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Uniform high-frequency description of singly, doubly, and vertex diffracted rays for a plane angular sector

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Abstract-A high-frequency analysis of the scattered field at a plane angular sector is presented, for the scalar case in which hard boundary conditions are imposed on the two faces. In this formulation, the ordinary UTD field is augmented by uniform vertex diffraction contributions that provide the compensation of the UTD ray field when the first order diffraction points disappear from the tip. Furthermore, an expression of the doubly diffracted rays from the two edges is derived, that provides a uniform description of the total field at the ordinary double diffraction transition regions, including their possible overlapping. Moreover, a new transition function is introduced, which uniformly describes the transition field between doubly diffracted and vertex rays, that occurs when the double diffraction points merge in the tip. In spite of the complication of the physical mechanism, the final solution is simple and easy to implement.

I. INTRODUCTION

Within the framework of the Geometrical Theory of Diffraction (GTD) [1] and its uniform extension (UTD) [2], an important canonical problem is that of a corner at the interconnection of two straight edges, joined by a plane angular sector. For most practical purposes, the need of a vertex diffraction coefficient for this problem in the UTD scheme mainly arises when the leading edge diffracted field experiences a discontinuity, as it does when the diffraction point disappears from an edge or changes abruptly its location from one edge to the other. An even more serious impairment is encountered in applying GTD to RCS calculations, due to the fact that the leading edge contributions are restricted to lying on the pertinent diffraction cones.

The canonical problem of the plane angular sector was solved by Satterwhite and Kouyoumjian [3,4], but the series expansion of the solution is hard to compute and not well-suited for a practical asymptotic evaluation. Most of the literature on this topic presents formulations based on numerical or hybrid techniques [5–7] or on approximate, high-frequency methods [8–13]. In particular, a heuristic corner diffraction coefficient was conjectured in [8]; however, although it provided surprisingly good results in certain specific examples, its general applicability may be questionable [9]. First order vertex diffraction coefficients were presented in [10], which are formulated in order to compensate the UTD discontinuity, and do not include second order interaction between edges. A spectral approach has been developed by Ivrissimtzis and Marhefka in [11] by introducing the interaction between the edges in the physical theory of diffraction (PTD) framework.

Recently, vertex diffraction coefficients have been derived in the plane-wave far-field regime by using the induction theorem [12]. These coefficients account for second order interactions between the two edges and provide the physical properties to satisfy reciprocity; anyway, they are derived in the plane wave-far field limit, so that they are cast in nonuniform expressions that exhibit the expected singularities at the caustics of single and doubly diffracted rays. The same formulation has been used in [13] for deriving a uniform solution to apply at a finite distance from the tip. There, the scalar problem of the plane angular sector with soft boundary conditions (BCs) was treated; the double diffraction contributions were neglected, since they are of higher asymptotic order with respect to the tip contribution.

In this paper, an asymptotic solution for hard BCs is presented, which is based on the application of a spectral synthesis [14–18] to the formulation presented in [12]. The asymptotic treatment of the hard plane angular sector up to k^{-1} asymptotic order is more elaborate than the corresponding soft problem. Indeed, in this case the doubly diffracted rays have the same k^{-1} asymptotic order of the vertex ray, so that they should be necessarily included in a rigorous analysis. In this paper, expressions of both the doubly diffracted and the vertex rays are derived, and a transition function for describing the transition region between them is introduced. Together with the formulation for the soft case previously presented [13], this provides the basic step for finding a uniform solution for the electromagnetic case.

This paper is structured as follows. In Sect. 2, the nonuniform formulation presented in [12] is summarized for the hard case; furthermore, a particular decomposition of this solution is introduced, that is useful for the final uniform asymptotic evaluation. In Sect. 3, a double integral describing the near field of the plane angular sector is derived by spectral synthesis. This integral is asymptotically evaluated in Sect. 4. The asymptotic solution has been derived in such a way that the first and the second order GTD ray field structure is easily recognizable far from the transition regions. This provides physical insight into the diffraction mechanism and gives simplicity to the solution.

Three different transition functions are defined in the high-frequency formulation; these involve special canonical functions very simple to calculate. In particular, the vertex contribution is multiplied by the same transition function introduced in [10] and used in [13] for the soft case. It involves the generalized Fresnel integral. The same transition function is also used, by introducing different arguments, to treat the double diffraction transition mechanism. Furthermore, a new transition function that involves cylinder parabolic functions is defined to describe the second order transition mechanisms.

In Sect. 5, numerical examples are shown to demonstrate the effectiveness of this solution.

II. PLANE-WAVE FAR-FIELD SOLUTION

The geometry at a vertex interconnecting the two edges of a plane angular sector is shown in Fig. 1, in which Ω denotes the angle between the two edges. The scalar case is considered, in which hard boundary conditions are imposed on the faces. A plane wave illumination is assumed, coming from the direction (β'_i, ϕ'_i) (i = 1, 2), in which β'_i denotes the angle between the direction of incidence and the *i*-th edge, and ϕ'_i is the aspect of incidence in the plane transverse with respect to the *i*-th edge; this latter is measured starting from the common face (Fig. 1a). Spherical coordinate systems (r, β_i, ϕ_i) are defined at each edge i = 1, 2, with their origin at the tip (Fig. 1b).



Figure 1. Coordinate systems at edges 1 and 2: (a) incidence aspects; (b) observation aspects.

The plane angular sector, joining edges 1 and 2, is thought of as the intersection of two half-planes with overlapping 0 and 2π faces. The plane-wave far field solution presented in [12] is constructed as the superposition of two analogous mechanisms; a field contribution emanating from edge 2 when it is illuminated by the field from edge 1 ($\psi_{21\infty}$), and that from 1 when it is illuminated by 2 ($\psi_{12\infty}$). The procedure [12] for deriving a solution for $\psi_{12\infty}$, consists of three basic steps that are summarized hereinafter: a) first, the total field scattered from edge 1 is used to find a plane wave spectral representation of the field which illuminates edge 2, and the field that effectively impinges on half-plane 2 is represented as the superposition of the two spectral contributions that define the incident field at the upper (0) and the lower (2π) face of half-plane 2, respectively; b) next, according to the induction theorem, each plane wave of the incident field spectral representation is used to define a current distribution on the two faces of the plane angular sector; it is assumed that the pertinent Green's function, is dictated by edge 2; c) then, by spectral synthesis, an integral representation is obtained, which is calculated by its residue contribution. This provides a closed-form, plane-wave/far-field solution for mechanism 21

$$\psi_{12\infty} \sim \frac{\exp(-jkr)}{4\pi r} 2D_{21}(\beta_2, \phi_2, \beta_1', \phi_1')$$
 (1)

where the far field pattern D_{21} is expressed by

$$D_{21}(\beta_2, \phi_2, \beta_1', \phi_1') = \frac{G(\overline{\alpha}_{21}, \phi_1')G(\overline{\alpha}_{21}, \phi_2)}{jk\sin\Omega\sin\beta_1'\sin\overline{\alpha}_{21}}$$
(2)

where

$$\cos \overline{\alpha}_{21} = \frac{-\cos \beta_1' \cos \Omega + \cos \beta_2}{\sin \beta_1' \sin \Omega}$$
(3)

$$\cos\overline{\overline{\alpha}}_{21} = \frac{\cos\beta_2\cos\Omega - \cos\beta_1'}{\sin\beta_2\sin\Omega}$$
(4)

and, for the hard case

$$G(\alpha, \phi) = \frac{2\cos\frac{\phi}{2}\cos\frac{\alpha}{2}}{\cos\phi + \cos\alpha}.$$
 (5)

In (1), the zero-phase of the incident plane wave is assumed to be at the tip. It is worth noting that D_{21} shows the symmetry property

$$D_{21}(\beta_2,\phi_2,\beta_1',\phi_1') = D_{21}(\pi - \beta_1',\phi_1',\pi - \beta_2,\phi_2)$$
(6)

with respect to the direction of incidence (β'_1, ϕ'_1) and that of observation (β_2, ϕ_2) . This property, which follows from the identity $\sin \beta'_1 \sin \overline{\alpha}_{21} = \sin \beta_2 \sin \overline{\overline{\alpha}}_{21}$, emphasizes that this solution explicitly satisfies reciprocity.

The field $\psi_{21\infty}$ represents a global contribution of the entire, infinite structure, in the extreme far zone. Explicit ray-contributions cannot be present in the planewave/far-field limit. Indeed, since the canonical structure exhibits straight edges, the GTD ray-field contributions are zero everywhere, except for the cone $\beta_i = \beta'_i$, where a caustic occurs due to the coherence of infinite diffracted rays. Analogously, the doubly diffracted rays are restricted to lie on the cone $\beta_2 = \beta'_1 - \Omega$, where a caustic of doubly diffracted rays occurs in the far region. Also, due to the flatness of the face, GO reflected rays are restricted to the specular direction, that is a caustic of the same rays.

Since $\psi_{21\infty}$ is a global contribution, it should inherently contain information on GO, first order and one of the two second order diffraction mechanisms (in particular, that mechanism relevant to the field that diffracts first at edge 1 and next at edge 2). These information are contained in the singularities of $\psi_{21\infty}$, that occur at $\beta_i = \beta'_i$ and at $\beta_2 = \beta'_1 - \Omega$. These singularities are clearly expressed in the following decomposition, that can be obtained by means of algebraic manipulations on Eq. (2):

$$D_{21}(\beta_2, \phi_2, \beta_1', \phi_1') = D_{21}'(\beta_2, \phi_2) + D_{21}''(\beta_2, \phi_2)$$
(7)

where

$$D'_{21}(\beta_2, \phi_2) = \frac{\overline{c}_{21}\left(c'_{21} - \frac{1}{2}\overline{s}_{21}\right) + \overline{\overline{c}}_{21}\left(c_{21} - \frac{1}{2}\overline{\overline{s}}_{21}\right)}{jk\left(\cos\beta'_1 - \cos\beta_1\right)\left(\cos\beta'_2 - \cos\beta_2\right)}$$
(8)

and

$$D_{21}''(\beta_2,\phi_2) = \frac{p_{21}(\beta_2)}{\overline{d}_{21}(\beta_2)}$$
(9)

in which

$$\overline{d}_{21}(\beta_2) = \sqrt{\sin\left(\frac{1}{2}(-\beta_1' + \beta_2 + \Omega)\right)}$$
(10)

$$p_{21}(\beta_2) = \frac{d_{21}}{jk\left(c_2 1' + \overline{s}_{21}\right)\left(c_2 1 + \overline{\overline{s}}_{21}\right)} \tag{11}$$

The various symbols are listed below:

$$\overline{\overline{d}}_{21} = \sqrt{\sin\left(\frac{1}{2}(-\beta_2 + \beta_1' + \Omega)\right)}$$
(12)

$$\overline{c}_{21} = \cos\frac{\overline{\alpha}_{21}}{2}\sqrt{2\sin\beta_1'\sin\Omega} = \sqrt{\cos\beta_2 - \cos(\beta_1' + \Omega)}$$
(13)

$$\overline{\overline{c}}_{21} = \cos\frac{\overline{\overline{\alpha}}_{21}}{2}\sqrt{2\sin\beta_2\sin\Omega} = \sqrt{\cos(\beta_2 - \Omega) - \cos\beta_1'}$$
(14)

$$\overline{s}_{21} = \sin \frac{\overline{\alpha}_{21}}{2} \sqrt{2 \sin \beta_1' \sin \Omega} = \sqrt{\cos(\beta_1' - \Omega) - \cos \beta_2}$$
(15)

$$\overline{\overline{s}}_{21} = \sin \frac{\overline{\overline{\alpha}}_{21}}{2} \sqrt{2 \sin \beta_2 \sin \Omega} = \sqrt{\cos \beta_1' - \cos(\beta_2 + \Omega)}$$
(16)

$$c'_{21} = \cos\frac{\phi'_1}{2}\sqrt{2\sin\beta'_1\sin\Omega} = \pm\sqrt{\cos(\beta'_1 - \Omega) - \cos\beta'_2}$$
(17)

$$c_{21} = \cos\frac{\phi_2}{2}\sqrt{2\sin\beta_2\sin\Omega} = \pm\sqrt{\cos\beta_1 - \cos(\beta_2 + \Omega)}$$
(18)

where the upper (lower) sign applies to $\phi'_1 < \pi$ (> π) in (17) and to $\phi_2 < \pi$ ($\phi_2 > \pi$) in (18), respectively. In the above equations, $\overline{\alpha}_{21}$ and $\overline{\overline{\alpha}}_{21}$ have been obtained by the inversion of Eqs. (3) and (4) on the locus $(-j\infty, \pi + j\infty)$. This corresponds to take the negative imaginary part of the square roots in (10)-(16) for negative values of their arguments. The decomposition in Eq. (7) allows the separation of singularities of different nature. In particular, in D'_{21} the first order diffraction caustic singularities occur at the cone $\beta_i = \beta'_i$ and in D''_{21} , a square-root type singularity occurs at the cone $\beta_2 = \beta'_1 - \Omega$. For reasons that will be clarified in the following, the cones defined by $\beta_i = \beta'_i$ and $\beta_2 = \beta'_1 - \Omega$ are denoted by the first and the second order shadow boundary cone (SBC), respectively.

When the far field pattern in (2) is used for the spectral synthesis procedure that defines the field at finite distance, the above singularities become poles and branch-point, respectively. From these spectral singularities arises contributions that will be interpreted as singly and doubly diffracted rays, respectively.

III. SPECTRAL SYNTHESIS

The plane-wave far field solution is used to construct, via spectral synthesis, an integral representation which is valid for source or observation point at finite distance from the vertex. To this end, let us first assume that the plane angular sector is illuminated by a spherical-wave point source located at $P' \equiv (r', \beta'_1, \phi'_1)$. The scalar field of this source can be represented as a superposition of spectral plane-waves; i.e.,

$$\frac{e^{-jkR}}{4\pi R} = \frac{k}{8\pi^2 j} \int_{C_{\alpha'}} \int_{C_{\theta'}} \sin\theta' e^{-jkrg(\beta_1,\theta',\alpha'-\phi_1)} e^{jkr'g(\pi-\beta_1',\theta',\alpha'-\phi_1')} d\alpha' d\theta'$$
(19)

in which

$$g(\beta, \theta', \alpha) = \cos\beta\cos\theta' + \sin\beta\sin\theta'\cos\alpha \tag{20}$$

Both the contours $C_{\alpha'}$ and $C_{\theta'}$ are defined along $(-j\infty, \pi + j\infty)$. The first exponential term represents a spectral plane wave coming from a direction (θ', α') . Each plane wave of the spectrum in Eq. (19) is now used to illuminate the plane angular sector. By spectral synthesis, the far-field pattern $P_{21}(\beta_2, \phi_2)$ (normalized with respect to $\exp(-jkr)/(4\pi r)$) due to an incident spherical wave from the point source at P', is obtained by replacing the first exponential term in (19) by the analytical continuation of the far-field pattern $2D_{21}(\beta_2, \phi_2, \beta'_1, \phi'_1)$ for complex incident angles (θ', α') ; i.e.,

$$P_{21}(\beta_2, \phi_2) = \frac{k}{4\pi^2 j} \int_{C_{\alpha'}} \int_{C_{\theta'}} \sin \theta' D_{21}(\beta_2, \phi_2, \theta', \alpha') e^{jkr'g(\pi - \beta_1', \theta', \alpha' - \phi_1')} d\alpha' d\theta'$$
(21)

Since our solution explicitly satisfies reciprocity, a similar integral representation can also be used for describing the field at a point $P \equiv (r, \beta_2, \phi_2)$ placed at finite distance from the vertex, when it is illuminated by a plane wave coming from (β'_1, ϕ'_1) . To this end, the formal substitution

$$(r', \pi - \beta'_1, \phi'_1) \to (r, \beta_2, \phi_2)$$
 (22)

can be used, thus leading to the near field expression

$$\psi_{21} = \psi'_{21} + \psi''_{21}$$

$$\psi_{21}^{''} = \frac{k}{4j\pi^2} \int_{C_{\alpha_2}} \int_{C_{\theta_2}} D_{21}^{''}(\theta_2, \alpha_2) \sin \theta_2 e^{-jkrg(\beta_2, \theta_2, \alpha_2 - \phi_2)} d\alpha_2 d\theta_2$$
(23)

in which, for convenience, the integration variables are now indicated by α_2 and θ_2 , and the decomposition in Eq. (7) has been used. In the following, explicit reference to the plane/wave - near field expression (23) will be made. Anyway, the final high-frequency closed-form solution of (23) that will be presented in the following sections can be easily reconverted into the corresponding near source far field by re-using the transformation (22).

IV. HIGH-FREQUENCY SOLUTION

To provide physical insight into the complex diffraction mechanism, the expected ray field contributions at finite distance r from the vertex are now discussed. These contributions are represented in Fig. 2. A reflected GO ray field originates at the specular point Q. The dominant UTD diffraction contributions ψ_1^d and ψ_2^d from edges 1 and 2 arise from Q_1' and Q_2' , respectively. Furthermore, a contribution ψ^v arises from vertex V and a double diffraction (DD) contribution ψ_{21}^{dd} arises from a point Q_2'' on edge 2 after diffracting at the point Q_1'' on edge 1. The other double diffraction term ψ_{12}^{dd} from edge 1 after the diffraction at edge 2 is not depicted for avoiding to overcrowd the figure. The first order diffraction contributions ψ_i^d (i = 1, 2) or the DD contribution are discontinuous when the observation point P moves in such a way that Q'_i or Q''_i disappears from V, respectively. This occurs when P crosses the first and second order SBC, respectively. This discontinuity should be compensated by the vertex contribution ψ^v . This latter asymptotically decays as $(kr)^{-1}$; this is due to the fact that the flux of scattered energy through a spherical surface centered at the tip must keep constant for any radius of the same surface. In the transition region where the vertex contribution ψ^v should compensate the discontinuity of ψ^d_i , namely close to first order SBC $\beta_i = \beta'_i$, it should become of order $(kr)^{-1/2}$ like ψ_i^d . When the reflection point is close to the tip, it should become of order $(kr)^0$, as the reflected ray field.



Figure 2. Ray contributions at a plane angular sector; (a) GO and singly diffracted rays; (b) doubly diffracted and vertex rays.

At variance, the asymptotic behavior of the DD contribution ψ_{21}^{dd} depends on the kind of BCs that are imposed on the plane angular sector. For soft BCs, only the derivative of ψ_1^d is discontinuous when $\phi_2 = \pi$ or $\phi_1' = \pi$ so that ψ_{21}^{dd} is a slope contribution that decays as $(kr)^{-2}$. Then, it is of higher asymptotic order with respect to the vertex contribution and it can be neglected in a second

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order analysis, as done in [13]. On the other hand, for the hard case we are presently concerned with, ψ_{21}^{dd} decays as $(kr)^{-1}$, and it should compensate for the discontinuities of ψ_1^d when the observation point or the incidence direction crosses the plane of the angular sector ($\phi_2 = \pi$ or $\phi_1' = \pi$, respectively). Thus, this contribution should become $(kr)^{-1/2}$ for $\phi_2 = \pi$ ($\phi_1' = \pi$) and $(kr)^0$ when simultaneously $\phi_2 = \pi$ and $\phi_1' = \pi$. It has an asymptotic behavior of the same order as that of the vertex contribution, and it must be included in a rigorous asymptotic analysis.

All the contributions described above arise from the asymptotic evaluation of the integral representation in (23). In particular, the integrand exhibits poles and branch-point singularities that yield the dominant UTD field and the DD contributions. Furthermore, it exhibits a saddle point at ($\alpha_2 = \phi_2$, $\theta_2 = \beta_2$). From the nature of the spectral formulation itself, the nonuniform asymptotic evaluation at this saddle point recovers the nonuniform vertex contribution in Eq. (1).

The nature of the singularities that appears in D'_{21} and D''_{21} suggests a separate treatment of the two relevant terms. In the following subsections, the second order (vertex and DD) contributions will be derived from the spectral integration. First (Sect. 4.1), the transition field that occurs when the reflection and/or first-order diffraction points merge in the tip is treated; this field is found to be the sum of the GO rays, the ordinary UTD singly diffracted rays, and a vertex ray, all arising from the term D'_{21} . Second (Sect. 4.2), a DD contribution and another vertex contribution are derived from the integral relevant to D''_{21} . It is shown that this additional vertex ray-contribution provides the uniform description of the transition field that occurs when the DD points merge at the tip. Third (Sect. 4.3), a proper asymptotic evaluation of the DD contribution is derived to provide the uniform description of the transition field at the ordinary DD transition regions relevant to the skewed edges configurations. Finally (Sect. 4.4), the total ray field structure is summarized.

4.1. Transition Among GO, Singly Diffracted, and Vertex Rays

Consider the term ψ'_{21} in (23); its integrand does not contain any branch-point singularities at the denominator, so that its uniform asymptotic evaluation at the saddle point can be treated with the same method as that presented in [10] and used in [13] for soft BCs. This yields

$$\psi_{21}' \sim \psi_{21}^{\nu'} + U_1^d \psi_1^d + U_2^d \psi_2^d + \psi^i U^i + \psi^r U^r$$
(24)

where

$$\psi_{21}^{v'} = \frac{\exp(-jkr)}{2\pi r} D'_{21}(\beta_2, \phi_2) T_{21}$$
(25)

 ψ^i and ψ^r are the incident and reflected plane waves, ψ^d_n are the UTD diffraction contribution from the two edges,

$$U^{\bar{r}} = U(\pi - \phi_2 \mp \phi'_2) U(\pi - \phi_1 \mp \phi'_1)$$
(26)

$$U_i^d = U(\beta_i' - \beta_i) \qquad i = 1,2 \tag{27}$$

in which U(x) is the Heaviside unit step function, and

$$T_{21} = T(\overline{\delta}_1, \delta_1, \overline{\delta}_2, \delta_2, kr) \tag{28}$$

where T is the generalized transition function (GTF), first introduced in [10], that is defined in terms of the generalized Fresnel integral. The complete definition of this function is given in [13], and reported here for convenience:

$$T(\overline{\delta}_{1}, \delta_{1}, \overline{\delta}_{2}, \delta_{2}, K) = 4j\pi K \frac{(\overline{\delta}_{1}^{2} + \delta_{1}^{2})\delta_{1}\delta_{2}}{(\delta_{1}\overline{\delta}_{2} + \delta_{2}\overline{\delta}_{1})} \left(G(K^{1/2}\delta_{1}, K^{1/2}\overline{\delta}_{1}) + G(K^{1/2}\delta_{2}, K^{1/2}\overline{\delta}_{2}) \right)$$

$$(29)$$

in which the arguments are

$$\delta_i = \sqrt{2} \sin\left(\frac{\beta_i' - \beta_i}{2}\right) \tag{30}$$

and

$$\overline{\delta}_{i} = \begin{cases} \overline{\delta}_{i}^{+} & \text{for } \phi_{1} < \pi \\ -\overline{\delta}_{i}^{-} & \text{for } \phi_{1} > \pi \end{cases}$$
(31)

where

$$\overline{\delta}_i^{\pm} = \sqrt{2\sin\beta_i \sin\beta_i'} \cos\left(\frac{\phi_i' \pm \phi_i}{2}\right)$$
(32)

the function G in (29) is the generalized Fresnel integral

$$G(x,y) \stackrel{\Delta}{=} \frac{y}{2\pi} e^{jx^2} \int_x^\infty \frac{e^{-jt^2}}{t^2 + y^2} dt.$$
 (33)

The GTF in (29), reduces to unity for large kr, namely at a far-field distance from the vertex, as usually happens in the standard UTD transition function. In [13], a simple algorithm is suggested for calculating the generalized Fresnel integral. This algorithm reduces to combinations of ordinary Fresnel integrals. As discussed in [13] for the soft case, the term in (25) provides by itself the compensation of the GO and the leading UTD ray contributions. When the observation point crosses the first order SBCs ($\beta_i = \beta'_i$), the first order diffraction point on the i-th edge disappears from the tip and a discontinuity occurs in the dominant asymptotic contribution. This discontinuity is described by the unit step function in (26). The peculiarity of the GTF is that of changing the spreading factor of $\psi_{21}^{v'}$ together with the observation point. Close to a SBC, one of the parameters δ_i vanishes and the GTF provides a cylindrical spreading factor in $\psi_{21}^{v'}$ that allows the proper compensation of the discontinuities of the singly diffracted contributions. When the observation point approaches the intersection between the two SBCs associated with the two edges, also the GO contribution disappears. In this case, both the parameters δ_i and $\overline{\delta}_i$ vanish and the GTF produces a plane wave, unit spreading factor, that provides the compensation of the GO discontinuity.

In the next subsections, it will be shown that the asymptotic treatment of the term D_{21}'' provides a further vertex contribution and a doubly diffracted (DD) field contribution.

4.2. Transition Among Doubly Diffracted and Vertex Rays.

In order to evaluate the contribution ψ_{21}'' in (23), let us first suppose that both the observation point and the incidence direction are far from grazing. In such a case the unique singularity that has to be accounted for is the branch point associated to $\overline{d}_{21}(\theta_2)$ in the θ_2 plane. After rewriting the exponential term as

$$g(\beta_2, \theta_2, \alpha_2 - \phi_2) = \cos(\beta_2 - \theta_2) + \sin\beta_2 \sin\theta_2 (\cos(\alpha_2 - \phi_2) - 1)$$
(34)

the integral in α_2 can be simply evaluated by its stationary phase contribution at $\alpha_2 = \phi_2$ so that

$$\psi_{21}'' \sim \frac{jk\sqrt{2\pi j}}{4\pi^2\sqrt{kr}} \int_{C_{\theta_2}} \frac{\sqrt{\sin\theta_2}}{\sqrt{\sin\beta_2}} \frac{p_{21}(\theta_2)}{d_{21}(\theta_2)} e^{-jkr\cos(\beta_2 - \theta_2)} d\theta_2$$
(35)

where $p_{21}(\theta_2)$ is a regular function of its argument close to $\theta_2 = \beta'_1 - \Omega$. The integrand in (35) shows a branch point at $\theta_2 = \beta'_1 - \Omega$ due to the term $\overline{d}_{21}(\theta_2)$, defined in Eq. (10); as shown in Fig. 3, the branch cut is chosen in such a way that $\operatorname{Im} \left[\sqrt{\cos \theta_2 - \cos(\beta'_1 - \Omega)} \right] > 0$ in the top Rieman sheet.

The integral in (35) is now asymptotically evaluated. To this end, consider first the case $\beta_2 < \beta'_1 - \Omega$, in which one expects to find a DD contribution for the mechanism 21, since the two diffraction points Q''_1 and Q''_2 (see Fig. 3a) are located on the real edges. The contour is deformed into a steepest descent path (SDP) $V_{\theta_2}^-$ through the point $\theta_2 = \beta_2$ on the top Rieman sheet. In this deformation, an integration on the contour D_{θ_2} along the branch cut has to be included, which is asymptotically dominated by the branch point $\beta'_1 - \Omega$ (Fig. 3a). The two integrands along the SDP and around the branch cut are interpreted as a vertex and the DD contribution, respectively.

When $\beta_2 > \beta'_1 - \Omega$ (Fig. 3b), the original contour is deformed into the SDP $V_{\theta_2}^+$ through $\theta_2 = \beta_2$. This SDP runs on the bottom and top Rieman sheets for $\text{Im}\theta_2 > 0$ and $\text{Im}\theta_2 < 0$, respectively. In such a way, the integral around the branch cut does not need to be included. This is consistent with the physical mechanism, since the DD contribution does not exist when $\beta_2 > \beta'_1 - \Omega$.

Finally, for all values of β_2, ψ_{21}'' in (35) can be rewritten as

$$\psi_{21}'' \sim \frac{jk\sqrt{2\pi j}}{4\pi^2\sqrt{kr}} \left[I(V_{\theta_2}^{\pm}) + U_{21}^{dd}I(D_{\theta_2}) \right]$$
(36)

where

$$I(L_{\theta_2}) = \int_{L_{\theta_2}} \frac{\sqrt{\sin\theta_2}}{\sqrt{\sin\beta_2}} \frac{p_{21}(\theta_2)}{\overline{d}_{21}(\theta_2)} e^{-jkr\cos(\beta_2 - \theta_2)} d\theta_2$$
(37)

where L_{θ_2} denotes both $V_{\theta_2}^{\pm}$ or D_{θ_2} , and the +(-) sign apply to $\beta_2 > \beta'_1 - \Omega$ $(\beta_2 < \beta'_1 - \Omega)$; furthermore

$$U_{21}^{dd} = U(\beta_1' - \Omega - \beta_2)$$
(38)

in which U(x) is the Heaviside unit step function. The asymptotic evaluation of the two integrals in (36) is performed by mapping the θ_2 plane into the z plane,

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which is defined by the transformation in Eqs. (56) and (57) of Appendix A. The topology of this plane is shown in Fig. 4. In the same appendix it is shown that

$$\psi_{21}'' \sim \psi_{21}^{\upsilon \prime \prime} + \psi_{21}^{dd} \ U_{21}^{dd} \tag{39}$$

in which

$$\psi_{21}^{\nu \prime \prime} = \Psi_{21}^{\nu \prime \prime} W_{21} \tag{40a}$$

$$\psi_{21}^{dd} = \Psi_{21}^{dd} W_{21}^{dd} \tag{40b}$$

where

$$\Psi_{21}^{\nu \prime \prime} = \frac{e^{-jkr}}{2\pi r} D_{21}^{\prime \prime}(\beta_2, \phi_2) \tag{41}$$

$$\psi_{21}^{dd} = \frac{e^{-jkr\cos(\beta_2 - \beta_1' + \Omega)}\sqrt{\sin\Omega}}{\pi jkrc_{21}c_{21}'\sqrt{\sin(\beta_1' - \beta_2 - \Omega)}}$$
(42)

Furthermore, new transition functions are introduced, that are defined as

$$W_{21} = W(\pm j\sqrt{kr}z_{21})$$
(43)

in which the upper (lower) sign applies to $\beta_2 > \beta_1' - \Omega \ (\beta_2 < \beta_1' - \Omega)$, and

$$W_{21}^{dd} = W(-\sqrt{kr}z_{21}) \tag{44}$$

respectively, where

į

$$z_{21} = -2e^{-j\frac{\pi}{4}}\sin\left(\frac{\beta_1' - \Omega - \beta_2}{2}\right)$$
(45)

and

$$W(x) = e^{x^2/4} \sqrt{x} D_{-\frac{1}{2}}(x)$$
(46)

where the phase $\Phi(x)$ of x is such that $-\pi < \Phi(x) \le \pi$ and $D_{-1/2}(x)$ is the cylinder parabolic function defined in Appendix A. A simple algorithm for the numerical implementation is also suggested in the same appendix. The function W(x) is defined in such a way that W_{21} and W_{21}^{dd} become unity for a large argument, namely far from the second order SBCs.

Consider the transition region close to the direction $\beta_2 = \beta'_1 - \Omega$. In this case, the two second order diffraction points merge each other and z_{21} in (45) tends to vanish so that both W_{21} and W_{21}^{dd} vanish. At the same time, the square-root type singularity within the multiplying coefficients $\Psi_{21}^{v''}$ and Ψ_{21}^{dd} provides both the contributions in (40a) and (40b) to be finite, in such a way that the sum of the two contributions is continuous during the transition. In order to better understand the actual compensation mechanism of the two contributions and to provide a guideline to the implementation, the continuity of $\psi_{21}^{v''}$ is demonstrated in Appendix B, where the explicit expression at $\beta_2 = \beta'_1 - \Omega$ is also reported. From this expression it is seen that the transition field on the second order SBC is of asymptotic order $(kr)^{-3/4}$ at variance with the normal behavior $(kr)^{-1}$ of $\psi_{21}^{v''}$.



(b)

Figure 3. Topology of the complex θ_2 plane; observation point inside (a) and outside (b) second order SBC. Contours $V_{\theta_2}^{\pm}$ and D_{θ_2} are relevant to vertex and double diffraction contribution, respectively. The top Rieman sheet corresponds to $\operatorname{Im}\left[\sqrt{\cos\theta_2 - \cos(\beta_1' - \Omega)}\right] > 0$; the dashed portion of $V_{\theta_2}^+$ runs on the bottom Rieman sheet.



Figure 4. Topology of the complex plane; $z_b^{\pm} = -z_{21} \pm 2e^{-j\frac{\pi}{4}};$ $z_{\pi} = -z_{21} \pm 2e^{-j\frac{\pi}{4}}\cos\frac{\beta_2}{2}.$

4.3. Transition Among GO, Singly and Doubly Diffracted Rays

The formulation in (40b) is nonuniform when $\phi'_1 = \pi$ and $\phi_2 = \pi$, namely does not provide the proper compensation of the jump discontinuity of the first order UTD contributions when the observation point and/or the incidence direction cross the plane of the angular sector. In order to provide a uniform solution also at these grazing aspects, another transition function should be introduced. This is derived with the same technique as that used for obtaining (25). In particular, in applying this technique, the observation point is supposed to be far from the second order SBC $\beta_2 = \beta'_1 - \Omega$, so that the two diffraction points are far from the vertex. In such a way, the contribution in Eq. (40a, b) modifies as

$$\psi_{21}^{\nu \prime \prime} = \Psi_{21}^{\nu \prime \prime} T_{21}^{\prime \prime} \tag{47a}$$

$$\psi_{21}^{dd} = \Psi_{21}^{dd} T_{21}^{dd} \tag{47b}$$

where,

$$T_{21}^{\prime\prime} = T(\pm\overline{\delta}_1^+, \delta_1, \pm\overline{\delta}_2^+, \delta_2, kr)$$
(48)

$$T_{21}^{dd} = T(\pm \overline{\delta}_1^+, \gamma_{21}, \pm \overline{\delta}_2^+, \gamma'_{21}, kr)$$
(49)

in which T, $\overline{\delta}_i^+$ and δ_i are the same as that in (29), (30) and (32), respectively, the upper (lower) sign applies to $\phi_i < (>)\pi$, and

$$\gamma_{21} = \sqrt{2\sin\left(\frac{\beta_1 - \beta_2 - \Omega}{2}\right)\sin\left(\frac{\beta_2 + \beta_1 - 2\beta_1' + \Omega}{2}\right)} \tag{50}$$

$$\gamma_{21}' = \sqrt{2\sin\left(\frac{-\beta_2' + \beta_1' - \Omega}{2}\right)\sin\left(\frac{2\beta_2 - \beta_1' - \beta_2' + \Omega}{2}\right)}$$
(51)

The new transition function T_{21}^{dd} becomes unity for large γ_{21} and γ'_{21} , so that the DD contribution in (47b) is essentially the same as that in (40b) for ϕ'_1 and

 ϕ_2 far from π and for observation point far from the second order SBC. Also, the transition function T_{21}'' is very similar to that in (28), except for the fact that $\overline{\delta}_i^+$ take the place of $\overline{\delta}_i$. Then T_{21}'' becomes unity far from the first order SBC and vanishes on these cones.

The double diffraction term ψ_{21}^{dd} provides continuity to the first order diffraction contributions when the observation point and/or the incidence direction cross the plane of the angular sector. The compensation mechanism is ensured by the GTF T_{21}^{dd} , whose arguments γ_{21} and γ'_{21} vanish at $\phi_2 = \pi$ and $\phi'_1 = \pi$, respectively. Again, T_{21}^{dd} changes the spreading factor of the DD contribution from spherical to cylindrical; when both ϕ_2 and ϕ'_1 approach π , all the arguments of the GTF vanish, thus providing a unit, plane wave spreading factor that provides the compensation of the GO discontinuity.

The term $\psi_{21}^{v''}$ vanishes at the first order SBC, owing to the absence of singularity in the multiplying factor $\Psi_{21}^{v''}$, so that its contribution is not actively involved in the transition mechanism described in Sect. 4.1. Its presence is only relevant to the transition at the second order SBC, which is discussed in Sect. 4.2.

Before proceeding further, let us compare the expression of $\psi_{21}^{v''}$ and ψ_{21}^{dd} in (40) and that in (47). The first expressions are valid far from grazing aspects (both ϕ_1' and ϕ_2 far from π), while the second are valid far from the second order SBC (β_2 far from $\beta_1' - \Omega$). Since the new transition functions that are introduced in (43) and (44) reduce to unity far from the above regions, a more complete expression of $\psi_{21}^{v''}$ and ψ_{21}^{dd} is suggested

$$\psi_{21}^{\nu \prime \prime} = \Psi_{21}^{\nu \prime \prime} T_{21}^{\prime \prime} W_{21} \tag{52a}$$

$$\psi_{21}^{dd} = \Psi_{21}^{dd} T_{21}^{dd} W_{21}^{dd} \tag{52b}$$

in which the simple product of the two transition functions which are introduced separately is applied to the final solution. For those aspects where the transition regions of a different kind overlap, the asymptotic expressions (52) are not rigorous. This happens, for example, when the first and the second order SBCs merge ($\phi'_1 = \pi$ and $\beta_2 = \beta'_2 = \beta'_1 - \Omega$), or where they intersect ($\phi_2 = \pi$, $\beta_2 = \beta'_2 = \beta'_1 - \Omega$). Anyway, for $\beta_2 = \beta'_1 - \Omega$, γ_{21} and γ'_{21} tend to δ_1 and δ_2 , respectively, so that T_{21}^{dd} tends to $T_{21}^{''}$ and the continuity of $\psi_{21}^{v''} + \psi_{21}^{dd}$ described in the previous subsection is preserved also in these pathological circumstances. Consequently, even though the composition of the two transition functions adopted in (52) is not based on a solid ground, it results applicable for all the observation aspects, as shown in the numerical examples presented in Sect. 5.

4.4. Total Ray Field

To complete the response of the scattered field from the plane angular sector, the DD contribution from the opposite interaction mechanism 12 should be added, namely the field contribution excited by edge 1 that diffracts at edge 2. This mechanism can be treated by the same formulation presented so far, except for the fact that only the contribution ψ_{12}'' has to be accounted for, since the vertex

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contribution $\psi_{21}^{\nu\prime}$ provides itself the right compensation of the GO and of the singly diffracted rays discontinuities. Finally, the total asymptotic field ψ is represented as

$$\psi = \psi^{v} + \sum_{\substack{m,n=1\\m \neq n}}^{2} \left[U_{n}^{d} \psi_{n}^{d} + U_{mn}^{dd} \psi_{mn}^{dd} \right] + \psi^{i} U^{i} + \psi^{r} U^{r}$$
(53)

where ψ^i and ψ^r are the incident and reflected plane waves, respectively, ψ_n^d is the first order UTD diffraction coefficient at the n^{th} edge, ψ_{mn}^{dd} are the double diffraction contributions from the two analogous mechanisms 12 and 21, and

$$\psi^{\nu} = \psi_{21}^{\nu\prime} + \psi_{21}^{\nu\prime\prime} + \psi_{12}^{\nu\prime\prime} \tag{54}$$

is the complete vertex diffraction contribution. All the contributions labeled by 12 can be obtained from those 21 defined in the previous paragraphs by using the formal substitution $1 \stackrel{\leftarrow}{\to} 2$ inside each formula. Furthermore, $\psi_{mn}^{vl'}$ and ψ_{mn}^{dd} are obtained from (52), $U^{r,i}$ is defined in (26) and U_{mn}^{dd} and U_n^d are obtained from (27) and (38). It is worth noting that due to symmetry properties of the GTF, $T_{12} = T_{21}$.

To obtain a more symmetrical definition of the vertex contribution, the term $\psi_{21}^{v'}$ in (25) can be replaced by

$$\psi^{\nu\prime} = \frac{1}{2}(\psi_{21}^{\nu\prime} + \psi_{12}^{\nu\prime}). \tag{55}$$

This change is not suggested by the need to enforce reciprocity, since (25) is reciprocal itself, but it seems to provide a result more close to the physical response, since it avoids privileging edge 1 with respect to edge 2. Anyway, we have found that the above mentioned change does not give significant differences from the numerical point of view.

V. NUMERICAL EXAMPLES

The following numerical examples are devoted to demonstrate the effectiveness of the formulation in the various transition regions discussed in Sects. 4.1–4.3. For the sake of convenience, the formulas that have been implemented are labeled as follows

 $\begin{array}{l} U_1^d \psi_1^d + U_2^d \psi_2^d + \psi^r U^r: \mbox{ UTD plus reflected field (UTD - R)} \\ \psi^{\nu\prime\prime}({\rm Eq. 55}): \mbox{ vertex field, first contribution (VF')} \\ \psi_{21}^{\nu\prime\prime}({\rm Eq. 52a}): \mbox{ vertex field, second contribution (VF'')} \\ \psi^{\nu}({\rm Eq. 54}): \mbox{ total vertex field (VF)} \\ U_{21}^{dd} \psi_{21}^{dd}({\rm Eq. 38, 52b}): \mbox{ double diffracted field (DDF)} \\ \psi_s = \psi({\rm Eq. 53}) - \psi^i \mbox{ (incident field): total scattered field (SF)} \end{array}$

Figure 5 shows the amplitude of the scattered field at a distance $r = 2\lambda$ from the tip of an $\Omega = 90^{\circ}$ plane angular sector; the plane wave is incident from $\beta'_1 = 120^{\circ}$, $\phi'_1 = 30^{\circ}$ (corresponding to $\beta'_2 = 138^{\circ}$, $\phi'_2 = 40^{\circ}$). The scan plane ($\phi_2 = 170^{\circ}$, β_2 from 0 to 180°) and the SBCs are depicted in the inset of the same figure. In particular, the first order SBCs are for $\beta_2 = 138^{\circ}$ and $\beta_1 = 120^{\circ}$ and the second order SBCs for $\beta_2 = 30^\circ$ and $\beta_1 = 48^\circ$. Dashed, dotted and continuous lines represent UTD-R, VF and SF, respectively. The UTD-R curve is first discontinuous at $\beta_2 = 31^{\circ}$ when the observation point P passes through the first order SBCs of edge 1; after, only the reflected field occurs until $\beta_2 = 138^\circ$, where the first order SBC of edge 2 is crossed; proceeding further, the UTD-R abruptly vanishes, and at 150° appears again, when the second intersection with the first order SBC of edge 1 occurs. The VF shows discontinuity at the same SBCs that compensate those of the UTD-R curve. It is worth noting that the total field is well-behaved even though the two transition regions close to 138° , and 151° overlap (by the way, this partial overlapping is due to the fact that the observation point passes close to the specular direction, which corresponds to the intersection of the two first order SBCs; here the reflection point and the first order diffraction points merge in the tip). The smooth curve of the SF demonstrates the effectiveness of the function T_{mn} in describing the transition field also when all its arguments vanishes.



Figure 5. Amplitude of diffraction contributions versus β_2 ($r = 2\lambda$, $\Omega = 90^\circ$, $\beta'_1 = 120^\circ$, $\phi'_1 = 30^\circ$, $\phi_2 = 170^\circ$); the first order SBCs are at $\beta_2 = 138^\circ$ and $\beta_1 = 120^\circ$; the second order SBCs are at $\beta_2 = 30^\circ$ and $\beta_1 = 48^\circ$; SF (continuous line), UTD-R (dashed line), DDF (dash-dotted line), VF (dotted line).



Figure 6. Amplitude of diffraction contributions versus ϕ_2 ($r = 1.5\lambda$, $\Omega = 60^{\circ}$, $\beta'_1 = 140^{\circ}$, $\phi'_1 = 120^{\circ}$, $\beta_2 = 40^{\circ}$); SF (continuous line), UTD-R (dashed line), DDF (dash-dotted line), VF (dotted line).

The scan plane in Fig. 6 is chosen to check the effectiveness of the transition function T_{21}^{dd} . This figure shows the amplitude of SF (continuous line), UTD-R (dashed line), DDF (dash-dotted line), versus ϕ_2 at a distance $r = 1.5\lambda$ for a plane wave incident from $\beta'_1 = 140^\circ$, $\phi'_1 = 120^\circ$, $\beta_2 = 40^\circ$ on a hard plane angular sector with $\Omega = 60^\circ$. As expected, the field predicted by UTD-R does not vanish at grazing aspects, thus showing here a phase jump of 180° . The introduction of DDF provides the amplitude of SF to be more close to zero, value at which the physical response should be. The incorrect residual field at grazing in the SF plot (-22 dB) is due to the absence of the higher order diffraction contributions.

The amplitudes of various diffraction contributions versus β_2 at $r = 2\lambda$ in the scan plane $\phi_2 = 150^\circ$ are plotted in Figs. 7(a-b). In this case, the incidence aspects are $\beta'_1 = 140^\circ$, $\phi'_1 = 30^\circ$ and the angle of the plane angular sector is rather small ($\Omega = 45^\circ$). The observation point crosses the first order SBC at $\beta_2 = 104^\circ$, $\beta_2 = 159^\circ$, and $\beta_2 = 172^\circ$, while the second order SBCs at $\beta_2 = 76^\circ$ and $\beta_2 = 95^\circ$.



Figure 7. Amplitude of diffraction contributions versus β_2 ($r = 2\lambda$, $\Omega = 45^{\circ}$, $\beta'_1 = 140^{\circ}$, $\phi'_1 = 30^{\circ}$, $\phi_2 = 150^{\circ}$); the first order SBCs are at $\beta_2 = 160^{\circ}$ and $\beta_1 = 140^{\circ}$; the second order SBCs are at $\beta_2 = 95^{\circ}$ and $\beta_1 = 114^{\circ}$; (a) SF (continuous line), UTD-R (dashed line), DDF (dash-dotted line), VF (dotted line); (b) VF" (dotted line), DDF (dash- dotted line) DDF+VF" (dotted line).

In Fig. 7a the relevance of DDF (dash-dotted) is noticeable in the region $0 < \beta_2 < 45^\circ$, so that UTD-R (dashed) is significantly corrected. Again, the first order transition fields are described with the proper continuity by VF (short dashed), so that SF (continuous) exhibits the desirable smooth behavior.

In order to show the effectiveness of the second order transition field described in Sect. 4.2, the contributions VF" (short dashed), DDF (dash-dotted), and VF"+DDF (continuous) are plotted in Fig. 7b. The description of the transition field between vertex and DD rays is quite satisfactory. It is also worth noting that, in this case, the observation point is also close to a first order SBC ($\beta_2 = 33^{\circ}$), so that the above transition region overlaps with the transition region between vertex and the leading diffracted rays. This result confirms what was mentioned at the end of Sect. 4.3, thus providing a partial numerical validation of the product between the two transition functions used in Eqs. (52). It is also worth noting that a small glitch occurs at 76°, owing to the absence of the higher order (slope) contributions in the asymptotic expansion.

VI. CONCLUDING REMARKS

A uniform solution for the field produced by a plane wave at a hard, plane angular sector has been formulated. This solution is expressed in terms of a) the standard GO plus the UTD leading rays; b) a vertex contribution, that provides the uniform description of the ray-field when the reflection and/or leading diffraction points merge in the tip; c) a double diffraction contribution, that uniformly compensates the discontinuity of the singly diffracted rays at the ordinary double diffraction transition regions, including the possible overlapping of them; d) a further vertex ray, that provides a uniform description of the field at the transition region in which double diffraction points merge at the tip. The mechanism b) is treated by the same transition function introduced in [10] and used in a previous work [13] for the soft case. The same transition function with different arguments, is also used to treat the the double diffraction transition mechanism c). Furthermore, a new transition function that involves cylinder parabolic functions is introduced to describe the transition mechanism d).

Despite the conceptual difficulty of the physical mechanism, the solution is quite simple, since it is structured in such a way to recover the ordinary singly and doubly diffracted GTD rays far from the transition regions. Furthermore, the various transition functions of the formulation are easy to implement.

Together with the soft problem previously treated [13], this hard case provides the basic step to study the more general electromagnetic problem.

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APPENDIX A

To give a uniform evaluation of the integrals in (37), the change of variable

$$j\cos(\beta_2 - \theta_2) = \frac{1}{2}(z - z_{21})^2 + j$$
(56)

is introduced, so that

$$z = 2e^{-j\frac{\pi}{4}}\sin\frac{\theta_2 - \beta_2}{2} + z_{21}$$
(57)

in which z_{21} is defined in (45). The topology of the z plane is depicted in Fig.4. The mapping (29) transforms the saddle point $\theta_2 = \beta_2$ into the saddle point $z = z_{21}$ and the branch point $\theta_2 = \beta'_1 - \Omega$ into the branch point z = 0. Furthermore, the contours $V_{\theta_2}^{\pm}$ and D_{θ_2} are mapped into V_z^{\pm} (through $z = z_{21}$) and D_z (through z = 0), respectively. The asymptotic evaluation is performed by multiplying and dividing the integrand by \sqrt{z} and by evaluating the slowly varying part of the integrand at the relevant critical point, namely $z = z_{21}$ and z = 0 for V_z^{\pm} and D_z , respectively. This leads to

$$I(V_{\theta_2}^{\pm}) \sim a_1 \overline{I}(V_{z^{\pm}}) \tag{58a}$$

$$I(D_{\theta_2}) \sim a_2 \overline{I}(D_z) \tag{58b}$$

where

$$\overline{I}(L_z) = \int_{L_z} \frac{1}{\sqrt{z}} e^{-kr\frac{1}{2}(z^2 - 2z_{21}z)} dz \qquad L_z = V_z^{\pm}, \ D_z$$
(59)

and

$$a_{\frac{1}{2}} = e^{-jkr} e^{-\frac{1}{2}krz_{21}^2} \left[p_{21}(\theta_2) \frac{\sqrt{\sin\theta_2}}{\sqrt{\sin\beta_2}} \frac{\sqrt{z}}{\overline{d}_{21}(\theta_2)} \frac{d\theta_2}{dz} \right]_{\substack{\theta_2 = \theta_2\\ \theta_2 = \theta_1' - \Omega}}$$
(60)

It is worth noting that the evaluation of the regular, slowly varying part of the integrand of (37) at the two separated points z = 0 and $z = z_{21}$, allows the asymptotic evaluation to be provided by the asymptotic structure of the doubly diffracted plus vertex rays far out from the transition regions. This is the reason why this technique has been preferred to a more conventional [19-20] asymptotic evaluation in which the regular part of the integrand is evaluated at the same point for the two contributions. The integrals in (59) are simply expressed in terms of the canonical cylinder parabolic function $D_{-\frac{1}{2}}(x)$ of order $-\frac{1}{2}$, defined as

$$D_{-\frac{1}{2}}(x) = e^{j\frac{\pi}{4}} \frac{e^{\frac{x^2}{4}}}{\sqrt{2\pi}} \int_{-\infty}^{\infty+j0^+} \frac{e^{-(\frac{t^2}{2}-jxt)}}{\sqrt{t}} dt$$
(61)

where the extremes of the integrations denote the choice of the branch cut in the integrand, that goes from 0 to ∞ , following the definition given in [19, Eqs. 9.4.27-28]. Alternative, integral representations of the same function are

$$D_{-\frac{1}{2}}(x) = -e^{-j\frac{\pi}{4}} \frac{e^{\frac{x^2}{4}}}{\sqrt{2\pi}} \int_{-\infty}^{\infty - j0^+} \frac{e^{-(\frac{t^2}{2} + jxt)}}{\sqrt{t}} dt$$
(62)

that can be obtained directly from (61) by using a simple change of variable, and [19, Eqs. 9.4.-29]

$$D_{-\frac{1}{2}}(x) = -\frac{e^{-\frac{x^2}{4}}}{2\sqrt{\pi}} \int_{\infty+j0^+}^{\infty-j0^+} \frac{e^{-(\frac{t^2}{2}+xt)}}{\sqrt{t}} dt$$
(63)

By using (61)-(63) it is straightforward to obtain

$$\overline{I}(V_z^{\pm}) = \pm e^{\mp j\frac{\pi}{4}} b D_{-\frac{1}{2}}(\pm j\sqrt{kr}z_{21})$$
(64a)

$$\overline{I}(D_z) = -\sqrt{2}bD_{-\frac{1}{2}}(-\sqrt{kr}z_{21})$$
(64b)

where

$$b = \sqrt{2\pi} (kr)^{-1/4} e^{\frac{1}{4}krz_{21}^2}$$
(64c)

After a proper normalization, expressions (58), (60) and (64) leads to (40-44). For small values of its argument (|x| < 2.5), the function in (61-63) can be calculated by [21, Eq. 13.6.36, pag. 510]

$$D_{-\frac{1}{2}}(x) = 2^{-\frac{1}{4}} e^{-\frac{x^2}{4}} \left[\left[\Gamma\left(\frac{1}{2}\right) \Gamma\left(\frac{3}{4}\right) \right]^{-1} M\left(\frac{1}{4}, \frac{1}{2}, \frac{x^2}{2}\right) - \frac{x}{\sqrt{2}} \left[\Gamma\left(\frac{1}{4}\right) \Gamma\left(\frac{3}{2}\right) \right]^{-1} M\left(\frac{3}{4}, \frac{3}{2}, \frac{x^2}{2}\right) \right]$$
(65)

in which M(a, b, y) is the Kummer's function defined in [21, Eq. 13.1.2, pag. 504], in which the first N terms, in which N = int(22.6|x|), and simply expressed in terms of fast convergent power expansion. For higher values of its argument (|x| > 2.5),

$$D_{-\frac{1}{2}}(x) = x^{-\frac{1}{2}}e^{-\frac{x^2}{4}} \sum_{n=0}^{3} \frac{(-2)^n}{x^{2n}n!} \left(\frac{1}{4}\right)_n \left(\frac{3}{4}\right)_n \tag{66}$$

where $(a)_n = a(a+1)(a+2)(a+3)...(a+n-1), (a) = 1$, in which the phase $\Phi(x)$ of x is such that $-\pi < \Phi(x) \le \pi$.

APPENDIX B

In this appendix, the continuity of the field ψ_{21} in Eq. (39) for small value of

$$\epsilon = \beta_1' - \beta_2 - \Omega \tag{67}$$

is demonstrated. From (40)-(46) it is straightforward to obtain that

$$\lim_{\epsilon \to 0^{\pm}} \psi_{21}^{\nu \prime \prime} = \begin{bmatrix} j \\ 1 \end{bmatrix} \frac{A(kr)^{-1}}{\sqrt{2}\sqrt{|\epsilon|}} \left(\sqrt{j|\epsilon|} (kr)^{1/4} e^{-j\frac{\pi}{8}} D_{-\frac{1}{2}}(0) \right)$$
(68)

where the upper (lower) sign, and the upper (lower) value into square brackets denote positive (negative) values of ϵ in the relevant limit, respectively, and

$$\lim_{\epsilon \to 0^+} \psi_{21}^{dd} = \frac{A(kr)^{-1}}{\sqrt{\epsilon}} \left(\sqrt{\epsilon} (kr)^{1/4} e^{-j\frac{\pi}{8}} D_{-\frac{1}{2}}(0) \right)$$
(69)

in which

$$A = \frac{e^{-jkr}\sqrt{\sin\Omega}}{\pi j c_{21} c_{21}'} \tag{70}$$

The constant $D_{-\frac{1}{2}}(0)$ assumes the value 1.216... The terms into parentheses in (68) and (69) represent the limit of the transition functions W_{21} and W_{21}^{dd} , defined in (43) and (44), respectively. Then

$$\lim_{\epsilon \to 0^{\pm}} \psi_{21}'' = \lim_{\epsilon \to 0^{\pm}} \psi_{21}^{\nu \prime \prime} + U(\epsilon) \psi_{21}^{dd} = \frac{1.241A}{\sqrt{2}} (kr)^{-3/4} e^{-j\frac{\pi}{8}} \begin{bmatrix} j\sqrt{j} + \sqrt{2} \\ \sqrt{j} \end{bmatrix}$$
(71)

where the upper (lower) term is relevant to the upper (lower) sign. By observing that $e^{j3\pi/4} + \sqrt{2} = -e^{-j\pi/4} + \sqrt{2} = e^{j\pi/4}$, the two limits are equal to each other, so that

$$\lim_{\epsilon \to 0} \psi_{21}'' = \frac{1.216A}{\sqrt{2}} (kr)^{-3/4} e^{j\frac{\pi}{4}}$$
(72)

The existence of this limit confirms that the ψ_{21}'' is continuous across the second order SBC.

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