Title
A NOTE ON BEG'S APPROACH TO PERATIZATION

Author
Leader, Elliot.

Publication Date
1964-10-22

UCRL-11762

University of California

Ernest O. Lawrence Radiation Laboratory

A NOTE ON BEǴ'S APPROACH TO PERATIZATION

Berkeley, California

DISCLAIMER

Rept. submitted for pub. in the
Journal of Mathematical Physics.

UCRL-11762

UNIVERSITY OF CALIFORNIA
Lawrence Radiation Laboratory
Berkeley, California

AEC Contract No. W-7405-eng-48

A NOTE ON BEǴ'S APPROACH TO PERATIZATION

Elliot Leader

October 22, 1964

A NOTE ON BEǴ'S APPROACH TO PERATIZATION*

Elliot Leader

Lawrence Radiation Laboratory
University of California
Berkeley, California

October 22, 1964

ABSTRACT

It is shown that Beǵ's elegant formulation of the peratization result of Feinberg and Pais can be rigoriously justified.

I. INTRODUCTION

Recently M. A. Baqi Bég[1] has given a very elegant derivation of the main result of the peratization theory of Feinberg and Pais.[2] However, Bég's method involves a certain amount of juggling with divergent series, and also the introduction of a regulator mass which is allowed to go to infinity after the completion of the manipulations.

We show in the following that Bég's result (with a minor qualification) is rigorously correct and can be obtained without the use of a regulator.

II. MAIN RESULT OF PERTIZATION THEORY AND BÉG'S METHOD

Consider leptons interacting via the exchange of a massive, charged vector boson, with leading order matrix element (Born approximation) given by

$$B_{\mu\nu}(p) = \frac{-ig^2}{p^2 + m^2}\left(g_{\mu\nu} + \frac{p_\mu p_\nu}{m^2}\right) \quad . \tag{1}$$

Here p is the four-momentum transfer, m the boson mass, and g the boson-lepton coupling constant, and the metric is $p^2 = \underset{\sim}{p}^2 - p_0{}^2$. This indices μ,ν are to be contracted with the usual V - A lepton currents.

Feinberg and Pais[2] have shown that it is possible to give a meaning to the sum of the uncrossed ladder graphs in this theory, with the result that the leading order matrix element becomes modified or peratized to

$$B_{\mu\nu}{}^P(p) = \frac{-ig^2}{p^2 + m^2}\left(g_{\mu\nu} + \frac{p_\mu p_\nu}{m^2}\right) + \frac{ig^2}{4m^2}\, g_{\mu\nu} \quad . \tag{2}$$

For simplicity we shall pretend in what follows that we are dealing with neutral vector bosons and that the leptons are massless. The relationship between this and the realistic situation is discussed in detail in Bég's paper.[1]

Following Bég we write, for the scattering amplitude,

$$T_{\mu\nu} = T_a g_{\mu\nu} + T_b \left[\frac{p^2}{4} g_{\mu\nu} - p_\mu p_\nu \right] . \tag{3}$$

From (1) the scalar functions T_a, T_b have Born approximations given by

$$B_a(p) = -\frac{3}{4} \frac{ig^2}{p^2 + m^2} - \frac{ig^2}{4m^2} , \tag{4}$$

and

$$B_b(p) = \frac{ig^2}{m^2(p^2 + m^2)} .$$

They satisfy the integral equations

$$T_a(p^2) = B_a(p) - \frac{16}{(2\pi)^4} \int B_a(p - q) \frac{1}{q^2} T_a(q^2) d^4q \tag{5}$$

and

$$T_b(p^2) = B_b(p) + \frac{4}{3(2\pi)^4 p^4} \int B_b(p - q)[3p^4 - p^2q^2 - 6p^2 q\cdot p + 4(p\cdot q)^2] \times T_b(q^2) d^4q . \tag{6}$$

In Eqs. (5) and (6) we have further specialized to the situation in which the leptons are scattering at zero energy. The final momenta are zero, and the initial momenta, p and -p, are of course off the mass shell. They are to be put onto the mass shell after the equations are solved.

In this situation the on-shell amplitude is determined entirely by $T_a(0)$ (see footnote 4 of reference 1), and we shall see that we will recover the result of Feinberg and Pais [Eq. (2)] if we can show that to leading order in g^2 Eq. (5) has the solution

$$T_a(p^2) \simeq B_a{}^P(p) \quad , \tag{7}$$

where

$$B_a{}^P(p) = -\frac{3}{4}\frac{ig^2}{p^2 + m^2} \equiv B_a(p) + \frac{ig^2}{4m^2} \quad . \tag{8}$$

Let us write Eq. (5) symbolically as

$$T_a = B_a + B_a G T_a \quad . \tag{9}$$

Consider now the continuous family of integral equations

$$T(c) = \mathcal{B} + c + [\mathcal{B} + c]GT(c) \quad , \tag{10}$$

where c is an arbitrary constant and $\mathcal{B}$ is a Born term such that

$$\mathcal{B}(p) \underset{p^2\to\infty}{=} O(p^{-2}) \quad . \tag{11}$$

It is clear that Eq. (9) is precisely of this form, with

$$\mathcal{B}(p) = -\frac{3}{4}\frac{ig^2}{p^2 + m^2} \neq B_a{}^P(p) \tag{12}$$

and

$$c = -ig^2/4m^2 \quad .$$

Bég's derivation of the peratization result hinges on the following remarkable result derived by him: There exists a single function

$$T_{\mathcal{B}}(p^2) \equiv T(p^2; c = 0) \quad , \tag{13}$$

which is independent of c and which satisfies Eqs. (10) for <u>all</u> values of c. Moreover $T_{\mathcal{B}}(p^2)$ is given by the <u>iterative</u> solution of Eq. (10) with c = 0.

Once this result is accepted we immediately get the peratized result [Eqs. (7) and (8)] by writing Eq. (9) in the form of Eq. (10), with $\mathcal{B}(p)$ given by Eq. (12), i.e., in leading order we have

$$T_a(p^2) \equiv T_{\mathcal{B}}(p^2) \simeq \mathcal{B}(p) \equiv B_a{}^P(p) \quad . \qquad (14)$$

In Bég's work the above-mentioned result is derived with the use of a regulator by invoking the self-damping properties of the Fermi interaction in the chain approximation.

We shall see in the next section that this result can be obtained directly in a very simple manner and without need of a regulator. However, it will turn out that the validity of the result depends upon the sign of $\mathcal{B}(p)$ as $p^2 \to \infty$, and that the sign in Eq. (12) is "good" in this sense. In this respect the result reached in reference 1 is not quite correct, since there the proof was given for the "singular" part of T_a, called T_s, whose $\mathcal{B}(p)$ term has the opposite sign to that in Eq. (12).

It is thus interesting to note that the result is valid only when $\mathcal{B}(p)$ corresponds to a potential which is repulsive at small distances.

III. SOLUTION OF INTEGRAL EQUATION FOR T(c)

We shall solve Eq.(10) only for the case of direct interest, in which $\mathcal{B}(p)$ is given by Eq. (12). Writing out the equation in detail we have

$$T(p^2;c) = -\frac{3}{4}\frac{ig^2}{p^2+m^2} + c - \frac{16}{(2\pi)^4}\int\left[c - \frac{3}{4}\frac{ig^2}{q^2+m^2}\right]\frac{T(q^2;c)}{q^2}\,d^4q \quad . \qquad (15)$$

If a solution exists to Eq. (15) then *a fortiori* the integral

$$\int\left[c - \frac{3}{4}\frac{ig^2}{q^2 + m^2}\right]\frac{T(q^2;c)}{q^2}\ d^4q$$

converges, and since the terms in parenthesis cannot possibly cancel against each other, we will have that the integral

$$\int\frac{T(q^2;c)}{q^2}\ d^4q$$

also converges.

We can now rewrite Eq. (15) as

$$T(p^2;c) = -\frac{3}{4}\frac{ig^2}{p^2 + m^2} - \frac{16}{(2\pi)^4}\int\left[-\frac{3}{4}\frac{ig^2}{q^2 + m^2}\right]\frac{T(q^2;c)}{q^2}\,d^4q$$
$$+ c - \frac{16c}{(2\pi)^4}\int\frac{T(q^2;c)}{q^2}\ d^4q \quad . \tag{16}$$

The function $T(p^2;c = 0)$, which is the solution of (15) when $c = 0$, will now satisfy (15) for <u>all</u> c provided only that

$$c - \frac{16c}{(2\pi)^4}\int\frac{T(q^2;c = 0)}{q^2}\ d^4q = 0 \quad ,$$

i.e., provided that

$$\frac{1}{(2\pi)^4}\int\frac{T(q^2;c = 0)}{q^2}\,d^4q = \frac{1}{16} \quad . \tag{17}$$

The condition (17) takes on a less formidable aspect if we look at it in coordinate space.

Defining

$$\mathcal{L}(x) = \frac{1}{(2\pi)^4}\int e^{-iq\cdot x}\ \frac{T(q^2;c = 0)}{q^2 - i\varepsilon}\ d^4q \quad , \tag{18}$$

we see that condition (17) is nothing more than the boundary condition

$$\mathcal{L}(0) = \frac{1}{16} . \tag{19}$$

We must not examine the Fourier transform of Eq. (15) and show that it possesses a solution with the required properties.

By taking the Fourier transform of Eq. (15) we get that $\mathcal{L}(x)$ satisfies the differential equation

$$\Box^2 \mathcal{L}(x) = -\frac{3}{4} ig^2 \Delta_F(x)[16\mathcal{L}(x) - 1] , \tag{20}$$

where $\Delta_F(x)$ is the usual Feynman propagator function

$$\Delta_F(x) = \frac{1}{(2\pi)^4}\int \frac{e^{-iq\cdot x}}{q^2 + m^2 - i\varepsilon}\, d^4q . \tag{21}$$

Introducting the complex variable η defined by $\eta^2 = x^2$, we can reduce Eq. (20 to

$$\left(\frac{d^2}{d\eta^2} + \frac{3}{\eta}\frac{d}{d\eta}\right)\mathcal{L}(\eta) = -\frac{3}{4} ig^2 \Delta_F(\eta)[16\mathcal{L}(\eta) - 1] . \tag{22}$$

This equation is of the type studied by Pwu and Wu,[3] so we may take over their results directly. For convenience we outline here the essential steps.

Firstly the Feinberg-Pais reduction formula[2] for taking Fourier transforms allows us to consider Eq. (22), restricting η to lie only in the first quadrant of the complex η plane. We may then replace $\Delta_F(\eta)$ by

$$(im/4\pi^2\eta)K_1(m\eta) ,$$

so that Eq. (22) becomes

$$\left(\frac{d^2}{d\eta^2} + \frac{3}{\eta}\frac{d}{d\eta}\right)\mathcal{I}(\eta) = \frac{3g^2m}{\pi^2\eta} K_1(m\eta)\left[\mathcal{I}(\eta) - \frac{1}{16}\right] \quad , \tag{23}$$

As a result of the analyticity properties of (23) we can ignore the contour integral in the reduction formula and we may thus further restrict ourselves to η real and positive.

Secondly we see that for η real and positive the solutions of (23) have the behavior

$$\mathcal{I}(\eta) \underset{\eta\to 0}{\longrightarrow} \frac{1}{16} + \eta^{\lambda_\pm} \quad ; \tag{24}$$

where

$$\lambda_\pm = -1 \pm (1 + 3g^2/\pi^2)^{1/2} \tag{25}$$

and

$$\mathcal{I}(\eta) \underset{\eta\to\infty}{=} 0(\eta^\mu) \quad , \tag{26}$$

where

$$\mu = 0 \quad \text{or} \quad -2 \quad . \tag{27}$$

Finally, it can be shown that an <u>iterative</u> solution exists provided that

$$3g^2/\pi^2 < 1 \quad , \tag{28}$$

and which has the behavior given by taking λ_+ in Eq. (24) and $\mu = -2$ in Eq. (26).

Since $\lambda_+ > 0$, the condition (19) is satisfied.

Since $\mu = -2$, the Fourier transform $T(p^2;c = 0)$ exists.

This completes the proof of Beg's assertion and justifies his method of obtaining the Feinberg-Pais peratization result.

ACKNOWLEDGMENTS

The author is indebted to M. A. Baqi Bég for many enlightening disucssions. He wishes to thank Profesor G. C. Wick for his kind hospitality at the Brookhaven National Laboratory, where this work was performed.

REFERENCES

* This work was performed under the auspices of the U.S. Atomic Energy Commission.

1. Mirza A. Baqi Bég, Ann. Phys. 27, 183 (1964).
2. G. Feinberg and A. Pais, Phys. Rev. 131, 2724 (1963), and Phys. Rev. 133, B 477 (1964).
3. Y. Pwu and T. T. Wu, Phys. Rev. 133, B 778 (1964).