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Monotonicity Properties of Regenerative Sets and Lorden's Inequality

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Dedicated to Professor Masatoshi Fukushima

Abstract. Lorden's inequality asserts that the mean return time in a renewal process with (iid) interarrival times $Y_1, Y_2, \ldots$, is bounded above by $2\mathbf{E}[Y_1]/\mathbf{E}[Y_1^2]$. We establish this result in the context of regenerative sets, and remove the factor of 2 when the regenerative set enjoys a certain monotonicity property. This property occurs precisely when the Lévy exponent of the associated subordinator is a *special Bernstein* function. Several equivalent stochastic monotonicity properties of such a regenerative set are demonstrated.

Keywords: Renewal process, Regenerative set, Subordinator, Bernstein function

2020 *Mathematics Subject Classification:* 60K05, 60J55, 60J30

1. Introduction. Let $Y_1, Y_2, \ldots$ be i.i.d. positive random variables with finite variance, and use their partial sums $W_n := \sum_{k=1}^n Y_k$, to form a renewal process $W = (W_n)_{n\ge 1}$. For $t > 0$, define $N(t) := \#\{n \ge 1 : W_n \le t\}$, the number of renewals up to time t, and let $R_t := W_{N(t)+1} - t$ denote the time until the next renewal after time t. Although the distribution of R_t is not particularly simple to express, Lorden [15] has shown that

$$\mathbf{E}[R_t] \le \frac{\mathbf{E}[Y_1^2]}{\mathbf{E}[Y_1]}, \qquad \forall t > 0. \tag{1}$$

In view of Wald's Identity

$$\mathbf{E}[W_{N(t)+1}] = \mathbf{E}[Y_1] \cdot \mathbf{E}[N(t) + 1], \tag{2}$$

the inequality (1) also provides an upper bound on the *renewal function* $\mathbf{E}[N(t)]$. In this paper we examine the analog of (1) in the context of regenerative sets (a continuous analog of renewal processes), and look at a class of such sets in possession of a monotonicity property that leads to an improvement of Lorden's inequality that is sharp in a certain sense.

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbf{P})$ be a filtered probability space. We fix once and for all a *regenerative set* M. This is an $(\mathcal{F}_t)$-progressively measurable set $M \subset \Omega \times [0,\infty[$, with closed sections $M(\omega) := \{t \geq 0 : (\omega, t) \in M\}$, such that $\mathbf{P}[0 \in M] = 1$ and such that for each $(\mathcal{F}_t)$-stopping time T with $T(\omega) \in M(\omega)$ for each $\omega \in \{T < \infty\}$, the (shifted) post-$T$ portion of M, defined by its sections

$$\theta_T M(\omega) := (M(\omega) - T(\omega)) \cap [0,\infty[, \tag{3}$$

is independent of $\mathcal{F}_T$ and has the same distribution as M, on the event $\{T < \infty\}$. The reader is referred to [6] or [13] for more details on such random sets.

It is known that the Lebesgue measure of M is either a.s. strictly positive or a.s. null. This is Kingman's *heavy/light* dichotomy [13, pp. 74–76]. The results presented here are true with slight modifications in the heavy case, but for definiteness we assume M to be light; that is $\int_0^\infty 1_{\{t \in M(\omega)\}}\, dt = 0$ for $\mathbf{P}$-a.e. $\omega \in \Omega$. There is a second dichotomy, according as $T_0 := \inf(M \cap]0,\infty[)$ satisfies $\mathbf{P}[T_0 = 0] = 0$ or 1. (It must be one or the other because of Blumenthal's 0–1 law.) In the former case, the random set M is discrete; this is the renewal process case. We shall stick to the latter situation (the *unstable* case in Kingman's terminology), in which case M has perfect sections $M(\omega)$ for $\mathbf{P}$-a.e. $\omega \in \Omega$. The generic example of such a regenerative set is the closure of the level set

$$\{t \geq 0 : X_t = x_0\} \tag{4}$$

of a right-continuous strong Markov process $X = (X_t)_{t\geq 0}$ started in a regular point x_0.

There are several stochastic processes associated with M that facilitate its study. First is the *last exit* process $G = (G_t)_{t\geq 0}$,

$$G_t := \sup(M \cap [0,t]), \qquad t \geq 0, \tag{5}$$

and the associated *age* process $A = (A_t)_{t\geq 0}$,

$$A_t := t - G_t, \qquad t \geq 0. \tag{6}$$

Both A and G are right continuous and adapted to $(\mathcal{F}_t)$, and

$$M = \{t : A_t = 0\}. \tag{7}$$

Moreover A is a time-homogeneous strong Markov process.

Next is the *return time* process

$$D_t := \inf\{s > t : s \in M\} = \inf(M \cap]t,\infty[), \qquad t \geq 0, (\inf \emptyset := \infty), \tag{8}$$

and the related *remaining life* process

$$R_t := D_t - t, \qquad t \geq 0. \tag{9}$$

These are also right-continuous processes. Notice that each D_t is an $(\mathcal{F}_t)$-stopping time and that R and D are optional with respect to the "advance" filtration

$(\mathcal{F}_{D_t})_{t\geq 0}$, and (R_t) is a strong Markov process with respect to this larger filtration, with values in $[0,\infty]$.

Finally, there is the *local time* process $L = L(M) = (L_t)_{t\geq 0}$; this is the unique continuous increasing process adapted to the filtration of A, increasing precisely on M, a.s., normalized so that $\mathbf{E}\int_0^\infty e^{-s}\,dL_s = 1$, and additive in the sense that

$$L_{t+s} = L_t + L_s(\theta_t M), \qquad \forall s,t \geq 0, \text{ a.s.} \tag{10}$$

Here $\theta_t M := (M-t)\cap[0,\infty[$. Thus $L_s(M)$ is a functional of the part $M\cap[0,s]$ of M, while $L_s(\theta_t M)$ is the same functional of $\theta_t M$. One can access L through the general theory of the additive functionals of a Markov process, but Kingman [12] has provided a direct construction that will guide intuition. Before getting to that we need to introduce one more associated process.

The right-continuous inverse process $\tau = (\tau_r)_{r\geq 0}$ defined by

$$\tau_r = \tau(r) := \inf\{t : L_t > r\}, \qquad r \geq 0, \tag{11}$$

is a strictly increasing, pure jump process—the subordinator associated with M. Notice that M coincides with the closure of the range $\{\tau_r : r \geq 0\}$ of τ. The process τ has stationary independent increments (an increasing Lévy process) with Laplace transforms

$$\mathbf{E}[\exp(-\alpha\tau_r)] = \exp(-r\phi(\alpha)], \qquad \alpha > 0, r \geq 0, \tag{12}$$

where the *Lévy exponent* ϕ admits the representation

$$\phi(\alpha) = \int_{]0,\infty]} (1-e^{-\alpha x})\,\nu(dx), \qquad \alpha > 0, \tag{13}$$

for a Borel measure ν on $]0,\infty]$ satisfying

$$\int_{]0,\infty]} (x\wedge 1)\,\nu(dx) < \infty, \tag{14}$$

which ensures that the integral in (13) is finite for each $\alpha > 0$. We write $h(x)$ for the tail $\nu\,(]x,\infty])$, and note that $\phi(\alpha) = \alpha\hat{h}(\alpha)$, where the hat indicates Laplace transform.

We now turn to Kingman's construction of the local time L: for $\delta > 0$, define

$$M_t(\delta) := \{(\omega,s)\in\Omega\times[0,\infty[: |s-v| < \delta \text{ for some } v\in M(\omega)\cap[0,t]\}, \tag{15}$$

and

$$\ell(\delta) := \int_0^\delta h(s)\,ds, \qquad \delta > 0. \tag{16}$$

Then [12, Thm. 3], there is a constant $c > 0$ such that

$$L_t = c\cdot\lim_{\delta\downarrow 0}\frac{\lambda(M_t(\delta))}{\ell(\delta)}, \qquad \forall t \geq 0, \text{ a.s.}, \tag{17}$$

where λ denotes Lebesgue measure on $[0,\infty[$.

The potential measure U associated with M is the mean occupation time of τ:

$$U(B) := \mathbf{E}\int_0^\infty 1_B(\tau_r)\,dr = \mathbf{E}\int_B dL_t, \qquad \forall B\in\mathbf{B}(\mathbf{R}_+), \tag{18}$$

and the associated distribution function

$$V(t) := \mathbf{E}[L_t], \qquad t\ge 0,$$

plays the role of the renewal function. The Laplace transform of the measure U is given by

$$\hat{U}(\alpha) := \int_0^\infty e^{-\alpha t}U(dt) = \mathbf{E}\int_0^\infty e^{-\alpha\tau(r)}\,dr = \int_0^\infty e^{-r\phi(\alpha)}\,dr = 1/\phi(\alpha), \tag{19}$$

for $\alpha>0$. For later reference we note that U is related to the potential kernel of the strong Markov process τ: writing $\mathbf{E}^r$ for expectation under the initial condition $\tau_0=r$, we have

$$\mathbf{E}^r\int_0^\infty f(\tau_s)\,ds = \int_0^\infty f(r+x)\,U(dx), \qquad r\ge 0, \tag{20}$$

for f non-negative and Borel.

2. Lorden's inequality.

By using the regeneration property of M at the stopping time D_t ($t>0$ fixed) and the fact that L is flat off M, one sees that

$$\mathbf{E}[\exp(-\alpha D_t)] = \phi(\alpha)\cdot\int_t^\infty e^{-\alpha s}\,U(ds). \tag{21}$$

Inverting this we can obtain the distribution of D_t or, equivalently, that of R_t. In fact,

$$\mathbf{E}\int_0^\infty g(x+R_t)\,U(dx) = \int_0^\infty g(y)\,U(dy+t), \tag{22}$$

provided g is a positive Borel function on $[0,\infty]$. This follows immediately from (21) for g of the form $g(x)=e^{-\alpha x}$, and then for general g by Weierstrass's theorem followed by the monotone class theorem. Another direct consequence of (21) is Wald's Identity for regenerative sets:

$$\mathbf{E}[D_t] = \mu\cdot V(t), \tag{23}$$

where $\mu := \int_0^\infty x\,\nu(dx)$.

If the mean μ is finite then (as is well known in the renewal theory context) the random variable R_t converges in distribution, as $t\to\infty$, to a random variable R_∞ whose law has density

$$\frac{h(x)}{\mu}, \qquad x>0, \tag{24}$$

with respect to Lebesgue measure on $]0,\infty[$; see [3, Thm. 1]. Observe that $\mathbf{E}[R_\infty] = (2\mu)^{-1}\int_0^\infty x^2\,\nu(dx)$.

The following proposition states Lorden's inequality [15] in our context. We reproduce the proof found in [5].

Proposition. *Assume that* $\mu := \int_0^\infty x\,\nu(dx) < \infty$. *Then* $\mathbf{E}[R_t] \le 2\mathbf{E}[R_\infty]$ *for all* $t > 0$.

Before turning to the proof we need the following lemma, both parts of which are well known.

Lemma 1. *(a)* V *is subadditive:* $V(t+s) \le V(t) + V(s)$, *for all* $s, t > 0$.

(b) $\mathbf{E}[V(t - R_\infty)] = t/\mu$ *for* $t > 0$, *with the understanding that* $V(s) = 0$ *for* $s \le 0$.

Proof. (a) We have, using (21) with $g = 1_{[0,s]}$ for the first equality below,

$$V(t+s) - V(t) = \mathbf{E}[V(s - R_t); R_t \le s] \le \mathbf{E}[V(s); R_t \le s] \le V(s). \tag{25}$$

(b) The Laplace transform of the left side of this identity is easily seen to be $\hat{U}(\alpha)\hat{h}(\alpha)/(\alpha\mu) = 1/(\alpha^2\mu)$ because of (19). This coincides with the Laplace transform of the right side, so the assertion follows by inversion because both sides are continuous in $t > 0$. □

Proof of the Proposition. Let Z_1 and Z_2 by independent random variables with the same distribution as R_∞. The subadditivity asserted in Lemma 1(a) persists for negative values of s, t provided we agree that V vanishes on $]-\infty, 0]$. Thus,

$$V(t) \le V(t + Z_1 - Z_2) + V(Z_2 - Z_1). \tag{26}$$

By Lemma 1(b), the conditional expectation of the first term on the right of (26), given Z_1, is $(t + Z_1)/\mu$. Likewise, the conditional expectation of the second term, given Z_2, is Z_2/μ. It follows that

$$\mu \cdot V(t) \le \mathbf{E}[t + Z_1] + \mathbf{E}[Z_2] = t + 2\mathbf{E}[R_\infty], \tag{27}$$

and the assertion follows because $\mathbf{E}[R_t] = \mathbf{E}[D_t] - t = \mu \cdot V(t) - t$ by (23). □

3. Monotone potential density.

The exponent ϕ is an example of what is called a *Bernstein function* (non-negative, completely monotone derivative). Such a ϕ is a *special Bernstein function* provided $\phi^* : \alpha \mapsto \alpha/\phi(\alpha)$ is also a Bernstein function. In this case, because $\hat{U}(\alpha)\phi(\alpha) = 1$, the measure U admits a Lebesgue density given by

$$u(x) := \nu^*\left(]x,\infty]\right), \tag{28}$$

where ν^* is the Lévy measure in the representation (13) of ϕ^*. Notice that u is right-continuous, and (more importunely) monotone decreasing. Conversely, if U admits a monotone density with respect to Lebesgue measure, then ϕ is a special Bernstein function. For discussion of special Bernstein functions see chapter 10 of [17].

Our main result contains further (stochastic) characterizations of the class of special Bernstein functions. For partial results in this vein, in the context of renewal processes, see [4] and [14].

Theorem. *For a light unstable regenerative set M the following are equivalent:*
(a) U is absolutely continuous and admits a monotone decreasing density.
(b) $t \mapsto R_t$ is stochastically increasing.
(c) $t \mapsto A_t$ is stochastically increasing.
(d) $t \mapsto \theta_t M$ is stochastically decreasing.
(e) $t \mapsto \theta_t L$ is stochastically decreasing.

[By (d) is meant that for each $0 < s < t$ there is some probability space carrying random sets M^s and M^t such that $M^s \stackrel{d}{=} \theta_s M$, $M^t \stackrel{d}{=} \theta_t M$, and $M^t \subset M^s$ almost surely. Point (e) should be interpreted in an analogous fashion, the local time being thought of as a random measure dL_s, and $(\theta_t L)_b := L_{t+b} - L_t = L_b(\theta_t M)$.]

Proof. (a)⇒(b). If U has a monotone density, then the left side of (22), which is nothing but $(\pi_t * U)(g)$ (π_t being the distribution of R_t), is monotone decreasing in t. It follows that if $s < t$ then π_t is "downstream" from π_s in the balayage order of the subordinator τ. By a theorem of H. Rost [16] there are (randomized) stopping times $T(s)$ and $T(t)$ of τ with $T(s) \le T(t)$ such that $\tau(T(s))$ has the same distribution as R_s and $\tau(T(t))$ has the same distribution as R_t. Since τ is increasing, $\mathbf{P}[R_s > x] \le \mathbf{P}[R_t > x]$ for each $x > 0$; that is, R_t is stochastically larger than R_s.

(b)⇒(a). Conversely, if $t \mapsto R_t$ is stochastically increasing, then from (22) with $g = 1_{[0,b]}$ we see that $t \mapsto U[t, t+b]$ is decreasing for each $b > 0$. In particular, V is midpoint concave, hence concave (because $x \mapsto V(x) = \mathbf{E}[L_x]$ is continuous). This implies that V is concave, so the righthand derivative $u := V'_+$ exists and is decreasing. Moreover, again by the concavity of V, the measure U is absolutely continuous with density u.

(c)⇔(b). $\mathbf{P}(R_t > x) = P(M \cap]t, t+x] = \emptyset) = \mathbf{P}(A_{t+x} > x)$.

(a)⇒(d). From the proof of (a)⇒(b) we know that if $0 < s < t$ then there are (randomized) stopping times $T(s)$ and $T(t)$ of τ such that $T(s) \le T(t)$, $\tau(T(s)) \stackrel{d}{=} R_s$, and $\tau(T(t)) \stackrel{d}{=} R_t$. In particular, $\tau(T(s)) \le \tau(T(t))$. Define $M^s := M \cap [\tau(T(s)), \infty[$ and $M^t := M \cap [\tau(T(t)), \infty[$. Then $M^t \subset M^s$, and the required distributional equalities hold by regeneration at the stopping times $\tau(T(s))$ and $\tau(T(t))$

(d)⇒(e). This follows immediately from Kingman's construction (17): For fixed $0 \le s < t$, we have $L_b(M^t) - L_a(M^t) \le L_b(M^s) - L_a(M^s)$, for all $0 \le a < b$, almost surely. This means that the measure with distribution function $b \mapsto L_b(M^t)$ is dominated setwise by the measure with distribution function $b \mapsto L_b(M^s)$, a.s.

(e) ⇒(a). $U\,(]t, t+b]) = \mathbf{E}[(\theta_t L)_b]$. □

Remarks. (a) It is shown in [4, Thm. 3], in the renewal context, that if the tail probability $\mathbf{P}[Y_k > y]$ (the analog of h) is *log-convex* then (a), (b), (c), and the "counting" version of (e) in the Theorem hold true. This log-convexity is equivalent to the *decreasing failure rate* property (DFR). Expressed in the present context this amounts to the statement that

$$y \mapsto \frac{h(x+y)}{h(y)} \tag{29}$$

is non-increasing on the interval where $h(y) > 0$, for each $x > 0$. It was shown by J. Hawkes [9, Thm. 2.1] that in our context, the log-convexity of the Lévy tail function h implies that U has a decreasing density. For more on this class of subordinators see [17]. Brown conjectured in [4] that the DFR property is equivalent to the concavity of the renewal function; a counterexample was found (after 31 years) by Y. Yu [18].

(b) The use of Rost's theorem (on Skorokhod stopping) in the proof of (a)⇒(b) (and again in (a)⇒(d)) was suggested by an argument of J. Bertoin, [1, p. 568].

Observe that when U has a monotone density, because R_t stochastically increasing in t, each random variable R_t is stochastically dominated by R_∞. This yields the following improvement on Lorden's inequality.

Corollary. *Under any of the conditions listed in the Theorem, we have*

$$\mathbf{E}[R_t] \le \frac{\int_0^\infty x^2\,\nu(dx)}{2\mu} = \mathbf{E}[R_\infty], \qquad \forall t > 0, \tag{30}$$

and the inequality is sharp.

Whether the constant 2 in Lorden's original inequality can be improved in the general case is an open question.

4. Concluding Remarks.

A regenerative set M is *infinitely divisible* (ID) provided for each positive integer n there are i.i.d. regenerative sets $M_{n,k}$, $1 \le k \le n$, such that $\cap_{k=1}^n M_{n,k}$ has the same distribution as M. A large class of such sets ("Poisson random cutout sets") is discussed and characterized in [7]. It has long been conjectured by the author that this class exhausts (at least among light unstable regenerative sets) all of the ID regenerative sets. In unpublished work the author has shown that an ID regenerative set whose potential measure admits a monotone density is, in fact, a Poisson cutout set. Somewhat irritatingly, this supplementary monotonicity condition *is* satisfied by all Poisson cutout sets. It should be noted that the parallel results for heavy ID sets, and for the discrete-time situation, have been established by D.G. Kendall [10, 11]; see also [8] for a detailed discussion of these matters and further references.

From the proof of the Theorem in section 3, we know that when U has a monotone density then for each $t > 0$ we have

$$\theta_t M \stackrel{d}{=} M \setminus I_t, \tag{31}$$

where I_t is a random interval $[0, \tau(T(t))\,[$ growing stochastically larger as t increases. In the Poisson cutout case, $\theta_t M$ and I_t are independent, because of the independence properties of the Poisson process. Does this independence characterize ID regenerative sets?

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