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Localization-type Results for Singular Random Schrödinger Operators

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by

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DEDICATION

To my family.

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ABSTRACT OF THE DISSERTATION

Localization-type Results for Singular Random Schrödinger Operators

By

Nishant Rangamani

Doctor of Philosophy in Mathematics

University of California, Irvine, 2021

Professor Svetlana Jitomirskaya, Chair

In this thesis we will prove various types of localization for some classes of one-dimensional random Schrödinger operators. The central theme for all models considered will be singularity. Here, we use the term *singularity* mainly to refer to the possible lack of continuity in the probability distribution governing the randomness of the potential terms; although, we also deal with the other notion of singularity: that of Jacobi matrices and its counterpart, the unboundedness of the potential.

In particular, we will prove spectral localization for unbounded one dimensional random Jacobi operators. Such operators are obtained by incorporating independent and identically distributed randomness into the off-diagonal terms of the standard Anderson model. The operators exhibit spectral localization if almost surely the spectrum is pure point and all of the eigenfunctions decay exponentially.

We also consider so-called random word models. These generalizations of random Schrödinger operators have potential terms given by (row) vectors of bounded but random length which permits consideration of local correlations within the potential. These operators sometimes have a finite set of critical energies where the rate of localization tends to zero as one approaches these energies. Nevertheless, we will prove that these operators exhibit exponential dynamical localization in expectation on compact sets not containing any critical energies.

Chapter 1

Introduction

In the following two sections we define the models that will be considered in this work.

1.1 Random Jacobi Operators

We first define a family of random Jacobi operators on $\ell^2(\mathbb{Z})$ by:

$$H_\omega \psi(n) = t_\omega(n-1)\psi(n-1) + t_\omega(n)\psi(n+1) + V_\omega(n)\psi(n), \tag{1.1}$$

where $\{V_\omega(n)\}_{n=-\infty}^{\infty}$ and $\{t_\omega(n)\}_{n=-\infty}^{\infty}$ are two i.i.d processes, independent of each other on some probability space Ω.

More specifically, let Ω_0 be $\mathbb{R}^+ \times \mathbb{R}$ with probability measure μ_1 on $\mathbb{R}^+$, μ_2 on $\mathbb{R}$ and $\mu := \mu_1 \times \mu_2$ on Ω_0. Then, with $\Omega = \Omega_0^{\mathbb{Z}}$, $\mathbb{P} = \mu^{\mathbb{Z}}$, and $\omega(n) = (\omega_1(n), \omega_2(n))$, we let $t_\omega(n) = \omega_1(n)$ and $V_\omega(n) = \omega_2(n)$. Additionally, we have the associated shift operator on Ω given by $T(\omega(n)) = \omega(n-1)$, which is ergodic. Ergodicity provides the foundation for the study of such random models.

We shall further suppose that $V_\omega(0)$ or $t_\omega(0)$ is almost surely (a.s.) non-constant. The only other conditions will be given by finiteness of certain moments of these processes. In particular, we require $\mathbb{E}[|V_\omega(0)|^\alpha] < \infty$, $\mathbb{E}[(1/t_\omega(0))^\alpha] < \infty$, and $\mathbb{E}[(t_\omega(0))^\alpha] < \infty$, for some $\alpha > 0$. Our central result for this model states that for almost every (a.e.) $\omega \in \Omega$, H_ω has pure point spectrum and all of its eigenfunctions decay exponentially. This phenomenon is known as spectral localization.

We now describe some of the historical localization results for the standard Anderson model in one dimension (when $t_\omega(n) = 1$ for all n and all ω). Firstly, it is worth pointing out that many proofs exist when the probability measure governing the $V_\omega(n)$'s is regular. Here, *regularity* refers to the existence of $f \in L^1(\mathbb{R})$ such that $\mu_2(E) = \int_E f(x)dx$ for all Lebesgue measurable subsets E of $\mathbb{R}$. The first proof under these conditions (in addition to the assumption that $f \in L^\infty$ with compact support) was obtained by Kunz and Souillard in 1980 [49]. In 1987, Carmona, Klein, and Martinelli [10] resolved the spectral localization problem for the one-dimensional Anderson model with singular potentials using a technique known as multi-scale analysis (MSA). The MSA is a versatile multidimensional technique that we will elaborate on below. Another technique developed by Simon and Wolff, which can only be applied in the one dimensional setting when the underlying distribution has a non-trivial absolutely continuous component, involves spectral averaging [57]. Finally, while the fractional moment method originally developed by Aizenman and Molchanov in [3] (and extended in [4] to cover the case of Hölder continuous probability distribution) does not require one-dimensionality, it does not solve the singular distribution issue in one dimension.

The multi-scale analysis presented in [10] is a robust method based on the work by Frölich and Spencer in [26]. As mentioned above, the techniques in [26] are multi-dimensional and in [10], these techniques are applied in the presence of high disorder in dimension greater than one. The argument in [26] proves almost sure expoential decay of the Green's function (in terms of the distance between two points in $\mathbb{Z}$) for a single energy. The results in [26] were significantly improved in [25], with [25] providing a proof of localization in the multidimensional high disorder regime, but absolute continuity of the single-site distribution remained a requirement. The MSA is based on Wegner's lemma [60] which requires regularity and the authors in [10] were able to overcome this requirement in one dimension by obtaining Hölder regularity of the integrated density of states based on LePage's results on large deviations for random matrix products [50]. The arguments from [26, 25] were substantially simplified by von Dreifus and Klein in [48] while still allowing for singular potentials as in [10] and these simplifications played an important role in the subsequent extensions discussed below. Indeed, Klein and various collaborators have further developed the MSA over several papers to not only cover a wide variety of models, but also to provide an axiomatic framework for deriving stronger dynamical statements [33, 34, 31, 32]. In fact, the MSA has also been applied in conjunction with a unique continuation principle for harmonic functions on $\mathbb{Z}^2$ to obtain localization at the bottom of the spectrum in dimensions two and three [18, 51] and in [62] the results from [18, 51] are taken as a starting hypothesis to obtain dynamical results following the MSA framework from [34]. Finally, we mention that an alternative method to obtain Hölder regularity of the integrated density of states has been explored from a harmonic analysis perspective in [54].

The MSA-based proofs did not take full advantage of one-dimensionality and the existence of well-defined dynamical quantities, such as the Lyaunov exponent. Recently, in 2017, three new proofs of spectral localization for the one-dimensional Anderson model requiring boundedness (but no regularity of the potential) were found. All three works rely on the uniform positivity of the Lyapunov exponent but employ different approaches to leverage said quantity. In particular, the approach in [9] adapts rather sophisticated techniques developed in [8] for the deterministic case and is similar in length and complexity to an MSA-based proof. In [37], Gorodetski and Kleptsyn employ a purely geometric approach based on the action of transfer matrices on the boundary of the unit circle in $\mathbb{R}^2$. Finally, the techniques in [44] are adapted from the so-called non-perturbative approach developed in [45] to prove spectral localization for the quasi-periodic Almost-Mathieu operator.

The proof presented in [44] provides the most direct route to spectral localization for singular potentials beginning with uniform positivity of the Lyapunov exponent. The techniques developed therein lend credence to the idea that whenever appropriate analogs of quasi-periodic techniques can be developed for the random case, proofs in the random setting can be greatly simplified. In fact, this is accomplished by using the positivity of the Lyapunov exponent to effectively replace parts of the multi-scale analysis; yet, it requires boundedness as an important input and so do the other proofs described above.

Since the proof presented in [10] requires only a moment assumption on the potential rather than boundedness, it is natural to question whether the argument in [44] extends under these assumptions. In this thesis, we prove that said arguments can in fact be extended to cover the following cases:

i) $V_\omega(0)$ is unbounded and $t_\omega(0) = 1$ a.s.,

ii) $V_\omega(0)$ is bounded (but not a.s. constant), while $t_\omega(0)$ is unbounded and/or singular i.e. $t_\omega(0) \in (0, \infty)$ a.s. (as opposed to $t_\omega(0) \in [M_1, M_2]$ a.s. with $0 < M_1 \leq M_2$) and, finally,

iii) $V_\omega(0)$ is unbounded, $t_\omega(0)$ is unbounded and/or singular in the same sense as case ii).

We note that case i) is simply the Anderson model with an unbounded potential, recovering the result of Carmona, Klein, and Martinelli [10]. The precise statement of the main theorem is as follows:

Theorem 1.1.1. Suppose $\mathbb{E}[|V_\omega(0)|^\alpha] < \infty$, $\mathbb{E}[(1/t_\omega(0))^\alpha] < \infty$, $\mathbb{E}[(t_\omega(0))^\alpha] < \infty$ for some $\alpha > 0$ and $V_\omega(0)$ or $t_\omega(0)$ is almost surely not constant. Then, with H_ω defined as in eq. (1.1), for almost every (a.e.) ω, the spectrum of H_ω is pure point and its eigenfunctions decay exponentially.

1.2 Random Word Models

In the fourth chapter of this work, we consider random word models on $\ell^2(\mathbb{Z})$ given by

$$H_\omega\psi(n) = \psi(n+1) + \psi(n-1) + V_\omega(n)\psi(n).$$

The potential is a family of random variables defined on a probability space Ω. To construct the potential V above, we fix an $m \in \mathbb{N}$ (a maximum word length) then consider words $\ldots, \omega_{-1}, \omega_0, \omega_1, \ldots$ which are vectors in $\mathbb{R}^n$ with $1 \leq n \leq m$, so that $V_\omega(0)$ corresponds to the kth entry in ω_0. A precise construction of the probability space Ω and the random variables $V_\omega(n)$ is carried out in Section 4.1.1 and the precise definition of H_ω is given in eq. (4.4).

These models are of particular interest not only because they cover a wide class of generalizations of the Anderson model such as the random dimer model, random polymer models, and generalized Anderson models [13], but also because they provide natural examples of the subtleties involved in the various forms of localization: spectral, dynamical, and exponential dynamical (in expectation).

As mentioned in the previous section, spectral localization occurs when almost surely, the spectrum is pure point and all of its eigenfunctions decay exponentially. Another type of localization, dynamical localization, is best understood by first discussing the solutions to the Schrödinger equation $i\partial_t\psi = H_\omega\psi$. In this case, the functional calculus implies that $\psi(t) = e^{-itH_\omega}\psi(0)$. Thus, the study of e^{-itH_w} is essential for understanding the time evolution of the state ψ and it is dynamical localization (defined below) which implies that the wave packet is suitably localized and there is an absence of transport in the corresponding medium. We say that H_ω exhibits dynamical localization on the interval I if for a.e. ω, and any $\psi \in \ell^2(\mathbb{Z})$ which decays exponentially,

$$\sup_t \langle P_I(H_\omega)e^{-iH_\omega t}\psi, |X|^q P_I(H_\omega)e^{-iH_\omega t}\psi\rangle < \infty, \tag{1.2}$$

where $P_I(H_\omega)$ is the spectral projection of H_ω onto the set I and $q > 0$.

While it was well known that dynamical localization implies spectral localization, that dynamical localization was a strictly stronger notion was not understood until the authors in [15] constructed an artificial model which was spectrally but not dynamically localized. Indeed, the example in [15] showed that pure point spectrum with exponentially decaying eigenfunctions (spectral localization) could coexist with $\limsup_{t\to\infty} \frac{||xe^{-itH}\delta_0||^2}{t^\alpha} = \infty$ for all $\alpha < 2$. In summary, spectral localization is not a sufficient condition to ensure an absence of transport, while the stronger notion of dynamical localization does imply this absence.

A well-known physically relevant model which sheds more light on these phenomena is the random dimer model. This model was first introduced in [19] and in the random word context introduced above, ω_i takes values (λ, λ) or $(-\lambda, -\lambda)$ with Bernoulli probability. It is known that the spectrum of the operator H_ω is almost surely pure point with exponentially decaying eigenfunctions. On the other hand, when $0 < \lambda \leq 1$ (with $\lambda \neq \frac{1}{\sqrt{2}}$), there are critical energies at $E = \pm\lambda$ where the Lyapunov exponent vanishes [14]. These so-called critical energies are precisely what prevent the absence of transport and lead to localization-delocalization phenomena.

In particular, the vanishing Lyapunov exponent at these energies can be exploited to prove lower bounds on quantum transport resulting in almost sure overdiffusive behavior [43]. The authors in [43] show that for almost every ω and for every $\alpha > 0$ there is a positive constant

C_α such that

$$\frac{1}{T}\int_0^T \langle \delta_0, e^{iH_\omega t}|X|^q e^{-iH_\omega t}\delta_0\rangle dt \geq C_\alpha T^{q-\frac{1}{2}-\alpha}. \tag{1.3}$$

This was later extended to a sharp estimate in [42].

In light of these remarks, the over-diffusive behavior above contrasts with the fact that not only does the random dimer model display spectral localization, but also dynamical localization on any compact set I not containing the critical energies $\pm\lambda$ [14].

We strengthen this last result by showing that there is exponential dynamical localization in expectation (EDL) on any compact set I with $\pm\lambda \notin I$. We say the family of operators H_ω display EDL on the interval I if there are $C, \alpha > 0$ such that for any $p, q \in \mathbb{Z}$,

$$\mathbb{E}\left[\sup_{t\in\mathbb{R}} |\langle \delta_p, P_I(H_\omega)e^{-itH_\omega}\delta_q\rangle|\right] \leq Ce^{-\alpha|p-q|}. \tag{1.4}$$

EDL has several interesting physical consequences including exponential decay of the two point function in the ground state [2] and thus it is important to prove such results in physically relevant contexts such as the dimer and random polymer cases. Our results, however, when taken in conjunction with the over-diffusive behavior above illustrate that the strength of localization does not necessarily impact transport when the localization regime excludes critical energies.

One of the central challenges in dealing with random word models is the lack of regularity of the single-site distribution. The absence of regularity is exactly what allows random word models to encompass singular Anderson models, random dimer models, and more generally, random polymer models. The issues presented by singularity were previously overcome using multi-scale analysis in various stages; first, in the Anderson setting [10], then in the dimer case [14, 30], and finally for random word models themselves in [13]. The multi-scale approach leads to weaker dynamical localization results than those where sufficient regularity of the single-site distribution allows one to instead appeal to the fractional moment method (e.g. [21], [4]). In particular, EDL always follows in the framework of the fractional moment method and the one dimensional techniques in [49, 16], but of course regularity is required.

Loosely speaking, the multi-scale analysis shows that the complement of the event where one has exponential decay of the Green's function has small probability. One of the consequences of this method is that while this event does have small probability, it can only be made sub-exponentially small.

A recent new proof of spectral and dynamical localization for the one-dimensional Anderson model for arbitrary single-site distributions [44] uses positivity and large deviations of the Lyapunov exponent to replace parts of the multi-scale analysis. The major improvement in this regard (aside from a shortening of the length and complexity of localization proofs in one-dimension) is that the complement of the event where the Green's function decays exponentially can be shown to have exponentially (rather than sub-exponentially) small probability. These estimates were implicit in the proofs of spectral and dynamical localization given in [44] and were made explicit in [29]. The authors in [29] then used these estimates to prove EDL for the Anderson model and we extend those techniques to the random word case.

There are, however, several issues one encounters when adapting the techniques developed for the Anderson model in [29, 44] to the random word case. Firstly, in the Anderson setting, a uniform large deviation estimate is immediately available using a theorem in [59]. Since random word models exhibit local correlations, there are additional steps that need to be taken in order to obtain suitable analogs of large deviation estimates used in [29, 44]. Secondly, random word models may have a finite set of energies where the Lyapunov exponent vanishes and this phenomena demands care in obtaining estimates on the Green's functions analogous to those in [29, 44]. Dealing with these issues does however, produce an unexpected benefit. Since we must consider Green's functions for non-symmetric intervals, we are able to obtain exponential decay of the Green's function centered around even and odd points simultaneously, while the arguments in [29, 44] require separate considerations.

Theorem 1.2.1. With H_ω defined in eq. (4.4) (and satisfying eq. (4.1)), for a.e. ω, the spectrum of H_ω is pure point and all of its eigenfunctions decay exponentially.

Our main result is:

Theorem 1.2.2. For H_ω defined in eq. (4.4) (and satisfying eq. (4.1)), there is a finite $D \subset \mathbb{R}$ such that if I is a compact set and $D \cap I = \emptyset$, then H_ω exhibits exponential dynamical localization in expectation in the interval I.

Chapter 2

Preliminaries

Before we can prove our desired results, we must first establish some preliminaries. We divide this task into two sections: deterministic preliminaries and random preliminaries. In the first section we describe the various notations and conventions one needs to understand the eigenvectors of our operator. We also provide the details on relationships between these objects which are not found in the literature for the off-diagonal case. As these relationships hold for the deterministic analog of our random operators (eq. (1.1)), the results will be stated in this context. As suggested by its name, the results in the random section rely heavily on randomness. Additionally, the results contained therein are of an advanced nature as they are proven in various well-known papers within the field.

2.1 Deterministic Preliminaries

We begin with the deterministic analog of our random Jacobi operators:

$$H\psi(n) = t(n-1)\psi(n-1) + t(n)\psi(n+1) + V(n)\psi(n), \tag{2.1}$$

where $\{V(n)\}_{n=-\infty}^{\infty}$ and $\{t(n)\}_{n=-\infty}^{\infty}$ are real-valued and non-negative sequences on $\mathbb{Z}$.

Remark 1. The operator H above acts on $\ell^2(\mathbb{Z})$.

For this operator on $\ell^2(\mathbb{Z})$, we will study its eigenvectors and define the objects needed to prove localization in the random setting. In particular, we will eventually apply the results below to H_ω where $\omega \in \Omega$ is fixed. We note that we include the proofs of several elementary results in this section largely because they do not appear in the literature for the off-diagonal case.

Before proceeding with the task described above, we describe the setting under which H is self-adjoint.

2.1.1 Self-Adjointness of H

Based on the Weyl m-function theory, Lemma 2.16 in [58] gives a simple criterion for self-adjointness of H. Namely, H is self-adjoint if $\sum_{n=-\infty}^{\infty} 1/t(n) = \infty$. We will impose this condition on $\{t(n)\}_{n=-\infty}^{\infty}$ for the remainder of this section.

Remark 2. We will later observe in Section 2.2.2 that the conditions provided in the random case imply almost sure self-adjointness.

2.1.2 Generalized Eigenfunctions

Definition 2.1.1. We call $\psi : \mathbb{Z} \to \mathbb{R}$ a generalized eigenfunction of H with generalized eigenvalue E if $H\psi = E\psi$ and $|\psi(n)| \leq (1 + |n|)$ for all $n \in \mathbb{Z}$.

Generalized eigenfunctions are best understood via the transfer matrices defined below. That is for $k \in \mathbb{Z}$ and $E \in \mathbb{R}$, we define a one-step transfer matrix:

Definition 2.1.2.

$$T_{k,E} := \begin{bmatrix} \frac{E-V(k)}{t(k)} & \frac{-1}{t(k)} \\ t(k) & 0 \end{bmatrix}. \tag{2.2}$$

The connection between transfer matrices and generalized eigenfunctions follows from the fact that:

$$T_{k,E} \begin{pmatrix} \psi(k) \\ \psi(k-1)t(k-1) \end{pmatrix} = \begin{pmatrix} \psi(k+1) \\ \psi(k)t(k) \end{pmatrix}. \tag{2.3}$$

By setting $S_{[a,b],E} = \prod_{k=b}^{a} T_{k,E}$ where $a \leq b$, $a, b \in \mathbb{Z}$, we can understanding the evolution of ψ from a to b since:

$$S_{[a,b],E} \begin{pmatrix} \psi(a) \\ \psi(a-1)t(a-1) \end{pmatrix} = \begin{pmatrix} \psi(b+1) \\ \psi(b)t(b) \end{pmatrix}. \tag{2.4}$$

Finally, the concepts above can be connected by considering the restriction of H to intervals $[a, b] \cap \mathbb{Z}$ where $a, b \in \mathbb{Z}$.

We let $\{\delta_k\}$ denote the standard basis on $\ell^2(\mathbb{Z})$ (i.e. $\delta_k(n) = 1$ if $k = n$ and 0 otherwise) and let $I_{[a,b]}$ be the operator defined by $I_{[a,b]}(\delta_k) = \delta_k$ if $k \in [a, b] \cap \mathbb{Z}$ and 0 otherwise. Now we define:

$$H_{[a,b]} := I_{[a,b]} H I_{[a,b]}.$$

so that $H_{[a,b]}$ is the restriction of H to $[a,b] \cap \mathbb{Z}$ with Dirichlet (i.e. zero) boundary condition.

For $E \notin \sigma(H_{[a,b]})$ (the spectrum of $H_{[a,b]}$), we let $G_{[a,b],E} = (H_{[a,b]} - E)^{-1}$.

Finally, we let:

$$P_{[a,b],E} = \det(H_{[a,b]} - E) \tag{2.5}$$

and

$$\tilde{P}_{[a,b],E} = \det(E - H_{[a,b]}). \tag{2.6}$$

Note that $\tilde{P}_{[a,b],E} = (-1)^{b-a+1} P_{[a,b],E}$.

2.1.3 Transfer Matrices - Determinants

Below, we express the products of transfer matrices along the interval $[a,b] \cap \mathbb{Z}$ through determinants of $H_{[a,b]} - E$.

We prove the following lemma by induction.

Lemma 2.1.1. If $a, b \in \mathbb{Z}$ with $a+1 \le b$ and $c = t(a)t(a+1)\cdots t(b)$, then

$$S_{[a,b],E} = \begin{bmatrix} \frac{\tilde{P}_{[a,b],E}}{c} & \frac{-\tilde{P}_{[a+1,b],E}}{c} \\ \frac{\tilde{P}_{[a,b-1]}(t(b))^2}{c} & \frac{-\tilde{P}_{[a+1,b-1]}(t(b))^2}{c} \end{bmatrix}. \tag{2.7}$$

Proof. For the base case, $(b = a+1)$, we have:

$$\begin{aligned} S_{[a,b],E} = S_{[a,a+1],E} &= T_{a+1,E}T_{a,E} \\ &= \begin{bmatrix} \frac{E-V(a+1)}{t(a+1)} & \frac{-1}{t(a+1)} \\ t(a+1) & 0 \end{bmatrix} \begin{bmatrix} \frac{E-V(a)}{t(a)} & \frac{-1}{t(a)} \\ t(a) & 0 \end{bmatrix} \\ &= \begin{bmatrix} \frac{(E-V(a+1))(E-V(a))}{t(a+1)t(a)} - \frac{t(a)}{t(a+1)} & -\frac{E-V(a+1)}{t(a+1)t(a)} \\ \frac{t(a+1)}{t(a)}(E-V(a)) & -\frac{t(a+1)}{t(a)} \end{bmatrix}. \end{aligned}$$

Now,

$$E - H_{[a,a+1]} = \begin{bmatrix} E - V(a) & -t(a) \\ -t(a) & E - V(a+1) \end{bmatrix}.$$

Thus,

$$\begin{aligned}
\tilde{P}_{[a,a+1],E} &= (E - V(a))(E - V(a+1)) + (t(a))^2 \\
-\tilde{P}_{[a+1,a+1],E} &= -(E - V(a)) \\
-\tilde{P}_{[a,a],E}(t(a+1))^2 &= (E - V(a))(t(a+1))^2 \\
-\tilde{P}_{[a+1,a],E}(t(a+1))^2 &= -(t(a+1))^2.
\end{aligned}$$

Dividing the quantities above by $t(a)t(a+1)$ shows that the base case holds.

Now suppose the result holds for $[a, b]$. We wish to show it holds for $[a, b+1]$. By the inductive hypothesis, we have:

$$\begin{aligned}
S_{[a,b+1]} &= \begin{bmatrix} \frac{E-V(b+1)}{t(b+1)} & -\frac{1}{t(b+1)} \\ t(b+1) & 0 \end{bmatrix} \cdot S_{[a,b]} \\
&= \begin{bmatrix} \frac{E-V(b+1)}{t(b+1)} & -\frac{1}{t(b+1)} \\ t(b+1) & 0 \end{bmatrix} \begin{bmatrix} \frac{\tilde{P}_{[a,b],E}}{c} & -\frac{\tilde{P}_{[a+1,b],E}}{c} \\ \frac{\tilde{P}_{[a,b-1],E}(t(b))^2}{c} & -\frac{\tilde{P}_{[a+1,b-1],E}(t(b))2}{c} \end{bmatrix} \\
&= \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix},
\end{aligned}$$

where

$$
\begin{aligned}
a_{11} &= \frac{E - V(b+1)}{t(b+1)} \frac{\tilde{P}_{[a,b],E}}{c} - \frac{\tilde{P}_{[a,b-1],E}(t(b))^2}{t(b+1)c}, \\
a_{12} &= \left(\frac{E - V(b+1)}{t(b+1)}\right)\left(\frac{-\tilde{P}_{[a+1,b],E}}{c}\right) + \frac{\tilde{P}_{[a+1,b-1],E}(t(b))^2}{t(b+1)c}, \\
a_{21} &= \frac{\tilde{P}_{[a,b],E}t(b+1)}{c}, \text{ and} \\
a_{22} &= \frac{-\tilde{P}_{[a+1,b],E}t(b+1)}{c}.
\end{aligned}
$$

Note that

$$
E - H_{[a,b+1]} = \begin{bmatrix}
E - V(a) & -t(a) & 0 & 0 & \cdots & 0 \\
-t(a) & E - V(a+1) & -t(a+1) & 0 & \cdots & 0 \\
0 & -t(a+1) & E - V(a+2) & -t(a+2) & \cdots & 0 \\
0 & 0 & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \vdots & -t(b-1) & E - V(b) & -t(b) \\
0 & 0 & \cdots & 0 & -t(b) & E - V(b)
\end{bmatrix}.
$$

We compute the determinant of the above matrix via the cofactor expansion along the last row to obtain:

$$
\tilde{P}_{[a,b+1],E} = (E - V(b+1)) \cdot \tilde{P}_{[a,b]} - (-t(b))^2 \cdot \tilde{P}_{[a,b-1],E}.
$$

Similarly,

$$\tilde{P}_{[a+1,b+1],E} = (E - V(b+1))\tilde{P}_{[a+1,b],E} - (-t(b))^2 \cdot \tilde{P}_{[a+1,b-1],E}.$$

Dividing the determinants above by $t(a) \cdots t(b)t(b+1)$ gives our desired result in the first row. Multiplying the expressions for a_{21} and a_{22} by $t(b+1)/t(b+1)$ gives the desired result in the second row and completes the proof.

□

2.1.4 Green's Function - Transfer Matrix

Since $G_{[a,b],E}$ is a linear operator on a finite-dimensional vector space (Span($\{\delta_a, ..., \delta_b\}$)), we can express $G_{[a,b],E}$ as a matrix with respect to the above basis. For $a \le x \le y \le b$ ($x, y \in \mathbb{Z}$), we let $G_{[a,b],E}(x,y)$ denote the corresponding matrix element of said representation. We express these matrix elements via determinants of $H_{[a,b]} - E$ through the following lemma.

Lemma 2.1.2. If $a, b, x, y \in \mathbb{Z}$ with $a \le x \le y \le b$, then

$$|G_{[a,b],E}(x,y)| = \frac{|P_{[a,x-1],E}t(x)\cdots t(y-1)P_{[y+1,b],E}|}{|P_{[a,b],E}|}. \tag{2.8}$$

where

$P_{[a,b],E} := 1$, if $b < a$,

$t(a) \cdots t(b) := 1$, if $b < a$, and

$t(a) \cdots t(b) := t(a)$, if $a = b$.

Proof. $G_{[a,b],E}(x,y)$ is given by the xth coordinate of $(G_{[a,b],E})\delta_y$. By Cramer's rule,

$$G_{[a,b],E}(x,y) = \frac{\det(H^{(x,y)}_{[a,b]} - E)}{\det(H_{[a,b]} - E)}.$$

Here, $H^{(x,y)}_{[a,b]} - E$ represents the matrix obtained by replacing the xth column of $H_{[a,b]} - E$ with δ_y.

We have:

$$H^{(x,y)}_{[a,b]} - E = \begin{bmatrix} \begin{bmatrix} & & \\ & H_{[a,x-1]} - E & \\ & & \end{bmatrix} & & \\ t(x-1) & \begin{bmatrix} t(x) & & \\ V(x+1) - E & \ddots & \\ & & t(y-1) \end{bmatrix} & \\ & t(y) & \begin{bmatrix} & & \\ & H_{[y+1,b]} - E & \\ & & \end{bmatrix} \end{bmatrix}.$$

We again employ the cofactor expansion along the row containing $t(x-1)$ and $t(x)$ to compute the determinant of the above matrix. We show that the minor corresponding to the $t(x-1)$ entry does not contribute to the determinant. Successive cofactor expansions using $t(x-1)$, $t(x-2)$, and so on produces a matrix with first row consisting of all zeros after $(x-a)$-many steps. We can deal with the minor corresponding to $t(x)$ similarly by expanding along the row containing $t(x)$, $t(x+1), \ldots$, and $t(y-1)$. Thus, the absolute value of the determinant in question is (up to a sign) equal to the product of $t(x)\cdots t(y-1)$ and the determinant of the block matrix:

$$\begin{bmatrix} \left[H_{[a,x-1]} - E\right] & \\ & \left[H_{[y+1,b]} - E\right] \end{bmatrix}. \tag{2.9}$$

The result now follows. □

2.1.5 Green's Function and Generalized Eigenvectors

Finally, we establish the relationship between the inverse of $H_{[a,b]} - E$ and generalized eigenfunctions (with generalized eigenvalue E).

Lemma 2.1.3. For a generalized eigenfunction ψ of H with generalized eigenvalue E, and $x \in [a, b]$,

$$\psi(x) = -G_{[a,b],E}(x, a)\psi(a-1)t(a-1) - G_{[a,b],E}(x, b)\psi(b+1)t(b).$$

Proof. We "decouple" the operator H by defining

$$\tilde{H}\delta_k(m) = \begin{cases} t(b) & \text{if } k = b \text{ and } m = b+1, \\ t(b) & \text{if } k = b+1 \text{ and } m = b, \\ t(a-1) & \text{if } k = a \text{ and } m = a-1, \\ t(a-1) & \text{if } k = a-1 \text{ and } m = a, \\ 0 & \text{otherwise.} \end{cases}$$

$$H_1\delta_k(m) = \begin{cases} (H\delta_k)(m) & \text{if } k, m \in [a,b], \\ 0 & \text{otherwise.} \end{cases}$$

$$H_2\delta_k(m) = \begin{cases} (H\delta_k)(m) & \text{if } k, m \notin [a,b], \\ 0 & \text{otherwise.} \end{cases}$$

We then have $H = H_1 + H_2 + \tilde{H}$. If $H\psi = E\psi$ with $x \in [a,b]$, then $(H_1 - E)\psi(x) = -\tilde{H}\psi(x)$ since H_2 vanishes on $[a,b]$.

Applying $G_{[a,b],E}$ on both sides of the above equality yields:

$$\begin{aligned} \psi(x) &= -G_{[a,b],E}\tilde{H}\psi(x) \\ &= -G_{[a,b],E}(x,a)\psi(x)t(a-1) - G_{[a,b],E}(x,b)\psi(x)t(b). \end{aligned}$$

□

2.2 Random Preliminaries

2.2.1 Notations and Conventions

We let $H_{\omega,[a,b]}$ denote the operator H_ω restricted to the interval $[a,b]$ with zero boundary condition, $\sigma(H_{\omega,[a,b]})$ denote its spectrum, and for $j \in [1, b-a+1] \cap \mathbb{N}$, $E_{j,[a,b],\omega}$ be the jth eigenvalue of $H_{\omega,[a,b]}$.

For a discrete Jacobi operator H_ω, we denote the *Green's function* on the interval $[a,b]$ with energy $E \notin \sigma(H_{\omega,[a,b]})$ and zero boundary condition as

$$G_{[a,b],E,\omega} = (H_{\omega,[a,b]} - E)^{-1}. \tag{2.10}$$

$G_{[a,b],E,\omega}$ can be viewed as a $(b-a+1)\times(b-a+1)$ matrix and we denote the x,y entry of this matrix as $G_{[a,b],E,\omega}(x,y)$.

We also let $P_{[a,b],E,\omega} = \det(H_{\omega,[a,b]} - E)$ and $\mathbb{P}_{[a,b]}$ be $\mu^{[a,b]\cap\mathbb{Z}}$ on $\Omega_0^{[a,b]\cap\mathbb{Z}}$.

Definition 2.2.1. For $c > 0$ and $n \in \mathbb{N}$, we say $x \in \mathbb{Z}$ is (c,n,E,ω)-*regular* if

$$|G_{[x-n,x+n],E,\omega}(x, x-n)| \leq \frac{1}{t_\omega(x)} e^{-cn}$$

and

$$|G_{[x-n,x+n],E,\omega}(x, x+n)| \leq \frac{1}{t_\omega(x+n)} e^{-cn}.$$

Definition 2.2.2. We say $x \in \mathbb{Z}$ is (c,n,E,ω)-*singular* if it is not (c,n,E,ω)-*regular*.

We first discuss some consequences of ergodicity, then set some conventions and list some additional objects and formulas below.

We let

$$T_{k,E,\omega} := \begin{pmatrix} \frac{E-V_\omega(k)}{t_\omega(k)} & \frac{-1}{t_\omega(k)} \\ t_\omega(k) & 0 \end{pmatrix},$$

so that for a generalized eigenfunction ψ_ω of H_ω, we have:

$$T_{k,E,\omega,} \begin{pmatrix} \psi_\omega(k) \\ \psi_\omega(k-1)t_\omega(k-1) \end{pmatrix} = \begin{pmatrix} \psi_\omega(k+1) \\ \psi_\omega(k)t_\omega(k) \end{pmatrix}.$$

Moreover, for any interval $[a,b]$, we set $S_{[a,b],E,\omega} = \prod_{k=b}^{a} T_{k,E,\omega}$,

so that

$$S_{[a,b],E,\omega} \begin{pmatrix} \psi_\omega(a) \\ \psi_\omega(a)t_\omega(a-1) \end{pmatrix} = \begin{pmatrix} \psi_\omega(b+1) \\ \psi_\omega(b)t_\omega(b) \end{pmatrix}. \tag{2.11}$$

Since the shift operator T on Ω is ergodic and $\mathbb{E}[\ln^+ ||T_{k,E,\omega}||] < \infty$, we can apply Kingman's subadditive ergodic theorem [46] to obtain the Lyapunov exponent:

$$\gamma(E) \overset{a.e.\ \omega}{:=} \lim_{n\to\infty} \frac{\ln ||S_{[1,n],E,\omega}||}{n}.$$

Remark 3. Note that the above limit exists a.e. ω for a fixed energy E.

We now apply the formulas from the previous section to H_ω.

For a generalized eigenfunction ψ of H_ω, and $x \in [a,b]$, by Lemma 2.1.3 we have

$$\psi(x) = -G_{[a,b],E,\omega}(x,a)\psi(a-1)t_\omega(a-1) - G_{[a,b],E,\omega}(x,b)\psi(b+1)t_\omega(b). \tag{2.12}$$

For $x \le y$, by Lemma 2.1.2 we have:

$$|G_{[a,b],E,\omega}(x,y)| = \frac{|P_{[a,x-1],E,\omega}|t_\omega(x)\cdots t_\omega(y-1)|P_{[y+1,b],E,\omega}|}{|P_{[a,b],E,\omega}|},$$

where

$P_{[a,b],E,\omega} := 1$, if $b < a$,

$t_\omega(a)\cdots t_\omega(b) := 1$, if $b < a$, and

$t_\omega(a)\cdots t_\omega(b) := t_\omega(a)$, if $a = b$.

By Lemma 2.1.1,

$$S_{[a,b],E,\omega} = \begin{pmatrix} \frac{P_{[a,b],E,\omega}}{t_\omega(a)\cdots t_\omega(b)} & \frac{-P_{[a+1,b],E,\omega}}{t_\omega(a)\cdots t_\omega(b)} \\ \frac{P_{[a,b-1]}(t_\omega(b))^2}{t_\omega(a)\cdots t_\omega(b)} & \frac{-P_{[a+1,b-1]}(t_\omega(b))^2}{t_\omega(a)\cdots t_\omega(b)} \end{pmatrix}. \tag{2.13}$$

2.2.2 Unboundedness

There are a variety of results that the argument given in [44] relies on; however, the most important results pertain to the positivity of the Lyapunov exponent, an estimate by Craig-Simon [11] that provides uniform bounds on transfer matrices, and certain large deviation estimates [59]. That these theorems can be applied to bounded random Schrödinger operators is well known and described here. In the following section, we describe their extension to the singular-unbounded Jacobi case.

We begin with self-adjointess of the operators H_ω, almost sure constancy of the spectrum, and Schnol's theorem.

1. As noted in Section 2.1.1, a sufficient condition for almost sure self-adjointness of the

H_ω's is $\sum_{n=-\infty}^{\infty} \frac{1}{t_\omega(n)} = \infty$ for a.e. ω. This condition holds by our assumptions on the $t_\omega(n)$'s. In particular, since the $t_\omega(n)$'s are positive i.i.d. random variables, the law of large numbers ensures that H_ω is self-adjoint for a.e. ω.

2. A classical result known as Schnol's theorem states that the spectral measures of the standard Anderson model are supported by the set of generalized eigenvalues. These results were extended by Han in [39] to long range operators with off-diagonal decay. In particular, by [39], we can analyze the generalized eigenfunctions en route to localization as in the bounded case (i.e. for a.e. ω, the spectral measures of H_ω are supported by the set of generalized eigenvalues).

3. With regards to the almost sure constancy of the spectrum, Kirsch and Martinelli [47] extended the result of Pastur [53] to cover unbounded ergodic potentials. Thus, by [47], we have a non-random set Σ such that $\sigma(H_\omega) = \Sigma$ for a.e. ω.

We now continue our discussion with the extension of various estimates on the Lyapunov exponent from the bounded to unbounded case.

1. (Unbounded) Craig-Simon Estimates.

   We begin by describing the exact conditions under which Craig and Simon [11] prove their upper-bounds on products of transfer matrices.

   Suppose $M_\omega = \Delta + V_\omega$ is a one-dimensional discrete random Schrödinger operator where the potential V is a bounded ergodic process, with

$$A_{k,E,\omega} := \begin{pmatrix} E - V_\omega(k) & -1 \\ 1 & 0 \end{pmatrix},$$

and $R_{[a,b],E,\omega} := \prod_{k=b}^{a} A_{k,E,\omega}$. Additionally, let

$$\overline{\gamma}^{+}(\omega, E) = \limsup_{n\to\infty} \frac{\ln ||R_{[1,n],E,\omega}||}{|n|},$$

$$\overline{\gamma}^{-}(\omega, E) = \limsup_{n\to\infty} \frac{\ln ||R^{-1}_{[-n,-1],E,\omega}||}{|n|},$$

and $\gamma(E)$ denote the associated Lyapunov exponent.

Theorem 2.2.1 (Craig-Simon [11])**.** In the above setting, for a.e. ω and all E, $\overline{\gamma}^{\pm}(\omega, E) \leq \gamma(E)$.

The proof given in [11] only requires that the random process through which the operator is defined is ergodic, in addition to the finiteness of $\mathbb{E}[\ln^{+} ||A_{k,E,\omega}||]$. Both these requirements hold for H_ω, given that the diagonal and off-diagonal elements are i.i.d. and our assumptions on the expectations of $t_\omega(0), 1/t_\omega(0)$,and $V_\omega(0)$. Specifically, the finiteness of the above quantity is used to ensure the application of Kingman's subadditive ergodic theorem which not only results in the almost sure existence of the Lyapunov exponent for each energy E, but also that $\overline{\gamma}^{\pm}(\omega, E)$ is submean and $\gamma(E)$ is subharmonic. It is this last fact that is proved in the Craig-Simon paper and is then used to give a proof of the theorem as stated.

With $T_{k,E,\omega}$ and $S_{[a,b],E,\omega}$ defined as in Section 3, $\overline{\gamma}^{+}(\omega, E) = \limsup_{n\to\infty} \frac{\ln ||S_{[1,n],E,\omega}||}{|n|}$, $\overline{\gamma}^{-}(\omega, E) = \limsup_{n\to\infty} \frac{\ln ||S^{-1}_{[-n,1],E,\omega}||}{|n|}$, and $\gamma(E)$ the Lyapunov exponent, we have:

Theorem 2.2.2 (Unbounded Craig-Simon [11])**.** For a.e. ω and all E, $\overline{\gamma}^{\pm}(\omega, E) \leq \gamma(E)$.

This result's primary role in our argument is through the following restatement and subsequent corollary.

Corollary 2.2.1. For a.e. ω, for all E, we have

$$\max\left\{\limsup_{n\to\infty}\frac{\ln||S^{-1}_{[-n,-1],E,\omega}||}{n},\ \limsup_{n\to\infty}\frac{\ln||S_{[1,n],E,\omega}||}{n}\right\}\le\gamma(E) \tag{2.14}$$

and

$$\max\left\{\limsup_{n\to\infty}\frac{\ln||S_{[n+1,2n],E,\omega}||}{n},\ \limsup_{n\to\infty}\frac{\ln||S_{[2n+2,3n+1],E,\omega}||}{n}\right\}\le\gamma(E). \tag{2.15}$$

Remark 4. The first statement in Corollary 2.2.1 is an immediate consequence of Theorem 2.2.2 and the second statement can be obtained by the same proof.

Corollary 2.2.2. For a.e. ω, for every E and any $\varepsilon>0$, there is $N_2(\omega,E,\varepsilon)$ such that for every $n>N_2$ we have

$$\max\{||S^{-1}_{[-n,-1],E,\omega}||,||S_{[1,n],E,\omega}||\}<e^{(\gamma(E)+\varepsilon)(n)} \tag{2.16}$$

and

$$\max\{||S_{[n+1,2n],E,\omega}||,||S_{[2n+2,3n+1],E,\omega}||\}<e^{(\gamma(E)+\varepsilon)(n)}. \tag{2.17}$$

We now turn to the positivity of the Lyapunov exponent.

2. Uniformly Positive Lyapunov Exponent .

As above, we describe the setting of the theorem first. Suppose $\{t_\omega(n)\}$ and $\{V_\omega(n)\}$ are two i.i.d. processes independent of each other with $V_\omega(0)$ a.s. non-constant, for some $c>0$, $t_\omega(0)>c$ a.s., and $\mathbb{E}[\ln(1+t_\omega(0)+|V_\omega(0)|)]<\infty$.

Theorem 2.2.3 (Figotin-Pastur [52])**.** Suppose the H_ω's are random Jacobi operators as in eq. (1.1) defined through the processes $\{t_\omega(n)\}_{n=-\infty}^{\infty}$ and $\{V_\omega(n)\}_{n=-\infty}^{\infty}$, satisfying the above conditions, then the corresponding Lyapunov exponent $\gamma(E)$ is strictly positive for any $E \in \mathbb{R}$.

We first note that condition on the expectation in the above theorem is required to ensure that $\mathbb{E}[\ln^+ ||T_{k,E,\omega}||] < \infty$. While they suppose $t_\omega(0)$ is bounded from below, this is only needed to ensure $\mathbb{E}[\ln^+ ||T_{k,E,\omega}||] < \infty$. As such, their result applies in our setting since we have a condition on the expectation of $1/t_\omega(0)$ which implies the finiteness of $\mathbb{E}[\ln^+ ||T_{k,E,\omega}||])$ as well.

Moreover, we remark that the argument of [52] proceeds by showing that the conditions from a theorem by Fürstenberg [27], which guarantees positivity of the Lyapunov exponent, hold. In particular, they show that the transfer matrices $(T_{k,E,\omega})$ with common distribution in $SL(2,\mathbb{R})$ denoted by ρ and the smallest closed subgroup containing the support of ρ denoted by G_ρ satisfy

ii) G_ρ is not compact.

iii) There is no non-trivial G_ρ invariant probability measure on $\mathbb{RP}^1$.

These two facts will become relevant in showing not only that $\nu > 0$ (see Theorem 2.2.4 and Theorem 2.2.5 below), but also that the result on large deviations of matrix elements [59] applies in our setting.

Theorem 2.2.4. $\gamma(E)$ is continuous on $\mathbb{R}$.

Proof. Fix $E \in \mathbb{R}$ and let E_k be a sequence in $\mathbb{R}$ such that $E_k \to E$ as $k \to \infty$. Let μ_E denote the probability measure on $SL(2,\mathbb{R})$ obtained through the transfer matrices $T_{n,E,\omega}$. That is, let $\mathcal{B}$ denote the Borel σ-algebra on $SL(2,\mathbb{R})$ and for any $O \in \mathcal{B}$, $\mu_E(O) = \mathbb{P}[\{\omega : T_{1,E,\omega} \in O\}]$. Moreover, let G_{μ_E} denote the smallest closed subgroup containing the support of μ_E. Note that by iii) above, there is no non-trivial subspace $W \subset \mathbb{R}^2$ such that W is G_{μ_E}-invariant. Thus, the hypothesis of Theorem B in [28] holds (i.e. that there can be at most one such W).

Now let $X_k : \Omega \to SL(2,\mathbb{R})$ be defined by $X_k(\omega) = T_{0,E_k,\omega}$ and $X : \Omega \to SL(2,\mathbb{R})$ be defined by $X(\omega) = T_{0,E,\omega}$. By Theorem B in [28], to prove $\gamma(E_k) \to \gamma(E)$ as $k \to \infty$, it suffices to show:

(1) For any $h : SL(2,\mathbb{R}) \to \mathbb{C}$ with h continuous and of compact support, $\mathbb{E}[h(X_k)] \to \mathbb{E}[h(X)]$ as $k \to \infty$,

(2) $\mathbb{E}[\log^+(||X_k||\chi_{\{||X_k|| \geq n\}})] \to 0$ as $n \to \infty$ uniformly in k,

(3) $\mathbb{E}[\log^+(||X^{-1}||\chi_{\{||X^{-1}|| \geq n\}})] \to 0$ as $n \to \infty$.

Remark 5. We note that μ_E is well defined since measurability of V, t, and $\frac{1}{t}$ implies measurability of the maps $\omega \to T_{n,E,\omega}$. Additionally, μ_E is independent of n since V, t, and $\frac{1}{t}$ are each i.i.d. processes.

Remark 6. Condition (1) above is known as *weak convergence* and conditions (2) and (3) together are known as *bounded convergence*.

Returning to the proof, (1) follows by dominated convergence since $X_k \to X$ for a.e. ω and h is continuous and of compact support. Now choose $M \in \mathbb{R}$ so that $|E_k| \leq M$ for all k. Note we have $||X_k(\omega)|| \leq Y(\omega) = \sqrt{2}\max\{(M + |V_\omega(0)|)/t_\omega(0), 1/t_\omega(0), t_\omega(0)\}$. Thus, $\alpha \log^+ ||X_k|| \leq Y^\alpha$ and we have $\mathbb{E}[Y^\alpha] < \infty$ by our hypotheses. It follows that $\{\log^+ ||X_k||\}$ is a uniformly integrable family, so (2) holds. Finally, (3) follows since $\mathbb{E}[\log^+ ||X^{-1}||] < \infty$. This completes the proof.

□

Theorem 2.2.5. If $\nu = \inf\{\gamma(E) : E \in I\})$, then $\nu > 0$.

Proof. By Theorem 2.2.3 (in particular, the discussion following the theorem), we have $\gamma(E) > 0$ for all $E \in \mathbb{R}$. The result now follows by Theorem 2.2.4 and compactness of I.

□

We now deal with the aforementioned large deviation result.

3. Large Deviation Theorems for the Lyapunov Exponent.

Again, we begin with the setting of the theorem.

Suppose $\{Y_k\}$ are i.i.d. 2×2 matrices with common distribution μ, where μ is a probability measure on $SL(2, \mathbb{R})$. Let $l(M) = \max\{\ln^+ ||M||, \ln^+ ||M^{-1}||\}$ and suppose for some $\tau > 0, \int \exp(\tau l(M))\ d(\mu(M)) < \infty$. Moreover, suppose G_μ, the smallest subgroup containing the support of μ, is both strongly irreducible and contracting. Finally, let γ denote the Lyapunov exponent.

Theorem 2.2.6 (Tsay [59])**.** In the above setting, there is $L > 0$ such that for each $\varepsilon > 0$, there is an $a > 0$ so that for all unit vectors $u, v \in \mathbb{R}^2$,

$$\mathbb{P}\left\{|\frac{1}{n}\log|\langle\prod_{k=1}^{n} Y_k u, v\rangle| - \gamma| \geq \varepsilon\right\} \leq e^{-an} \tag{2.18}$$

for sufficiently large n.

Tsay goes on to extend this result when the matrices (and distributions) depend on a real parameter E in a fixed compact set F. That is, suppose $Y_{k,E}$ are i.i.d. 2×2 matrices in $SL(2, \mathbb{R})$ with respective probability measures μ_E and Lyapunov exponent $\gamma(E)$. If there is $C > 0$ and $\tau > 0$ such that $\int \exp(\tau l(M))\ d(\mu_E(M)) < C$ for all $E \in F$ (in addition to $Y_{k,E}$ satisfying the conditions in Theorem 2.2.6), then Theorem 2.2.6 holds uniformly in E.

Theorem 2.2.7 (Tsay [59])**.** In the above setting, for each $\varepsilon > 0$, there is an $a > 0$ so that for all unit vectors $u, v \in \mathbb{R}^2$, $\mathbb{P}\{|\frac{1}{n}\log|\langle\prod_{k=1}^{n} Y_{k,E} u, v\rangle| - \gamma(E)| \geq \varepsilon\} \leq e^{-an}$ for sufficiently large n uniformly in E.

We call the a in the above theorem the 'large deviation parameter' associated with ε.

We now show that the conditions of Theorem 2.2.6 and Theorem 2.2.7 hold in our setting. We begin by explaining the terminology used in the statement. The term *strongly irreducible* means that there is no finite union of proper subspaces $W \subset \mathbb{R}^2$ such that $M(W) = W$ for all $M \in G_\mu$. The term *contracting* means that there is a sequence in G_μ say $\{M_n\}$ such that $M_n/||M_n||$ converges to a rank one matrix. Firstly, by taking M_n in G_μ with $||M_n|| \to \infty$ and considering a convergent subsequence of $M_n/||M_n||$, it follows that condition ii) (non-compactness of G_μ) implies contracting. Next, if strong irreducibility does not hold, this implies the existence of a non-empty, finite $L \subset \mathbb{RP}^1$ such that $M(L) = L$ for all $M \in G_\mu$. Indeed, taking the sum of point masses with weight $1/|L|$ at each of the points of L gives a non-trivial G_μ invariant probability measure on $\mathbb{RP}^1$ and we conclude iii) implies strong irreducibility. Finally, the required moment condition is easily seen to be satisfied given our assumptions on the various moments of $V_\omega(0), t_\omega(0)$,and $1/t_\omega(0)$ and compactness of I.

Theorem 2.2.7 finds its use in our argument through the following corollary:

Corollary 2.2.3. (Large Deviations [59]) For any $\varepsilon > 0$, there is $\eta > 0$ and $N_0 \in \mathbb{N}$ such that for $b - a > N_0$,

$$\mathbb{P}\left\{\omega : \left|\left(\frac{1}{b-a+1}\log\frac{|P_{[a,b],E,\omega}|}{t_\omega(a)\cdots t_\omega(b)}\right) - \gamma(E)\right| \geq \varepsilon\right\} \leq e^{-\eta(b-a+1)}. \tag{2.19}$$

Proof. The transfer matrices $T_{k,E,\omega}$ are certainly i.i.d. Additionally, by the above discussion, they also satisfy the irreducibility, contracting, and expectation condition. The corollary now follows by taking $u = v = (1, 0)$ and applying formula eq. (2.13).

□

Chapter 3

Spectral Localization

3.1 Preliminaries

We use this section to reestablish certain notations and conventions in addition to providing a simpler precursor to the 'good' and 'bad' sets that appear in the multi-scale analysis.

Since we aim to prove localization at energies in a finite interval $\tilde{I}$, we fix $\tilde{I} = [s, t]$, a compact interval with non-empty interior in $\mathbb{R}$ and let $I = [s-1, t+1]$. We then define the following 'large deviation' sets:

$$B^{+}{}_{[a,b],\varepsilon} = \left\{ (E, \omega) : E \in I,\ \frac{|P_{[a,b],E,\omega}|}{t_\omega(a) \cdots t_\omega(b)} \geq e^{(\gamma(E)+\varepsilon)(b-a+1)} \right\}, \tag{3.1}$$

$$B^{-}{}_{[a,b],\varepsilon} = \left\{ (E, \omega) : E \in I,\ \frac{|P_{[a,b],E,\omega}|}{t_\omega(a) \cdots t_\omega(b)} \leq e^{(\gamma(E)-\varepsilon)(b-a+1)} \right\}, \tag{3.2}$$

and denote the corresponding sections by

$$B^{\pm}{}_{[a,b],\varepsilon,\omega} = \left\{E : (E,\omega) \in B^{\pm}_{[a,b],\varepsilon}\right\} \tag{3.3}$$

and

$$B^{\pm}{}_{[a,b],\varepsilon,E} = \left\{\omega : (E,\omega) \in B^{\pm}_{[a,b],\varepsilon}\right\}. \tag{3.4}$$

Additionally, we let

$B_{[a,b],*} = B^{+}_{[a,b],*} \cup B^{-}_{[a,b],*}$.

By rescaling the operator in question, we may assume that $\mathbb{E}[(1/t_\omega(0))^\alpha] = c_1 < 1$. Furthermore, we let $\mathbb{E}[t_\omega(0)^\alpha] = c_2$ and $\mathbb{E}[|V_\omega(0))|^\alpha] = c_3$.

Lastly, we let $\nu = \inf\{\gamma(E) : E \in I\}$ and note here that $\nu > 0$. For the proof, see Theorem 2.2.5.

3.2 Lemmas

Given that we can express Green's function via ratio of determinants, a viable strategy appears to be showing that the numerator of eq. (2.8) cannot be too large and the denominator cannot be too small. The lemmas in this section formalize these heuristics and pave the road for the proof of spectral localization found in the next section.

We begin with an elementary lemma that shows singularity forces points into the 'bad' deviation sets.

Lemma 3.2.1. Let $n \geq 2$ and suppose $0 < 8\varepsilon_0 < \nu$. If x is $(\gamma(E) - 8\varepsilon_0, n, E, \omega)$-singular, then $(E, \omega) \in {B^-}_{[x-n,x+n],\varepsilon_0} \cup {B^+}_{[x-n,x-1],\varepsilon_0} \cup {B^+}_{[x+1,x+n],\varepsilon_0}$.

Proof. This follows by the definition of $(\gamma(E) - 8\varepsilon_0, n, E, \omega)$- *singularity* and the definitions given in eq. (3.1) and eq. (3.2). □

Remark 7. Lemma 3.2.2 and Lemma 3.2.3 follow [44] very closely with some minor modifications needed to deal with unbounded and/or singular $\{t_\omega(n)\}$ and unbounded $\{V_\omega(n)\}$.

Roughly speaking, the first of the two lemmas below controls the Lebesgue measure of the so-called 'bad' deviation sets and the second shows that eigenvalues of disjoint boxes with length are rarely in 'far enough' bad deviation sets.

Let m denote Lebesgue measure on $\mathbb{R}$.

Lemma 3.2.2. Suppose $0 < \varepsilon_0 < \nu$, η_0 is the corresponding large deviation parameter (from Corollary 2.2.3), and $0 < \delta_0 < \eta_0$, then for a.e. ω, there is $N_1(\omega)$ such that for $n > N_1$, $\max\left\{m(B^-_{[n+1,3n+1],\varepsilon_0,\omega}), m(B^-_{[-n,n],\varepsilon_0,\omega})\right\} \leq e^{-(\eta_0-\delta_0)(2n+1)}$.

Proof. We have

$$\begin{aligned} m \times \mathbb{P}\left(B^-_{[a,b],\varepsilon_0}\right) &= \mathbb{E}\left(m\left(B^-_{[a,b],\varepsilon_0,\omega}\right)\right) \\ &= \int_{\mathbb{R}} \mathbb{P}\left(B^-_{[a,b],\varepsilon_0,E}\right)\, dm(E) \\ &\leq m(I)e^{-\eta_0(b-a+1)}. \end{aligned}$$

The first two equalities are simply Fubini's theorem, and the inequality follows by Corollary 2.2.3.

Let

$$F_n = \left\{\omega : m\left(B^-_{[n+1,3n+1],\varepsilon_0,\omega}\right) \geq e^{-(\eta_0-\delta_0)(2n+1)}\right\},$$

and

$$G_n = \left\{\omega : m\left(B^-_{[-n,n],\varepsilon_0,\omega}\right) \geq e^{-(\eta_0-\delta_0)(2n+1)}\right\}.$$

We have, $e^{-(\eta_0-\delta_0)(2n+1)}\mathbb{P}(F_n) \leq \mathbb{E}\left(m\left(B^-_{[n+1,3n+1],\varepsilon_0,\omega}\right)\right) \leq m(I)e^{-\eta_0(2n+1)}$, with a similar estimate holding for G_n. The first inequality is Chebyshev and the second follows by the first line of the proof.

Thus,

$$\mathbb{P}(F_n \cup G_n) \leq 2m(I)e^{-\delta_0(2n+1)},$$

and the result follows by Borel-Cantelli. □

Lemma 3.2.3. Suppose $0 < \varepsilon < \nu$, $\eta_\varepsilon > 0$ is the corresponding large deviation parameter and $p > 6/\eta_\varepsilon$. For $n \in \mathbb{N}$, put

$$C_n = \{\omega : \exists y \in [-n,n], |-n-y| \geq \ln(n^p), \text{ and } E_{j,[n+1,3n+1],\omega} \in B_{[-n,y],\varepsilon,\omega} \text{ for some } 1 \leq j \leq 2n+1\}$$

and

$$D_n = \{\omega : \exists y \in [-n,n], |n-y| \geq \ln(n^p), E_{j,[n+1,3n+1],\omega} \in B_{[y,n],\varepsilon,\omega} \text{ for some} 1 \leq j \leq 2n+1\}.$$

Then $\mathbb{P}[C_n \cup D_n \text{ infinitely often }] = 0$.

Proof. Fix $n \in \mathbb{N}$, y with $|-n-y| \geq \ln(n^p)$, and $1 \leq j \leq 2n+1$, and put

$$A_{n,y,j} = \{\omega : E_{j,[n+1,3n+1],\omega} \in B_{[-n,y]}\}.$$

Since $[n+1, 3n+1] \cap [-n, n] = \emptyset$, by independence and Corollary 2.2.3 we have

$$\begin{aligned}\mathbb{P}\left(B_{[-n,y],\varepsilon,E_{j,[n+1,3n+1],\omega}}\right) &= \mathbb{P}_{[n+1,3n+1]^c}\left(B_{[-n,y],\varepsilon,E_{j,[n+1,3n+1],\omega}}\right) \\ &\leq e^{-\eta_\varepsilon|-n-y|}.\end{aligned} \tag{3.5}$$

Indeed, for each n, if $Q_n^{'} = \{y \in [-n, n] : |-n-y| \geq \ln(n^p)\}$,

$$C_n = \bigcup_{y \in Q_n^{'},\ 1 \leq j \leq 2n+1} A_{n,y,j}.$$

By the above, we have $\mathbb{P}[C_n] \leq (2n+1)^2 e^{-\eta_\varepsilon \ln(n^p)}$. Thus $\mathbb{P}[C_n$ infinitely often $] = 0$ by Borel-Cantelli. The result follows by applying the same argument to D_n. □

The remaining lemmas provide bounds on the growth of $V_\omega(n)$s and $t_\omega(n)$s.

Lemma 3.2.4. Suppose $p > 0$ and $r > 1$. Let $J_n = \{\omega : \exists k \in [-n, n], |-n-k| \leq \ln(n^p)$ or $|k-n| \leq \ln(n^p)$ and $|V_\omega(k)| \geq n^{r/\alpha}$ or $|t_\omega(k)| \geq n^{r/\alpha}\}$. Then $\mathbb{P}[J_n$ infinitely often $] = 0$.

Proof. Put $Q_n = \{k \in [-n, n] : |-n-k| \leq \ln(n^p)$ or $|n-k| \leq \ln(n^p)\}$, and $A_k = \{\omega : |V_\omega(k)| \geq n^{r/\alpha}$ or $t_\omega(k) \geq n^{r/\alpha}\}$. Then

$$J_n = \bigcup_{k \in Q_n} A_k.$$

By Chebyshev and stationarity, for any $k \in Q_n$, $\mathbb{P}[A_k] \leq (c_2 + c_3)/n^r$. Thus,
$\mathbb{P}(J_n) \leq 2(c_2 + c_3)(\ln(n^p) + 1)n^{-r}$. By Borel-Cantelli, $\mathbb{P}[J_n$ infinitely often $] = 0$. □

Corollary 3.2.1. If $p > 0$ and $r > 1$, for a.e. ω, there is $N(\omega)$ such that for $n > N$ and any $k \in [-n, n]$ st. $|-n-k| \leq \ln(n^p)$ (respectively, $|n-k| \leq \ln(n^p)$),

$$|P_{[-n,k],\omega}| \leq n^{\frac{2r}{\alpha}(\ln(n^p)+1)}$$

(respectively, $|P_{[k,n],\omega}| \leq n^{\frac{2r}{\alpha}(\ln(n^p)+1)}$).

Remark 8. Proceeding in the same manner as in the above lemma, we can obtain the following three lemmas. We prove Lemma 3.2.5, Lemma 3.2.6 and exclude the proof of Lemma 3.2.7, as its proof is identical to the proof of Lemma 3.2.6.

Lemma 3.2.5. Suppose $r > 2$. If

$$A_n = \left\{ \omega : \exists k \in [-n, n] \text{ s.t. } \frac{1}{t_\omega(k)} > n^{\frac{r}{\alpha}} \right\},$$

then $\mathbb{P}[A_n \text{ infinitely often }] = 0$.

Proof. Put $J_{k,n} = \{\omega : 1/t_\omega(k) \geq n^{r/\alpha}\}$, then by Chebyshev, $n^r \mathbb{P}[J_{k,n}] \leq \mathbb{E}[1/(t_\omega(k))^\alpha] = c_1$ and this holds for all $n \in \mathbb{N}$ and $k \in [-n, n]$ because the process $\{t_\omega(m)\}_{m=-\infty}^{\infty}$ is stationary. Since

$$A_n = \bigcup_{k \in [-n,n]} J_{k,n},$$

we obtain,

$$\mathbb{P}[A_n] \leq \frac{c_1(2n+1)}{n^r}.$$

The result now follows by Borel-Cantelli. □

Lemma 3.2.6. Suppose $r > 1$. If

$$A_n = \left\{ \omega : \exists k \in [-n, n], |-n-k| \leq \ln(n^p) \text{and } 1/(t_\omega(-n) \cdots t_\omega(k-1)) \geq n^{r/\alpha} \right\}, \tag{3.6}$$

then $\mathbb{P}[A_n \text{ infinitely often }] = 0$.

Proof. Put $Q_n' = \{k \in [-n, n] : |-n-k| \leq \ln(n^p)\}$, and

$$J_{k,n} = \left\{ \omega : \frac{1}{t_\omega(-n) \cdots t_\omega(k-1)} \geq n^{\frac{r}{\alpha}} \right\}.$$

Then $A_n = \bigcup_{k \in Q_n'} J_{k,n}$.

Hence,

$$n^r \mathbb{P}[J_{k,n}] \leq \mathbb{E}\left[1/(t_\omega(-n)\cdots t_\omega(k-1))^\alpha\right]$$

$$= (c_1)^{|-n-k+1|} \leq 1,$$

for all $k \in Q_n^{'}$. The first inequality is Chebyshev, the equality follows by stationarity together with independence, and the final inequality follows as $c_1 < 1$. Thus, $\mathbb{P}(A_n) \leq (\ln(n^p)+1)n^{-r}$ and by Borel-Cantelli, $\mathbb{P}[A_n$ infinitely often $] = 0$.

□

Lemma 3.2.7. Suppose $r > 1$. If

$$A_n = \{\omega : \exists k \in [-n, n], |k-n| \leq \ln(n^p) \text{ and } 1/(t_\omega(k+1)\cdots t_\omega(n)) \geq n^{\frac{r}{\alpha}}\}, \tag{3.7}$$

then $\mathbb{P}[A_n$ infinitely often $] = 0$.

Proof. The argument is identical to the one given for Lemma 3.2.6.

□

3.3 Proof of Theorem 1.1.1

We recall the main result (Theorem 1.1.1) and then present two reductions.

Theorem 1.1.1 For almost every (a.e.) ω, the spectrum of H_ω is pure point and its eigenfunctions decay exponentially.

Theorem 3.3.1. For a.e. ω, for every generalized eigenvalue E, the corresponding generalized eigenfunction $\psi_{\omega,E}(n)$ decays exponentially in n.

We now prove Theorem 3.3.1 implies Theorem 1.1.1.

Proof. (Theorem 3.3.1 implies Theorem 1.1.1)

By Section 2.2.2 (e.g. [39]), the spectral measures are supported by the generalized eigenvalues. Thus, Theorem 3.3.1 implies every generalized eigenfunction is in fact an $\ell^2(\mathbb{Z})$ eigenfunction which decays exponentially and the result follows. $\square$

Theorem 3.3.2. For *a.e.* ω, for every generalized eigenvalue E of H_ω, there is $C(E) > 0$ and $N(\omega, E)$ such that for $n > N$, $2n$ and $2n+1$ are $(C(E), n, E, \omega)$-regular.

We now show that Theorem 3.3.2 implies Theorem 3.3.1, but due to the presence of off-diagonal terms in the definition of regularity, we will need a preliminary lemma to control the growth of these terms.

Lemma 3.3.1. If $r > 1$ and

$$A_n = \left\{\omega : \frac{t_\omega(n)}{t_\omega(2n+1)} \geq n^{\frac{r}{\alpha}} \text{ or } \frac{t_\omega(n)}{t_\omega(2(n+1))} \geq n^{\frac{r}{\alpha}}\right\},$$

then $\mathbb{P}[A_n \text{ infinitely often }] = 0$.

Proof. Put

$$J_n = \left\{\omega : \frac{t_\omega(n)}{t_\omega(2n+1)} \geq n^{\frac{r}{\alpha}}\right\}$$

and

$$K_n = \left\{\omega : \frac{t_\omega(n)}{t_\omega(2(n+1))} \geq n^{\frac{r}{\alpha}}\right\}.$$

Then we have:

$$n^r \mathbb{P}[J_n] \leq \mathbb{E}[(t_\omega(n))^\alpha]\mathbb{E}\left[\frac{1}{(t_\omega(2n+1))^\alpha}\right] = c_1 c_2.$$

The inequality follows by Chebyshev, together with independence. Applying the same argument to K_n, we have $n^r \mathbb{P}[K_n] \leq c_1 c_2$. Since A_n is the union of K_n and J_n, $\mathbb{P}[A_n] \leq (2c_1c_2)/n^r$ and the result follows by Borel-Cantelli. $\square$

Proof. (Theorem 3.3.2 implies Theorem 3.3.1)

To show each generalized eigenfunction decays exponentially, it suffices to show exponential decay of the corresponding Green's function (e.g. regularity) by virtue of eq. (2.12) and that fact that generalized eigenfunctions are polynomially bounded.

$\square$

We are now ready to give the proof of Theorem 3.3.2. We note that it essentially goes along the lines of the proof in [44] with minor adjustments.

Proof. (Theorem 1.1.1 via Theorem 3.3.2)

We will show $2n+1$ is $(\gamma(\tilde{E}) - 8\varepsilon_0, n, \tilde{E}, \omega)$-*regular* for all sufficiently large n. The proof that $2n$ is $(\gamma(\tilde{E}) - 8\varepsilon_0, n, \tilde{E}, \omega)$-*regular* (for sufficiently large n) is similar. Fix $0 < \varepsilon_0 < \nu/8$ and obtain a corresponding $\eta_0 > 0$ through Corollary 2.2.3. Choose $0 < \delta_0 < \eta_0$, $0 < \varepsilon < \min\{(\eta_0 - \delta_0)/3, \varepsilon_0\}$, $p > 6/\eta_\varepsilon$, and $r > 2$. Using Lemma 3.3.1 and Lemmas 3.2.2 to 3.2.7 with the above $\varepsilon_0, \varepsilon, \delta_0$, and p and using Corollary 2.2.2, we obtain $\tilde{\Omega}$ with $\mathbb{P}(\tilde{\Omega}) = 1$ such that the conclusion of Corollary 2.2.2 along with the conclusions of Lemma 3.3.1 and Lemmas 3.2.2 to 3.2.7 hold for all $\omega \in \tilde{\Omega}$.

Now let $\omega \in \tilde{\Omega}$ and let $\tilde{E} \in \tilde{I}$ be a generalized eigenvalue for H_ω with generalized eigenfunction ψ. We assume without loss of generality that $\psi(0) \neq 0$. Finally, we may choose N so that for $n > N$ the conclusions of Lemmas 3.2.2 to 3.2.7 along with the conclusion of Corollary 2.2.2 (with the above ε and $\tilde{E}$) hold for this ω and 0 is $(\gamma(\tilde{E}) - 8\varepsilon_0, n, \tilde{E}, \omega)$-*singular*.

Suppose that for infinitely many n $(n > N)$, $2n+1$ is $(\gamma(\tilde{E}) - 8\varepsilon_0, n, \tilde{E}, \omega)$-*singular*. By Lemma 3.2.1 and Corollary 2.2.2, $\tilde{E} \in B^-_{[n+1,3n+1],\varepsilon_0,\omega}$. We claim that there is E_j, an eigenvalue of $H_{\omega,[n+1,3n+1]}$, so that $E_j \in I$ and $|\tilde{E} - E_j| \le e^{-(\eta_0-\delta_0)(2n+1)}$. If no eigenvalue of $H_{\omega,[n+1,3n+1]}$ is in I, then there are essentially two cases: i) some eigenvalues are on the left and some eigenvalues are on the right of I, or ii) all eigenvalues lie to the left (or to the right) of I. We provide the details in the first case as the other case can be handled similarly. Suppose there are some eigenvalues on the left and some on the right. Let E_{j_1} denote the smallest eigenvalue to the right of I and E_{j_2} the largest eigenvalue to the left of I. Note that all $2n+1$ eigenvalues of $H_{\omega,[n+1,3n+1]}$ are real and are the zeros of $P_{[n+1,3n+1],E,\omega}$, a polynomial in E of degree $2n+1$. It follows that $P_{[n+1,3n+1],E,\omega}$ is monotone on $(\tilde{E}, E_{j_1})$ or $(E_{j_2}, \tilde{E})$. As $\tilde{E} \in B^-_{[n+1,3n+1],\varepsilon_0,\omega}$ and $P_{[n+1,3n+1],E,\omega}$ vanishes at E_{j_1} and E_{j_2}, this implies

$$1 \le \min\{E_{j_1} - \tilde{E}, \tilde{E} - E_{j_2}\} \le m(B^-_{[n+1,3n+1],\varepsilon_0,\omega}) \le e^{-(\eta_0-\delta_0)(2n+1)} < 1.$$

The first inequality is from the fact that $\tilde{E} \in \tilde{I}$, $E_{j_1}, E_{j_2} \notin I$, and I is obtained by adding and subtracting 1 from the right and left points of $\tilde{I}$ (respectively). The second to last inequality is from Lemma 3.2.2. The contradiction shows we have the existence of E_j, an eigenvalue of $H_{\omega,[n+1,3n+1]}$ so that $E_j \in I$. A similar argument shows that at least one eigenvalue $E_j \in I$ satisfies $|E_j - \tilde{E}| \le e^{-(\eta_0-\delta_0)(2n+1)}$ (for otherwise we contradict the measure condition from Lemma 3.2.2).

Applying the above argument with 0 in place of $2n+1$ yields E_i, an eigenvalue of $H_{\omega,[-n,n]}$ which lies in $B^-_{[-n,n],\varepsilon_0,\omega}$ such that $|\tilde{E} - E_i| \le e^{-(\eta_0-\delta_0)(2n+1)}$. Thus,
$|E_i - E_j| \le 2e^{-(\eta_0-\delta_0)(2n+1)}$.
By the previous line and the fact that $E_j \notin B_{[-n,n],\varepsilon,\omega}$ (by Lemma 3.2.3),
$||G_{[-n,n],E_j,\omega}|| \ge \frac{1}{2}e^{(\eta_0-\delta_0)(2n+1)}$ and hence there exist $y_1, y_2 \in [-n,n]$, (WLOG $y_1 \le y_2$), so that

$$|G_{[-n,n],E_j,\omega}(y_1,y_2)| \ge \frac{1}{2\sqrt{2n+1}}e^{(\eta_0-\delta_0)(2n+1)}.$$

Again, using $E_j \notin B_{[-n,n],\varepsilon,\omega}$, we obtain:

$$\frac{|P_{[-n,n],E_j,\omega}|}{t_\omega(-n)\cdots t_\omega(n)} \geq e^{(\gamma(E_j)-\varepsilon)(2n+1)}.$$

By recalling $|G_{[-n,n],E_j,\omega}(y_1,y_2)| = \dfrac{|P_{[-n,y-1],E_j,\omega} t_\omega(y_1)\cdots t_\omega(y_2-1)P_{[y_2+1,n],E_j,\omega}|}{|P_{[-n,n],E_j,\omega}|}$,

we have:

$$\frac{|P_{[-n,y_1-1],E_j,\omega}|t_\omega(y_1)\cdots t_\omega(y_2-1)|P_{[y_2+1,n],E_j,\omega}|}{\prod_{k=-n}^{n} t_\omega(k)} \geq \frac{e^{(\eta_0-\delta_0)(2n+1)}e^{(\gamma(E_j)-\varepsilon)(2n+1)}}{2\sqrt{2n+1}}.$$

We rewrite the left hand side of our inequality as:

$$\frac{|P_{[-n,y_1-1],E_j,\omega}|}{t_\omega(-n)\cdots t_\omega(y_1-1)}\frac{1}{t_\omega(y_2)}\frac{|P_{[y_2+1,n],E_j,\omega}|}{t_\omega(y_2+1)\cdots t_\omega(n)}. \tag{3.8}$$

Recall we have $y_1 \leq y_2$, so there are effectively three cases to consider: the first is $|-n-y_1| \geq \ln(n^p)$ and $|n-y_2| \geq \ln(n^p)$, the second is $|-n-y_1| \geq \ln(n^p)$ while $|n-y_2| \leq \ln(n^p)$, and the third is $|-n-y_1| \leq \ln(n^p)$ and $|n-y_2| \leq \ln(n^p)$.

For the first case, we apply Lemma 3.2.5 to the middle term in eq. (3.8) and Lemma 3.2.3 to the remaining two terms to obtain:

$$n^{r/\alpha}e^{(\gamma(E_j)+\varepsilon)(2n+1)} \geq \frac{1}{2\sqrt{2n+1}}e^{(\eta_0-\delta_0)(2n+1)}e^{(\gamma(E_j)-\varepsilon)(2n+1)}. \tag{3.9}$$

For the second case, we again apply Lemma 3.2.5 to the middle term in eq. (3.8). We then apply Corollary 3.2.1 to the numerator of right-most term, Lemma 3.2.7 to the denominator, and Lemma 3.2.3 to the left-hand term to obtain:

$$n^{\frac{2r}{\alpha}+(\frac{2r}{\alpha}(\ln(n^p)+1))}e^{(\gamma(E_j)+\varepsilon)(2n+1)} \geq \frac{1}{2\sqrt{2n+1}}e^{(\eta_0-\delta_0)(2n+1)}e^{(\gamma(E_j)-\varepsilon)(2n+1)}. \tag{3.10}$$

And finally, for the third case, we again apply Lemma 3.2.5 to the middle term. Then, we apply Corollary 3.2.1 to the numerators of the terms on the left and the right, Lemma 3.2.6 and Lemma 3.2.7 to the denominators to obtain:

$$n^{\frac{3r}{\alpha}+\frac{4r}{\alpha}(\ln(n^p+1))} \geq \frac{1}{2\sqrt{2n+1}}e^{(\eta_0-\delta_0)(2n+1)}e^{(\gamma(E_j)-\varepsilon)(2n+1)}. \tag{3.11}$$

The first case leads to a contradiction by letting $n \to \infty$, since $(\gamma(E_j)-\varepsilon)+(\eta_0-\delta_0) > \gamma(E_j)+\varepsilon$.

For the second and third cases, the ratio of the RHS to the LHS in the above inequalities tends to ∞ as $n \to \infty$, providing the desired contradiction. We conclude that for $n > N$, $2n+1$ is $(\gamma(\tilde{E})-8\varepsilon_0, n, \tilde{E}, \omega)$-*regular*.

Finally, since the interval $\tilde{I}$ was arbitrary, the proof is complete. □

Chapter 4

Random Word Models

4.1 Preliminaries

4.1.1 Model Set-up

We begin by providing details on the construction of Ω and $V_\omega(n)$ by following [13].

Fix $m \in \mathbb{N}$ (the maximum word length) and $M > 0$. Set $\mathcal{W} = \bigcup_{j=1}^{m} \mathcal{W}_j$ where $\mathcal{W}_j = [-M, M]^j$ and ν_j are finite Borel measures on $\mathcal{W}_j$ so that $\sum_{j=1}^{m} \nu_j(\mathcal{W}_j) = 1$. Let ν denote the direct sum of the measures ν_j, a probability measure on $\mathcal{W}$.

Additionally, we assume that $(\mathcal{W}, \nu)$ has two words which do not commute. That is,

$$\begin{cases} \text{For } i = 0, 1 \text{ there exist } w_i \in \mathcal{W}_{j_i} \in supp(\nu) \text{ such that} \\ (w_0(1), w_0(2), ..., w_0(j_0), w_1(1), w_1(2), ..., w_1(j_1)) \\ \text{and } (w_1(1), w_1(2), ..., w_1(j_1), w_0(1), w_0(2), ..., w_0(j_0)) \text{ are distinct.} \end{cases} \tag{4.1}$$

Here, $supp(\nu)$ refers to the support of the measure ν.

Set $\Omega_0 = \mathcal{W}^{\mathbb{Z}}$ and $\mathbb{P}_0 = \otimes_{\mathbb{Z}} \nu$ on the σ-algebra generated by the cylinder sets in Ω_0.

The average length of a word is defined by $\langle L \rangle = \sum_{j=1}^{m} j\nu(\mathcal{W}_j)$ and if $w \in \mathcal{W} \cap \mathcal{W}_j$, we say w has length j and write $|w| = j$.

We define $\Omega = \bigcup_{j=1}^{m} \Omega_j \subset \Omega_0 \times \{1, ..., m\}$ where $\Omega_j = \{\omega \in \Omega_0 : |\omega_0| = j\} \times \{1, ..., j\}$. We define the probability measure $\mathbb{P}$ on the σ-algebra generated by the sets $A \times \{k\}$ where $A \subset \Omega_0$ such that for all $\omega \in A$, $|\omega_0| = j$ and $1 \leq k \leq j$.

For such sets we set

$$\mathbb{P}[A \times \{k\}] = \frac{\mathbb{P}_0(A)}{\langle L \rangle}. \tag{4.2}$$

Remark 9. The above construction implies that every event $A \subset \Omega_0$ gives rise to an event $\tilde{A} \subset \Omega$ with the same probability (up to multiplication by $\langle L \rangle$).

The shifts T_0 and T on Ω_0 and Ω (respectively) are given by:

$(T_0\omega)_n = \omega_{n+1}$ and

$$T(\omega, k) = \begin{cases} (\omega, k+1) & \text{if } k < |\omega_0| \\ (T_0(\omega), 1) & \text{if } k = |\omega_0|. \end{cases} \tag{4.3}$$

With this set-up, the shift T is ergodic and the potential $V_{\omega,k}$ is obtained through $..., \omega_{-1}, \omega_0, \omega_1, ...$ so that $V_{\omega,k}(0) = \omega_0(k)$.

That is,

$$(H_{(\omega,k)}u)(n) = u(n+1) + u(n-1) + V_{(\omega,k)}(n)u(n) \tag{4.4}$$

for all $u \in \ell^2(\mathbb{Z})$.

Thus, the ergodicity of the shift T implies the results from [53] can be used to show the spectrum of $H_{(\omega,k)}$ is almost surely a non-random set.

Remark 10. For notational convenience, we will often drop the k from the subscript on $H_{(\omega,k)}$.

4.1.2 Basic Definitions and Notations

Definition 4.1.1. We call $\psi_{\omega,E}$ a *generalized eigenfunction* with *generalized eigenvalue* E if $H_\omega \psi_{\omega,E} = E\psi_{\omega,E}$ and $|\psi_{\omega,E}(n)| \leq (1+|n|)$.

We denote the restriction of H_ω to the interval $[a,b]\cap\mathbb{Z}$ where $a,b\in\mathbb{Z}$ by $H_{\omega,[a,b]}$ and for $E\notin\sigma(H_{\omega,[a,b]})$ the corresponding Green's function by

$$G_{[a,b],E,\omega} = (H_{\omega,[a,b]} - E)^{-1}.$$

Additionally, we let

$$P_{[a,b],E,\omega} = \det(H_{\omega,[a,b]} - E)$$

and

$$\tilde{P}_{[a,b],E,\omega,} = \det(E - H_{\omega,[a,b]}).$$

We also let $E_{j,[a,b],\omega}$ denote the jth eigenvalue of the operator $H_{\omega,[a,b]}$ and note that there are $b-a+1$ many of them (counting multiplicity).

Definition 4.1.2. $x\in\mathbb{Z}$ is called (c,n_1,n_2,E,ω)-*regular* if there is a $c>0$ so that:

1. $|G_{[x-n_1,x+n_2],E,\omega}(x,x-n_1)| \leq e^{-cn_1}$ and
2. $|G_{[x-n_1,x+n_2],E,\omega}(x,x+n_2)| \leq e^{-cn_2}$.

By the results in Section 2.1.4 and Section 2.1.5, we have for any generalized eigenfunction $\psi_{\omega,E}$ and any $x \in [a,b]$,

$$\psi_{\omega,E}(x) = -G_{[a,b],E,\omega}(x,a)\psi_{\omega,E}(a-1) - G_{[a,b],E,\omega}(x,b)\psi_{\omega,E}(b+1), \tag{4.5}$$

and

$$|G_{[a,b],E,\omega}(x,y)| = \frac{|P_{[a,x-1],E,\omega}P_{[y+1,b],E,\omega}|}{|P_{[a,b],E,\omega}|}. \tag{4.6}$$

4.1.3 Transfer Matrices and the Lyapunov Exponent

For $w \in \mathcal{W}$, with $w = (w_1, ..., w_j)$, we define word transfer matrices by $T_{w,E} = T_{w_j,E} \cdots T_{w_1,E}$ where $T_{v,E} = \begin{pmatrix} E - v & -1 \\ 1 & 0 \end{pmatrix}$.

The transfer matrices over several words are given by

$$T_{\omega,E}(k,l) = \begin{cases} T_{\omega_k,E} \cdots T_{\omega_l,E} & \text{if } k > l, \\ \mathbb{I} & \text{if } k = l, \\ {T_{\omega,E}}^{-1}(l,k) & \text{if } k < l. \end{cases} \tag{4.7}$$

and $T_{[a,b],E,\omega}$ denotes the product of the transfer matrices so that for any generalized eigenfunction ψ with generalized eigenvalue E:

$$T_{[a,b],E,\omega} \begin{pmatrix} \psi(a) \\ \psi(a-1) \end{pmatrix} = \begin{pmatrix} \psi(b+1) \\ \psi(b) \end{pmatrix}. \tag{4.8}$$

Again by Section 2.1.3, we have:

$$T_{[a,b],E,\omega} = \begin{pmatrix} \tilde{P}_{[a,b],E,\omega} & -\tilde{P}_{[a+1,b],E,\omega} \\ \tilde{P}_{[a,b-1],E,\omega} & -\tilde{P}_{[a+1,b-1],E,\omega} \end{pmatrix}. \tag{4.9}$$

We now define two Lyapunov exponents, one via matrix products obtained from Ω_0 and the other from Ω. We will see that the two quantities are essentially the same (up to multiplication by a positive constant) and hence provide the same information. It is worth noting, however, that the former is obtained through products of i.i.d. matrices while the latter is not. In fact, we will need to utilize independence to obtain various estimates on the matrix products and it is therefore important to verify the relationship between the two Lyapunov exponents.

Since both T_0 and T are ergodic, Kingman's subadditive ergodic theorem [46] allows us to define the Lyapunov exponent. Recalling that $\langle L \rangle$ denotes the average length of a word, by the arguments in [13], we have:

in Ω_0,

$$\gamma_0(E) := \lim_{k\to\infty} \frac{1}{k} \log ||T_{\omega,E}(k,1)||, \tag{4.10}$$

and in Ω,

$$\gamma(E) := \lim_{k\to\infty} \frac{1}{k\langle L\rangle} \log \left\| \begin{pmatrix} E - V_\omega(n) & -1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} E - V_\omega(1) & -1 \\ 1 & 0 \end{pmatrix} \right\|. \tag{4.11}$$

In both cases, the limit exists for fixed E on a full measure set.

Remark 11. We note that the limit in Ω is defined via one-step transfer matrices while the limit in Ω_0 is defined via word transfer matrices.

In [13], the authors prove the relationship between the two Lyapunov exponents described in the following theorem.

Theorem 4.1.1. $\frac{\gamma_0(E)}{\langle L\rangle} = \gamma(E)$.

Let μ_E denote the smallest closed subgroup of $SL(2,\mathbb{R})$ generated by the 'word'-step transfer matrices. It is shown in [13] that μ_E is strongly irreducible and contracting for all E outside of a finite set $D \subset \mathbb{R}$ and hence, Furstenberg's theorem implies $\gamma(E) > 0$ for all such E. Since the Lyapunov exponent is defined as a product of i.i.d. matrices, γ is continuous. So, if I is a compact set such that $D \cap I = \emptyset$ and

$$\nu := \inf\{\gamma(E) : E \in I\},$$

then $\nu > 0$.

Motivated by eq. (4.9) above and large deviation theorems, we define:

$$B^+{}_{[a,b],\varepsilon} = \left\{(E,\omega) : E \in I, |P_{[a,b],E,\omega}| \geq e^{(\gamma(E)+\varepsilon)(b-a+1)}\right\}, \tag{4.12}$$

and

$$B^{-}{}_{[a,b],\varepsilon} = \left\{(E,\omega) : E \in I, |P_{[a,b],E,\omega}| \le e^{(\gamma(E)-\varepsilon)(b-a+1)}\right\}, \tag{4.13}$$

and the corresponding sections:

$$B^{\pm}{}_{[a,b],\varepsilon,\omega} = \left\{E : (E,\omega) \in B^{\pm}_{[a,b],\varepsilon}\right\}, \tag{4.14}$$

and

$$B^{\pm}{}_{[a,b],\varepsilon,E} = \left\{\omega : (E,\omega) \in B^{\pm}_{[a,b],\varepsilon}\right\}. \tag{4.15}$$

4.2 Large Deviation Theorems

The goal of this section is to obtain a uniform large deviation estimate for $P_{[a,b],E}$. In the Anderson model, a direct application of Tsay's theorem for matrix elements of products of i.i.d. matrices results in both an upper and lower bound for the above determinants. In the general random word case, there are two issues. Firstly, the one-step transfer matrices are not independent. This issue is naturally resolved by considering ω_k-step transfer matrices and treating products over each word as a single step. However, in this case, both the randomness in the length of the chain as well as products involving partial words need to be accounted for. Since matrix elements are majorized by the norm of the matrix and all matrices in question are uniformly bounded, we can obtain an upper bound identical to the one obtained in the Anderson case. Lower bounds on the matrix elements are more delicate and require the introduction of random scales. For the reader's convenience, we first recall Tsay's theorem and then give the precise statements and proofs of the results alluded to above.

As remarked above, μ_E is strongly irreducible and contracting for $E \in I$. In addition, the 'word'-step transfer matrices (defined in eq. (4.7)) are bounded, independent, and identically distributed. These conditions are sufficient to apply Tsay's theorem on large deviations of matrix elements for products of i.i.d. matrices.

Theorem 4.2.1 ([59])**.** Suppose I is a compact interval and for each $E \in I$, $Z_1^E, ..., Z_n^E, ...$ are bounded i.i.d random matrices such that the smallest closed subgroup of $SL(2,\mathbb{R})$ generated by the matrices is strongly irreducible and contracting. Then for any $\varepsilon > 0$, there is an $\eta > 0$ and an $N \in \mathbb{N}$ such that for any $E \in I$, for any unit vectors u, v and $n > N$,

$$\mathbb{P}\left[e^{(\gamma(E)-\varepsilon)n} \leq |\langle Z_n^E \dots Z_1^E u, v\rangle| \leq e^{(\gamma(E)+\varepsilon)n}\right] \geq 1 - e^{-\eta n}.$$

Lemma 4.2.1. If I is compact and $I \cap D = \emptyset$, then for any $\varepsilon > 0$ there is an $\eta > 0$ and an N such that if a,$b \in \mathbb{Z}$ such that $b - a + 1 > N$ and $E \in I$, then

$$\mathbb{P}[B^+_{[a,b],\varepsilon,E}] \leq e^{-\eta(b-a+1)}$$

.

Proof. Let $Y_i = |\omega_i|$, so Y_i is the length of the ith word and let $S_n = Y_1 + \cdots + Y_n$.

Let $u = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and let $P_{(\omega_1,\omega_n),E} = \det(H_{(\omega_1,\omega_n)} - E)$ where $H_{(\omega_1,\omega_n)}$ denotes H_ω restricted to the interval where V takes values determined by ω_1 through ω_n. By eq. (4.9) from the previous section, $|P_{(\omega_1,\omega_n),E}| = |\langle T_{\omega,E}(n,1)u, u\rangle|$.

Letting $\varepsilon > 0$ and applying Theorem 4.2.1 to the random products $T_{\omega,E}(k,1)$, we obtain an

$\eta_1 > 0$ and an N_1 such that for $n > N_1$, $\mathbb{P}_0[\{\omega \in \Omega_0 : |P_{(\omega_1,\omega_n),E}| \leq e^{(\gamma(E)+\varepsilon)n\langle L\rangle}\}] \geq 1 - e^{-\eta_1 n}$.

Now let $\varepsilon_1 > 0$ so that $\varepsilon_1 \sup\{\gamma(E) : E \in I\} < \varepsilon$. We apply large deviation estimates (e.g. [20]) to the real, bounded, i.i.d. random variables Y_i to obtain an N_2 and an $\eta_2 > 0$ such that for $n > N_2$, $\mathbb{P}_0[S_n - n\varepsilon_1 < n\langle L\rangle < S_n + n\varepsilon_1] \geq 1 - e^{-\eta_2 n}$.

Denoting the intersection of the above events by A_n, we have, on A_n,

$$\begin{aligned}
|P_{(\omega_1,\omega_n),E}| &\leq e^{(\gamma(E)+\varepsilon)n\langle L\rangle} \\
&\leq e^{(\gamma(E)+\varepsilon)(S_n+n\varepsilon_1)} \\
&= e^{(\gamma(E)+\varepsilon)S_n+\gamma(E)\varepsilon_1 n+n\varepsilon\varepsilon_1} \\
&\leq e^{(\gamma(E)+\varepsilon)S_n+\gamma(E)\varepsilon_1 S_n+S_n\varepsilon\varepsilon_1} \\
&\leq e^{(\gamma(E)+3\varepsilon)S_n}.
\end{aligned}$$

Thus, we have an event in Ω_0 where the Lyapunov behavior is a true reflection of the length of the interval and we can obtain an estimate in between two words.

That is, for any $1 \leq k \leq S_{n+1} - S_n$, let $P_{(\omega_1,\omega_n+k),E} = \det(H_{(\omega_1,\omega_n+k)} - E)$ where $H_{(\omega_1,\omega_n+k)}$ denotes H_ω restricted to the interval where V_ω takes values determined by ω_1 through the kth letter of ω_{n+1}.

Since the one-step transfer matrices are uniformly bounded, eq. (4.9) and the last inequality imply for any $1 \leq k \leq S_{n+1} - S_n$, $|P_{[1,S_n+k],E,\omega}| \leq Ce^{(\gamma(E)+3\varepsilon)S_n} \leq e^{(\gamma(E)+4\varepsilon)S_n}$ on A_n.

Let $\eta_3 = \min\{\eta_1, \eta_2\}$, and choose $0 < \eta < \frac{\eta_3}{2m}$.

Since every event in Ω_0 gives rise to an event in Ω (e.g. eq. (4.2)), we obtain an estimate where the Lyapunov behavior and the probability of the event reflect the true length of the interval, so we can apply the shift T to conclude that for any sufficiently large n,

$$\mathbb{P}[\{|P_{[1,n],E,\omega}| \leq e^{(\gamma(E)+4\varepsilon)n}\}] \geq 1 - e^{-\eta n}.$$

The result now follows for any interval $[a,b]$ (with $b-a+1$ sufficiently large) since T preserves the probability of events.

□

We finish the section with a lemma that relies crucially on independence. As above, we will work in Ω_0 and 'lift' our results to Ω.

The lemma below holds for any fixed $K > 1$ and this K will be chosen in the next section.

Lemma 4.2.2. There are real-valued random variables R_n, $R_n^{'}$, Q_n, $Q_n^{'}$, and $\tilde{Q}_n$ such that:

if I is compact with $I \cap D = \emptyset$ and $\varepsilon > 0$ with

$$F^3_{l,n,\varepsilon} = \bigcap_j \left\{\omega : E_{j,[l+Q_n+k,l+Q_n'],\omega} \notin B^-_{[l+R_n,l+R_n'],\varepsilon,\omega}\ \forall k,\ 0 \leq k \leq 2m\right\}, \tag{4.16}$$

$$F^{2+}_{l,n,\varepsilon} = \bigcap_j \left\{\omega : E_{j,[l+Q_n+k,l+Q_n'],\omega} \notin B^+_{[y,l+R_n'],\varepsilon,\omega}\ \forall y \in \left[l+R_n, l+R_n^{'} - \frac{n}{K}\right], 0 \leq k \leq 2m\right\}, \tag{4.17}$$

$$F^{2-}_{l,n,\varepsilon} = \bigcap_j \left\{ \omega : E_{j,[l+Q_n+k,l+Q'_n],\omega} \notin B^{+}_{[l+R_n,y],\varepsilon,\omega} \forall y \in \left[l + R_n + \frac{n}{K}, l + R'_n \right], 0 \leq k \leq 2m \right\}, \tag{4.18}$$

and

$$F^{2}_{l,n,\varepsilon} = F^{2+}_{l,n,\varepsilon} \cap F^{2-}_{l,n,\varepsilon}, \tag{4.19}$$

then there is N and an $\eta' > 0$ such that if $n > N$, $l \in \mathbb{Z}$, and $E \in I$:

$$\mathbb{P}[B^{-}_{[l+R_n,l+R'_n],\varepsilon,E}] \leq e^{-\eta'(2n+1)}, \tag{4.20}$$

$$\mathbb{P}[B^{-}_{[l+Q_n,l+Q'_n],\varepsilon,E}] \leq e^{-\eta'(2n+1)}, \tag{4.21}$$

$$\mathbb{P}[F^{3}_{l,n,\varepsilon}] \geq 1 - 2m^2(2n+3)^2 e^{-\eta'(2n+1)}, \tag{4.22}$$

and

$$\mathbb{P}[F^{2}_{l,n,\varepsilon}] \geq 1 - 2m^4(2n+3)^3 e^{-\eta'(\frac{n}{K})}. \tag{4.23}$$

Proof. For $\omega \in \Omega_0$, and $a, b \in \mathbb{Z}$, let $H_{(\omega_a,\omega_b)}$ denote the restriction of H to the interval where the potential is given by the words ω_a through ω_b. Take $u = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, and let $P_{(\omega_a,\omega_b),E} = \det(H_{(\omega_a,\omega_b)} - E)$ where $H_{(\omega_a,\omega_b)}$ denotes H_ω restricted to the interval in which V takes values determined by ω_a through ω_b. By eq. (4.9) from the previous section, $|P_{(\omega_{-n},\omega_n),E}| = |\langle T_{\omega,E}(n,-n)u, u\rangle|$. Finally, let $S_{(a,b)} = |\omega_a| + \cdots + |\omega_b|$. Letting $\varepsilon > 0$ and applying Theorem 4.2.1 to the random products $T_{\omega,E}(k,1)$, we obtain an $\eta_1 > 0$ and an N_1 such that for $n > N_1$ and any $E \in I$,

$$\mathbb{P}_0[\{\omega \in \Omega_0 : |P_{(\omega_{-n},\omega_n),E}| \geq e^{(\gamma(E)-\varepsilon)(2n+1)\langle L\rangle}\}] \geq 1 - e^{-\eta_1(2n+1)}. \tag{4.24}$$

and

$$\mathbb{P}_0[\{\omega \in \Omega_0 : |P_{(\omega_{n+1},\omega_{3n+1}),E}| \geq e^{(\gamma(E)-\varepsilon)(2n+1)\langle L\rangle}\}] \geq 1 - e^{-\eta_1(2n+1)}. \tag{4.25}$$

If E_j denotes an eigenvalue corresponding to $H_{(\omega_{n+1},\omega_{3n+1})}$, then by independence

$$\mathbb{P}_0[\{\omega \in \Omega_0 : |P_{(\omega_{-n},\omega_n),E_j}| \geq e^{(\gamma(E_j)+\varepsilon)(2n+1)\langle L\rangle}\}] \geq 1 - e^{-\eta_1(2n+1)}$$

whenever $n > N$. We can also apply large deviation theorems (e.g. [20]) to the (real) random products $S_{(a,b)}$ where $0 < \varepsilon_1 < \min\{1, \varepsilon, \varepsilon \sup\{\gamma(E) : E \in I\}\}$ to obtain an N and an $\eta_2 > 0$ such that whenever $b - a + 1 > N$,

$$(b-a+1)(\langle L\rangle - \varepsilon_1) \leq S_{(a,b)} \leq (b-a+1)(\langle L\rangle + \varepsilon_1) \tag{4.26}$$

with probability greater than $1 - e^{-\eta_2(b-a+1)}$.

Thus, we have an event A where:

$$
\begin{aligned}
|P_{(\omega_{-n},\omega_n),E_j}| &\geq e^{(\gamma(E)-\varepsilon)(2n+1)\langle L\rangle} \\
&\geq e^{(\gamma(E)-\varepsilon)(S_{(-n,n)}-(2n+1)\varepsilon_1)} \\
&= e^{(\gamma(E)-\varepsilon)S_{(-n,n)}-(\gamma(E)\varepsilon_1)(2n+1)+(2n+1)\varepsilon\varepsilon_1} \\
&\geq e^{(\gamma(E)-\varepsilon)S_{(-n,n)}-(\gamma(E)\varepsilon_1)S_{(-n,n)}+(2n+1)\varepsilon\varepsilon_1} \\
&\geq e^{(\gamma(E)-2\varepsilon)S_n}.
\end{aligned}
\tag{4.27}
$$

By using a similar argument to deal with the upper-bound, we have:

$|P_{(\omega_{-n},\omega_n),E_j}| \leq e^{(\gamma(E)+3\varepsilon)S_{(-n,n)}}$.

In particular, whenever $\frac{n}{K} > N$, and $y \in [-n+\frac{n}{K}, n]$, we have:

$$
|P_{E_j,(\omega_y,\omega_n)}| \leq e^{(\gamma(E)+3\varepsilon)S_{(-n+\frac{n}{K},n)}} \tag{4.28}
$$

with probability greater than $1-e^{\eta_1\frac{n}{K}}$ with a similar estimate holding whenever $y \in [-n, n-\frac{n}{K}$.

Since the single-step transfer matrices are uniformly bounded and there are at most m single-step transfer matrices in a word transfer matrix, we have a $C > 0$ such that:

$$
\mathbb{P}_0[\{\omega \in \Omega_0 : |P_{(\omega_y+k,\omega_n),E_j}| \leq Ce^{(\gamma(E_j)+\varepsilon)(n-y)}\}] \leq e^{-\eta_1(n-y-1)}, \tag{4.29}
$$

where $P_{(\omega_y+k,\omega_n)}$ denotes the determinant obtained by restricting H from the kth letter of ω_y to ω_n.

Note that the products $S_{(a,b)}$ are also well-defined on Ω and we now are able to define the random variables from the statement of the lemma.

For $(\omega, k) \in \Omega$ we put,

1. $R_n' = S_{(0,n)} - k$,
2. $R_n = -S_{(-n,-1)} - k + 1$,
3. $Q_n = R_n' + 1$,
4. $Q_n' = R_n' + S_{(n+1,3n+1)}$, and
5. $\tilde{Q}_n = R_n' + S_{(n+1,2n+1)}$.

Choosing $0 < \tilde{\eta_2} < \eta_2$ and applying eq. (4.26), we obtain an event with probability greater than $1 - e^{-\tilde{\eta_2} n}$ where we have estimates on all of the random variables defined above.

We choose $\eta' > 0$ to be smaller than η_1 and $\tilde{\eta_2}$ so that by the remark after eq. (4.2), the definitions of the random variables above, and section 4.2 and eq. (4.25), we have for any $E \in I$:

$$\mathbb{P}[B^-_{[l+R_n, l+R_n'], E}] \leq e^{-\eta'(2n+1)}$$

and

$$\mathbb{P}[B^-_{[l+Q_n, l+Q_n'], E}] \leq e^{-\eta'(2n+1)}.$$

Moreover, by the same reasoning, after taking the union over all of the possible eigenvalues E_j and the ordered pairs (y, n) (and $(-n, y)$), eq. (4.27) and eq. (4.28) provide the desired estimates on

$$F^3_{0,n,\varepsilon} = \bigcap_j \left\{ \omega : E_{j,[Q_n+k,Q'_n],\omega} \notin B^-_{[R_n,R'_n],\varepsilon,\omega} \text{ for all } k \text{ with } 0 \le k \le 2m \right\}$$

and $F^2_{0,n,\varepsilon}$.

The result now follows by applying the (measure-preserving) shift T so that the intervals are centered around l rather than 0.

□

Remark 12. Each of the lemmas above furnish a positive constant: η, $\eta^{'}$. For a fixed ε, we call the the minimum of these constants the 'large deviation parameter' associated with ε and denote it by η_ε.

4.3 Lemmas

We prove localization results on a compact interval I where $D \cap I = \emptyset$. In order to do so, we fix a larger interval $\tilde{I}$ such that I is properly contained in $\tilde{I}$ and $D \cap \tilde{I} = \emptyset$, then apply the large deviation theorems from the previous section to $\tilde{I}$.

The following lemmas involve parameters $\varepsilon_0, \varepsilon, \eta_0, \delta_0, \eta_\varepsilon$,$K$, and the intervals $I, \tilde{I}$. The lemmas hold for any values satisfying the constraints below:

1. Let $\nu = \inf\{\gamma(E) : E \in \tilde{I}\}$, take $0 < \varepsilon_0 < \nu/8$ and let η_0 denote the large deviation parameter corresponding to ε_0. Choose any $0 < \delta_0 < \eta_0$ and let $0 < \varepsilon < \min\{(\eta_0 - \delta_0)/3m, \varepsilon_0/4\}$. Choose $\tilde{M} > 0$ so that $|P_{[a,b],E,\omega}| \le \tilde{M}^{b-a+1}$ for all intervals $[a,b]$, $E \in \tilde{I}$, and $\omega \in \Omega$. Lastly, choose K so that $\tilde{M}^{1/K} < e^{\nu/2}$ and let η_ε, $\eta_{\frac{\varepsilon}{4}}$ denote the large deviation parameters corresponding to ε and $\frac{\varepsilon}{4}$ respectively.

2. Any N's and constants furnished by the lemmas below depend only on the parameters above (i.e. they are independent of $l \in \mathbb{Z}$ and ω).

Thus, for the remainder of the work, $\varepsilon_0, \varepsilon, \eta_0, \delta_0, \eta_\varepsilon$, and K will be treated as fixed parameters chosen in the manner outlined above.

Remark 13. Note that the sets $B^{\pm}_{[a,b],\varepsilon}$ are hereafter defined in terms of $\tilde{I}$ rather than I.

Following [44] and [29], we define subsets of Ω below on which we have regularity of the Green's functions. This is the key to the proof of all the localization results. As mentioned in the introduction, the proofs of spectral and dynamical localization given in [44] show that an event formed by the complement of the sets below has exponentially small probability. These estimates were exploited in [29] to provide a proof of exponential dynamical localization for the one-dimensional Anderson model. We follow the example set in these two papers with appropriate modifications needed to handle the presence of critical energies and the varying length of words.

Let m_L denote Lebesgue measure on $\mathbb{R}$.

Lemma 4.3.1. If $n \geq 2$ and x is $(\gamma(E) - 8\varepsilon_0, n_1, n_2, E, \omega)$-singular, then

$$(E, \omega) \in B^{-}{}_{[x-n_1, x+n_2],\varepsilon_0} \cup B^{+}{}_{[x-n_1, x-1],\varepsilon_0} \cup B^{+}{}_{[x+1, x+n_2],\varepsilon_0}.$$

Proof. The result follows by eq. (4.6) and the definition of singularity.

□

Let $R_n, R_n^{'}, Q_n, Q_n^{'}$, and $\tilde{Q}_n$ be the random variables from Lemma 4.2.2 and for $l \in \mathbb{Z}$ set

$$F^1_{l,n,\varepsilon_0} = \{\omega : \max\{m_L(B^-_{[l+R_n, l+R_n^{'}],\varepsilon_0,\omega}), m_L(B^-_{[l+Q_n, l+Q_n^{'}],\varepsilon_0,\omega})\} \leq e^{-(\eta_0-\delta_0)(2n+1)}\}.$$

Lemma 4.3.2. There is an N such that for $n > N$ and any $l \in \mathbb{Z}$,

$$\mathbb{P}[F^1_{l,n,\varepsilon_0}] \geq 1 - 2m_L(\tilde{I})e^{-\delta_0(2n+1)}.$$

Proof. With $0 < \varepsilon_0 < 8\nu$ as above, choose N such that the conclusion of Lemma 4.2.2 holds. Then for $n > N$,

$$\begin{aligned} m_L \times \mathbb{P}(B^-_{[l+R_n,l+R'_n],\varepsilon_0}) &= \mathbb{E}(m_L(B^-_{[l+R_n,l+R'_n],\varepsilon_0,\omega})) \\ &= \int_{\mathbb{R}} \mathbb{P}(B^-_{[l+R_n,l+R'_n],\varepsilon_0,E})\, dm_L(E) \\ &\leq m_L(\tilde{I})e^{-\eta_0(2n+1)}. \end{aligned}$$

Applying the same reasoning to$B^-_{[l+Q_n,l+Q'_n],\varepsilon_0}$, by the estimate above and Chebyshev's inequality,

$$e^{-(\eta_0-\delta_0)(2n+1)}\mathbb{P}[(F^1_{l,n,\varepsilon_0})^c] \leq 2m_L(\tilde{I})e^{-\eta_0(2n+1)}.$$

The result follows by multiplying both sides of the last inequality by $e^{(\eta_0-\delta_0)(2n+1)}$.

□

Remark 14. Lemma 4.3.3 is proved in [44] and used there to give a uniform (and quantitative) Craig-Simon estimate similar to the one in [29].

Lemma 4.3.3. Let $Q(x)$ be a polynomial of degree $n-1$. Let $x_i = \cos\frac{2\pi(i+\theta)}{n}$, for $0 < \theta < \frac{1}{2}, i = 1, 2, \ldots, n$. If $Q(x_i) \leq a^n$, for all i, then $Q(x) \leq Cna^n$, for all $x \in [-1, 1]$, where $C = C(\theta)$ is a constant.

Set

$$F_{[a,b],\varepsilon} = \{\omega : |P_{[a,b],E,\omega}| \leq e^{(\gamma(E)+4\varepsilon)(b-a+1)} \text{ for all } E \in \tilde{I}\}. \tag{4.30}$$

Lemma 4.3.4. There are $C > 0$ and N such that for $b - a + 1 > N$,

$$\mathbb{P}[F_{[a,b],\varepsilon}] \geq 1 - C(b - a + 2)e^{-\eta_{\frac{\varepsilon}{4}}(b-a+1)}.$$

Proof. Since $\tilde{I}$ is compact and $\tilde{I} \cap D = \emptyset$, $\tilde{I}$ is contained in the union of finitely many compact intervals which all intersect D trivially. Hence, it suffices to prove the result for all E in one of these intervals. So fix one of these intervals, call it I_1. By continuity of γ and compactness of I_1, if $\varepsilon > 0$, there is $\delta > 0$ such that if $E,E^{'} \in I_1$ with $|E - E^{'}| < \delta$, then $|\gamma(E) - \gamma(E^{'})| < \frac{1}{4}\varepsilon$. Divide I_1 into sub-intervals of size δ, denoted by $J_k = [E_k^n, E_{k+1}^n]$ where $k = 1, 2, ...C$. Additionally, let $E_{k,i}^n = E_k^n + (x_i + 1)\frac{\delta}{2}$.

By Lemma 4.2.1, there is an N such that for $b - a + 1 > N$,

$$\mathbb{P}[\{\omega : |P_{[a,b],E_{k,i}^n,\omega}| \geq e^{(\gamma(E_{k,i}^n)+\frac{1}{4}\varepsilon)(b-a+1)}\}] \leq e^{-\eta_{\frac{\varepsilon}{4}}(b-a+1)}.$$

Put $F_{[a,b],k,\varepsilon} = \bigcup_{i=1}^{n}\{\omega : |P_{[a,b],E_{k,i}^n,\omega}| \geq e^{(\gamma(E_{k,i}^n)+\frac{1}{4}\varepsilon)(b-a+1)}\}$ and $\gamma_k = \inf_{E\in J_k} \gamma(E)$. For $\omega \notin F_{[a,b],k,\varepsilon}$, $|P_{[a,b],E_{k,i}^n,\omega}| \leq e^{(\gamma_k+\frac{1}{2}\varepsilon)(b-a+1)}$. Thus, an application of Lemma 4.3.3 yields, for any such ω, $|P_{[a,b],E,\omega}| \leq e^{(\gamma(E)+\frac{3}{4}\varepsilon)(b-a+1)}$.

We have

$$\mathbb{P}\left[\bigcup_{k=1}^{C} F_{[a,b],k,\varepsilon}\right] \geq 1 - C(b - a + 2)e^{\eta_{\frac{\varepsilon}{4}}(b-a+2)}.$$

Thus, since

$$\bigcup_{k=1}^{C} F_{[a,b],k,\varepsilon} \subset F_{[a,b],\varepsilon},$$

the result follows.

□

Let $n_{r,\omega}$ denote the center of localization (if it exists) for $\psi_{\omega,E}$ (i.e. $|\psi_{\omega,E}(n)| \leq |\psi_{\omega,E}(n_{r,\omega})|$). Note that by the results in [36], $n_{r,\omega}$ can be chosen as a measurable function of ω.

Put $F^4_{l,n,\varepsilon} = \Big((F_{[l+R_n,l-1],\varepsilon} \cap F_{[l+1,l+R_n'],\varepsilon} \Big) \cap \Big(\bigcup_{0\leq k\leq 2m} (F_{[l+\tilde{Q}_n+k+1,l+Q_n'],\varepsilon} \cap F_{[l+Q_n,l+\tilde{Q}_n+k-1],\varepsilon} \Big)$

and

$$J_{l,n,\varepsilon} = F^1_{l,n,\varepsilon_0} \cap F^2_{l,n,\varepsilon} \cap F^3_{l,n,\varepsilon} \cap F^4_{l,n,\varepsilon}. \tag{4.31}$$

Lemma 4.3.5. There is N such that if $n > N$, $\omega \in J_{l,n,\varepsilon}$, with a generalized eigenfunction $\psi_{\omega,E}$ satisfying either

1. $n_{r,\omega} = l$, or
2. $|\psi_\omega(l)| \geq \frac{1}{2}$,

then if $l + \tilde{Q}_n + k$ is $(\gamma(E) - 8\varepsilon_0, \tilde{Q}_n + k - Q_n, Q_n' - \tilde{Q}_n - k, E, \omega)$-singular (with $0 \leq k \leq 2m$), there exist

$$l + R_n \leq y_1 \leq y_2 \leq l + R_n'$$

and $E_j = E_{j,[l+Q_n+k,l+Q_n'],\omega}$ such that

$$|P_{[l+R_n,y_1],E_j,\tilde{\omega}} P_{[y_2,l+R_n'],E_j,\omega}| \geq \frac{1}{2m_L(\tilde{I})\sqrt{m(2n+1)}} e^{(\gamma(E_j)-\varepsilon)(R_n'-R_n+1)+(\eta_0-\delta_0)(2n+1)}. \tag{4.32}$$

Remark 15. Note that y_1 and y_2 depend on ω and l but we do not include this subscript for notational convenience. In particular, this is done when the other terms in expressions involving y_1 or y_2 have the correct subscript and indicate the appropriate dependence.

Proof. Firstly, if $|\psi_\omega(l)| \geq \frac{1}{2}$, we may choose N_1 such that l is $(\gamma(E) - 8\varepsilon_0, -R_n, R_n^{'}, E, \omega)$-singular for $n > N_1$. In the case that $n_{r,\omega} = l$, then there is an N_2 such that l is naturally, $(\nu - 8\varepsilon_0, -R_n, R_n^{'}, E, \omega)$-singular for all $n > N_2$. Choose N_3 so that $e^{\frac{-\nu}{2}n} < \mathrm{dist}(I, \tilde{I})$ for $n > N_3$ and finally choose N to be larger than N_1, N_2, N_3 and the N's from Lemma 4.3.2, Lemma 4.2.2, and Lemma 4.3.4.

Suppose that for some $n > N$, $l + \tilde{Q}_n + k$ is $(\gamma(E) - 8\varepsilon_0, \tilde{Q}_n + k - Q_n, Q_n^{'} - \tilde{Q}_n - k, E, \omega)$-singular. By Lemma 4.3.1 and Lemma 4.3.4, $E \in B^-_{[l+Q_n, l+Q_n'], \varepsilon_0, \omega}$. Note that all eigenvalues of $H_{[l+Q_n, l+Q_n'], \omega}$ belong to $B^-_{[l+Q_n, l+Q_n'], \varepsilon_0, \omega}$. Since $P_{[l+Q_n, l+Q_n'], \tilde{E}, \omega}$ is a polynomial in $\tilde{E}$, it follows that $B^-_{[l+Q_n, l+Q_n'], \omega, \varepsilon_0}$ is contained in the union of sufficiently small intervals centered at the eigenvalues of $H_{\omega, [l+Q_n, l+Q_n']}$. Moreover, Lemma 4.3.2 gives

$$m(B^-_{[l+Q_n, l+Q_n'], \omega, \varepsilon_0}) \leq 2m_L(\tilde{I})e^{-(\eta_0 - \delta_0)(2n+1)},$$

so we have the existence of $E_j = E_{j, [l+Q_n, l+Q_n'], \omega}$ so that $|E - E_j| \leq 2m_L(\tilde{I})e^{-(\eta_0 - \delta_0)(2n+1)}$.

Applying the above argument with l in place of $l + \tilde{Q}_n + k$ yields an eigenvalue $E_i = E_{i, \omega, [l+R_n, l+R_n']}$ such that $E_i \in B^-_{[l+R_n, l+R_n'], \varepsilon_0, \omega}$ and $|E - E_i| \leq 2m_L(\tilde{I})e^{-(\eta_0 - \delta_0)(2n+1)}$. Hence, $|E_i - E_j| \leq 4m_L(\tilde{I})e^{-(\eta_0 - \delta_0)(2n+1)}$. By the previous line and the fact that $E_j \notin B^-_{[l+R_n, l+R_n'], \varepsilon, \omega}$, we see that $||G_{[l+R_n, l+R_n'], E_j, \omega}|| \geq \frac{1}{4m_L(\tilde{I})}e^{(\eta_0 - \delta_0)(2n+1)}$ so that for some y_1, y_2 with $l + R_n \leq y_1 \leq y_2 \leq l + R_n^{'}$,

$$|G_{[l+R_n, l+R_n'], E_j, \omega}(y_1, y_2)| \geq \frac{1}{4m_L(\tilde{I})\sqrt{m(2n+1)}}e^{(\eta_0 - \delta_0)(2n+1)}.$$

Additionally, another application of Lemma 4.2.2 yields, $|P_{[l+R_n, l+R_n'], E_j, \omega}| \geq e^{(\gamma(E_j) - \varepsilon)(R_n^{'} - R_n + 1)}$.

Thus, by eq. (4.6) we obtain

$$|P_{[l+R_n,y_1],E_j,\omega}P_{[y_2,l+R'_n],E_j,\omega}| \geq \frac{1}{4m_L(\tilde{I})\sqrt{m(2n+1)}}e^{(\gamma(E_j)-\varepsilon)(R'_n-R_n+1)+(\eta_0-\delta_0)(2n+1)}.$$

□

Lemma 4.3.6. There is a $\tilde{\eta} > 0$ and N such that $n > N$ implies $\mathbb{P}(J_{l,n,\varepsilon}) \geq 1 - e^{-\tilde{\eta}n}$.

Proof. Let $\mathcal{A}_1 = [2n+1-m, (2n+3)m]$, $\mathcal{A}_2 = [3n+1-m, (3n+3)m]$, and $\mathcal{A}_3 = [n+1-m, (n+3)m]$.

Thus,

$$\left(\bigcup_{j_1\in\mathcal{A}_1, j_2\in\mathcal{A}_2, j_2-j_1\geq n/2} F_{[j_1,j_2],\varepsilon}\right) \cup \left(\bigcup_{j_3\in\mathcal{A}_3, j_4\in\mathcal{A}_1, j_4-j_3\geq n/2} F_{[j_3,j_4],\varepsilon}\right)$$

is contained in

$$\left(\bigcup_{0\leq k\leq 2m} (F_{[l+\tilde{Q}_n+k+1, l+Q'_n],\varepsilon} \cap F_{[l+Q_n, l+\tilde{Q}_n+k-1],\varepsilon})\right).$$

We also have

$$\mathbb{P}\left[\left(\bigcup_{j_1\in\mathcal{A}_1, j_2\in\mathcal{A}_2, j_2-j_1\geq n/2} F_{[j_1,j_2],\varepsilon}\right) \cup \left(\bigcup_{j_3\in\mathcal{A}_3, j_4\in\mathcal{A}_1, j_4-j_3\geq n/2} F_{[j_3,j_4],\varepsilon}\right)\right] \geq 1 - 2n^2 e^{-\eta\frac{n}{2}}$$

for sufficiently large n.

Note that the same reasoning provides a similar estimate on $\mathbb{P}[F_{[l+R_n,l-1],\varepsilon} \cap F_{[l+1,l+R'_n],\varepsilon}]$.

Choose N as in Lemma 4.3.5, and note that by Lemma 4.3.2, Lemma 4.2.2, Lemma 4.3.4 and the argument above, for $n > N$,

$$\mathbb{P}[J_{l,n,\varepsilon}] \geq 1-2m_L(\tilde{I})e^{-\delta_0(2n+1)}-2m^4(2n+3)^3e^{-\eta_\varepsilon(\frac{n}{K})}-2m^2(2n+3)^2e^{-\eta_\varepsilon(2n+1)}-4n^2e^{-\eta_{\frac{\varepsilon}{4}}\frac{n}{2}}.$$

We may choose $\tilde{\eta}$ sufficiently close to 0 and increase N such that for $n > N$, we have

$$2m_L(\tilde{I})e^{-\delta_0(2n+1)} + 2m^4(2n+3)^3e^{-\eta_\varepsilon(\frac{n}{K})} + 2m^2(2n+3)^2e^{-\eta_\varepsilon(2n+1)} + 4n^2e^{-\eta_{\frac{\varepsilon}{4}}\frac{n}{2}} \leq e^{-\tilde{\eta}n},$$

and the result follows.

□

Lemma 4.3.7. There is N such that for $n > N$, any $\omega \in J_{l,n,\varepsilon}$, any y_1, y_2 with $l + R_n \leq y_1 \leq y_2 \leq l + R_n^{'}$ and any $E_j = E_{j,[l+k+Q_n,l+Q_n^{'}],\omega}$ (with $0 \leq k \leq 2m$),

$$|P_{[l+R_n,y_1],E_j,\omega}P_{[y_2,l+R_n^{'}],E_j,\omega}| \leq e^{(\gamma(E_j)+\varepsilon)(R_n^{'}-R_n+1)}.$$

Proof. By choosing N so that Lemma 4.2.2 holds for $n > N$, we are led to consider three cases:

1. $l + R_n + \frac{n}{K} \leq y_1 \leq y_2 \leq l + R_n^{'} - \frac{n}{K}$,
2. $l + R_n + \frac{n}{K} \leq y_1 \leq l + R_n$, while $l + R_n^{'} - \frac{n}{K} \leq y_2 \leq l + R_n^{'}$, and finally,
3. $l + R_n \leq y_1 \leq l + R_n + \frac{n}{K}$ and $l + R_n + \frac{n}{K} \leq y_2 \leq l + R_n^{'}$.

In the first case, Lemma 4.2.2 immediately yields:

$$|P_{[l+R_n,y_1],E_j,\omega}P_{[y_2,l+R_n^{'}],E_j,\omega}| \leq e^{(\gamma(E_j)+\varepsilon)(R_n^{'}-R_n+1)}.$$

In the second case, we have $|P_{[y_2,l+R_n'],E_j,\omega}| \leq \tilde{M}^{\frac{n}{K}}$, while Lemma 4.2.2 gives

$$|P_{[l+R_n,y_1],E_j,\omega}| \leq e^{(\gamma(E_j)+\varepsilon)(\frac{n}{K})}.$$

By our choice of K, $\tilde{M}^{\frac{1}{K}} \leq e^{\frac{\nu}{2}} \leq e^{(\gamma(E_j)+\varepsilon)}$, so we again obtain the desired result.

Finally, in the third case, $|P_{[l+R_n,y_1],E_j,\omega}P_{[y_2,l+R_n'],E_j,\omega}| \leq \tilde{M}^{\frac{2n}{K}} \leq e^{(\gamma(E_j)+\varepsilon)(R_n^{'}-R_n+1)}$ (again by our choice of K). □

4.4 Spectral Localization

Theorem 4.4.1. There is N such that if $n > N$, $0 \leq k \leq 2m$, and $\omega \in J_{l,n,\varepsilon}$, with a generalized eigenfunction $\psi_{\omega,E}$ satisfying either

1. $n_{r,\omega} = l$, or
2. $|\psi_\omega(l)| \geq \frac{1}{2}$,

then $l + \tilde{Q}_n + k$ is $(\gamma(E) - 8\varepsilon_0, \tilde{Q}_n + k - Q_n, Q_n^{'} - \tilde{Q}_n - k, E, \omega)$-regular.

Proof. Choose N so that Lemma 4.3.5 and Lemma 4.3.7 hold and

$$\frac{1}{4m_L(\tilde{I})\sqrt{m(2n+1)}}e^{(\gamma(E_j)-\varepsilon)(R_n^{'}-R_n+1)+(\eta_0-\delta_0)(2n+1)} > e^{(\gamma(E_j)+\varepsilon)(R_n^{'}-R_n+1)}$$

for $n > N$. This can be done since $\varepsilon < \frac{\eta_0-\delta_0}{3m}$.

For $n > N$, we obtain the conclusion of the theorem. For if $l + Q_n + k$ was not $(\gamma(E) -$

$8\varepsilon_0, \tilde{Q}_n + k - Q_n, Q_n^{'} - \tilde{Q}_n - k, E, \omega)$-regular, then by Lemma 4.3.5

$$|P_{[l+R_n, y_1], E_j, \omega} P_{[y_2, l+R_n^{'}], E_j, \omega}| \geq \frac{1}{4m_L(\tilde{I})\sqrt{m(2n+1)}} e^{(\gamma(E_j)-\varepsilon)(R_n^{'}-R_n+1)+(\eta_0-\delta_0)(2n+1)}.$$

On the other hand, by Lemma 4.3.7, we have

$$|P_{[l+R_n, y_1], E_j, \omega} P_{[y_2, l+R_n^{'}], E_j, \omega}| \leq e^{(\gamma(E_j)+\varepsilon)(R_n^{'}-R_n+1)}.$$

Our choice of N in the first line of the proof yields a contradiction and completes the argument.

□

We are now ready to give the proof of Theorem 1.2.1. Again, R_n, $R_n^{'}, Q_n, Q_n^{'}$, and $\tilde{Q}_n$ are the scales from Lemma 4.2.2.

Proof. By Lemma 4.3.6, $\mathbb{P}[J_{0,n,\varepsilon}$ eventually $] = 1$. Thus, we obtain $\tilde{\Omega}$ with $\mathbb{P}[\tilde{\Omega}] = 1$ and for $\omega \in \tilde{\Omega}$, there is $N(\omega)$ such that for $n > N(\omega)$, $\omega \in J_{0,n,\varepsilon}$.

Since the spectral measures are supported by the set of generalized eigenvalues (e.g. [39]), it suffices to show for all $\omega \in \tilde{\Omega}$, every generalized eigenfunction with generalized eigenvalue $E \in I$ is in fact an $\ell^2(\mathbb{Z})$ eigenfunction which decays exponentially.

Fix an ω in $\tilde{\Omega}$ and let $\psi = \psi_{\omega,E}$ be a generalized eigenfunction for H_ω with generalized eigenvalue E. Employing the same reasoning as in the proof of Theorem 1.1.1, and the bounds established on $R_n, R_n^{'}, Q_n, Q_n^{'}$, and $\tilde{Q}_n$ from Lemma 4.2.2, it suffices to show that there is $N(\omega)$ such that for $n > N(\omega)$, if $0 \leq k < 2m$, then $\tilde{Q}_n + k$ is $(\gamma(E) - 8\varepsilon_0, \tilde{Q}_n + k - Q_n, Q_n^{'} - \tilde{Q}_n - k, E, \omega)$-regular. We may assume $\psi(0) \neq 0$, and moreover, by rescaling ψ, $|\psi(0)| \geq \frac{1}{2}$. Choose N so that for $n > N$, the conclusions of Theorem 4.4.1 hold. Additionally, we may choose $N(\omega)$ such that for $n > N(\omega)$, $\omega \in J_{0,n,\varepsilon}$. For $n > \max\{N, N(\omega)\}$, the hypotheses of Theorem 4.4.1 are met, and hence $\tilde{Q}_n + k$ is $(\gamma(E) - 8\varepsilon_0, \tilde{Q}_n + k - Q_n, Q_n' - \tilde{Q}_n - k, E, \omega)$-regular. □

4.5 Exponential Dynamical Localization

The strategy used in this section follows [29] with appropriate modifications needed to deal with the fact that single-step transfer matrices were not used in the large deviation estimates. In particular, the randomness in the conclusion of Theorem 4.4.1 will need to be accounted for.

The following lemma was shown in [40] and we state a version below suitable for obtaining EDL on the interval I.

Let $u_{k,\omega}$ denote an orthonormal basis of eigenvectors for $Ran(P_I(H_\omega))$, the range of the spectral projection of H_ω onto the interval I.

Lemma 4.5.1. [40] Suppose there is $\tilde{C} > 0$ and $\tilde{\gamma} > 0$ such that for any s, $l \in \mathbb{Z}$,

$$\mathbb{E}\left[\sum_{n_{r,\omega}=l} |u_{k,\omega}(s)|^2\right] \leq \tilde{C} e^{-\tilde{\gamma}|s-l|}.$$

Then there are $C > 0$ and $\gamma > 0$ such that for any $p, q \in \mathbb{Z}$,

$$\mathbb{E}[\sup_{t\in\mathbb{R}} |\langle \delta_p, P_I(H_\omega) e^{itH_\omega} \delta_q \rangle|] \leq C(|p-q|+1)e^{-\gamma|p-q|}.$$

By Lemma 4.5.1, Theorem 1.2.2 follows from Theorem 4.5.1.

Theorem 4.5.1. There is $\tilde{C} > 0$ and $\tilde{\gamma} > 0$ such that for any s,$l \in \mathbb{Z}$,

$$\mathbb{E}\left[\sum_{n_{r,\omega}=l} |u_{k,\omega}(s)|^2\right] \leq \tilde{C}e^{-\tilde{\gamma}|s-l|}.$$

Proof. We choose N so that Theorem 4.4.1 and Lemma 4.3.6 hold and for $0 \leq k \leq 2m$, set $\zeta_{j,k} = \min\{\tilde{Q}_j + k - Q_j, Q_j^{'} - \tilde{Q}_j - k\}$. We then choose $0 < c < (\nu - 8\varepsilon_0)$ and increase N such that if $j > N$ and $0 \leq k \leq 2m$, $c(\tilde{Q}_j + k) < (\nu - 8\varepsilon_0)2\zeta_{j,k}$. This can be done using the bounds on $Q_n, Q_n^{'}$, and $\tilde{Q}_n$ established in Lemma 4.2.2. Finally, we choose $0 < \tilde{\eta}_1 < \frac{\tilde{\eta}}{(\langle L\rangle+\varepsilon)3}$ and increase N such that if $j > N$, $(\frac{\tilde{\eta}}{(\langle L\rangle+\varepsilon)3} - \tilde{\eta}_1)j > \ln(j)$.

Now consider s and l in $\mathbb{Z}$.

There are two cases to consider:

1. $s - l > 2m(N + 1)$,
2. $s - l \leq 2m(N + 1)$.

Now suppose $n_{r,\omega} = l$ and $l + \tilde{Q}_j \leq s < l + \tilde{Q}_{j+1}$, and $\omega \in J_{l,j,\varepsilon}$, since $j > N$, using Theorem 4.4.1 and eq. (4.5),

$$\begin{aligned} |u_{r,\omega}(s)| &\leq 2|u_{r,\omega}(l)|e^{-(\gamma(E_{r,\omega})-8\varepsilon_0)\zeta} \\ &\leq 2|u_{r,\omega}(l)|e^{-(\nu-8\varepsilon_0)\zeta}. \end{aligned} \tag{4.33}$$

By orthonormality and Hölder's inequality,

$$\begin{aligned}\sum_{n_{r,\omega}=l} |u_{r,\omega}(s)|^2 &\leq 4 \sum_{n_{r,\omega}=l} |u_{r,\omega}(l)|^2 e^{-(\nu-8\varepsilon_0)2\zeta_{j,k}} \\ &\leq 4 \sum_{n_{r,\omega}=l} e^{-(\nu-8\varepsilon_0)2\zeta_{j,k}}.\end{aligned} \tag{4.34}$$

We need to replace the randomness in the exponent above with an estimate that depends only on the point s.

Since $s = l + \tilde{Q}_j + k$ with $0 \leq k \leq 2m$, by our choice of c and N,

$$\sum_{n_{r,\omega}=l} e^{-(\nu-8\varepsilon_0)2\zeta_{j,k}} \leq \sum_{n_{r,\omega}=l} e^{-c|s-l|}.$$

Now let $\mathcal{A}$ denote the set of $j \in \mathbb{Z}$ so that $l + \tilde{Q}_j \leq s < l + \tilde{Q}_{j+1}$.

Finally, letting $J = \bigcup_{j\in\mathcal{A}} J_{l,j,\varepsilon}$, using the estimate provided by Lemma 4.3.6 on $\mathbb{P}[J_{l,j,\varepsilon}]$ and our choice of $\tilde{\eta}_1$,

$$\begin{aligned}\mathbb{E}\left[\sum_{n_{r,\omega}=l} |u_{k,\omega}|^2\right] &= \mathbb{E}\left[\sum_{n_{r,\omega}=l} |u_{r,\omega}|^2 \chi_J + \sum_{n_{r,\omega}=l} |u_{r,\omega}|^2 \chi_{J^c}\right] \\ &\leq Ce^{-c|s-l|} + Ce^{-\tilde{\eta}_1|s-l|}.\end{aligned} \tag{4.35}$$

In the second case, again by orthonormality, $\mathbb{E}\left[\sum_{n_{r,\omega}=l} |u_{r,\omega}(s)|^2\right] \leq 1$.

By letting $\tilde{\gamma} = \min\{c, \tilde{\eta}_1\}$ and choosing a sufficiently large $\tilde{C} > 0$, we obtain:

$$\mathbb{E}\left[\sum_{n_{r,\omega}=l} |u_{r,\omega}(s)|^2\right] \leq \tilde{C}e^{-\tilde{\gamma}|s-l|}.$$

□

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