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Abstract. The paper studies the issue of stability of solutions to the Navier-Stokes and damped Euler systems in periodic boxes. We show that under action of fast oscillating-in-time external forces all two dimensional regular solutions converge to a time periodic flow. Unexpectedly, effects of stabilization can be also obtained for systems with stationary forces with large total momentum (average of the velocity). Thanks to the Galilean transformation and space boundary conditions, the stationary force changes into one with time oscillations. In the three dimensional case we show an analogical result for weak solutions to the Navier-Stokes equations.

1. Introduction

In many analytical and computational studies of the forced Navier-Stokes or Euler equations, subject to periodic boundary conditions, it is usually assumed that spatial average of the forcing term is zero. This in turn implies that the spatial average of the solution is invariant, and for simplicity it is also taken to be zero. In this paper we investigate the long-time behavior of these systems when the spatial average of the initial velocity is taken to be large. By using the Galilean transformation of such systems the problem is transformed into a similar system with fast time-oscillating forcing term. Therefore, we investigate instead the long-time dynamics of the transformed Navier-Stokes equations and the damped Euler equations under the action of fast time-oscillating force. Specifically, we show that the fast time-oscillating forces have a stabilization effect; and that the long-time dynamics consists of a globally attracting unique time-periodic solution. This result is consistent with other results concerning the investigation of the Navier-Stokes and Euler equation with high oscillations. Fast rotation is, for example, a stabilizing mechanism of inviscid turbulent flows

[BMN, CG, GIM, MN]. Moreover, in the case of the forced two-dimensional Navier-Stokes equations it is observed that fast rotation is trivializing the long-time dynamics, i.e., the global attractor is a single stable steady state at the limit of large rotation rate [W].

Naturally, we distinguish in our proofs between the two-dimensional and the three-dimensional cases. For the two-dimensional models we show that the long-time dynamics of regular solutions is trivial; specifically, that all solutions tend to a unique time-periodic solution generated by the fast time-oscillating forcing term. For the three-dimensional case we only consider the Navier-Stokes equations, and we show that all Leray-Hopf weak solutions converge, as time tends to infinity, to a unique time periodic solution generated by the fast time-oscillating forcing term. Notably, we do not impose any restriction on the magnitude of spatial norms of the forcing term, and we only assume that the size spatial average of the initial data (or equivalently the rate of oscillation in the forcing term) is large enough. In the last section of this paper we also provide some numerical results supporting our qualitative theoretical results.

The paper is organized as follows. In the next section we motivate our study and introduce the relevant models. In section 3 we treat the two-dimensional models, and in section 4 we consider the three-dimensional Navier-Stokes system. Numerical results are reported in section 5. We also provide an Appendix section in which we sketch the construction of time periodic solutions.

2. Settings and motivation

In this paper we consider the Navier-Stokes and the damped Euler systems of equations

$$\begin{aligned} & v_t + v\cdot\nabla v - \epsilon\Delta v + \alpha[v] + \nabla p = F, \\ & \operatorname{div} v = 0, \end{aligned} \tag{2.1}$$

subject to periodic boundary conditions in the n-dimensional torus $\mathbb{T}^n$, for $n = 2, 3$, and with divergence-free initial datum v_0. Here we denote by

$$[v] = v - \frac{1}{|\mathbb{T}^n|}\int_{\mathbb{T}^n} v dx, \tag{2.2}$$

and assume that

$$\int_{\mathbb{T}^n} F(x,t)dx = 0, \quad \text{for all } \ t > 0. \tag{2.3}$$

Assumptions (2.2) and (2.3) imply the conservation of the total momentum of the flow, i.e.,

$$\int_{\mathbb{T}^n} v(x,t)dx = \int_{\mathbb{T}^n} v_0(x)dx. \tag{2.4}$$

Throughout this work we assume that $\max\{\alpha,\epsilon\} > 0$.

In this section we consider two special cases of system (2.1):

Case (A): In the first case we consider a very fast (i.e., Ω in (2.5) below is very large) constant background flow motion in the x_1-direction, given by the initial data, i.e.,

$$\frac{1}{|\mathbb{T}^n|}\int_{\mathbb{T}^n} v_0 dx = \Omega\hat{e}_1. \tag{2.5}$$

In addition, we assume that the external force in (2.1) is time independent, i.e., $F(x,t) = F(x)$; and that the Fourier coefficients $\hat{F}_k = 0$, for $k = (0, k_2)$ in the 2d case, and for $k = (0, k_2, k_3)$ in the 3d case. As it will become clear later, this assumption implies that the forcing term does not resonate with the background constant flow given in (2.5).

The choice of the direction x_1 for the background flow is not important, but it simplifies the presentation. The above assumptions allow us in this case to make the following change of the variables, using the Galilean transformation,

$$x \to x + \Omega t \hat{e}_1 \quad \text{and} \quad v \to v - \Omega \hat{e}_1 \quad \text{with} \quad \Omega \in \mathbb{R}_+. \tag{2.6}$$

Consequently, we arrive to the following equivalent system to (2.1)

$$\begin{aligned} &v_t + v \cdot \nabla v - \epsilon \Delta v + \alpha v + \nabla p = F(x_1 + \Omega t, x'), \\ &\operatorname{div} v = 0, \end{aligned} \tag{2.7}$$

with initial datum v_0 such that $\int_{\mathbb{T}^n} v_0 dx = 0$. Here $x' = x_2$ for the 2d case, and $x' = (x_2, x_3)$ for the 3d case. For large values of Ω, the speed of the background flow, transformation (2.6) yields a new system, (2.7), that governs the perturbation about the background flow, with fast oscillating in time forcing term, with period $T_{per} = \frac{2\pi}{\Omega}$, but with zero total momentum.

Case (B): In the second case we consider system (2.1) with a special type of fast oscillating forcing term. Specifically, we consider fast oscillating force of the form

$$F(x,t) = f(x) \sin \Omega t. \tag{2.8}$$

We also assume that $\displaystyle\int_{\mathbb{T}^n} f(x) dx = \int_{\mathbb{T}^n} v_0(x) dx = 0$. Here $[v] = v$.

The key observation in both, case (A) and case (B), is that we consider systems with fast oscillating forcing terms in (2.7) and (2.8), respectively. The main purpose of this study is to take advantage of these fast oscillating forcing terms to stabilize the long-time behavior of the solutions of the corresponding systems. Consequently, our analysis will concentrate on the limit, as $\Omega \to \infty$. Note that the force (2.8) is, roughly speaking, a particular case of the force considered in (2.7), since $F(\cdot)$ is spatially periodic.

In both cases, (A) and (B), we require the force F to be sufficiently smooth and we do not restrict the magnitude of its spatial norms. In particular, the $L_2(\mathbb{T}^n)$ and $H^{-1}(\mathbb{T}^n)$ type norms can be arbitrary large and fixed. The same we assume about the size of the initial data.

Next, we provide a rough description of the main results presented in this article.

Results for the two-dimensional case:

1. Periodic in time solutions. Let $\delta > 0$ be sufficiently small, and let F be sufficiently smooth, fulfilling the forms in case (A) or case (B). Let $\epsilon \geq 0$ and $\alpha \geq 0$ with $\max\{\epsilon, \alpha\} > 0$. Then there exists a periodic in time solution, v_{per}, to (2.1) such that

$$\|\nabla v_{per}\|_{L_\infty(\mathbb{T}^2 \times T_{per}\mathbb{S}^1)} \leq \delta, \tag{2.9}$$

provided Ω is sufficiently large. The period is $T_{per} = 2\pi/\Omega$. Theorem 1 from Section 3.

2. Global stability and uniqueness of the periodic solution. Let v be a solution to (2.1), with arbitrary, divergence-free, initial datum $v_0 \in L_2$. Then for every Ω large enough, such that the above statement is valid, we have

$$\|v(t) - v_{per}(t)\|_{L_2(\mathbb{T}^2)} \to 0, \text{ as } t \to \infty, \tag{2.10}$$

where v_{per} is as above. In particular, and under the above assumptions, it follows from (2.10) that v_{per} is unique. Theorem 2 from Section 3.

Here we shall mention about a current result from [CZ] concerning dissipative PDEs which intersects with the case A for some class of external forces. Note however that methods used there are significantly different from ones we apply in the present paper.

Results for the three-dimensional case:

We also investigate the long-time behavior of the three-dimensional Navier-Stokes equations, for large values of Ω. However, due to our inability to prove global existence of strong solutions, or the uniqueness of weak solutions for the 3d Navier-Stokes equations, we will focus on the long-time behavior of the Leray-Hopf class of weak solutions of the 3d Navier-Stokes system. Thus, by taking $\alpha = 0$ and $\epsilon = \nu > 0$ in (2.1), we consider the following 3d Navier-Stokes system:

$$\begin{aligned} &v_t + v \cdot \nabla v - \nu \Delta v + \nabla p = F, \\ &\operatorname{div} v = 0, \end{aligned} \tag{2.11}$$

subject to periodic boundary conditions on the torus $\mathbb{T}^3$, and with the divergence-free initial datum $v_0 \in L_2(\mathbb{T}^3)$ that has zero spatial average on $\mathbb{T}^3$. Moreover, the forcing term in (2.11), F, is assumed to satisfy the conditions in either case (A) or case (B), above.

3. Periodic in time solutions. Let $\delta > 0$ be sufficiently small. Let F be sufficiently smooth satisfying either case (A) or case (B). Then there exists a periodic in time solution, v_{per}, to (2.11) such that

$$\|v_{per}(t)\|_{L_\infty(\mathbb{T}^3 \times T_{per}\mathbb{S}^1)} \leq \delta, \tag{2.12}$$

provided Ω is sufficiently large. Theorem 3 from Section 4.

4. Global stability and uniqueness of the periodic solution. Let v be a Leray-Hopf weak solution to (2.11), with arbitrary divergence-free initial datum $v_0 \in L_2(\mathbb{T}^3)$. Then, for every Ω sufficiently large, we have

$$\|v(t) - v_{per}(t)\|_{L_2(\mathbb{T}^3)} \to 0, \text{ as } t \to \infty, \tag{2.13}$$

where v_{per} is as above. In particular, and under the above assumptions, it follows from (2.13) that v_{per} is unique. Theorem 4 from Section 4.

Numerical results for the two-dimensional case:

We conclude the paper with numerical tests illustrating our analytical results for certain class of flows. Specifically, we consider the 2D Kolmogorov flows in the flat torus $\mathbb{T}^2_\beta = [0, 2\pi] \times [0, 2\pi\beta]$, for $\beta > 0$; which is the 2D NS equations, subject to periodic boundary condition, forced by an eigen-function of the Stokes operator. This problem is on the one hand simple, but on the other hand is dynamically rich enough to illustrate the phenomena at hand.

We consider the vorticity formulation of the 2D version of (2.7) with the specific forcing term $F = (-\lambda\beta \sin(\frac{y}{\beta} + \Omega t), 0)$ which yields

$$\omega_t + v \cdot \nabla\omega - \epsilon\Delta\omega + \alpha\omega = \lambda \cos(\frac{y}{\beta} + \Omega t). \tag{2.14}$$

Observe that $\omega(z, t)$ is a solution of (2.14), for $z = (x, y) \in \mathbb{T}^2_\beta$, if and only if $\tilde{\omega}(z, t) = \omega(z + \beta\Omega t\hat{e}_2, t)$ is a solution to the evolution equation

$$\tilde{\omega}_t + (\tilde{v} + \beta\Omega\hat{e}_2) \cdot \nabla\tilde{\omega} - \epsilon\Delta\tilde{\omega} + \alpha\tilde{\omega} = \lambda \cos(\frac{y}{\beta}). \tag{2.15}$$

In particular, $\omega^*(z)$ is a stationary solution of (2.15) if and only if $\omega(z, t) = \omega^*(z + \beta\Omega t\hat{e}_2)$, for $z = (x, y) \in \mathbb{T}^2_\beta$, is a time periodic solution to (2.14). Moreover, $\omega^*(z)$ is globally stable for the dynamics of (2.15) if and only if $\omega^*(z + \beta\Omega t\hat{e}_2)$ is global stable time periodic solution of (2.14).

- Based on the above observation we present a bifurcation analysis of the stationary solutions to (2.15). In particular, investigate the bifurcation diagram of (2.15) for large values of Ω. Moreover, we show that the range of λ's for which system (2.15) admits a unique stationary solution increases proportionally to Ω.
- We investigate the stationary solutions of (2.15) rather than direct numerical integration in order to avoid working with rapidly oscillating in time functions.
- We also investigate the rate of convergence to the globally stable solution of (2.15). The purpose of this study is to provide an evidence of the exponential convergence rate, which we show in Theorem 2.

All of the numerical results were derived using a finite dimensional Galerkin approximations, we argue that the dimensions we used are sufficient.

An illustrative linear toy model with friction – Newton's second law

To illustrate the stabilization mechanism, due to the fast oscillations in the forcing term, we focus here on the following simple linear equation with friction/damping/drag term:

$$w_t + \alpha w = F(x, \Omega t), \tag{2.16}$$

in the torus $\mathbb{T}^n$. Here F is time periodic, with period $T_{per} = \frac{2\pi}{\Omega}$. First, we observe that the solution to system (2.16) does not involve dynamically the spatial variable, x, so the solution will treat x as a parameter (label), i.e., we have a parameterized system of simple ODEs.

We assume that the forcing term $F(x, \Omega t)$, in (2.16), enjoys specific structure, namely, there exists a smooth function $g(x, \Omega t)$, periodic in space and time, such that

$$\partial_t g(x, t) = F(x, t), \text{ consequently } \quad \frac{1}{\Omega}\partial_t g(x, \Omega t) = F(x, \Omega t). \tag{2.17}$$

We have two prototypical examples in mind of the forcing terms, $F(x, \Omega t)$, satisfying the above structure. Specifically, let $f(x)$, for $x \in \mathbb{T}^n$, be a smooth spatially periodic function. We consider again the cases:

$$(A) \quad g(x, \Omega t) = \frac{1}{\Omega} D_1^{-1} f(z)|_{z=x+\Omega t\hat{e}_1}, \qquad (B) \quad g(x, \Omega t) = -\frac{1}{\Omega} f(x) \cos \Omega t. \tag{2.18}$$

Here we put $D_1^{-1}f(z)$ as the primitive function of f with respect to the first variable, i.e., we have $\frac{1}{\Omega}\partial_t D_1^{-1}f(x+\Omega t\hat{e}_1) = f(x+\Omega t\hat{e_1})$. Put it in other words we define

$$g(x,\Omega t) = \sum_{k\in\mathbb{Z}^n} \frac{\hat{f}_k}{ik_1} e^{ik\cdot x} e^{i\Omega t k_1} \tag{2.19}$$

via Fourier series. Here we see that the assumption $\hat{f}_k = 0$, for $k = (0, k')$, is required to justify the above form of g.

As a result of the previous assumptions on f we have in both cases that

$$\|g(\cdot,\Omega t)\|_{C^2(\mathbb{T}^n)} \leq \frac{C}{\Omega}, \tag{2.20}$$

where C depends on the spatial norms of f, but is independent of Ω. Notice that only the time derivatives of g will add multiplication by factors of Ω. Set

$$w(x,t) = W(x,t) + g(x,\Omega t), \quad \text{then } W \text{ satisfies} \quad W_t + \alpha W = \alpha g. \tag{2.21}$$

Therefore, from (2.21), by (2.20) we have

$$W(x,t) = \exp\{-\alpha t\}W_0(x) + \int_0^t \exp\{-\alpha(t-s)\}\alpha g(x,\Omega s)ds \sim e^{-\alpha t} + \frac{1}{\Omega}. \tag{2.22}$$

Thus, the solution to (2.16) satisfies

$$\|w(t)\|_{L_\infty(\mathbb{T}^n)} \sim e^{-\alpha t} + \frac{1}{\Omega}. \tag{2.23}$$

Since the problem is linear, the above structure holds for arbitrary positive α and Ω.

Next, let us consider the time periodic solutions to (2.21):

$$W_{per,t} + \alpha W_{per} = \alpha g. \tag{2.24}$$

The construction of periodic solutions to the (2.24) can be done explicitly through the Fourier series in time, on the time periodic interval $T_{per}\mathbb{S}$. Using the energy methods we immediately obtain the following bound

$$\|W_{per}\|_{L_\infty(T_{per}\mathbb{S}^1)} \leq \|g\|_{L_\infty(T_{per}\mathbb{S}^1)}. \tag{2.25}$$

Comparing the solutions to (2.21) and to (2.24) we obtain the trivial identity

$$\partial_t(W - W_{per}) + \alpha(W - W_{per}) = 0. \tag{2.26}$$

The above identity implies

$$|W(x,t) - W_{per}(x,t)| = |W_0(x) - W_{per}(x,0)|e^{-\alpha t} \sim e^{-\alpha t}. \tag{2.27}$$

Summing up we obtain the following:

Proposition 1. *Let $\alpha > 0$, and g has one of the forms in (2.18), then every solution to (2.16) admits the following structure*

$$\|w(\cdot,t)\|_{L_\infty(\mathbb{T}^n)} \lesssim e^{-\alpha t}\|w_0\|_{L_\infty(\mathbb{T}^n)} + \frac{1}{\Omega}\|f\|_{L_\infty(\mathbb{T}^n)}. \tag{2.28}$$

and

$$\|w(\cdot,t) - w_{per}(\cdot,t)\|_{L_\infty(\mathbb{T}^n)} \leq e^{-\alpha t}\|w_0 - w_{per}(\cdot,0)\|_{L_\infty(\mathbb{T}^n)}, \tag{2.29}$$

where $w_{per} = W_{per} - g$.

3. The two-dimensional case

System (2.1), that we consider here, is a modification of the Navier-Stokes and Euler equations, by basically adding a linear friction/damping/drag force with coefficient $\alpha \geq 0$. We require in addition that $\max\{\alpha, \epsilon\} > 0$, thus, (2.1) is a dissipative form of the Euler system. We use the special proprieties which are valid in the 2d case. Namely, we analyze system (2.1) in the vorticity formulation which takes the form

$$\omega_t + v \cdot \nabla\omega - \epsilon\Delta\omega + \alpha\omega = \operatorname{rot} F, \tag{3.1}$$

where

$$\omega = \operatorname{rot} v. \tag{3.2}$$

Our result concerning system (3.1)-(3.2) is the following

Theorem 1. *Let* $\alpha \geq 0$, $\epsilon \geq 0$, *with* $\max\{\alpha, \epsilon\} > 0$, *and let* F *be sufficiently smooth force of form (A) or (B). In addition, for case (A) let us assume that there exists a scalar function* $g(x, \Omega t)$ *such that*

$$\frac{1}{\Omega}\partial_t g(x, \Omega t) = \operatorname{rot} F(x_1 + \Omega t, x_2) \quad \text{and} \quad \sup_s \|g(\cdot, s)\|_{C^3(\mathbb{T}^2)} \leq G,$$

where G *is independent of* Ω. *Choose* δ *small enough such that*

$$0 < \delta \leq \frac{1}{4}(\alpha + a\epsilon),$$

where a *is an absolute constant. Then there exists a regular periodic in time solution to problem (2.1) such that*

$$\|v\|_{L_\infty(T_{per}\mathbb{S}^1; W^1_\infty(\mathbb{T}^2))} \leq \delta, \tag{3.3}$$

provided Ω *is sufficiently large, depending on* α, ϵ, G *and* δ, *with* $T_{per} = 2\pi/\Omega$.

Proof. The construction of time periodic solutions is based on the domain $\mathbb{T}^2 \times T_{per}\mathbb{S}^1$. We consider system (2.1) in the form of (3.1). Let $\bar{v}$ be a given smooth enough time periodic velocity field satisfying:

$$\operatorname{div} \bar{v} = 0 \quad \text{with} \quad \|\bar{v}\|_{L_\infty(T_{per}\mathbb{S}^1; W^1_\infty(\mathbb{T}^2))} \leq \delta. \tag{3.4}$$

We then look for a time periodic vorticity ω that solves the following "linearized" transport, by the velocity field $\bar{v}$, version of problem (3.1):

$$\omega_t + \bar{v} \cdot \nabla\omega - \epsilon\Delta\omega + \alpha\omega = \operatorname{rot} F. \tag{3.5}$$

Then we construct the time periodic velocity field, v, corresponding to the vorticity ω such that

$$\operatorname{rot} v = \omega, \qquad \operatorname{div} v = 0, \qquad \int_{\mathbb{T}^2} v dx = 0. \tag{3.6}$$

We show that the above procedure defines a map

$$\mathcal{T} : L_\infty(T_{per}\mathbb{S}^1; W^1_\infty(\mathbb{T}^2) \cap \{\operatorname{div} v = 0\}) \to L_\infty(T_{per}\mathbb{S}^1; W^1_\infty(\mathbb{T}^2) \cap \{\operatorname{div} v = 0\}),$$

such that $\mathcal{T}(\bar{v}) = v$. And we look for a fixed point of this map, which will in turn define a time periodic solution to (3.1). Indeed, we show that $\mathcal{T}$ maps the set

$$\Xi = \{\operatorname{div} v = 0 \text{ with } \|v\|_{L_\infty(T_{per}\mathbb{S}^1; W^1_\infty(\mathbb{T}^2))} \leq \delta\}$$

into itself and that $\mathcal{T}$ is a compact map. Then the assumptions of the Schauder fixed point theorem are fulfilled, which will imply the existence of a fixed point of map $\mathcal{T}$. Given $\bar{v}$ satisfying (3.4), the existence of time periodic solution to the linearized system (3.5) is not difficult, and it can be proved easily by the Galerkin method – see the Appendix. Next, we establish the required estimates.

We split our proof into two cases, distinguishing influences of $\epsilon\Delta w$ and αw.

The dominant-damping case is when $\alpha \geq a\epsilon$, for some positive absolute constant a to be specified later. This is the case when the damping $\alpha > 0$ is dominating the viscosity ϵ which is very small or maybe even equal to zero. In the latter case we essentially have the damped Euler system. Recalling (2.18) we set

$$W = \omega - \frac{1}{\Omega} g(x, \Omega t). \tag{3.7}$$

For case (A), the function g is given by (2.19); and for case (B) we take, as before, $g(x, \Omega t) = -\cos \Omega t \cdot f(x)$. Then $\partial_t \frac{1}{\Omega} g(x, \Omega t) = \operatorname{rot} F$. Then W fulfills

$$W_t + \bar{v} \cdot \nabla W - \epsilon \Delta W + \alpha W = \bar{v} \cdot \nabla \frac{1}{\Omega} g - \epsilon \Delta \frac{1}{\Omega} g + \alpha \frac{1}{\Omega} g. \tag{3.8}$$

Multiplying (3.8) by $|W|^{p-2}W$ and integrating over $\mathbb{T}^2$ we obtain

$$\frac{1}{p}\frac{d}{dt}\|W\|^p_{L_p(\mathbb{T}^2)} + \epsilon \int_{\mathbb{T}^2} (p-1)|\nabla W|^2 |W|^{p-2} dx + \alpha \|W\|^p_{L_p(\mathbb{T}^2)} dx \leq \frac{1}{\Omega} \int_{\mathbb{T}^2} \big(|\bar{v}||\nabla g| + \epsilon|\Delta g| + \alpha|g|\big)|W|^{p-1} dx. \tag{3.9}$$

Integrating with respect to time, over $T_{per}\mathbb{S}^1$, and using the periodicity in time, we get

$$\epsilon \int_{T_{per}\mathbb{S}^1} \int_{\mathbb{T}^2} (p-1)|\nabla W|^2 |W|^{p-2} dx dt + \alpha \|W\|^p_{L_p(\mathbb{T}^2 \times T_{per}\mathbb{S}^1)} \leq C\Big(\frac{G}{\Omega}\|\bar{v}\|_{L_p(T_{per}\mathbb{S}^1; L_\infty(\mathbb{T}^2))} \|W\|^{(p-1)/p}_{L_p(T_{per}\mathbb{S}^1; L_p(\mathbb{T}^2))} + \frac{G}{\Omega}(\epsilon + \alpha) T_{per}^{1/p} \|W\|^{(p-1)/p}_{L_p(T_{per}\mathbb{S}^1; L_p(\mathbb{T}^2))}\Big). \tag{3.10}$$

In particular, we have

$$\alpha \|W\|_{L_p(T_{per}\mathbb{S}^1; L_p(\mathbb{T}^2))} \leq C \frac{G}{\Omega} T_{per}^{1/p} (\|\bar{v}\|_{L_\infty(T_{per}\mathbb{S}^1; L_\infty(\mathbb{T}^2))} + \alpha). \tag{3.11}$$

However this regularity is not enough, one more spatial derivative is required, so we differentiate (3.8) with respect to $x_i \in \{x_1, x_2\}$ getting

$$W_{x_i,t} + \bar{v}\cdot\nabla W_{x_i} - \epsilon\Delta W_{x_i} + \alpha W_{x_i} = -\bar{v}_{x_i}\cdot\nabla W + \bar{v}_{x_i}\cdot\nabla\frac{1}{\Omega}g + \bar{v}\cdot\nabla\frac{1}{\Omega}g_{x_i} - \epsilon\Delta\frac{1}{\Omega}g_{x_i} + \alpha\frac{1}{\Omega}g_{x_i}. \tag{3.12}$$

Multiplying (3.12) by $|W_{x_i}|^{p-2}W_{x_i}$, integrating over $\mathbb{T}^2\times T_{per}\mathbb{S}^1$, and using the periodicity in time, we get

$$\sum_{i=1}^{2}\Big[(p-1)\epsilon\int_{T_{per}\mathbb{S}^1}\int_{\mathbb{T}^2}|\nabla W_{x_i}|^2|W_{x_i}|^{p-2}dxdt + \alpha\int_{T_{per}\mathbb{S}^1}\int_{\mathbb{T}^2}|W_{x_i}|^p dxdt\Big] \tag{3.13}$$
$$\le \|\nabla\bar{v}\|_{L_\infty}\|\nabla W\|^p_{L_p(T_{per}\mathbb{S}^1;L_p(\mathbb{T}^2))} + \frac{G}{\Omega}T_{per}^{1/p}(\|\bar{v}\|_{L_\infty(T_{per}\mathbb{S}^1;W^1_\infty(\mathbb{T}^2))} + \alpha)\|\nabla W\|^{(p-1)/p}_{L_p(T_{per}\mathbb{S}^1;L_p(\mathbb{T}^2))}.$$

Assuming, as required in (3.3), that $\|\nabla\bar{v}\|_{L_\infty} \le \delta$, and observing that in this case we have $\delta \le \frac{1}{2}\alpha$, we conclude that

$$\alpha\|\nabla W\|_{L_p(T_{per}\mathbb{S}^1;L_p(\mathbb{T}^2))} \le C\frac{G}{\Omega}T_{per}^{1/p}(\|\bar{v}\|_{L_\infty(T_{per}\mathbb{S}^1;W^1_\infty(\mathbb{T}^2))} + \alpha) \le C\frac{\alpha G}{\Omega}T_{per}^{1/p}. \tag{3.14}$$

Substituting estimates (3.14) and (3.11) into equation (3.8) we find that

$$W_t - \epsilon\Delta W \in L_p(T_{per}\mathbb{S}^1; L_p(\mathbb{T}^2)). \tag{3.15}$$

Based on the classical result for the heat equation [A, LSU] of the maximal regularity estimates for the L_p spaces, we obtain information with no dependence from ϵ. Hence

$$\|W_t\|_{L_p(T_{per}\mathbb{S}^1;L_p(\mathbb{T}^2))} \le C\frac{G}{\Omega}T_{per}^{1/p}(\|\bar{v}\|_{L_\infty(T_{per}\mathbb{S}^1;W^1_\infty(\mathbb{T}^2))} + \alpha) \le C\frac{\alpha G}{\Omega}T_{per}^{1/p}. \tag{3.16}$$

Concerning estimates (3.8)-(3.16), we observe that the term $-\epsilon\Delta W$, for $\epsilon > 0$, has the right sign. Moreover, it also has the right good sign even in the maximal regularity (3.15).

The dominant-viscosity case when $a\epsilon \ge \alpha$, with possibly $\alpha = 0$. This case allows us to take full advantage of the parabolicity of (3.8), which reads

$$W_t - \epsilon\Delta W + \alpha W = -\bar{v}\cdot\nabla W + \bar{v}\cdot\nabla\frac{1}{\Omega}g - \epsilon\Delta\frac{1}{\Omega}g + \alpha\frac{1}{\Omega}g. \tag{3.17}$$

The maximal regularity estimate for the heat equation [LSU] or more direct [Mu2] implies that

$$\|W_t, \epsilon\nabla^2 W, \alpha W\|_{L_p(\mathbb{T}^2\times T_{per}\mathbb{S}^1)} \le C_p\|\bar{v}\cdot\nabla W, \bar{v}\cdot\nabla\frac{1}{\Omega}g, \epsilon\Delta\frac{1}{\Omega}g, \alpha\frac{1}{\Omega}g\|_{L_p(\mathbb{T}^2\times T_{per}\mathbb{S}^1)}, \tag{3.18}$$

where the constant C_p depends only on p, there is no dependence on T_{per}, since we consider only homogeneous norms in (3.18). Observe that

$$C_p\|\bar{v}\cdot\nabla W, \bar{v}\cdot\nabla\frac{1}{\Omega}g, \epsilon\Delta\frac{1}{\Omega}g, \alpha\frac{1}{\Omega}g\|_{L_p(\mathbb{T}^2\times T_{per}\mathbb{S}^1)} \le C_p\delta\|\nabla W\|_{L_p\mathbb{T}^2\times T_{per}\mathbb{S}^1)} + C\frac{\epsilon G}{\Omega}T_{per}^{1/p},$$

since $\bar{v} \in \Xi$ we have $\|\bar{v}\|_{L_\infty} \leq \delta$. Furthermore, since in this case we have $\delta \leq \frac{1}{2}a\epsilon$ then the first term above can be absorbed by the left-hand side of (3.18), thanks to the facts $\int_{\mathbb{T}^2} W dx = 0$ and $\|\nabla W\|_{L_p(\mathbb{T}^2)} \leq C\|\nabla^2 W\|_{L_p(\mathbb{T}^2)}$. Hence we conclude the following bound

$$\|W_t, \epsilon\nabla^2 W, \alpha W\|_{L_p(\mathbb{T}^2 \times T_{per}\mathbb{S}^1)} \leq C\frac{\epsilon G}{\Omega} T_{per}^{1/p}, \tag{3.19}$$

for the case $a\epsilon \geq \alpha$, establishing the analogue of (3.14) and (3.16) for this case.

Now we return to studying properties of the map $\mathcal{T}$ treated for both cases. Before we establish the L_∞-bound for ∇v, a comment is in order. The key problem is the length of T_{per}. In general the constant in the Sobolev imbeddings may highly depend on T_{per} in a bad way. Hence a solution, which here seems to be most natural, is to consider $\bar{v}$ over several periods of time. Here we think about $\mathbb{S}^* \sim [\frac{1}{T_{per}}] T_{per}\mathbb{S}^1$, where $[t]$ denotes the integer part of t. Then $\mathbb{S}^*$ is close to $\mathbb{S}^1$. The functions are defined over the domain $\mathbb{T}^2 \times \mathbb{S}^*$, so the problems with thinness of domain will disappear.

For this purpose we set $v = v_2 - v_1$, where v_2 is given as a solution to the following problem

$$\operatorname{rot} v_2 = W, \qquad \operatorname{div} v_2 = 0 \tag{3.20}$$

and v_1 is given by

$$\operatorname{rot} v_1 = \frac{1}{\Omega} g(x, \Omega t), \qquad \operatorname{div} v_1 = 0. \tag{3.21}$$

The functions are considered on time interval $\mathbb{S}^*$, since we assumed that Ω is large, hence $T_{per} << 1$. Then from (3.14) and (3.16), together with (3.11), and from (3.19), we get

$$\|\nabla v_2\|_{W_p^{1,1}(\mathbb{T}^2 \times \mathbb{S}^*)} \lesssim \frac{1}{T_{per}^{1/p}} \|\operatorname{rot} v_2\|_{W_p^{1,1}(\mathbb{T}^2 \times T_{per}\mathbb{S}^1)} \leq C(1+\alpha)\frac{G}{\Omega} \lesssim \frac{G}{\Omega}. \tag{3.22}$$

So the Sobolev imbedding $W_p^{1,1}(\mathbb{T}^2 \times \mathbb{S}^*) \subset L_\infty(\mathbb{T}^2 \times \mathbb{S}^*)$ gives

$$\|\nabla v_2\|_{L_\infty(0,1;L_\infty(\mathbb{T}^2))} \leq C\frac{G}{\Omega}, \tag{3.23}$$

where C is independent of T_{per}. Thus,

$$\|\nabla v\|_{L_\infty(\mathbb{T}^2 \times \mathbb{S}^*)} \leq \|\nabla v_2\|_{L_\infty(\mathbb{T}^2 \times \mathbb{S}^*)} + \|\nabla v_1\|_{L_\infty(\mathbb{T}^2 \times \mathbb{S}^*)} \leq C\frac{G}{\Omega} \leq \delta, \tag{3.24}$$

provided Ω large enough.

We showed that $\mathcal{T}$ maps the set Ξ into itself, and the imbedding (for the case $\alpha \geq a\epsilon$)

$$W_p^{1,1}(\mathbb{T}^2 \times T_{per}\mathbb{S}^1) \subset L_\infty(\mathbb{T}^2 \times T_{per}\mathbb{S}^1) \tag{3.25}$$

for $p > 3$ is compact – (3.14) and (3.16). The space $W_p^{1,1}(\mathbb{T}^2 \times T_{per}\mathbb{S}^1)$ is defined as a set of functions f such that $\nabla_x f \in L_p(\mathbb{T}^2 \times T_{per}\mathbb{S}^1)$ and $\partial_t f \in L_p(\mathbb{T}^2 \times T_{per}\mathbb{S}^1)$. The case $a\epsilon \geq \alpha$ is simpler.

By the Schauder fixed theorem we obtain existence of at least one fixed point of the map $\mathcal{T}$ fulfilling (3.3). This implies existence of time periodic solutions to the nonlinear system (3.1)-(3.2) satisfying bound (3.3). Theorem 1 is proved.

Global stability of the time periodic solutions

Theorem 2. *Let $p>3$, let $v(t)$ be a solution to (3.1-3.2) corresponding to the initial data $v_0\in W^2_p(\mathbb{T}^2)$ and F fulfills the assumptions from Theorem 1, then*

$$\|v(t)-v_{per}(t)\|_{L_2}\to Ce^{-\frac{(\alpha+\epsilon)t}{2}},\ \text{as } t\to\infty, \tag{3.26}$$

where v_{per} is the time periodic solution established by Theorem 1. Moreover v_{per} must be unique.

Proof. Note that for smooth enough initial datum v_0 we have the global in time existence of solutions to system (2.1). Consider the difference $\delta v(t)=v(t)-v_{per}(t)$, it fulfills the following system

$$\begin{aligned}&\delta v_t+v\cdot\nabla\delta v-\epsilon\Delta\delta v+\alpha\delta v+\nabla\delta p=-\delta v\cdot\nabla v_{per},\\&\operatorname{div}\delta v=0,\\&\delta v|_{t=0}=v_0-v_{per}(\cdot,0).\end{aligned} \tag{3.27}$$

Multiplying (3.27) by δv and integrating over $\mathbb{T}^2$ yields

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}^2}(\delta v)^2dx+\int_{\mathbb{T}^2}\big(\epsilon|\nabla\delta v|^2+\alpha(\delta v)^2\big)dx\le\|\nabla v_{per}\|_{L_\infty}\int_{\mathbb{T}^2}(\delta v)^2dx. \tag{3.28}$$

Applying the Poincaré inequality we get

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}^2}(\delta v)^2dx+(\epsilon+\alpha)\int_{\mathbb{T}^2}(\delta v)^2dx\le\delta\int_{\mathbb{T}^2}(\delta v)^2dx. \tag{3.29}$$

In our setting the constant from the Poincar'e inequality (in the L_2 spaces) is equal 1. Our choice of δ guaranteed that $\delta\le\frac{1}{2}(\epsilon+\alpha)$, so we get

$$\frac{d}{dt}\int_{\mathbb{T}^2}(\delta v)^2dx+(\epsilon+\alpha)\int_{\mathbb{T}^2}(\delta v)^2dx\le 0. \tag{3.30}$$

We immediately conclude (3.26). In particular (3.26) shows that constructed time periodic solution by Theorem 1 is unique.

4. The 3D case

Theorem 3. *Let $\nu>0$, and F be sufficiently smooth divergence-free vector field of form (A) or (B). Suppose that for case (A) there exists a divergence-free vector field H satisfying*

$$\partial_t\frac{1}{\Omega}H(x,\Omega t)=F(x_1+\Omega t,x')\quad\text{and}\quad\sup_t\|H(\cdot,\Omega t)\|_{C^1(\mathbb{T}^3)}\le G. \tag{4.1}$$

There exists $\Omega_0(\nu,G)>0$ such that there exists a time periodic solution, with period $T_{per}=2\pi/\Omega$ satifying

$$\|v_{per}\|_{L_\infty(\mathbb{T}^3\times T_{per}\mathbb{S}^1)}\le\frac{C^*(1+\nu)G}{\Omega}, \tag{4.2}$$

provided $\Omega\ge\Omega_0$, where C^* depends on ν, only.

Proof. Let $\delta=\frac{C^*(1+\nu)G}{\Omega}$. Introduce the set

$$\mathcal{X}=\{v\in L_\infty(\mathbb{T}^3\times T_{per}\mathbb{S}^1):\operatorname{div}v=0\text{ and }\|v\|_{L_\infty}\le\delta\}. \tag{4.3}$$

Let $\bar{v} \in \mathcal{X}$, then we consider the linearization of (2.11)

$$v_t - \nu\Delta v + \nabla p = -\mathrm{div}\,(\bar{v} \otimes v) + F, \qquad \mathrm{div}\, v = 0, \qquad \int_{\mathbb{T}^3} v(x,t)dx = 0. \tag{4.4}$$

This process introduces a map $\mathcal{T}(\bar{v}) = v$. We will show $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ and $\mathcal{T}$ is compact. As a result this will establish existence of a time periodic solution to the nonlinear system (2.11).

Similar to the 2d case we set

$$v = V - \frac{1}{\Omega} H(x, \Omega t). \tag{4.5}$$

In case (A) we take $H = \sum_{k\in\mathbb{Z}^3} \frac{\hat{F}_k}{ik_1} e^{ikx} e^{i\Omega t k_1}$ and for case (B) we take $H = -\cos\Omega t f(x)$, thus $\partial_t \frac{1}{\Omega} H = F$. Then we get

$$V_t - \nu\Delta V + \nabla p = -\mathrm{div}\,(\bar{v}\otimes V) + \mathrm{div}\,(\bar{v} \otimes \frac{1}{\Omega}H) + \frac{\nu}{\Omega}\Delta H. \qquad \mathrm{div}\, V = 0, \tag{4.6}$$

The existence of solutions to (4.6) is sketched in Appendix. The estimates for solutions to (4.6) are done in the domain $\mathbb{T}^3 \times \mathbb{S}^*$ with $\mathbb{S}^* = ([\frac{1}{T_{per}}] + 1) T_{per}\mathbb{S}^1$ just in order to avoid the possible problem with smallness of T_{per}. Provided we solved system (4.4) with $\bar{v} \in \mathcal{X}$, we want to find a suitable estimate guaranteeing v in L_∞. Here we work with the Slobodeckii spaces $W^{1,1/2}_p(\mathbb{T}^3 \times \mathbb{S}^*)$ [A, LSU]. In the Appendix we explain the details. Then we find the following inequality for system (4.6)

$$\|V\|_{W^{1,1/2}_p(\mathbb{T}^3\times\mathbb{S}^*)} \le C_\nu \|\bar{v}V,\ \frac{1}{\Omega}\bar{v}H,\ \frac{\nu}{\Omega}\nabla H\|_{L_p(\mathbb{T}^3\times\mathbb{S}^*)}. \tag{4.7}$$

Now we use Ω large enough to guarantee the smallness of δ such that $C_\nu\|\bar{v}\|_{L_\infty} \le 1/2$, (observe the norm $\|V\|_{W^{1,1/2}_p}$ contains $\|V\|_{L_p}$, as well), then

$$\|V\|_{W^{1,1/2}_p(\mathbb{T}^3\times\mathbb{S}^*)} \le \frac{C_\nu(\delta+\nu)}{\Omega} G. \tag{4.8}$$

Next, we note that if $p > 5$ then the space $W^{1,1/2}_p(\mathbb{T}^3 \times \mathbb{S}^*)$ is compactly imbedded in $L_\infty(\mathbb{T}^3 \times \mathbb{S}^*)$ [BIN], Chap XII. Consequently we have

$$\|V\|_{L_\infty(\mathbb{T}^3\times T_{per}\mathbb{S}^1)} \le \frac{C(\delta+\nu)}{\Omega} G. \tag{4.9}$$

The constant is independent from T_{per}, since (4.7) is considered on $\mathbb{T}^3 \times \mathbb{S}^*$. Therefore (4.5) implies that

$$\|v\|_{L_\infty(\mathbb{T}^3\times T_{per}\mathbb{S}^1)} \le C(\frac{(\delta+\nu)}{\Omega} + \frac{1}{\Omega})G \le \delta = C^* \frac{1+\nu}{\Omega} G \tag{4.10}$$

which is guaranteed for Ω larger than $\Omega_0(\nu, G)$.

Thus, we $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ is compact. Hence the Schauder theorem implies existence of a fixed point of the map $\mathcal{T}$, what yields existence of a time periodic solution to the original system (2.11). Theorem 3 is proved.

Attraction of weak solutions – the 3d case

Theorem 4. *Let $v_0 \in L_2(\mathbb{T}^3)$ be a divergence-free vector field, and let $v(t)$ be a Leray-Hopf weak solution to (2.11) with initial datum v_0. Then*

$$\|v(t) - v_{per}(t)\|_{L_2(\mathbb{T}^3)} \to 0, \ \text{as } t \to \infty, \tag{4.11}$$

where v_{per} is the time periodic solution given by Theorem 3, provided Ω is large enough.

Proof. Since weak solutions are not known whether they satisfy the energy equality, in the three-dimensional case, it will not be possible for us to follow the same arguments as in the proof of Theorem 2. However, since we are considering here Leray-Hopf weak solutions, then, on one hand, $v(t)$ satisfies following strong energy inequality

$$\|v(t)\|^2_{L_2(\mathbb{T}^3)} + 2\nu \int_s^t \|\nabla v(\tau)\|^2_{L_2(\mathbb{T}^3)} d\tau \le \|v(s)\|^2_{L_2(\mathbb{T}^3)} + 2\int_s^t (F, v) d\tau, \tag{4.12}$$

for all $t > 0$ and a.e. s such that $t > s \ge 0$. On the other hand, time periodic solutions are regular solutions, thus they do satisfy the energy equality

$$\|v_{per}(t)\|^2_{L_2(\mathbb{T}^3)} + 2\nu \int_s^t \|\nabla v_{per}(\tau)\|^2_{L_2(\mathbb{T}^3)} d\tau = \|v_{per}(s)\|^2_{L_2(\mathbb{T}^3)} + 2\int_s^t (F, v_{per}) d\tau, \tag{4.13}$$

for every $t \ge s \ge 0$. To obtain an estimate for $\|v(t) - v_{per}(t)\|_{L_2(\mathbb{T}^3)}$ we use the observation that

$$\begin{aligned} \|v(t) - v_{per}(t)\|^2_{L_2(\mathbb{T}^3)} &= (v(t) - v_{per}(t), v(t) - v_{per}(t)) = \\ &(v(t), v(t)) + (v_{per}(t), v_{per}(t)) - (v(t), v_{per}(t)) - (v_{per}(t), v(t)). \end{aligned} \tag{4.14}$$

To control the last terms we use the weak formulation for the solutions v and v_{per}. Specifically, since v_{per} is a sufficiently smooth we are allowed, on the one hand, to use it as a test function in the weak formulation for the weak solution v to obtain

$$2\partial_t(v, v_{per}) - 2(v, v_{per,t}) + 2\nu(\nabla v, \nabla v_{per}) + 2(v \cdot \nabla v, v_{per}) = 2(F, v_{per}). \tag{4.15}$$

On other hand, since v_{per} is regular enough solution and the equation holds in $L_2(\mathbb{T}^3 \times T_{per}\mathbb{S}^1)$, we can multiply by v and integrate over $\mathbb{T}^3$ to infer

$$2(v_{per,t}, v) + 2\nu(\nabla v_{per}, \nabla v) + 2(v_{per} \cdot \nabla v_{per}, v) = 2(F, v). \tag{4.16}$$

Both (4.15) and (4.16) are meant in the distributional sense in time. We add (4.15) and (4.16) and integrate over time interval (s, t), and obtain

$$\begin{aligned} 2(v(t), v_{per}(t)) &+ 2\nu \int_s^t (\nabla v, \nabla v_{per}) + (\nabla v_{per}, \nabla v) d\tau \\ &+ 2\int_s^t (v \cdot \nabla v, v_{per}) + (v_{per} \cdot \nabla v_{per}, v) d\tau = \\ &\qquad 2(v(s), v_{per}(s)) + 2\int_s^t (F, v + v_{per}) d\tau \end{aligned} \tag{4.17}$$

Taking (4.12)+(4.13)-(4.17) we obtain

$$\|v(t)-v_{per}(t)\|^2_{L_2(\mathbb{T}^3)}+2\nu\int_s^t\|\nabla(v(t)-v_{per}(t))\|^2_{L_2(\mathbb{T}^3)}d\tau \leq \|v(s)-v_{per}(s)\|^2_{L_2(\mathbb{T}^3)}+2\int_s^t(v\cdot\nabla v,v_{per})+(v_{per}\cdot\nabla v_{per},v)d\tau. \tag{4.18}$$

Let $\delta v(t)=v(t)-v_{per}(t)$. Consider the term

$$(v\cdot\nabla v,v_{per})+(v_{per}\cdot\nabla v_{per},v). \tag{4.19}$$

We have

$$(v\cdot\nabla v,v_{per})+(v_{per}\cdot\nabla v_{per},v)=(v\cdot\nabla v,v_{per})-(v_{per}\cdot\nabla v,v_{per})= (v\cdot\nabla v_{per},v_{per})-(v_{per}\cdot\nabla v_{per},v_{per})+(v\cdot\nabla\delta v,v_{per})-(v_{per}\cdot\nabla\delta v,v_{per})= 0+0+(\delta v\cdot\nabla\delta v,v_{per}). \tag{4.20}$$

Two first terms vanished. Next we note that

$$|\int_{\mathbb{T}^3}\delta v\cdot\nabla\delta v\, v_{per}dx|\leq\frac{\nu}{2}\|\nabla\delta v\|^2_{L_2(\mathbb{T}^3)}+\frac{C}{\nu}\|v_{per}\|^2_{L_\infty}\|\delta v\|^2_{L_2(\mathbb{T}^3)}. \tag{4.21}$$

Next, the Poincaré inequality yields $\|\delta u\|_{L_2(\mathbb{T}^3)}\leq\|\nabla\delta u\|_{L_2(\mathbb{T}^3)}$. Moreover, we observe that $\|v_{per}\|_{L_\infty}\leq\delta$ – see the proof of Theorem 3, with δ very small so that $\frac{C}{\nu}\delta^2\leq\frac{\nu}{4}$ which holds as Ω is sufficiently large. Using the above to finally obtain

$$\|\delta v(t)\|^2_{L_2(\mathbb{T}^3)}+\nu\int_s^t\|\nabla\delta v(t)\|^2_{L_2(\mathbb{T}^3)}d\tau\leq\|\delta v(s)\|_{L_2(\mathbb{T}^3)}. \tag{4.22}$$

Again using the Poincaré inequality we obtain

$$\|\delta v(t)\|^2_{L_2(\mathbb{T}^3)}+\nu\int_s^t\|\delta v(t)\|^2_{L_2(\mathbb{T}^3)}d\tau\leq\|\delta v(s)\|^2_{L_2}, \tag{4.23}$$

for all $t>0$ and a.e. $s\geq 0$. Simple analysis of (4.23) implies

$$\|\delta v(t)\|_{L_2(\mathbb{T}^3)}\leq Ce^{-c\nu t}. \tag{4.24}$$

Theorem 4 is proved. As a corollary we obtain the fact that time periodic established by Theorem 3 are unique for sufficiently large Ω.

5. Numerical results for 2D case

The numerical results presented in this section focus on the system (2.1), i.e.

$$v_t+v\cdot\nabla v-\epsilon\Delta v+\alpha(v-\Omega\hat{e}_2)+\nabla p=F(z), \tag{5.1a}$$

$$\operatorname{div} v=0, \tag{5.1b}$$

$$\frac{1}{|\mathbb{T}^2|}\int_{\mathbb{T}^2}v_0\,dz=\Omega\hat{e}_2, \tag{5.1c}$$

where $v\colon[0,\infty)\times\mathbb{T}^2\to\mathbb{R}^2$, $z=(x,y)$. In the following sections we are concerned with the numerical investigation of the dependence of the long-time qualitative behavior of the solutions of the above system on the parameter Ω, for a given particular external forcing term $F(z)$. In the sequel we are going to assume (2.3), i.e., that average of the forcing term over $\mathbb{T}^2$ is zero. In turn, this implies that the spatial average of the solutions remain constant.

In view of the theoretical analysis, presented in the previous sections, it follows that when the values of Ω exceed certain critical value implies the stabilization of the evolutionary problem (5.1a). More precisely, for a given particular forcing $F(z)$ the stationary problem (5.1a) might have multiple solutions, however, after increasing Ω beyond certain critical value one obtains a unique stationary solution.

5.1. **Numerical investigation particular setting.** In our numerical investigation we focus on a particular case study of the Kolmogorov flow that was discussed in details at the end of section 2. Specifically we consider system (2.14) in the flat torus $\mathbb{T}^2_\beta$. As we have discussed earlier, in the end of section 2, the global stability of time period solutions of (2.14) is equivalent to the global stability of stationary solutions of (2.15). For this reason we focus in the next section the study of the bifurcation diagram of stationary solutions of (2.15).

5.2. **Stationary problem bifurcation analysis.** After dropping the tilde system (2.15) is given by

$$\omega_t + (v + \beta\Omega\hat{e}_2)\cdot\nabla\omega - \epsilon\Delta\omega + \alpha\omega = \lambda\cos(\frac{y}{\beta}), \tag{5.2}$$

subject to periodic boundary condition, with basic domain $\mathbb{T}^2_\beta = [0, 2\pi] \times [0, 2\pi\beta]$. In this section we present our numerical investigations of the stationary problem of (5.2):

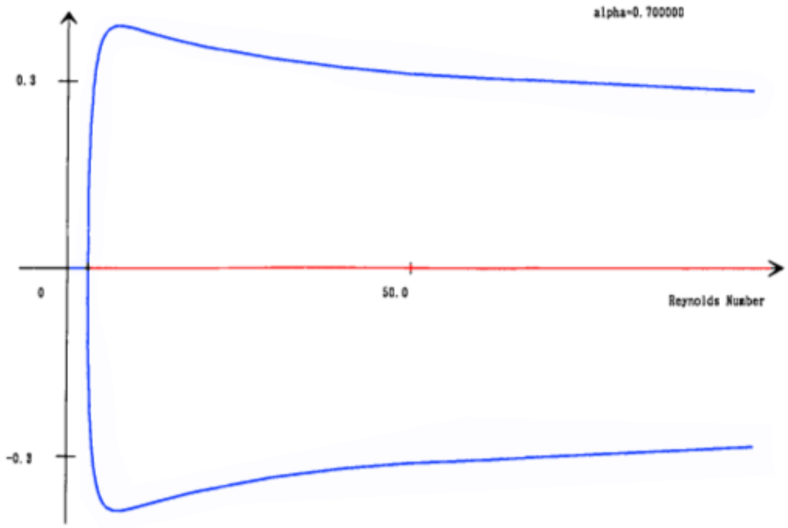
$$(v + \beta\Omega\hat{e}_2)\cdot\nabla\omega - \epsilon\Delta\omega + \alpha\omega = \lambda\cos(\frac{y}{\beta}), \tag{5.3}$$

subject to periodic boundary condition, with basic domain $\mathbb{T}^2_\beta = [0, 2\pi] \times [0, 2\pi\beta]$.

In the case $\beta = 1$ (the square) and when $\Omega = 0$ system (5.2) admits a globally stable stationary solution (called trivial solution) [CFT, M]. Consequently this globally stable stationary solution does not undergo any bifurcation when λ is increased. On the other hand, numerical experiments in [OS] show that for $\beta \in (0, 1)$ system (5.3), when $\Omega = 0$, the trivial stationary solution undergoes a pitchfork bifurcation (see also [BV]). In the remaining part of this section we restrict our attention to the particular case $\beta = 0.7$, for which we present bifurcation diagram on Figure 1 (reproduced from [OS]).

Looking at Figure 1 it is evident that problem (5.3), with $\Omega = 0$, exhibits unique stationary solution for λ values smaller than a critical value (we denote it by λ_0) – the point of the pitchfork bifurcation, at which two branches of stable solutions are born. Let $\lambda_0(\Omega)$ denote the point of the pitchfork bifurcation in problem (5.3), and let $\omega(\lambda_0(\Omega))$ denote the solution at bifurcation point.

We investigate here the dependence of $\lambda_0(\Omega)$, and the dependence of $\|\omega(\lambda_0(\Omega))\|$ – the solution at bifurcation point L_2 norm on Ω. The numerical tests are in agreement with the theory presented in the theoretical part of this paper, from which it follows that the region of the parameter λ, for which one has unique stable stationary solution of (5.3), is enlarged while Ω increases, and that $\|\omega(\lambda_0(\Omega))\|$ should increase with the order of magnitude lower than that of λ_0. Figure 2 agrees with theoretical derivations of (3.23) for equation (5.3)



Bifurcation diagram when $\alpha = 0.7$: $a(1,0)$ vs. Reynolds number.

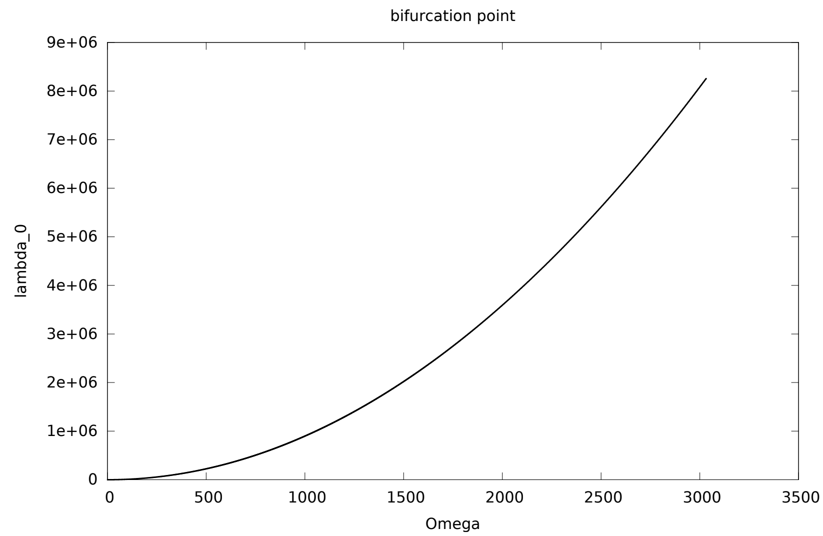
FIGURE 1. The bifurcation diagram for the problem (5.2) with $\beta = 0.7, \Omega = 0, \epsilon = 1, \alpha = 0$ (reproduced from [OS]), the scaling parameter here is the Reynolds number defined by the authors $Re \simeq \frac{\lambda}{\epsilon^2 \beta^3}$. Let $a(1,0)$ denotes the ω's Fourier coefficient corresponding to $\exp ix$ basis function. The stable solution (in blue) having $a(1,0) = 0$ becomes unstable (in red) at particular critical value of the Reynolds number, where two new stationary solutions are being born.

showing $\|\omega\|_{L_2} \sim \lambda\Omega^{-1}$ (G from (3.23) is proportional to λ). Presented numerical results indicate also that the lost of uniqueness occur for $\lambda \gtrsim \Omega^2$ hence the norm of the solution at bifurcation point is of linear growth in Ω.

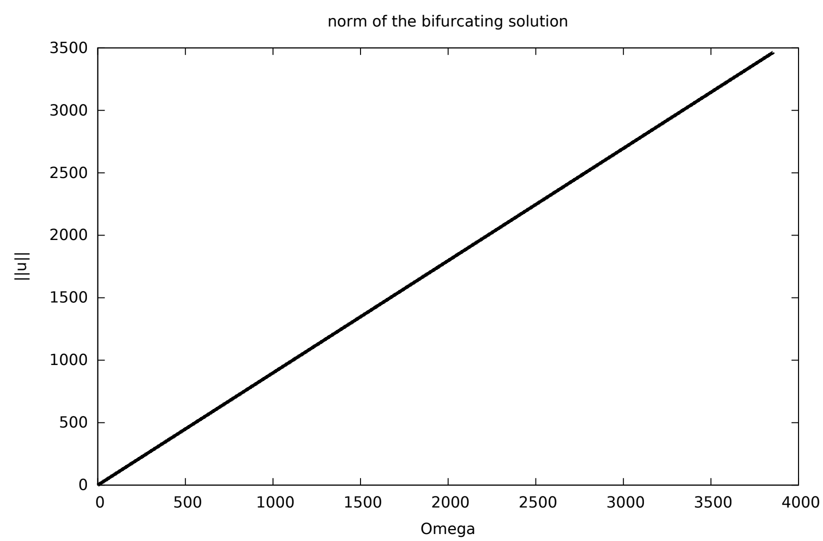
In Figure 2 we present the calculated results for the problem (5.3) with $(\epsilon, \alpha) = (1, 0)$, we skip here results for other choices of (ϵ, α), as we did not observe any qualitative difference in this case.

5.3. **Effect of stabilization.** This part illustrates the stabilization effect for problem (5.2). For particular initial conditions provided later on we integrate in time the evolution equations (5.2). In (5.2) we force the second mode (the forcing is $\lambda \cos(\frac{2y}{\beta})$), as we observe a rich dynamics for that case. For a fixed λ we compare $\Omega = 0$ case with Ω large. As a result we obtain the stabilization effect with exponential convergence rate for the latter case, as Theorem 2 predicts.

In order to numerically integrate (5.2) forward in time we invoke standard numerical integrator. We write ω in (complex) Fourier basis, i.e., $\omega(t,z) = \sum a_k(t) \exp i(k,z)$. We consider a Galerkin approximation of the infinite system of ODEs including only modes a_k with k such that $|k|_\infty \leq N$. We call N the approximation dimension. In the presented experiment



(A) diagram showing the bifurcation point λ_0 with respect to Ω. The presented graph is approximately $\lambda_0(\Omega) = 0.89851 \cdot \Omega^2 + 0.0014294 \cdot \Omega + 1$, obtained by least squares fitting with root mean square of residuals equal to 0.612266



(B) diagram showing $\|\omega(\lambda_0)\|$ (L_2 norm of the solution at bifurcating point) with respect to Ω

FIGURE 2. Bifurcation diagrams for the problem (5.2) with $\alpha = 0$, and $\epsilon = 1$.

we fixe $N = 13$, this choice is motivated by the fact that this approximation dimension represents well the dynamics of the PDE, we validate this by checking that for a larger dimension ($N = 19$) the obtained results are qualitatively the same (not provided here). To perform time integration procedure we use the Taylor method, the time step is selected adaptively, is maximized under constraint such that the local error do not exceed the machine precision. In our actual computations we fix Taylor's method order to 15, which is relatively high order as for a high dimensional system, however, in our case it provides an efficient procedure.

We describe the following numerical experiment.

We fix $\lambda = 100$, $\beta = 0.75$, $N = 13$, order of the Taylor method is 15. We pick four initial conditions and integrate the equation on the time interval $[0, 5]$.

(1) Initial condition I – $\omega_0^I(x, y) = 60\cos(x + y)$; it is attracted by a periodic orbit of L_2 norm in $(19.7, 20)$,
(2) Initial condition II – $\omega_0^{II}(x, y) = 2\cos(x)$; it is attracted by a stationary solution of L_2 norm approximately 9.68043,
(3) Initial condition III – $\omega_0^{III}(x, y) = 2\cos(2x)$; it is attracted by a stationary solution of L_2 norm approximately 14.2384,
(4) Initial condition IV – $\omega_0^{IV}(x, y) = 0$; it is attracted by a stationary solution of L_2 norm approximately 25.

Observe that for $\Omega = 100$ all of the initial conditions, even the periodic orbit case, are being attracted by the same stationary solution, so the stabilization is achieved for $\Omega \gtrsim \sqrt{\lambda}$, which agrees with our expectations.

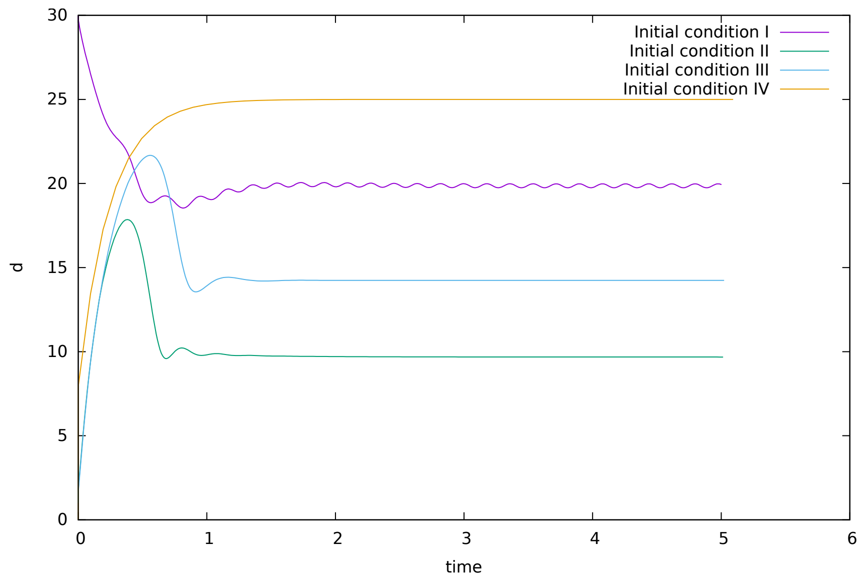
5.4. **Evolutionary problem convergence rate analysis.** In this section we present the results of our investigation of the global convergence to the unique stationary solution, for large initial values, of the general time dependent solutions of the evolution equation (5.2).

First we fix the forcing amplitude $\lambda = 1$, in (5.2), and we consider the initial value $\omega_0 = \Omega(\sin x + \cos x)$, Ω here is the large parameter and is the amplitude of the initial value. We recorded the time needed for ω_0 to be attracted by the stationary solution of (5.2), with $\lambda = 1$. We stopped our numerical integration procedure at time T when

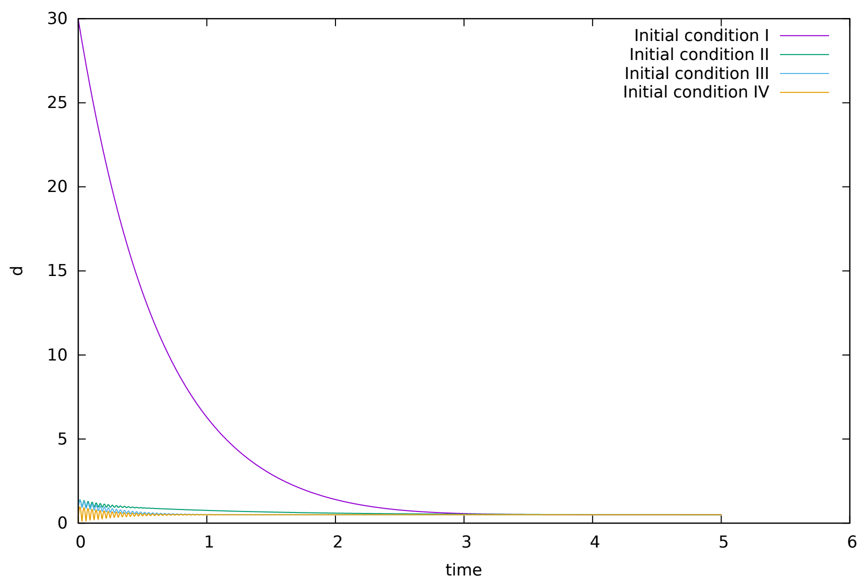
$$\|v(T) \cdot \nabla\omega(T) - \epsilon\Delta\omega(T) + \alpha\omega(T) - \lambda\cos(\frac{y}{\beta})\|_{L_2} \leq 10^{-5}.$$

In Figure 4a we present the results for the case $(\epsilon, \alpha) = (1, 0)$ (other cases were qualitatively very similar). Figure 4a is plotted in the logscale, and the apparent linear growth of T matches the exponential convergence established in Theorem 2.

To perform the numerical time integration we invoke the same techniques as in the previous section, however, for this particular choice of initial condition and $\lambda = 1$ the dynamics is apparently low dimensional, so we fix $N = 3$. This low Galerkin approximation may seem not sufficient. Therefore to argue that the results for larger Galerkin approximation, in this case, do not differ qualitatively from the obtained results, we perform an additional numerical test in which we measure the relative difference between two approximation of dimensions N and M by $E(\Omega, N, M) = |(\lambda_0(\Omega, N) - \lambda_0(\Omega, M))/\lambda_0(\Omega, N)|$, where $\lambda_0(\Omega, N)$ is the bifurcation point for particular Ω, calculated using the approximation dimension N. We present the obtained diagrams for $N = 3$, and $M = 5$ in Figure 4b. Clearly, the values



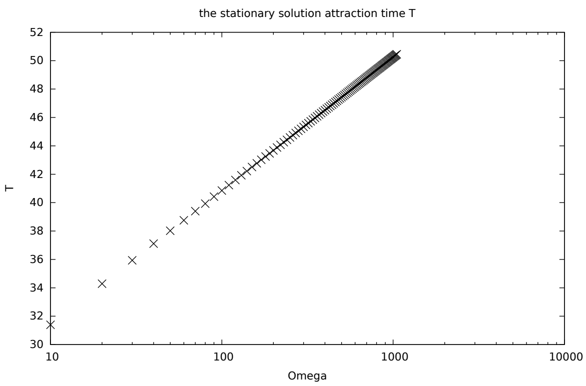
(A) Case of $\Omega = 0$



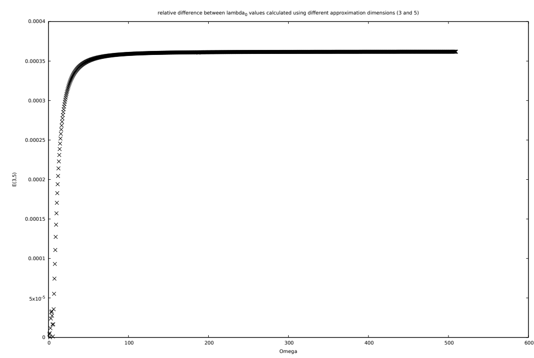
(B) Case of $\Omega = 100$

FIGURE 3. Results from integrating in time the equation (5.2) with $\Omega = 0$, and $\Omega = 100$, using four different initial conditions $\{\omega_0^I, \omega_0^{II}, \omega_0^{III}, \omega_0^{IV}\}$. The L_2 norm of the solution with respect to time is plotted, we denote $d = \|\omega(t)\|_{L_2}$.

shown remain essentially constant for larger Ω values, and does not exceed 0.0004, which supports our claim that it is enough to use a small approximation dimension to provide a qualitatively correct illustration.



(A) Logscale plot of the stationary solution attraction time of (denoted T) with respect to Ω ($\lambda = 1$ is fixed). The initial condition for a fixed Ω is $\omega_0 = \Omega(\sin x + \cos x)$. Apparent linear growth of T matches the exponential convergence established in Theorem 2.



(B) Diagram presenting $E(\Omega, 3, 5) = |(\lambda_0(\Omega, 3) - \lambda_0(\Omega, 5))/\lambda_0(\Omega, 3)|$, is the relative difference between $\lambda_0(\Omega, 3)$, and $\lambda_0(\Omega, 5)$, the parameter here is Ω.

5.5. **Conclusions from numerical experiments and future work.** The goal of this section was to present a numerical investigations of a simple case, as an illustration of the theoretical results presented in the theoretical sections of this paper. Obviously, the numerical results match the theoretical predictions. All theorems in this paper are about periodic solutions, but in order to obtain the numerical results we always reduce the problem to the stationary case. This is imposed by the fact that for high values of Ω the periodic solutions oscillate rapidly, which is a major obstacle for numerical integration in time. Due to the equivalence between the two problems, as we have indicated in this section, the conclusions from the numerical results are meaningful for the case of oscillating rapidly periodic solutions, although the computations are performed for the stationary case. There exist several numerical methods which probably allow to treat the case with rapid oscillations directly, but our current goal was solely to provide an illustration for the theoretical results established in this paper, rather than invoking sophisticated numerical methods to deal with rapid oscillations generated by large values of Ω. We leave the task for future research.

6. Appendix

In this part we explain the construction of time periodic solutions for the "linearized" problem. We establish this for the three-dimensional case, which is an essential step in the proof of Theorem 3. The case for Theorem 1 is almost the same.

Having $\bar{v} \in \mathcal{X}$ we consider (4.6) in the following form

$$
\begin{aligned}
&V_t - \nu\Delta V + \nabla p = -\mathrm{div}\,(\bar{v} \otimes V) + \mathrm{div}\,(\bar{V} \otimes \tfrac{1}{\Omega} H) + \tfrac{\nu}{\Omega}\Delta H, \\
&\mathrm{div}\, V = 0.
\end{aligned} \tag{6.1}
$$

The issue of existence for the above system lays in the classical theory. The simplest approach is through Fourier methods using series in time and space to the linear system with given right-hand side. We acts on the domain $\mathbb{T}^3 \times T_{per}\mathbb{S}^1$ and we represent the solution in the form

$$
V^{(j)}(x,t) \sim \sum_{l \in \mathbb{Z}, k \in \mathbb{Z}^3} V^{(j)}_{lk} e^{iltT_{per}} e^{ikx}, \qquad j = 1, 2, 3. \tag{6.2}
$$

The solvability of the system for the finite dimensional approximation is clear, so we need just a good estimate which allows to pass to the limit. But the energy estimate is allowed to be used in the chosen framework, so we get

$$
V \in L_2(T_{per}\mathbb{S}^1; H^1(\mathbb{T}^3)), \tag{6.3}
$$

with the a priori estimate

$$
\|V\|_{L_2(T_{per}\mathbb{S}^1;H^1(\mathbb{T}^3))} \le C(\frac{1}{\Omega}\|\bar{v}\|_{L_\infty}\|H\|_{L_2(\mathbb{T}^3 \times T_{per}\mathbb{S}^1)} + \frac{\nu}{\Omega}\|\nabla H\|_{L_2(\mathbb{T}^3 \times T_{per}\mathbb{S}^1)}). \tag{6.4}
$$

The construction by approximation in the time periodic functions ensures the solution V is T_{per} - periodic. The form of (6.2) guarantees the periodicity in time and space. The term $\mathrm{div}\,(\bar{v} \otimes V)$ can be treated as a perturbation, and thanks to the estimate (6.4) we obtain the existence to system (6.1), too.

Next, we improve the regularity of solutions V. Here we apply the maximal regularity result for the Stokes operator in the Slobodeckii spaces of type $W^{1,1/2}_p(\mathbb{T}^3 \times \mathbb{S}^*)$ [A, BIN, Tr]. For solutions to

$$
\begin{aligned}
&V_t - \nu\Delta V + \nabla p = \mathrm{div}\, F, \\
&\mathrm{div}\, V = 0
\end{aligned} \tag{6.5}
$$

the following estimate holds

$$
\|V\|_{W^{1,1/2}_p(\mathbb{T}^3 \times \mathbb{S}^*)} \le C_\nu \|F\|_{L_p(\mathbb{T}^3 \times \mathbb{S}^*)}. \tag{6.6}
$$

The definition of the Slobodeckii space [Sol, Tr] is by the following norm

$$
\|V\|^p_{W^{1,1/2}_p(\mathbb{T}^3 \times \mathbb{S}^*)} = \|V\|^p_{L_p(\mathbb{T}^3 \times \mathbb{S}^*)} + \|\nabla V\|^p_{L_p(\mathbb{T}^3 \times \mathbb{S}^*)} + \int_\Omega \int_{\mathbb{S}^*} \int_{\mathbb{S}^*} \frac{|V(x,t) - V(x,t')|^p}{|t-t'|^{1+\frac{1}{2}p}} dt dt' dx. \tag{6.7}
$$

Thus we justify estimate (4.7).

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