

UNIVERSITY OF CALIFORNIA
Los Angeles

2-categorical Brown representability
and the relation between derivators
and infinity-categories

A dissertation submitted in partial satisfaction
of the requirements for the degree
Doctor of Philosophy in Mathematics

by

Kevin David Arlin

2020

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ABSTRACT OF THE DISSERTATION

2-categorical Brown representability
and the relation between derivators
and infinity-categories

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Doctor of Philosophy in Mathematics

University of California, Los Angeles, 2020

Professor Paul Balmer, Chair

In this thesis, we study the 2-category of infinity-categories, largely with attention to its relationships with the 2-category of prederivators. We prove that the 2-category of infinity-categories admits a small set of objects detecting equivalences and satisfies a Brown representability theorem, which we formulate using a new notion of compactly generated 2-category. We show that the canonical 2-functor from the 2-category of infinity-categories into the 2-category of prederivators detects equivalences and, under appropriate size conditions, induces an equivalence on hom-categories. We explain how to extend prederivators defined on the 2-category of ordinary categories to the domain of all infinity-categories using the delocalization theorem. We use the Brown representability theorem to give conditions under which a prederivator is representable by an infinity-category. We also show how to extend derivators defined on categories and satisfying a mild size condition to derivators on infinity-categories, using an extension of Cisinski's theorem on the universality of derivators of spaces. This extension allows us to give conditions under which the small sub-prederivators of quite general derivators are all representable by infinity-categories.

The dissertation of Kevin David Arlin is approved.

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*For Sam
sine qua non*

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ACKNOWLEDGMENTS

Thanks are due to Paul Balmer, for his advice and support during a research program full of twists and turns. George Raptis repeatedly exhibited a gratifying interest in the topics of this thesis, asked many helpful questions, and served as a key driving force behind the study of derivators in Europe. Denis-Charles Cisinski gave a highly influential suggestion on each of my visits to Regensburg during the degree. Dan Christensen invited me for a highly productive visit to UWO. Thanks to Emily Riehl for several helpful conversations, as well as for organizing with Dominic Verity the 2018 Talbot workshop, a rare opportunity to fill a room with higher categorists. For other helpful discussions surrounding the topics in this thesis over the past years, thanks are due to Moritz Groth, Mike Shulman, Martin Gallauer, Fernando Muro, Martina Rovelli, and Daniel Fuentes-Keuthan. Thanks to the Balminar crew, Andy Smith, Ian Coley, Ioannis Lagkas, and John Zhang, for countless hours spent in pursuit of glory.

Thanks to Eric Auld and Michael Andrews, and my cats, who were all reliable company even if they were unwilling to learn the definition of a derivator. Mike Miller, if less tolerant of rants technical and otherwise than might have been expected, was always more so than was deserved. Thanks to Nancy Carlson, for her constant unwavering support, and to Ken Carlson, whose thinly disguised impatience for this thing to be over with was a constant source of motivation. And thanks to Sam, for being my primary source of concrete nonsense to balance out all the below.

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Kevin Arlin. (né Carlson.) “A higher Whitehead theorem and the embedding of quasicategories in prederivators.” *Homotopy, Homology, and Applications* 22(1):117-139,2020.

Kevin Arlin and Dan Christensen. “Detecting isomorphisms in the homotopy category.”
Submitted.

CHAPTER 1

Introduction

In this introduction, we briefly describe the background of this work from two perspectives. Both emerge naturally from the successes in studying stable abstract homotopy theories via their homotopy categories. This work is focused on two intertwined extensions to this project: first (1.1), we give results indicating that its closest generalization to unstable homotopy theory should operate via the homotopy 2-category, rather than the homotopy category, and second (1.2), we give results indicating to what extent an abstract homotopy theory can be modeled using the *family* of homotopy categories called a prederivator.

1.1 Unstable homotopy theory via 2-categories

The story of abstract homotopy theory has been that of a continuing dialectic between the *maximalist* approach of developing an axiomatic capable of grandly encompassing the entirety of homotopy theory and the *minimalist* approach that aims for an axiomatic just strong enough to capture key phenomena, with the goal of moving quickly past foundational work.

Triangulated categories

As an exemplar of the minimalist approach, we might take the theory of triangulated categories due to Verdier [Ver96]. From the perspective of abstract homotopy, a triangulated category \mathcal{T} is the homotopy category of a stable homotopy theory. In \mathcal{T} , one directly accesses only a fragment of the total resources of the stable homotopy theory, effectively cones and extension groups. Yet it is a hopeless task to give even a representative sample of ap-

plications of triangulated categories in algebraic geometry, representation theory, algebraic topology, and beyond; one might mention Verdier duality, mirror symmetry, Bridgeland stability, tilting theory, the Balmer spectrum, and the category of mixed motives.

In terms of general theory, perhaps the most important indication of the strength of the triangulated axiomatic is the Brown representability theorem, as exposed for instance in [Nee01]. This allows us to construct objects of many triangulated categories \mathcal{T} in terms of the functors they represent, even though triangulated categories admit few colimits. As it happens, they admit enough *weak* colimits to characterize the representable functors, at least if \mathcal{T} admits an appropriately generating set.

Unstable homotopy categories

However, efforts to formulate unstable homotopy theory in terms of homotopy categories have been generally much less successful. A suggestion of a reason why: the archetypal homotopy category **Hot**, given for instance by spaces of the homotopy type of a CW complex together with homotopy classes of continuous functions, lacks the good properties of the most useful triangulated categories. In particular, the author proved with Christensen in [CC19] that **Hot** admits no set of objects that jointly detect isomorphisms, unlike the familiar case of spheres in the *pointed* homotopy category. This argument is reproduced as Theorem 4.3.7.

Based on experience from triangulated categories and with the original Brown representability theorem for pointed connected spaces, this suggests that **Hot** may not satisfy a Brown representability theorem. And in fact, this was already known: Heller gave in [Hel81] an example of a functor $N : \mathbf{Hot}^{\text{op}} \rightarrow \mathbf{Set}$ which preserves all the weak colimits available but is not representable. Roughly speaking, N sends a connected space X to the set of normal subgroups of its fundamental group.

Furthermore, the weak colimits in the form of cones and suspensions that serve triangulated categories so well are effectively unavailable in unstable homotopy categories. Weak colimits, which we recall are cocones through which every cocone factors, but perhaps not uniquely, are not generally determined up to even non-unique isomorphism. The long exact

sequence arguments that establish cones as determined up to isomorphism in triangulated categories are not available unstably. Between the absence of representability (and thus adjoint functor) theorems and of essentially any usable (co)limit concepts, it becomes clear why the abstract study of unstable homotopy categories has had so many fewer notable successes than its stable analogue.

Homotopy 2-categories

One of the main goals of this thesis is a suggestion for a fix to the above-mentioned deficiencies of unstable homotopy categories. In [RV15], Riehl and Verity introduced a notion of weak (co)limit in a 2-category that is *always* determined up to (not necessarily unique) equivalence. For the precise definition, see Definition 2.1.5. They give examples of such weak (co)limits in the 2-category $\infty\text{-}\mathbf{Cat}$ of ∞ -categories, notably including comma and cocomma objects.

In Definition 2.1.9, we axiomatize a notion of weakly cocomplete 2-category. The weak cocommas mentioned above play a central role. Beyond that, a weakly cocomplete 2-category has certain legitimate colimits: firstly, it has coproducts, as with many large triangulated categories. However, there is a novel example of legitimate colimit arising at the 2-categorical level, namely, the *coinverter*, which plays a central role in much of this thesis.

A coinverter in a 2-category is the universal map $q : Y \rightarrow Y[\alpha^{-1}]$ inverting a 2-morphism α with codomain Y . That is, the category of maps $Y[\alpha^{-1}] \rightarrow Z$, for any Z , is *equivalent* (this is not a weak colimit!) to the category of maps $Y \rightarrow Z$ inverting α . We show that coinverters in the 2-category of ∞ -categories may be constructed out of localizations, which are functors of ∞ -categories universally inverting a class of arrows. The existence of localizations is rather well known—see for instance [Ste17], or Chapter 7 of [Cis19]—but they do not appear in [Lur09] and their 2-categorical characterization in $\infty\text{-}\mathbf{Cat}$ has not apparently been remarked on before.

In fact, there are many more examples of such *weakly cocomplete* 2-categories arising from homotopy theory. Most notably, if Q is any ∞ -category, then there is a construction (see [Lur09]) of the universal 2-category $\mathrm{Ho}_2(Q)$ admitting a map from Q , and we show that

the homotopy 2-category $\mathrm{Ho}_2 Q$ is weakly cocomplete if Q is cocomplete, see Proposition 5.3.7. In particular, we get in this way the weakly cocomplete *homotopy 2-category* **Hot** of CW complexes, maps, and homotopy classes of homotopies.

Furthermore, we show in Theorem 4.3.13 that **Hot** does admit a set of objects jointly detecting equivalences, unlike for **Hot**. The same holds for $\infty\text{-Cat}$ (see Theorem 4.3.14) and also for the homotopy 2-category of any locally presentable ∞ -category (see Corollary 4.3.18).

As for representability theorems, a weakly cocomplete 2-category \mathcal{K} admits a natural notion of cohomological 2-functor $H : \mathcal{K}^{\mathrm{op}} \rightarrow \mathbf{Cat}$ valued in categories (see Definition 5.2.1). Furthermore, \mathcal{K} admits weak colimits (which are, again, determined up to equivalence) of sequences. This leads to a definition of compactly generated 2-category (5.1.1) and a Brown representability theorem:

Theorem 1.1.1 (5.2.2). *Every cohomological 2-functor on a compactly generated 2-category is representable.*

The 2-categories **Hot** and $\infty\text{-Cat}$ are compactly generated, as is the homotopy 2-category $\mathrm{Ho}_2(Q)$ of any locally finitely presentable ∞ -category (5.1.7). Furthermore, Brown representability descends along localizations.

Thus the picture in stable homotopy theory, in which the homotopy category is always triangulated, and, if large, usually satisfies Brown representability, is closely duplicated in unstable homotopy theory once we increment from homotopy categories to homotopy 2-categories. This indicates the potential for an extension of the accomplishments of triangulated category theory into unstable homotopy theory via the systematic study of homotopy 2-categories.

1.2 ∞ -categories versus derivators

Triangulated categories in the stable case, and homotopy 2-categories in the unstable, do not suffice to study the entirety of homotopy theory. For this, one needs one of the top-down

axiomatics. The most established example is Quillen’s model category theory [Qui67], while since the publication of [Lur09], the theory of ∞ -categories has rapidly gained in usability and use.

The second major prong of this thesis is to study the 2-category $\underline{\infty - \mathbf{Cat}}$ of ∞ -categories via its Yoneda embedding into the 2-category of 2-functors $\underline{\infty - \mathbf{Cat}}^{\text{op}} \rightarrow \underline{\mathbf{Cat}}$. We call such a prestack of categories on $\underline{\infty - \mathbf{Cat}}$ an ∞ -prederivator. As motivation for this move, we note that an ∞ -prederivator, consisting essentially of a family of ordinary categories, is more directly amenable than an ∞ -category to study via pre-existing categorical techniques.

Furthermore, the ∞ -prederivator of a ∞ -category Q knows about the homotopy category of Q , but also about the homotopy categories $\text{Ho}(Q^R)$ of R -indexed diagrams in Q for every ∞ -category R . All these categories, viewed as objects of the 2-category $\underline{\mathbf{Cat}}$, are really invariants of Q . Thus to understand Q via its associated prederivator is a more natively homotopical approach than to understand it, for instance, via sets of simplices.

(Pre)derivators

To try to turn ∞ -category theory yet further into a special case of category theory, or at least 2-category theory, one may restrict the domain of an ∞ -prederivator to those ∞ -categories arising as the nerves of categories. This produces a 2-functor $\underline{\mathbf{Cat}}^{\text{op}} \rightarrow \underline{\mathbf{Cat}}$, called a *prederivator*. Such an object has no obvious connection to homotopy theory, but when improved with the addition of a few simple axioms to a *derivator*, it was known already to Grothendieck [Gro90] and Heller [Hel88] to be capable of capturing a considerable proportion of the homotopy-theoretic phenomena missed by triangulated categories. Most notably, derivators give perhaps the simplest functorial theory of homotopy limits and colimits. Furthermore, Cisinski proved in [Cis08] the extraordinary result that the apparently entirely categorical-axioms of derivators suffice to reinvent homotopy theory: specifically, the free (left) derivator on a point is that associated to the ∞ -category of spaces.

While authors including Cisinski as well as Maltiniotis [Mal05], Groth [Gro13], Shulman and Ponto [GPS14], Muro and Raptis [MR11] [MR17], and Coley [Col19] have considerably

increased our knowledge about derivators over the past decade or so, the list of results explicating the extent to which (pre)derivators can be used as a top-down model, capturing the whole of the homotopy theory, is shorter. Renaudin [Ren09] gave an equivalence between a certain 2-category of combinatorial model categories—in effect, the 2-category of locally presentable ∞ -categories—and a 2-category of derivators. More recently, Tobias Lenz [Len18] showed that the prederivator detects equivalences at least for ∞ -categories associated to cofibration categories, while Rovelli, Fuentes-Keuthan, and Kedziorek gave in [FKR19] a characterization up to isomorphism of those prederivators arising from ∞ -categories.

We take as a main focus the 2-functor $\text{HO} : \underline{\infty - \mathbf{Cat}} \rightarrow \mathbf{PDer}$ mapping an ∞ -category to a prederivator.

Morphisms of ∞ -categories versus morphisms of prederivators

For small ∞ -categories, it follows quickly from the delocalization theorem that every (2-)morphism between the prederivators $\text{HO}(Q), \text{HO}(R)$ associated to an ∞ -category arises essentially uniquely from a (2-)morphism between Q and R . This is Theorem 4.2.1.

When large ∞ -categories are allowed in $\underline{\infty - \mathbf{Cat}}$, things are a bit trickier. At least if the ∞ -categories are locally small, then we show in Theorem 4.4.1 that HO detects when an ∞ -category is complete or cocomplete and more generally when a functor of ∞ -categories has a left or a right adjoint. This is of some interest, even when compared to the previous result, since complete ∞ -categories cannot be expected to be small.

Finally, for totally arbitrary ∞ -categories in the domain, all we can guarantee is that HO detects equivalences. This is a reframing of the result Theorem 4.3.14 advertised above as saying that $\underline{\infty - \mathbf{Cat}}$ admits a small set of objects detecting equivalences.

Representability of prederivators

Our main application of the Brown representability theorem is to explain when a prederivator is in the image of ∞ -categories under HO . Brown representability applies directly to ∞ -prederivators to characterize the representables as those preserving coproducts, coinverters,

and weak cocomma objects, see Proposition 5.4.1. But our real interest is in representability of prederivators. Thus the question arises: When does a prederivator extend to an ∞ -prederivator satisfying the assumptions of the Brown representability theorem?

It is, in fact, not too hard to extend many prederivators canonically to ∞ -prederivators. The reason is that every ∞ -category Q arises as a localization of a category, namely, its category of simplices $\Delta \downarrow Q$. The analogue for simplicially enriched categories was known to Dwyer and Kan, while the quasicategorical version is originally due to Joyal. The upshot is that, if \mathbb{D} is a prederivator *which preserves* the coinverters of the form $\Delta \downarrow J \rightarrow J$ where J is a category, then \mathbb{D} extends canonically to an ∞ -prederivator $\overline{\mathbb{D}}$, as is shown in Theorem 3.1.3.

It is, furthermore, not too difficult to give conditions under which the extension $\overline{\mathbb{D}}$ to an ∞ -prederivator will preserve coproducts and cocommas. In essence, we need only ask the same of \mathbb{D} , although there is a technical subtlety with the distinction between cocommas and lax pushouts, for which we direct the reader to Proposition 2.3.8. Given this, the rest of the claim is Theorem 3.2.2.

What is more difficult is to explain when the prederivator \mathbb{D} will have an extension $\overline{\mathbb{D}}$ which respects coinverters. In fact, our route toward this last representability question occupies perhaps a third of the entire thesis, detouring some distance through some developments in *derivator* theory.

Homotopically locally small derivators

Not to leave the reader utterly bereft of specifics, if our running example of a prederivator \mathbb{D} is a 2-functor of the form $J \mapsto \mathrm{Ho}(Q^J)$ for some ∞ -category Q , then for a morphism $u : J \rightarrow K$ the action $\mathbb{D}(u) : \mathrm{Ho}(Q^K) \rightarrow \mathrm{Ho}(Q^J)$ is effectively just pre-composition with u .

In a left derivator, these pre-composition functors all have left adjoints $u_!$, which should be thought of as homotopy left Kan extensions. In particular, if u is a functor into the terminal category, then $u_!$ should be thought of as a homotopy colimit functor. As in the prederivator case, we refer to a “derivator” with domain $\underline{\infty - \mathbf{Cat}}$ as an ∞ -derivator.

We give in Theorem 3.4.7 conditions under which a derivator \mathbb{D} extends to an ∞ -derivator. The result depends crucially on the assumption that \mathbb{D} should be *homotopically locally small*. This says, roughly, that the categories $\mathbb{D}(J)$ must all be enriched over the homotopy category **Hot** of spaces—for a precise statement along this line, see Proposition 3.3.12. This would be an apparently arbitrary condition to assume on a *prederivator*, but any left derivator \mathbb{D} is naturally closely related to the derivator $\mathcal{H}ot$ of spaces by Cisinski’s theorem mentioned above. In fact, each category $\mathbb{D}(J)$ is canonically identified with the category of cocontinuous derivator morphisms into \mathbb{D} from the derivator $\mathcal{H}ot^{J^{op}}$ associated to the ∞ -category of presheaves of spaces on J . Thus our precise definition of homotopical local smallness (Definition 3.3.9) is not in terms of enrichment in spaces, but instead asks that every such cocontinuous morphism should have a right adjoint.

To explain the yoga here, we provide a table of analogies. First, we recall from Freyd’s special adjoint functor theorem (SAFT) that, if \mathcal{C} is a locally small category and J is a small category, every cocontinuous functor $\mathbf{Set}^{J^{op}} \rightarrow \mathcal{C}$ admits a right adjoint. Thus we have:

	Ordinary category theory	Derivator theory
Free cocompletion of $*$	Set	$\mathcal{H}ot$
Free cocompletion of J	$\mathbf{Set}^{J^{op}}$	$\mathcal{H}ot^{J^{op}}$
Codomain for SAFT	Locally small \mathcal{C}	Ho. locally small \mathbb{D}

In this light, homotopical local smallness may appear as a natural notion. And indeed we are unable to give a single example of a left derivator \mathbb{D} which is *not* homotopically locally small, except trivially by allowing the values $\mathbb{D}(J)$ not to be locally small.

As a payoff for introducing the new notion, Theorem 3.4.7 shows that any homotopically locally small left derivator automatically admits an extension to a left ∞ -derivator respecting all coinverters under no further assumptions.

In particular, this gives a solution to our question above of when a prederivator \mathbb{D} ’s canonical extension to an ∞ -prederivator respects coinverters: it suffices that \mathbb{D} embed nicely in a homotopically locally small derivator which is “moderate” in overall size (see Definition 5.4.4). Every prederivator associated to an ∞ -category admits such an embedding, so that we can prove:

Theorem 1.2.1 (5.4.7). *A prederivator $\mathbb{D} : \mathbf{Cat}^{\text{op}} \rightarrow \mathbf{Cat}$ is representable by an ∞ -category if and only if it preserves lax pushouts and coproducts and embeds fully faithfully in some homotopically locally small left derivator.*

1.3 Remaining questions

We list some remaining questions for future work.

1. *Can one characterize precisely which weak colimits exist in $\underline{\infty - \mathbf{Cat}}$?*
2. *What is an example of a morphism between prederivators associated to ∞ -categories which does not arise from a morphism of the ∞ -categories? We give a suggestion in Conjecture 4.3.5.*
3. *Does every left derivator extend to a left ∞ -derivator? Does every left (∞ -)derivator respect coinverters, regardless of homotopical local smallness? This seems likely, but requires some novel argument regarding the fibers of arbitrary localizations.*
4. *Does the forgetful functor from small ∞ -prederivators respecting coinverters to all small ∞ -prederivators admit a left adjoint? Since the domain of a prederivator is a large 2-category, an adjoint functor theorem-type answer to this would tend to produce a *large* prederivator. However, a positive answer would greatly simplify the characterization of prederivators representable by ∞ -categories.*
5. *Is there a model of ∞ -categories allowing more direct application of Brown representability to $\underline{\infty - \mathbf{Cat}}$? The difficulty, as discussed below Proposition 5.3.10, is in the description of cocomma objects in $\underline{\infty - \mathbf{Cat}}$. These might be better behaved in a model based on spaces, rather than simplicial sets.*

CHAPTER 2

Definitions and generalities

2.1 2-categorical generalities

We generally denote 1-categories in **Bold** and 2-categories in **UnderlinedBold**. When size issues are apropos, by **Cat** we denote the category of categories small with respect to the smallest Grothendieck universe \mathcal{U} , and by **CAT** that of categories small with respect to a second Grothendieck universe \mathcal{V} .

Below we recall the various 2-categorical definitions we shall require.

We shall write horizontal compositions in 2-categories in diagrammatic order, so that the pasting

$$\begin{array}{ccccc} & \curvearrowright & & \curvearrowright & \\ x & \Downarrow \alpha & y & \Downarrow \beta & z \\ & \curvearrowleft & & \curvearrowleft & \end{array}$$

will be denoted $\alpha * \beta$. If f is a 1-morphism for which the composite $\alpha * \text{id}_f$, is defined, we shall write the latter as $\alpha * f$.

Definition 2.1.1. We make use of both strictly 2-natural and pseudonatural transformations between 2-functors. Let us recall that, if \mathcal{K}, \mathcal{L} are 2-categories and $F, G : \mathcal{K} \rightarrow \mathcal{L}$ are 2-functors, a *pseudonatural transformation* $\Lambda : F \Rightarrow G$ consists of

- Morphisms $\Lambda_x : F(x) \rightarrow G(x)$ associated to every object $x \in \mathcal{K}$
- 2-morphisms $\Lambda_f : \Lambda_y \circ F(f) \Rightarrow G(f) \circ \Lambda_x$ for every morphism $f : x \rightarrow y$ in \mathcal{K}

satisfying the coherence conditions

- (Pseudonaturality) Λ_f is an isomorphism, for every f .

- (Coherence) Λ is a functor from the underlying 1-category of \mathcal{K} to the category of pseudo-commutative squares in \mathcal{L} , that is, squares commuting up to a chosen isomorphism, where composition is by pasting.
- (Respect for 2-morphisms) For every 2-morphism $\alpha : f \Rightarrow g : x \rightarrow y$ in \mathcal{K} , we have the equality of 2-morphisms

$$\Lambda_g \circ (\Lambda_y * F(\alpha)) = (G(\alpha) * \Lambda_x) \circ \Lambda_f : \Lambda_y \circ F(f) \Rightarrow G(g) \circ \Lambda_x.$$

In case all the Λ_f are identities, we say that Λ is strictly 2-natural. In this case, the axiom of coherence is redundant, and that of respect for 2-morphisms simplifies to $\Lambda_y * F(\alpha) = G(\alpha) * \Lambda_x$.

If, instead, the pseudonaturality assumption is completely eliminated, then we have a *lax* natural transformation.

Definition 2.1.2. The morphisms between pseudonatural transformations, are called *modifications*. A modification $\Xi : \Lambda \Rightarrow \Gamma : F \Rightarrow G : \mathcal{K} \rightarrow \mathcal{L}$ consists of 2-morphisms

$$\Xi_x : \Lambda_x \rightarrow \Gamma_x$$

for each object $x \in \mathcal{K}$, subject to the sole condition

$$(G(f) * \Xi_x) \circ \Lambda_f = \Gamma_f \circ (\Xi_y * F(f)) : \Lambda_y \circ F(f) \Rightarrow G(f) \circ \Gamma_x,$$

for any morphism $f : x \rightarrow y$ in \mathcal{K} . When F and G are strict, this simplifies to

$$G(f) * \Xi_x = \Xi_y * F(f).$$

An *equivalence* between the objects $x, y \in \mathcal{K}$ consists of two morphisms $f : x \leftrightarrow y : g$ together with invertible 2-morphisms $\alpha : g \circ f \cong \text{id}_x$ and $\beta : f \circ g \cong \text{id}_y$.

If $F : \mathcal{K} \rightarrow \mathcal{L}$ is a 2-functor between 2-categories, then in general we say F is “locally φ ” if φ is a predicate applicable to functors between 1-categories which holds of each functor $\mathcal{K}(x, y) \rightarrow \mathcal{L}(F(x), F(y))$ induced by F . For instance, we can in this way ask that F be locally essentially surjective, locally fully faithful, or locally an equivalence.

We shall use the phrase “bicategorically φ ” for global properties of F that categorify the property φ as applied to a single functor of categories. For instance, we shall use “bicategorically fully faithful” as a synonym for the potentially misleading term “local equivalence”: a bicategorically fully faithful 2-functor is one inducing equivalences on hom-categories.

Similarly, F will be said to be “bicategorically conservative” if it reflects equivalences, so that whenever we have $f : x \rightarrow y$ in \mathcal{K} such that $F(f)$ is an equivalence in \mathcal{L} , we can conclude f is an equivalence in \mathcal{K} .

Weak cocompleteness of 2-categories

The notion of weak colimit in a 2-category, essentially as we use it here, was introduced in [RV15]. We depart slightly from Riehl and Verity by requiring only essential surjectivity where they require strict surjectivity. This is all we need for Brown representability, and allows simpler choices of especially the coinverters in $\infty - \mathbf{Cat}$.

Definition 2.1.3. A functor $F : J \rightarrow K$ of categories will be called *weakly smothering* if it is full, conservative, and essentially surjective.

We now explicitly define the colimits and weak colimits required for the 2-categorical Brown representability theorem.

Definition 2.1.4. Consider a 2-category \mathcal{K} .

1. A *2-coproduct* of a family $(X_i)_{i \in I}$ of objects in \mathcal{K} is an object $\coprod X_i$ equipped for every Y in \mathcal{K} with a natural equivalence

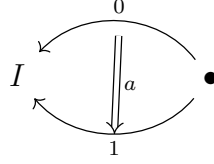
$$\mathcal{K}(\coprod X_i, Y) \rightarrow \prod (X_i, Y)$$

of categories.

2. Given a diagram

$$\begin{array}{ccc} & f & \\ X & \begin{array}{c} \curvearrowright \\ \downarrow \alpha \\ \curvearrowleft \end{array} & Y \\ & g & \end{array} \quad Y \xrightarrow{q} Z$$

in \mathcal{K} , we say q is a *coinverter* of α if it is the weighted colimit of the diagram $\alpha : f \Rightarrow g : X \rightarrow Y$ weighted by the diagram



in which I denotes the category freely generated by an isomorphism a .

Concretely, this means that $\alpha * q$ is invertible and initial with that property. In other words, composition with q induces an equivalence $\mathcal{K}(Z, W) \cong \mathcal{K}(Y, W)_\alpha$ with image the full subcategory of $\mathcal{K}(Y, W)$ on those maps $r : Y \rightarrow Z$ such that $\alpha * r$ is invertible.

3. Given a span $A \leftarrow B \rightarrow C$ in \mathcal{K} , to give P the structure of a *weak cocomma object*¹ for the span is to give a square

$$\begin{array}{ccc} B & \longrightarrow & C \\ \downarrow & \swarrow & \downarrow \\ A & \longrightarrow & P \end{array}$$

satisfying the following weak universal property:

Given any X in \mathcal{K} , let T denotes the comma object in **Cat** of the cospan $\mathcal{K}(A, X) \rightarrow \mathcal{K}(B, X) \leftarrow \mathcal{K}(C, X)$. Then the induced functor $\mathcal{K}(P, X) \rightarrow T$ is weakly smothering.

4. A *weak tensor* of an object A of \mathcal{K} by a category J consists of:

- An object $A \otimes J$ of \mathcal{K} .
- A lax cocone $T : c_A \Rightarrow c_{A \otimes J} : J \rightarrow \mathcal{K}$, where c_A denotes the diagram constant at A . Equivalently, T is given by a functor $J \rightarrow \mathcal{K}(A, A \otimes J)$.

These data satisfy the weak universal property that the functor

$$\mathcal{K}(A \otimes J, X) \rightarrow \mathcal{K}(A, X)^J$$

induced by T is weakly smothering for every $X \in \mathcal{K}$.

¹In a $(2, 1)$ -category, a weak cocomma is equivalent to a lax or oplax pushout, but this is not quite the case in a general 2-category.

All the above concepts fit under the general definition:

Definition 2.1.5. In general, consider a diagram $D : J \rightarrow \mathcal{K}$ indexed by a 2-category and a weight $W : J^{\text{op}} \rightarrow \mathbf{Cat}$. A W -weighted cylinder under D with base X is given by a pseudonatural transformation $W \rightarrow \mathcal{K}(D(-), X)$; for instance, if W is the terminal weight, then a W -weighted cylinder is simply a (pseudo) cocone.

A (*weighted*) *colimit* for D weighted by W , denoted $W \otimes_J^w D$, is an object X equipped with a *universal* cocylinder $\eta : W \rightarrow \mathcal{K}(D(-), X)$. This means that the induced functors $\mathcal{K}(X, Y) \rightarrow \text{PsNat}(W, \mathcal{K}(D(-), Y))$ are equivalences, for every Y .²

Similarly, a *weak (weighted) colimit* for D weighted by W , denoted $W \otimes_J^w D$, is an object X equipped with a *weakly universal* cocylinder $\eta : W \rightarrow \mathcal{K}(D(-), X)$. This means that the induced functors $\mathcal{K}(X, Y) \rightarrow \text{PsNat}(W, \mathcal{K}(D(-), Y))$ are weakly smothering, for every Y .

Note that, under our definitions, coproducts and coinverters are legitimate colimits, not merely weak ones. *A fortiori*, a weighted colimit is a weak weighted colimit, as in the 1-categorical case. Let us now give a few examples of weak weighted colimits, mostly closely related to the explicit colimits defined above.

- Example 2.1.6.*
1. When W is taken to be the terminal weight, one gets a notion of conical weak colimit.
 2. For instance, the conical weak colimit of a diagram indexed by the span category $\bullet \leftarrow \bullet \rightarrow \bullet$ is simply an iso-comma or pseudo-pushout. The conical pseudo-colimit of a diagram indexed by a discrete category is a 2-coproduct.
 3. When W is taken to be the weight in (2) of Definition 2.1.4, a W -weighted weak colimit is a *weak* coinverter; note that we shall only have use for coinverters which are not weak.

²It is common in the 2-categorical literature to define weighted colimits using strictly 2-natural cylinders and inducing isomorphisms of categories, rather than the pseudo ones inducing equivalences we define here. While by cofibrantly replacing the weights one sees that the strict notion is technically more general, the homotopical nature of 2-categories like $\underline{\infty} - \mathbf{Cat}$ makes the use of strict colimits less natural, and we shall have no need for them.

4. If $J = \bullet \leftarrow \bullet \rightarrow \bullet$ and $W = \bullet \rightarrow (\bullet \cong \bullet \leftarrow \bullet) \leftarrow \bullet$, then W -weighted (weak) colimits are (weak) cocomma objects.
5. If we take instead $W = \bullet \rightarrow (\bullet \rightarrow \bullet \leftarrow \bullet) \leftarrow \bullet$ in the above example, then W -weighted (weak) colimits are called (weak) lax pushouts, a concept which is subtly but meaningfully distinct from that of weak cocomma.

In contrast to the 1-categorical case, there are no other weak colimits when colimits exist. Intuitively, the conservativity condition is enough to make weak colimits essentially unique—just not in an essentially unique way.

Proposition 2.1.7 ([RV15],(3.3.5)). *Weak colimits are determined up to a (not necessarily essentially unique) equivalence.*

Notation 2.1.8. We shall denote a (weak) cocomma object for a span $A \leftarrow B \rightarrow C$ by $A \overset{\leftarrow}{\sqcup}_B C$ and a (weak) comma object for a cospan $X \rightarrow Z \leftarrow Y$ by $X \overset{\leftarrow}{\times}_Z Y$. By Proposition 2.1.7, there is no risk of ambiguity in whether the (co)commas are weak. We shall denote a (weak) inverter of a 2-morphism α of domain A by A_α , and a (weak) coinverter of ζ with codomain Z by $Z[\zeta^{-1}]$. Note that the notation A_α was already used on $\mathcal{K}(Y, W)_\alpha$ in (2) of Definition 2.1.4.

For the Brown representability theorem 5.2.2 and for the study of the relationship between the 2-category $\underline{\infty}\text{-Cat}$ of ∞ -categories and the 2-category $\underline{\mathbf{PDer}}$ of prederivators, a key notion will be that of a 2-category admitting the homotopically reasonable colimits and weak colimits.

Definition 2.1.9. A *weakly cocomplete* 2-category \mathcal{K} is one admitting:

- 2-coproducts.
- Coinverters.
- Weak cocomma objects.

Let us emphasize that the coproducts and coinverters are to be fully legitimate bicategorical colimits: the comparison functors are equivalences of categories, where those involved in the weak cocomma objects need not be faithful.

Any such 2-category admits further colimits which shall be of use to us:

Proposition 2.1.10. *A weakly cocomplete 2-category \mathcal{K} admits the following:*

1. *Weak (pseudo) coequalizers, also known as iso-inserters.*
2. *Weak colimits of countable sequences $X_0 \rightarrow X_1 \rightarrow \dots$*
3. *Weak tensors by the category $\bullet \rightrightarrows \bullet$ freely generated by two parallel arrows.*

Proof. For iso-inserters, given parallel arrows $f, g : x \rightrightarrows y$ in \mathcal{K} , we must construct $i : y \rightarrow z$ and a weakly universal isomorphism $\alpha : if \rightrightarrows ig$. We first construct the weak cocomma object c of the span $y \leftarrow x \sqcup y \rightarrow y$ in which the morphisms are (f, id_y) and (g, id_y) respectively. Next, we construct the converter d of the induced 2-morphism $\hat{\alpha} : (f, i_1) \rightrightarrows (g, i_2) : x \sqcup y \rightarrow c$. We define either of the isomorphic induced morphisms $y \rightarrow x \sqcup y \rightarrow c \rightarrow d$ as q , which defines an isomorphism $\alpha : q \circ f \rightrightarrows q \circ g$ as desired.

Weak sequential colimits are constructed as mapping telescopes. Suppose given a sequence $D = (X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} \dots)$. We are looking for some $X \in \mathcal{K}$ so that maps $X \rightarrow W$ weakly represent pseudo-cocones from D to W . Now, a pseudo-cocone from D to W is uniquely determined by its components $c_i : X_i \rightarrow W$ and pseudo-naturality morphisms $\lambda_i : c_{i+1}f_i \rightarrow c_i$. Similarly, a morphism of pseudo-cones $(c, \lambda) \rightarrow (d, \mu)$ is uniquely determined by 2-morphisms $\gamma_i : c_i \rightrightarrows d_i$ such that the following squares commute in $\mathcal{K}(X_i, W)$:

$$\begin{array}{ccc} c_{i+1}f_i & \xrightarrow{\gamma_{i+1}f_i} & d_{i+1}f_i \\ \downarrow \lambda_i & & \downarrow \mu_i \\ c_i & \xrightarrow{\gamma_i} & d_i \end{array}$$

In other words, if we construct the diagram $D' = (\coprod X_i \rightrightarrows \coprod X_i)$ in which one arrow is the identity and the other has components f_i , then the category of pseudo-cocones from D to W is equivalent to the category of pseudo-cocones from D' to W . Thus we may take a weak coequalizer of D' as our weak sequential colimit.

For the weak tensors by $\bullet \rightrightarrows \bullet$, given $X \in \mathcal{K}$ we need an object $X \otimes \bullet \rightrightarrows \bullet$ equipped with natural full and essentially surjective functors

$$\mathcal{K}(X \otimes \bullet \rightrightarrows \bullet, Y) \rightarrow \mathcal{K}(X, Y)^{\bullet \rightrightarrows \bullet}.$$

The right-hand side is simply the groupoid of quadruples $(h, k : X \rightrightarrows Y, \gamma, \delta : h \rightrightarrows k)$. This is equivalent to the groupoid of pseudo-cocones from the span $X \leftarrow X \coprod X \rightarrow X$ to ΔY , where both maps in the span are the codiagonal. Thus, we can define $X \otimes \bullet \rightrightarrows \bullet$ as the weak pushout of $X \leftarrow X \coprod X \rightarrow X$. \square

2.2 The 2-category of ∞ -categories

This section is dedicated to introducing the 2-category $\underline{\infty} - \mathbf{Cat}$ that will be our main object of study.

Basic notions on simplicial sets and ∞ -categories

We denote the category associated to the poset $0 < 1 < \dots < n$ by $[n]$, so that $[0]$ is the terminal category. The simplex category Δ is the full subcategory of \mathbf{Cat} on the categories $[n]$.

If S is a simplicial set, that is, a functor $\Delta^{\text{op}} \rightarrow \mathbf{Set}$, then we denote its set of n -simplices by $S([n]) = S_n$. The face map $S_n \rightarrow S_{n-1}$ which forgets the i^{th} vertex will be denoted d_i^n or just d_i . We denote by Δ^n the simplicial set represented by $[n] \in \Delta$. Equivalently, $\Delta^n = N([n])$, where we recall that the nerve $N(J)$ of a category J is the simplicial set defined by the formula $N(J)_n = \mathbf{Cat}([n], J)$. The natural extension of N to a functor is a fully faithful embedding of categories in simplicial sets. See [Joy08, Proposition B.0.13].

We recall that a *quasicategory* [Joy08] is a simplicial set Q in which every inner horn has a filler.³ That is, every map $\Lambda_i^n \rightarrow Q$ extends to an n -simplex $\Delta^n \rightarrow Q$ when $0 < i < n$, where $\Lambda_i^n \subset \Delta^n$ is the simplicial subset generated by all faces $d_j \Delta^n$ with $j \neq i$. For instance, when

³We shall use the word “quasicategory” in general when specific combinatorial properties arising from simplicial sets are needed, and “ ∞ -category” when we discuss model-independent notions.

$n = 2$, the only inner horn is Λ_1^2 , and then the filler condition simply says we may compose “arrows” (that is, 1-simplices) in Q , though not uniquely. Morphisms of quasicategories are simply morphisms of simplicial sets. The quasicategories in which every inner horn has a *unique* filler are, up to isomorphism, the nerves of categories; in particular the nerve functor $N : \mathbf{Cat} \rightarrow \mathbf{SSet}$ factors through the full subcategory spanned by quasicategories, which we denote by \mathbf{QCat} .

Every quasicategory Q has a homotopy category $\mathrm{Ho}(Q)$, the 1-category defined as follows. The objects of $\mathrm{Ho}(Q)$ are simply the 0-simplices of Q . For two 0-simplices q_1, q_2 , temporarily define $Q_{q_1, q_2} \subset Q_1$ to be the set of 1-simplices f with initial vertex q_1 and final vertex q_2 . Then the hom-set $\mathrm{Ho}(Q)(q_1, q_2)$ is the quotient of Q_{q_1, q_2} which identifies *homotopic* 1-simplices. Here two 1-simplices $f_1, f_2 \in Q_{q_1, q_2}$ are said to be homotopic if f_1, f_2 are two faces of some 2-simplex in which the third face is both outer and degenerate. We have a functor $\mathrm{Ho} : \mathbf{QCat} \rightarrow \mathbf{Cat}$ from quasicategories to categories, left adjoint to the nerve $N : \mathbf{Cat} \rightarrow \mathbf{QCat}$. This follows from the fact that a morphism $f : Q \rightarrow R$ of quasicategories preserves the homotopy relation between 1-simplices, so that it descends to a well defined functor $\mathrm{Ho}(f) : \mathrm{Ho}(Q) \rightarrow \mathrm{Ho}(R)$. In fact, $\mathrm{Ho} : \mathbf{QCat} \rightarrow \mathbf{Cat}$ admits an extension, sometimes denoted τ_1 , to all of \mathbf{SSet} , which is still left adjoint to N . But it is not amenable to computation.

The fact that the Joyal model structure is Cartesian and has the quasicategories as its the fibrant objects implies (see [RV15, 2.2.8]) that Q^S is a quasicategory for every simplicial set S and quasicategory Q . In particular, the category of quasicategories is enriched over itself via the usual simplicial exponential

$$(R^Q)_n = \mathbf{SSet}(Q \times \Delta^n, R).$$

It is immediately checked that the homotopy category functor Ho preserves finite products, so that by change of enrichment we get finally *the 2-category of ∞ -categories*, $\underline{\infty} - \mathbf{Cat}$. Its objects are quasicategories, and for quasicategories Q, R , the hom-category $\underline{\infty} - \mathbf{Cat}(Q, R)$ is simply the homotopy category $\mathrm{Ho}(R^Q)$ of the hom-quasicategory R^Q . This permits the following tautological definition of equivalence of ∞ -categories.

Definition 2.2.1. An *equivalence of ∞ -categories* is an equivalence in the 2-category $\infty - \mathbf{Cat}$.

Remark 2.2.2. Thus an equivalence of ∞ -categories is a pair of maps $f : Q \rightleftarrows R : g$ together with two homotopy classes $a = [\alpha], b = [\beta]$ of morphisms $\alpha : Q \rightarrow Q^{\Delta^1}, \beta : R \rightarrow R^{\Delta^1}$, with endpoints gf and id_Q , respectively, fg and id_R , such that a is an isomorphism in $\text{Ho}(Q^{\mathcal{Q}})$, as is b in $\text{Ho}(R^{\mathcal{R}})$. We can make the definition yet more explicit by noting that, for each $q \in Q_0$, the map α sends q to some $\alpha(q) \in Q_1$, and recalling that the invertibility of a is equivalent to that of each homotopy class $[\alpha(q)]$, as explicated for instance in the statement below:

Lemma 2.2.3 ([RV15], 2.3.10). *The equivalence class $[\alpha]$ of a map $\alpha : Q \rightarrow R^{[1]}$ of ∞ -categories is an isomorphism in the homotopy category $\text{Ho}(R^{\mathcal{Q}})$ if and only if, for every vertex $q \in Q_0$ of Q , the equivalence class $[\alpha(q)]$ is an isomorphism in $\text{Ho}(R)$.*

Recalling that a Kan complex is a simplicial set in which *every* inner horn has a filler, we have the full subcategory **Kan** of **QC**at.

Definition 2.2.4. The *homotopy category of spaces* **Hot** is the homotopy category of **Kan**: the category of Kan complexes (or equivalently of CW complexes), and homotopy classes of morphisms.

The corresponding full sub-2-category of $\infty - \mathbf{Cat}$ will be denoted **Hot**. Note that **Hot** is the homotopy category, not the underlying category, of **Hot**.

We next recall that an equivalence of ∞ -categories is nothing more than an essentially surjective and fully faithful functor, once these words are defined. First, an ∞ -category Q has mapping spaces $Q(x, y)$ for each $x, y \in Q$, which can be given various quasicategorical models. When necessary, we shall use the balanced model in which we have $Q(x, y) = \{(x, y)\} \times_{Q \times Q} Q^{\Delta^1}$, so that an n -simplex of $Q(x, y)$ is a prism $\Delta^n \times \Delta^1$ in Q which is degenerate on x and y at its respective endpoints.

Definition 2.2.5. We say that a map $f : Q \rightarrow R$ of ∞ -categories is *fully faithful* if it induces an isomorphism $Q(x, y) \rightarrow R(f(x), f(y))$ in **Hot** for every $x, y \in Q$.

The map f is *essentially surjective* if, for every $z \in R$, there exists $x \in Q$ and an edge $a : f(x) \rightarrow z$ which becomes an isomorphism in $\text{Ho}(R)$.

Then we have

Theorem 2.2.6 (Joyal). *A map $f : Q \rightarrow R$ of quasicategories is an equivalence in the sense of Definition 2.2.1 if and only if it is fully faithful and essentially surjective.*

It is sometimes convenient to note that, just as with equivalences, fully faithful maps of ∞ -categories may be characterized in $\infty - \mathbf{Cat}$.

Proposition 2.2.7 (Riehl-Verity). *A map $f : Q \rightarrow R$ of ∞ -categories is fully faithful in the sense of Definition 2.2.5 if and only if, as a morphism of $\infty - \mathbf{Cat}$, it is representably fully faithful. This means that, for any ∞ -category S , the functor $\infty - \mathbf{Cat}(S, Q)$ \rightarrow $\infty - \mathbf{Cat}(S, R)$ induced by f is a fully faithful functor of categories.*

Proof. This is the equivalence (iii) \iff (iv) in [RV18], Corollary 3.5.6. □

(Co)limits in $\infty - \mathbf{Cat}$

We now describe some of the universal and weakly universal constructions permitted by $\infty - \mathbf{Cat}$. Except for inverters and coinverters, the limits and colimits constructed in the proposition below are all given also in [RV15]. The coinverters are also essentially already known, but not under that name:

Definition 2.2.8 (Joyal). If $f : S \rightarrow T$ is a map of ∞ -categories and \mathcal{W} is a class of edges in S , then we say that f is a *localization at \mathcal{W}* if, for every ∞ -category Q , the induced map of ∞ -categories $Q^T \rightarrow Q^S$ is a fully faithful embedding (see Definition 2.2.5) with image the full sub- ∞ -category $Q_{\mathcal{W}}^S$ of Q^S spanned by those maps $S \rightarrow Q$ sending \mathcal{W} to equivalences of Q .

We can also characterize localizations 2-categorically.

Lemma 2.2.9. *A morphism $f : S \rightarrow T$ of ∞ -categories is a localization at \mathcal{W} in the sense of Definition 2.2.8 if and only if it is a coinverter of the following 2-morphism:*

$$\begin{array}{ccc} & \curvearrowright & \\ \mathcal{W} & \Downarrow \alpha & S \\ & \curvearrowleft & \end{array}$$

Here the class \mathcal{W} is viewed as a discrete ∞ -category, and the 2-morphism α is defined by mapping $w \in \mathcal{W}$ to its homotopy class in $\underline{\infty - \mathbf{Cat}}([0], S) = \mathrm{Ho}(S)$.

Proof. For any ∞ -category Q , consider the natural transformation

$$\begin{array}{ccc} & \curvearrowright & \\ \underline{\infty - \mathbf{Cat}}(S, Q) & \Downarrow Q^\alpha & \underline{\infty - \mathbf{Cat}}(\mathcal{W}, Q) \\ & \curvearrowleft & \end{array}$$

induced by α . Then by definition f is a coinverter of α if and only if Then we see that $f : S \rightarrow T$ is a localization at \mathcal{W} if and only if, for each such Q , the induced functor $\underline{\infty - \mathbf{Cat}}(T, Q) \rightarrow \underline{\infty - \mathbf{Cat}}(S, Q)$ is an equivalence onto the inverter $\underline{\infty - \mathbf{Cat}}(S, Q)_{Q^\alpha}$ of Q^α .

By Proposition 2.2.7, each functor $Q^T \rightarrow Q^S$ is fully faithful with image $Q_{\mathcal{W}}^S$ if and only if each functor $\underline{\infty - \mathbf{Cat}}(T, Q) \rightarrow \underline{\infty - \mathbf{Cat}}(S, Q)$ is fully faithful, with the corresponding image $\underline{\infty - \mathbf{Cat}}(S, Q)_{\mathcal{W}}$.

Finally, $\underline{\infty - \mathbf{Cat}}(S, Q)_{\mathcal{W}}$ coincides with $\underline{\infty - \mathbf{Cat}}(S, Q)_{Q^\alpha}$, since for a map $g : S \rightarrow Q$, we have $Q_g^\alpha(w) = [g(w)]$ is the homotopy class of $g(w)$ in $\mathrm{Ho}(Q) = \underline{\infty - \mathbf{Cat}}([0], Q)$. Thus Q_g^α is invertible if and only if g sends each w to an invertible morphism of Q .

□

We now give the desired limits and colimits in $\underline{\infty - \mathbf{Cat}}$.

Proposition 2.2.10. *The 2-category of ∞ -categories, $\underline{\infty - \mathbf{Cat}}$, is weakly complete and cocomplete. It furthermore admits all weak tensors and cotensors by free categories, as well as weak lax pushouts. The nerve 2-functor $N : \mathbf{Cat} \rightarrow \underline{\infty - \mathbf{Cat}}$ preserves these (weak) limits, coproducts, and weak lax pushouts, but not coinverters or cocomma objects.*

Proof. Products and coproducts: The 2-coproduct in $\infty\text{-Cat}$ is represented by the coproduct of simplicial sets, and similarly for the 2-product.

Coinverters and inverters: Since invertible morphisms in functor ∞ -categories are detected pointwise (see Lemma 2.2.3), a map $q : Y \rightarrow Z$ is a coinverter of

$$\begin{array}{ccc} & f & \\ X & \begin{array}{c} \curvearrowright \\ \parallel \\ \alpha \\ \parallel \\ \curvearrowleft \end{array} & Y \\ & g & \end{array}$$

in $\infty\text{-Cat}$ if and only if it is a coinverter of

$$\begin{array}{ccc} & & \\ \mathcal{W} & \begin{array}{c} \parallel \\ \parallel \\ \alpha' \\ \parallel \\ \parallel \end{array} & Y \\ & & \end{array}$$

where $\mathcal{W} := \{\alpha_x\}_{x \in X}$ and as above we define α'_w as $[w]$ for each $w \in \mathcal{W}$. By Lemma 2.2.9, q is such a coinverter if and only if it is the localization of Y at \mathcal{W} in the sense of Definition 2.2.8. Finally, such localizations always exist. See for instance Chapter 7 of [Cis19].

The inverter of a 2-morphism α with domain A is given simply as the full sub- ∞ -category of A on those objects a such that α_a is an isomorphism, as follows straightforwardly again from pointwise detection of isomorphisms for natural transformations between functors of ∞ -categories.

Weak commas and cocommas: The weak comma of a cospan $X \rightarrow Y \leftarrow Z$ can be constructed as the strict pullback of simplicial sets $Y^{\Delta^1} \times_{Y \times Y} (X \times Z)$, while the weak cocomma of $A \leftarrow B \rightarrow C$ may be constructed dually as $(C \sqcup A) \bigsqcup_{B \sqcup B} (B \times \Delta^1)$, where again the pushout is given in simplicial sets. A proof of the universal properties is given in [RV15, 3.3.18].

Weak tensors and cotensors: Given a free category J^4 , we claim that the simplicial sets $Q \times N(J)$ and $Q^{N(J)}$ model weak tensors and cotensors of Q by J , respectively, for any ∞ -category Q .

⁴that is, J is in the image of the left adjoint of the forgetful functor from \mathbf{Cat} to the category of directed graphs

Indeed, since

$$\underline{\infty - \mathbf{Cat}}(Q \times N(J), R) = \mathrm{Ho}(R^{Q \times N(J)}) = \mathrm{Ho}((R^Q)^{N(J)}),$$

and similarly $\underline{\infty - \mathbf{Cat}}(Q, R)^J = \mathrm{Ho}(Q^R)^J$, we have to show that the canonical map $\mathrm{Ho}((R^Q)^{N(J)}) \rightarrow \mathrm{Ho}(R^Q)^J$ is weakly smothering. This is precisely the claim of Lemma 2.2.11, with X set to R^Q .

Weak lax pushouts: Given a span $Q \xleftarrow{q} S \xrightarrow{r} R$ of ∞ -categories, we define the weak lax pushout as the colimit in \mathbf{SSet} of the diagram D :

$$\begin{array}{ccccc} & & S & & \\ & \swarrow & & \searrow & \\ Q & & & & R \\ & \nwarrow & & \swarrow & \\ & & S \times (0 \leftarrow 2 \rightarrow 1) & & \end{array}$$

Let us denote the colimit by P . We verify the universal property of P . Given some T , maps $f : Q \rightarrow T, g : R \rightarrow T, h : S \rightarrow T$, and 2-morphisms $\alpha : h \Rightarrow f \circ q$ and $\beta : h \Rightarrow g \circ r$, by choosing representatives of α and β we obtain a morphism $S \times (0 \leftarrow 2 \rightarrow 1) \rightarrow T$ of simplicial sets inducing a cocone $D \Rightarrow T$ in \mathbf{SSet} and thus a morphism $P \rightarrow T$, showing the essential surjectivity clause in the weakly universal property.

For fullness, consider given morphisms $p_1, p_2 : P \rightarrow T$ restricting to weighted cocones $(f_i, g_i, h_i, \alpha_i, \beta_i)$. A morphism between the cocones is determined by 2-morphisms $\gamma : f_1 \Rightarrow f_2, \delta : g_1 \Rightarrow g_2, \varepsilon : h_1 \Rightarrow h_2$ such that $\gamma * q \circ \alpha_1 = \alpha_2 \circ \varepsilon$ and $\delta * r \circ \beta_1 = \beta_2 \circ \varepsilon$. We may lift $\gamma, \delta, \varepsilon$, and the given commutative squares to produce morphisms $Q \rightarrow T^{[1]}, R \rightarrow T^{[1]}$, and $S \times (0 \leftarrow 2 \rightarrow 1) \rightarrow T^{[1]}$ giving rise to the desired map $P \rightarrow T^{[1]}$.

As usual, the conservativity clause is easier, following as it does from the joint surjectivity of the maps $Q \rightarrow P, R \rightarrow P$, and $S \times (0 \leftarrow 2 \rightarrow 1) \rightarrow P$ on objects.

Preservation of limits and colimits by the nerve: Limits and weak limits are preserved by $N : \mathbf{Cat} \rightarrow \underline{\infty - \mathbf{Cat}}$ as N is a right 2-adjoint. This also accounts for preservation of weak tensors by free categories, those being modeled by products. Preservation of weak tensors follows from the fact that N preserves exponentials.

The functor N sends coproducts of categories to coproducts of simplicial sets, so preserves coproducts.

The lax pushout of a span in \mathbf{Cat} is the total category of its Grothendieck construction, while by the construction about the lax pushout of a span in $\underline{\infty}\text{-Cat}$ is again the total ∞ -category of the Grothendieck construction. The Grothendieck construction of a functor $D : J \rightarrow \mathbf{Cat}$ is given by the pullback of the universal Grothendieck fibration $\mathbf{Cat}_* \rightarrow \mathbf{Cat}$ along D . Since the nerve preserves pullbacks and $N(\mathbf{Cat}_*)$ is the pullback of $\mathbf{QCat}_* \rightarrow \mathbf{QCat}$ along $\mathbf{Cat} \rightarrow \mathbf{QCat}$, we see that the nerve preserves Grothendieck constructions, including the lax pushout.⁵

Finally, N fails to preserve coinverters and cocommas: if J is a category with NJ representing a homotopy type X which is not 1-truncated, then a Kan complex model for X is the ∞ -localization of NJ at all arrows, while the nerve of the localization of J at all arrows is a groupoid, thus not equivalent to X . If J denotes the noncommutative triangle, that is, the category freely generated by $\partial[2]$, then the cocomma object of $[0] \leftarrow J \rightarrow [0]$ in \mathbf{Cat} is isomorphic to $[1]$, while the cocomma object of $[0] \leftarrow N(J) \rightarrow [0]$ in $\underline{\infty}\text{-Cat}$ has the homotopy type of a 2-sphere.

□

Lemma 2.2.11. *For any ∞ -category X and free category J , the canonical functor*

$$\mathrm{Ho}(X^{N(J)}) \rightarrow \mathrm{Ho}(X)^J$$

is weakly smothering.

Proof. We first show that, for a free category J , $N(J)$ is also the free quasicategory generated by the directed graph $\mathrm{ndc}(J)$ of indecomposable arrows of J . Indeed, consider an n -simplex x of $N(J)$, say with spine (f_1, f_2, \dots, f_n) . Each edge f_i decomposes uniquely as a composite of m_i edges from $\mathrm{ndc}(J)$, and then x arises as a facet of a unique $\sum_{i=1}^n m_i$ -simplex \bar{x} with spine consisting of indecomposable morphisms. Thus we may construct $N(J)$ from $\mathrm{ndc}(J)$ inductively by filling inner horns whose spine consists of indecomposable arrows. Here the spine of the n -simplex is the 1-skeletal simplicial set $0 \rightarrow 1 \rightarrow 2 \rightarrow \dots \rightarrow n-1 \rightarrow n$.

⁵Indeed, Gepner, Haugseng, and Nikolaus show [GHN17] that any Grothendieck construction produces a lax colimit. All such are thus preserved by N viewed as a functor of ∞ -categories, and thus $N : \mathbf{Cat} \rightarrow \underline{\infty}\text{-Cat}$ preserves whatever 2-categorical lax colimits arise from ∞ -categorical ones.

From the previous paragraph, we deduce that for any ∞ -category X we have an equivalence of ∞ -categories $X^{N(J)} \simeq X^{\text{ndc}J}$. Then we may lift a given functor $F : J \rightarrow \text{Ho}(X)$ to a functor $N(J) \rightarrow X$ by freely choosing edges in X representing the image of each irreducible morphism of J under F .

Given a natural transformation $\alpha : F \Rightarrow G : J \rightarrow \text{Ho}(X)$ and lifts $\bar{F}, \bar{G} : N(J) \rightarrow X$ of F and G , for each indecomposable morphism $m : x \rightarrow y$ of J we first freely choose squares $\bar{\alpha}_m : [1] \times [1] \rightarrow X$ lifting the squares

$$\begin{array}{ccc} F(x) & \xrightarrow{\alpha_x} & G(x) \\ \downarrow F(m) & & \downarrow G(m) \\ F(y) & \xrightarrow{\alpha_y} & G(y) \end{array}$$

assumed to commute in $\text{Ho}(X)$. These choices together comprise a morphism $\text{ndc}J \rightarrow X^{[1]}$, which by the above argument extends essentially uniquely to a morphism $\bar{\alpha} : N(J) \rightarrow X^{[1]}$. By assumption, the evaluation of $\bar{\alpha}'$ at 0 is a morphism $\bar{F}' : N(J) \rightarrow X$ whose restriction to $\text{ndc}(J)$ coincides with that of \bar{F} , and similarly for the evaluation at 1, \bar{G}' and \bar{G} . Thus we may compose $\bar{\alpha}'$ with isomorphisms $\bar{F}' \cong \bar{F}$ and $\bar{G}' \cong \bar{G}$ to produce the desired lift of α .

Finally, conservativity follows from the pointwise detection of isomorphisms in functor ∞ -categories. □

We shall frequently use the *delocalization theorem*, explaining a precise sense in which ∞ -categories are determined by categories.

Definition 2.2.12. For a simplicial set S , we denote by $\Delta \downarrow S$ the category of elements of S , which comes equipped with a natural map $\ell_S : \Delta \downarrow S \rightarrow S$ evaluating a simplex at its last vertex.

The following is the delocalization theorem itself.

Theorem 2.2.13 (Joyal). *For any ∞ -category Q , the last-vertex mapping $\ell_Q : \Delta \downarrow Q \rightarrow Q$ is a localization at the class of edges it inverts. This class is denoted \mathcal{L}_Q and will be referred to as the class of “last-vertex maps.”*

Proof. (Sketch) We follow [Ste17, 1.3] (but see [Cis19, 7.3.15].) It suffices to consider $Q = \Delta^n$. In that case, ℓ_Q is a split epimorphism, with splitting $s(m) = (0 < 1 < \dots < m)$. There is a morphism $\Delta \downarrow \Delta^n \times \Delta^1 \rightarrow \Delta \downarrow \Delta^n$ with first endpoint the identity and last endpoint $s \circ q$, sending an object x of $\Delta \downarrow \Delta^n$ to the unique simplex x' in the image of s admitting a last-vertex map $x \rightarrow x'$ induced by a monomorphism in Δ . Localizing at the last vertex maps turns this 2-morphism into an isomorphism, as desired. \square

2.3 Derivators, prederivators, semiderivators

Let **Dia** be a 2-category admitting a terminal object $[0]$, small coproducts, weak comma and cocomma objects, weak tensors with free categories, a nerve 2-functor $N : \mathbf{Dia} \rightarrow \infty - \mathbf{CatCat}$, and an isomorphism $(-)^{\text{op}} : \mathbf{Dia}^{\text{co}} \rightarrow \mathbf{Dia}$.

There are many examples of such 2-categories. We shall generally be interested in the choices $\infty - \mathbf{Cat}$, with $N = \text{id}$, as well as \mathbf{Cat} with the usual nerve functor, as well as briefly the 2-category **HFin** of homotopically finite categories, which we now define:

Definition 2.3.1. A category J is *homotopically finite* if, either of the equivalent conditions holds:

1. The nerve $N(J)$ is a finitely presentable simplicial set, that is, one containing only finitely many nondegenerate simplices.
2. The category J is finite, and every endomorphism in J is an identity.

Synonyms for "homotopically finite" include "finite direct" and "finite inverse".

We say a 2-functor $\mathbb{D} : \mathbf{Dia}^{\text{op}} \rightarrow \mathbf{CAT}$ is a *prederivator*, or **Dia**-prederivator for emphasis. A prederivator may satisfy various axioms, as follows. The axioms without primes are well-established, while the primed axioms are variants introduced here.

(Der1) Let $(J_i)_{i \in I}$ be a family of objects of **Dia** such that $\coprod_I J_i \in \mathbf{Dia}$. Then the canonical map $\mathbb{D}(\coprod_I J_i) \rightarrow \prod_I \mathbb{D}(J_i)$ is an equivalence.

(Der1)' \mathbb{D} satisfies (Der1) and also respects coinverters of the form $\ell_J : \Delta \downarrow J \rightarrow J$ for all J in **Dia**. That is, the induced functor $\mathbb{D}(J) \rightarrow \mathbb{D}(\Delta \downarrow J)$ must be an equivalence onto $\mathbb{D}(\Delta \downarrow J)_{\mathcal{L}_J}$, the full subcategory of "diagrams" inverting the last-vertex maps. Formally, this is the inverter of the natural transformation $\mathbb{D}(\alpha)$, for α the natural transformation

$$\begin{array}{ccc} & \curvearrowright & \\ \mathcal{L}_J & \Downarrow \alpha & \Delta \downarrow J \\ & \curvearrowleft & \end{array}$$

sending each $m \in \mathcal{L}_J$, viewed as a discrete category, to itself viewed as an arrow.

(Der1)'' \mathbb{D} satisfies (Der1) and also respects all *homotopically correct* coinverters: if $q : K \rightarrow L$ is a converter of the 2-morphism $\alpha : f \Rightarrow g : J \rightarrow K$ and Nq remains a converter of $N\alpha$, then the induced map $\mathbb{D}(L) \rightarrow \mathbb{D}_\alpha(K)$ is an equivalence, where $\mathbb{D}_\alpha(K)$ is shorthand for the inverter $[\mathbb{D}(K)]_{\mathbb{D}(\alpha)}$ (see Notation 2.1.8.)

(Der2) For every $J \in \mathbf{Dia}$, the underlying diagram functor

$$\text{dia}^J : \mathbb{D}(J) \rightarrow \mathbb{D}([0])^{\text{Ho}(N(J))}$$

is conservative, i.e., reflects equivalences.

(Der3) For every functor $u : J \rightarrow K$ in **Dia**, $u^* = \mathbb{D}(u) : \mathbb{D}(K) \rightarrow \mathbb{D}(J)$ has both a left and a right adjoint, denoted by $u_!$ and u_* respectively.

We refer to the !-half of this axiom as (Der3L), and (Der3R) for the * half.

(Der4) For every (weak) comma square

$$\begin{array}{ccc} J & \xrightarrow{u} & K \\ \downarrow v & \swarrow & \downarrow w \\ L & \xrightarrow{x} & M \end{array}$$

in **Dia**, so that $J = K \overset{\leftarrow}{\times}_M L$, the canonical maps $v_! u^* X \rightarrow x^* w_! X$ and $w^* x_* \rightarrow u_* v^*$ are isomorphisms.

We refer to the !-half of this axiom as (Der4L), respectively, (Der4R) for the * half.

(Der5) Every canonical functor $\text{dia} : \mathbb{D}(J \times [1]) \rightarrow \mathbb{D}(J)^{\text{Ho}(N[1])}$ is full, conservative, and essentially surjective.

(Der5)' Given a span $J \leftarrow K \rightarrow L$ in **Dia** with weak cocomma object $P = J \overset{\leftarrow}{\sqcup}_K L$, if $N(P)$ remains a weak cocomma object for $N(J) \leftarrow N(K) \rightarrow N(L)$, then the canonical functor from $\mathbb{D}(P)$ to the comma category $\mathbb{D}(J) \overset{\leftarrow}{\times}_{\mathbb{D}(K)} \mathbb{D}(L)$ is weakly smothering (see Definition 2.1.3.)

Remark 2.3.2. We may summary (Der5)' as saying that such a \mathbb{D} must respect "homotopically correct" weak cocommas.

Note that, at least if N preserves weak tensors with $[1]$, then (Der5)' implies (Der5). Indeed, we can consider the weak tensor $J \times [1]$ as the weak cocomma object of the span $J \leftarrow J \rightarrow J$. Furthermore, (Der1)" implies (Der1)' since last-vertex projections are homotopically correct coinverters.

Definition 2.3.3. We have the following terminology for prederivators satisfying various combinations of the axioms:

- We call a prederivator satisfying (Der1) and (Der2) a *semiderivator*.
- We call a prederivator satisfying the first four axioms a *derivator*, respectively a *left* or a *right* derivator if just the L, respectively R, forms of (Der3) and (Der4) hold.
- We call a prederivator satisfying (Der5)' *strong*.
- Finally we call a prederivator satisfying (Der1)' *localizing*.

We have the following 2-categories of (pre)derivators:

Definition 2.3.4. We denote by **PDer** or, for emphasis, **PDer_{Dia}** the 2-category of **Dia**-prederivators, pseudonatural transformations, and modifications.

A morphism $F : \mathbb{D}_1 \rightarrow \mathbb{D}_2$ between left derivators is *cocontinuous* if for every $u : J \rightarrow K$ in **Dia** the canonical morphism $F_K \circ u_! \rightarrow u_! \circ F_J$ is an isomorphism. This gives rise to the

2-category $\underline{\mathbf{LDer}}_1$ of left derivators, cocontinuous morphisms, and modifications. Dually, we have the 2-category $\underline{\mathbf{RDer}}_*$ of right derivators and continuous morphisms.

We shall also have occasion to use the 2-categories $\underline{\mathbf{LDer}}_{\text{Ladj}}$, $\underline{\mathbf{LDer}}_{\text{Radj}}$, $\underline{\mathbf{RDer}}_{\text{Ladj}}$, and $\underline{\mathbf{RDer}}_{\text{Radj}}$ of respectively left and right derivators with left and right adjoint pseudonatural transformations as morphisms.

Definition 2.3.5. The notation \mathbb{D}^J , where J is in $\underline{\mathbf{Dia}}$, will denote the *shifted* prederivator on $\underline{\mathbf{Dia}}$ defined by $K \mapsto \mathbb{D}(J \times K)$. We remark that \mathbb{D}^J satisfies whatever axioms \mathbb{D} does.

For any prederivator \mathbb{D} one has an *opposite* prederivator \mathbb{D}^{op} defined as the composite

$$\underline{\mathbf{Dia}}^{\text{op}} \xrightarrow{(-)^{\text{op}}} \underline{\mathbf{Dia}}^{\text{coop}} \xrightarrow{\mathbb{D}^{\text{co}}} \underline{\mathbf{CAT}}^{\text{co}} \xrightarrow{(-)^{\text{op}}} \underline{\mathbf{CAT}}.$$

Proposition 2.3.6. *The operation $\mathbb{D} \mapsto \mathbb{D}^{\text{op}}$ extends to an equivalence of 2-categories $\underline{\mathbf{PDer}}^{\text{co}} \rightarrow \underline{\mathbf{PDer}}$ which restricts to an equivalence $\underline{\mathbf{LDer}}^{\text{co}} \rightarrow \underline{\mathbf{RDer}}$.*

Proof. We define the action of $(-)^{\text{op}}$ on morphisms and 2-morphisms of prederivators using the 3-equivalence $(-)^{\text{co}} : \underline{\mathbf{2Cat}}^{3\text{-op}} \rightarrow \underline{\mathbf{2Cat}}$, which is covariant on 1- and 2-morphisms but contravariant on 3-morphisms. Thus for instance if $F : \mathbb{D}_1 \rightarrow \mathbb{D}_2$, we have $F^{\text{co}} : \mathbb{D}_1^{\text{co}} \Rightarrow \mathbb{D}_2^{\text{co}} : \underline{\mathbf{Dia}}^{\text{co-op}} \rightarrow \underline{\mathbf{CAT}}^{\text{co}}$, which we whisker with $(-)^{\text{op}}$ on both sides, as in the definition of \mathbb{D}^{op} , to give F^{op} .

Like any equivalence of 2-categories, $(-)^{\text{op}} : \underline{\mathbf{PDer}}^{\text{co}} \rightarrow \underline{\mathbf{PDer}}$ defined in this manner will preserve and reflect adjunctions, so we have only to note that an adjunction $(L, R, \eta, \varepsilon)$ in a 2-category \mathcal{K} corresponds to an adjunction with flipped chirality, $(R, L, \eta, \varepsilon)$, in \mathcal{K}^{co} .

Now given $u : J \rightarrow K$, consider the morphism $u^* : \mathbb{D}^K \rightarrow \mathbb{D}^J$, whose component at L is $(u^* \times \text{id}_L) : \mathbb{D}(K \times L) \rightarrow \mathbb{D}(J \times L)$. Since $(\mathbb{D}^K)^{\text{op}}$ coincides with $(\mathbb{D}^{\text{op}})^{K^{\text{op}}}$, we get $(u^*)^{\text{op}} = (u^{\text{op}})^*$. Then the above argument implies that if $u_! : \mathbb{D}^J \rightarrow \mathbb{D}^K$ is left adjoint to u^* , then $u_!^{\text{op}} = u_*^{\text{op}}$ is a right Kan extension morphism in \mathbb{D}^{op} . In particular, \mathbb{D} satisfies (Der3L) if and only if \mathbb{D}^{op} satisfies (Der3R).

For (Der4), we thus have that \mathbb{D} satisfies (Der4L) with respect to the comma square

$$\begin{array}{ccc} J & \xrightarrow{u} & K \\ \downarrow v & \swarrow & \downarrow w \\ L & \xrightarrow{x} & M \end{array}$$

if and only if \mathbb{D}^{op} satisfies (Der4R) with respect to the comma square

$$\begin{array}{ccc} J^{\text{op}} & \xrightarrow{v^{\text{op}}} & L^{\text{op}} \\ \downarrow u^{\text{op}} & \swarrow & \downarrow x^{\text{op}} \\ K^{\text{op}} & \xrightarrow{w^{\text{op}}} & M^{\text{op}} \end{array}$$

Indeed, the latter means the canonical map $\chi : (x^{\text{op}})^* \circ w_*^{\text{op}} \rightarrow v_*^{\text{op}} \circ (u^{\text{op}})^*$ is an isomorphism in $\mathbb{D}^{\text{op}}(L^{\text{op}}) = \mathbb{D}(L)^{\text{op}}$. Then the above identification of $(u^{\text{op}})_*$ in \mathbb{D}^{op} with $u_!$ in \mathbb{D} identifies χ^{op} with the canonical comparison map $v_! u^* \rightarrow x^* w_!$ in \mathbb{D} , as desired. \square

Remark 2.3.7. We should note that our notion of strong prederivator is more demanding than either of the standard notions, which require either (Der5) or an intermediate version involving tensors with all free categories. In our defense, there has been no consensus in the literature about the “correct” amount of strongness to assume, and we argue that the Brown representability theorem shows that (Der5)’ is just right.

While the nerve functor $N : \mathbf{Cat} \rightarrow \underline{\infty - \mathbf{Cat}}$ behaves so poorly with respect to inverting morphisms that there is no hope of avoiding the somewhat awkward requirements for (Der1)’, the same is not the case for (Der5)’.

Proposition 2.3.8. *Let \mathbb{D} be a prederivator on \mathbf{Cat} . Then for \mathbb{D} to be strong, it suffices that \mathbb{D} satisfy (Der1)’ and that \mathbb{D} preserve lax pushouts in the following sense: for every span $A \leftarrow C \rightarrow B$ in \mathbf{Dia} with weak pushout P , the induced map from $\mathbb{D}(P)$ to the weak pullback H of the cospan $\mathbb{D}(A) \rightarrow \mathbb{D}(C) \leftarrow \mathbb{D}(B)$ is weakly smothering.*

Proof. First, we recall that $N : \mathbf{Cat} \rightarrow \underline{\infty - \mathbf{Cat}}$ preserves weak lax pushouts. Suppose $A \xleftarrow{a} C \xrightarrow{b} B$ is a span for which N preserves the weak cocomma object P , and let P' be the lax pushout. We get a canonical map $q : P' \rightarrow P$, and we claim $N(a)$ is a localization. Indeed, $N(P)$ weakly represents the functor given by triples (f, g, α) of a map f out of A ,

a map g out of B , and a 2-morphism $\alpha : g \circ b \Rightarrow f \circ a$. This is equivalent to the functor given by quintuples (f, g, h, α, β) of maps f, g as before, a map h out of C , a 2-morphism $\alpha : g \circ b \Rightarrow h$, and an invertible 2-morphism $\beta : f \circ a \Rightarrow h$.

If we write $N(P)$ as the colimit of the diagram

$$\begin{array}{ccccc}
 & N(C) & & N(C) & \\
 & \swarrow & \searrow & \swarrow & \searrow \\
 N(A) & & N(C) \times (0 \leftarrow 2 \rightarrow 1) & & N(B) \\
 & \swarrow^{N(a)} & \xrightarrow{0} & \swarrow^2 & \searrow^{N(b)}
 \end{array}$$

then we see that the functor (f, g, h, α, β) described above is represented by the localization of $N(P)$ at the image of the morphisms in the class $\mathcal{W} = \{(\text{id}_c, 0 \leftarrow 2)\}$ in $N(C) \times (0 \leftarrow 2 \rightarrow 1)$, as c runs over objects of C .

In short, we have shown that $\mathbb{D}(P') \rightarrow \mathbb{D}(P)$ is fully faithful, with essential image given by $\mathbb{D}_{\mathcal{W}}(P)$. There is a natural fully faithful functor t from the lax comma $\mathbb{D}(A) \overset{\leftarrow}{\times}_{\mathbb{D}(C)} \mathbb{D}(B)$ to the lax pullback, mapping $(X, Y, m : b^*Y \rightarrow a^*X)$ to $(X, Y, a^*X, m, \text{id}_{a^*X})$. Thus given an object $M = (X, Y, m : b^*Y \rightarrow a^*X)$ of the lax comma, we may lift $t(M)$ to $\overline{t(M)} \in \mathbb{D}(P)$. Now since M inverts the maps of \mathcal{W} , by definition $\overline{t(M)} \in \mathbb{D}(P)_{\mathcal{W}}$, so under (Der1)' we may conclude that $\overline{t(M)}$ arises from some $\bar{M} \in \mathbb{D}(P')$. This shows that $\mathbb{D}(P') \rightarrow \mathbb{D}(A) \overset{\leftarrow}{\times}_{\mathbb{D}(C)} \mathbb{D}(B)$ is essentially surjective. That it is full and conservative follows from full faithfulness of $\mathbb{D}(P') \rightarrow \mathbb{D}(P)$. \square

CHAPTER 3

Extending the domain of (pre)derivators

This chapter will involve considerable alternation between $\mathbf{Dia} = \mathbf{Cat}$ and $\mathbf{Dia} = \infty - \mathbf{Cat}$. In this context, when we refer a *prederivator* or *derivator* we shall generally intend \mathbf{Cat} as \mathbf{Dia} , while we will use ∞ -*prederivator* and ∞ -*derivator* when $\infty - \mathbf{Cat}$ is intended. We shall also write $\infty - \mathbf{PDer}$ for $\mathbf{PDer}_{\infty - \mathbf{Cat}}$.

The goal of the following sections is to explain that the 2-category of ∞ -derivators is almost exactly equivalent to that of derivators.

3.1 Restriction of ∞ -prederivators

We study the restriction 2-functor $\infty - \mathbf{PDer} \rightarrow \mathbf{PDer}$. For an ∞ -prederivator \mathbb{D} , let \mathbb{D}_1 denote its restriction to a prederivator.

Proposition 3.1.1. *Let \mathbb{D} and \mathbb{E} be ∞ -prederivators satisfying $(Der1)'$. Then the restriction functor $\infty - \mathbf{PDer}(\mathbb{D}, \mathbb{E}) \rightarrow \mathbf{PDer}(\mathbb{D}_1, \mathbb{E}_1)$ is an equivalence of categories.*

Proof. Given a morphism $F : \mathbb{D}_1 \rightarrow \mathbb{E}_1$, there is an essentially unique $G : \mathbb{D} \rightarrow \mathbb{E}$ with $G_1 \cong F$. To verify uniqueness, we use the commutativity of the squares

$$\begin{array}{ccc} \mathbb{D}(\Delta \downarrow J) & \xrightarrow{F_{\Delta \downarrow J}} & \mathbb{E}(\Delta \downarrow J) \\ \ell_J^* \uparrow & & \ell_J^* \uparrow \\ \mathbb{D}(J) & \xrightarrow{G_J} & \mathbb{E}(J) \end{array}$$

for each $J \in \infty - \mathbf{Cat}$, up to isomorphism determined by the canonical identifications $\mathbb{D}(J) \simeq \mathbb{D}(\Delta \downarrow J)_{\mathcal{L}_J}$ and $\mathbb{E}(J) \simeq \mathbb{E}(\Delta \downarrow J)_{\mathcal{L}_J}$. This shows that G_J is essentially uniquely determined by F . To show that a choice of G_J is possible, we have only to note that $F_{\Delta \downarrow J}$ sends $\mathbb{D}(\Delta \downarrow J)_{\mathcal{L}_J}$ into $\mathbb{E}(\Delta \downarrow J)_{\mathcal{L}_J}$, as is proven in Lemma 3.1.6.

To define the pseudonaturality constraints of G , given $u : J \rightarrow K$, full faithfulness of ℓ_J^* and ℓ_K^* implies that

$$G_u : \mathbb{D}(u) \circ G_K \cong G_J \circ \mathbb{E}(u) : \mathbb{E}(K) \rightarrow \mathbb{D}(J)$$

is uniquely determined by the cube it coinhabits with $F_{\Delta \downarrow u}$. Then the functoriality of G_u in u follows from that of $F_{\Delta \downarrow u}$.

It now remains only to verify 2-naturality of G . For this, by Lemma 3.1.5, it suffices to show that G respects dia. Given $X \in \mathbb{E}(J \times [1])$, by 2-functoriality we have

$$\ell_J^* \text{dia}_{\mathbb{E}}^J(X) = \text{dia}_{\mathbb{E}}^{\Delta \downarrow J}((\ell_J \times [1])^* X)$$

so we get an isomorphism

$$\begin{aligned} \ell_J^* G_J(\text{dia}_{\mathbb{E}}^J(X)) &\cong F_{\Delta \downarrow J} \ell_J^*(\text{dia}_{\mathbb{E}}^J(X)) \\ &= F_{\Delta \downarrow J} \text{dia}_{\mathbb{E}}^{\Delta \downarrow J}((\ell_J \times [1])^* X) \\ &\cong \text{dia}_{\mathbb{D}}^{\Delta \downarrow J} F_{\Delta \downarrow J \times [1]}((\ell_J \times [1])^* X) \\ &\cong \text{dia}_{\mathbb{D}}^{\Delta \downarrow J}(\ell_J \times [1])^* G_{J \times [1]}(X) \\ &= \ell_J^* \text{dia}_{\mathbb{D}}^J G_{J \times [1]}(X) \end{aligned}$$

which reflects into $\mathbb{D}(J)$ to show G respects dia, as desired.

Full faithfulness of $\infty - \mathbf{PDer}(\mathbb{D}, \mathbb{E}) \rightarrow \mathbf{PDer}(\mathbb{D}_1, \mathbb{E}_1)$ is easier. Given $\Xi : F_1 \Rightarrow F'_1 : \mathbb{D}_1 \rightarrow \mathbb{E}_1$, we construct Υ with $\Upsilon_1 = \Xi$ by defining $\Upsilon_J : F_J \Rightarrow F'_J$ such that

$$\mathbb{E}(\ell_J) * \Upsilon_J = F'_u \circ (\Xi_{\Delta \downarrow J} * \mathbb{D}(\ell_J)) \circ F_u^{-1}.$$

This is the unique possible definition of Υ_J , as follows from full faithfulness of $\mathbb{E}(\ell_J)$. \square

We now consider the surjectivity of $\infty - \mathbf{PDer} \rightarrow \mathbf{PDer}$ on objects.

Proposition 3.1.2. *If \mathbb{D} is any prederivator satisfying $(Der1)'$, then there exists an ∞ -prederivator \mathbb{D}' satisfying $(Der1)'$ admitting an equivalence $\mathbb{D}'_1 \simeq \mathbb{D}$.*

Proof. We must define $\mathbb{D}'(J)$ as $\mathbb{D}(\Delta \downarrow J)_{\mathcal{L}_J}$ and $\mathbb{D}'(J \xrightarrow{u} K)$ by restriction from $\mathbb{D}(\Delta \downarrow u)$. For this, we must use that $\mathbb{D}(\Delta \downarrow u)$ maps $\mathbb{D}(\Delta \downarrow K)_{\mathcal{L}_K}$ into $\mathbb{D}(\Delta \downarrow J)_{\mathcal{L}_J}$, which is shown in Lemma 3.1.6.

Thus far we have constructed \mathbb{D}' as a 1-functor with domain $\infty - \mathbf{Cat}$. Given $\alpha : u \Rightarrow v : J \rightarrow K$, we will define $\mathbb{D}'(\alpha)$ as $\text{dia}\mathbb{D}'(\bar{\alpha})$ for any lift $\bar{\alpha} : J \times [1] \rightarrow K$ of α to a functor. We first note that this choice will be well defined. Indeed, if D is the ∞ -category

$$\begin{array}{ccc} & f & \\ 0 & \begin{array}{c} \curvearrowright \\ \Downarrow \alpha \\ \curvearrowleft \end{array} & 1 \\ & g & \end{array} ,$$

then any other lift $\bar{\alpha}'$ is the image of f under a functor $H : D \rightarrow K^J$ so that $\bar{\alpha} = H(g)$. Furthermore, it is immediate that such a choice will respect vertical compositions $\beta \circ \alpha : u \Rightarrow v \Rightarrow w : J \rightarrow K$, so it remains only to show that dia may be defined, consistent with whiskering.

We define $\text{dia}_J^{\mathbb{D}'}$ as the composite

$$\begin{aligned} D'(J \times [1]) &= \mathbb{D}(\Delta \downarrow (J \times [1]))_{\mathcal{L}_{J \times [1]}} \simeq \mathbb{D}(\Delta \downarrow J \times [1])_{\mathcal{L}_J \times \mathcal{I}_{[1]}} \\ &\xrightarrow{\text{dia}} \mathbb{D}(\Delta \downarrow J)_{\mathcal{W}_J}^{[1]} \simeq \mathbb{D}(J)^{[1]}. \end{aligned}$$

Here \mathcal{I} denotes the class of identity maps. Then the first equivalence follows from (Der1)' and Lemma 3.1.4, which shows that the map

$$\Delta \downarrow J \times \Delta \downarrow [1] \rightarrow \Delta \downarrow (J \times [1])$$

is a localization at $\mathcal{I}_J \times \mathcal{L}_{[1]}$. This definition is natural in J , from the naturality of $\text{dia}^{\mathbb{D}}$. \square

We have shown:

Theorem 3.1.3. *The restriction 2-functor $\infty - \mathbf{PDer} \rightarrow \mathbf{PDer}$ induces a 2-equivalence when restricted to those (∞) -prederivators satisfying (Der1)'.*

Below are the lemmata used in the above arguments.

Lemma 3.1.4. *Let $\ell : J \rightarrow K$ and $\ell' : J' \rightarrow K'$ be localizations of ∞ -categories given by inverting the classes \mathcal{W} and \mathcal{W}' of maps, respectively. If \mathcal{W} and \mathcal{W}' contain the identity maps of their domains, then $\ell \times \ell' : J \times J' \rightarrow K \times K'$ is a localization at the class $\mathcal{W} \times \mathcal{W}'$.*

Proof. This follows easily from the Cartesian closure of $\underline{\infty - \mathbf{Cat}}$ together with the pointwise detection of isomorphisms.

For any L , we have

$$\begin{aligned} \underline{\infty - \mathbf{Cat}}(K \times K', L) &\simeq \underline{\infty - \mathbf{Cat}}(K, L^{K'}) \simeq \underline{\infty - \mathbf{Cat}}(J, L^{K'})_{\mathcal{W}} \\ &\simeq \underline{\infty - \mathbf{Cat}}(K', (L^J)_{\mathcal{W}}) \simeq \underline{\infty - \mathbf{Cat}}(J', (L^J)_{\mathcal{W}})_{\mathcal{W}'} \\ &\simeq \underline{\infty - \mathbf{Cat}}(J' \times J, L)_{\mathcal{W}' \times \mathcal{W}} \end{aligned}$$

as desired. □

Lemma 3.1.5. *Suppose given two prederivators $\mathbb{D}, \mathbb{E} : \mathbf{Dia} \rightarrow \mathbf{CAT}$ and a family of functors $F_J : \mathbb{D}(J) \rightarrow \mathbb{E}(J)$ for $J \in \mathbf{Dia}$ together with constraint isomorphisms $F_u : \mathbb{E}(u) \circ F_K \cong F_J \circ \mathbb{D}(u)$ functorially in $u : J \rightarrow K$. The given data assemble to a pseudonatural transformation, that is, satisfy 2-naturality, if and only if F respects dia in the sense that*

$$F_J * \text{dia}_J^{\mathbb{D}} = \text{dia}_J^{\mathbb{E}} * F_{J \times [1]}$$

for every $J \in \mathbf{Dia}$.

Proof. The given condition is an instance of 2-naturality, so is certainly necessary. We assume it holds and aim to prove sufficiency.

Given $\alpha : u \Rightarrow v : J \rightarrow K$, we first observe that

$$\text{dia}_J^{\mathbb{E}} * (F_{J \times [1]} \circ \mathbb{D}(\bar{\alpha})) \circ F_u = F_v^{-1} \circ \text{dia}_J^{\mathbb{E}}(\mathbb{E}(\bar{\alpha}) \circ F_J)$$

where $\bar{\alpha} : J \times [1] \rightarrow K$ is any lift of α . Indeed, we have the isomorphism $F_{\bar{\alpha}} : F_{J \times [1]} \circ \mathbb{D}(\bar{\alpha}) \cong \mathbb{E}(\bar{\alpha}) \circ F_K$, which gives the desired identification upon applying $\text{dia}_J^{\mathbb{E}}$.

With that done, we may write

$$\begin{aligned} (F_J * \mathbb{D}(\alpha)) \circ F_u &= F_J * (\text{dia}_J^{\mathbb{D}} \circ \mathbb{D}(\bar{\alpha})) \circ F_u = \text{dia}_J^{\mathbb{E}} * (F_{J \times [1]} \circ \mathbb{D}(\bar{\alpha})) \circ F_u \\ &= F_v^{-1} \circ \text{dia}_J^{\mathbb{E}}(\mathbb{E}(\bar{\alpha}) \circ F_J) = F_v \circ \mathbb{E}(\alpha) * F_J \end{aligned}$$

as desired. \square

Lemma 3.1.6. *For any morphism $F : \mathbb{D} \rightarrow \mathbb{E}$ of prederivators and any ∞ -category J , the functor $F_{\Delta \downarrow J} : \mathbb{D}(\Delta \downarrow J) \rightarrow \mathbb{E}(\Delta \downarrow J)$ sends the category $\mathbb{D}(\Delta \downarrow J)_{\mathcal{L}_J}$ of diagrams inverting the last-vertex maps into $\mathbb{E}(\Delta \downarrow J)_{\mathcal{L}_J}$.*

Furthermore, if $u : J \rightarrow K$, then $\mathbb{D}(u)$ sends $\mathbb{D}(\Delta \downarrow K)_{\mathcal{L}_K}$ into $\mathbb{D}(\Delta \downarrow J)_{\mathcal{L}_J}$.

Proof. For the first claim, given $X \in \mathbb{D}(\Delta \downarrow J)_{\mathcal{L}_J}$ and $f \in \mathcal{L}_J$, we have by assumption that $\text{dia}(f^*X)$ is an isomorphism. Now by 2-naturality of F , $\text{dia}(f^*F(X))$ is isomorphic as an arrow to $F(\text{dia}(f^*(X)))$, from which the claim follows.

For the second claim, given $X \in \mathbb{D}(\Delta \downarrow K)_{\mathcal{L}_K}$ and $f = (x, a, y) : (x, m) \rightarrow (y, n) \in \mathcal{L}_J$, we want to show

$$\text{dia}f^*(\Delta \downarrow u)^*X : (x, m)^*(\Delta \downarrow u)^*X \rightarrow (y, n)^*(\Delta \downarrow u)^*X$$

is an isomorphism. Under the identifications

$$(x, m)^*(\Delta \downarrow u)^*X = (u \circ x, m)^*X \text{ and } (y, n)^*(\Delta \downarrow u)^*X = (u \circ y, n)^*X,$$

we see $\text{dia}f^*(\Delta \downarrow u)^*$ is identified with $\text{dia}(u \circ x, a, u \circ y)^*X$, which is invertible by assumption. \square

3.2 Semiderivators and ∞ -semiderivators

The strong semiderivator axioms are also detected by restriction from $\underline{\infty} - \mathbf{Cat}$ to \mathbf{Cat} .

Proposition 3.2.1. *If \mathbb{D} is an ∞ -prederivator satisfying $(\text{Der1})'$, then \mathbb{D} satisfies (Der5) , $(\text{Der5})'$, or (Der2) if and only if the restriction \mathbb{D}_1 does.*

Proof. (Der2) : From (Der2) for \mathbb{D}_1 , one knows that a morphism f in $\mathbb{D}(J)$ is an isomorphism if and only if $\text{dia}(\ell_J^*f)$ is an isomorphism in $\mathbb{D}([0]^{\Delta \downarrow J})$. However, since ℓ_J^* factors through $\mathbb{D}(\Delta \downarrow J)_{\mathcal{L}_J}$, for $\text{dia}(\ell_J^*f)$ to be an isomorphism it is sufficient that it be an isomorphism on the 0-simplices of J , which is simply the condition that f itself be an isomorphism on the objects of J .

(Der5'): Consider $B \xleftarrow{f} A \xrightarrow{g} C$ and a weak cocomma $P = B \overleftarrow{\sqcup}_A C$ in $\underline{\infty - \mathbf{Cat}}$. Consider furthermore the weak cocomma

$$P' = (\Delta \downarrow B) \overleftarrow{\sqcup}_{\Delta \downarrow A} (\Delta \downarrow C).$$

We claim the induced projection $P' \rightarrow P$ is a localization at the image of $\mathcal{L}_B \sqcup \mathcal{L}_C$ in P' . Since $\Delta \downarrow B$ and $\Delta \downarrow C$ embed in P' , we will abuse notation by writing $\mathcal{L}_B \sqcup \mathcal{L}_C$ for this image.

We have, for any Q , $Q^P \simeq Q^B \overleftarrow{\times}_{Q^A} Q^C$. Now by Lemma 3.2.3, the functor $Q^P \rightarrow Q^{P'} \simeq Q^{\Delta \downarrow B} \overleftarrow{\times}_{Q^{\Delta \downarrow A}} Q^{\Delta \downarrow C}$ is fully faithful, with image

$$Q_{\mathcal{L}_B}^{\Delta \downarrow B} \overleftarrow{\times}_{Q^{\Delta \downarrow A}} Q_{\mathcal{L}_C}^{\Delta \downarrow C} \simeq \left(Q^{\Delta \downarrow B} \overleftarrow{\times}_{Q^{\Delta \downarrow A}} Q^{\Delta \downarrow C} \right)_{\mathcal{L}_B \sqcup \mathcal{L}_C},$$

as desired.

Now to show that the natural functor $\text{dia}_P : \mathbb{D}(P) \rightarrow \mathbb{D}(B) \overleftarrow{\times}_{\mathbb{D}(A)} \mathbb{D}(C)$ is full and essentially surjective, we consider the square

$$\begin{array}{ccc} \mathbb{D}(P) & \xrightarrow{\text{dia}_P} & \mathbb{D}(B) \overleftarrow{\times}_{\mathbb{D}(A)} \mathbb{D}(C) \\ \downarrow & & \downarrow \\ \mathbb{D}(P') & \xrightarrow{\text{dia}_{P'}} & \mathbb{D}(\Delta \downarrow B) \overleftarrow{\times}_{\mathbb{D}(\Delta \downarrow A)} \mathbb{D}(\Delta \downarrow C) \end{array}$$

in which the vertical arrows are fully faithful. Furthermore, the lower arrow is full and essentially surjective, by (Der5)' for \mathbb{D}_1 . Since dia_P factors through a full functor, it is full.

For essential surjectivity, given

$$(X, Y, t : f^* X \leftarrow g^* Y) \in \mathbb{D}(A) \overleftarrow{\times}_{\mathbb{D}(B)} \mathbb{D}(C),$$

by (Der5)' for \mathbb{D}_1 there exists $T \in \mathbb{D}(P')$ with $\text{dia}_{P'}(T) \cong (\ell_B^* X, \ell_C^* Y, \ell_A^* t)$. Furthermore, T is in $\mathbb{D}(P')_{\mathcal{L}_A \sqcup \mathcal{L}_B}$, since if we denote i_B the inclusion $\Delta \downarrow B \rightarrow P'$ and similarly $i_C : \Delta \downarrow C \rightarrow P'$, we have $i_B^* T = \ell_B^* X \in \mathbb{D}(\Delta \downarrow B)_{\mathcal{L}_B}$ and $i_C^* T = \ell_C^* Y \in \mathbb{D}(\Delta \downarrow C)_{\mathcal{L}_C}$. Thus dia_P is essentially surjective, as desired.

It is clear that we can deduce (Der5) for \mathbb{D} from (Der5) for \mathbb{D}_1 as above. \square

Altogether, treating semiderivators as a full and locally full sub-2-category of prederivators, we have

Theorem 3.2.2. *Restriction of **Dia** from $\infty - \mathbf{Cat}$ to \mathbf{Cat} induces an equivalence of 2-categories between ∞ -semiderivators satisfying $(Der1)'$ and semiderivators satisfying $(Der1)'$, and also between strong ∞ -prederivators satisfying $(Der1)'$ and strong prederivators satisfying $(Der1)'$.*

We have used the following lemma above:

Lemma 3.2.3. *Suppose given cospans $X \xrightarrow{f} Z \xleftarrow{g} Y$, $X' \xrightarrow{f'} Z' \xleftarrow{g'} Y'$ of ∞ -categories, together with a morphism of cospans (strictly natural) with fully faithful components $X \xrightarrow{a} X'$, $Y \xrightarrow{b} Y'$, $Z \xrightarrow{c} Z'$. Then the induced morphism $a \overleftarrow{\times}_c b : X \overleftarrow{\times}_Z Y \rightarrow X' \overleftarrow{\times}_{Z'} Y'$ is also fully faithful.*

Furthermore, an object

$$(x', y', t' : f'(x') \leftarrow g'(y'))$$

of $X' \overleftarrow{\times}_{Z'} Y'$ is in the image of $X \overleftarrow{\times}_Z Y$ if and only if x' is in the image of X and y' is in the image of Y .

Proof. Given (x_1, y_1, t_1) and (x_2, y_2, t_2) in $X \overleftarrow{\times}_Z Y$, the mapping space $X \overleftarrow{\times}_Z Y((x_1, y_1, t_1), (x_2, y_2, t_2))$ may be modeled by the simplicial set

$$X(x_1, x_2) \times_{T(f(x_1), f(x_2))} Z^{\Delta^1}(t_1, t_2) \times_{T(g(y_1), g(y_2))} Y(y_1, y_2).$$

This follows from so that $(X \overleftarrow{\times}_Z Y)^{[1]} \simeq X^{[1]} \overleftarrow{\times}_{Z^{[1]}} Y^{[1]}$. The desired mapping space is the pullback of $(X \overleftarrow{\times}_Z Y)^{[1]}$ along the map

$$((x_1, y_1, t_1), (x_2, y_2, t_2)) : [0] \rightarrow (X \overleftarrow{\times}_Z Y) \times (X \overleftarrow{\times}_Z Y),$$

which proves the given formula upon identifying $(X \overleftarrow{\times}_Z Y)^{[1]}$ with $X^{[1]} \overleftarrow{\times}_{Z^{[1]}} Y^{[1]}$, using the fact that right 2-adjoints preserve weak limits, which is proved just as for ordinary limits.

Since $Z^{\Delta^1} \rightarrow (Z')^{\Delta^1}$ is fully faithful when $Z \rightarrow Z'$ is (see Proposition 2.2.7), we see that the induced functor $X \overleftarrow{\times}_Z Y \rightarrow X' \overleftarrow{\times}_{Z'} Y'$ is fully faithful as claimed. Given $(a(x), b(y), t' :$

$f'(a(x)) \leftarrow g'(b(y)) \in X' \overset{\leftarrow}{\times}_{Z'} Y'$, we see that $t' = c(t)$ for an essentially unique t since c is full, so $(a(x), b(y), t')$ is in the image of $a \overset{\leftarrow}{\times}_c b$. The converse containment on images is clear from the definition of $a \overset{\leftarrow}{\times}_c b$. \square

3.3 The derivator of spaces

Recall that by a left ∞ -derivator we mean a left derivator with **Dia** = $\underline{\infty - \mathbf{Cat}}$. By $\underline{\infty - \mathbf{Der}}$, we denote the 2-category of left ∞ -derivators, cocontinuous morphisms, and arbitrary modifications.

Proposition 3.3.1. *If \mathbb{D} and \mathbb{D}' are left ∞ -derivators satisfying $(\text{Der1})'$, then the restriction functor $\underline{\infty - \mathbf{Der}}(\mathbb{D}, \mathbb{D}') \rightarrow \underline{\mathbf{Der}}(\mathbb{D}_1, \mathbb{D}'_1)$ is an equivalence.*

Proof. By Theorem 3.1.3, we have only to show essential surjectivity. That is, given $F : \mathbb{D} \rightarrow \mathbb{D}'$ such that $F_1 : \mathbb{D}_1 \rightarrow \mathbb{D}'_1$ is cocontinuous, we must show that F is cocontinuous. It suffices to show that F preserves colimits indexed by any $J \in \underline{\infty - \mathbf{Cat}}$. Here by colimits we intend the functors $(p_J)_! : \mathbb{D}(J) \rightarrow \mathbb{D}([0])$ left adjoint to p_J^* , where $p_J : J \rightarrow [0]$ is the unique morphism. Indeed, this follows from (Der4) and (Der2), since for any $u : J \rightarrow K$ we may rewrite $j^*u_!X$ in terms of a colimit.

We verify that F preserves colimits as follows:

$$F((p_J)_!X) \cong F((p_{\Delta \downarrow J})_!\ell_J^*X) \cong (p_{\Delta \downarrow J})_!\ell_J^*F(X) \cong (p_J)_!F(X)$$

using (Der1)'. \square

While the semiderivator axioms were addressed above, it is not as clear how to handle (Der3) and (Der4) under restriction, to address the question of essential surjectivity of the forgetful functor from ∞ -derivators to derivators. Our solution appears in Definition 3.3.9 and below. First we recall and extend some details of Cisinski's work in [Cis08] on the characterization of the free left derivator on the prederivator represented by a category.

The universal derivator

Let $\mathbf{Dia} = \mathbf{Cat}$ until further notice, and let $\mathcal{H}ot$ denote the derivator sending a category A to the localization of the category \mathbf{Cat}^A at those natural transformations whose nerves are levelwise weak equivalences (in the Kan-Quillen sense) of simplicial sets. By [Cis08], $\mathcal{H}ot$ is a derivator.

Definition 3.3.2. If $F : A^{\text{op}} \rightarrow \mathbf{Cat}$ is any functor, then we write $\int_A F$ for the Grothendieck construction of F , which comes equipped with the natural Grothendieck fibration $p : \int_A F \rightarrow A$.

Similarly, if $G : B \rightarrow \mathbf{Cat}$, then by $q : \nabla_B G \rightarrow B$ we denote the Grothendieck *opfibration* associated to G .

For convenient reference, we recall that $\int_A F$ has as objects the pairs $(a \in A, x \in F(a))$ and morphisms $(f, u) : (a, x) \rightarrow (a', x')$ given by maps $u : x \rightarrow F(f)(x')$ in $F(a)$. The Cartesian morphisms are those for which u is an identity. Similarly, the opfibration q has domain with objects $(b \in B, x \in G(b))$ and morphisms $(g, u) : (b, x) \rightarrow (b', x')$ given by $g : b \rightarrow b'$ and $u : G(g)(x) \rightarrow x'$. In other words, $\nabla_B G = (\int_{B^{\text{op}}} G)^{\text{op}}$, where on the right-hand side we view the domain of F as $(B^{\text{op}})^{\text{op}}$.

Furthermore, if $\alpha : F \Rightarrow F' : A^{\text{op}} \rightarrow \mathbf{Cat}$, we have an induced Cartesian functor over A , $\int_A \alpha : \int_A F \rightarrow \int_A F'$, given by $\int_A \alpha(a, x) = (a, \alpha_a(x))$ and $\int_A \alpha(f, u) = (f, \alpha_a(u))$, and similarly for $\beta : G \rightarrow G' : B \rightarrow \mathbf{Cat}$. Finally, the fiber $(\int_A F)_a$ is clearly isomorphic to $F(a)$.

Let us note that, if $\Delta_A : B \rightarrow \mathbf{Cat}$ denotes the constant functor valued at A , we have $\nabla_B \Delta_A \cong B \times A$, with $q : \nabla_B \Delta_A \rightarrow B$ identified with the product projection. Similarly, considering $\Delta_B : A^{\text{op}} \rightarrow \mathbf{Cat}$, we have $\int_A \Delta_B = A \times B$, with $p : \int_A \Delta_B \rightarrow A$ identified with the projection.

Definition 3.3.3. If $F : A^{\text{op}} \times B \rightarrow \mathbf{Cat}$ is any functor, then we have $\int_A F : B \rightarrow \mathbf{Cat}$ sending $b \mapsto \int_A F(-, b)$, together with a natural levelwise Grothendieck fibration $p : \int_A F \rightarrow \Delta_A$. Applying ∇_B gives a bifibration $(p, q) : \nabla_B \int_A F \rightarrow A \times B$. Similarly, we can produce $(p, q) : \int_A \nabla_B F \rightarrow A \times B$.

Proposition 3.3.4. *We have a canonical isomorphism $\nabla_B \int_A F \rightarrow \int_A \nabla_B F$ over $A \times B$.*

Proof. The objects of $\nabla_B \int_A F$ are given by a pair $(b, x \in \int_A F(b))$, which can be identified with triples $(b, a, y \in F(a, b))$. The morphisms in $\nabla_B \int_A F$ between (b, x) and (b', x') are given by $g : b \rightarrow b'$ together with $u : x \rightarrow \int_A F(g)(x')$ in $\int_A F(-, b)$, so that if $x = (a, y \in F(a, b))$ and $x' = (a', y' \in F(a', b'))$, u is given by $f : a \rightarrow a'$ and $v : y \rightarrow F(f, b)(F(a', g)(y')) = F(f, g)(y')$. Under the identification above on objects, the morphisms $(b, x) \rightarrow (b', x')$ are thus identified with triples $(f, g, v) : (a, b, y) \rightarrow (a', b', y')$ given by $v : y \rightarrow F(f, g)(y')$. Dually, $\int_A \nabla_B F$ is isomorphic to the latter category, as desired. \square

Theorem 3.3.5 (Cisinski). *Let \mathbb{D} be a left derivator on \mathbf{Cat} , $B \in \mathbf{Cat}$. Then we have an equivalence of categories $\mathbf{Der}_!(\mathcal{H}ot^{B^{op}}, \mathbb{D}) \rightarrow \mathbb{D}(B)$ given by evaluation at the Yoneda embedding $yo_B \in \mathcal{H}ot^{B^{op}}(B)$.*

Proof. (Sketch:) To define a quasi-inverse of evaluation at yo_B , fix $X \in \mathbb{D}(B)$. Given an object $F : A \times B^{op} \rightarrow \mathbf{Cat}$ of $\mathcal{H}ot^{B^{op}}(A)$, define $T_X(F)$ in $\mathbb{D}(A)$ as $p!q^*X$, where $(p, q) : \int_A \nabla_B F \rightarrow A \times B$ is the canonical bifibration. Cisinski shows that T_X gives in this way a well-defined, cocontinuous morphism with $T_X yo_B = X$, essentially unique with this property, and functorial in F . \square

Cisinski's theorem as an identification of prederivators

We now give an extension of Cisinski's theorem for later applications.

Definition 3.3.6. We define a 2-functor $\mathbb{Y} : \mathbf{Dia}^{coop} \rightarrow \mathbf{LDer}_{\text{Radj}}$ (see Definition 2.3.4) as follows:

- If $J \in \mathbf{Dia}$, then $\mathbb{Y}(J) = \mathcal{H}ot^{J^{op}}$
- If $u : J \rightarrow K$, then $\mathbb{Y}(u)_L$ is the functor, strictly 2-natural in L , given by

$$(u^{op} \times id_L)^* : \mathcal{H}ot(K^{op} \times L) \rightarrow \mathcal{H}ot(J^{op} \times L)$$

- If $\alpha : u \Rightarrow v : J \rightarrow K$, then $\mathbb{Y}(\alpha)_L$ is given by

$$(\alpha^{op} \times id_L)^* : (v^{op} \times id_L)^* \Rightarrow (u^{op} \times id_L)^*.$$

The identity-on-objects equivalence $\underline{\mathbf{LDer}}_{\text{Radj}}^{\text{coop}} \rightarrow \underline{\mathbf{LDer}}_{\text{Ladj}}$ given on morphisms by taking the left adjoint and on 2-morphisms by taking the mate then yields a pseudo-functor $\mathbb{Y}' : \mathbf{Dia} \rightarrow \underline{\mathbf{LDer}}_{\text{Ladj}}$ mapping $u \mapsto u_!^{\text{op}}$ and $\alpha \mapsto \alpha_!^{\text{op}}$.

Proposition 3.3.7. *Consider a left derivator \mathbb{D} and the prederivator $\underline{\mathbf{LDer}}_!(\mathcal{H}\text{ot}^{(-)\text{op}}, \mathbb{D})$ defined as the composite*

$$\underline{\mathbf{Cat}}^{\text{op}} \xrightarrow{(\mathbb{Y}')^{\text{op}}} \underline{\mathbf{LDer}}_!^{\text{op}} \xrightarrow{\underline{\mathbf{LDer}}_!((-), \mathbb{D})} \underline{\mathbf{CAT}}.$$

Evaluation at $\text{yo}_{(-)}$ gives an equivalence of prederivators $\underline{\mathbf{LDer}}_!(\mathcal{H}\text{ot}^{(-)\text{op}}, \mathbb{D}) \rightarrow \mathbb{D}$.

Proof. Given Theorem 3.3.5, it remains only to show that evaluation at $\text{yo}_{(-)}$ gives a pseudo-natural transformation between the given prederivators. Given any cocontinuous $T : \mathcal{H}\text{ot}^{B^{\text{op}}} \rightarrow \mathbb{D}$ and $u : A \rightarrow B$, we have $\mathbb{D}(u)(T_B(\text{yo}_B)) = T_A(\mathcal{H}\text{ot}^{B^{\text{op}}}(u)(\text{yo}_B))$, which we must show coincides with $(T \circ u_!^{\text{op}})_A(\text{yo}_A) = T_A((u_!^{\text{op}})_A(\text{yo}_A))$.

Thus it suffices to give isomorphisms $\mathcal{H}\text{ot}^{B^{\text{op}}}(u)(\text{yo}_B) \cong (u_!^{\text{op}})_A(\text{yo}_A)$. The left-hand side evaluates to the functor $B(-, u) : B^{\text{op}} \times A \rightarrow \mathbf{Set} \subset \mathbf{Cat}$.

As for the right-hand side, the component $(u_!^{\text{op}})_A : \mathcal{H}\text{ot}(A^{\text{op}} \times A) \rightarrow \mathcal{H}\text{ot}(B^{\text{op}} \times A)$ is simply $(u^{\text{op}} \times \text{id}_A)_!$, which by [Cis08, 1.9,1.13] is represented by the composite

$$\begin{aligned} \mathbf{Cat}(A^{\text{op}} \times A, \mathbf{Cat}) &\cong \mathbf{Cat}(A, \mathbf{Cat}(A^{\text{op}}, \mathbf{Cat})) \xrightarrow{f_A} \mathbf{Cat}(A, \mathbf{Cat} \downarrow A) \\ &\xrightarrow{u^{\text{op}}(-)} \mathbf{Cat}(A, \mathbf{Cat} \downarrow B) \xrightarrow{\varphi^{\text{op}}(-)} \mathbf{Cat}(A, \mathbf{Cat}(B^{\text{op}}, \mathbf{Cat})) \cong \mathbf{Cat}(B^{\text{op}} \times A, \mathbf{Cat}). \end{aligned}$$

Here $\varphi : \mathbf{Cat} \downarrow B \rightarrow \mathbf{Cat}(B^{\text{op}}, \mathbf{Cat})$ sends $F : C \rightarrow B$ to the functor $b \mapsto b \downarrow C$.

In the case of $\text{yo}_A : A^{\text{op}} \times A \rightarrow \mathbf{Cat}$, we have $\int_A \text{yo}_A(a) = A \downarrow a$, which lands in $\mathbf{Cat}(A, \mathbf{Cat} \downarrow B)$ on the functor

$$a \mapsto (A \downarrow a \rightarrow B, f : a' \rightarrow a \mapsto u(a')).$$

Now post-composing with φ , we get the functor $(a, b) \mapsto b \downarrow (A \downarrow a)$ which has objects (g, f) with $g : b \rightarrow u(a')$ and $f : a' \rightarrow a$. The projection $A \downarrow a \rightarrow *$ admitting the right adjoint, $\text{id}_A : * \rightarrow A \downarrow a$, we have also a natural right adjoint to the projection $b \downarrow (A \downarrow a) \rightarrow B(b, u(a))$. This produces a natural Thomason equivalence $(u^{\text{op}} \times \text{id}_A)_!(\text{yo}_A) \cong (\text{id}_{B^{\text{op}}} \times u)^*(\text{yo}_B)$, as desired. \square

We now derive another extension of Cisinski's theorem, in the ∞ -context.

Let now $\mathbf{Dia} = \infty - \mathbf{Cat}$, and let $\mathcal{H}ot$ denote the ∞ -derivator of spaces, as constructed for instance in 4.4.8 and below of [Cis19]. The particular model given there lets $\mathcal{H}ot(A)$ be the category of right fibrations over A and homotopy classes of morphisms; if A is a category then Thomason's theorem shows that $\mathcal{H}ot_1$, in this sense, coincides with $\mathcal{H}ot$ viewed as an ordinary derivator above.

Corollary 3.3.8. *In this context, $\mathcal{H}ot$ is again the free cocompletion of a point, at least among left ∞ -derivators satisfying (Der1)'.*

Indeed, every such left ∞ -derivator \mathbb{D} is equivalent to the prederivator $\infty - \mathbf{LDer}_1(\mathcal{H}ot^{(-)^{op}}, \mathbb{D})$, as for ordinary derivators.

Proof. Immediately from Proposition 3.3.1 together with Proposition 3.3.7 we conclude that \mathbb{D}_1 coincides with the restriction of the desired prederivator. From Theorem 3.1.3, we conclude the same for \mathbb{D} . □

Homotopically locally small derivators

We now consider the essential surjectivity of the restriction 2-functor

$$(-)_1 : \infty - \mathbf{Der} \rightarrow \mathbf{Der}.$$

We shall have to somewhat strengthen our notion of derivator to proceed:

Definition 3.3.9. We say that a derivator (or ∞ -derivator) \mathbb{D} is *homotopically locally small* if, equivalently:

1. Every cocontinuous morphism $F : \mathcal{H}ot^J \rightarrow \mathbb{D}$, with J any small category, admits a right adjoint.
2. Every cocontinuous morphism $F : \mathcal{H}ot^J \rightarrow \mathbb{D}$, with J any small ∞ -category, admits a right adjoint.
3. Every cocontinuous morphism $F : \mathbb{D}' \rightarrow \mathbb{D}$, where \mathbb{D}' is any reflective subderivator of $\mathcal{H}ot^J$ for some ∞ -category J , admits a right adjoint.

4. Every cocontinuous morphism $F : \mathrm{HO}(\mathcal{M}) \rightarrow \mathbb{D}$, where \mathcal{M} is a combinatorial model category, admits a right adjoint.

Let us prove the equivalence of the clauses in the definition:

Proof. For (4) \implies (1), we need only note that $\mathcal{H}\mathrm{ot}^J \simeq \mathrm{HO}(\mathcal{M})$ for \mathcal{M} the projective model structure on $\mathbf{S}\mathrm{Set}^J$, as shown in [Lur09, 2.2.1.2]. For (3) \implies (4), we observe that for such a $\mathrm{HO}(\mathcal{M})$ we may assume that \mathcal{M} arises from a left Bousfield localization of a projective model structure on some category of simplicial presheaves $\mathbf{S}\mathrm{Set}^J$, so that $\mathrm{HO}(\mathcal{M})$ is reflective in $\mathrm{HO}(\mathbf{S}\mathrm{Set}^J)$.

That (2) \implies (3) is verified directly: the right adjoint of F is the right adjoint of $F \circ L$, $L : \mathbf{H}\mathrm{ot}^J \rightarrow \mathbb{D}'$ being left adjoint to a fully faithful inclusion. The usual argument, applied levelwise, implies that such a right adjoint takes values in the essential image of \mathbb{D}' . For (1) \implies (2), we apply the previous argument, noting that $\mathcal{H}\mathrm{ot}^J$ is reflective in $\mathcal{H}\mathrm{ot}^{\Delta \downarrow J}$. \square

Remark 3.3.10. It will follow from later chapters' work on the embedding of ∞ -categories in prederivators that the derivator associated to any cocomplete, locally small ∞ -category—in particular that associated to *any* model category—is homotopically locally small. Thus homotopical local smallness is strictly more general than arising as a localization of $\mathcal{H}\mathrm{ot}^J$ for some J .

To be clear, we have no examples of derivators valued in locally small categories which are not homotopically locally small, though it seems likely that examples exist, analogous to Heller's example of a non-representable cohomological functor $\mathbf{H}\mathrm{ot} \rightarrow \mathbf{S}\mathrm{et}$.¹

To justify the terminology "homotopically locally small," recall Freyd's special adjoint functor theorem: a cocontinuous functor out of a sufficiently nice category C , for instance a reflective subcategory of a presheaf category, admits a right adjoint as soon as the codomain is locally small. Thus the analogy at hand is between derivators $\mathcal{H}\mathrm{ot}^J$ and presheaf categories.

We can justify the terminology more precisely as follows.

¹The obstacle to constructing an example straightforwardly relying on Heller's lies in producing an example of a large space Ω such that not only $\mathbf{H}\mathrm{OT}(X, \Omega)$ is small for all small spaces X , but also $\mathbf{H}\mathrm{OT}(\Omega, \Omega)$ is small.

Definition 3.3.11. Given a left derivator \mathbb{D} and $X \in \mathbb{D}(J)$, denote the induced morphism $\mathcal{H}ot^{J^{op}} \rightarrow \mathbb{D}$ by $(-) \otimes X$. For each K , we thus get bifunctors $\otimes : \mathbb{D}(J) \times \mathcal{H}ot^{J^{op}}(K) \rightarrow \mathbb{D}(K)$.

We say that a derivator \mathbb{D} is *enriched over $\mathcal{H}ot$* if, for each J and K in **Dia**, the tensoring bifunctor above admits a right adjoint in its second variable, denoted $(-)^{(-)} : \mathbb{D}(J)^{op} \times \mathbb{D}(K) \rightarrow \mathcal{H}ot^{J^{op}}(K)$. Concretely, for $X \in \mathbb{D}(J)$, $Y \in \mathbb{D}(K)$, and $Z \in \mathcal{H}ot^{J^{op}}(K)$, we get natural isomorphisms $\mathbb{D}(K)(Z \otimes X, Y) \cong \mathcal{H}ot^{J^{op}}(K)(Z, Y^X)$.

Note that we cannot necessarily expect the tensoring functors to be left adjoints in both variables for \mathbb{D} merely a left derivator, as that would correspond to \mathbb{D} being cotensored over $\mathcal{H}ot$.

Proposition 3.3.12. *A derivator \mathbb{D} is homotopically locally small if and only if it is enriched over $\mathcal{H}ot$ in the sense above. In particular, each category $\mathbb{D}(J)$ of a homotopically locally small derivator is enriched over the homotopy category $\mathbf{Hot} = \mathcal{H}ot([0])$ of spaces. (See Definition 2.2.4)*

Proof. This follows immediately from the fact that any $X \in \mathbb{D}(J)$, the morphism $(-) \otimes X : \mathcal{H}ot^{J^{op}} \rightarrow \mathbb{D}$ has a right adjoint if and only if its components $(-) \otimes X : \mathcal{H}ot(J^{op} \times K) \rightarrow \mathbb{D}(K)$ do. See [Gro13, 2.11].

For the second claim, by shifting, it suffices to consider $J = [0]$, and then the mapping space functor $(-)^x$ is defined as the right adjoint to

$$(-) \otimes x : \mathbf{Hot} \rightarrow \mathbb{D}([0]).$$

□

Remark 3.3.13. Note that the above proposition gives two distinct senses in which a category $\mathbb{D}(J)$ is “enriched over spaces”: via the two-variable right adjoint

$$\mathbb{D}(J)^{op} \times \mathbb{D}(J) \rightarrow \mathcal{H}ot(J \times J^{op})$$

to the tensoring functor

$$\mathcal{H}ot(J \times J^{op}) \times \mathbb{D}(J) \rightarrow \mathbb{D}(J^{op}),$$

and respectively via the right adjoint

$$\mathbb{D}(J)^{\text{op}} \times \mathbb{D}(J) \rightarrow \mathbf{Hot}$$

to the tensoring functor

$$\mathcal{H}\text{ot}([0]) \times \mathbb{D}^J([0]) \rightarrow \mathbb{D}^J([0])$$

induced by shifting. For $X, Y \in \mathbb{D}(J)$, the mapping diagram $Y^X \in \mathbb{D}(J \times J^{\text{op}})$ should be thought of as the functor $(j_1, j_2) \mapsto Y(j_2)^{X(j_1)}$, while the mapping space $Y^X \in \mathbf{Hot}$ should be thought of as the space of natural transformations $X \rightarrow Y$.

If J admits product decompositions other than $J \times [0]$, then other shifts produce yet more intermediate notions of internal homs for $\mathbb{D}(J)$.

In the homotopically locally small case, the identification in Proposition 3.3.7 may have its variance flipped.

Proposition 3.3.14. *Let \mathbb{D} be a homotopically locally small left derivator, and recall the 2-functor $\mathbb{Y} : \mathbf{Dia}^{\text{coop}} \rightarrow \mathbf{LDer}_{\text{Radj}}$ defined in Definition 3.3.6.*

Then we have a natural identification of \mathbb{D} with the prederivator

$$\mathbf{Dia}^{\text{op}} \xrightarrow{\mathbb{Y}^{\text{co}}} \mathbf{LDer}_{\text{Radj}}^{\text{co}} \xrightarrow{(-)^{\text{op}}} \mathbf{RDer}_{\text{Ladj}} \xrightarrow{\mathbf{RDer}_{\text{Ladj}}^{\mathbb{D}^{\text{op}}, (-)}} \mathbf{CAT}.$$

Proof. Recall that in Proposition 3.3.7 we have identified \mathbb{D} with the composite

$$\mathbf{Dia}^{\text{op}} \xrightarrow{(\mathbb{Y}')^{\text{op}}} \mathbf{LDer}_{\text{Ladj}}^{\text{op}} \xrightarrow{\mathbf{LDer}_{\text{Ladj}}^{\text{op}}(\mathbb{D}, (-))} \mathbf{CAT}.$$

Furthermore, we defined $\mathbb{Y}' : \mathbf{Dia} \rightarrow \mathbf{LDer}_{\text{Ladj}}$ as the composite

$$\mathbf{Dia} \xrightarrow{\mathbb{Y}^{\text{coop}}} \mathbf{LDer}_{\text{Radj}}^{\text{coop}} \xrightarrow{L} \mathbf{LDer}_{\text{Ladj}}.$$

Thus $(\mathbb{Y}')^{\text{op}} = L^{\text{op}} \circ \mathbb{Y}^{\text{co}}$. Furthermore, the composite

$$\mathbf{LDer}_{\text{Radj}}^{\text{co}} \xrightarrow{L^{\text{op}}} \mathbf{LDer}_{\text{Ladj}}^{\text{op}} \xrightarrow{\mathbf{LDer}_{\text{Ladj}}^{\text{op}}(\mathbb{D}, (-))} \mathbf{CAT}$$

is naturally identified with $\underline{\mathbf{LDer}}_{\mathbf{Radj}}^{\mathbf{co}}(\mathbb{D}, (-))$, since $L^{\mathbf{op}}$ is an equivalence of 2-categories. Thus \mathbb{D} is identified with

$$\underline{\mathbf{Dia}}^{\mathbf{op}} \xrightarrow{\mathbb{Y}^{\mathbf{co}}} \underline{\mathbf{LDer}}_{\mathbf{Radj}}^{\mathbf{co}} \xrightarrow{\underline{\mathbf{LDer}}_{\mathbf{Radj}}^{\mathbf{co}}(\mathbb{D}, (-))} \underline{\mathbf{CAT}}.$$

We conclude by identifying the representable 2-functor $\underline{\mathbf{LDer}}_{\mathbf{Radj}}^{\mathbf{co}} \xrightarrow{\underline{\mathbf{LDer}}_{\mathbf{Radj}}^{\mathbf{co}}(\mathbb{D}, (-))} \underline{\mathbf{CAT}}$ with the composite

$$\underline{\mathbf{LDer}}_{\mathbf{Radj}}^{\mathbf{co}} \xrightarrow{(-)^{\mathbf{op}}} \underline{\mathbf{RDer}}_{\mathbf{Ladj}} \xrightarrow{\underline{\mathbf{RDer}}_{\mathbf{Ladj}}(\mathbb{D}^{\mathbf{op}}, (-))} \underline{\mathbf{CAT}}$$

making use of the equivalence $(-)^{\mathbf{op}} : \underline{\mathbf{LDer}}_{\mathbf{Radj}}^{\mathbf{co}} \rightarrow \underline{\mathbf{RDer}}_{\mathbf{Ladj}}$ constructed in Proposition 2.3.6. □

We now give the key examples of homotopically locally small derivators using as little information external to derivator theory as possible. Namely, we shall have to use that $\mathcal{H}ot^J$ is both a left and a right derivator, as follows easily from its presentation via a model category but can be derived somewhat more directly as in Cisinski's thesis.

Proposition 3.3.15. *The derivators $\mathcal{H}ot^J$ of J -diagrams of spaces are homotopically locally small.*

Proof. We recall that an object $A \in \mathcal{H}ot(J^{\mathbf{op}} \times K)$ gives rise to the category of elements $\int \nabla A$, which comes equipped with a fibration $\phi_A : \int \nabla A \rightarrow J$ and an opfibration $\omega_A : \int \nabla A \rightarrow K$. We recall also the result from the proof of Proposition 3.3.7 that, if $u : J \rightarrow K$, then one has a natural identification

$$(1_{K^{\mathbf{op}}} \times u)^*(y_{\mathbf{O}_K}) \cong (u^{\mathbf{op}} \times 1_J)_!(y_{\mathbf{O}_J})$$

in $\mathcal{H}ot(K^{\mathbf{op}} \times J)$. This implies the analogous identification

$$\begin{aligned} (1_{J^{\mathbf{op}}} \times u)_!(y_{\mathbf{O}_J}) &= ((u^{\mathbf{op}})^{\mathbf{op}} \times 1_{J^{\mathbf{op}}})_!(y_{\mathbf{O}_{J^{\mathbf{op}}}}) \\ &= (1_{(K^{\mathbf{op}})^{\mathbf{op}}} \times u^{\mathbf{op}})^*(y_{\mathbf{O}_{K^{\mathbf{op}}}}) = (u^{\mathbf{op}} \times 1_K)^*(y_{\mathbf{O}_K}). \end{aligned}$$

A cocontinuous morphism $F : \mathcal{H}ot^{K^{\mathbf{op}}} \rightarrow \mathcal{H}ot^{J^{\mathbf{op}}}$ corresponds to a diagram $A = F(y_{\mathbf{O}_K}) \in \mathcal{H}ot^{J^{\mathbf{op}}}(K) = \mathcal{H}ot(J^{\mathbf{op}} \times K)$. For readability, let us denote $(\omega_A)_!$ by ω_1^A , and similarly for

ϕ . Then we recall from [Cis08, 1.15] that A is naturally identified with $(1_{J^{\text{op}}} \times \omega_1^A) \circ (1_{J^{\text{op}}} \times \phi_A)^* y_{\text{O}_J}$. Similarly, any $X \in \mathcal{H}\text{ot}^{K^{\text{op}}}(L)$ is identified with $(1_{K^{\text{op}}} \times \omega_X)_! \circ (1_{K^{\text{op}}} \times \phi_X)^* y_{\text{O}_K}$. Thus by cocontinuity, we must have

$$\begin{aligned}
F(X) &= (1_{J^{\text{op}}} \times \omega_X)_! \circ (1_{J^{\text{op}}} \times \phi_X)^* A \\
&= (1_{J^{\text{op}}} \times \omega_X)_! \circ (1_{J^{\text{op}}} \times \phi_X)^* (1_{J^{\text{op}}} \times \omega_A)_! \circ (1_{J^{\text{op}}} \times \phi_A)^* y_{\text{O}_J} \\
&= (1_{J^{\text{op}}} \times \omega_X)_! \circ (1_{J^{\text{op}}} \times \phi_X)^* (1_{J^{\text{op}}} \times \omega_A)_! \circ (\phi_A^{\text{op}} \times 1_{f \nabla A})_! y_{\text{O}_J \nabla A} \\
&= (1_{J^{\text{op}}} \times \omega_X)_! \circ (1_{J^{\text{op}}} \times \phi_X)^* (\phi_A^{\text{op}} \times 1_K)_! \circ (1_{f \nabla A^{\text{op}}} \times \omega_A)_! y_{\text{O}_J \nabla A} \\
&= (1_{J^{\text{op}}} \times \omega_X)_! \circ (\phi_A^{\text{op}} \times 1_{f \nabla X})_! (1_{f \nabla A^{\text{op}}} \times \phi_X)^* \circ (1_{f \nabla A^{\text{op}}} \times \omega_A)_! y_{\text{O}_J \nabla A} \\
&= (\phi_A^{\text{op}} \times 1_L)_! (1_{f \nabla A^{\text{op}}} \times \omega_X)_! (1_{f \nabla A^{\text{op}}} \times \phi_X)^* (\omega_A^{\text{op}} \times 1_K)^* y_{\text{O}_K} \\
&= (\phi_A^{\text{op}} \times 1_L)_! (1_{f \nabla A^{\text{op}}} \times \omega_X)_! (\omega_A^{\text{op}} \times 1_{f \nabla X})^* (1_{K^{\text{op}}} \times \phi_X)^* y_{\text{O}_K} \\
&= (\phi_A^{\text{op}} \times 1_L)_! (\omega_A^{\text{op}} \times 1_L)^* (1_{K^{\text{op}}} \times \omega_X)_! (1_{K^{\text{op}}} \times \phi_X)^* y_{\text{O}_K} \\
&= (\phi_A^{\text{op}} \times 1_L)_! (\omega_A^{\text{op}} \times 1_L)^* X
\end{aligned}$$

For checking these computations, we reference the signatures $\phi_A : f \nabla A \rightarrow J, \omega_A : f \nabla A \rightarrow K, \phi_X : f \nabla X \rightarrow K, \omega_X : f \nabla X \rightarrow L$, as well as the shriek-to-star facts recalled above, the pseudofunctoriality of shrieks and 2-functoriality of stars, and the homotopy exactness of product squares.

This being done, it is easily verified that F admits a right adjoint, granted that we are willing to use the fact that $\mathcal{H}\text{ot}^J$ is a right derivator.

Namely, for any Y we have

$$\begin{aligned}
\mathcal{H}\text{ot}^{J^{\text{op}}}(L)(F(X), Y) &= \mathcal{H}\text{ot}(J^{\text{op}} \times L)((\phi_A^{\text{op}} \times 1_L)_! (\omega_A^{\text{op}} \times 1_L)^* X, Y) \\
&= \mathcal{H}\text{ot}(K^{\text{op}} \times L)(X, (\omega_A^{\text{op}} \times 1_L)_* (\phi_A^{\text{op}} \times 1_L)^* Y) \\
&=: \mathcal{H}\text{ot}^{K^{\text{op}}}(L)(X, G(Y)).
\end{aligned}$$

□

Thus in particular, we have confirmed that $\mathcal{H}\text{ot}$ is enriched over itself.

3.4 Extending homotopically locally small derivators

Now we want to construct an ∞ -derivator from a homotopically locally small derivator. We shall use a different construction for the extended derivator than we did above in extending prederivators.

First, we examine the cocontinuity properties of the 2-functor $\mathbb{Y}' : \underline{\infty - \mathbf{Cat}} \rightarrow \underline{\infty - \mathbf{LDer}}_{\mathbf{Ladj}}$ defined in Definition 3.3.6.

Lemma 3.4.1. *The 2-functor $\mathbb{Y}' : \underline{\infty - \mathbf{Cat}} \rightarrow \underline{\mathbf{LDer}}_{\mathbf{Ladj}}$ preserves 2-coproducts and coinverters.*

Proof. To show that some ∞ -derivator \mathbb{D} is a 2-coproduct or a coinverter in $\underline{\mathbf{LDer}}_{\mathbf{Ladj}}$ is to show that it is a 2-product or an inverter in $\underline{\mathbf{LDer}}_{\mathbf{Radj}} \cong \underline{\mathbf{LDer}}_{\mathbf{Ladj}}^{\text{coop}}$. We claim that 2-products and inverters are created by the forgetful 2-functor $\underline{\mathbf{LDer}}_{\mathbf{Radj}} \rightarrow \underline{\mathbf{PDer}}$. Intuitively, this is because the domain is 2-categorically algebraic over the codomain, and indeed the forgetful 2-functor admits a left 2-adjoint.² However, we prefer to verify the desired claims directly in this case, to avoid a detour through 2-categorical universal algebra.

For the case of products, we need only verify that given a prederivator morphism $F : \mathbb{D} \rightarrow \prod \mathbb{E}_i$, where \mathbb{D} and each \mathbb{E}_i are derivators and each component $F_i : \mathbb{D} \rightarrow \mathbb{E}_i$ admits a left adjoint, F itself has a left adjoint. Choosing left adjoints G_i to F_i , we have have

$$\left(\prod \mathbb{E}_i \right) ((X_i), F(Y)) \cong \prod \mathbb{D}(G_i(X_i), Y) \cong \mathbb{D}(\prod G_i(X_i), Y),$$

so that $\prod G_i : \prod \mathbb{E}_i \rightarrow \mathbb{D}$ gives the desired left adjoint.

For inverters, it suffices to show that if $F : \mathbb{D} \rightarrow \mathbb{E}'$ is a morphism of prederivators, \mathbb{D} and \mathbb{E}' are derivators, and $I : \mathbb{E}' \rightarrow \mathbb{E}$ is an inverter in $\underline{\mathbf{PDer}}$ admitting a left adjoint L , then F has a left adjoint when $I \circ F$ does. This needs, in fact, only that I is fully faithful. Indeed, if $I \circ F$ has a left adjoint $G : \mathbb{E} \rightarrow \mathbb{D}$, then $G \circ I$ is left adjoint to F , insofar as for any $J \in \underline{\mathbf{Dia}}$,

$$\mathbb{D}(J)(G \circ I(X), Y) \cong \mathbb{E}(J)(I(X), I(F(Y))) \cong \mathbb{E}'(J)(X, F(Y)).$$

²In particular, the forgetful functor actually preserves all PIE limits.

To show that \mathbb{Y} preserves 2-coproducts, we are thus to show that $\mathcal{H}ot^{\amalg J_i}(K) \simeq \amalg \mathcal{H}ot^{J_i}(K)$, which follows immediately from distributivity of products over coproducts in \mathbf{Cat} and the fact that $\mathcal{H}ot$ satisfies (Der1). Similarly, for coinverters, if

$$\begin{array}{ccc}
 & f & \\
 J & \begin{array}{c} \curvearrowright \\ \parallel \\ \alpha \\ \curvearrowleft \end{array} & K \xrightarrow{q} L \\
 & g &
 \end{array}$$

is a coinverter diagram in $\infty\text{-}\mathbf{Cat}$, then we have to show that for every M , the following diagram is an equifier in \mathbf{Cat} :

$$\begin{array}{ccc}
 & g^* & \\
 \mathcal{H}ot^J(M) & \begin{array}{c} \curvearrowright \\ \parallel \\ \alpha^* \\ \curvearrowleft \end{array} & \mathcal{H}ot^K(M) \xleftarrow{q^*} \mathcal{H}ot^L(M) \\
 & f^* &
 \end{array}$$

This is, similarly, immediate from the facts that the 2-functor $(-) \times M : \mathbf{Cat} \rightarrow \mathbf{Cat}$ preserves coinverters and that $\mathcal{H}ot$ satisfies (Der1)".

□

(Der1)' and (Der1)" for derivators

In the homotopically locally small context, the above implies that (Der1)" is redundant.

Corollary 3.4.2. *Let \mathbb{D} be a homotopically locally small left derivator. Then \mathbb{D} satisfies (Der1)".*

Proof. This follows immediately from Lemma 3.4.1, since $\mathbb{D}(J)$ is identified with $\mathbf{Dia} - \mathbf{Der}_{\text{Ladj}}(\mathcal{H}ot^{J^{\text{op}}}, \mathbb{D})$. □

It is possible to show that *any* left derivator whatsoever satisfies (Der1)'. Though as mentioned we have no examples of non-homotopically small left derivators, we include the argument to demonstrate a more elementary approach.

Proposition 3.4.3. *Every left derivator satisfies (Der1)'.*

Proof. Given $\ell_J : \Delta \downarrow J \rightarrow J$, fix $j \in J$ and denote by A the comma category $\{j\} \overset{\leftarrow}{\times}_J \Delta \downarrow J$ and by $p : A \rightarrow \Delta \downarrow J$ the canonical projection. We have also a functor $\{j\} \overset{\leftarrow}{\times}_J \ell_J : A \rightarrow \{j\} \overset{\leftarrow}{\times}_J J$, whose codomain has the terminal object id_j . Let B denote the (non-full) subcategory of the fiber A_{id_j} allowing only those morphisms which, when projected to Δ , fix the last object. Lemma 3.4.5 claims that $i : B \rightarrow A$ admits a left adjoint u .

To compute the functor colim_A in \mathbb{D} , the functoriality of left adjoints implies that we may as well compute $\text{colim}_B \circ u_!$. The 2-functoriality of \mathbb{D} implies that $u_!$ is identified with i^* . Then (Der4) implies that, for any $X \in \mathbb{D}(J)$, we may calculate $j^*(\ell_J)_! \ell_J^* X$ as $\text{colim}_B i^* p^* \ell_J^* X = \text{colim}_B (\ell_J p i)^* X$. Now the functor $\ell_J p i : B \rightarrow J$ is constant at j , so equivalently we are to compute $\text{colim}_B \pi_B^* j^* X$, with $\pi_B : B \rightarrow [0]$ the terminal morphism.

Finally, since B has an initial object $((j), (j \xrightarrow{\text{id}_j} j))$, it is contractible and the latter colimit is simply $j^* X$. Thus the counit of $(\ell_J)_! \dashv \ell_J^*$ is an isomorphism. The same argument shows that the unit is an isomorphism on $\mathbb{D}(\ell_J)_{\mathcal{L}_J}$, which gives the result. \square

We note that an in-principle similar argument should work for left ∞ -derivators, but that the direct technical manipulation of A and B would be less possible. That said, we are happy to conjecture it. Furthermore, the fact that a free left derivator satisfies (Der1)" strongly suggests that every left derivator does so, though we have only been able to show this for the homotopically locally small case. The obstruction to an argument similar to the above for an arbitrary localization $J \rightarrow J[W^{-1}]$ is in the ability to write colimits over the weak fiber in terms of colimits over the strict fiber. This is most easily done, as above, in the case of proper functors ([Cis19]), a condition not every localization satisfies.

Conjecture 3.4.4. *Every left ∞ -derivator \mathbb{D} satisfies (Der1)'. Every left derivator or left ∞ -derivator satisfies (Der1)".*

Lemma 3.4.5. *If $i : B \rightarrow A$ is as in the first paragraph of the proof of Proposition 3.4.3, then i admits a left adjoint u .*

Proof. Consider an object $x = ((x_0 \rightarrow \dots \rightarrow x_n), (x_n \rightarrow j))$ of A . We propose the value $u(x) = ((x_0 \rightarrow \dots \rightarrow x_n \rightarrow j), (j \xrightarrow{\text{id}_j} j))$ in B . We abusively identify a map in A with its

image in Δ under the faithful projection $A \rightarrow \Delta$. Then given $\alpha : x \rightarrow x'$, with x' a k -simplex, the simplicial operation $[n + 1] \rightarrow [k + 1]$ for $u(\alpha)$ is defined to restrict to α and map $n + 1$ to $k + 1$, as it must to land in B .

The unit $\eta_x : x \rightarrow iu(x)$ is determined by the face map

$$d_{n+1} : (x_0 \rightarrow \dots \rightarrow x_n) \rightarrow (x_0 \rightarrow \dots \rightarrow x_n \rightarrow j).$$

Given $\alpha : x \rightarrow x'$, the map $\eta_{x'} \circ \alpha$ is given by $d_{k+1} \circ \alpha$, the corestriction of α to $[k + 1]$, while the map $iu(\alpha) \circ \eta_x$ is the restriction of $iu(\alpha)$ to $[n]$, which coincides with α by definition. Thus η is natural.

Given $y = ((y_0 \rightarrow \dots \rightarrow y_m \rightarrow j), (j \xrightarrow{\text{id}_j} j))$ in B , we have $ui(y) = ((y_0 \rightarrow \dots \rightarrow y_m \rightarrow j \xrightarrow{\text{id}_j} j), (j \xrightarrow{\text{id}_j} j))$, and the counit $\varepsilon_y : ui(y) \rightarrow y$ is determined by the degeneracy map $s_{m+1} : (y_0 \rightarrow \dots \rightarrow y_m \rightarrow j \xrightarrow{\text{id}_j} j) \rightarrow (y_0 \rightarrow \dots \rightarrow y_m \rightarrow j)$, which we note is in B . Given $y' = ((y_0 \rightarrow \dots \rightarrow y_p \rightarrow j), (j \xrightarrow{\text{id}_j} j))$ and a map $\beta : y \rightarrow y'$ in B , we see $\beta \circ \varepsilon_y$ is determined by the simplicial map $[m + 2] \rightarrow [p + 1]$ which agrees with β on $[m + 1]$ and maps $m + 2$ to $\beta(m + 1)$. On the other hand $ui(\beta) : [m + 2] \rightarrow [p + 2]$ agrees with β on $[m + 1]$ and sends $m + 2$ to $p + 2$, so $\varepsilon_{y'} \circ ui(\beta)$ agrees with β on $[m + 1]$ and sends $m + 2$ to $p + 1$. Since $\beta \in B$, we have $\beta(m + 1) = p + 1$, so ε is natural, which was the reason for taking B non-full.

Turning finally to the triangle identities, the composite

$$i(y) \xrightarrow{\eta_{i(y)}} iui(y) \xrightarrow{i\varepsilon_y} i(y)$$

is determined by the composite

$$[m + 1] \xrightarrow{d_{m+2}} [m + 2] \xrightarrow{s_{m+1}} [m + 1]$$

in Δ , which is the identity of $[m + 1]$. Similarly, the composite

$$u(x) \xrightarrow{u\eta_x} uui(x) \xrightarrow{\varepsilon_{u(x)}} u(x)$$

is determined by the composite

$$[n + 1] \xrightarrow{d_{n+1}} [n + 2] \xrightarrow{s_{n+1}} [n + 1]$$

in Δ , which is again the identity of $[n + 1]$. □

The extension theorem

We can now give a rather formal proof that homotopically locally small derivators extend to ∞ -derivators.

Proposition 3.4.6. *Let \mathbb{D} be a homotopically locally small left derivator. Define the ∞ -prederivator \mathbb{D}' as the composite*

$$\underline{\infty - \mathbf{Cat}}^{\text{op}} \xrightarrow{\mathbb{Y}^{\text{op}}} \underline{\mathbf{Der}}_{\text{Ladj}}^{\text{op}} \xrightarrow{\underline{\mathbf{Der}}_{\text{Ladj}}^{(-, \mathbb{D})}} \underline{\mathbf{Cat}}.$$

So, in particular, we have $\mathbb{D}'(J) = \underline{\mathbf{Der}}_{\text{Ladj}}(\mathcal{H}\text{ot}^{J^{\text{op}}}, \mathbb{D})$. Then \mathbb{D}' is a localizing left ∞ -derivator equipped with a natural equivalence $\mathbb{D}'_1 \simeq \mathbb{D}$.

Proof. That \mathbb{D}' , defined in this way, satisfies (Der1)'' is immediate from the fact that \mathbb{Y} and $\underline{\mathbf{Der}}_{\text{Ladj}}(-, \mathbb{D})$ preserve 2-coproducts and coinverters (see Lemma 3.4.1). Furthermore, $\underline{\mathbf{Der}}_{\text{Ladj}}(-, \mathbb{D})$ preserves adjunctions, like any 2-functor, so \mathbb{D}' satisfies (Der3L).

The axiom (Der4) is also readily dispensed with. Given a comma square

$$\begin{array}{ccc} J & \xrightarrow{u} & K \\ \downarrow v & \swarrow \alpha & \downarrow w \\ L & \xrightarrow{t} & M \end{array}$$

so that $J = K \overset{\leftarrow}{\times}_M L$ with u and v the canonical projections, the following square is also a comma:

$$\begin{array}{ccc} J^{\text{op}} & \xrightarrow{v^{\text{op}}} & L^{\text{op}} \\ \downarrow u^{\text{op}} & \swarrow \alpha^{\text{op}} & \downarrow t^{\text{op}} \\ K^{\text{op}} & \xrightarrow{w^{\text{op}}} & M^{\text{op}} \end{array}$$

This implies that, in $(\underline{\mathbf{Der}}_{\text{Ladj}})^{\text{op}}$, the 2-morphism

$$\alpha_1^{\text{op}} : (v^{\text{op}})^* u_1^{\text{op}} \Rightarrow t_1^{\text{op}} (w^{\text{op}})^* : \mathcal{H}\text{ot}^{K^{\text{op}}} \rightarrow \mathcal{H}\text{ot}^{L^{\text{op}}}$$

is invertible. Indeed, (Der4) for $\mathcal{H}\text{ot}$ implies that it has invertible components.

To show that \mathbb{D}' satisfies (Der4), we must show that $\mathbb{D}'(\alpha)_1$, the mate of the natural transformation $\mathbb{D}'(\alpha)$, is an isomorphism. As 2-functors preserve mates and α_1^{op} is the mate

of $(\alpha^{\text{op}})^*$ in $\underline{\mathbf{LDer}}_{\text{Ladj}}^{\text{op}}$, we have that $\mathbb{D}'(\alpha)_! = \underline{\mathbf{Der}}_{\text{Ladj}}(\alpha_!^{\text{op}}, \mathbb{D})$. Thus $\mathbb{D}'(\alpha)_!$ is an isomorphism as desired.

Finally, (Der2), (Der5), or (Der5)' for \mathbb{D}' follows from the corresponding axiom for \mathbb{D} and Theorem 3.2.2. □

All in all, Proposition 3.3.1, Proposition 3.4.6, Corollary 3.4.2, and Theorem 3.1.3 combine to prove:

Theorem 3.4.7. *The restriction 2-functor $\infty - \underline{\mathbf{Der}}_! \rightarrow \underline{\mathbf{Der}}_!$ induces an equivalence when restricted to the homotopically locally small (∞) -derivators, and also when restricted to the strong homotopically locally small (∞) -derivators.*

CHAPTER 4

The prederivator associated to a homotopy theory

We now construct the 2-functor $\mathbf{HO} : \underline{\infty} - \mathbf{Cat} \rightarrow \mathbf{PDer}$ giving the canonical prederivator associated to an ∞ -category, and investigate various of its properties.

4.1 The homotopy prederivator

We first extend the homotopy category functor $\mathbf{Ho} : \underline{\infty} - \mathbf{Cat} \rightarrow \mathbf{Cat}$ (see Definition 2.2.4) to a 2-functor of the same name, $\mathbf{Ho} : \underline{\infty} - \mathbf{Cat} \rightarrow \mathbf{Cat}$. This still sends an ∞ -category to its homotopy category; we must define the action on morphism categories. This will be for each R and Q a functor

$$\mathbf{Ho}_{Q,R} : \mathbf{QCAT}(Q, R) = \mathbf{Ho}(R^Q) \rightarrow \mathbf{Ho}(R)^{\mathbf{Ho}(Q)} = \mathbf{CAT}(\mathbf{Ho}(Q), \mathbf{Ho}(R)).$$

The functor $\mathbf{Ho}_{Q,R}$ is defined as the transpose of the composition

$$\mathbf{Ho}(R^Q) \times \mathbf{Ho}(Q) \cong \mathbf{Ho}(R^Q \times Q) \xrightarrow{\mathbf{Ho}(\text{ev})} \mathbf{Ho}(R)$$

across the product-hom adjunction in the 1-category \mathbf{CAT} . For this isomorphism we have used again the preservation of finite products by \mathbf{Ho} . The morphism $\text{ev} : R^Q \times Q \rightarrow R$ is evaluation, the counit of the adjunction $(-) \times Q \dashv (-)^Q$ between endofunctors of $\underline{\infty} - \mathbf{Cat}$.

We also need a 2-functor $N : \mathbf{Cat} \rightarrow \underline{\infty} - \mathbf{Cat}$ sending a category $J \in \mathbf{Cat}$ to $N(J)$. The map on hom-categories is the composition $J^K \cong \mathbf{Ho}(N(J^K)) \cong \mathbf{Ho}(N(J)^{N(K)})$. The first isomorphism is the inverse of the counit of the adjunction $\mathbf{Ho} \dashv N$, which is an isomorphism by full faithfulness of the nerve. The second uses the fact that N preserves exponentials, see [Joy08, Proposition B.0.16].

Finally, we require the following fact: a monoidal functor $F : \mathcal{V} \rightarrow \mathcal{W}$ induces a 2-functor $F_*(-) : \mathcal{V} - \mathbf{Cat} \rightarrow \mathcal{W} - \mathbf{Cat}$ between 2-categories of \mathcal{V} - and \mathcal{W} -enriched categories. The fully general version of this claim was apparently not published until recently; it comprises Chapter 4 of [Cru09]. In our case, the functor Ho is monoidal insofar as it preserves products and thus it induces the 2-functor $\mathrm{Ho}_*(-)$ sending simplicially enriched categories, simplicial functors, and simplicial natural transformations to 2-categories, 2-functors, and 2-natural transformations.

Now we define the homotopy prederivator.

Definition 4.1.1. Let Q be an ∞ -category. Then the *homotopy prederivator* $\mathrm{HO}(Q)$ is given as the composition

$$\mathbf{Dia}^{\mathrm{op}} \xrightarrow{N^{\mathrm{op}}} \underline{\infty - \mathbf{Cat}}^{\mathrm{op}} \xrightarrow{Q^{(-)}} \underline{\infty - \mathbf{Cat}} \xrightarrow{\mathrm{Ho}} \mathbf{Cat}.$$

In particular, $\mathrm{HO}(Q)$ maps a category J to the homotopy category of J -shaped diagrams in Q , that is, to $\mathrm{Ho}(Q^{N(J)})$.

Given a morphism of quasicategories $f : Q \rightarrow R$, we have a strictly 2-natural morphism of prederivators $\mathrm{HO}(f) : \mathrm{HO}(Q) \rightarrow \mathrm{HO}(R)$ given as the analogous composition

$$\mathrm{HO}(f) = \mathrm{Ho} \circ f^{(-)} \circ N,$$

so that for each category J the functor $\mathrm{HO}(f)_J$ is given by post-composition with f , that is, by $\mathrm{Ho}(f^{N(J)}) : \mathrm{Ho}(Q^{N(J)}) \rightarrow \mathrm{Ho}(R^{N(J)})$.

Note that if we should start with a large ∞ -category Q , we should simply end up with a 2-functor $\mathbf{Dia}^{\mathrm{op}} \rightarrow \mathbf{CAT}$, following the same recipe as above.

We now record the axioms which are satisfied by the homotopy prederivator of any ∞ -category.

Proposition 4.1.2. *For any ∞ -category Q , the homotopy prederivator $\mathrm{HO}(Q)$ satisfies the axioms (Der1)'', (Der2), and (Der5)'.*

Proof. The axiom (Der2) is an application of Lemma 2.2.3, with Q specialized to $N(J)$ for some $J \in \mathbf{Dia}$. The other axioms follow immediately from the identification of $\mathrm{HO}(Q)(\sqcup J_i)$, $\mathrm{HO}(Q)(J[\mathcal{W}^{-1}]$

and $\mathrm{HO}(Q)(J \overset{\leftarrow}{\sqcup}_L K)$ with $\underline{\infty - \mathbf{Cat}}(\sqcup J_i, Q)$, $\underline{\infty - \mathbf{Cat}}(J[\mathcal{W}^{-1}], Q)$, and $\underline{\infty - \mathbf{Cat}}(J \overset{\leftarrow}{\sqcup}_L K, Q)$, respectively. \square

It may be worth noting that, while it is possible to define a 2-category \mathbf{SSet} of simplicial sets using τ_1 and extend HO to \mathbf{SSet} , the prederivator associated to an arbitrary simplicial set S will not, in general, satisfy any of the three axioms. It is straightforward to see that $\mathrm{HO}(S)$ need not satisfy (Der2) or (Der5), while the reason (Der1) may fail is that τ_1 , unlike Ho , need not preserve infinite products.

4.2 Small phenomena

In this section, we prove the following:

Theorem 4.2.1. *For any small ∞ -category Q and any ∞ -category R , the 2-functor HO induces an equivalence $\underline{\infty - \mathbf{Cat}}(Q, R) \rightarrow \mathbf{PDer}(\mathrm{HO}(Q), \mathrm{HO}(R))$. In particular, the 2-functor $\mathrm{HO} : \underline{\infty - \mathbf{Cat}} \rightarrow \mathbf{PDer}$ is bicategorically fully faithful.*

It is crucial that we insist on small ∞ -categories in the domain.

We first determine the image of morphisms of ∞ -categories in the categories of maps between the associated prederivators.

Proposition 4.2.2. *Given quasicategories Q, R , the map $\mathbf{QCat}(Q, R) \rightarrow \mathbf{PDer}(Q, R)$ is an isomorphism onto the subset of strictly 2-natural transformations.*

The proof has the following outline:

- (1) Eliminate most of the data of a prederivator map by showing strict maps $\mathrm{HO}(Q) \rightarrow \mathrm{HO}(R)$ are determined by their restriction to natural transformations between ordinary functors $\mathbf{Cat}^{\mathrm{op}} \rightarrow \mathbf{Set}$. This is Lemma 4.2.4.
- (2) Show that $\mathrm{HO}(Q)$ and $\mathrm{HO}(R)$ recover Q and R upon restricting the domain to Δ^{op} and the codomain to \mathbf{Set} , and that natural transformations as in the previous step are in bijection with maps $Q \rightarrow R$. This is Lemma 4.2.6.

- (3) Show that $\text{HO}(f)$ restricts back to f for a map $f : Q \rightarrow R$, which implies that HO is faithful, and that a map $F : \text{HO}(Q) \rightarrow \text{HO}(R)$ is exactly HO applied to its restriction, which implies that HO is full.

Let us begin with step (1).

Definition 4.2.3. A **Dia**-set is a large presheaf on **Dia** that is, an ordinary functor $\mathbf{Dia}^{\text{op}} \rightarrow \mathbf{SET}$.

Given a prederivator \mathbb{D} , let $\mathbb{D}^{\text{ob}} : \mathbf{Dia}^{\text{op}} \rightarrow \mathbf{SET}$ be its underlying **Dia**-set, so that \mathbb{D}^{ob} sends a small category J to the set of objects $\text{ob}(\mathbb{D}(J))$ and a functor $u : I \rightarrow J$ to the action of $\mathbb{D}(u)$ on objects.

Whereas (Der5) requires that $\text{dia} : \mathbb{D}(J \times [1]) \rightarrow \mathbb{D}(J)^{[1]}$ be (full and) essentially surjective, we say a prederivator is *smothering* if in fact dia is strictly surjective. Note that by definition of Ho , the homotopy prederivator of a quasicategory is smothering.

The following lemma shows that under this assumption most of the apparent structure of a strict prederivator map is redundant.

Lemma 4.2.4. *A strict morphism $F : \mathbb{D}_1 \rightarrow \mathbb{D}_2$ between smothering prederivators is determined by its restriction to the underlying **Dia**-sets $\mathbb{D}_1^{\text{ob}}, \mathbb{D}_2^{\text{ob}}$. That is, the restriction functor from smothering prederivators to **Dia**-sets is faithful.*

Proof. The data of a strict morphism $F : \mathbb{D}_1 \rightarrow \mathbb{D}_2$ is that of a functor $F_J : \mathbb{D}_1(J) \rightarrow \mathbb{D}_2(J)$ for every J .¹

The induced map $F^{\text{ob}} : \mathbb{D}_1^{\text{ob}} \rightarrow \mathbb{D}_2^{\text{ob}}$ is given by the action of F on objects. So to show faithfulness it is enough to show that, given a family of functions $r_J : \text{ob}(\mathbb{D}_1(J)) \rightarrow \text{ob}(\mathbb{D}_2(J))$, that is, the data required in a natural transformation between **Dia**-sets, there is at most one 2-natural transformation with components $F_J : \mathbb{D}_1(J) \rightarrow \mathbb{D}_2(J)$ and object parts $\text{ob}(F_J) = r_J$.

¹Note the simplification here over pseudonatural transformations, which require also a natural transformation associated to every functor and do not induce maps of **Dia**-sets. That is the fundamental difficulty leading to the dramatically different techniques of the next sections.

Indeed, suppose F is given with object parts $r_J = \text{ob}(F_J)$ and let $f : X \rightarrow Y$ be a morphism in $\mathbb{D}_1(J)$. Then since \mathbb{D}_1 is smothering, f is (strictly equal to) the underlying diagram of some $\widehat{f} \in \text{ob}\mathbb{D}_1(J \times [1])$. By 2-naturality, the following square must commute:

$$\begin{array}{ccc} \mathbb{D}_1(J \times [1]) & \xrightarrow{F_{J \times [1]}} & \mathbb{D}_2(J \times [1]) \\ \downarrow \text{dia}_J^{[1]} & & \downarrow \text{dia}_J^{[1]} \\ \mathbb{D}_1(J)^{[1]} & \xrightarrow{F_J} & \mathbb{D}_2(J)^{[1]} \end{array}$$

Indeed, $\text{dia}_J^{[1]}$ is the action of a prederivator on the unique natural transformation between the two functors $0, 1 : [0] \rightarrow [1]$ from the terminal category to the arrow category, as is described in full detail below [Gro13, Proposition 1.7]. Thus the square above is an instance of the axiom of respect for 2-morphisms. It follows that we must have $F_J(f) = F_J(\text{dia}_J^{[1]}\widehat{f}) = \text{dia}_J^{[1]}(r_{J \times [1]}(\widehat{f}))$.

Thus if F and G are two strict morphisms $\mathbb{D}_1 \rightarrow \mathbb{D}_2$ with the same restrictions to the underlying **Dia**-sets, they must coincide, as claimed. \square

Note the above does not claim that the restriction functor is full: the structure of a strict prederivator map is determined by the action on objects of each $\mathbb{D}_1(J), \mathbb{D}_2(J)$, but it is not generally true that an arbitrary map of **Dia**-sets will admit a well defined extension to morphisms.

We proceed to step (2) of the proof.

Let us recall the theory of pointwise Kan extensions for 1-categories. Let $F : \mathcal{C} \rightarrow \mathcal{D}$ and $G : \mathcal{C} \rightarrow \mathcal{E}$ be functors. At least if \mathcal{C} and \mathcal{D} are small and \mathcal{E} is complete, then we always have a right Kan extension $F_*G : \mathcal{D} \rightarrow \mathcal{E}$ characterized by the adjunction formula $\mathcal{E}^{\mathcal{D}}(H, F_*G) \cong \mathcal{E}^{\mathcal{C}}(H \circ F, G)$ and computed on objects by

$$F_*G(d) = \lim_{d \downarrow F} G \circ q \tag{4.2.5}$$

Here $d \downarrow F$ is the comma category with objects $(c, f : d \rightarrow F(c))$ and morphisms the maps in \mathcal{C} making the appropriate triangle commute, and $q : d \downarrow F \rightarrow \mathcal{C}$ is the projection.

Lemma 4.2.6. *Let $j : \Delta^{\text{op}} \rightarrow \mathbf{Dia}^{\text{op}}$ be the inclusion. Then for any quasicategory R , the **Dia**-set $\text{HO}(R)^{\text{ob}}$ underlying $\text{HO}(R)$ is the right Kan extension of R along j .*

Proof. For any small category J , the **Dia**-set $\text{HO}(R)^{\text{ob}}$ takes J to the set of simplicial set maps from J to R :

$$\text{HO}(R)^{\text{ob}}(J) = \text{ob}(\text{Ho}(R^{N(J)})) = \mathbf{SSET}(N(J), R).$$

We shall show that the latter is the value required of j_*R at J , which exists and is calculated via Equation 4.2.5 since **SET** is complete (in the sense of a universe in which its objects constitute the small sets).

First, one of the basic properties of presheaf categories implies that $N(J)$ is a colimit over its category of simplices. That is, $N(J) = \text{colim}_{\Delta \downarrow NJ} y \circ q$, where $q : \Delta \downarrow NJ \rightarrow \Delta$ is the projection and $y : \Delta \rightarrow \mathbf{SSet}$ is the Yoneda embedding.

Then we can rewrite the values of $\text{HO}(R)^{\text{ob}}$ as follows:

$$\text{HO}(R)^{\text{ob}}(J) = \mathbf{SSET}(N(J), R) = \mathbf{SSET}(\text{colim}_{\Delta \downarrow NJ} y \circ q, R) \cong$$

$$\lim_{(\Delta \downarrow NJ)^{\text{op}}} \mathbf{SSET}(y \circ q, R) \cong \lim_{(\Delta \downarrow NJ)^{\text{op}}} R \circ q^{\text{op}}$$

The last isomorphism follows from the Yoneda lemma.

The indexing category $(\Delta \downarrow N(J))^{\text{op}}$ has as objects pairs $(n, f : \Delta^n \rightarrow N(J))$ and as morphisms $\bar{a} : (n, f) \rightarrow (m, g)$, the maps $a : \Delta^m \rightarrow \Delta^n$ such that $f \circ a = g$. That is, $(\Delta \downarrow N(J))^{\text{op}} \cong N(J) \downarrow \Delta^{\text{op}}$, where on the right-hand side $N(J)$ is viewed as an object of $\mathbf{SSET}^{\text{op}}$. Using the full faithfulness of the nerve functor N , we see $(\Delta \downarrow N(J))^{\text{op}} \cong J \downarrow \Delta^{\text{op}}$, where again $J \in \mathbf{Dia}^{\text{op}}$.

Thus, if q^{op} serves also to name the projection $J/\Delta^{\text{op}} \rightarrow \Delta^{\text{op}}$, we may continue the computation above with

$$\text{HO}(R)^{\text{ob}}(N(J)) \cong \lim_{N(J) \downarrow \Delta^{\text{op}}} R \circ q^{\text{op}}$$

This is exactly the formula for $j_*R(J)$ recalled above. The isomorphism thus constructed is certainly natural with respect to the action on maps of the Kan extension, so the lemma is established. \square

We arrive at step (3).

Proof of Proposition 4.2.2. Note that, by Lemma 4.2.6, the restriction of $\mathrm{HO}(Q)^{\mathrm{ob}}$ to a functor $\Delta^{\mathrm{op}} \rightarrow \mathbf{SET}$ is canonically isomorphic to Q , since Kan extensions along fully faithful functors are splittings of restriction. Thus a map $F : \mathrm{HO}(Q) \rightarrow \mathrm{HO}(R)$ restricts to a map $\rho(F) : Q \rightarrow R$. In fact, we have a natural isomorphism $\rho \circ \mathrm{HO} \cong \mathrm{id}_{\mathbf{QCAT}}$, so that $\rho \circ \mathrm{HO}(f)$ is again f , up to this isomorphism. Indeed, given $f : Q \rightarrow R$, we already know how to compute $\mathrm{HO}(f)$ as $\mathrm{Ho} \circ (f^{N(-)})$. Then the restriction $\rho(\mathrm{HO}(f)) : Q \rightarrow R$, which we are to show coincides with f , is given by $\rho(\mathrm{HO}(f))_n = \mathrm{ob} \circ \mathrm{Ho} \circ f^{\Delta^n}$. That is, $\rho(\mathrm{HO}(f))$ acts by the action of f on the objects of the homotopy categories of Q^{Δ^n} and R^{Δ^n} . In other words, it acts by the action of f on the sets $\mathbf{SSET}(\Delta^n, Q)$ and $\mathbf{SSET}(\Delta^n, R)$; via Yoneda, $\rho(\mathrm{HO}(f))$ acts by f itself.

It remains to show that $\mathrm{HO}(\rho(F)) = F$ for any $F : \mathrm{HO}(Q) \rightarrow \mathrm{HO}(R)$. By Lemma 4.2.4 it suffices to show that the restrictions of $\mathrm{HO}(\rho(F))$ and F to the underlying **Dia**-sets coincide. Using Lemma 4.2.6 and the adjunction characterizing the Kan extension, we have

$$\begin{aligned} \mathbf{SET}^{\mathbf{Dia}^{\mathrm{op}}}(\mathrm{HO}(Q)^{\mathrm{ob}}, \mathrm{HO}(R)^{\mathrm{ob}}) &= \mathbf{SET}^{\mathbf{Dia}^{\mathrm{op}}}(j_*Q, j_*R) \\ &\cong \mathbf{SSET}(j^*j_*Q, R) \cong \mathbf{SSET}(Q, R). \end{aligned}$$

In particular, maps between $\mathrm{HO}(Q)^{\mathrm{ob}}$ and $\mathrm{HO}(R)^{\mathrm{ob}}$ agree when their restrictions to Q and R do. Thus we are left to show that $\rho(\mathrm{HO}(\rho(F))) = \rho(F)$. But as we showed above, $\rho \circ \mathrm{HO}$ is the identity map on $\mathbf{SSET}(Q, R)$, so the proof is complete. \square

We are now prepared to prove the theorem that is the aim of this section. Recall the delocalization and its constituents from Definition 2.2.12 and below.

Proof of Theorem 4.2.1. First, we must show that if $F : \mathrm{HO}(Q) \rightarrow \mathrm{HO}(R)$ is a pseudonatural transformation, then there exists $h : Q \rightarrow R$ and an isomorphism $\Lambda : \mathrm{HO}(h) \cong F$. Observe that, since Q is small, $\Delta \downarrow Q$ is in **Cat**. Now we claim that $F_{\Delta \downarrow Q}(p_Q) : \Delta \downarrow Q \rightarrow R$ sends the class \mathcal{L}_Q of last-vertex maps into equivalences in R . Indeed, if $\ell : \Delta^1 \rightarrow \Delta \downarrow Q$ is in \mathcal{L}_Q , then we have, using F 's respect for 2-morphisms and the structure isomorphism F_ℓ ,

$$F_{[0]}(\mathrm{dia}(\ell^*p_Q)) = \mathrm{dia}(F_{[1]}(\ell^*p_Q)) \cong \mathrm{dia}(\ell^*F_{\Delta \downarrow J}(p_Q)).$$

Thus $\text{dia}(\ell^* F_{\Delta \downarrow J}(p_Q))$ is an isomorphism in $\text{Ho}(R)$, since $\text{dia}(\ell^* p_Q)$ is an isomorphism in $\text{Ho}(Q)$. From (Der2) for $\text{Ho}(R)$, it follows that $F_{\Delta \downarrow Q}(p_Q)$ inverts the last-vertex maps as desired.

Then using the delocalization theorem, we can define $h : Q \rightarrow R$ as any map admitting an isomorphism $\sigma : h \circ p_Q \cong F_{\Delta \downarrow Q}(p_Q)$ in $\text{Ho}(R^{\Delta \downarrow Q})$. From σ , we get an invertible modification

$$\text{HO}(\sigma) : \text{HO}(h \circ p_Q) \Rightarrow \text{HO}(F_{\Delta \downarrow Q}(p_Q)) : \text{HO}(\Delta \downarrow Q) \rightarrow \text{HO}(R).$$

We now construct an invertible modification

$$\Lambda : \text{HO}(h) \Rightarrow F : \text{HO}(Q) \rightarrow \text{HO}(R).$$

Fixing $J \in \mathbf{Cat}$ and $X : N(J) \rightarrow Q$, since $\ell_J^* : \text{HO}(R)(J) \rightarrow \text{HO}(R)(\Delta \downarrow J)$ is fully faithful we can uniquely define $\Lambda_{X,J} : \text{HO}(h)_J(X) \cong F_J(X)$ by giving $\ell_J^*(\Lambda_{X,J})$. To wit, we require $\ell_J^*(\Lambda_{X,J})$ to be the composition

$$\begin{aligned} \ell_J^* \text{HO}(h)_J(X) &= h \circ X \circ \ell_J = h \circ p_Q \circ \Delta \downarrow X \\ &\cong F_{\Delta \downarrow Q}(p_Q) \circ \Delta \downarrow X \cong F_{\Delta \downarrow J}(p_Q \circ \Delta \downarrow X) = F_{\Delta \downarrow J}(X \circ \ell_J) \\ &\cong F_J(X) \circ \ell_J = \ell_J^* F_J(X). \end{aligned}$$

The first isomorphism is a component of $\text{HO}(\sigma)$, while the latter two are components of F . The naturality of $\Lambda_{J,X}$ in X thus follows from the facts that F is pseudonatural and that $\text{HO}(\sigma)$ is a modification. So we have natural isomorphisms $\Lambda_J : \text{HO}(h)_J \Rightarrow F_J$ for each J . To verify that the Λ_J assemble into a modification, consider any $u : K \rightarrow J$. Then we must show that, for any $X : N(J) \rightarrow Q$, the diagram

$$\begin{array}{ccc} \text{HO}(h)_J(X) \circ u & \xrightarrow{\Lambda_{J,X \circ u}} & F_J(X) \circ u \\ \parallel & & \downarrow F_u \\ \text{HO}(h)_K(X \circ u) & \xrightarrow{\Lambda_{K,X \circ u}} & F_K(X \circ u) \end{array}$$

commutes. Using, as always, full faithfulness of the pullback along a localization, we may precompose with p_K . Then the modification axiom is verified by the commutativity of the following diagram:

$$\begin{array}{ccccc}
hXup_K & \xrightarrow{\Lambda_{J,X}*up_K} & F_J(X)up_K & & \\
\parallel & & \parallel & & \\
hX\ell_J\Delta \downarrow u & \xrightarrow{\Lambda_{J,X}*\ell_J\Delta \downarrow u} & F_J(X)\ell_J\Delta \downarrow u & \xlongequal{\quad} & F_J(X)up_K \\
\parallel & & F_{\ell_J*\Delta \downarrow u} \uparrow & & F_u*p_K \uparrow \\
hp_Q\Delta \downarrow Xu & & F_{\Delta \downarrow J}(X\ell_J)\Delta \downarrow u & & F_K(Xu)p_K \\
\parallel & & \parallel & & F_{p_K} \uparrow \\
F_{\Delta \downarrow Q}(p_Q)\Delta \downarrow Xu & \xrightarrow{F_{\Delta \downarrow X}^{-1}*\Delta \downarrow u} & F_{\Delta \downarrow J}(p_Q\Delta \downarrow X)\Delta \downarrow u & & F_{\Delta \downarrow K}(Xup_K) \\
& \searrow^{F_{\Delta \downarrow Xu}^{-1}} & \uparrow F_{\Delta \downarrow u} & \swarrow & \\
& & F_{\Delta \downarrow K}(p_Q\Delta \downarrow Xu) & &
\end{array}$$

The upper left square commutes since $up_K = \ell_J\Delta \downarrow u$. The left central hexagon commutes by definition of $\Lambda_{J,X}$, and the lower left triangle and right-hand heptagon commute by functoriality of the pseudonaturality isomorphisms of F . Meanwhile, the outer route around the diagram from $hXup_K$ to $F_J(X)up_K$ is $F_u\Lambda_{K,Xu}$, while the inner route is $\Lambda_{J,X} * up_K$. So Λ is an invertible modification $\text{HO}(h) \cong F$, as desired.

We have shown that HO induces an essentially surjective functor $\underline{\infty} - \mathbf{Cat}(Q, R) \rightarrow \mathbf{PDer}(\text{HO}(Q), \text{HO}(R))$. We next consider full faithfulness. So, assume given a modification

$$\Xi : \text{HO}(f) \Rightarrow \text{HO}(g) : \text{HO}(Q) \rightarrow \text{HO}(R).$$

We must show there exists a unique $\xi : f \Rightarrow g$ with $\text{HO}(\xi) = \Xi$. First, we consider

$$\Xi_{p_Q} : f \circ p_Q \rightarrow g \circ p_Q,$$

which is a morphism in $\text{HO}(R)(\Delta \downarrow Q)$. According to (Der5), we can lift this to a map $\widehat{\Xi}_{p_Q} : \Delta \downarrow Q \rightarrow R^{\Delta^1}$ with $\text{dia}(\widehat{\Xi}_{p_Q}) = \Xi_{p_Q}$.

Since the domain and codomain $f \circ p_Q$ and $g \circ p_Q$ of $\widehat{\Xi}_{p_Q}$ invert the last-vertex maps \mathcal{L}_Q , by (Der2) so does $\widehat{\Xi}_{p_Q}$ itself. Thus by the delocalization theorem we get $\widehat{\Xi}' : Q \rightarrow R^{\Delta^1}$ with an isomorphism

$$a : \widehat{\Xi}' \circ p_Q \cong \widehat{\Xi}_{p_Q}.$$

The domain and codomain

$$0^*a : 0^*(\widehat{\Xi}' \circ p_Q) \cong fp_Q \text{ and } 1^*a : 1^*(\widehat{\Xi}' \circ p_Q) \cong gp_Q$$

give rise to unique isomorphisms

$$i : 0^*\widehat{\Xi}' \cong f \text{ and } j : 1^*\widehat{\Xi}' \cong g.$$

Now we can construct $\widehat{\Xi} : Q \rightarrow R^{\Delta^1}$ as a lift of the composite

$$f \xrightarrow{i^{-1}} 0^*\widehat{\Xi}' \xrightarrow{\text{dia}(\widehat{\Xi}')} 1^*\widehat{\Xi}' \xrightarrow{j} g$$

in $\text{Ho}(R^Q)$. Using the fullness clause of (Der5), we can choose an isomorphism $b : \widehat{\Xi} \cong \widehat{\Xi}'$ in $\text{Ho}((R^{\Delta^1})^Q)$ lifting $(i^{-1}, j^{-1}) : \text{dia}(\widehat{\Xi}) \rightarrow \text{dia}(\widehat{\Xi}')$.

Then $a \circ (b * p_Q) : \widehat{\Xi} \circ p_Q \rightarrow \widehat{\Xi}_{p_Q}$ is an isomorphism with endpoints fixed, insofar as $0^*(b * p_Q) = i^{-1} * p_Q = 0^*a^{-1}$ and similarly $1^*(b * p_Q) = 1^*a^{-1}$. Thus $\text{dia}(\widehat{\Xi} \circ p_Q) = \text{dia}(\widehat{\Xi}_{p_Q}) = \Xi_{p_Q}$ in $\text{Ho}(R^{\Delta^1 \downarrow Q})$.

Notice that if $\widehat{\Xi}_2 : Q \rightarrow R^{\Delta^1}$ is any other morphism satisfying $\text{dia}(\widehat{\Xi}_2 \circ p_Q) = \Xi_{p_Q}$, then $\text{dia}(\widehat{\Xi}_2) = \text{dia}(\widehat{\Xi})$, since pullback along p_Q is faithful. So we have a unique candidate $\xi := \text{dia}(\widehat{\Xi}) : f \Rightarrow g$; it remains to show that $\text{HO}(\xi) = \Xi$.

To that end, we claim that for every $X : J \rightarrow Q$, we have $\text{HO}(\xi)_X = \xi * X = \Xi_X$. As above, it suffices to precompose X with ℓ_J , and then we have

$$\begin{aligned} \xi * X * \ell_J &= \text{dia}(\widehat{\Xi}) * p_Q * \Delta \downarrow X = \text{dia}(\widehat{\Xi} \circ p_Q) \circ \Delta \downarrow X \\ &= \Xi_{p_Q} * \Delta \downarrow X = \Xi_{p_Q \circ \Delta \downarrow X} = \Xi_{X \circ \ell_J} = \Xi_X * \ell_J \end{aligned}$$

as desired. In the equations above we have used the 2-functoriality of $\text{HO}(R)$, naturality of p , and the modification property of Ξ . So $\text{HO}(\xi) = \Xi$, as was to be shown. \square

Thus morphisms and 2-morphisms between small ∞ -categories are optimally detected by their associated prederivators, at least with **Dia** = **Cat**. The above theorem implies that HO detects equivalences in this case, but in fact this is true in greater generality; see Theorem 4.3.14.

Application: the Yoneda embedding

Recall from Corollary 3.3.8 that for any small ∞ -category J , the derivator $\mathcal{H}ot^{J^{op}}$ comes with a morphism $y : \text{HO}(J) \rightarrow \mathcal{H}ot^{J^{op}}$ making $\mathcal{H}ot^{J^{op}}$ into the free left derivator on $\text{HO}(J)$.

By Theorem 4.2.1, the morphism y arises from a morphism of ∞ -categories, also denoted $y : J \rightarrow \hat{J}$, where $\mathrm{HO}(\hat{J}) = \mathcal{H}\mathrm{ot}^{J^{\mathrm{op}}}$. We have:

Corollary 4.2.7. *The Yoneda embedding exists for small ∞ -categories. That is, every small ∞ -category J admits a free cocompletion $y : J \rightarrow \hat{J}$.*

We hasten to emphasize that we have not thus far constructed \hat{J} out of $\mathcal{H}\mathrm{ot}^{J^{\mathrm{op}}}$.

4.3 Large phenomena

In general, HO appears to be neither locally essentially surjective nor locally full. That is, one may give ∞ -categories Q, R together with a pseudo-natural transformation $F : \mathrm{HO}(Q) \rightarrow \mathrm{HO}(R)$ not isomorphic to $\mathrm{HO}(f)$ for any $f : Q \rightarrow R$, and similarly for 2-morphisms. In this section, we show that HO is at least bicategorically conservative no matter what assumptions are placed on the domain and codomain. This relies on a fundamental superiority of the homotopy 2-category **Hot** of spaces to the homotopy category **Hot**, as we discuss after presenting a conjectural example of an F as above.

A conjectural unrectifiable morphism of prederivators

For this section, we set **Dia** = **HFin**. The local non-fullness of HO should arise from the failure of homotopy finite categories to detect high-dimensional coherence homotopies between morphisms of infinite ∞ -categories.

To make this concrete, we define an incoherent notion of homotopy colimit. Take any derivator \mathbb{D} with $D \in \mathbb{D}(J)$.

Definition 4.3.1. Denote by $B(\mathbb{D}, D) \in \mathbb{D}(\Delta^{\mathrm{op}})$ the object $q_!^J \iota_J^* D$, where $\iota_J : \Delta^{\mathrm{op}} \downarrow J \rightarrow J$ is the initial vertex projection and $q : \Delta^{\mathrm{op}} \downarrow J \rightarrow \Delta^{\mathrm{op}}$ is the forgetful functor. We refer to $B(\mathbb{D}, D)$ as the *simplicial bar construction* of D .

It is shown in [PS16] that the colimit $p_!^J D$ may be computed as $p_!^{\Delta^{\mathrm{op}}} B(\mathbb{D}, D)$. We now generalize:

Definition 4.3.2. If $0 \leq n \leq \infty$ and $i_n : \Delta_{\leq n}^{\text{op}} \rightarrow \Delta^{\text{op}}$ denotes the canonical inclusion, then we define by $p_{!}^{\Delta_{\leq n}^{\text{op}}} i_n^* B(\mathbb{D}, D)$ the A_n -colimit of D .

Thus the A_{∞} -colimit is identified with the ordinary (homotopy) colimit of D . For n finite, the A_n -colimit is a partly coherent colimit; intuitively, the A_{∞} -colimit glues together the values of D via homotopies determined by the morphisms of J , which are themselves glued together by higher and higher homotopies determined by the higher-dimensional simplices of $N(J)$. The A_n -colimit simply truncates this process at dimension n .

We shall require a point-set construction. For this, let \mathcal{M} denote a simplicial model category with tensoring over $\mathbf{SSet}_{\text{Quillen}}$ denoted by \otimes . We refine the above definitions to be well-defined up to isomorphism in this case.

Definition 4.3.3. If $D : J \rightarrow \mathcal{M}$ is a diagram in \mathcal{M} , then the *simplicial bar construction* of D , $B(\mathcal{M}, D) : \Delta^{\text{op}} \rightarrow \mathcal{M}$, is the simplicial object in \mathcal{M} given by

$$\coprod_{j_0 \in J} D(j_0) \rightrightarrows \coprod_{j_0 \rightarrow j_1} D(j_0) \rightarrow \dots$$

That is, $B(\mathcal{M}, D)$ is the left Kan extension of the restriction of D to $\Delta^{\text{op}} \downarrow J$ along the projection to Δ^{op} .

For any $n \geq 0$, denote by $B(\mathcal{M}, D)_{\leq n}$ the *n-truncated simplicial bar construction* given as the restriction of $B(\mathcal{M}, D)$ to the full subcategory $\Delta_{\leq n}^{\text{op}}$ of Δ^{op} on the objects $0, 1, \dots, n$.

It is well known (see for instance [Dug08]) that the homotopy colimit of D may be modeled as $\text{colim}_{\Delta^{\text{op}}} B(\mathcal{M}, D)$, so that the point-set left Kan extension taken above is homotopically correct. We now generalize:

Definition 4.3.4. If $0 \leq n \leq \infty$, the A_n -colimit of D is defined as $\text{colim}_{\Delta_{\leq n}^{\text{op}}} B(\mathcal{M}, D)_{\leq n}$.

Thus, again, the A_{∞} -colimit of D recovers the standard local model for the usual homotopy colimit.

We now let $\mathcal{M} = \mathbf{SSet}^+$ denote the simplicial model category of marked simplicial sets and recall that ω_1 is the least uncountable ordinal. Recall the natural marking functor

$(-)^{\natural} : \mathbf{QCat} \rightarrow \mathbf{SSet}^+$ and let $D : \omega_1 \rightarrow \mathbf{SSet}^+$ be defined by $D(\alpha) = N(\alpha)^{\natural}$. Write $\omega_1^{A_2}$ for the A_2 -colimit of D .

$\omega_1^{A_2}$ admits two canonical projections $p_1 : \omega_1^{A_2} \rightarrow N(\omega_1)$ and $p_2 : \omega_1^{A_2} \rightarrow P$, where P is the A_2 -colimit in $\mathbf{SSet}_{\text{Quillen}}$ of the constant diagram $\omega_1 \rightarrow \mathbf{SSet}$ valued at Δ^0 . We conjecture that it may be shown that there exist no maps $N(\omega_1) \rightarrow \widetilde{\omega_1^{A_2}}$ into a Joyal fibrant replacement of $\omega_1^{A_2}$ whose composite with p_1 has unbounded image. However, there do exist analogous maps of homotopically finite prederivators.

Conjecture 4.3.5. *If we consider $\text{HO} : \underline{\infty} - \mathbf{Cat} \rightarrow \mathbf{PDer}_{\mathbf{HFin}}$, then there is a map $F : \text{HO}(N(\omega_1)) \rightarrow \text{HO}(\widetilde{\omega_1^{A_2}})$ splitting $\text{HO}(p_1)$, not isomorphic to $\text{HO}(f)$ for any $f : N(\omega_1) \rightarrow \widetilde{\omega_1^{A_2}}$.*

The author and Christensen have proven an analogous result in [CC19], for an A_1 -homotopy colimit of an ω_1 -indexed family of wedges of circles in \mathbf{Hot} . However, the 2-categorical aspect of the current situation appears to forestall a similar argument.

Whitehead's theorem fails for the homotopy category of spaces

Our next aim is to prove a Whitehead theorem for spaces, specifically, Theorem 4.3.13. Of course, the classical Whitehead theorem, in the form that a map $f : X \rightarrow Y$ of spaces is a homotopy equivalence if it induces isomorphisms on all homotopy groups at all base points, is already about spaces. We are going, instead, for the stronger form the Whitehead theorem takes in pointed connected spaces. Let \mathbf{Hot} denote the category of CW complexes and homotopy classes of continuous maps. Let $\mathbf{Hot}_{*,c}$ denote the category of pointed connected CW complexes and equivalence classes of pointed maps up to homotopies through pointed maps. Then the Whitehead theorem can be interpreted as follows:

Theorem 4.3.6 (Whitehead). *In the category $\mathbf{Hot}_{*,c}$, the spheres S^n jointly detect isomorphisms.*

This version of Whitehead's theorem is at the heart of various aspects of modern homotopy theory. For instance, the triangulated categories underlying presentable stable model

categories or stable ∞ -categories are known to admit such sets of objects. In that context and in pointed, connected spaces, the existence of a set of objects detecting isomorphisms is crucial to the proof of the Brown representability theorem. Thus it is an important flaw of the category **Hot** that it does *not* admit a generator. This was claimed in [Hel81], but Heller's proof only goes through to show that there exists no generator consisting of *finite* spaces.

A correct proof follows here, due to the author and J. Daniel Christensen in [CC19].

We make the following definitions. For an ordinal α , write Σ_α for the group of all bijections of the set α , ignoring order. When $\beta < \alpha$, there is a natural inclusion $\Sigma_\beta \hookrightarrow \Sigma_\alpha$, and we define Σ_α^c to be the union of the images of Σ_β for all $\beta < \alpha$. We typically consider Σ_α^c when α is a cardinal, considered as the smallest ordinal with that cardinality, and we call the elements of Σ_α^c *essentially constant* permutations.

Theorem 4.3.7. *The category **Hot** contains no set \mathcal{G} of spaces that jointly reflect isomorphisms.*

Proof. Let \mathcal{G} be a set of CW complexes and let α be an uncountable regular cardinal larger than the number of cells in each $S \in \mathcal{G}$. We must construct a map $f : X \rightarrow Y$ which is not a homotopy equivalence but which induces bijections on homotopy classes of maps from spaces in \mathcal{G} .

Our example will be $Bs : B\Sigma_\alpha^c \rightarrow B\Sigma_\alpha^c$, where $s : \Sigma_\alpha^c \rightarrow \Sigma_\alpha^c$ is the shift homomorphism given by

$$(s\sigma)(\gamma) = \begin{cases} \sigma(\gamma') + 1, & \gamma = \gamma' + 1 \\ \gamma, & \gamma \text{ a limit ordinal,} \end{cases}$$

for $\sigma \in \Sigma_\alpha^c$. (Here and in what follows, if γ is a successor ordinal, we write γ' for its predecessor.) We must check that $s\sigma \in \Sigma_\alpha^c$. First, it is essentially constant: if $\beta < \alpha$ and σ fixes each $\gamma \geq \beta$, then for $\gamma > \beta$ we have $(s\sigma)(\gamma) = \gamma$, if γ is a limit ordinal, and $(s\sigma)(\gamma) = \sigma(\gamma') + 1 = \gamma' + 1 = \gamma$, if γ is a successor. Next, we see that s is a homomorphism: $s(\sigma\tau)$ and $(s\sigma)(s\tau)$ both fix all limit ordinals, while for successors we have

$$(s\sigma)((s\tau)(\gamma)) = \sigma([\tau(\gamma') + 1]') + 1 = \sigma\tau(\gamma') + 1 = s(\sigma\tau)(\gamma),$$

as desired. Note that setting $\tau = \sigma^{-1}$, respectively $\sigma = \tau^{-1}$, we confirm that $s\sigma$ is indeed a bijection.

Recall that for groups G and H , $\mathbf{Hot}(BG, BH)$ is isomorphic to $\text{Hom}(G, H)$ modulo conjugation by elements of H , while an element of $\mathbf{Hot}(BG, BH)$ is a homotopy equivalence if and only if it is represented by an isomorphism. Also, for X connected, we have a natural isomorphism $\mathbf{Hot}(X, BH) \cong \mathbf{Hot}(B\pi_1(X), BH)$.

Note that s is not surjective, since $s\sigma$ always preserves limit ordinals. Therefore, $Bs : B\Sigma_\alpha^c \rightarrow B\Sigma_\alpha^c$ is not a homotopy equivalence. However, we will show that it induces an isomorphism on \mathcal{G} . First observe that it suffices to prove this for connected components of spaces in \mathcal{G} . It follows that it is enough to prove this for spaces of the form BG , where G is a group of cardinality less than α . (This uses that α is uncountable.)

Any map $BG \rightarrow B\Sigma_\alpha^c$ arises from a homomorphism $\varphi : G \rightarrow \Sigma_\alpha^c$, well-defined up to conjugation. Since α is regular, there is a limit ordinal $\beta < \alpha$ so that $\varphi(g) \in \Sigma_\beta$ for every $g \in G$. We claim that $s \circ \varphi$ is conjugate to φ by an element $\tau \in \Sigma_\alpha^c$ defined as follows:

$$\tau(\gamma) = \begin{cases} \gamma', & \gamma < \beta \text{ a successor ordinal} \\ \beta + \gamma, & \gamma < \beta \text{ a limit ordinal} \\ \gamma + 1, & \beta \leq \gamma < \beta + \beta \\ \gamma, & \text{otherwise.} \end{cases}$$

It is straightforward to check that τ is a permutation, and it clearly fixes ordinals greater than or equal to $\beta + \beta$, which is less than α . For $g \in G$, let $\sigma = \varphi(g)$. Then, noting that

$\tau^{-1}(\gamma) = \gamma + 1$ for any $\gamma < \beta$, we have

$$\begin{aligned}
(\tau^{-1}\sigma\tau)(\gamma) &= \begin{cases} \tau^{-1}(\sigma(\gamma')), & \gamma < \beta \text{ a successor ordinal} \\ \tau^{-1}(\sigma(\beta + \gamma)), & \gamma < \beta \text{ a limit ordinal} \\ \tau^{-1}(\sigma(\gamma + 1)), & \beta \leq \gamma < \beta + \beta \\ \tau^{-1}(\sigma(\gamma)), & \text{otherwise} \end{cases} \\
&= \begin{cases} \tau^{-1}(\sigma(\gamma')), & \gamma < \beta \text{ a successor ordinal} \\ \tau^{-1}(\beta + \gamma), & \gamma < \beta \text{ a limit ordinal} \\ \tau^{-1}(\gamma + 1), & \beta \leq \gamma < \beta + \beta \\ \tau^{-1}(\gamma), & \text{otherwise} \end{cases} \\
&= \begin{cases} \sigma(\gamma') + 1, & \gamma < \beta \text{ a successor ordinal} \\ \gamma, & \gamma < \beta \text{ a limit ordinal} \\ \gamma, & \beta \leq \gamma < \beta + \beta \\ \gamma, & \text{otherwise} \end{cases} \\
&= s(\sigma)(\gamma).
\end{aligned}$$

We have used that if $\gamma \geq \beta$, then $\sigma(\gamma) = \gamma$, and the consequence that if $\gamma < \beta$, then $\sigma(\gamma) < \beta$.

In summary, we have shown that Bs induces the identity on $\mathbf{Hot}(S, B\Sigma_\alpha^c)$ for every $S \in \mathcal{G}$, proving the claim. \square

Remark 4.3.8. Since the map $Bs : B\Sigma_\alpha^c \rightarrow B\Sigma_\alpha^c$ used in the proof has connected domain and codomain, it follows that there is no set of connected spaces that jointly reflect isomorphisms in the homotopy category of connected spaces.

We end this section with a remark about the origin of the maps s and τ . Morally, s is conjugation by the successor operation on ordinals, with limit ordinals handled specially. The map τ implements this by “making room” for the relevant limit ordinals in a range

outside of the support of a particular permutation σ . In fact, if we denote the map τ above by τ_β , then s itself is conjugation by τ_α in Σ_γ^c for a regular cardinal $\gamma > \alpha$.

Whitehead's theorem for the homotopy 2-category of spaces

We will now show that, while **Hot** lacks any set of spaces detecting isomorphisms, the homotopy 2-category of spaces **Hot** admits a very manageable such set, namely, the good old spheres. By **Hot**, we mean the full sub-2-category of $\infty\text{-Cat}$ spanned by the ∞ -groupoids, that is, by those ∞ -categories which may be modelled by Kan complexes. That is, $\mathbf{Hot}(X, Y) = \Pi_1(Y^X)$ is the fundamental groupoid of the usual mapping space.

We now introduce terminology for a set of objects satisfying Whitehead's theorem:

Definition 4.3.9. A small set \mathcal{G} of objects in a 2-category \mathcal{K} constitutes a *conservative-generating set* for \mathcal{K} if \mathcal{G} jointly detects equivalences. That is, whenever $f : X \rightarrow Y$ is a morphism in \mathcal{K} such that, for every $G \in \mathcal{G}$, $\mathcal{K}(G, f) : \mathcal{K}(G, X) \rightarrow \mathcal{K}(G, Y)$ is an equivalence of categories, we may conclude that f is an equivalence in \mathcal{K} .

We shall show in Theorem 4.3.11 that the 2-category **Hot** admits a small set \mathcal{G} of objects jointly detecting equivalences, namely $\mathcal{G} = \{S^n : n \in \mathbf{N}\}$.

First, we shall compute some homotopy groups of mapping spaces X^{S^k} . We abusively denote the constant map $S^k \rightarrow X$ valued at x by x . Then we have:

Lemma 4.3.10. *For any space X , any $x \in X$, and any $n > 0$, the homotopy group $\pi_n(X^{S^k}, x)$ is given by a semidirect product $\pi_n(X, x) \ltimes \pi_{n+k}(X, x)$.*

Of course, the semidirect product is direct in case $n > 1$.

Proof. The map $e : X^{S^k} \rightarrow X$ given by evaluation at a fixed point $*$ is a fibration with fiber over x given by $\Omega^k(X, x)$, the space of based maps $(S^k, *) \rightarrow (X, x)$. Furthermore, e is a split epimorphism, with splitting the map $X \rightarrow X^{S^k}$ assigning to $x \in X$ the constant map valued at x .

Thus if we consider X, X^{S^k} , and $\Omega^k(X, x)$ to be pointed by (the constant map valued at) x , the long exact sequence in homotopy groups determined by e degenerates into split short exact sequences

$$1 \rightarrow \pi_n \Omega^k(X, x) \rightarrow \pi_n(X^{S^k}, x) \rightarrow \pi_n(X, x) \rightarrow 1.$$

Since $\pi_n \Omega^k(X, x)$ is naturally identified with $\pi_{n+k}(X, x)$, the result follows. \square

With this, we are prepared to show the spheres satisfy the analogue of Whitehead's theorem for **Hot**.

Theorem 4.3.11. *The set $\mathcal{G} = \{S^n\}$ of spheres jointly detect equivalences in the 2-category **Hot** of spaces.*

Proof. Let $f : X \rightarrow Y$ be such that $\mathbf{Hot}(S^n, f) : \mathbf{Hot}(S^n, X) \rightarrow \mathbf{Hot}(S^n, Y)$ is an equivalence of groupoids, for every n . Firstly, the equivalences $\mathbf{Hot}(S^0, X) \rightarrow \mathbf{Hot}(S^0, Y)$ induce equivalences $\mathbf{Hot}(*, X) \rightarrow \mathbf{Hot}(*, Y)$, that is to say, equivalences $\Pi_1(X) \rightarrow \Pi_1(Y)$ of fundamental groupoids. Thus f induces an isomorphism on π_0 and on every π_1 , and we have only to show that it induces an isomorphism on every π_n .

Now, we also have by assumption that f induces isomorphisms $\mathbf{Hot}(S^n, X)(x, x) \rightarrow \mathbf{Hot}(S^n, Y)(f(x), f(x))$ for every $x \in X$, where again x denotes the constant map valued at x . In other words, f induces isomorphisms $\pi_1(X^{S^n}, x) \rightarrow \pi_1(Y^{S^n}, f(x))$ for each $x \in X$, and thus using Lemma 4.3.10, isomorphisms $\pi_1(X, x) \times \pi_{n+1}(X, x) \rightarrow \pi_1(Y, f(x)) \times \pi_{n+1}(Y, f(x))$ arising from maps from the short exact sequences

$$1 \rightarrow \pi_{n+1}(X, x) \rightarrow \pi_1(X, x) \times \pi_{n+1}(X, x) \rightarrow \pi_1(X, x) \rightarrow 1$$

to their analogues over Y . Thus since we have already shown f induces isomorphisms on π_1 , it induces isomorphisms of π_n as well, and by the classical form of Whitehead's theorem, f is a homotopy equivalence, and thus an equivalence in **Hot**. \square

Remark 4.3.12. In fact, for any cofibrant based space (A, a) , the map $e : X^A \rightarrow X$ from the space of unbased maps given by evaluation at a has the same properties as the evaluation map $X^{S^n} \rightarrow X$. That is, e is a fibration which admits a splitting by constant maps and whose

fiber is the space $(X, x)^{(A, a)}$ of based maps $(A, a) \rightarrow (X, x)$. Thus $\pi_n(X^A, x)$ is identified with a (semi)direct product of $\pi_n(X, x)$ and $\pi_n((X, x)^{(A, a)}, x)$. When the latter is understood, we get a recipe for producing sets detecting equivalences in **Hot**. In fact, our original example exhibited let $A = (S^1)^n$ be a finite-dimensional torus and computed homotopy groups of free loop spaces via this recipe.

Whitehead's theorem for ∞ -categories

We now rephrase Theorem 4.3.11 in a form more convenient for our purposes:

Theorem 4.3.13. *The restriction of $\text{HO} : \underline{\text{QCAT}} \rightarrow \underline{\text{PDer}}_{\underline{\text{HFin}}}$ to the 2-category **HOT** spanned by (possibly large) spaces reflects equivalences.*

Recall that equivalences in $\underline{\text{PDer}}_{\underline{\text{HFin}}}$ in the abstract 2-categorical sense coincide with pseudonatural transformations which induce equivalences of categories levelwise.

Proof. Given $f : X \rightarrow Y$ in **HOT**, the image $\text{HO}(f)$ is an equivalence in $\underline{\text{PDer}}_{\underline{\text{HFin}}}$ if and only if, for every homotopically finite category J , the induced functor $\text{Ho}(f^{N(J)}) : \text{Ho}(X^{N(J)}) \rightarrow \text{Ho}(Y^{N(J)})$ is an equivalence. Since the classical model structure on simplicial sets is Cartesian, we have equivalences $\text{Ho}(X^{N(J)}) \simeq \text{Ho}(X^{\text{Ex}^\infty(N(J))})$, and similarly for Y , where Ex^∞ is Kan's fibrant replacement functor. Now, by Thomason's theorem [Tho80], as J varies, $\text{Ex}^\infty(N(J))$ runs through all finite homotopy types. In particular, if $\text{HO}(f)$ is an equivalence in $\underline{\text{PDer}}$, then f induces equivalences $\text{Ho}(X^{S^n}) \rightarrow \text{Ho}(Y^{S^n})$ for every n , which is to say, $\underline{\text{HOT}}(S^n, f)$ is an equivalence. Thus f must be an equivalence, by Theorem 4.3.11. □

Now we can prove our Whitehead theorem for ∞ -categories.

Theorem 4.3.14. *Let $f : Q \rightarrow R$ be a map of ∞ -categories, and suppose that $\text{HO}(f)$ is an equivalence in $\underline{\text{PDer}}_{\underline{\text{HFin}}}$. Then f is an equivalence of ∞ -categories.*

Proof. Since $\text{HO}(f)$ is an equivalence by assumption, f is essentially surjective. Thus we have only to show f is fully faithful. By Theorem 4.3.13, it suffices to show that $\text{HO}(f)$ induces

an equivalence $\mathrm{HO}(Q(x, y)) \cong \mathrm{HO}(R(f(x), f(y)))$ in **PDer** for every x and y in Q . What is more, since for any J we have $Q(x, y)^{NJ} \cong Q^{NJ}(\ell_J^*x, \ell_J^*y)$, it suffices at last to show that f induces equivalences $f_{x,y} : \mathrm{Ho}(Q(x, y)) \rightarrow \mathrm{Ho}(R(f(x), f(y)))$ on the homotopy categories of mapping spaces.

Essential surjectivity is proved via an argument that also appeared in the construction of $\widehat{\Xi}$ in the proof of Theorem 4.2.1. Namely, from essential surjectivity of $\mathrm{HO}(f)$, given any $X \in \mathrm{Ho}(R(f(x), f(y)))$ and any $Y \in \mathrm{HO}(Q)([1])$ with an isomorphism $s : \mathrm{HO}(f)(Y) \cong X$ in $\mathrm{HO}(R)([1])$, we see by conservativity and fullness of $\mathrm{HO}(f)$ that we have isomorphisms $0^*Y \cong x \in \mathrm{HO}(Q)(J)$ and, similarly, $1^*Y \cong y$. Composing these isomorphisms and $\mathrm{dia}Y$ in $\mathrm{Ho}(Q)$ gives a morphism $x \rightarrow y$ in $\mathrm{Ho}(Q)$ isomorphic to $\mathrm{dia}Y$ in $\mathrm{Ho}(Q)^{[1]}$. By (Der5) and (Der2) we can lift this to an isomorphism $r : Y' \cong Y$ in $\mathrm{HO}(Q)([1])$ such that $0^*(s \circ \mathrm{HO}(f)(r)) = \mathrm{id}_x$ and $1^*(s \circ \mathrm{HO}(f)(r)) = \mathrm{id}_y$. This implies that $s \circ \mathrm{HO}(f)(r)$ may be lifted to an isomorphism $\mathrm{HO}(f)(Y') \cong X$ in $\mathrm{Ho}(R(f(x), f(y)))$. Thus $f_{x,y}$ is essentially surjective.

For fullness, we observe that if $a : Y_1 \rightarrow Y_2 \in \mathrm{HO}(Q)([1])$ verifies $Y_1, Y_2 : x \rightarrow y$, $0^*\mathrm{HO}(f)(a) = \mathrm{id}_{f(x)}$, and $1^*\mathrm{HO}(f)(a) = \mathrm{id}_{f(y)}$, then we have also $0^*(a) = \mathrm{id}_x$ and $1^*a = \mathrm{id}_y$, since $\mathrm{HO}(f)$ is faithful. This implies that a can be lifted to a morphism $a' : Y_1 \rightarrow Y_2$ in $\mathrm{Ho}(Q(x, y))$ with $f_{x,y}(a) = \mathrm{HO}(f)(a)$. And since $\mathrm{HO}(f)$ is full, every morphism $\mathrm{HO}(f)(Y_1) \rightarrow \mathrm{HO}(f)(Y_2)$ in $\mathrm{Ho}(R(f(x), f(y)))$ is equal to $\mathrm{HO}(f)(a)$ in $\mathrm{HO}(R)([1])$, for some a .

Finally, we turn to faithfulness. Suppose we have morphisms $a, b : Y_1 \rightarrow Y_2$ in $\mathrm{Ho}(Q(x, y))$ with $f_{x,y}(a) = f_{x,y}(b)$ in $\mathrm{Ho}(R(f(x), f(y)))$. We wish to show $a = b$. First, we may represent a and b by $\hat{a}, \hat{b} \in \mathrm{HO}(Q)([1] \times [1])$, each with boundary

$$\begin{array}{ccc} x & \xrightarrow{Y_1} & y \\ \parallel & & \parallel \\ x & \xrightarrow{Y_2} & y \end{array}$$

Let $\partial[2]$ denote the category on objects $0, 1, 2$ freely generated by *three* arrows $0 \rightarrow 1, 1 \rightarrow 2, 0 \rightarrow 2$, so that $N\partial[2]$ is Joyal equivalent to $\partial\Delta^2$. The lifts \hat{a} and \hat{b} fit together in a diagram

$W \in \text{HO}(Q)([1] \times \partial[2])$ of the following form:

$$\begin{array}{ccc} Y_1 & \xrightarrow{\hat{a}} & Y_2 \\ & \searrow^{\pi_1^* Y_1} & \nearrow^{\hat{b}} \\ & & Y_1 \end{array}$$

where $\pi_1 : [1] \times [1] \rightarrow [1]$ projects out the last coordinate.

The significance of W is that we have $a = b$ in $\text{Ho}(Q(x, y))(Y_1, Y_2)$ if and only if W admits an extension Z to $\text{HO}(Q)([1] \times [2])$ such that $Z|_{\{0\} \times [2]} = p_{[2]}^* x$ and $Z|_{\{1\} \times [2]} = p_{[2]}^* y$. It suffices to exhibit $W' \in \text{HO}(Q)([1] \times \partial[2])$ with $W'|_{0 \times \partial[2]} = p_{\partial[2]}^* x$ and $W'|_{1 \times \partial[2]} = p_{\partial[2]}^* y$ admitting such an extension Z' , together with an isomorphism $t : W \rightarrow W'$ in $\text{HO}(Q)([1] \times \partial[2])$ such that $t|_{0 \times \partial[2]} = \text{id}_{p_{\partial[2]}^* x}$ and $t|_{1 \times \partial[2]} = \text{id}_{p_{\partial[2]}^* y}$. Indeed, in this situation W and W' both represent maps from S^1 to the Kan complex $Q(x, y)$, Z and Z' represent putative extensions to Δ^2 , and t represents a homotopy between them.

In particular, since by assumption $\text{HO}(f)(a) = \text{HO}(f)(b)$ in $\text{Ho}(R(f(x), f(y)))$, there exists an extension T of $\text{HO}(f)(W)$ to $\text{HO}(R)([1] \times [2])$ with trivial endpoints, as above. Now take $\hat{T} \in \text{HO}(Q)([1] \times [2])$ with an isomorphism $s : \text{HO}(f)(\hat{T}) \cong T$. In particular, this gives isomorphisms $\text{HO}(f)(\hat{T})|_{\{0\} \times [2]} \cong p_{[2]}^* f(x)$ and $\text{HO}(f)(\hat{T})|_{\{1\} \times [2]} \cong p_{[2]}^* f(y)$ in $\text{HO}(R)([2])$, which lift uniquely to isomorphisms $\hat{T}|_{\{0\} \times [2]} \cong p_{[2]}^* x$ and $\hat{T}|_{\{1\} \times [2]} \cong p_{[2]}^* y$ in $\text{HO}(Q)([2])$.

Composing these isomorphisms with $\text{dia}\hat{T}$ and lifting into $\text{HO}(Q)([1] \times [2])$ gives $Z' \in \text{HO}(Q)([1] \times [2])$ with $Z'|_{\{0\} \times [2]} = p_{[2]}^* x$ and $Z'|_{\{1\} \times [2]} = p_{[2]}^* y$, together with an isomorphism $t' : \text{HO}(f)(Z') \cong T$ in $\text{HO}(R)([1] \times [2])$ inducing the identity on $p_{[2]}^* f(x)$ and $p_{[2]}^* f(y)$, respectively. Restricting t' to $[1] \times \partial[2]$ and lifting to $\text{HO}(Q)([1] \times \partial[2])$ specifies an isomorphism $t : Z'|_{[1] \times \partial[2]} \cong W$ such that $t|_{0 \times \partial[2]} = \text{id}_{p_{\partial[2]}^* x}$ and $t|_{1 \times \partial[2]} = \text{id}_{p_{\partial[2]}^* y}$. As we saw above, this suffices to guarantee that W admits an extension Z as desired. \square

Whitehead's theorem for the objects of a locally presentable ∞ -category

By making use of a bit more ∞ -categorical machinery, we can give another proof of Theorem 4.3.14 with more general applicability.

Definition 4.3.15. For any ∞ -category Q , let $\text{Ho}_2(Q)$ denote any $(2, 1)$ -category admitting

a map $Q \rightarrow N(\mathrm{Ho}_2(Q))$ inducing an equivalence $\underline{\infty - \mathbf{Cat}}(N(\mathrm{Ho}_2(Q)), R) \rightarrow \underline{\infty - \mathbf{Cat}}(Q, R)$ whenever R is the nerve of a $(2, 1)$ -category.

One construction of $\mathrm{Ho}_2 Q$ proceeds by taking the homotopy coherent realization $\mathfrak{C}Q$, a cofibrant simplicial category, taking its Bergner fibrant replacement $R\mathfrak{C}Q$, and then hitting each mapping Kan complex with the fundamental groupoid functor. For another construction, see [Lur09, 2.3.4.12]. By [Dus02, 8.6], $\mathrm{Ho}_2 Q$ constructed a la Lurie is isomorphic to the nerve of a $(2, 1)$ -category in the ordinary sense, and in fact the $(2, 1)$ -categories produced in this way are isomorphic: the mapping spaces in $R\mathfrak{C}Q$ are well known to be homotopy equivalent to those in Q , while the functors above preserve the sets of arrows between any two objects up to isomorphism.

Lemma 4.3.16. *Let $i : Q \rightarrow R$ be a fully faithful functor of ∞ -categories admitting a left adjoint L . If $\mathrm{Ho}_2(R)$ admits a conservative-generating set \mathcal{G} , then $L(\mathcal{G})$ is a conservative-generating set for $\mathrm{Ho}_2(Q)$.*

Proof. A map $f : x \rightarrow y$ in $\mathrm{Ho}_2(Q)$ is an equivalence if and only if $i(f)$ is an equivalence in $\mathrm{Ho}_2(R)$, if and only if $\mathrm{Ho}_2(R)(S, i(f))$ is an equivalence of groupoids for every $S \in \mathcal{G}$, if and only if $\mathrm{Ho}_2(Q)(L(S), f)$ is an equivalence for every $L(S) \in L(\mathcal{G})$. \square

Lemma 4.3.17. *If J is any small ∞ -category and \mathcal{S} is the ∞ -category of spaces, then $\mathrm{Ho}_2(\mathcal{S}^J)$ has a conservative-generating set.*

Proof. A map $f : x \rightarrow y$ in $\mathrm{Ho}_2(\mathcal{S}^J)$ is an equivalence if and only if each $f_j : x(j) \rightarrow y(j)$ is an equivalence in **Hot**, if and only if each $\mathbf{Hot}(S^n, f_j)$ is an equivalence of groupoids. Now f_j is identified with $\mathrm{Ho}_2(\mathcal{S}^J)(j_! *, f)$, $j_!$ denoting the left Kan extension along the functor $j : * \rightarrow J$ picking out j ; thus the set $\{j_! * \otimes (S^1)^n\}$ detects equivalences in $\mathrm{Ho}_2(\mathcal{S}^J)$, as was to be shown. \square

It may be worth recalling at this point that like **Hot**, the 1-categories $\mathrm{Ho}(\mathcal{S}^J)$ never admit conservative-generating sets, unless J is empty.

We immediately conclude:

Corollary 4.3.18. *Any reflective subcategory of a presheaf ∞ -category $\mathcal{S}^{J^{\text{op}}}$ admits a conservative-generating set.*

As in ordinary category theory, most familiar large ∞ -categories arise in this way.

Proposition 4.3.19. *The ∞ -category \mathcal{I} of small ∞ -categories is a reflective subcategory of a presheaf ∞ -category. Specifically, \mathcal{I} embeds reflectively in the ∞ -category $\mathcal{S}^{\Delta^{\text{op}}}$ of simplicial spaces.*

Proof. This is well-known and may be proven by showing that the full subcategory of \mathcal{I} spanned by $\{[n]\}_{n \in \mathbf{N}}$ is a dense generator composed of finitely presentable objects, so that \mathcal{I} is locally presentable. Concretely, the equivalence of \mathcal{I} with the complete Segal objects of $\mathcal{S}^{\Delta^{\text{op}}}$ is one choice of the desired reflective embedding. \square

We now can give the promised alternative proof of Whitehead's theorem for ∞ -categories.

Corollary 4.3.20. *The 2-category of ∞ -categories admits a conservative-generating set. In particular, $\text{HO} : \underline{\infty - \mathbf{CAT}} \rightarrow \mathbf{PDer}_{\mathbf{HFin}}$ is bicategorically conservative.*

Proof. By Proposition 4.3.19, such a set may be given by $\{[n] \times S^m\}$ as n and m vary over \mathbf{N} .

For the corollary regarding HO , for each m let J_m be a homotopically finite category whose nerve is weakly equivalent to S^m . Then the set $\{[n] \times N(J_m)\}$ will also serve as a conservative-generating set. Indeed, given a map $f : Q \rightarrow R$, the induced functor $\underline{\infty - \mathbf{CAT}}([n] \times N(J_m), f)$ preserves the category of functors out of $N(J_m) \rightarrow Q^{[n]}$ inverting all morphisms of J_m , which is equivalent to $\underline{\infty - \mathbf{CAT}}([n] \times S^m, Q)$.

Thus if the functor $\underline{\infty - \mathbf{CAT}}([n] \times J_m, f)$ is an equivalence, then so is $\underline{\infty - \mathbf{CAT}}([n] \times S^m, f)$. If $\text{HO}(f)$ is an equivalence, then so is each $\underline{\infty - \mathbf{CAT}}([n] \times J_m, f)$, each $\underline{\infty - \mathbf{CAT}}([n] \times S^m, f)$, and the restriction to the underlying groupoids $\text{Ho}_2(\mathcal{I})([n] \times S^m, f)$, so that f is an equivalence as desired. \square

We note that essentially the same argument would give an equivalence-detecting 2-functor $\mathrm{Ho}_2(Q) \rightarrow [\mathbf{Cat}^{\mathrm{op}}, \mathbf{GPD}]$ for any Q tensored over spaces and admitting a conservative-generating set. However, such a prederivator valued in groupoids misses the value of the derivator axioms; the proper setting for such a general result would let Q be an $(\infty, 2)$ -category.

4.4 Locally small phenomena

When the categories in \mathbf{Dia} are at least as big as the *mapping spaces* in the ∞ -categories under consideration, HO has strong positive properties falling short of bicategorical full faithfulness.

For the remainder of this section, fix a choice of \mathbf{Dia} and let $\underline{\infty - \mathbf{Cat}}$ denote the 2-category of those ∞ -categories whose mapping spaces are all equivalent to the geometric realization of a category in \mathbf{Dia} . Finally, let \mathbf{Hot} denote the homotopy 2-category of spaces equivalent to such a geometric realization. So when $\mathbf{Dia} = \mathbf{Cat}$, we are taking $\underline{\infty - \mathbf{Cat}}$ to be the *locally* small ∞ -categories and \mathbf{Hot} to be the *small* spaces, while for $\mathbf{Dia} = \mathbf{HFin}$ we consider the locally finite ∞ -categories, which we should note are very rare in practice, finite spaces not being closed under internal homs.

We find that these size conditions allow HO to capture all information about adjunctions in \mathbf{QCat} , and in particular that (co)complete ∞ -categories and their morphisms are detected by HO . Note that there are no presentability conditions in force below.

Theorem 4.4.1. *Fix \mathbf{Dia} -locally small ∞ -categories Q, R and a map $f : Q \rightarrow R$. Then we have the following:*

1. *There exists a right (or a left) adjoint to f in $\underline{\infty - \mathbf{Cat}}$ if and only if $\mathrm{HO}(f)$ admits a right (left) adjoint in \mathbf{PDer} .*
2. *The ∞ -category Q admits (co)limits of shape J if and only if pullback along each projection $\pi_2 : J \times K \rightarrow K$ admits a right (left) adjoint in $\mathrm{HO}(Q)$.*
3. *Assuming they exist, the morphism f preserves (co)limits of shape J if and only if*

$\mathrm{HO}(f)$ does, in the sense that the canonical comparison map $\mathrm{colim}_J \circ \mathrm{HO}(f)_J \Rightarrow \mathrm{HO}(f)_{[0]} \circ \mathrm{colim}(J) : \mathrm{Ho}(Q^{NJ}) \rightarrow \mathrm{Ho}(R)$ is an isomorphism.

4. More generally, f preserves left (right) Kan extensions along $u : J \rightarrow J'$ if and only if $\mathrm{HO}(f)$ does.

Proof. Point (2) follows from (1) as soon as we interpret its assumption on $\mathrm{HO}(Q)$ as the existence of a right (left) adjoint to the diagonal morphism $\mathrm{HO}(Q) \rightarrow \mathrm{HO}(Q)^J$.

The proofs of (1) and (3) are very similar. Suppose that $\mathrm{HO}(f)$ has a right adjoint G . Then in attempting to construct a right adjoint g to f , we have a candidate on objects given by $g(y) = G(y)$ for each $y \in R_0$. It suffices, then, to show that we have equivalences $Q(x, g(y)) \simeq R(f(x), y)$ which are natural in x . We shall construct these equivalences as transformations, natural in X , $\mathbf{Hot}(X, Q(x, g(y))) \simeq \mathbf{Hot}(X, R(f(x), y))$.

Let K be a category in **Dia** with nerve equivalent to X . Then we have isomorphisms, natural in X and x :

$$\begin{aligned} \mathbf{Hot}(X, Q(x, g(y))) &\cong \mathbf{Hot}(X, \widehat{Q}(\hat{x}, \widehat{g(y)})) \\ &\cong \mathrm{Ho}(\widehat{Q})(X \otimes \hat{x}, \widehat{g(y)}) \cong \mathrm{Ho}(\widehat{Q}^{NK})(p_K^* \hat{x}, p_K^* \widehat{g(y)}) \\ &\cong \mathrm{Ho}(\widehat{Q}^{NK})(\widehat{p_K^* x}, \widehat{p_K^* g(y)}) \cong \mathrm{Ho}(Q^{NK})(p_K^* x, p_K^* g(y)). \end{aligned}$$

Here we passed through the Yoneda embedding $(\widehat{-})$ in case Q itself lacks tensors, in which case the argument can be simplified; we have used the full faithfulness of the Yoneda embedding and the construction of tensoring with a space as $p_i^J p_j^*$ in a derivator, due to [Cis08].

Similarly, we find $\mathbf{Hot}(X, R(f(x), y)) \cong \mathrm{Ho}(R^{NK})(p_K^* f(x), p_K^* g(y))$. Now we use the assumption that G is right adjoint to $\mathrm{HO}(f)$ to find:

$$\begin{aligned} \mathbf{Hot}(X, Q(x, g(y))) &\cong \mathrm{Ho}(Q^{NK})(p_J^* x, p_J^* g(y)) \cong \mathrm{Ho}(Q^{NK})(p_K^* x, G(p_K^* y)) \\ &\cong \mathrm{Ho}(R^{NK})(f \circ p_K^* x, p_K^* y) \cong \mathrm{Ho}(R^{NK})(p_K^* f(x), p_K^* y) \cong \mathbf{Hot}(X, R(f(x), y)). \end{aligned}$$

Thus g may be extended to a right adjoint of f , as desired.

Turning to point (3), given a J -shaped diagram $D : NJ \rightarrow Q$ we must show that the canonical comparison map $\text{colim}(f \circ D) \rightarrow f(\text{colim}D)$ is an isomorphism in $\text{Ho}(R)$. We shall show that we have an induced equivalence of spaces $R(y, \text{colim}(f \circ D)) \rightarrow R(y, f(\text{colim}D))$ for every $y \in R_0$, by showing finally that $\mathbf{Hot}(X, R(y, \text{colim}(f \circ D))) \rightarrow \mathbf{Hot}(X, R(y, f(\text{colim}D)))$ is an isomorphism of sets for every space $X \simeq N(K)$.

As above, we have a natural isomorphism $\mathbf{Hot}(X, R(y, \text{colim}(f \circ D))) \cong \text{Ho}(R^{NK})(p_K^*y, p_K^*\text{colim}(f \circ D))$ and similarly, $\mathbf{Hot}(X, R(y, f(\text{colim}D))) \cong \text{Ho}(R^{NK})(p_K^*y, p_K^*f(\text{colim}D))$, so it suffices to show that the induced maps

$$\text{Ho}(R^{NK})(p_K^*y, p_K^*\text{colim}(f \circ D)) \rightarrow \text{Ho}(R^{NK})(p_K^*y, p_K^*f(\text{colim}D))$$

are all isomorphisms. By assumption, $\text{HO}(f)$ preserves colimits of shape J , so we are done.

The proof of (4) is a straightforward generalization of the proof of (3). \square

Remark 4.4.2. There is a simpler argument for (3) in case R admits all tensors by spaces, in which we use instead the isomorphism

$$\mathbf{Hot}(X, R(y, \text{colim}(f \circ D))) \cong \text{Ho}(R)(X \otimes y, \text{colim}(f \circ D))$$

to see the same result: preserving colimits on the homotopy category is enough. Our argument permits the generalization to the case in which R has only those colimits forced to exist by lying in the image of f . This exhibits a common phenomenon in homotopy theory: certain coherence conditions required to show that colimits actually exist, namely, the existence of left Kan extensions along projections, are redundant to show that given colimits are preserved by a functor.

CHAPTER 5

Brown representability and its consequences

We prove a Brown representability theorem for a class of 2-categories broad enough to include the main examples arising from homotopy theory, and use it to give conditions under which a prederivator is represented by an ∞ -category.

5.1 Compactly generated 2-categories

The 2-categories at interest will be, in particular, weakly cocomplete, see Definition 2.1.9. A weakly cocomplete 2-category is missing a last endowment necessary for Brown representability: a set of compact generators.

Definition 5.1.1. Let \mathcal{K} be a weakly cocomplete 2-category. We say that an object $X \in \mathcal{K}$ is *compact* if for every sequential diagram $D = (Y_0 \rightarrow Y_1 \rightarrow \dots)$ with weak colimit Y , the induced functor $\text{colim} \mathcal{K}(X, Y_i) \rightarrow \mathcal{K}(X, Y)$ is an equivalence.

Remark 5.1.2. While our (weak) colimits are generally *pseudo*, in the definition of compactness it is equivalent to consider the *strict* colimit $\text{colim}^{\text{str}} \mathcal{K}(X, Y_i)$. The reason is that strict filtered colimits of categories are always equivalent to pseudo filtered colimits. A homotopy-theoretic explanation for this phenomenon is that the canonical model structure on categories is combinatorial, with the domains and codomains of generating cofibrations and trivial cofibrations all finitely presentable.

Note that we cannot, in a weakly cocomplete 2-category, give any obvious notion of λ -compactness for uncountable cardinals λ , insofar as we cannot expect weak λ -filtered colimits to exist. We leave the question of formulating, if it should exist, a theory of well generated (λ -compactly generated) homotopy 2-categories to future work, and for now define:

Definition 5.1.3. A weakly cocomplete 2-category \mathcal{K} is called *compactly generated* if it admits a conservative-generating set \mathcal{G} , each object of which is compact.

Examples of compactly generated 2-categories

Our next goal is to build weak colimits in Ho_2Q (see Definition 4.3.15) from colimits in Q .

Proposition 5.1.4. *If Q is a cocomplete ∞ -category, then Ho_2Q is weakly cocomplete.*

In particular, this holds for the underlying $(2, 1)$ -category of a completely arbitrary model category.

Proof. We first note that for a $(2, 1)$ -category to be weakly cocomplete, it suffices to construct coproducts and either weak cocommas or weak pseudo-coequalizers, coinverters being vacuous.

As with the ordinary homotopy category, it is straightforward to show that a coproduct in Q represents a coproduct in Ho_2Q . Indeed, a coproduct in Q is characterized by the condition that the natural map $Q(\coprod x_i, y) \rightarrow \prod Q(x_i, y)$ be a weak equivalence of Kan complexes. Since the fundamental groupoid functor Π_1 preserves products of Kan complexes, such a coproduct also has the universal property that $\Pi_1Q(\coprod x_i, y) \simeq \prod \Pi_1Q(x_i, y)$ is an equivalence of groupoids.

If we have a parallel pair $f, g : x \rightrightarrows y$ in Q with coequalizer $q : y \rightarrow z$ and canonical isomorphism $\alpha : qf \rightarrow qg$, then we have an equivalence $Q(z, w) \rightarrow \mathrm{hoeq}(Q(y, w) \rightrightarrows Q(x, w))$, where we are taking a homotopy equalizer in spaces. It is by no means the case that Π_1 preserves homotopy equalizers.¹

Instead, what we would like to show is that the comparison map

$$\mathrm{Ho}_2Q(z, w) \rightarrow \mathrm{ps} - \mathrm{eq}(\mathrm{Ho}_2Q(y, w) \rightrightarrows \mathrm{Ho}_2Q(x, w))$$

¹ For instance, the free loop space LX is given as the homotopy equalizer of id_X with itself. If X has trivial fundamental groups, then Π_1X is a discrete groupoid, so that $L\Pi_1X$ (that is, $\mathbf{Gpd}(\Pi_1S^1, X)$) is equivalent to Π_1X and is, in particular, simply connected. But as $\pi_1(LX)$ is a semidirect product of π_1X and π_2X , so if X has nontrivial second homotopy group then we see Π_1 does not preserve L .

is full and essentially surjective. For essential surjectivity, given a map $y \xrightarrow{t} w$ together with an isomorphism $\lambda : tf \cong tg$ in $\mathrm{Ho}_2Q(x, w)$, lifting λ to a particular edge in $Q(x, w)$ determines a vertex of $\mathrm{hoeq}(Q(y, w) \rightrightarrows Q(x, w))$, which by assumption lifts to $Q(z, w)$. Taking the image of such a lift back down in $\mathrm{Ho}_2Q(z, w)$, we have proved essential surjectivity.

For fullness, consider maps $u, v : z \rightarrow w$ inducing the objects $(uq, uq\alpha)$ and $(vq, vq\alpha)$ of $\mathrm{ps} - \mathrm{eq}(\mathrm{Ho}_2Q(y, w) \rightrightarrows \mathrm{Ho}_2Q(x, w))$. Suppose we are given a morphism between the latter objects, which amounts to a map $\mu : uq \rightarrow vq$ that makes the square

$$\begin{array}{ccc} uqf & \xrightarrow{uq\alpha} & uqg \\ \downarrow \mu f & & \downarrow \mu g \\ vqf & \xrightarrow{vq\alpha} & vqg \end{array}$$

commute in $\mathrm{Ho}_2Q(x, w)$. Then any lift of μ to $Q(y, w)_1$ and any choice of a map $\Delta^1 \times \Delta^1 \rightarrow Q$ witnessing the assumed commutativity produces an edge of $\mathrm{hoeq}(Q(y, w) \rightrightarrows Q(x, w))$, which again lifts to an edge of $Q(z, w)$ which represents the desired morphism in $\mathrm{Ho}_2Q(z, w)$.

Thus we have shown what we needed: if Q has coproducts and coequalizers, which is equivalent to it being cocomplete, then Ho_2Q is weakly cocomplete. \square

Now we consider the relationship between generators of ∞ -categories and of their underlying $(2, 1)$ -categories.

Definition 5.1.5. A conservative-generating set for an ∞ -category Q is a small set \mathcal{G} of objects such that a map $X \rightarrow Y$ in Q is an equivalence if and only if the induced map $Q(G, X) \rightarrow Q(G, Y)$ is an equivalence of Kan complexes for every $G \in \mathcal{G}$.

Proposition 5.1.6. *Let Q be an ∞ -category with a generating set \mathcal{G} . If Q admits finite colimits, then Ho_2Q admits a conservative-generating set given by $\{G \otimes S^n\}$ as n runs over \mathbf{N} and G runs over \mathcal{G} .*

Proof. A map $f : X \rightarrow Y$ in Ho_2Q is an equivalence if and only if for each $G \in \mathcal{G}$, each $Q(G, f)$ is an equivalence in **Hot**, if and only if each $\mathbf{Hot}(S^n, Q(G, f))$ is an equivalence of groupoids, if and only if each $\mathrm{Ho}_2Q(S^n \otimes G, f)$ is an equivalence of groupoids. \square

Recall that an ∞ -category Q is *locally finitely presented* by a small set \mathcal{G} if Q has small colimits, every object of Q is a colimit of objects of \mathcal{G} , and maps out of the objects of \mathcal{G} commute with filtered colimits. (See [Lur09, Chapter 5].) In particular, it follows easily that \mathcal{G} is a conservative-generating set for Q . We now see that locally finitely presented ∞ -categories have compactly generated homotopy 2-categories.²

Proposition 5.1.7. *Suppose Q is an ∞ -category locally finitely presented by the set \mathcal{G} . Then $\mathcal{G}' := \{S^n \otimes G : S \in \mathcal{G}\}$ gives a compact generating set for Ho_2Q .*

Proof. By Proposition 5.1.6, \mathcal{G} is a conservative-generating set for Ho_2Q . We must show its elements are compact in the sense of Definition 5.1.1.

Given $S^n \otimes G \in \mathcal{G}'$ and a weak colimit X of a sequence $(X_i)_{i \in \mathbf{N}}$ in Ho_2Q , we first observe that, using the freeness of \mathbf{N} as a category, the sequence X_i lifts to a map $\mathbf{N} \rightarrow Q$, whose colimit may be identified with X . Then we have

$$\begin{aligned} \mathrm{Ho}_2Q(S^n \otimes G, X) &\simeq \underline{\mathbf{Hot}}(S^n, Q(G, X)) = \underline{\mathbf{Hot}}(S^n, \mathrm{colim}_i Q(G, X_i)) = \\ &\mathrm{colim} \underline{\mathbf{Hot}}(S^n, Q(G, X_i)) = \mathrm{colim} \mathrm{Ho}_2Q(S^n \otimes G, X_i). \end{aligned}$$

Here we have used compactness of S^n in $\underline{\mathbf{Hot}}$. □

Finally, some more specific examples:

Example 5.1.8. Most familiar “large” ∞ -categories fall under the scope of Proposition 5.1.7. For instance, the ∞ -categories of spaces, of ∞ -categories, of A_∞ and E_∞ ring spaces and spectra, of functors from a fixed small ∞ -category into any of the above, and of slices over or under an object in any of the above are all locally finitely presentable. This is not to mention the stable cases of spectra, the derived categories of classical rings, et cetera, and the intermediate case of pointed connected spaces, which also have compactly generated homotopy 1-categories.

Note that the homotopy 2-category of each example above can also be constructed from the model category that models it, and so all these examples can be interpreted in that language.

²We repeat once more that no such result holds for the homotopy 1-category.

The 2-category of ∞ -categories is compactly generated

The examples above, arising from $(\infty, 1)$ -categories as they do, are all in fact compactly generated $(2, 1)$ -categories. There is presumably a more natural result along the line that “any locally finitely presentable $(\infty, 2)$ -category has a compactly generated homotopy 2-category.” Lacking a general definition of presentability for an $(\infty, 2)$ -category, we content ourselves here with the leading example:

Proposition 5.1.9. *The 2-category $\underline{\infty - \mathbf{Cat}}$ is compactly generated.*

Proof. It was proved in Proposition 2.2.10 that $\underline{\infty - \mathbf{Cat}}$ is weakly cocomplete. It was proved in Theorem 4.3.14 that the homotopically finite categories constitute a conservative-generating set in $\underline{\infty - \mathbf{Cat}}$. It thus suffices to show that for a homotopically finite category J , the nerve $N(J)$ is a compact object of $\underline{\infty - \mathbf{Cat}}$, which follows from the fact that by definition both $N(J)$ and $N(J) \times [1]$ admit only finitely many nondegenerate simplices. \square

5.2 Formulation and proof of Brown representability

We now turn to Brown representability proper.

Definition 5.2.1. If \mathcal{K} is a weakly cocomplete 2-category, then we call a 2-functor $H : \mathcal{K}^{\text{op}} \rightarrow \underline{\mathbf{Cat}}$ a *cohomological 2-functor* if:

1. H sends coproducts to products and coinverters to inverters.
2. Whenever Z is a weak coequalizer of the parallel pair $X \rightrightarrows Y$ and $E \rightarrow H(Y) \rightrightarrows H(X)$ is the pseudo-equalizer³ of the induced diagram of categories, the canonical comparison functor $H(Z) \rightarrow E$ is full, conservative, and essentially surjective.

The complexity of the second clause in the definition of a cohomological 2-functor is necessitated by the uniqueness of weak colimits in 2-category theory (see Proposition 2.1.7.)

³a.k.a inserter

We cannot ask that $H(Z)$ simply be a weak equalizer, because this would force the map to E to be an equivalence, which will essentially never occur.

The obvious examples of cohomological 2-functors are the representable 2-functors. The primary aim of this chapter is to demonstrate the converse in the compactly generated case.

As a notational convention, if $f : A \rightarrow B$ is a map in a weakly cocomplete 2-category \mathcal{K} and we are given a cohomological 2-functor $H : \mathcal{K}^{\text{op}} \rightarrow \mathbf{Cat}$ together with $y \in H(B)$, we will denote $H(f)(y)$ by $y \cdot f$, and if $\alpha : f \Rightarrow g$, $H(\alpha)_y : H(f)(y) \rightarrow H(g)(y)$ by $y \cdot \alpha : y \cdot f \rightarrow y \cdot g$.

We now give the Brown representability theorem proper.

Theorem 5.2.2. *Let \mathcal{K} be a compactly generated 2-category. Then every cohomological 2-functor $H : \mathcal{K}^{\text{op}} \rightarrow \mathbf{Cat}$ is equivalent to a representable 2-functor.*

Proof. We construct an object which represents H 's restriction to the compact generators by applying Proposition 5.2.3 to the initial object \emptyset of \mathcal{K} , together with the essentially unique object $0 \in H(\emptyset)$. Then Proposition 5.2.5 implies that in fact X represents H on the entirety of \mathcal{K} . \square

Proposition 5.2.3. *Consider a cohomological functor H on a 2-category \mathcal{K} , compactly generated by \mathcal{G} . Given an object Y of \mathcal{K} and an object $y \in H(Y)$, there exist another object $X \in \mathcal{K}$, an object $x \in H(X)$, and a map $f : Y \rightarrow X$ such that:*

1. *There exists an isomorphism $x \cdot f \cong y$ and*
2. *For every $S \in \mathcal{G}$, the map $x \cdot (-) : \mathcal{K}(S, X) \rightarrow H(S)$ is an equivalence.*

We shall summarize condition (2) by saying that X "represents H on the generators."

Proof. We will construct X as the weak colimit of a countable chain. To start, we define

$$X_0 = Y \sqcup \coprod_{S \in \mathcal{G}, s \in H(S)} S.$$

Since H preserves coproducts, there exists $x_0 \in H(X_0)$ such that, if $i_{S,s} : S \rightarrow X_0$ is one of the canonical inclusions, we have $x_0 \cdot i_{S,s} \cong s$, while $x_0 \cdot i_Y \cong y$.

We now proceed by induction. If i is even, then let $t_i : X_i \rightarrow X_{i+1}$ and $x_{i+1} \in H(X_{i+1})$ be as in Lemma 5.2.6. If i is odd, then let them be as in Lemma 5.2.7. Finally, let X be the weak colimit of the sequence thus constructed. If $c_i : X_i \rightarrow X$ is the canonical map, then since H ‘‘preserves weak coequalizers’’ in the sense of Definition 5.2.1, we can find $x \in H(X)$ with isomorphisms $x \cdot c_i \cong x_i$ such that the squares

$$\begin{array}{ccc} x \cdot c_{i+1} \circ t_i & \longrightarrow & x \cdot c_i \\ \downarrow & & \downarrow \\ x_{i+1} \cdot t_i & \longrightarrow & x_i \end{array} \quad (5.2.4)$$

commute.

We now must prove that X represents H on the generators. Fix $S \in \mathcal{G}$. The essential surjectivity of the functor $x \cdot (-) : \mathcal{K}(S, X) \rightarrow H(S)$ follows from the fact that it factors through the essentially surjective functor $x_0 \cdot (-) : \mathcal{K}(S, X_0) \rightarrow H(S)$, up to isomorphism.

To prove fullness and faithfulness, it is expedient to assume that $\mathcal{K}(S, X)$ is isomorphic to the strict colimit $\text{colim}_i \mathcal{K}(S, X_i)$ of categories. As was justified in Remark 5.1.2, the strict colimit is certainly equivalent to $\mathcal{K}(S, X)$, and $x \cdot (-) : \mathcal{K}(S, X) \rightarrow H(S)$ is an equivalence if it factors through an equivalence $\text{colim}_i \mathcal{K}(S, X_i) \rightarrow H(S)$ via that equivalence.

This permits us to write $c_i \circ (-) : \mathcal{K}(S, X_i) \rightarrow \mathcal{K}(S, X)$ and $t_i \circ (-) : \mathcal{K}(S, X_i) \rightarrow \mathcal{K}(S, X_{i+1})$ as the identity. Now given $f, g : S \rightarrow X$ with $u : x \cdot f \rightarrow x \cdot g$, we can assume f and g both factor strictly through X_i , say $f = c_i \circ f'$ and $g = c_i \circ g'$. We construct $u' : x_i \cdot f' \rightarrow x_i \cdot g'$ by requiring the commutativity of the following diagram:

$$\begin{array}{ccccccc} x \cdot f & \xlongequal{\quad} & x \cdot (c_i \circ f') & \xlongequal{\quad} & (x \cdot c_i) \cdot f' & \xrightarrow{\cong} & x_i \cdot f' \\ \downarrow u & & \downarrow u & & \downarrow u & & \downarrow u' \\ x \cdot g & \xlongequal{\quad} & x \cdot (c_i \circ g') & \xlongequal{\quad} & (x \cdot c_i) \cdot g' & \xrightarrow{\cong} & x_i \cdot g' \end{array}$$

Now define $f'' : S \rightarrow X_{i+2}$ as $t_{i+1} t_i f'$ and similarly for g'' . Then by construction we have $\alpha : f'' \Rightarrow g'' : S \rightarrow X_{i+2}$ such that the diagram

$$\begin{array}{ccccc} x_i \cdot f' & \xrightarrow{\cong} & (x_{i+2} \cdot t_{i+1} t_i) \cdot f' & \xlongequal{\quad} & x_{i+2} \cdot f'' \\ \downarrow u' & & \downarrow & & \downarrow x_{i+2} \cdot \alpha \\ x_i \cdot g' & \xrightarrow{\cong} & (x_{i+2} \cdot t_{i+1} t_i) \cdot g' & \xlongequal{\quad} & x_{i+2} \cdot g'' \end{array}$$

commutes. Since by assumption $c_{i+2}f'' = c_{i+2}t_{i+1}t_i f' = c_i f' = f$, we can finish by showing the composed square

$$\begin{array}{ccc} x \cdot f & \xlongequal{\quad} & x \cdot c_{i+2}f'' \\ \downarrow u & & \downarrow x \cdot c_{i+2}\alpha \\ x \cdot g & \xlongequal{\quad} & x \cdot c_{i+2}g'' \end{array}$$

commutes. This follows from the fact that the composition

$$x \cdot c_i \rightarrow x_i \rightarrow x_{i+2} \cdot t_{i+1}t_i \rightarrow x \cdot c_{i+2} \cdot t_{i+1}t_i$$

is the identity, as can be seen by juxtaposing two copies of Diagram 5.2.4.

Faithfulness is more straightforward. Given $\alpha, \beta : f \Rightarrow g : S \rightarrow X$ with $x \cdot \alpha = x \cdot \beta$, we have that α, β, f and g all factor through some X_i on the nose via f', g', α', β' . We have a commutative diagram

$$\begin{array}{ccc} x \cdot f & \xrightarrow{\quad} & x_i \cdot f' \\ \begin{array}{c} x \cdot \beta \left(\downarrow \right) \\ \downarrow \end{array} x \cdot \alpha & & \begin{array}{c} x_i \cdot \beta' \left(\downarrow \right) \\ \downarrow \end{array} x_i \cdot \alpha' \\ x \cdot g' & \xrightarrow{\quad} & x_i \cdot g \end{array}$$

induced by the canonical map $x \cdot c_i \rightarrow x_i$, where we use again the identification of $x \cdot f$ with $(x \cdot c_i) \cdot f'$, as well as the similar identifications involving α, β , and g . This implies that $x_i \cdot \beta' = x_i \cdot \alpha'$, whence α' and β' must become equal in X_{i+2} , and thus in X .

□

The above proposition says that every cohomological functor is representable on the set of generators, while the following proposition says that this is all we need for the theorem.

Proposition 5.2.5. *Suppose given a cohomological functor $H : \mathcal{K}^{\text{op}} \rightarrow \mathbf{CAT}$ with \mathcal{K} compactly generated together with $X \in \mathcal{K}$ and $x \in H(X)$. If the induced functor $x \cdot (-) : \mathcal{K}(S, X) \rightarrow H(S)$ is an equivalence for each object S in the conservative-generating set \mathcal{G} , then $x \cdot (-) : \mathcal{K}(-, X) \rightarrow H$ is an equivalence of 2-functors.*

Proof. Consider some W and $w \in H(W)$. The reason for the relative aspect of Proposition 5.2.3 is to justify why there should be any interesting maps $W \rightarrow X$ at all, as is necessary for the claim.

So let us take Z with $z \in H(Z)$ representing H on the generators and a map $g : W \rightarrow Z$ with $z \cdot g \cong w$, as in Proposition 5.2.3. Then we observe that any map $f : Z \rightarrow X$ such that $x \cdot f \cong z$ must be an equivalence. Indeed, in that case the triangle

$$\begin{array}{ccc} \mathcal{K}(S, Z) & \xrightarrow{f} & \mathcal{K}(S, X) \\ & \searrow^{z \cdot} & \swarrow_{x \cdot} \\ & & H(S^n) \end{array}$$

commutes up to isomorphism for every $S \in \mathcal{G}$, while the diagonal arrows are equivalences.

We claim next that such a map f exists. We first take $Z \sqcup X$, together with the object $(z, x) \in H(Z \sqcup X)$. Now we apply the construction of Proposition 5.2.3 again, producing a space U with $u \in H(u)$ representing H on the generators and equipped with a map $Z \sqcup X \rightarrow U$ pulling u back to (z, x) up to isomorphism. By the argument of the previous paragraph, the induced maps $Z \rightarrow Z \sqcup X \rightarrow U$ and $X \rightarrow Z \sqcup X \rightarrow U$ are both equivalences. Thus by composing with a quasi-inverse, we get a map $f : Z \rightarrow X$ with $x \cdot f \cong z$ as desired. In particular, $f \circ g : W \rightarrow X$ satisfies $x \cdot f \circ g \cong w$, so $\mathcal{K}(W, X) \rightarrow H(W)$ is essentially surjective.

Now consider maps $f, g : W \rightarrow X$ together with a morphism $u : x \cdot f \rightarrow x \cdot g$ in $H(W)$. We must produce a 2-morphism $\alpha : f \Rightarrow g$ with $x \cdot \alpha = u$. We begin by taking the weak coequalizer $q : X \rightarrow C$ of f and g , which comes with a canonical 2-morphism $\gamma : qf \Rightarrow qg$. Since H is cohomological, there exists $c \in H(C)$ together with a commutative square

$$\begin{array}{ccc} x \cdot f & \longrightarrow & c \cdot q \cdot f \\ \downarrow u & & \downarrow c \cdot \gamma \\ x \cdot g & \longrightarrow & c \cdot q \cdot g \end{array}$$

induced by an isomorphism $x \rightarrow c \cdot q$ in $H(X)$. Now perform the construction of Proposition 5.2.3 on C and c , getting D with $d \in H(D)$, together with a map $h : C \rightarrow D$ with $d \cdot h \cong c$. The naturality of $(-) \cdot \gamma$ yields the commutative square

$$\begin{array}{ccc} c \cdot qf & \longrightarrow & d \cdot h \cdot qf \\ \downarrow c \cdot \gamma & & \downarrow d \cdot h \cdot \gamma \\ c \cdot qg & \longrightarrow & d \cdot h \cdot qg \end{array}$$

Now setting $m := hq : X \rightarrow D$, since $d \cdot m \cong x$, we may conclude that m is an equivalence. Denote an inverse equivalence by k ; using the unit $\eta : \text{id}_D \rightarrow mk$ of the equivalence $k \dashv m$, we get an induced isomorphism $d \rightarrow d \cdot mk \rightarrow x \cdot k$. Again using naturality of $(-) \cdot \gamma$, we get another commutative square

$$\begin{array}{ccc} d \cdot mf & \longrightarrow & x \cdot kmf \\ \downarrow d \cdot h \cdot \gamma & & \downarrow x \cdot kh \cdot \gamma \\ d \cdot mg & \longrightarrow & x \cdot kmg \end{array}$$

Pasting the previous three commutative squares, we get

$$\begin{array}{ccc} x \cdot f & \longrightarrow & x \cdot kmf \\ \downarrow u & & \downarrow x \cdot kh \cdot \gamma \\ x \cdot g & \longrightarrow & x \cdot kmg \end{array}$$

A straightforward computation shows that the horizontal maps in the square above can be identified with $x \cdot \varepsilon^{-1}f$, where $\varepsilon : km \rightarrow \text{id}_X$ is the counit of the equivalence. Thus we have that u is in the image of $x \cdot (-) : \mathcal{K}(W, X) \rightarrow H(W)$, as desired.

Finally, we settle faithfulness. So suppose that we have $\alpha, \beta : f \rightrightarrows g : W \rightarrow X$ such that $x \cdot \alpha = x \cdot \beta$. We must show $\alpha = \beta$. For this, consider a map $(W \otimes \bullet \rightrightarrows \bullet) \rightarrow X$ restricting to (α, β, f, g) up to isomorphism. Let L denote the weak cocomma $X \overset{\leftarrow}{\sqcup}_{W \otimes \bullet \rightrightarrows \bullet} W \otimes (\bullet \rightarrow \bullet)$ and let $r : X \rightarrow L$ be the induced map.

Since H is cohomological, there exists $\ell \in H(L)$ with $\ell \cdot r \cong x$. If we now apply Proposition 5.2.3 to (L, ℓ) , then we get P with $p \in H(P)$ and $t : L \rightarrow P$. In particular, $t \circ r \circ \alpha = t \circ r \circ \beta$. Since $p \cdot t \cdot r \cong x$, we see that $t \circ r : X \rightarrow P$ is an equivalence. Since post-composition with an equivalence is faithful, we conclude that $\alpha = \beta$ as desired. \square

Lemma 5.2.6. *Given an object Y of a compactly generated $(2, 1)$ -category \mathcal{K} and a cohomological functor H on \mathcal{K} with a fixed $y \in H(Y)$, there exists a map $q : Y \rightarrow Y'$ and a $y' \in H(Y')$ satisfying the following properties:*

- *There exists an isomorphism $y' \cdot q \rightarrow y$.*
- *Whenever $S \in \mathcal{G}$ and we have $\alpha, \beta : f \rightrightarrows g : S \rightarrow Y$ with $y \cdot \alpha = y \cdot \beta$, then also $\alpha * q = \beta * q$.*

Proof. Let I be the set of quintuples (S, f, g, α, β) with $S \in \mathcal{G}$, $f, g : S \rightarrow Y$, $\alpha, \beta : f \Rightarrow g$ such that $y \cdot \alpha = y \cdot \beta$. Then there is a canonical parallel pair of 2-morphisms

$$\coprod_I S \begin{array}{c} \xrightarrow{\sqcup f} \\ \left(\begin{array}{c} \xrightarrow{\sqcup \beta} \\ \xrightarrow{\sqcup \alpha} \end{array} \right) \\ \xrightarrow{\sqcup g} \end{array} Y.$$

We may choose a morphism $(\coprod_I S \otimes \bullet \rightrightarrows \bullet) \rightarrow Y$ representing the above data, and then we let Y' denote the weak cocomma of the span

$$\left(\coprod_I S \otimes \bullet \rightarrow \bullet \right) \leftarrow \left(\coprod_I S \otimes \bullet \rightrightarrows \bullet \right) \rightarrow Y.$$

We choose the desired $y' \in H(Y)$ using the preservation of weak cocommas by H . \square

Lemma 5.2.7. *Given an object Y of a compactly generated 2-category \mathcal{K} generated by \mathcal{G} and a cohomological functor H on \mathcal{K} with a fixed $y \in H(Y)$, there exists a map $q : Y \rightarrow Y'$ and a $y' \in H(Y')$ satisfying the following properties:*

- *There exists an isomorphism $i : y' \cdot q \rightarrow y$.*
- *Whenever $S \in \mathcal{G}$ and we have $f, g : S \rightarrow Y$ and $u : y \cdot f \rightarrow y \cdot g$ in $H(S)$, there exists $\alpha : q \circ f \Rightarrow q \circ g$ such that the square*

$$\begin{array}{ccc} y \cdot f & \xrightarrow{i \cdot f} & y' \cdot qf \\ \downarrow u & & \downarrow y' \cdot \alpha \\ y \cdot g & \xrightarrow{i \cdot g} & y' \cdot qg \end{array}$$

commutes in $H(S)$.

Proof. Consider the parallel pair $\coprod f, \coprod g : \coprod_T S \rightrightarrows Y$, where T runs over the triples (f, g, u) , where $f, g : S \rightarrow Y$ and $u : y \cdot f \rightarrow y \cdot g$. We let $q : Y \rightarrow Y'$ be the weak coequalizer of this pair, and choose y' so that the required squares commute when we take α to be restricted from the canonical morphism $\alpha' : q \circ \coprod f \rightarrow q \circ \coprod g$. \square

5.3 Extensions and first applications

To assuage any coherence-related concerns, let us note:

Proposition 5.3.1. *Brown representability holds also for mere pseudofunctors $H : \mathcal{K}^{\text{op}} \rightarrow \mathbf{Cat}$.*

Proof. Let $\mathcal{K}' \rightarrow \mathcal{K}$ be an equivalence such that every pseudofunctor out of \mathcal{K}' is equivalent to a strict 2-functor. For instance, we could let \mathcal{K}' be a cofibrant replacement of \mathcal{K} in Lack's model structure for 2-categories. For the sufficiency of this choice, see [Lac04, Lemma 5].

Then H also induces a cohomological 2-functor on the compactly generated 2-category \mathcal{K}' , which is thus representable on \mathcal{K}' . Now an inverse equivalence $\mathcal{K} \rightarrow \mathcal{K}'$ gives a representation for H . □

As usual, a representability theorem is tantamount to an adjoint functor theorem.

Proposition 5.3.2. *Let $F : \mathcal{K} \rightarrow \mathcal{L}$ be a pseudofunctor between locally small 2-categories such that \mathcal{K} is weakly cocomplete and satisfies Brown representability and F preserves coproducts, coinverters, and weak cocommas. Then F admits a right adjoint.*

Proof. It suffices to show that each functor $\mathcal{L}(F(-), Y) : \mathcal{K}^{\text{op}} \rightarrow \mathbf{Cat}$ is representable. This follows immediately from cocontinuity of F and Brown representability for \mathcal{K} . □

In particular, we can show that compactly generated 2-categories admit products.

Corollary 5.3.3. *Every weakly cocomplete 2-category satisfying Brown representability admits products.*

Proof. If \mathcal{K} is compactly generated, then so is the 2-category $\prod_I \mathcal{K}$ for any small set I : the colimits are constructed termwise while the generator is $\prod_I \mathcal{G}$, where \mathcal{G} generates \mathcal{K} . The diagonal 2-functor $\Delta_I : \mathcal{K} \rightarrow \prod_I \mathcal{K}$ preserves coproducts, coinverters, and weak cocommas, so by Proposition 5.3.2 it admits a right adjoint, which is an I -indexed product pseudofunctor for \mathcal{K} . □

Every conceivable 2-category satisfies Brown representability

It is well known that Brown representability passes along localizations. This situation persists in the 2-categorical setting, though localizations do not preserve compact objects and so a localization of a compactly generated 2-category need not be compactly generated.

Corollary 5.3.4. *Suppose \mathcal{L} is any 2-category admitting a fully faithful pseudo-functor $i : \mathcal{L} \rightarrow \mathcal{K}$ such that i admits a left 2-adjoint L and \mathcal{K} is weakly cocomplete and satisfies Brown representability. Then \mathcal{L} is weakly cocomplete and satisfies Brown representability.*

Proof. First, we must verify that \mathcal{L} is weakly cocomplete. This follows as usual for coproducts and coinverters. Now given a span $Y \xleftarrow{f} X \xrightarrow{g} Z$ in \mathcal{L} , take the weak cocomma W of if and ig in \mathcal{K} . We claim that LW is a weak cocomma of Lif and Lig , thus of f and g . Indeed, for any T we have a full, conservative, and essentially surjective functor

$$\begin{aligned} \mathcal{L}(LW, T) &\simeq \mathcal{K}(W, iT) \rightarrow \mathcal{K}(Y, iT) \overset{\leftarrow}{\times}_{\mathcal{K}(X, iT)} \mathcal{K}(Z, iT) \\ &\simeq \mathcal{L}(LY, T) \overset{\leftarrow}{\times}_{\mathcal{L}(LX, T)} \mathcal{L}(LZ, T) \end{aligned}$$

as desired.

Now then, given a cohomological functor $H : \mathcal{L}^{\text{op}} \rightarrow \mathbf{Cat}$, since L preserves weak colimits, HL is also a cohomological functor on \mathcal{K} , which by Brown representability for \mathcal{K} is representable by some object, say X . We must show that X is equivalent to iY for some Y : then for any $Z \in \mathcal{L}$ we will have

$$\mathcal{L}(Z, Y) \simeq \mathcal{K}(iZ, iY) \simeq \mathcal{K}(iZ, X) \simeq HL(iZ) = H(LiZ) \simeq H(Z),$$

as desired. As in ordinary category theory, we can identify \mathcal{L} with the local objects with respect to the unit maps $\eta_W : W \rightarrow iLW$ as W runs over objects of \mathcal{K} . That is, X is in the essential image of i if and only if η_W induces equivalences $\mathcal{K}(iLW, X) \rightarrow \mathcal{K}(W, X)$ for every W . Since $HL(\eta_W) = H(L\eta_W)$ is an equivalence for every W , so is $\mathcal{K}(\eta_W, X)$. Thus X is local and, in particular, in the essential image of i , as was to be shown. \square

This leads to numerous examples of 2-categories satisfying Brown representability.

Corollary 5.3.5. *The homotopy 2-category of any locally finitely presentable ∞ -category satisfies Brown representability.*

Proof. This follows immediately from Proposition 5.1.4, Proposition 5.1.7, and Theorem 5.2.2. □

Corollary 5.3.6. *The homotopy 2-category of any locally presentable ∞ -category satisfies Brown representability. Indeed, this holds for any localization of a presheaf ∞ -category.*

Proof. The second claim holds because a localization of ∞ -categories induces a localization of the homotopy 2-categories, and thus from Corollary 5.3.4. The first claim is a special case. □

The relationship with compactly generated triangulated categories

We next consider implications of Brown representability for the underlying 1-category.

If $I : \mathbf{Cat} \rightarrow \mathbf{Set}$ denotes the functor of isomorphism classes, then from any 2-category \mathcal{K} we may construct the underlying category $I_*\mathcal{K}$.

Proposition 5.3.7. *If \mathcal{K} is weakly cocomplete, then $I_*\mathcal{K}$ has coproducts and distinguished weak coequalizers, well defined up to isomorphism. Compact objects in \mathcal{K} are compact in $I_*\mathcal{K}$ with respect to the resulting distinguished weak colimits of countable sequences.*

Proof. The coproducts in $I_*\mathcal{K}$ are constructed as in \mathcal{K} ; that they are still coproducts follows from the fact that K preserves products.

By “distinguished weak coequalizers” we mean a choice of isomorphism class of weak coequalizer for every parallel pair $f, g : x \rightrightarrows y$. We choose the isomorphism class of a weak pseudo-coequalizer of f and g in \mathcal{K} , which is well defined since equivalences in \mathcal{K} become isomorphisms in $I_*\mathcal{K}$. The last claim follows from the fact that K preserves filtered colimits. □

On the other hand, $I_*\mathcal{K}$ will generally not have a generating set, compact or no, even if \mathcal{K} has one. If \mathcal{G} generates \mathcal{K} and $IK(S, f)$ is an isomorphism for every $S \in \mathcal{G}$, we only know

that $\mathcal{K}(S, f)$ is essentially surjective and reflects the existence of an isomorphism between two objects. The resulting gap cannot be gotten around by any technical trick as was used in Theorem 5.2.2, as indicated by the fact that the 1-category **Hot** admits no small generating set whatsoever.

However, something can be said by stabilizing the situation. Say a set \mathcal{G} is a *weak generator* for \mathcal{K} if whenever $\mathcal{K}(S, X)$ is equivalent to the terminal category for every $S \in \mathcal{G}$, then X must be a terminal object of \mathcal{K} , and analogously for a 1-category.

Proposition 5.3.8. *Let the 2-category \mathcal{K} admit a terminal object 1 and weak tensors with $\bullet \rightrightarrows \bullet$. Then a class \mathcal{G} of objects of \mathcal{K} is a weak generator if and only if the closure of \mathcal{G} under tensors with $\bullet \rightrightarrows \bullet$ is a weak generator of $I_*\mathcal{K}$.*

Proof. Let us assume that \mathcal{G} is a class of objects in \mathcal{K} closed under weak tensors with $\bullet \rightrightarrows \bullet$. If \mathcal{G} is a weak generator of $I_*\mathcal{K}$, then it is certainly a weak generator of \mathcal{K} , since if $\mathcal{K}(S, X)$ is trivial for every S then the same holds for $IK(S, X)$.

For the converse, suppose \mathcal{G} is a weak generator for \mathcal{K} and that for each $S \in \mathcal{G}$, the set $IK(S, X)$ is a singleton, so that each category $\mathcal{K}(S, X)$ admits a single isomorphism class of objects. Up to equivalence, that is, $\mathcal{K}(S, X)$ is a monoid. To show that X is terminal, we must thus show that any two endomorphisms in $\mathcal{K}(S, X)$ are equal.

Given $\alpha, \beta : f \rightrightarrows f : S \rightarrow X$, we may choose a map $A : S \otimes \bullet \rightrightarrows \bullet \rightarrow X$ restricting to (α, β, f, f) up to isomorphism. By assumption, A is isomorphic to to the composite $(S \otimes \bullet \rightrightarrows \bullet) \rightarrow S \xrightarrow{f} X$ representing two copies of id_f . Thus $(\alpha, \beta, f, f) \cong (\text{id}_f, \text{id}_f, f, f)$ in $\mathcal{K}(S, X)^{\bullet \rightrightarrows \bullet}$, implying $\alpha = \beta$ as desired. \square

A particularly consequential case of the above result is when \mathcal{K} is a weakly cocomplete 2-category equipped with a generating set such that $I_*\mathcal{K}$ is triangulated by the candidate triangles coming from weak iso-cocoma squares in \mathcal{K} . For in that case, the weakly generating set for $I_*\mathcal{K}$ is in fact conservatively generating.

We have shown:

Corollary 5.3.9. *Let \mathcal{T} be a triangulated category which has a model in the weak sense that $\mathcal{T} \simeq I_*\mathcal{K}$ for some weakly cocomplete 2-category \mathcal{K} , and that distinguished triangles $x \rightarrow y \rightarrow z \rightarrow x[1]$ are those for which $z = y \overset{\leftrightarrow}{\sqcup}_0 x$ and the map $z \rightarrow x[1]$ is consistent with the cocone giving z the isococoma structure. If \mathcal{K} is compactly generated, then so is \mathcal{T} .*

We can thus consider the theory of compactly generated $(2, 1)$ -categories as an extension of the theory of compactly generated triangulated categories.

The classifying space of a topological group

As an example application, we give a new construction of the classifying space of a topological group.

Proposition 5.3.10. *Given a topological group G , consider the 2-functor $BG : \mathbf{Hot}^{\text{op}} \rightarrow \mathbf{Gpd}$ sending a space X to its groupoid of principal G -bundles and isomorphisms between them. Then BG is a cohomological functor, and in particular is representable by a space.*

Proof. It is obvious that BG preserves coproducts and vacuous that it preserves coinverters. To show that it preserves weak cocommas, consider a span $Z \leftarrow X \rightarrow Y$, in which we may assume the maps are cofibrations between cofibrant topological spaces.

Given principal G -bundles $E \rightarrow Z, F \rightarrow Y$, and an isomorphism $i : E|_X \rightarrow F|_X$, cover X by opens on which both E and F are trivial. Then define a principal G -bundle T on $X \times I$ as a quotient of $E|_X \times [0, 1] \sqcup F|_X \times (0, 1]$ via the relation that identifies $E|_X \times (0, 1)$ with $F|_X \times (0, 1)$ via i . Then T, E , and F glue together to define a space S over the homotopy pushout $W = Z \sqcup_{X \times I} Y$ which we claim is a principal G -bundle. Indeed, the only fact left to prove is that S can be trivialized over points on the intersections of Z and Y with $X \times I$ in W . It suffices to consider a point of the form $(x, 0)$. If U is a neighborhood of x in Z over which E is trivial, then $E|_X$ is trivial over $U \cap X$, so that $U \cup (U \cap X) \times [0, \frac{1}{2})$ is a trivializing neighborhood in W .

The above has shown that BG preserves weak pushouts on the level of objects. Turning to fullness of the canonical comparison map, we must show that a map of principal G -bundles

over W can always be constructed from a coherent family of maps of the restrictions to X, Y , and Z . First note that, given a principal G -bundle T over $X \times I$, there exists a unique isomorphism $T \rightarrow T|_{X \times \{0\}} \times I$ in $BG(X \times I)$ which restricts to the identity over $X \times \{0\}$. Indeed, this follows from the fact that $p : X \times I \rightarrow X$ and $i_0 : X \rightarrow X \times I$ induce mutually inverse equivalences under BG .

Now given G -bundles S and S' over W , we consider an isomorphism between the induced spans of G -bundles over $Z \leftarrow X \rightarrow Y$. Concretely, let $i_0 : S|_{X \times I} \rightarrow S|_{X \times \{0\}} \times I, i'_0 : S'|_{X \times \{0\}} \times I \rightarrow S'|_{X \times I}$, and similarly i_1, i'_1 , be determined as above. Then such a map amounts to isomorphisms $j : S|_Z \rightarrow S'|_Z$ and $k : S|_Y \rightarrow S'|_Y$ such that the morphisms $s, t : S|_{X \times I} \rightarrow S'|_{X \times I}$ defined below are equal in $BG(X \times I)$. Here we define s as

$$S|_{X \times I} \xrightarrow{i_0} S|_{X \times \{0\}} \times I \xrightarrow{j|_{X \times I}} S'|_{X \times \{0\}} \times I \xrightarrow{i'_0} S'|_{X \times I}$$

and t as

$$S|_{X \times I} \xrightarrow{i_1} S|_{X \times \{0\}} \times I \xrightarrow{k|_{X \times I}} S'|_{X \times \{0\}} \times I \xrightarrow{i'_1} S'|_{X \times I}.$$

In particular, there is a homotopy between s and t over $X \times I$. So, we have a map $S|_{Z \cup X \times I} \rightarrow S'|_{Z \cup X \times I}$ which extends, up to homotopy, to a map $S \rightarrow S'$. Since $S|_{Z \cup X \times I} \rightarrow S$ is a cofibration of spaces over W , we can extend to get an actual map as desired. Thus BG preserves weak pushouts, and thus weak coequalizers, and thus it is representable, as was to be shown. \square

Remark 5.3.11. It would be very interesting to imitate this example for, say, right fibrations in the context of $\infty\text{-Cat}$, to directly construct the ∞ -category of spaces. Unfortunately, the standard way to show that the functor of right fibrations respects cocommas of ∞ -categories relies on its representability, see [Cis19]. This is what allows one to show that a right fibration over a simplicial set is equivalent to a right fibration over its fibrant replacement, which is a kind of locality condition getting around the fact that a pushout of quasicategories is much bigger than the pushout of the underlying simplicial sets. What one seems to need is, contrary to the usual trend in thought surrounding the homotopy hypothesis, is a *less* algebraic model of ∞ -categories.

For instance, we might ask:

Question: Is there a model category for ∞ -categories equipped with a forgetful functor to **Set** which creates colimits and in which the fibrant objects are closed under colimits?

It might be hoped that a model category of directed topological spaces could be designed for this purpose.

5.4 Representability of (pre)derivators

Our immediate motivation for studying Brown representability for $\underline{\infty} - \mathbf{Cat}$ was to represent (pre)derivators, and it is to that topic we now turn.

Representable prederivators

By a *small* prederivator we mean one taking values in the 2-category \mathbf{Cat} of small categories.

Proposition 5.4.1. *Let \mathbb{D} be a small, strong, localizing ∞ -prederivator. Then \mathbb{D} is representable by an ∞ -category: there exists an ∞ -category Q together with an equivalence $\mathrm{HO}(Q) \simeq \mathbb{D}$.*

Proof. We have precisely assumed that \mathbb{D} preserves coproducts, coinverters, and weak comma objects. Thus the claim follows from Theorem 5.2.2. \square

We shall now consider how to extend this result, to an extent, to large strong ∞ -prederivators. It is common that we are presented with a large prederivator and wish to see that it arises from an ∞ -category. The Brown representability theorem is not immediately helpful here, as a large ∞ -prederivator will never be representable by a small ∞ -category. However, in most natural examples, the prederivator of interest may be approximated by small prederivators, and it is to this point we now turn. We first consider closed sub-prederivators, in analogy to closed subfunctors of a presheaf.

Definition 5.4.2. Let \mathbb{D} be an ∞ -prederivator. We say a sub-prederivator \mathbb{D}' is *closed* if:

1. Whenever $a \in \mathbb{D}(\coprod A_i)$ and for each i we have $a|_{A_i} \in \mathbb{D}'(A_i)$, then also $a \in \mathbb{D}'(\coprod A_i)$.

2. Whenever $p : J \rightarrow J[\mathcal{W}^{-1}]$ is an ∞ -localization and $x \in \mathbb{D}(J[\mathcal{W}^{-1}])$ with $p^*x \in \mathbb{D}'(J)$, then $x \in \mathbb{D}'(J[\mathcal{W}^{-1}])$.
3. Whenever $P = B \overset{\leftarrow}{\sqcup}_A C$ is a weak cocomma, $l \in \mathbb{D}(P)$, and we have both $l|_B \in \mathbb{D}'(B)$ and $l|_C \in \mathbb{D}'(C)$, then $l \in \mathbb{D}'(P)$ as well.
4. \mathbb{D}' is full and replete in \mathbb{D} .

Corollary 5.4.3. *Let \mathbb{D} be a strong, localizing ∞ -prederivator and $\mathbb{D}' \subset \mathbb{D}$ be a small closed sub-prederivator. Then \mathbb{D}' is representable by an ∞ -category.*

Proof. Under the assumptions, \mathbb{D}' is a small, strong, localizing ∞ -prederivator, so the claim follows from Proposition 5.4.1. □

Note that closed sub-prederivators are closed under intersection, so that each sub-prederivator admits a closure. However, for representability purposes, we need to know furthermore that this closure is small. Unfortunately, it is far from clear that this always holds, in particular with respect to point (2). Instead, we define mild conditions under which small pre-derivators to admit small closures.

Definition 5.4.4. We say a set A is *moderate* if it may be written as an **Ord**-indexed union of small subsets. We say a category is moderate if it is locally small, with a moderate set of isomorphism classes.

Finally, we call a prederivator \mathbb{D} moderate if each category $\mathbb{D}(J)$ is moderate and if the fibers of the underlying diagram functors $\text{dia} : \mathbb{D}(J) \rightarrow \mathbb{D}([0])^J$ are essentially small.

While we know well that the underlying diagram functors $\text{dia}^J : \mathbb{D}(J) \rightarrow \mathbb{D}([0])^J$ are often far from full, and thus need not be nearly injective in isomorphism classes even though they are conservative, here we require only that the degree of failure of this injectivity be small.

We can now show that in every moderate prederivator, small sub-prederivators admit small closures.

Proposition 5.4.5. *Let $\mathbb{D} : \infty\text{-Cat}^{\text{op}} \rightarrow \mathbf{CAT}$ be a moderate ∞ -prederivator. Then for every small sub-prederivator \mathbb{D}' of \mathbb{D} , the closure $\overline{\mathbb{D}'}$ remains small.*

Proof. Let us write $\mathbb{D}([0])$ as an **Ord**-indexed union of small full replete subcategories $\mathbb{D}_\alpha([0])$ and define, for each α , the full subcategory $\mathbb{D}_\alpha(J)$ of $\mathbb{D}(J)$ as the inverse image under $\text{dia}^J : \mathbb{D}(J) \rightarrow \mathbb{D}([0])^J$ of $\mathbb{D}_\alpha([0])^J$. Then each \mathbb{D}_α is a sub-prederivator of \mathbb{D} , which is small by assumption on the fibers of dia . Furthermore, \mathbb{D}_α is closed, since

Let $\widehat{\mathbb{D}'}$ of \mathbb{D} denote the small sub-prederivator of \mathbb{D} so that $\widehat{\mathbb{D}'}(J)$ consists of the inverse image under dia^J of the full replete subcategory of $\mathbb{D}([0])$ generated by $\mathbb{D}'([0])$. Then $\widehat{\mathbb{D}'}$ is (essentially) small, by assumption on the fibers of dia .

Furthermore, $\widehat{\mathbb{D}'}$ is closed. Indeed, consider $u_i : J_i \rightarrow J$, any family of functors in **Dia**, jointly surjective on objects. We observe that, if $X \in \mathbb{D}(J)$ and each $u_i^* X \in \widehat{\mathbb{D}'}(J_i)$, then also $X \in \widehat{\mathbb{D}'}(J)$, since whether X is in $\widehat{\mathbb{D}'}$ is determined by the values $j^* X$ for each $j \in J$. Since all the conditions for closedness of a sub-prederivator depend precisely on respect for a jointly surjective-on-objects family of functors, (respectively $(A_i \rightarrow \sqcup A_i), J \rightarrow J[W^{-1}], (B, C \rightarrow B \overset{\leftarrow}{\sqcup}_A C)$) this justifies the claim.

Thus \mathbb{D}' is contained in a small closed sub-prederivator, so that its closure must be small. □

Example 5.4.6. If Q is any locally small ∞ -category with $\text{Ho}(Q)$ moderate, then the associated ∞ -prederivator $\text{HO}(Q)$ is moderate. Local smallness of Q implies that of $\text{HO}(Q)(J)$ for each J , while the fiber of $\text{dia} : \text{HO}(Q)(J) \rightarrow \text{HO}(Q)([0])^J$ over $F : J \rightarrow \text{HO}(Q)([0])$ is small. Indeed, it suffices to note that if $X \in \text{HO}(Q)(J)$ is in the fiber over $\text{ob}F$, then X factors through the small full subcategory of Q spanned by the objects in the image of F .

Beyond that, it is also clear that the moderate prederivators are closed under small limits and colimits. On the other hand, there are pathological examples of prederivators valued in moderate categories which are not moderate, easily produced for instance by requiring $\mathbb{D}([0])$ to be terminal.

Every reasonable derivator is locally representable

We have established that the left derivator associated to any moderate ∞ -category is strong, moderate, and homotopically locally small. We conclude by indicating that there are, at least locally, no other examples of strong, moderate, and homotopically locally small derivators.

Theorem 5.4.7. *Let $\mathbb{D} : \mathbf{Cat}^{\text{op}} \rightarrow \mathbf{CAT}$ be a left derivator which is moderate, strong, and homotopically locally small. Then every small full sub-prederivator of \mathbb{D} embeds fully faithfully in $\text{HO}(Q)$ for some small ∞ -category Q .*

Conversely, every full sub-prederivator of a prederivator of the form $\text{HO}(Q)$ embeds fully faithfully in a moderate, strong, homotopically locally small left derivator.

Proof. Given such a \mathbb{D} , let $\hat{\mathbb{D}}$ denote the localizing ∞ -derivator extending \mathbb{D} constructed in Theorem 3.4.7. Any small sub-prederivator $\mathbb{D}' \subset \mathbb{D}$ extends to a small sub-prederivator $\hat{\mathbb{D}}' \subset \hat{\mathbb{D}}$, which admits a small closure $\mathbb{E} := \overline{\hat{\mathbb{D}}}'$ by Proposition 5.4.5. Then by Corollary 5.4.3, \mathbb{E} is representable by a small ∞ -category Q , so that the same holds for \mathbb{D}' .

The converse follows from the fact that the Yoneda embedding gives a fully faithful embedding of a small ∞ -category Q into a moderate, cocomplete ∞ -category. \square

In particular, a small prederivator $\mathbb{D}' : \mathbf{Cat}^{\text{op}} \rightarrow \mathbf{Cat}$ is representable by an ∞ -category Q if and only if \mathbb{D}' embeds as a closed sub-prederivator of a moderate, strong, and homotopically locally small derivator \mathbb{D} , if and only if \mathbb{D}' extends to a strong, localizing ∞ -prederivator, and if and only if \mathbb{D}' admits an extension to an ∞ -prederivator embedding as a closed sub-prederivator of a strong, localizing left ∞ -derivator.

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