of the matrix

$$\begin{bmatrix} x & y & z \\ 1 & y_1 & z_1 \end{bmatrix}$$

is given by $y = xy_1$ and $z = xz_1$. Then our equation $x^2 + yz^2 + y^{n-1} = 0$ becomes $x^2(1 + xy_1 z_1^2 + x^{n-3} y_1^{n-1}) = 0$. This surface is reducible into two components. The first component given by $x = 0$ is our exceptional divisor of the blow up of $\mathbb{A}^3$. The second component is the strict transform of our surface given by equation $1 + xy_1 z_1^2 + x^{n-3} y_1^{n-1} = 0$. These two surfaces do not intersect since $n \geq 4$.

Consider the affine patch of $\mathbb{P}^2$ where $y_1 = 1$. The vanishing of all 2 by 2 minors of the matrix

$$\begin{bmatrix} x & y & z \\ x_1 & 1 & z_1 \end{bmatrix}$$

is given by $x = yx_1$ and $z = yz_1$. Then our equation $x^2 + yz^2 + y^{n-1} = 0$ becomes $y^2(x_1^2 + yz_1^2 + y^{n-3}) = 0$. This surface is reducible into two components. The first component given by $y^2 = 0$ is our exceptional divisor of the blow up of $\mathbb{A}^3$. The second component is the strict transform of our surface given by equation $x_1^2 + yz_1^2 + y^{n-3} = 0$. Note for $n \geq 6$, the equation is exactly the standard D_{n-2} equation. Therefore, we just need to show the case for $n = 4$ and $n = 5$ and induction proves the rest.

For the D_4 case, after relabeling, the equation is given by $y^2(x_1^2+yz_1^2+y) = 0$. This equation is singular at $z_1 = \pm\sqrt{-1}$. We can assume that $\pm\sqrt{-1}$ lies in $\mathbb{Z}_p$. If not, the same argument given below will work over a quadratic extension of $\mathbb{Z}_p$. Hence, we let $i = \sqrt{-1}$. We shift the equation to $y^2(x_1^2 + y(z_1 - i)^2 + y) = 0$. We blow up at the origin.

On the patch $x_1 = yx_2, z_1 = yz_2$, the equation becomes $y^4(yz_2^2 - 2z_2 i) = 0$. There is no intersection between the exceptional divisor and the strict transform.

On the patch $z_1 = x_1z_2, y = x_1y_2$, the equation becomes $x_1^3y_2^2(x_1 + y_2(x_1z_2 - i)^2 + y_2) = 0$. One more blow up on the same patch gives $x_1^4y_3^2(1 + y_3(xz_3 - i)^2 + y_3) = 0$. There is no intersection between the exceptional divisor and strict transform.

On the patch $y = z_1y_2, x_1 = z_1x_2$, the equation becomes $z_1^4y_2^2(x_2^2 + y_2z_1 - 2iy_2) = 0$. We shift to the origin so the equation becomes $z_1^4y_2^2(x_2^2 + y_2z_1) = 0$. This is an A_1 singularity which has been covered.

In the D_5 case, we have $y^2(x_1^2 + yz_1^2 + y^2) = 0$. We blow up the origin again.

On the patch $x_1 = yx_2, z_1 = yz_2$, the equation becomes $y^4(x_2^2 + z_2^2y + 1) = 0$. Let $a \in \mathbb{Z}_p$. Consider a point in the double intersection given by $y = 0$, $x_2 = i$, $z_2 = a$. We shift to the origin given by $y^4((x_2 - i)^2 + (z_2 - a)^2y + 1) = 0$. Let s be the origin. This corresponds to ideals (x_2, y, z_2) and (x_2, y, z_2, p). The blow up defines a map

$$\phi : V(x_2, y, z_2) \to \mathbb{A}^3(u, v, w)$$

$$\phi(x_2, y, z_2) = ((x_2 - i)^2 + (z_2 - a)^2y + 1, y, z) = (u, v, w)$$

where V is the neighborhood of the origin. The image of the origin under ϕ is the origin. For ideal (x_2, y, z_2), we have $\tilde{\mathcal{O}}_{X,s} = \mathbb{Q}_p[[x_2, y, z_2]]$ and $\tilde{\mathcal{O}}_{Y,\phi(s)} = \mathbb{Q}_p[[u, v, w]] = \mathbb{Q}_p[[(x_2 - i)^2 + (z_2 - a)^2 y + 1, y, z_2]]$. For ideal (x_2, y, z_2, p), $\tilde{\mathcal{O}}_{X,s} = \mathbb{F}_p[[x_2, y, z_2]]$ and $\tilde{\mathcal{O}}_{Y,y\phi(s)} = \mathbb{F}_p[[u, v, w]] = \mathbb{F}_p[[(x_2 - i)^2 + (z_2 - a)^2 y + 1, y, z_2]]$. In either case, Condition 2 of 4.1 is satisfied because the quotient field is either $\mathbb{F}_p$ or $\mathbb{Q}_p$ which is a finite separable extension of itself. For Condition 1, there is a natural action of u, v, w given by the action of $(x_2 - i)^2 + (z_2 - a)^2 y + 1, y, z_2$ which makes $\tilde{\mathcal{O}}_{X,s}$ into an $\tilde{\mathcal{O}}_{Y,\phi(s)}$ module. We check the 3x3 matrix of the coefficients linear terms of ϕ. The matrix is given by

$$\begin{bmatrix} -2i & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The determinant is nonzero so $\tilde{\mathcal{O}}_{X,x}$ is free over $\tilde{\mathcal{O}}_{Y,y}$.

On the patch $y = x_1 y_2$, $z_1 = x_1 z_2$, the equation is given $x_1^4 y_2^4 (1 + x_1 z_2^2 y_2 + y_2^2) = 0$. The nontrivial intersection is given by $x = 0$ and the strict transform. Let $a \in \mathbb{Z}_p$. Consider a point in the nontrivial intersection given by $x_1 = 0$, $y_2 = i$, $z_2 = a$. We shift to the origin so the strict transform becomes $1 + x_1(z_2 - a)^2(y_2 - i) + (y_2 - i)^2 = 0$. Let s be the origin.This corresponds to ideals (x_1, y_2, z_2) and (x_1, y_2, z_2, p). The blow up defines a map

$$\phi : V(x_1, y_2, z_2) \to \mathbb{A}^3(u, v, w)$$
$$\phi(x_1, y_2, z_2) = (x_1, 1 + x_1(z_2 - a)^2(y_2 - i) + (y_2 - i)^2, z_2) = (u, v, w)$$

where V is a neighborhood of the origin. The image of the origin under ϕ is the origin. For ideal (x_1, y_2, z_2), we have $\tilde{\mathcal{O}}_{X,s} = \mathbb{Q}_p[[x_1, y_2, z_2]]$ and $\tilde{\mathcal{O}}_{Y,\phi(s)} = \mathbb{Q}_p[[u, v, w]] = \mathbb{Q}_p[[x_1, 1 + x_1(z_2 - a)^2(y_2 - i) + (y_2 - i)^2, z_2]]$. For ideal (x_1, y_2, z_2, p), $\tilde{\mathcal{O}}_{X,s} = \mathbb{F}_p[[x_1, y_2, z_2]]$ and $\tilde{\mathcal{O}}_{Y,\phi(s)} = \mathbb{F}_p[[u, v, w]] = \mathbb{F}_p[[x_1, 1 + x_1(z_2 - a)^2(y_2 - i) + (y_2 - i)^2, z_2]]$. In either case, Condition 2 of 4.1 is satisfied because the quotient field is either $\mathbb{F}_p$ or $\mathbb{Q}_p$ which is a finite separable extension of itself. For Condition 1, there is a natural action of u, v, w given by the action of $x_1, 1 + x_1(z_2 - a)^2(y_2 - i) + (y_2 - i)^2, z_2$ which makes $\tilde{\mathcal{O}}_{X,s}$ into an $\tilde{\mathcal{O}}_{Y,\phi(s)}$ module. We check the 3x3 matrix of the coefficients of the linear terms of ϕ. The matrix is given by

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -2i & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The determinant is nonzero so $\tilde{\mathcal{O}}_{X,s}$ is free over $\tilde{\mathcal{O}}_{Y,\phi(s)}$.

On the patch $y = z_1 y_2, x_1 = z_1 x_2$, the equation is given by $z_1^4 y_2^2(x_2^2 + z_1 y_2 + y_2^2) = 0$. We do the change of coordinates where we replace z_1 with $z_1 - y_2$. The equation then becomes $z_1^4 y_2^2(x_2^2 + z_1 y_2) = 0$ which is a type A_1 equation, and this case has been covered.

Finally, we head back to the beginning where we blow up the origin of the type D_n singularity on the patch where $z_1 = 1$. The equation is given by $z^2(x_1^2 + z y_1 + z^{n-3} y_1^{n-1}) = 0$. One can check in all cases of n, the Hessian is non degenerate so this is a type A_1 singularity which we proved the result holds already.

4.1.3 Standard E_n case

Now let X be an affine hypersurface with standard E_6 equation given by $x^2 + y^3 + z^4 = 0$. Similar to before, the blowup is a subset of $\mathbb{A}^3 \times \mathbb{P}^2$ with coordinates x, y, z, x_1, y_1, z_1.

Consider the affine patch on $\mathbb{P}^2$ where $x_1 = 1$. Then the vanishing of all 2 by 2 minors of the matrix

$$\begin{bmatrix} x & y & z \\ 1 & y_1 & z_1 \end{bmatrix}$$

is given by $y = xy_1$ and $z = xz_1$. Then our equation $x^2 + y^3 + z^4 = 0$ becomes $x^2(1 + xy_1^3 + x^2z_1^2) = 0$. There is no intersection between the exceptional divisor and strict transform.

Consider the affine patch of $\mathbb{P}^2$ where $y_1 = 1$. The vanishing of all 2 by 2 minors of the matrix

$$\begin{bmatrix} x & y & z \\ x_1 & 1 & z_1 \end{bmatrix}$$

is given by $x = yx_1$ and $z = yz_1$. Then our equation $x^2 + y^3 + z^4 = 0$ becomes $y^2(x_1^2 + y + y^2z_1^4) = 0$. We now blow up at the z_1-axis given by the ideal (x_1, y).

On one patch, we have $x = yx_2$. The equation becomes $y^3(yx_2^2 + 1 + yz_1^4) = 0$. There is no intersection between the exceptional divisor and strict transform.

On another patch, we have $y = x_1y_2$. The equation becomes $x_1^3y_2^2(x_1 + y_2 + x_1y_2^2z_1^4) = 0$. We blow up the z_1-axis again.

On one patch, $x_1 = y_2x_3$. So we get $y_2^6x_3^3(x_3 + 1 + y_2^2x_3z_1^4) = 0$. The nontrivial intersection is between $y_2 = 0$ and the strict transform. Let $a \in \mathbb{Z}_p$. Consider a point in the nontrivial intersection given by $y_2 = 0, x_3 = -1, z_1 = a$. We shift to the origin so the strict transform is given by $x_3 + y_2^2(x_3 - 1)(z_1 + a)^2 = 0$. Let s be the origin. This is given by ideals (x_3, y_2, z_1) and (x_3, y_2, z_1, p). The blow up defines a map

$$\phi : V(x_3, y_2, z_1) \to \mathbb{A}^3(u, v, w)$$
$$\phi(x_3, y_2, z_1) = (z_1, y_2, x_3 + y_2^2(x_3 - 1)(z_1 + a)^2) = (u, v, w)$$

where V is a neighborhood of the origin. The image of the origin is the origin which corresponds to ideals (u, v, w) and (u, v, w, p). For ideal (x_3, y_2, z_1), we have $\tilde{\mathcal{O}}_{X,s} = \mathbb{Q}_p[[x_3, y_2, z_1]]$ and $\tilde{\mathcal{O}}_{Y,\phi(s)} = \mathbb{Q}_p[[u, v, w]] = \mathbb{Q}_p[[z_1, y_2, x_3 + y_2^2(x_3 - 1)(z_1 + a)^2]]$. For ideal (x_3, y_2, z_1, p), we have $\tilde{\mathcal{O}}_{X,s} = \mathbb{F}_p[[x_3, y_2, z_1]]$ and $\tilde{\mathcal{O}}_{Y,\phi(s)} = \mathbb{F}_p[[u, v, w]] = \mathbb{F}_p[[z_1, y_2, x_3 + y_2^2(x_3 - 1)(z_1 + a)^2]]$. In either case, Condition 2 of Definition 4.1 is satisfied because the quotient field is either $\mathbb{F}_p$ or $\mathbb{Q}_p$ which is a finite separable extension of itself. For Condition 1, there is a natural action of u, v, w given by the action of $z_1, y_2, x_3 + y_2^2(x_3 - 1)(z_1 + a)^2$ which makes $\tilde{\mathcal{O}}_{X,s}$ into an $\tilde{\mathcal{O}}_{Y,\phi(s)}$ module. We look at the 3x3 matrix of the coefficient linear terms of the map ϕ. It is given by

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

The determinant is nonzero so $\tilde{\mathcal{O}}_{X,s}$ is free over $\tilde{\mathcal{O}}_{Y,\phi(s)}$.

On other patch, $y_2 = x_1 y_3$. So we get $x_1^6 y_3^2(1 + y_3 + x_1^2 y_3^2 z_1^4) = 0$. The nontrivial intersection is between $x_1 = 0$ and the strict transform. Let $a \in \mathbb{Z}_p$. Then consider a point in the nontrivial intersection given by $x_1 = 0, y_3 = -1, z_1 = a$. We shift to the origin so the equation of strict transform becomes $y_3 + x_1^2(y_3 - 1)^2(z_1 + a)^4 = 0$. Let s be the origin. The origin is given by ideals (x_1, y_3, z_1) and (x_1, y_3, z_1, p).The blow up defines a map

$$\phi : V(x, y, z) \to \mathbb{A}^3(u, v, w)$$
$$\phi(x, y, z) = (x, z, y + x^2(y - 1)^2(z + a)^4) = (u, v, w)$$

where V is neighborhood of the origin. For ideal (x_1, y_3, z_1), we have $\tilde{\mathcal{O}}_{X,s} = \mathbb{Q}_p[[x_1, y_3, z_1]]$ and $\tilde{\mathcal{O}}_{Y,\phi(s)} = \mathbb{Q}_p[[u, v, w]] = \mathbb{Q}_p[[x_1, z_1, x_3 + y_2^2(x_3 - 1)(z_1 + a)^2]]$. For ideal (x_1, y_3, z_1, p), we have $\tilde{\mathcal{O}}_{X,s} = \mathbb{F}_p[[x_1, y_3, z_1]]$ and $\tilde{\mathcal{O}}_{Y,\phi(s)} = \mathbb{F}_p[[u, v, w]] = \mathbb{F}_p[[x_1, z_1, x_3 + y_2^2(x_3 - 1)(z_1 + a)^2]]$. In either case, Condition 2 of Definition 4.1 is satisfied because the quotient field is either $\mathbb{F}_p$ or $\mathbb{Q}_p$ which is a finite separable extension of itself. For Condition 1, there is a natural action of u, v, w given by the action of $x_1, z_1, x_3 + y_2^2(x_3 - 1)(z_1 + a)^2$ which makes $\tilde{\mathcal{O}}_{X,s}$ into an $\tilde{\mathcal{O}}_{Y,\phi(s)}$ module. We consider the 3x3 matrix given by the coefficients of the linear terms of ϕ. It is given by

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

The determinant is nonzero so $\tilde{\mathcal{O}}_{X,s}$ is free over $\tilde{\mathcal{O}}_{Y,\phi(s)}$.

We go back to the third patch of the blow up at the origin. Consider the affine patch on $\mathbb{P}^2$ where $z_1 = 1$. Then the vanishing of all 2 by 2 minors of the matrix

$$\begin{bmatrix} x & y & z \\ x_1 & y_1 & 1 \end{bmatrix}$$

is given by $y = zy_1$ and $x = zx_1$. Then our equation $x^2 + y^3 + z^4 = 0$ becomes $z^2(x_1^2 + zy_2^3 + z^2) = 0$. This equation is an A_5 singularity which was proven earlier.

Now let X be an affine hypersurface standard E_7 equation given by $x^2 + y^3 + yz^3 = 0$. The blowup of the origin is a subset of $\mathbb{A}^3 \times \mathbb{P}^2$ with coordinates x, y, z, x_1, y_1, z_1.

Consider the affine patch on $\mathbb{P}^2$ where $x_1 = 1$. Then the vanishing of all 2 by 2 minors of the matrix

$$\begin{bmatrix} x & y & z \\ 1 & y_1 & z_1 \end{bmatrix}$$

is given by $y = xy_1$ and $z = xz_1$. Then our equation $x^2 + y^3 + yz^3 = 0$ becomes $x^2(1 + xy_1^3 + x^2y_1z_1^3) = 0$. There is no intersection of $x = 0$ and the strict transform.

Consider the affine patch of $\mathbb{P}^2$ where $y_1 = 1$. The vanishing of all 2 by 2 minors of the matrix

$$\begin{bmatrix} x & y & z \\ x_1 & 1 & z_1 \end{bmatrix}$$

is given by $x = yx_1$ and $z = yz_1$. Then our equation $x^2 + y^3 + yz^3 = 0$ becomes $y^2(x_1^2 + y + y^2z_1^2) = 0$. We blow up the equation at the z_1-axis given by ideal (x_1, y).

On one patch, $x_1 = yx_2$. The equation becomes $y^3(yx_2^2 + 1 + yz_1^2)$. There is no intersection between $y = 0$ and the strict transform.

On the other patch, $y = x_1y_2$. The equation becomes $x_1^3y_2^2(x_1 + y_2 + x_1y_2^2z_1^2) = 0$. We blow up at the z_1-axis again.

On the patch $x_1 = y_2x_3$, we have $y_2^6x_3^3(x_3 + 1 + y_2^2x_3z_1^2) = 0$. The interesting intersection is when $y_2 = 0$. Let $a \in \mathbb{Z}_p$ take a point in the intersection given by $x_3 = -1, y_2 = 0, z_1 = a$. We shift to the origin so the equation becomes $x_3 + y_2^2(x_3 - 1)(z_1 + a)^2 = 0$. Let s be the origin and s corresponds to ideals (x_3, y_2, z_1) and (x_3, y_2, z_1, p). The blow up defines a map

$$\phi : V(x_3, y_2, z_1) \to \mathbb{A}^3(u, v, w)$$
$$\phi(x_3, y_2, z_1) = (z_1, y_2, x_3 + y_2^2(x_3 - 1)(z_1 + a)^2) = (u, v, w)$$

where V is a neighborhood of the origin. For ideal (x_3, y_2, z_1), we have $\tilde{\mathcal{O}}_{X,s} = \mathbb{Q}_p[[x_3, y_2, z_1]]$ and $\tilde{\mathcal{O}}_{Y,\phi(s)} = \mathbb{Q}_p[[u, v, w]] = \mathbb{Q}_p[[z_1, y_2, x_3 + y_2^2(x_3 - 1)(z_1 + a)^2]]$. For ideal (x_3, y_2, z_1, p), we have $\tilde{\mathcal{O}}_{X,s} = \mathbb{F}_p[[x_3, y_2, z_1]]$ and $\tilde{\mathcal{O}}_{Y,\phi(s)} = \mathbb{F}_p[[u, v, w]] = \mathbb{F}_p[[z_1, y_2, x_3 + y_2^2(x_3 - 1)(z_1 + a)^2]]$. In either case, Condition 2 of 4.1 is satisfied because the quotient field is either $\mathbb{F}_p$ or $\mathbb{Q}_p$ which is a finite separable extension of itself. For Condition 1, there is a natural action of u, v, w given by the action of $z, y, x + y^2(x - 1)(z + a)^2$ which makes $\tilde{\mathcal{O}}_{X,s}$ into an $\tilde{\mathcal{O}}_{Y,\phi(s)}$ module. We look at the 3x3 matrix given by the coefficients of the linear terms of ϕ. The matrix is given by

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

The determinant is nonzero so $\tilde{\mathcal{O}}_{X,s}$ is free over $\tilde{\mathcal{O}}_{Y,\phi(s)}$.

On patch $y_2 = x_1y_3$, we have $x_1^6y_3^2(1 + y_3 + x_1^2y_3^2z_1^2) = 0$. The interesting intersection is when $x_1 = 0$. Let $a \in \mathbb{Z}_p$. Consider a point in the interesting intersection given by $x_1 = 0, y_3 = -1, z_1 = a$. We shift to the origin given by $y_3 + x_1^2(y_3 - 1)^2(z_1 - a)^2$. Let s be the origin and s corresponds to ideals (x_1, y_3, z_1) and (x_1, y_3, z_1, p). The blow up defines a map

$$\phi : V(x_1, y_3, z_1) \to \mathbb{A}^3(u, v, w)$$
$$\phi(x_1, y_3, z_1) = (x_1, y_3 + x_1^2(y_3 - 1)^2(z_1 - a)^2, z_1) = (u, v, w)$$

where V is a neighborhood of the origin. For ideal (x_1, y_3, z_1), we have $\tilde{\mathcal{O}}_{X,s} = \mathbb{Q}_p[[x_1, y_3, z_1]]$ and $\tilde{\mathcal{O}}_{Y,\phi(s)} = \mathbb{Q}_p[[u, v, w]] = \mathbb{Q}_p[[x_1, y_3 + x_1^2(y_3 - 1)^2(z_1 - a)^2, z_1]]$.For ideal (x_1, y_3, z_1, p), we have $\tilde{\mathcal{O}}_{X,s} = \mathbb{F}_p[[x_1, y_3 + x_1^2(y_3 - 1)^2(z_1 - a)^2, z_1]]$ and $\tilde{\mathcal{O}}_{Y,\phi(s)} = \mathbb{F}_p[[u, v, w]] = \mathbb{F}_p[[x_1, y_3 + x_1^2(y_3 - 1)^2(z_1 - a)^2, z_1]]$. In either case, Condition 2 of 4.1 is satisfied because the quotient field is either $\mathbb{F}_p$ or $\mathbb{Q}_p$ which is a finite separable extension of itself. For Condition 1, there is a natural action of u, v, w given by the action of $x_1, y_3 + x_1^2(y_3 - 1)^2(z_1 - a)^2, z_1$ which makes $\tilde{\mathcal{O}}_{X,s}$ into an $\tilde{\mathcal{O}}_{Y,\phi(s)}$ module. We look at the 3x3 matrix given by the coefficients of the linear terms of ϕ.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Consider the affine patch on $\mathbb{P}^2$ where $z_1 = 1$. Then the vanishing of all 2 by 2 minors of the matrix

$$\begin{bmatrix} x & y & z \\ x_1 & y_1 & 1 \end{bmatrix}$$

is given by $y = zy_1$ and $x = zx_1$. Then our equation $x^2+y^3+yz^3 = 0$ becomes $z^2(x_1^2+zy_1^3+z^2y_1) = 0$. The strict transform is a singularity of type D_6. The result has been proven for type D_6 singularities so we are done with the E_7 case.

Now let X be an affine hypersurface with a E_8 standard equation given by $x^2+y^3+z^5 = 0$. Similar to before, the blowup is a subset of $\mathbb{A}^3 \times \mathbb{P}^2$ with coordinates x, y, z, x_1, y_1, z_1.

Consider the affine patch on $\mathbb{P}^2$ where $a = 1$. Then the vanishing of all 2 by 2 minors of the matrix

$$\begin{bmatrix} x & y & z \\ 1 & y_1 & z_1 \end{bmatrix}$$

is given by $y = xy_1$ and $z = xz_1$. Then our equation $x^2+y^3+z^5 = 0$ becomes $x^2(1+xy_1^3+x^3z_1^5) = 0$. There is no intersection of $x = 0$ with the strict transform.

Consider the affine patch of $\mathbb{P}^2$ where $y_1 = 1$. The vanishing of all 2 by 2 minors of the matrix

$$\begin{bmatrix} x & y & z \\ x_1 & 1 & z_1 \end{bmatrix}$$

is given by $x = yx_1$ and $z = yz_1$. Then our equation $x^2+y^3+z^5 = 0$ becomes $y^2(x_1^2+y+y^3z_1^5) = 0$. We now blow up at the z_1-axis given by the ideal (x_1, y).

On one patch, we have $x_1 = yx_2$. The equation becomes $y^3(yx_2^2 + 1 + y^2z_1^5) = 0$. There is no intersection of the exceptional divisor with the strict transform.

On another patch, we have $y = x_1y_2$. The equation becomes $x_1^3y_2^2(x_1 + y_2 + x_1^2y_2^3z_1^5) = 0$. We blow up the z_1-axis given by (x_1, y_2).

On one patch, $x_1 = y_2x_3$. So we get $y_2^6x_3^3(x_3 + 1 + y_2^4x_3^2z_1^5) = 0$. The interesting intersection is given in the double intersection of $y_2 = 0$ and $x_3 + 1 + y_2^4x_3^2z_1^5 = 0$. Let $a \in \mathbb{Z}_p$. Consider a point in the interesting intersection given by $x_3 = -1, y_2 = 0, z_1 = a$. We shift to the origin denoted by s. The strict transform equation then becomes $x_3 + y_2^4(x_3 - 1)^2(z_1 - a)^5$. The origin corresponds to ideals (x_3, y_2, z_1) and (x_3, y_2, z_1, p). The blow up defines a map

$$\phi : V(x_3, y_2, z_1) \to \mathbb{A}^3(u, v, w)$$
$$\phi(x_3, y_2, z_1) = (x_3 + y_2^4(x_3 - 1)^2(z_1 - a)^5, y_2, z_1) = (u, v, w)$$

where V is a neighborhood of the origin. For ideal (x_3, y_2, z_1), we have $\tilde{\mathcal{O}}_{X,s} = \mathbb{Q}_p[[x_3, y_2, z_1]]$ and $\tilde{\mathcal{O}}_{Y,\phi(s)} = \mathbb{Q}_p[[u, v, w]] = \mathbb{Q}_p[[x_3 + y_2^4(x_3 - 1)^2(z_1 - a)^5, y_2, z_1]]$. For ideal (x_3, y_2, z_1, p), we have $\tilde{\mathcal{O}}_{X,s} = \mathbb{F}_p[[x_3, y_2, z_1]]$ and $\tilde{\mathcal{O}}_{Y,\phi(s)} = \mathbb{F}_p[[u, v, w]] = \mathbb{F}_p[[x_3 + y_2^4(x_3 - 1)^2(z_1 - a)^5, y_2, z_1]]$.In either case, Condition 2 of Definition 4.1 is satisfied because the quotient field is either $\mathbb{F}_p$ or $\mathbb{Q}_p$ which is a finite separable extension of itself. For Condition 1, there is a natural action of u, v, w given by the action of $x_3 + y_2^4(x_3 - 1)^2(z_1 - a)^5, y_2, z_1$ which makes $\tilde{\mathcal{O}}_{X,s}$ into an $\tilde{\mathcal{O}}_{Y,\phi(s)}$ module. We look at the 3x3 matrix given by the linear terms of ϕ. The matrix is given by

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The determinant is nonzero so $\tilde{\mathcal{O}}_{X,s}$ is free over $\tilde{\mathcal{O}}_{Y,\phi(s)}$.

On the other patch, $y_2 = x_1 y_3$. So we get $x_1^6 y_3^2(1 + y_3 + x_1^4 y_3^3 z_1^5) = 0$. The nontrivial intersection is when $x_1 = 0$. Let $a \in \mathbb{Z}_p$. Take a point in the intersection given by $x_1 = 0, y_3 = -1, z_1 = a$. We shift to the origin so the strict transform becomes $y_3 + x_1^4(y_3 - 1)^3(z_1 - a)^5 = 0$. Let s be the origin given by ideals (x_1, y_3, z_1) and (x_1, y_3, z_1, p). The blow up defines a map

$$\phi : V(x_1, y_3, z_1) \to \mathbb{A}^3(u, v, w)$$

$$\phi(x_1, y_3, z_1) = (x_1, y_3 + x_1^4(y_3 - 1)^3(z_1 - a)^5, z_1) = (u, v, w)$$

where V is a neighborhood of the origin. For ideal (x_1, y_3, z_1), we have $\tilde{\mathcal{O}}_{X,s} = \mathbb{Q}_p[[x_1, y_3, z_1]]$ and $\tilde{\mathcal{O}}_{Y,\phi(s)} = \mathbb{Q}_p[[u, v, w]] = \mathbb{Q}_p[[x_1, y_3 + x_1^4(y_3 - 1)^3(z_1 - a)^5, z_1]]$. For ideal (x_1, y_3, z_1, p), we have $\tilde{\mathcal{O}}_{X,s} = \mathbb{F}_p[[x_1, y_3, z_1]]$ and $\tilde{\mathcal{O}}_{Y,\phi(s)} = \mathbb{F}_p[[u, v, w]] = \mathbb{F}_p[[x_1, y_3 + x_1^4(y_3 - 1)^3(z_1 - a)^5, z_1]]$.In either case, Condition 2 of Definition 4.1 is satisfied because the quotient field is either $\mathbb{F}_p$ or $\mathbb{Q}_p$ which is a finite separable extension of itself. For Condition 1, there is a natural action of u, v, w given by the action of $x_1, y_3 + x_1^4(y_3 - 1)^3(z_1 - a)^5, z_1$ which makes $\tilde{\mathcal{O}}_{X,s}$ into an $\tilde{\mathcal{O}}_{Y,\phi(s)}$ module. We look at the 3x3 matrix given by the linear terms of ϕ. The matrix is given by

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The determinant is nonzero so $\tilde{\mathcal{O}}_{X,s}$ is free over $\tilde{\mathcal{O}}_{Y,\phi(s)}$.

Consider the affine patch on $\mathbb{P}^2$ where $z_1 = 1$. Then the vanishing of all 2 by 2 minors of the matrix

$$\begin{bmatrix} x & y & z \\ x_1 & y_1 & 1 \end{bmatrix}$$

is given by $y = zy_1$ and $x = zx_1$. Then our equation $x^2 + y^3 + z^5 = 0$ becomes $z^2(x_1^2 + zy_1^3 + z^3) = 0$. The strict transform is the standard E_7 equation which we showed the result held earlier. So we have shown the result for ADE singularities in standard form.

4.1.4 Transition to General Equation

We have shown that there is a sequence of blow ups on the standard equations that lead to a divisor with simple normal crossings. We now show the same can be done before analytic change of coordinates. On any stage of the blow up, we have a union of 2-dimensional surfaces given by $E_0 \cup E_1 \cup ... \cup E_k$, where E_0 is the strict transform of the divisor given before the blow up and E_i for $i > 1$ are the irreducible components of the exceptional divisor. Suppose we have a standard equation, and there are k total blow-ups to obtain a divisor with simple normal crossings. Let X_k denote the strict transforms on the kth blow up with X_0 being the original surface. Let Z_k denote the union of the irreducible components of exceptional divisors on the kth blow up (in notation above, this is the union of E_i for $i > 1$). Z_0 will be the singular point. Let $\tilde{X}_0$ denote the surface before the change to standard form. Let $\tilde{Z}_0$ denote the singular point. Let

$$X_0|_{Z_0} = \varprojlim X_0/Z_0^k \cong \varprojlim \tilde{X}_0/\tilde{Z}_0^k$$

where the limit runs as $k \to \infty$. Since Z_0 and $\tilde{Z}_0$ denote the origin which is the ideal (x, y, z), the two spaces above are isomorphic by Definition 1.2. These are isomorphic by assumption of equisingular lift. For simplicity, given a scheme X and ideal I, we will denote this inverse limit as the formal completion of X along I given by $X|_I$. We let

$$\phi|_{(X,I)} : X|_I \to X$$

denote the natural map into the formal completion. We let the blow up maps from X_i to X_{i+1} be denoted as

$$\psi_i : X_{i+1} \to X_i$$

Base Case: We show that the blow up of the original of the standard equation and applying formal completion gives a blow up procedure on the side with the original equation and taking formal completion such that the formal completions are isomorphic. We have shown the sequence of blow ups work on the side of the standard equation side so we have the first blow up map

$$\psi_0 : X_1 \to X_0$$

Let S_0 be the formal scheme that is isomorphic to the formal completion of both the original equation and the standard equation. We then consider the formal completion along Z_1 with map given by

$$\phi|_{(X_1,Z_1)} : X_1|_{Z_1} \to X_1$$

But by Proposition 6 of Bosch [3], this is equivalent to taking the formal completion

$$\phi|_{(X_0,Z_0)} : S \to X_0$$

and then blowing up along Z_0 to get a map

$$\eta_0 : S_1 \to X_0|_{Z_0}$$

where S_1 is the blow up of S along Z_0. Then we have that $S_1 \cong X_1|_{Z_1}$.

$$\begin{tikzcd}
X_1 \arrow[d, "\psi_0"] & & X_1|_{Z_1} \cong S_1 \arrow[ll, "\phi|_{(X_1,Z_1)}"'] \arrow[d, "\eta_0"] & \\
X_0 & & S \arrow[ll, "\phi_{(X_0,Z_0)}"] \arrow[r, "\psi|_{(\tilde{X_0},\tilde{Z_0})}"] & \tilde{X}_0
\end{tikzcd}$$

But then by Proposition 6 of Bosch [3], formal completion and blow up commute so there exists $\tilde{Z}_1$, blow up map $\tilde{\psi}_0$, and formal completion map $\phi|_{(\tilde{X}_1,\tilde{Z}_1)}$ such that the following diagram commutes.

$$\begin{tikzcd}
X_1 \arrow[d, "\psi_0"] & X_1|_{Z_1} \cong S_1 \cong \tilde{X}_1|_{\tilde{Z}_1} \arrow[l, "\phi|_{(X_1,Z_1)}"'] \arrow[d, "\eta_0"] \arrow[r, "\phi|_{(\tilde{X_1},\tilde{Z_1})}"] & \tilde{X}_1 \arrow[d, "\tilde{\psi_0}"] \\
X_0 & S \arrow[l, "\phi_{(X_0,Z_0)}"] \arrow[r, "\psi|_{(\tilde{X_0},\tilde{Z_0})}"] & \tilde{X}_0
\end{tikzcd}$$

Therefore, we have given a blow up after analytic change of coordinates , there exist a blow up procedure on the original equation such that the formal completions of the corresponding strict transforms are isomorphic, i.e. $X_1|_{Z_1} \cong S_1\tilde{X}_1|_{\tilde{Z}_1}$.

Suppose there exists a blow up procedure on before analytic change of coordinates and after analytic change of coordinates such that the formal completions along the irreducible components of the exceptional divisor are isomorphic up the the jth blow up. So $X_j|_{Z_j} \cong S_j\tilde{X}_j|_{\tilde{Z}_j}$. We show the result holds for the $(j+1)$ blow up. By same process, we have the blow up procedure on the side of the standard equation so we denote the blow up map as

$$\psi_j : X_{j+1} \to X_j$$

We then have the formal completion along the exceptional locus Z_{j+1} given by the map

$$\phi|_{(X_{j+1},Z_{j+1})} : X_{j+1}|_{Z_j} \to X_{j+1}$$

Note in the blow up procedure of the standard equation, we are always blowing up a point or a curve within Z_j so Proposition 6 of Bosch [3] applies. By Proposition 6 of Bosch [3], formal completion and blow up commute so the same result holds if we do formal completion along Z_j first and then blow up. We will denote the formal completions and blow up as

$$\phi|_{(X_j,Z_j)} : X_j|_{Z_j} \to X_j$$

But the formal completion of X_j along Z_j and blowing up a point or curve along S_j is the same as blowing up and completing. Denote S_{j+1} the blow up of S_j along the corresponding point or curve. By induction hypothesis, we have the following commuting diagram.

$$\begin{tikzcd}
X_{j+1} \arrow[d, "\psi_j"] & X_{j+1}|_{Z_{j+1}} \cong S_{j+1} \arrow[l, "\phi|_{(X_{j+1},Z_{j+1})}"'] \arrow[d, "\eta_j"] & \\
X_j & S_j \arrow[l, "\phi_{(X_j,Z_j)}"] \arrow[r, "\psi|_{(\tilde{X}_j,\tilde{Z}_j)}"] & \tilde{X}_j
\end{tikzcd}$$

But by Proposition 6 of Bosch [3], formal completion commutes with blow up so we have the following commuting diagram:

$$\begin{tikzcd}
X_{j+1} \arrow[d, "\psi_j"] & X_{j+1}|_{Z_{j+1}} \cong S_{j+1} \cong \tilde{X}_{j+1}|_{\tilde{Z}_{j+1}} \arrow[l, "\phi|_{(X_{j+1},Z_{j+1})}"'] \arrow[d, "\eta_j"] \arrow[r, "\phi|_{(\tilde{X_{j+1}},\tilde{Z_{j+1}})}"] & \tilde{X_{j+1}} \arrow[d, "\tilde{\psi_j}"] \\
X_j & S_j \arrow[l, "\phi_{(X_j,Z_j)}"] \arrow[r, "\psi|_{(\tilde{X}_j,\tilde{Z}_j)}"] & \tilde{X}_j
\end{tikzcd}$$

So we have that there is a sequence of blow up on the coordinates of the original equation such that $X_{j+1}|_{Z_{j+1}} \cong S_{j+1} \cong \tilde{X}_{j+1}|_{\tilde{Z}_{j+1}}$.

Lemma 4.1. *Smoothness and strict relative normally crossing divisor holds on the formal completion given above.*

At the end of the final blow up of the standard equation side, we have union of two dimensional pieces $E_0 \cup E_1 \cup ... \cup E_j$ where E_0 denotes the strict transform and E_i for $i > 1$ denote the irreducible components of exceptional divisor. Points not in any E_j do not need to be checked to satisfy the theorem by Baldassarri and Chiarellotto [2]. Points in only one E_i are smooth and normal crossings so they do not need to be checked. The interesting case is when points are in double intersections or triple intersections of E_i which we have checked normal crossings have worked on the standard equation side. For the strict normal crossings divisor, we needed to show the morphism given from the functions of the irreducible components of the blow up on the 2 dimensional strata is etale at each point s in the irreducible component. The etale condition is given in terms of maps of complete local rings. Since we complete along the union of the irreducible components, we have two ideals in place. We have ideal I given by the union of the irreducible components and ideal J given by the point s. Note $I \subset J$. Let R is our local ring at s. Then let $(R^I)^J$ be the completion of R along I and then along J. Let R^{I+J} be the completion of R along $I + J$. From Atiyah and MacDonald [1], $(R^I)^J \cong R^{I+J}$. But since $I \subset J$, $I + J = J$ so we have $(R^I)^J \cong R^J$. Hence, this says completing along the irreducible components and then completing along the point is the same as directly completing along the point. Hence, we have our result since we are completing the completion along the ideal correspond the point.

For smoothness, we show both directions. We work on an affine set.

Suppose all irreducible components on the 2 dimensional strata of the blow up are smooth. Since smoothness is a local property, we restrict to the localization at a smooth point, s. Let R, R_s be the ring corresponding to the surface the smooth point is in and the localization of that ring at a smooth point s respectively. Then R_s is regular since R is a smooth scheme. Let M denote the completion along the maximal ideal of the smooth point given by ideal I. Then since R_s is regular, M is a regular scheme since completion does not alter the Krull and essential dimension. Since the fraction field of our maximal ideal is either $\mathbb{Q}_p$ or $\mathbb{F}_p$, our field is perfect and regular schemes over perfect fields are smooth.

Now suppose we have a smooth point in the formal completion. Then s belongs to one of the irreducible components of the two dimensional strata of the blow up with ideal I. Let F denote the field of fractions of s which is either $\mathbb{Q}_p$ or $\mathbb{F}_p$. Denote the equation of the irreducible component as f. Let R, R_s, M be as before. $\Omega_{M/F}$ be the module of Kahler differentials over M , $\Omega_{R_s/F}$ be the module of Kahler differentials over R_s. Then

$$\Omega_{M/F} =< Mdx, Mdy, Mdz > /Mdf$$

$$\Omega_{R_s/F} =< R_s dx, R_s dy, R_s dz > /R_s df$$

But $\Omega_{M/F}$ is projective by assumption so Mdf is nonzero at s. But then this means that $R_s df$ is nonzero since there is an injection from R_s to M. The kernel of the map is given by the intersection of powers of I which is 0 by Krull's intersection theorem since we have a Noetherian local ring. So this means R_s is smooth as s. □

4.2 Zeta Function and Run Time Complexity

Below is the link to the Sage code. There is a code for computing the basis on the E_2 page. This is fully automated. The user has to input the degrees of the Koszul map and function f. The second code is the image of Frobenius and reduction. I attached videos in the README file of my code on GitHub and Zenodo. The link is : `https://zenodo.org/record/5810714#.YhhcVorMJyw`

Note my Frobenius reduction code is not fully automated. A human will need to compute the differential operators given in Theorem 3.1 and polynomial multiplier L used to find the basis on higher level of the vector spaces.

Below, we give the examples of zeta function for cubic and quartics equations. The cubic examples are for prime $p = 5$, and these can be computed without the differential operator approach. For the quartic examples, consider the case when $p = 11$. For a zeta function is of degree 6, we will need point counts up to $\mathbb{F}_{11^6}$. For the point count of $\mathbb{F}_{11^6}$, since there are 4 variables, we will need to brute force point count 11^{24} points. Assuming Sage takes 10^{-3} second to input the values, this would take Sage around $1.140015356 \cdot 10^{18}$ days to run. Using the method given in this paper, all the zeta functions given below have been done within a scope of a day to 3 weeks. I used the Sage cloud computing from CoCalc using a 31GB Virtual Machine.

Function	Singularity	E2 Basis	Zeta Function
$zx^2 - zwy + x^3$	1 A1, 2 A2	wy	$\frac{1}{(1-T)(1-5T)^2(1-25T)}$
$zx^2 - zwy + w^2x - wx^2$	1 A1, 1 A3	w^2, wx	$\frac{1}{(1-T)(1-5T)^3(1-25T)}$
$zx^2 - zwy + wx^2$	1 A1, 2 A2	wx	$\frac{1}{(1-T)(1-5T)^2(1-25T)}$
$zx^2 - zwy + wx^2 - x^3$	2 A1, 1 A2	w^2, wy	$\frac{1}{(1-T)(1-5T)^3(1-25T)}$
$zwx - yw^2 - y^3 - wy^2$	2 A2	w^2, wy	$\frac{1}{(1-T)(1-5T)^2(1+5T)(1-25T)}$
$zwx - y^3$	3 A2	no basis	$\frac{1}{(1-T)(1-5T)(1-25T)}$

In all the following examples for quartics, the identification of the singularity type has been done using Sage as well. The code from Sage borrows from Singular with implementation based off of Pfister, Marais, and Boehm [14].

Example 4.1. *A more interesting hypersurface is the degree* 4 *quartic given by*

$$f = w^3x + (x + y + z)(x - y - z)(x + y + 2z)(x - 2y + z).$$

This quartic has an A_5 *singularity at* $[0 : 0 : 1 : -1]$ *and* A_2 *singularities at* $[0 : 1 : 0 : -1], [0 : 1 : 3 : -2], [0 : 5 : 1 : -3], [0 : 3 : 2 : 1],$ *and* $[0 : 1 : -1 : 0]$*. Along with evaluation at the singular points, here are the operators that annihilate the Jacobian ideal. For simplicity of notation, I will only write the differential operators. Keep in mind one has to evaluate at the corresponding singular point after applying the differential operators. For the* A_5 *singularity,*

$$\begin{aligned}
D_1 &= \frac{\partial}{\partial w} + \frac{\partial}{\partial y} - \frac{\partial}{\partial z}\\
D_2 &= \frac{\partial^2}{\partial^2 w}\\
D_3 &= \frac{\partial^3}{\partial^3 w} - \frac{\partial}{\partial x}\\
D_4 &= \frac{\partial^4}{\partial^4 w} - 4\frac{\partial}{\partial w}\frac{\partial}{\partial x} + \frac{\partial}{\partial w}\frac{\partial}{\partial y} - \frac{\partial}{\partial w}\frac{\partial}{\partial z}.
\end{aligned}$$

For $[0 : 1 : 0 : -1]$*, the operator is*

$$D_5 = \frac{\partial}{\partial w} + \frac{\partial}{\partial x} - \frac{\partial}{\partial z}.$$

For $[0 : 1 : 3 : -2]$*, the operator is*

$$D_6 = \frac{\partial}{\partial w} + \frac{\partial}{\partial x} + 3\frac{\partial}{\partial y} - 2\frac{\partial}{\partial z}.$$

For $[0:5:1:-3]$*, the operator is*

$$D_7 = \frac{\partial}{\partial w} + 5\frac{\partial}{\partial x} + \frac{\partial}{\partial y} - 3\frac{\partial}{\partial z}.$$

For $[0:3:2:1]$*. the operator is*

$$D_8 = \frac{\partial}{\partial w} + 3\frac{\partial}{\partial x} + 2\frac{\partial}{\partial y} + \frac{\partial}{\partial z}.$$

For $[0:1:-1:0]$*, the operator is*

$$D_9 = \frac{\partial}{\partial w} + \frac{\partial}{\partial x} - \frac{\partial}{\partial y}.$$

In this example, one does not need to find L *since computation of inverse Frobenius operator involves only low degree terms. We first show Theorem 4.1 holds. We do this either by showing the change of coordinates to the standard equation in which case Theorem 4.1 applies or by blowing up the equation and show the blow up is a smooth strict relative normal crossing divisor.*

We use the following property: Suppose we have

$$w^3 = L_1(x_1, x_2, x_3)L_2(x_1, x_2, x_3)L_3(x_1, x_2, x_3)L_4(x_1, x_2, x_3)$$

where L_i *are general linear forms. Then we have 6 possible* A_2 *singularities. The singularities are given by solutions to equation* $w^3 = L_i = L_j = 0$*. This is because the other* L_k *are invertible in the neighborhood of the solution. So our* L_i *and* L_j *become our new variables* u *and* v *giving equation* $w^3 = uv$ *which is an* A_2 *singularity. Since* $w^3 = L_1 = L_2 = 0$ *has no solution, we end up with 5* A_2 *singularities.*

For the A_5 *singularity, we use the same concept. The equation* $w^3x+(x+z+1)(x-z-1)(x+$

$2z+1)(x+z-2)=0$ has singularity at $w=0, x=0, z=-1$. We replace z with $z-1$ to make singularity at the origin. The equation becomes $w^3x+(x+z)(x-z)(x-2z+3)(x-z-1)=0$. The linear forms $x-2z+3$ and $x-z-1$ are invertible in the neighborhood of origin. So we make substitutions $u=x+z$, $v=x-z$ to get equation $w^3(\frac{u+v}{2})+uv=0$ after adjusting w^3 by a cube root of $(x-2z+3)(x-z-1)$. We make a change of variables and replace u with $2u$ and v with $2v$. The equation becomes $w^3u+w^3u+4uv=0$. We relabel the equation as $w^3x+w^3z+4xz=0$. We blow up at the origin.

Since we know the leading differential operator corresponds to the w variable, we focus on the interesting patch where $x=wx_1, z=wz_1$. The equation becomes $w^2(w^2x_1+w^2z_1+4x_1z_1)=0$. We notice the equation is the same as before except the power of w was reduced by 1 in the first two terms of the strict transform. We blow up the origin two more times to obtain $w^6(x_3+z_3+4x_3z_3)=0$. This is a strict relative normal crossing divisor due to the fact that the map ϕ as in Theorem 4.1 constructed by the divisors gives a 3x3 matrix of the coefficients of the linear terms with nonzero determinant.

Using the code, one can compute that the zeta function is given by

$$Z(x)=(1-11x)^4(1+11x)^2.$$

The case when $p=13$ is more interesting with complex roots. The interesting part of the zeta function, $Z(x)$, is given by

$$Z(x)=4826809x^6-171366x^5-26364x^4+1690x^3-156x^2-6x+1.$$

Example 4.2. *A similar example is a hypersurface defined by equation*

$$f=w^2y^2-xy^3+xwz^2+w^2x^2.$$

This quartic has an A_7 singularity at $[0:0:0:1]$, A_3 singularity at $[1:0:0:0]$, and E_6 singularity at $[0:1:0:0]$. The interesting part of the zeta function, $Z(x)$ is of degree 5. Along with evaluation at the singular points, here are the differential operators that annihilate the Jacobian ideal.

For the A_7 singularity, the operators are

$$\begin{aligned}
D_1 &= \frac{\partial}{\partial y}\\
D_2 &= \frac{\partial^2}{\partial^2 y}\\
D_3 &= \frac{\partial^3}{\partial^3 y} + 6\frac{\partial}{\partial w}\\
D_4 &= \frac{\partial^4}{\partial^4 y} + 24\frac{\partial}{\partial y}\frac{\partial}{\partial w}\\
D_5 &= \frac{\partial^5}{\partial^5 y} + 60\frac{\partial^2}{\partial^2 y}\frac{\partial}{\partial w} - 240\frac{\partial}{\partial x}\\
D_6 &= \frac{\partial^6}{\partial^6 y} + 120\frac{\partial^3}{\partial^3 y}\frac{\partial}{\partial w} - 1440\frac{\partial}{\partial y}\frac{\partial}{\partial x} + 360\frac{\partial}{\partial w}\frac{\partial}{\partial w}.
\end{aligned}$$

For the A_3 singularity, the operators are

$$\begin{aligned}
D_7 &= \frac{\partial}{\partial z}\\
D_8 &= \frac{\partial^2}{\partial^2 z} - \frac{\partial}{\partial x}.
\end{aligned}$$

For the E_6 singularity, the operators are

$$\begin{aligned}
D_9 &= \frac{\partial}{\partial y} \\
D_{10} &= \frac{\partial}{\partial z} \\
D_{11} &= \frac{\partial}{\partial y}\frac{\partial}{\partial z} \\
D_{12} &= \frac{\partial^2}{\partial^2 z} - \frac{\partial}{\partial w} \\
D_{13} &= \frac{\partial^2}{\partial^2 z}\frac{\partial}{\partial y} - \frac{\partial}{\partial w}\frac{\partial}{\partial y}.
\end{aligned}$$

The operator L in Theorem 3.2 is given by $w^j + x^j + z^j$. We first show de Rham cohomology is rigid cohomology. We first show Theorem 4.1 holds. We do this either by showing the change of coordinates to the standard equation in which case Theorem 4.1 applies or by blowing up the equation and show the blow up is a smooth strict relative normal crossing divisor.

For the A_7 singularity, the equation becomes $w^2y^2 - xy^3 + xw + w^2x^2 = 0$. We blow up at the origin and focus on the interesting patch given by $x = yx_1, w = yw_1$. The equation becomes $y^2(y^2w_1^2 - y^2x_1 + x_1w_1 + y^2x_1^2w_1^2) = 0$.

The equation is still singular so we blow up at the origin again. We focus on the interesting patch where $x_1 = yx_2, w_1 = yw_2$. The equation becomes $y^4(y^2w_2^2 - yx_2 + x_2w_2 + y^4x_2^2w_2^2) = 0$.

We blow up at the origin again and focus on the patch where $x_2 = yx_3, w_2 = yw_3$. The equation is given by $y^6(y^2w_3^2 - x_3 + x_3w_3 + y^8x_3^2w_3^2) = 0$.

We blow up the w_3-axis given by the ideal (x_3, y).

On the patch where $y = x_3y_4$, the equation becomes $x_3^7y_4^6(xy_4^2w_3^2 - 1 + w_3 + x^9y_4^8w_3^2) = 0$. This is a strict relative normal crossing divisor due to the fact that the map ϕ constructed as in proof of Theorem 4.1 by the divisors gives a 3x3 matrix of the coefficients of the linear terms *with nonzero determinant.*

On the patch where $x_3 = yx_4$, *the equation becomes* $y^7(yw_3^2 - x_4 + x_4w_3 + y^9x_4^2w_3^2) = 0$. *We blow up the* w-*axis again.*

The patch $y = x_4y_5$ *gives a normal crossing divisor as argued before. On the patch* $x_4 = yx_5$, *the equation becomes* $y^8(w_3^2 - x_5 + x_5w_3 + y^{10}x_5^2w_3^2) = 0$. *This is a strict relative normal crossing divisor due to the fact that the map* ϕ *as in proof of Theorem 4.1 constructed by the divisors gives a 3x3 matrix of the coefficients of the linear terms with nonzero determinant.*

For the E_6 *singularity, the equation is* $w^2y^2 - y^3 + wz^2 + w^2 = 0$. *We blow up at the origin.*

On the patch where $y = wy_1, z = wz_1$, *the equation becomes* $w^2(w^2y_1^2 - wy_1^3 + wz_1^2 + 1) = 0$. *There is no intersection and the surface is smooth.*

On the patch where $z = yz_1, w = yw_1$, *the equation becomes* $y^2(y^2w_1^2 - y + yz_1^2w_1 - w_1^2) = 0$. *We blow up the* z_1-*axis given by the ideal* (y, w_1). *On the patch* $w_1 = yw_2$, *the equation becomes* $y^3(y^3w_2^2 - 1 - yz_1^2w_2 - yw_2^2) = 0$. *There is no intersection. On the patch* $y = w_1y_2$, *we have* $y_2^2w_1^3(y_2^2w_1^3 - y_2 - y_2w_1z_1^2 - w_1)$. *We see on this patch, the* w_1 *term as the end reduced in power. We blow this up again at the* z_1-*axis to turn the* $-w_1$ *to a* -1. *So the surfaces in question are given by* $y_2 = 0$, $w_1 = 0$, *and* $(y_2^2w_1^3 - y_2 - y_2w_1z_1^2 - 1) = 0$. *The interesting intersection is* $w_1 = 0$ *and the strict transform. We can map* y_2 *to* z_1 *for the map* ϕ *as in proof of Theorem 4.1 so we have a normal crossing divisor.*

We work on the third patch given by $y = zy_1, w = zw_1$. *The equation becomes* $z^2(z^2y_1^2w_1^2 - zy_1^3 + zw_1 + w_1^2) = 0$. *We blow up at the origin again and focus on the interesting patch where* $w_1 = zw_2$ *and* $y_1 = zy_2$. *Then we have* $z^4(z^4w_2^2y_2^2 - z^2y_2^3 + w_2 + w_2^2) = 0$. *We blow up at the* y_2-*axis given by ideal* (z, w_2).

On the patch $z = w_2z_3$, *the equation becomes* $w_2^5z_3^4(w_2^5z_3^4y_2^2 - w_2z_3^2y_2^3 + 1 + w_2) = 0$. *The interesting intersection is when* $z_3 = 0$ *and the strict transform. We can map* w_2 *to* y_2. *Then* *this is a strict relative normal crossing divisor due to the fact that the map ϕ as in Theorem 4.1 constructed by the divisors gives a 3x3 matrix of the coefficients of the linear terms with nonzero determinant.*

On the patch $w_2 = zw_3$, the equation becomes $z^5(z^5w_3^2y_1^2 - zy_1^3 + w_3 + zw_3^2) = 0$. We blow up at the y_1-axis again and focus on the interesting patch where $w_3 = zw_4$. Then the equation becomes $z^6(z^6w_4^2y_1^2 - y_1^3 + w_4 + z^2w_4^2) = 0$. This is a strict relative normal crossing divisor due to the fact that the map ϕ as given in Theorem 4.1 constructed by the divisors gives a 3x3 matrix of the coefficients of the linear terms with nonzero determinant.

For the A_3 singularity, the equation is $y^2 - xy^3 + xz^2 + x^2 = 0$. We blow up the equation at the origin.

On the patch where $y = xy_1, z = xz_1$, the equation becomes $x^2(y_1^2 - x^2y_1^3 + xz_1^2 + 1) = 0$. We shift a to $a + i$ so the intersection with the strict transform is at the origin. Then $(a + i)^2$ gives a linear term $2ai$ so this is a strict relative normal crossing divisor due to the fact that the map ϕ as in Theorem 4.1 constructed by the divisors gives a 3x3 matrix of its linear terms with nonzero determinant.

On the patch where $x = yx_1, z = yz_1$, the equation becomes $y^2(1-y^2x_1+yx_1z_1^2+x_1^2) = 0$. The same argument as the example in paragraph before shows we have a strict relative normal crossing divisor.

On the patch where $x = zx_1, y = zy_1$, the equation becomes $z^2(y_1^2 - z^2y_1^3x_1 + zx_1 + x_1^2) = 0$. We blow up at the origin again.

On the patch where $y = x_1y_2, z = x_1z_2$, the equation becomes $x_1^4z_2^2(y_2^2 - x_1^4z_2^2y_2^3 + z_2 + 1) = 0$. The same argument as before where we shift the intersection to the origin gives a strict relative normal crossing divisor.

On the patch where $x_1 = y_1x_2, z = y_1z_2$, the equation becomes $y_1^4z_2^2(1-y_1^4z_2^2x_2+x_2z_2+x_2^2) = 0$.

The same argument where we shift the intersection to the origin gives a strict relative normal crossing divisor.

On the patch where $x_1 = zx_2, y_1 = zy_2$, the equation becomes $z^4(y_2^2 - z^4y_2^3x_2 + x_2 + x_2^2) = 0$. This is a strict relative normal crossing divisor due to the fact that the map ϕ as in Theorem 4.1 constructed by the divisors gives a 3x3 matrix of the coefficients of the linear terms with nonzero determinant.

For $p = 7$, the zeta function is given by

$$Z(x) = -16807x^5 + 2401x^4 + 686x^3 - 98x^2 - 7x + 1.$$

For $p = 11$, the zeta function is given by

$$Z(x) = -161051x^5 + 14641x^4 + 1694x^3 - 154x^2 - 11x + 1.$$

The $p = 11$ case is more interesting as there are 4 complex eigenvalues and 1 real eigenvalue.

Example 4.3. *In this example, we give a degree 9 zeta function, $Z(x)$. Consider the hypersurface defined by equation*

$$f = -xy^3 + w^2x^2 + x^2z^2 + w^2z^2.$$

This hypersurface has 2 A_5 singularities at $[1 : 0 : 0 : 0]$ and $[0 : 0 : 0 : 1]$ and an A_2 singularity at $[0 : 1 : 0 : 0]$. Along with evaluation at the singular points, here are the differential operators that annihilate the Jacobian ideal. Again, as a reminder, for simplicity of notation, I will only provide the differential operators without the evaluation symbol. Keep in mind one has to evaluate at the singular points after applying the differential operators.

By symmetry, aside from evaluating at the singular points after, the operators for both A_5 singularities are the same. the operators are

$$D_1 = \frac{\partial}{\partial y}$$
$$D_2 = \frac{\partial^2}{\partial^2 y}$$
$$D_3 = \frac{\partial^3}{\partial^3 y} + 3\frac{\partial}{\partial x}$$
$$D_4 = \frac{\partial^4}{\partial^4 y} + 12\frac{\partial}{\partial x}\frac{\partial}{\partial y}.$$

For the A_2 singularity, the operator is given by

$$D_5 = \frac{\partial}{\partial y}.$$

The operator L in Theorem 3.2 is given by $w^j + x^j + z^j$. We first show de Rham cohomology is rigid cohomology, i.e. Theorem 4.1 holds. We do this either by showing the change of coordinates to the standard equation in which case Theorem 4.1 applies or by blowing up the equation and show the blow up is a smooth strict relative normal crossing divisor.

For the A_5 singularity, by symmetry, it suffices to show the result holds for one of the A_5 singularities. We make use of the fact that the leading term is y. We first do the change of coordinates $z = \frac{u}{\sqrt{x^2+1}}$ to get equation $-xy^3 + x^2 + u^2 = 0$. We relabel the equation as $-xy^3 + x^2 + z^2 = 0$. So we blow up at the origin. On the interesting patch where $x = yx_1, z = yz_1$, the equation becomes $y^2(-y^2x_1 + x_1^2 + z_1^2) = 0$. We now blow up at the origin again to obtain on the patch $x_1 = yx_2, z_1 = yz_2$ the equation $y^4(-yx_2 + x_2^2 + z_2^2) = 0$. We blow up at the origin one more time to get equation $y^6(-x_3 + x_3^2 + z_3^2) = 0$. This is a strict relative normal crossing divisor due to the fact that the map ϕ as in Theorem 4.1 constructed by the divisors gives a 3x3 matrix of the coefficients of the linear terms with *nonzero determinant.*

For the A_2 *singularity, the equation is* $-y^3+w^2+z^2+w^2z^2$. *We again do the same change of coordinates for the* A_5 *case to obtain the equation* $-y^3+w^2+z^2=0$. *We blow up at the origin again to get on the interesting patch the equation* $y^2(-y+w_1^2+z_1^2)$. *We blow up at the origin again to obtain* $y^3(-1+yw_2^2+yz_2^2)=0$. *There is no intersection so we have normal crossings.*

For $p=7$, *the zeta function is given by*

$$Z(x) = -40353607x^9 - 7411887x^8 + 1411788x^7 + 336140x^6 + 14406x^5 - 2058x^4 \\ - 980x^3 - 84x^2 + 9x + 1.$$

For $p=11$, *the zeta function is given by*

$$Z(x) = -2357947691x^9 + 214358881x^8 + 77948684x^7 - 7086244x^6 - 966306x^5 \\ + 87846x^4 + 5324x^3 - 484x^2 - 11x + 1.$$

For $p=13$, *the zeta function is given by*

$$Z(x) = -10604499373x^9 + 4329647673x^8 - 637138788x^7 + 31188612x^6 + 1199562x^5 \\ - 92274x^4 - 14196x^3 + 1716x^2 - 69x + 1.$$

For $p=17$, *the zeta function is given by*

$$Z(x) = -118587876497x^9 + 6975757441x^8 + 1641354692x^7 - 96550276x^6 \\ - 8519142x^5 + 501126x^4 + 19652x^3 - 1156x^2 - 17x + 1.$$

The examples for $p = 7$ and $p = 13$ has 2 complex roots and 7 real roots, and examples for $p = 11$ and $p = 17$ has all real roots.

Example 4.4. *In this example, we give a degree 10 zeta function, $Z(x)$, over 2 different primes. Consider the hypersurface defined by equation*

$$f = y^4 + x^2yw + w^2z^2 + yxz^2.$$

This hypersurface has an A_5 singularity at $[0:1:0:0]$, a D_5 singularity at $[1:0:0:0]$, and A_1 singularity at $[0:0:0:1]$. Here are the operators that annihilate the Jacobian ideal aside from evaluation at the singular points. Since the operator for the A_1 singularity is simply evaluation, I just need to provide the operators for the A_5 and D_5 singularity.

For the A_5 singularity, the operators are

$$\begin{aligned}
D_1 &= \frac{\partial}{\partial z} \\
D_2 &= \frac{\partial^2}{\partial^2 z} - 2\frac{\partial}{\partial w} \\
D_3 &= \frac{\partial^3}{\partial^3 z} - 6\frac{\partial}{\partial w}\frac{\partial}{\partial z} \\
D_4 &= \frac{\partial^4}{\partial^4 z} - 12\frac{\partial^2}{\partial^2 z}\frac{\partial}{\partial w} + 12\frac{\partial^2}{\partial^2 w} + 48\frac{\partial}{\partial y}.
\end{aligned}$$

For the D_5 singularity, the operators are

$$\begin{aligned}
D_5 &= \frac{\partial}{\partial y} \\
D_6 &= \frac{\partial}{\partial x} \\
D_7 &= \frac{\partial^2}{\partial^2 y} \\
D_8 &= \frac{\partial^3}{\partial^3 y} - 12\frac{\partial^2}{\partial^2 x}.
\end{aligned}$$

The operator L in Theorem 3.2 is given by $w^j + x^j + z^j$. We first show de Rham cohomology is rigid cohomology. We first show Theorem 4.1 holds. We do this either by showing the change of coordinates to the standard equation in which case Theorem 4.1 applies or by blowing up the equation and show the blow up is a smooth strict relative normal crossing divisor.

For the A_5 singularity at $x = 1$, the equation is given by $y^4 + yw + w^2z^2 + yz^2 = 0$. We show directly instead of change of coordinate this holds by doing the sequence of blow ups. We make use of the fact that we know the leading term is z from the operators given above.

We focus on the patch that require multiple blow ups. On the interesting patch where $w = zw_1, y = zy_1$, our equation becomes $z^2(z^2y_1^4 + w_1y_1 + z^2w_1^2 + zy_1) = 0$. We now blow up at the w_1-axis given by ideal (y_1, z). On the patch where $y_1 = zy_2$, the equation becomes $z^3(z^5y_2^4 + w_1y_2 + zy_2^2 + zy_2) = 0$.

We blow up at the origin and focus on the affine patch where $w_1 = zw_3, y_2 = zy_3$. The equation becomes $z^5(z^7y_3^4 + w_3y_3 + zw_3^2 + y_3) = 0$. We now blow up at the w_3-axis given by ideal (z, y_3). On the patch where $y_3 = zy_4$, the equation becomes $z^6(z^{10}y_4^4 + w_3y_4 + w_3^2 + y_4) = 0$. This is a normal crossings divisor due to the fact that the map ϕ as in Theorem 4.1 constructed by the divisors gives a 3x3 matrix of the coefficient of the linear terms with nonzero determinant.

For the D_5 singularity at $w = 1$, the equation is given by $y^4 + x^2y + z^2 + yxz^2 = 0$. We start with equation given by $u^2 + v^2t + t^4 = 0$.

We do the change of coordinates given by $v = y, t = z$, and $u = z\sqrt{1 + yx}$ to obtain the equation we want.

For the A_1 singularity at $z = 1$, the equation is given by $y^4 + x^2yw + w^2 + yx = 0$. We start off with $t^2 - uv = 0$.

We let $u = -u_0$ to get equation $t^2 + u_0v = 0$.

Then, we let $u_0 = u_1 + v^3$ *to get* $t^2 + u_1 v + v^4 = 0$.

Next, we let $u_1 = u_2^2 t + u_2$ *to get* $t^2 + u_2^2 tv + u_2 v + v^4 = 0$.

Finally, we let $u_2 = x, v = y$, *and* $t = w$ *to get the equation we had.*

For $p = 7$, *the zeta function is given by*

$$Z(x) = -282475249x^{10} + 5764801x^8 + 235298x^6 - 4802x^4 - 49x^2 + 1.$$

This zeta function is interesting as there are no odd powers. Furthermore, all roots are of the form $a + bi$ *where* $a = 0$ *or* $b = 0$.

For $p = 11$, *the zeta function is given by*

$$\begin{aligned} Z(x) = &-25937424601x^{10} + 12861532860x^9 - 2825639795x^8 + 354312200x^7 \\ &- 24157650x^6 + 199650x^4 - 24200x^3 + 1595x^2 - 60x + 1. \end{aligned}$$

This zeta function is also interesting as there is no x^5 *term. There are 6 real roots and 4 complex roots.*

Chapter 5

Conclusion and Future Research

5.1 Conclusion

To conclude, aside from the brute force point counting approach,from Remke [15], even Lauder's deformation method with Picard Fuchs equation may not apply in the singular case. Along with building on the algorithm from Stetson and Baranovsky [19], I have identified the Jacobian ideal as the zero set of differential operators as in Theorem 3.1. Furthermore, I have shown that given an equisingular lift, the theorem from Baldassarri and Chiarellottto [2]. In addition, I have shown for hypersurfaces with ADE singularities in $\mathbb{P}^3$, the subdiagonal on the E_2 page vanishes.

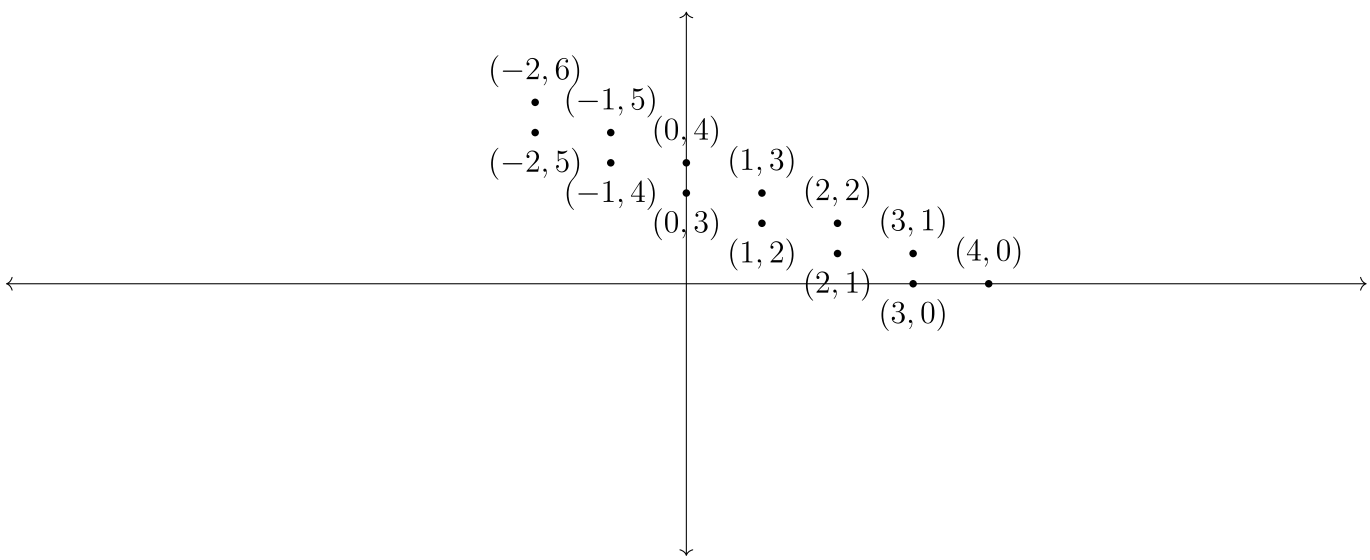
5.1.1 Future Work

While determining whether a polynomial in the stable range is in the Jacobian ideal does not require the use of a Gröbner basis, undoing the Koszul differential does. Therefore, the only remaining issue is the lifting of the Koszul differential using the Gröbner basis which takes up most of the run time for the code. From Theorem 3.1, one hypothesis is that instead of using a Gröbner basis, a lift may be linked to these differential operators in the ADE case. To clarify, given a polynomial h in the stable range that belongs to the Jacobian ideal, the lifts q_1, q_2, q_3, q_4 such that

$$h = q_1 f_w + q_2 f_x + q_3 f_y + q_4 f_z$$

can be found by applying some differential operators similar to Theorem 3.1.

Aside from run time improvements for the code, another option is to consider higher dimensions such as $\mathbb{P}^4$. The subdiagonal need not vanish now. To see this, part of the E_1 page is now given by the diagram below.



To remind the reader the meaning of the coordinates, in Definition 1.3, the double complex is given by $B^{s,t} = \Omega^{s+t}_{tN}$. The t coordinate gives the power of f while $s+t+1$ gives the type of differential form. For example, the coordinate $(2,1)$ represents $s=2$ and $t=1$. This denotes the that the power of f is 1 and the form on the numerator is a $2+1+1=4$ form. Hence, given a form $\frac{h\Omega}{f}$, where $\Omega = dx_0 \wedge dx_1 \wedge dx_2 \wedge dx_3$, by homogeneity, if $deg(f) = N$, then $deg(h) = N-4$. Hence, one important thing to note is that in $\mathbb{P}^3$, when $t=2$ which means the degree on denominator is of degree $2N$, the term on the subdiagonal lies on the coordinate axis which Dimca proves is 0. In $\mathbb{P}^4$, when $t = 2$, the point $(1, 2)$ lies in the first quadrant. This is the only place where the subdiagonal is nonzero. Hence, when we reprove the possible dimensions of the subdiagonal, we need to keep in mind that degree $2N$ is a possibility.

Saito [18] shows that for hypersurfaces with weighted homogeneous singularities, the spectral sequence degenerates on the E_2 page. While the space of all such hypersurfaces is too large, we can restrict to a subset. Aside from ADE singularities, a second path is to extend to the other singularities in Arnold's list given in Hikami [11]. In Arnold's classification of hypersurface singularities, along with ADE singularities, there are unimodal singularities. As the unimodal singularities still have normal forms, the theory of operators still holds. However, one needs to study the blow up of unimodal singularities and see if the isomorphism between de Rham cohomology and rigid cohomology still hold. Similar to the $\mathbb{P}^4$ case, the subdiagonal need not vanish. A harder path would be to consider singularities not in Arnold's list. The definition of a Milnor number is still well-defined in that case, but since there is no normal form to relate to, one has to consider a different approach as the theory of operators is no longer relevant.

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