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ABSTRACT

It is shown that many commonly used one-step algorithms which are unconditionally stable for linear transient heat conduction problems become conditionally stable in the nonlinear regime. Alternative algorithms are proposed which, for linear problems, are identical to those commonly used, whereas for nonlinear problems the unconditional stability behavior of the linear case is retained.

INTRODUCTION

In this paper we consider the stability of a commonly used family of one-step algorithms in linear and nonlinear heat conduction. The stability behavior of these algorithms is well-known in the linear regime. In particular, the family contains a second-order accurate unconditionally stable member (i.e., Crank-Nicholson method). However, when applying these algorithms to nonlinear problems, it is found that, with the exception of one method (backward difference method), unconditional stability is lost.

To remedy this situation an alternative family of one-step algorithms is proposed. For linear problems these methods coincide with the commonly used ones mentioned above. However, it is shown that, in the nonlinear regime, this family retains the same stability behavior as for the linear case. In particular, an unconditionally stable second-order accurate method is amongst the proposed algorithms.

ANALYSIS

Consider the discrete equations of nonlinear heat conduction:

$$\underset{\sim}{C}(\underset{\sim}{\theta},t)\dot{\underset{\sim}{\theta}} + \underset{\sim}{K}(\underset{\sim}{\theta},t)\underset{\sim}{\theta} = \underset{\sim}{R}(t), \tag{1}$$

in which $\underset{\sim}{C}$ is the capacity matrix, $\underset{\sim}{K}$ is the conductivity matrix, $\underset{\sim}{R}$ is the heat supply vector, $\underset{\sim}{\theta}$ is the temperature vector, t denotes time and a superposed dot indicates time differentiation. We assume throughout that $\underset{\sim}{C}$ is symmetric and positive-definite and that $\underset{\sim}{K}$ is symmetric and positive semi-definite. The initial value problem for (1) consists of finding a function $\underset{\sim}{\theta} = \underset{\sim}{\theta}(t)$, $t \in [0,\tau]$, $\tau > 0$, satisfying (1) and the initial condition

$$\underset{\sim}{\theta}(0) = \underset{\sim}{T}, \tag{2}$$

where $\underset{\sim}{T}$ is the given initial data.

Various discrete algorithms have been proposed for the solution of the initial-value problem. Many of these algorithms are members of the following one-parameter (α) family of methods: Find $\underset{\sim}{T}_n$, $n \in \{0,1,\ldots,N\}$ such that

$$\underset{\sim}{C}_n \underset{\sim}{U}_n + \underset{\sim}{K}_n \underset{\sim}{T}_n = \underset{\sim}{R}_n, \qquad n \in \{0,1,\ldots,N\} \tag{3a}$$

$$\underset{\sim}{T}_{n+1} = \underset{\sim}{T}_n + \Delta t\, \underset{\sim}{U}_{n+\alpha}, \qquad n \in \{0,1,\ldots,N-1\}, \tag{3b}$$

$$\underset{\sim}{T}_o = \underset{\sim}{T}, \tag{3c}$$

in which

$$\underset{\sim}{C}_n = \underset{\sim}{C}\,(\underset{\sim}{T}_n, t_n), \tag{3d}$$

$$\underset{\sim}{K}_n = \underset{\sim}{K}\,(\underset{\sim}{T}_n, t_n), \tag{3e}$$

$$\underset{\sim}{R}_n = \underset{\sim}{R}\,(t_n), \tag{3f}$$

$$\underset{\sim}{U}_{n+\alpha} = (1-\alpha)\, \underset{\sim}{U}_n + \alpha \underset{\sim}{U}_{n+1}, \tag{3g}$$

where $\underset{\sim}{T}_n$ is the approximation to $\underset{\sim}{\theta}(t_n)$, N is the total number of time steps, $\Delta t = \tau/N$ and $t_n = n\,\Delta t$. In this paper we limit our attention to parameter values of α in the interval [0,1]. With the exception of the case $\alpha = 0$, at each time step the problem to be solved is a nonlinear algebraic one and techniques such as the Newton-Raphson method, with suitable notions of convergence, must be resorted to. If $\alpha = 0$, (forward difference or Euler method) the method is explicit and the solution may be constructed without solving systems of linear equations. For linear problems, in which $\underset{\sim}{C}$ and $\underset{\sim}{K}$ are constant matrices, the stability properties of this family of algorithms is well-known (see for example Wood and Lewis [1] and Taylor [2]). For instance, if $\alpha < 1/2$ the algorithm in question is conditionally stable, i.e., stability considerations limit the maximum size of the time step employed. To be precise the time step must satisfy the condition

$$\lambda\,\Delta t \leq 2/(1 - 2\alpha), \tag{4}$$

in which λ is the maximum eigenvalue of the matrix $\underset{\sim}{C}^{-1}\underset{\sim}{K}$. On the other hand, if $\alpha \geq 1/2$ the algorithm in question is unconditionally stable, i.e., there is no restriction on the maximum size of time step. Condition (4) is a stringent one in practice and for this reason unconditionally stable algorithms are generally preferred. We are interested in determining the values of α for which unconditional stability holds in the nonlinear case.

We shall deal with this issue by considering the single-degree-of-freedom nonlinear model equation

$$\dot{\theta} + \lambda(\theta,t)\theta = 0 \ , \tag{5}$$

in which it is assumed $\lambda > 0$. Applying the algorithm (3) to (5) and employing the obvious notation for the single-degree-of-freedom case, we obtain the recursion relation:

$$T_{n+1} = A\, T_n, \tag{6a}$$

where

$$A = \frac{1 - \Delta t(1-\alpha)\, \lambda_n}{1 + \Delta t\, \alpha\, \lambda_{n+1}}, \tag{6b}$$

in which $\lambda_n = \lambda(T_n, t_n)$ and $\lambda_{n+1} = \lambda(T_{n+1}, t_{n+1})$. In keeping with the common definition of stability for equations of the type considered here, we require that

$$|A| \leq 1. \tag{7}$$

(In the linear case (7) leads to the stability conditions cited above.) In addition, we stipulate that (7) must hold for all possible combinations of λ_n and λ_{n+1}. For example, if $\alpha = 1/2$ (trapezoidal rule or Crank-Nicholson method) and $\lambda_n > \lambda_{n+1}$, then (7) imposes the time step restriction

$$\Delta t \leq 4/(\lambda_n - \lambda_{n+1}). \tag{8}$$

Thus the unconditional stability of the Crank-Nicholson method in linear problems does not carry over to the nonlinear regime. In fact, only the case $\alpha = 1$ (backward difference method) is unconditionally stable for nonlinear problems. Precise stability conditions for the various cases are summarized as follows:

$$\alpha = 0: \qquad \Delta t \leq 2/\lambda_n \tag{9a}$$

$$\alpha \in (0,1): \quad \begin{cases} \text{stable}, & \lambda_{n+1} \geq \dfrac{(1-\alpha)}{\alpha}\lambda_n \quad (9b) \\[2ex] \Delta t \leq \dfrac{2}{(1-\alpha)\lambda_n - \alpha\lambda_{n+1}}, & \lambda_{n+1} < \dfrac{(1-\alpha)}{\alpha}\lambda_n \quad (9c) \end{cases}$$

$\alpha = 1$: stable (9d)

As can be seen from (9b) and (9c), increasing α tends to stabilize the algorithm. Since conditions (9b) and (9c) involve λ_{n+1} they are not suitable for establishing a priori time step estimates. All algorithms for which $\alpha < 1$ become conditionally stable in the nonlinear regime.

The previous results are somewhat disconcerting despite the fact that at least one of the algorithms considered (backward differences) is unconditionally stable. The reasons for this are as follows: The only second-order accurate algorithm among those considered is the Crank-Nicholson method. The remaining algorithms all possess only first-order accuracy. The ones with the largest error constants are $\alpha = 0$ and $\alpha = 1$. Thus to attain unconditional stability within the framework delineated by equations (3), we must be content with a significant loss of accuracy.

To remedy this situation consider the following family of algorithms: Find $\underset{\sim}{T}_n$, $n \in \{0,1,...,N\}$, such that

$$\underset{\sim}{C}_{n+\alpha} \underset{\sim}{U}_{n+\alpha} + \underset{\sim}{K}_{n+\alpha} \underset{\sim}{T}_{n+\alpha} = \underset{\sim}{R}_{n+\alpha}, \qquad n \in \{0,1,...,N-1\}, \tag{10a}$$

$$\underset{\sim}{T}_{n+1} = \underset{\sim}{T}_n + \Delta t\, \underset{\sim}{U}_{n+\alpha}, \qquad n \in \{0,1,...,N-1\}, \tag{10b}$$

$$\underset{\sim}{T}_o = \underset{\sim}{T}, \tag{10c}$$

where

$$\underset{\sim}{C}_{n+\alpha} = \underset{\sim}{C}(\underset{\sim}{T}_{n+\alpha}, t_{n+\alpha}), \tag{10d}$$

$$\underset{\sim}{K}_{n+\alpha} = \underset{\sim}{K}(\underset{\sim}{T}_{n+\alpha}, t_{n+\alpha}), \tag{10e}$$

$$\underset{\sim}{T}_{n+\alpha} = (1-\alpha)\, \underset{\sim}{T}_n + \alpha\, \underset{\sim}{T}_{n+1}, \tag{10f}$$

$$\underset{\sim}{R}_{n+\alpha} = (1-\alpha)\ \underset{\sim}{R}_n + \alpha\ \underset{\sim}{R}_{n+1}, \tag{10g}$$

$$\underset{\sim}{U}_{n+\alpha} = (1-\alpha)\ \underset{\sim}{U}_n + \alpha\ \underset{\sim}{U}_{n+1}, \tag{10h}$$

$$t_{n+\alpha} = (n+\alpha)\ \Delta t. \tag{10i}$$

Notice that in the linear case this family of algorithms is identical to the preceding one. However, in the nonlinear case things are quite different.

Let us apply (10) to the model equation (5). In this case

$$A = \frac{1 - \Delta t(1-\alpha)\ \lambda_{n+\alpha}}{1 + \Delta t\ \alpha\ \lambda_{n+\alpha}}, \tag{11}$$

where $\lambda_{n+\alpha} = \lambda(T_{n+\alpha}, t_{n+\alpha})$ and condition (7) requires that

$$\lambda_{n+\alpha}\ \Delta t \leq 2/(1-2\alpha), \tag{12}$$

for $\alpha < 1/2$, whereas for $\alpha \geq 1/2$ the algorithm in question is unconditionally stable. Thus the family of algorithms (10) has the advantage that unconditionally stable methods for linear problems maintain this property in the nonlinear regime. In addition, for $\alpha = 1/2$ (midpoint rule) the method is second-order accurate.

We now shall show that for $\alpha \geq 1/2$ equations (10) are unconditionally stable for multi-degree-of-freedom systems. Our hypothesis on $\underset{\sim}{C}$ and $\underset{\sim}{K}$ insure that the matrix $\underset{\sim}{C}^{-1}_{n+\alpha}\ \underset{\sim}{K}_{n+\alpha}$ has a complete set of orthonormal eigenvectors $\underset{\sim}{\phi}^i_{n+\alpha}$ and corresponding eigenvalues $\lambda^i_{n+\alpha} \geq 0$. Thus we can write

$$\underset{\sim}{T}_n = \sum_i T^i_n\ \underset{\sim}{\phi}^i_{n+\alpha}, \tag{13a}$$

$$\underset{\sim}{T}_{n+1} = \sum_i T^i_{n+1}\ \underset{\sim}{\phi}^i_{n+\alpha}, \tag{13b}$$

where

$$T_n^i = \underset{\sim}{T}_n^T \underset{\sim}{\phi}_{n+\alpha}^i , \tag{13c}$$

$$T_{n+1}^i = \underset{\sim}{T}_{n+1}^T \underset{\sim}{\phi}_{n+\alpha}^i . \tag{13d}$$

Assuming R = 0 and employing (13) in (10a) and (10b) yields

$$\underset{\sim}{T}_{n+1} = \sum_i \left\{ \left[\frac{1 - \Delta t\ (1-\alpha)\ \lambda_{n+\alpha}^i}{1 + \Delta t\ \alpha\ \lambda_{n+\alpha}^i} \right] T_n^i\ \underset{\sim}{\phi}_{n+\alpha}^i \right\} , \tag{14}$$

from which it follows that

$$||\underset{\sim}{T}_{n+1}|| \leq ||\underset{\sim}{T}_n|| , \tag{15}$$

where $||\quad||$ denotes the euclidean norm. Thus if $\lambda_{n+\alpha}$ is interpreted as the maximum $\lambda_{n+\alpha}^i$, the results for the model problem provide both necessary and sufficient conditions for the stability of equation (10). Note, however, that a similar conclusion cannot be drawn for equations (3) since the matrices $\underset{\sim}{C}$ and $\underset{\sim}{K}$ must be evaluated at two different steps. Because of this the modal decomposition argument does not work for these equations and we can only conclude that the analysis of the model equation for $\alpha \in (0,1)$ provides necessary stability conditions - they may not be sufficient.

CONCLUSIONS

We have shown that many algorithms used in transient heat conduction which are unconditionally stable for linear problems lose this property when applied to nonlinear problems. To remedy this we have constructed a family of one-step methods for nonlinear heat conduction which possess the same stability properties in both linear and nonlinear problems. Amongst this family of methods is a second-order accurate, unconditionally stable method.

Similar concepts can be used in nonlinear structural dynamics (see Hughes and Hilber [3]).

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