Admittance Matrix Concentration Inequalities for Understanding Uncertain Power Networks

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Admittance Matrix Concentration Inequalities for Understanding Uncertain Power Networks

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***Abstract*—This paper presents conservative probabilistic bounds for the spectrum of the admittance matrix and classical linear power flow models under uncertain network parameters; for example, probabilistic line contingencies. Our proposed approach imports tools from probability theory, such as concentration inequalities for random matrices. This provides a theoretical framework for understanding error bounds of common approximations of the AC power flow equations under parameter uncertainty, including the DC and LinDistFlow approximations. Additionally, we show that the upper bounds scale as functions of nodal criticality. This network-theoretic quantity captures how uncertainty concentrates at critical nodes for use in contingency analysis. We validate these bounds on IEEE test networks, demonstrating that they correctly capture the scaling behavior of spectral perturbations up to conservative constants.**

***Index Terms*—admittance matrix, concentration inequalities, random matrix theory, uncertainty, power flow approximation**

I. Introduction

In *network problems*, the underlying graphical structure of the electric power system itself may be uncertain, estimated, controlled, or optimized. Such uncertainty is common when topology is estimated from measurements, altered by switching actions, or optimized in planning and reconfiguration problems. For example, distribution networks may have incomplete or time-varying network data. This source of uncertainty impacts power system computations, which may cause downstream impacts on decision-making tools.

At the same time, even in the case where the model is known with certainty, many network control problems—such as transmission switching or expansion planning—may have vast combinatorial solution spaces, which may be challenging to search through. In both settings, it is desirable to understand how such randomness *propagates through the power flow equations* via the admittance matrix. To this end, we propose *admittance matrix concentration inequalities* as a fundamental tool for working with random power networks models.

a) Proposed Approach: We consider a power network modeled by an undirected graph with n nodes and m possible (but not necessarily connected) lines. We reserve the index l for lines (edges), so that $l \in [m] := \{1, 2, \ldots, m\}$, and $l = (i, j)$ where $i, j \in [n]$ are the indices reserved for the nodes. We denote by $\boldsymbol{A} \in \{-1, 0, 1\}^{m \times n}$ the branch-to-bus incidence matrix, whose rows $\{\boldsymbol{a}_l\}_{l \in [m]}^{\top}$ are the *incidence vectors* associated with each line $l = (i, j) \in [m]$, where $\boldsymbol{a}_l := \boldsymbol{e}_i - \boldsymbol{e}_j$, with $\boldsymbol{e}_i$ denoting the i-th standard Euclidean basis vector in $\mathbb{R}^n$. We consider a vector $\boldsymbol{w} \in \mathbb{C}^m$ of *random line admittances* $w_l \sim \mathcal{D}_l$, where $\mathcal{D}_l$ is the *admittance distribution* for each line $l = (i, j) \in [m]$, which is not necessarily assumed to be known.

This work studies *random admittance matrices* of the form $\boldsymbol{Y} := \boldsymbol{A}^{\top} \boldsymbol{W} \boldsymbol{A}$, where $\boldsymbol{W} := \mathsf{diag}(\boldsymbol{w})$ is the diagonal matrix with the entries of the complex weight vector $w \in \mathbb{C}^m$ on the diagonal. There are several situations where this model is useful, as discussed earlier. How do such matrices $\boldsymbol{Y}$ behave? We provide precise answers to this question under an array of assumptions; those answers come in the form of upper bounds for the quantities

$$\mathsf{E}\left[\|\boldsymbol{Y}\|\right] \quad \text{and} \quad \mathsf{Pr}\left(\|\boldsymbol{Y}\| \geq t\right) \quad \text{for some } t > 0,$$

i.e., for the expectation and the tail probability of the operator norm $||\boldsymbol{Y}|| = \sqrt{\lambda_{\max}(\boldsymbol{Y}^* \boldsymbol{Y})}$ of the random admittance matrix. The admittance matrix is equivalently characterized as the *graph Laplacian* of an electric grid. We direct the reader to explore recent works on the concentration of general random graph Laplacians such as [1] and [2]. Modeling the line parameters as arbitrary bounded random variables provides a single framework that subsumes many physically relevant uncertainties (unknown parameters, probabilistic line outages, and configuration changes) directly at the level of the admittance matrix. It captures model uncertainty that other realizations of randomness considered, such as uncertainty in power injections, cannot obtain.

b) Contributions: This paper presents conservative probabilistic upper bounds on spectral perturbations in admittance matrices. Our results make use of progress in applied probability theory; in particular, matrix concentration inequalities due to [3]–[5]. The results address several linear power flow models used in the literature for transmission and distribution networks alike, via the linear AC power flow (LACPF) approximation due to [6], [7]. Our main contributions are:

1) *Random admittance matrix framework:* We introduce the perspective of considering the edge weights of an electric grid's lines as *arbitrary bounded random variables* to develop a general theory of the probabilistic behavior of power networks.
2) *Nodal criticality and scaling characterization:* We introduce the *contingency factors* c_l and *nodal criticality* $\Delta_{\boldsymbol{c}}$ as network-theoretic quantities that govern spectral concentration under uncertain contingencies. We prove that the centered admittance matrix concentrates as $O(\sqrt{\Delta_{\boldsymbol{c}} \log n})$, identifying how network topology and uncertainty jointly affect spectral behavior.
3) *Conservative bounds for risk analysis:* The bounds provide reliable upper bounds suitable for worst case and risk-aware analysis, with the sharpest bound achieving $\sim$2× conservatism on a 200-bus network. This framework allows us to model the DC and LinDistFlow approximations under uncertain network parameters.

In Section II we present concentration inequalities to control the spectrum of random admittance matrices under fixed and

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uncertain connectivity. We outline several applications of the proposed approach, such as bounding the error of linear power flow approximations under parameter uncertainty, including the LCPF formulation of the power flow equations, in Section III and bounds for popular linear approximates relative to the AC power flow manifold in Section IV. Additionally, in Section V we numerically validate an application on contingency analysis. We conclude in Section VI with limitations and future work.

II. Bounding the Spectrum of Admittance Matrices

In this section, we present bounds on the error of the admittance matrix $\boldsymbol{Y} \in \mathbb{C}^{n\times n}$. In particular, we will study the following block Laplacian matrix operator, also known as the *flat start Jacobian*. This matrix captures the spectral behavior of both the linear approximations of the power flow equations about the flat start and the admittance matrix.

Definition II.1 (Flat start Jacobian)**.** For an arbitrary power network modeled by an undirected graph $\mathcal{G} = (\mathcal{N}, \mathcal{E})$, where $n := |\mathcal{N}|$ and $m := |\mathcal{E}|$, define the *linear power flow operator*

$$\boldsymbol{F} := \begin{bmatrix} \boldsymbol{G} & -\boldsymbol{B} \\ -\boldsymbol{B} & -\boldsymbol{G} \end{bmatrix} \in \mathbb{R}^{2n\times 2n}, \tag{1}$$

where $\boldsymbol{G}, \boldsymbol{B} \in \mathbb{R}^{n\times n}$ are graph Laplacian matrices corresponding to networks with sufficiently many strictly positive (i.e., conductance) edge weights such that $\mathsf{rank}(\boldsymbol{G}) \geq n-1$ or real-valued (i.e., susceptance) edge weights, respectively.

The matrix $\boldsymbol{F}$ is the standard power flow Jacobian matrix evaluated at the flat start condition; see Appendix A for a concise derivation. The matrix $\boldsymbol{F}$ is symmetric-indefinite, due to the following practical assumption.

Assumption 1. The conductance and susceptance edges weights are such that

$$\boldsymbol{G} \succeq 0, \qquad \text{and} \qquad \boldsymbol{B} \preceq 0.$$

See [8, Thm 3.2] for the conditions on the susceptances required for $\boldsymbol{B} \preceq 0$ to hold. For intuition, it implies that any positive susceptance line must be weak relative to the effective negative susceptance coupling between its endpoints for this to hold.

In addition to being the flat start Jacobian, the matrix $\boldsymbol{F}$ can also be interpreted as being *equivalent to the admittance matrix up to phase shifts*. By this, we mean that the matrix $\boldsymbol{F}$ is equivalent to the lifted, real valued version of the standard admittance matrix $\boldsymbol{Y}$. In particular, if we let $\boldsymbol{G} = \mathsf{Re}(\boldsymbol{Y})$ and $\boldsymbol{B} = \mathsf{Im}(\boldsymbol{Y})$, and define

$$\bar{\boldsymbol{Y}} := \begin{bmatrix} \boldsymbol{G} & \boldsymbol{B} \\ \boldsymbol{B} & -\boldsymbol{G} \end{bmatrix} \in \mathbb{R}^{2n\times 2n}, \tag{2}$$

only the sign of $\boldsymbol{B}$ is flipped in (2) compared to $\boldsymbol{F}$, i.e. $\bar{\boldsymbol{Y}}$ is isomorphic to the Jacobian $\boldsymbol{F}$ under a phase shift of π.

Note that the matrix $\bar{\boldsymbol{Y}}$ is symmetric; it can be easily seen to have the same operator norm as $\boldsymbol{Y}$. Its 2×2 block structure suggests the decomposition as a sum of Kronecker products [9]:

$$\begin{aligned} \bar{\boldsymbol{Y}} = \sum_{(i,j)\in\mathcal{E}} \begin{bmatrix} g_{ij} & b_{ij} \\ b_{ij} & -g_{ij} \end{bmatrix} \otimes \boldsymbol{E}_{ij} &:= \sum_{(i,j)\in\mathcal{E}} \boldsymbol{\Upsilon}_{ij} \otimes \boldsymbol{E}_{ij} \\ &:= \sum_{(i,j)\in\mathcal{E}} \boldsymbol{M}_{ij}, \end{aligned}$$

where $\{\boldsymbol{\Upsilon}_{ij}\}_{(i,j)\in\mathcal{E}}$ is a sequence of 2×2 *random symmetric matrices* representing the uncertainty in the connection $(i,j) \in \mathcal{E}$, and the sequence of random matrices $\{\boldsymbol{M}_{ij}\}_{ij\in\mathcal{E}}$ decompose the entire network in terms of elementary Laplacian matrices, which we now define.

Definition II.2 (Elementary Laplacian Matrix)**.** For each line $(i,j) \in \mathcal{E}$, let $\boldsymbol{E}_{ij} \succeq 0$ denote the positive-semidefinite *elementary Laplacian matrix*

$$\boldsymbol{E}_{ij} := \boldsymbol{e}_{ij}\boldsymbol{e}_{ij}^\top := (\boldsymbol{e}_i - \boldsymbol{e}_j)(\boldsymbol{e}_i - \boldsymbol{e}_j)^\top,$$

describing the normalized subgraph between each pair i, j.

A. An illustrative example: Admittances bounded by 1 per-unit

In many applications, it is useful to understand how a power network will behave under uncertainty in the admittance parameters, where the uncertainty has a bounded size. A useful tool for analyzing matrices that have this property is the matrix Bernstein inequality.

Theorem 1 (A matrix Bernstein bound [3])**.** *Suppose that $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_k \in \mathbb{R}^{n\times n}$ are independent, symmetric, zero-mean random matrices such that $||\boldsymbol{X}_i|| \leq R$ always. Let $\boldsymbol{X} := \sum_{i=1}^k \boldsymbol{X}_i$ and define the matrix statistic*

$$\nu(\boldsymbol{X}) := ||\mathsf{var}(\boldsymbol{X})|| = \left|\left|\sum_{i=1}^k \mathsf{E}\left[\boldsymbol{X}_i^2\right]\right|\right|.$$

Then

$$\mathsf{E}\left[||\boldsymbol{X}||\right] \leq \sqrt{2\nu(\boldsymbol{X})\log(2n)} + \frac{R}{3}\log(2n)$$

Furthermore, for any $t > 0$, we have

$$\mathsf{Pr}\left(||\boldsymbol{X}|| \geq t\right) \leq 2n \exp\left(\frac{-t^2/2}{\nu(\boldsymbol{X}) + Rt/3}\right).$$

We now give an illustrative example of bounded admittance uncertainty in a network with fixed, known connectivity. In the sequel, we will allow for uncertain connectivity. Note that the matrix Bernstein bound can be extended to a sum of *uncentered* random matrices (see Section 6.1 of [3]), at the cost of an additional deterministic term for the mean. We impose the zero-mean hypothesis below to simplify exposition.

Theorem 2 (Concentration of the admittance matrix with fixed connectivity and bounded admittances)**.** *Consider a power system with n nodes and m lines. Let $\Delta = \mathsf{max}_i\, \mathsf{deg}(i)$ be the maximum degree of any node in the network. Suppose that the admittances are independent across lines and distributed according to any bounded, zero-mean distribution $w_{i,j} \sim \mathcal{D}$ that satisfies $|w_{i,j}| \leq 1$. Then, we have*

$$\mathsf{E}\left[||\boldsymbol{Y}||\right] \leq \sqrt{8\Delta\log(4n)} + \frac{2}{3}\log(4n).$$

The proof appears in Appendix B.

B. Uncertain contingencies

In this section, we present bounds on the spectrum of the admittance matrix under random contingencies, shown in Theorem 3. This bound is useful for analyzing the behavior of the power flow equations under uncertain changes in network topology. This has natural applications in many relevant settings, for example, during natural disasters, public safety power shut-offs, or faults. Throughout this section, we operate under the following assumption.

Assumption 2 (Uncertain contingencies)**.** Suppose that each line $l = (i, j) \in \mathcal{E}$ in a power network is switched closed (resp. switched open) with probability $\mathfrak{p}_l \in [0, 1]$ (resp. $1 - \mathfrak{p}_l$).

Assumption 2 is equivalent to modeling the power network as an *inhomogeneous* Erdős-Rényi graph. It is highly relevant in the context of risk-based optimal transmission switching; see [10] for example.

We will analyze how the admittance matrix behaves under the setting of Assumption 2. To achieve this, we will define the following notion of *contingency factors*, and the *criticality* of a node.

Definition II.3 (Contingency factors and nodal criticality)**.** Consider a power network in the context of Assumption 2 with line admittances $\{y_l\}$. Define the *contingency factors* $\{c_l\}$ of each line $l = (i, j) \in [m]$ as

$$c_l := 2 \cdot \mathfrak{p}_l \left(1 - \mathfrak{p}_l\right) |y_l|^2 \tag{3}$$

and the *degree of criticality* of each node i under the contingency factors $\boldsymbol{c} \in \mathbb{R}^m_+$ is defined as

$$d_i(\boldsymbol{c}) := \sum_{l:l\ni i} c_l = \sum_{l:l\ni i} 2 \cdot \mathfrak{p}_l \left(1 - \mathfrak{p}_l\right) |y_l|^2 . \tag{4}$$

Moreover, we denote the *maximum criticality* under $\boldsymbol{c}$ as

$$\Delta_{\boldsymbol{c}} := \max_{i \in [n]} d_i(\boldsymbol{c}).$$

In essence, the objects in Definition II.3 are the edge weights, the nodal degrees, and the maximum nodal degree, respectively, of a certain graph Laplacian matrix. In particular, it is the Laplacian matrix that arises from the matrix-valued variance of the admittance matrix under uncertain contingencies, which we define explicitly in the forthcoming result.

Theorem 3 (Concentration with fixed admittances and uncertain contingencies)**.** *Consider a power network in the context of Definition II.3. Let each line $l = (i, j) \in \mathcal{E}$ have admittance $y_{ij} \in \mathbb{C}$ with $|y_{ij}| \le 1$ per-unit. Define the random line edge weights*

$$w_{ij} := y_{ij} \cdot \xi_{ij}, \quad \xi_{ij} \sim \mathsf{Ber}(\mathfrak{p}_{ij}), \quad (i, j) \in E,$$

and the corresponding random admittance matrix

$$\boldsymbol{Y} := \sum_{(i,j)\in\mathcal{E}} \xi_{ij} y_{ij} \left(\boldsymbol{e}_i - \boldsymbol{e}_j\right) \left(\boldsymbol{e}_i - \boldsymbol{e}_j\right)^\top \tag{5}$$

and center as $\tilde{\boldsymbol{Y}} := \boldsymbol{Y} - \mathsf{E}\,\boldsymbol{Y}$. Define the normalized total degree of criticality:

$$\bar{D} := \Delta_{\boldsymbol{c}}^{-1} \sum_{i\in[n]} d_i(\boldsymbol{c}). \tag{6}$$

Then, for all $t \ge \sqrt{2\Delta_{\boldsymbol{c}}} + 2/3$, we have

$$\Pr(\|\tilde{\boldsymbol{Y}}\| \ge t) \le 8\bar{D} \cdot \exp\left(\frac{-t^2}{4\left(\Delta_{\boldsymbol{c}} + t/3\right)}\right); \tag{7}$$

moreover, there exists a constant $C > 0$ such that

$$\mathsf{E}\,\|\tilde{\boldsymbol{Y}}\| \le C\left(\sqrt{2\Delta_{\boldsymbol{c}} \log(1 + 2\bar{D})} + 2\log(1 + 2\bar{D})\right). \tag{8}$$

The proof of Theorem 3 appears in Appendix C.

Remark. The matrix (5) is not positive-semidefinite, except in extremely restrictive cases; e.g., $\boldsymbol{G}\boldsymbol{B} = \boldsymbol{B}\boldsymbol{G}$ is one sufficient condition. Hence, we must use the Bernstein inequality, as opposed to bounds with potentially simpler forms, like the matrix Chernoff inequality.

III. The Linear Coupled Power Flow Model: Formulation and Bounds

The LinDistFlow equations are known to be equivalent to the Linear Coupled Power Flow (LCPF) model in the special case that the network is a tree. We will continue analysis working with the LCPF model for this section as it is more general and results for LinDistFlow follow simply. See Appendix A for more details.

We can decompose $\boldsymbol{F}$, as defined in (17) into the sum of Kronecker products between a particular 2×2 block matrix of admittances and elementary graph Laplacian matrices (Def. II.2), as follows:

$$\begin{aligned}
\boldsymbol{F} &= \begin{bmatrix} \boldsymbol{A}^\top \mathsf{diag}(\boldsymbol{g})\boldsymbol{A} & -\boldsymbol{A}^\top \mathsf{diag}(\boldsymbol{b})\boldsymbol{A} \\ -\boldsymbol{A}^\top \mathsf{diag}(\boldsymbol{b})\boldsymbol{A} & -\boldsymbol{A}^\top \mathsf{diag}(\boldsymbol{g})\boldsymbol{A} \end{bmatrix} \\
&= \begin{bmatrix} \sum_{ij\in\mathcal{E}} \boldsymbol{E}_{ij} g_{ij} & -\sum_{ij\in\mathcal{E}} \boldsymbol{E}_{ij} b_{ij} \\ -\sum_{ij\in\mathcal{E}} \boldsymbol{E}_{ij} b_{ij} & -\sum_{ij\in\mathcal{E}} \boldsymbol{E}_{ij} g_{ij} \end{bmatrix} \\
&= \sum_{ij\in\mathcal{E}} \begin{bmatrix} \boldsymbol{E}_{ij} g_{ij} & -\boldsymbol{E}_{ij} b_{ij} \\ -\boldsymbol{E}_{ij} b_{ij} & -\boldsymbol{E}_{ij} g_{ij} \end{bmatrix} \\
&= \sum_{ij\in\mathcal{E}} \underbrace{\begin{bmatrix} g_{ij} & -b_{ij} \\ -b_{ij} & -g_{ij} \end{bmatrix}}_{:=\boldsymbol{\Upsilon}_{ij}} \otimes \boldsymbol{E}_{ij} \\
&= \sum_{ij\in\mathcal{E}} \boldsymbol{\Upsilon}_{ij} \otimes \boldsymbol{E}_{ij} := \sum_{ij\in\mathcal{E}} \boldsymbol{M}_{ij},
\end{aligned}$$

where $\boldsymbol{\Upsilon}_{ij}$ is a 2×2 symmetric matrix of admittances for a given line (i, j), as defined above.

A. Bounded spectra of elementary power flow Jacobians

Let $\boldsymbol{M}_{ij} = \boldsymbol{\Upsilon}_{ij} \otimes \boldsymbol{E}_{ij}$ be the elementary Jacobian corresponding to edge $(i, j) \in \mathcal{E}$. Note that the operator norm of $\boldsymbol{M}_{ij}$ is

$$||\boldsymbol{M}_{ij}|| \overset{(1)}{=} ||\boldsymbol{\Upsilon}_{ij}||\,||\boldsymbol{E}_{ij}|| \overset{(2)}{=} 2\sqrt{g_{ij}^2 + b_{ij}^2} = 2\,|y_{ij}|$$

where step (1) is due to the fact that $||\boldsymbol{A} \otimes \boldsymbol{B}|| = ||\boldsymbol{A}||\,||\boldsymbol{B}||$ for any matrices $\boldsymbol{A}, \boldsymbol{B}$, and step (2) is due to the fact that $||\boldsymbol{E}_{ij}|| = 2$, and furthermore,

$$\begin{aligned}
||\boldsymbol{\Upsilon}_{ij}|| &= \sqrt{\lambda_{\max}\left(\boldsymbol{\Upsilon}_{ij}^\top \boldsymbol{\Upsilon}_{ij}\right)} \\
&= \sqrt{\lambda_{\max}\left(\begin{bmatrix} g_{ij}^2 + b_{ij}^2 & 0 \\ 0 & g_{ij}^2 + b_{ij}^2 \end{bmatrix}\right)} \\
&= \sqrt{g_{ij}^2 + b_{ij}^2} = |y_{ij}|\,.
\end{aligned}$$

Remark. Note that the following identities hold:

$$||\boldsymbol{M}_{ij}|| = \sqrt{2\,\mathsf{tr}\left(\boldsymbol{\Upsilon}_{ij}^\top \boldsymbol{\Upsilon}_{ij}\right)} = \sqrt{2}\,||\boldsymbol{\Upsilon}_{ij}||_F\,, \tag{9}$$

and $||\boldsymbol{M}_{ij}||_F = 2\sqrt{2}\,||\boldsymbol{\Upsilon}_{ij}||\,.$

B. Matrix variance of the LCPF model under uncertainty

We now bound the matrix variance of the LCPF model.

Lemma 1. *Suppose that $\boldsymbol{g}, \boldsymbol{b} \in \mathbb{R}^m$ are independent and uniformly distributed on $(m-1)$-dimensional sphere of radius $1/2$; $\boldsymbol{g}, \boldsymbol{b} \overset{\text{iid}}{\sim}$ Uniform $\left(\{\boldsymbol{y} \in \mathbb{R}^n \,:\, \boldsymbol{y}^\top \boldsymbol{y} = \frac{1}{2}\}\right)$. Then* *the matrix-valued variance of the linear power flow operator (Def. II.1) is upper-bounded as*

$$\mathsf{E}\left[\boldsymbol{F}\boldsymbol{F}^*\right] \preceq \frac{2}{n}\mathbf{I}_2 \otimes \boldsymbol{A}^\top \boldsymbol{A} := \boldsymbol{V}.$$

The proof appears in Appendix D.

Theorem 4 (Spectral error of LCPF model under uncertain admittance parameters)**.** *Suppose that the conductance and susceptance parameters of an electric power system model are uncertain, and can be modeled for each line $(i,j) \in \mathcal{E}$ as*

$$g_{ij} = g_{ij}^{\bullet} + \Delta_{ij}^{g} \tag{10a}$$
$$b_{ij} = b_{ij}^{\bullet} + \Delta_{ij}^{b}, \tag{10b}$$

where $g_{ij}^{\bullet}, b_{ij}^{\bullet} \in \mathbb{R}$ are the true parameters, and Δ_{ij}^{g}, Δ_{ij}^{b} are bounded, zero-mean uncertainty random variables, independent across lines, such that

$$\max\left\{||\Delta_{ij}^{b}||, ||\Delta_{ij}^{g}||\right\} \leq \Delta \quad \forall (i,j) \in \mathcal{E}.$$

Then, for any $t \geq 0$, we have that

$$\Pr\left(||\boldsymbol{F} - \mathsf{E}\left[\boldsymbol{F}\right]|| \geq t\right) \lesssim n \exp\left(\frac{-t^2}{8\left(\Delta^2 n + \Delta t/3\right)}\right), \tag{11}$$

and moreover,

$$\mathsf{E}\left[||\boldsymbol{F} - \mathsf{E}\left[\boldsymbol{F}\right]||\right] \leq 2\Delta\sqrt{2}\left(\sqrt{n \log\left(4n\right)} + \frac{1}{3}\log\left(4n\right)\right) \tag{12}$$

The proof appears in Appendix E.

IV. Application: Bounding the Error of a Family of Power Flow Linearizations

In this section, we provide an error bound of linear approximations of the AC power flow equations under uncertain admittances. This serves as a useful primitive for further applications on the evaluation of the quality of DC power flow in practical problems such as contingency analysis and network reconfiguration. Following the very recent results of [11], we can utilize the perspective that the power flow equations admit a manifold interpretation to perform such analysis under uncertain admittances. Recounting the setup of [11], let the AC power flow manifold be

$$\mathcal{M}_{PF} = \mathrm{gph}(\Phi) = \left\{(\boldsymbol{x}, \boldsymbol{p}, \boldsymbol{q}) \in \mathbb{R}^{4n} : (\boldsymbol{p}, \boldsymbol{q}) = \Phi(x)\right\},$$

where $\boldsymbol{x} = (\boldsymbol{v}, \boldsymbol{\theta}) \in \mathbb{R}^{2n}$, $\Phi : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ is the power flow mapping, and graph is defined as

$$\mathrm{gph}(f) := \{(\boldsymbol{x}, f(\boldsymbol{x})) \in \mathbb{R}^{n+m} \mid \boldsymbol{x} \in X\}$$

where $f : X \to Y$, $X \subset \mathbb{R}^n$, and $Y \subset \mathbb{R}^m$. Equivalently, $\mathcal{M}_{PF} = F^{-1}(0)$ with $F(\boldsymbol{x}, \boldsymbol{p}, \boldsymbol{q}) = \Phi(\boldsymbol{x}) - (\boldsymbol{p}, \boldsymbol{q})$ and

$$\mathsf{D}F(\boldsymbol{x}, \boldsymbol{p}, \boldsymbol{q}) = [\mathsf{D}\Phi_{\boldsymbol{x}} \quad -I_{2n}]$$

is surjective everywhere. Fix a feasible base point

$$\boldsymbol{z}_\star = (\boldsymbol{x}_\star, \boldsymbol{p}_\star, \boldsymbol{q}_\star) = (\boldsymbol{x}_\star, \Phi(\boldsymbol{x}_\star)) \in \mathcal{M}_{PF}.$$

For a step $\boldsymbol{h} \in \mathbb{R}^{2n}$ tangent at $\boldsymbol{z}_*$ (i.e. implicit function theorem evaluated at $\boldsymbol{z}_*$ s.t. $(\boldsymbol{h}, \mathsf{D}\Phi_{\boldsymbol{x}_*}\boldsymbol{h}) \in T_{\boldsymbol{z}_*}\mathcal{M}_{PF}$), form the tangent point $\bar{\boldsymbol{z}} := \boldsymbol{z}_* + (\boldsymbol{h}, \mathsf{D}\Phi_{\boldsymbol{x}_*}\boldsymbol{h})$, where $T_{\boldsymbol{z}}\mathcal{M}_{PF} = \ker\left[\mathsf{D}\Phi_{\boldsymbol{x}} \quad -I_{2n}\right]$ is the tangent space at $\boldsymbol{z}$.

Proposition 1. *Let $\bar{z} \in T_{\boldsymbol{z}}\mathcal{M}_{PF}$ be a point in the linear tangent space of the power flow manifold about $\boldsymbol{z} \in \mathcal{M}_{PF}$. For a random admittance matrix $\boldsymbol{Y}$, defined in a similar manner as Theorem 2, the expected Euclidean distance (in its ambient space) of a tangent step from a random AC PF manifold is*

$$\begin{aligned}\mathsf{E}\left[\mathrm{dist}(\bar{\boldsymbol{z}}, \mathcal{M}_{PF})\right] &\leq 3\|\boldsymbol{h}\|_\infty \|\boldsymbol{h}\|_2 \,\mathsf{E}\left[||\boldsymbol{Y}||\right] \\ &\leq 3\|\boldsymbol{h}\|_2^2\left(\sqrt{8\Delta\log(4n)} + \frac{2}{3}\log(4n)\right).\end{aligned}$$

Proof. From [11, Prop III.1], we immediately get

$$\mathrm{dist}(\bar{\boldsymbol{z}}, \mathcal{M}_{PF}) \leq 3\,||F(\bar{\boldsymbol{z}})||.$$

By definition, $F(\boldsymbol{x}, \boldsymbol{p}, \boldsymbol{q}) = \Phi(\boldsymbol{x}) - (\boldsymbol{p}, \boldsymbol{q})$. Therefore,

$$F(\bar{\boldsymbol{z}}) = \Phi(\boldsymbol{x}_* + \boldsymbol{h}) - (\Phi(\boldsymbol{x}_*) + \mathsf{D}\Phi_{\boldsymbol{x}_*}\boldsymbol{h}) \tag{13}$$

The AC PF manifold can be equivalently represented as

$$\mathcal{M}_{PF} = \{(\boldsymbol{u}, \boldsymbol{s}) \in \mathbb{C}^n \times \mathbb{C}^n : \boldsymbol{s} = \Psi_{\boldsymbol{Y}}(\boldsymbol{u}) := \mathsf{diag}(\boldsymbol{u})\underline{\boldsymbol{Y}\boldsymbol{u}}\}$$

where $\underline{(\cdot)}$ denotes complex conjugate. Let $(\boldsymbol{u}_*, \boldsymbol{s}_*)$ be the complex form of $\boldsymbol{z}_*$, so the complex tangent point is

$$(\bar{\boldsymbol{u}}, \bar{\boldsymbol{s}}) := (\boldsymbol{u}_* + \boldsymbol{h}_u, \boldsymbol{s} + \mathsf{D}\Psi_{\boldsymbol{Y}}(\boldsymbol{u}_*)[\boldsymbol{h}_u])$$

where $\boldsymbol{h}_u \in \mathbb{C}^n$ is the complex voltage step. Since $\Psi_{\boldsymbol{Y}}$ is quadratic in $\boldsymbol{u}$,

$$\begin{aligned}\Psi_{\boldsymbol{Y}}(\boldsymbol{u}_* + \boldsymbol{h}_u) = \Psi_{\boldsymbol{Y}}(\boldsymbol{u}_*) &+ \mathsf{D}\Psi_{\boldsymbol{Y}}(\boldsymbol{u}_*)[\boldsymbol{h}_u] \\ &+ \tfrac{1}{2}\mathsf{D}^2\Psi_{\boldsymbol{Y}}(\boldsymbol{u}_*)[\boldsymbol{h}_u, \boldsymbol{h}_u]\end{aligned}$$

with first and second Fréchet derivatives as

$$\mathsf{D}\Psi_{\boldsymbol{Y}}(\boldsymbol{u})[\boldsymbol{h}_u] = \mathsf{diag}(\boldsymbol{h}_u)\underline{\boldsymbol{Y}\boldsymbol{u}} + \mathsf{diag}(\boldsymbol{u})\underline{\boldsymbol{Y}\boldsymbol{h}_u},$$
$$\mathsf{D}^2\Psi_{\boldsymbol{Y}}[\boldsymbol{h}_u, \boldsymbol{k}_u] = \mathsf{diag}(\boldsymbol{h}_u)\underline{\boldsymbol{Y}\boldsymbol{k}_u} + \mathsf{diag}(\boldsymbol{k}_u)\underline{\boldsymbol{Y}\boldsymbol{h}_u}.$$

Subtracting the linearization,

$$\begin{aligned}\Psi_{\boldsymbol{Y}}(\boldsymbol{u}_* + \boldsymbol{h}_u) - (\Psi(\boldsymbol{u}_*) + \mathsf{D}\Psi_{\boldsymbol{x}_*}[\boldsymbol{h}_u]) &= \frac{1}{2}\mathsf{D}^2\Psi_{\boldsymbol{Y}}(\boldsymbol{u}_*)[\boldsymbol{h}_u, \boldsymbol{h}_u] \\ &= \mathsf{diag}(\boldsymbol{h}_u)\underline{\boldsymbol{Y}\boldsymbol{h}_u}\end{aligned}$$

Since this is the same form as (13) and since $\mathbb{C}^n \cong \mathbb{R}^{2n}$,

$$\|F(\bar{\boldsymbol{z}})\|_2 = \|\,\mathsf{diag}(\boldsymbol{h}_u)\underline{\boldsymbol{Y}\boldsymbol{h}_u}\|_2$$

and by standard norm inequalities, we have

$$\begin{aligned}\|F(\bar{\boldsymbol{z}})\|_2 &= \|\,\mathsf{diag}(\boldsymbol{h}_u)\underline{\boldsymbol{Y}\boldsymbol{h}_u}\|_2 \\ &\leq ||\mathsf{diag}(\boldsymbol{h})||\,||\boldsymbol{Y}||\,\|\boldsymbol{h}\|_2 = \|\boldsymbol{h}\|_\infty\,||\boldsymbol{Y}||\,\|\boldsymbol{h}\|_2 \\ &\leq \|\boldsymbol{h}\|_2^2\,||\boldsymbol{Y}||.\end{aligned}$$

Plugging into [11, Prop III.1]

$$\begin{aligned}\mathrm{dist}(\bar{\boldsymbol{z}}, \mathcal{M}_{PF}) &\leq 3\|F(\bar{\boldsymbol{z}})\| \\ &\leq 3\|\boldsymbol{h}\|_\infty\,||\boldsymbol{Y}||\,\|\boldsymbol{h}\|_2 \\ &\leq 3\|\boldsymbol{h}\|_2^2\,||\boldsymbol{Y}||\end{aligned}$$

Taking expectations on both sides and using Thm 2 yields

$$\begin{aligned}\mathsf{E}\left[\mathrm{dist}(\bar{\boldsymbol{z}}, \mathcal{M}_{PF})\right] &\leq 3\|\boldsymbol{h}\|_\infty \|\boldsymbol{h}\|_2 \,\mathsf{E}\left[||\boldsymbol{Y}||\right] \\ &\leq 3\|\boldsymbol{h}\|_2^2\left(\sqrt{8\Delta\log(4n)} + \frac{2}{3}\log(4n)\right).\end{aligned}$$

□

There is a useful interpretation for the expression derived in Prop 1. It is an upper bound on the expected worst case deviation of a local tangent step from a given point of a *typical* ACPF manifold. We are able to describe a family of corresponding manifolds generated by this random admittance matrix that corresponding to a large class of physically realizable grids because we only assume the line admittances come from any bounded probability distribution. Our result also suggests that, in a sense, the geometry of the ACPF manifold itself concentrates in a manner described in Thm 2.

Proposition 2. *Suppose the special case where we assume a lossless network (i.e. $\boldsymbol{Y} = -j\boldsymbol{B}$) with the susceptances are random variables of the form $w_e = b_e \cdot s_e$ where b_e is the physical susceptance and $s_e \sim \mathsf{Ber}(p_e)$, where p_e is the rate at which line e is switched closed. From some constant $C > 0$, the expected distance from a local point on the linear tangent space to the random AC PF manifold is*

$$\begin{aligned}\mathsf{E}\left[\operatorname{dist}(\bar{\boldsymbol{z}}, \mathcal{M}_{PF})\right] \leq\, & 3C\|\boldsymbol{h}\|_\infty \|\boldsymbol{h}\|_2 \\ & \left(\sqrt{2\Delta_{\boldsymbol{c}} \log(1+2\bar{D})} + 2\log(1+2\bar{D})\right)\end{aligned}$$

Proof. Following directly from Prop 1, we know that

$$\mathsf{E}\left[\operatorname{dist}(\bar{\boldsymbol{z}}, \mathcal{M}_{PF})\right] \leq 3\|\boldsymbol{h}\|_\infty \|\boldsymbol{h}\|_2 \,\mathsf{E}\left[||\boldsymbol{Y}||\right]$$

so what's left is to bound this special case of $\mathsf{E}\left[||\boldsymbol{Y}||\right]$. By lossless assumption and norm properties,

$$||\boldsymbol{Y}|| = ||-j\boldsymbol{B}|| = ||\boldsymbol{B}|| \,.$$

From Theorem 3, we directly get that

$$\mathsf{E}\,\|\boldsymbol{B}\| \leq C\left(\sqrt{2\Delta_{\boldsymbol{c}} \log(1+2\bar{D})} + 2\log(1+2\bar{D})\right),$$

so the expected distance from the ACPF manifold is

$$\begin{aligned}\mathsf{E}\left[\operatorname{dist}(\bar{\boldsymbol{z}}, \mathcal{M}_{PF})\right] \leq\, & 3C\|\boldsymbol{h}\|_\infty \|\boldsymbol{h}\|_2 \\ & \left(\sqrt{2\Delta_{\boldsymbol{c}} \log(1+\bar{D})} + 2\log(1+\bar{D})\right)\end{aligned}$$

for some constant $C > 0$. □

Remark. Remember the error bounds presented in Prop 1 and 2 are measuring a distance of a vector $\bar{\boldsymbol{z}}$ between a linearization to the ACPF manifold in its implicit/ambient space, i.e. the space of all vectors $(\boldsymbol{v}, \boldsymbol{\theta}, \boldsymbol{p}, \boldsymbol{q}) \in \mathbb{R}^{4n}$. It may be of more direct interest in applications to consider this distance with respect to a specific quantity of interest, such as voltage magnitude. In these settings, it is sufficient to consider a projection of the vector $\bar{\boldsymbol{z}}$ to that quantity of interest. We defer proof for future work, but the distance of this projected vector likely has the same scaling as the distance of $\bar{\boldsymbol{z}}$, only differing by a constant.

V. Numerical Validation on IEEE Networks

We validate our theoretical bounds on the ACTIVSg200 synthetic transmission network (n = 200 buses, m = 245 lines) using Monte Carlo simulations, with the line admittances normalized so that $\max_l |y_l| = 1$ as the theorems assume. For each switching probability $p \in \{0.1, 0.3, 0.5, 0.7, 0.9\}$, we generate 500 random contingency samples and compute the empirical distribution of $\|\tilde{\boldsymbol{Y}}\| = \|\boldsymbol{Y} - \mathsf{E}\,\boldsymbol{Y}\|$.

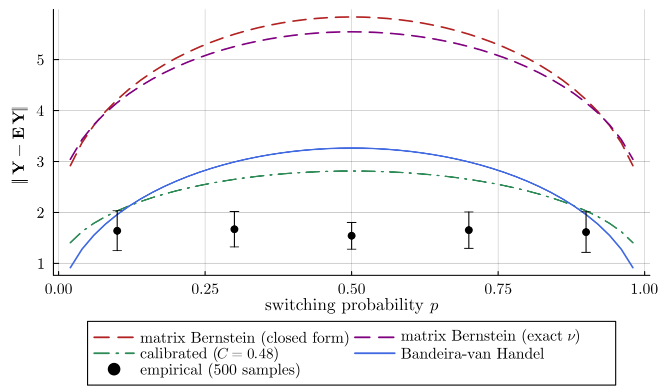

Fig. 1. Comparison of empirical $\mathsf{E}\,\|\tilde{\boldsymbol{Y}}\|$ (markers) versus theoretical bounds across switching probabilities on the ACTIVSg200 network. Error bars show ±1 sample standard deviation over 500 Monte Carlo samples. Red dashed line: the (14) bound from Theorem 3 with the universal constant set to one. Purple dashed line: the same (14) bound evaluated with the exact variance statistic ν and intrinsic dimension. Green dash-dotted line: the red bound with its universal constant calibrated as a tight upper envelope over the markers. Blue solid line: the (15) bound from [4]. The bounds correctly bound the empirical mean; the bounds are parabolic in p with maximum at p = 0.5, while the empirical mean is nearly constant in p on this network.

A. Bound Evaluation

We evaluate the expectation bound from Theorem 3 against the empirical distribution and a sharper bound from [4] that assumes independent entries of our random matrix.

Substituting $\nu = 2\Delta_{\boldsymbol{c}}$, $L = 2$, and setting the universal constant $C = 1$, the expectation bound from Theorem 3 gives:

$$\mathsf{E}\left[\|\tilde{\boldsymbol{Y}}\|\right] \leq \sqrt{4\Delta_{\boldsymbol{c}} \log(1+2\bar{D})} + \tfrac{2}{3}\log(1+2\bar{D}). \tag{14}$$

In [4], it states that for a symmetric random matrix $\boldsymbol{X}$ with independent entries,

$$\mathsf{E}\left[\|\boldsymbol{X}\|\right] \lesssim \sigma_{\mathsf{row}} + \sigma_{\mathsf{max}}\sqrt{\log n}, \tag{15}$$

where $\sigma_{\mathsf{row}} := \max_i \sqrt{\sum_j \mathsf{Var}\left[\boldsymbol{X}_{ij}\right]}$ is the maximum row standard deviation and $\sigma_{\mathsf{max}} := \max_{ij} \sqrt{\mathsf{Var}\left[\boldsymbol{X}_{ij}\right]}$ is the maximum entry standard deviation. For our centered admittance matrix $\tilde{\boldsymbol{Y}}$, we have $\sigma_{\mathsf{row}} = \sqrt{\Delta_{\boldsymbol{c}}}$. The key advantage of (15) is that the leading term $\sigma_{\mathsf{row}} = \sqrt{\Delta_{\boldsymbol{c}}}$ has no log factor, unlike (14). Note that the admittance matrix, and graph Laplacian matrices in general, trivially do not satisfy the requirements of the bound in (15). However, in section V-B we discuss why this bound is of potential interest and how it can make sense of the results.

Figure 1 shows that both bounds correctly bound the empirical mean. For the (14) bound, we get a ratio of ∼0.26, i.e., ∼3.8× conservative. For the (15) bound, the ratio is ∼0.47, i.e., ∼2× conservative. Evaluating (14) with the exact variance statistic and intrinsic dimension, rather than the worst case $\nu \leq 2\Delta_c$ and $2\bar{D}$, changes this ratio only slightly on the centered contingency matrix (to ~ 0.28). Here the closed form is already close to the sharp matrix Bernstein value, so the residual gap to the empirical mean is the irreducible matrix Bernstein slack as described in section V-B below.

B. Discussion

The numerical experiments confirm the following:

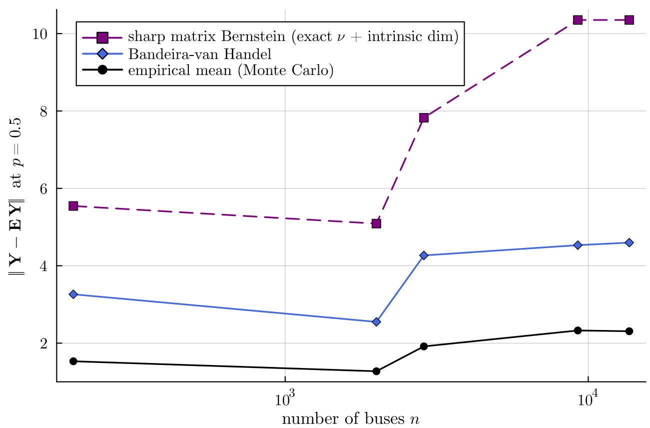

Fig. 2. Empirical $\mathsf{E}\,\|\tilde{\boldsymbol{Y}}\|$ at $p = 0.5$ and the contingency bounds across meshed transmission networks from $n = 200$ to $n = 13,659$ buses. The empirical deviation stays $O(1)$ while the matrix Bernstein bound grows like $\sqrt{\log n}$; the (15) bound from [4] remains near twice the empirical mean.

1) *Bound the expectation:* The empirical mean $\mathsf{E}\,\|\tilde{\boldsymbol{Y}}\|$ lies below both theoretical bounds in all configurations. Individual samples *can* exceed the expectation bound (as visible in error bars), which is expected since these are bounds on the mean, not individual realizations.
2) *Capture scaling:* Both bounds correctly capture the parabolic shape in p (maximum at $p = 0.5$) and the $O(\sqrt{\Delta_{\boldsymbol{c}}})$ scaling with network parameters.

However, the (14) bound developed in Theorem 3 seems to give empirically looser bounds than the (15) bound from [4]; why? There may be a sense that despite the tighter bound requiring independent entries of our random matrix, it is actually the right scaling for our structured problem. A key insight is that any looseness from invoking matrix Bernstein to solicit concentration results of structured random matrices may be attributed to the lack of exploiting noncommutativity. In recent years, there has been advancement in the use of *free probability theory* to develop sharpened concentration inequalities for random matrices utilizing "intrinsic freeness." This may allow us to argue that our structured random admittance matrix can have the same scaling as a random matrix with independent entries, see [5], [12]. We leave this direction of research for future work. For tighter estimates, the bound (15) from [4] appears numerically preferable, despite its theoretical difference from (14). For bounding individual realizations with high probability, use the tail bound (7).

Finally, we check that this picture persists at scale. Figure 2 repeats the contingency experiment at $p = 0.5$ on a sequence of meshed transmission networks spanning two orders of magnitude in size, from ACTIVSg200 ($n = 200$) to the PEGASE 13,659-bus system, using sparse power iteration for the operator norms. The empirical deviation $\mathsf{E}\,\|\tilde{\boldsymbol{Y}}\|$ stays between 1.3 and 2.3 across the entire range, so the fluctuation of the admittance spectrum does not grow with network size. The matrix Bernstein bound grows slowly, like $\sqrt{\log n}$, and its conservatism widens mildly with scale, from $\sim 3.6\times$ to $\sim 4.5\times$, while the (15) bound from [4] stays near $\sim 2\times$ at every size. A larger test case therefore does not tighten the matrix Bernstein constant, but it confirms that the predicted scaling and the advantage of the noncommutative bound are stable.

VI. Conclusions

In this paper, we presented a number of results applying matrix concentration inequalities to characterize behaviors of the power flow equations under uncertain admittances. We first derive an expectation bound on the spectrum of the admittance matrix under general distribution of bounded admittances that grows like $O(\sqrt{\Delta \log n} + \log n)$, where Δ is the maximum degree and n is the numbers of nodes. This quantity has the interpretation that the under light distributional assumptions on the admittances, the effective resistance of a power network concentrates. We use it to develop refined tail bounds of uncertain contingencies expressed through contingency factors and nodal criticality. We then lift these results to the linear coupled power flow (LCPF) operators and show how the induced spectral uncertainty propagates to linear approximations of the AC power flow manifold. This produces explicit error bounds for a family of linear power flow models.

A. Applications and Future Work

The framework developed in this paper allows the development for future work in the average case analysis of existing power system algorithms, formal justification of problem-specific randomized algorithms, and probabilistic analysis of grid operations, reliability, and planning problems. Some specific examples of such applications are given below.

1) Contingency Analysis: With a suitable Bernoulli parameter on the admittances, analyzing the distribution of power networks subsumes all possible $n-1$ configurations of the network. In particular, one can show that the simplex of all $n-1$ contingencies does not violate any line flow constraint. For example, in the context of the DC approximation, the results of the present paper can show that over the simplex $\boldsymbol{s} \in \Delta^n_{n-1}$,

$$\mathsf{Pr}(||\boldsymbol{SWA\theta}||_\infty > \epsilon) \le \delta(\epsilon),$$

for diagonal matrices $\boldsymbol{S}$ and $\boldsymbol{W}$ and for some choice of (ϵ, δ).

2) Network Reconfiguration: Network reconfiguration is of great interest in recent power systems research, particularly for modern applications, such as in congestion management and grid planning. The combinatorial solution space of such problems implies that analyzing or sampling from a family of possible network configurations is a potentially relevant task. The theory of the present paper is directly applicable to such a setting, such as in [13], as it describes the behavior of such nondeterministic power flow models.

3) Evaluation of Linearizations in Specialized Problems: Following up on the results on the error bounds developed in this paper, the evaluation of linearizations such as DC power flow in more specific problems such as contingency analysis and network reconfiguration are of great interest to practitioners. This probabilistic framework could allow screening of linearization choices conformal to specific parameters of the problem. Moreover, applications involving bounding the error of locational marginal prices (LMPs) are also tractable, leveraging results that connect admittance matrices to LMP sensitivities [14], [15].

AI Usage Disclosure

Claude Code with the Opus 4.5 model was used to help create a plotting interface to visualize the inequalities. No AI was used for prose generation/improvement nor mathematics.

Appendix

A. Fresh derivation of the Linear Coupled Power Flow Model

Theorem 5 (Linear Coupled Power Flow Model [6], [7], [16]). *Consider the flat start condition $\boldsymbol{u}_\star := \mathbf{1} + \jmath\mathbf{0}$, and suppose that $\boldsymbol{\omega} = \mathbf{0} + \jmath\mathbf{0}$. Then, the linear coupled power flow manifold around $\boldsymbol{u}_\star$ is the linear space*

$$\mathcal{M}_\star := \left\{ \boldsymbol{x} \in \mathbb{R}^{4n} : \boldsymbol{F}(\boldsymbol{x}_\star)(\boldsymbol{x} - \boldsymbol{x}_\star) = \mathbf{0}_{2n} \right\}, \qquad (16)$$

where $\boldsymbol{F} : \mathbb{R}^{4n} \to \mathbb{R}^{2n\times 4n}$ is the Jacobian of the power flow equations $\mathcal{F}$ at the nominal state $\boldsymbol{x}_\star$, which we write as

$$\boldsymbol{F}(\boldsymbol{x}_\star) = \begin{bmatrix} \boldsymbol{G} & -\boldsymbol{B} & -\mathbf{I}_{n\times n} & \mathbf{0}_{n\times n} \\ -\boldsymbol{B} & -\boldsymbol{G} & \mathbf{0}_{n\times n} & -\mathbf{I}_{n\times n} \end{bmatrix} = \begin{bmatrix} \boldsymbol{M} & -\mathbf{I}_{2n\times 2n} \end{bmatrix}. \qquad (17)$$

We define the $2n \times 2n$ matrix $\boldsymbol{M}$ as the linear power flow matrix. *This matrix defines the linear power flow model*

$$\boxed{\begin{bmatrix} \boldsymbol{p} \\ \boldsymbol{q} \end{bmatrix} = \begin{bmatrix} \boldsymbol{G} & -\boldsymbol{B} \\ -\boldsymbol{B} & -\boldsymbol{G} \end{bmatrix} \begin{bmatrix} \boldsymbol{\epsilon} \\ \boldsymbol{\theta} \end{bmatrix}} \qquad (18)$$

where $\boldsymbol{\epsilon} := \boldsymbol{v} - \mathbf{1}$. If the network is a tree with n non-reference nodes and n edges, the inverse of the linear power flow matrix $\boldsymbol{M}$ is given in closed form as

$$\boldsymbol{M}^{-1} := \begin{bmatrix} \boldsymbol{G} & -\boldsymbol{B} \\ -\boldsymbol{B} & -\boldsymbol{G} \end{bmatrix}^{-1} = \begin{bmatrix} \boldsymbol{R} & \boldsymbol{X} \\ \boldsymbol{X} & -\boldsymbol{R} \end{bmatrix}, \qquad (19)$$

where $\boldsymbol{R}, \boldsymbol{X} \succ 0$ are resistance and reactance matrices. Thus,

$$\boxed{\begin{bmatrix} \boldsymbol{\epsilon} \\ \boldsymbol{\theta} \end{bmatrix} = \begin{bmatrix} \boldsymbol{R} & \boldsymbol{X} \\ \boldsymbol{X} & -\boldsymbol{R} \end{bmatrix} \begin{bmatrix} \boldsymbol{p} \\ \boldsymbol{q} \end{bmatrix}}, \qquad (20)$$

where $\boldsymbol{\epsilon} := \boldsymbol{v} - \mathbf{1}$.

Proof. The *linear manifold tangent* to $\mathcal{M}$ at a nominal operating point $\boldsymbol{x}_\bullet$ is given by

$$\mathcal{M}_\bullet := \left\{ \boldsymbol{x} \in \mathbb{R}^{4n} : \boldsymbol{F}(\boldsymbol{x}_\bullet)\left(\boldsymbol{x} - \boldsymbol{x}_\bullet\right) = \mathbf{0}_{2n} \right\}, \qquad (21)$$

where

$$\boldsymbol{F}(\boldsymbol{x}_\bullet) = \begin{bmatrix} \frac{\partial \mathcal{F}}{\partial \boldsymbol{v}}(\boldsymbol{x}_\bullet) & \frac{\partial \mathcal{F}}{\partial \boldsymbol{\theta}}(\boldsymbol{x}_\bullet) & \frac{\partial \mathcal{F}}{\partial \boldsymbol{p}}(\boldsymbol{x}_\bullet) & \frac{\partial \mathcal{F}}{\partial \boldsymbol{q}}(\boldsymbol{x}_\bullet) \end{bmatrix} \qquad (22a)$$

$$= \begin{bmatrix} \mathsf{Re}\,\frac{\partial \boldsymbol{s}}{\partial \boldsymbol{v}}\left(\boldsymbol{u}_\bullet\right) & \mathsf{Re}\,\frac{\partial \boldsymbol{s}}{\partial \boldsymbol{\theta}}\left(\boldsymbol{u}_\bullet\right) & -\mathbf{I}_n & \mathbf{0}_n \\ \mathsf{Im}\,\frac{\partial \boldsymbol{s}}{\partial \boldsymbol{v}}\left(\boldsymbol{u}_\bullet\right) & \mathsf{Im}\,\frac{\partial \boldsymbol{s}}{\partial \boldsymbol{\theta}}\left(\boldsymbol{u}_\bullet\right) & \mathbf{0}_n & -\mathbf{I}_n \end{bmatrix} \qquad (22b)$$

$$= \begin{bmatrix} \frac{\partial \boldsymbol{p}}{\partial \boldsymbol{v}}(\boldsymbol{u}_\bullet) & \frac{\partial \boldsymbol{p}}{\partial \boldsymbol{\theta}}(\boldsymbol{u}_\bullet) & -\mathbf{I}_n & \mathbf{0}_n \\ \frac{\partial \boldsymbol{q}}{\partial \boldsymbol{v}}(\boldsymbol{u}_\bullet) & \frac{\partial \boldsymbol{q}}{\partial \boldsymbol{\theta}}(\boldsymbol{u}_\bullet) & \mathbf{0}_n & -\mathbf{I}_n \end{bmatrix}. \qquad (22c)$$

Let $\boldsymbol{\omega} := \boldsymbol{\gamma} + \jmath\boldsymbol{\beta} \in \mathbb{C}^n$ denote the vector of self-admittances of each node. Then, following [17, 5.10], the Jacobian of complex power injections with respect to voltage phase angles is

$$\begin{aligned} \frac{\partial \boldsymbol{s}}{\partial \boldsymbol{\theta}}\left(\boldsymbol{u}_\star\right) &= \jmath\,\mathsf{diag}\left(\boldsymbol{u}_\star\right)\left(\mathsf{diag}\left(\underline{\boldsymbol{Y}}\underline{\boldsymbol{u}_\star}\right) - \underline{\boldsymbol{Y}}\,\mathsf{diag}\left(\underline{\boldsymbol{u}_\star}\right)\right) \\ &= \jmath\mathbf{I}_{n\times n}\left(\mathsf{diag}\left(\underline{\boldsymbol{Y}}\mathbf{1}_n\right) - \underline{\boldsymbol{Y}}\mathbf{I}_{n\times n}\right) \\ &= \jmath\left(\mathsf{diag}\left(\underline{\boldsymbol{\omega}}\right) - \underline{\boldsymbol{Y}}\right); \end{aligned}$$

similarly, with respect to the voltage magnitudes

$$\begin{aligned} \frac{\partial \boldsymbol{s}}{\partial \boldsymbol{v}}(\boldsymbol{u}_\star) &= \mathsf{diag}\left(\boldsymbol{u}\right)\left(\mathsf{diag}\left(\underline{\boldsymbol{Y}}\underline{\boldsymbol{u}}\right) + \underline{\boldsymbol{Y}}\,\mathsf{diag}\left(\underline{\boldsymbol{u}}\right)\right)\mathsf{diag}\left(\boldsymbol{v}\right)^{-1} \\ &= \mathbf{I}_{n\times n}\left(\mathsf{diag}\left(\underline{\boldsymbol{Y}}\mathbf{1}_n\right) + \underline{\boldsymbol{Y}}\mathbf{I}_{n\times n}\right)\mathbf{I}_{n\times n}^{-1} \\ &= \mathsf{diag}\left(\underline{\boldsymbol{\omega}}\right) + \underline{\boldsymbol{Y}}. \end{aligned}$$

Substituting the above into (22) yields the desired result.

Now, we show that the assumption that the network is a tree, or radial, ensures that the linear power flow matrix $\boldsymbol{M}$ can be inverted in the analytical form given in (19). Note that as $\boldsymbol{G} \succ 0$, both Schur complements of $\boldsymbol{M}$ exist.

Moreover, setting $\boldsymbol{S}$ to be the Schur Complement of $\boldsymbol{M}$ in $-\boldsymbol{G}$, we obtain that the inverse of $\boldsymbol{S}$ is, in fact, the *resistance matrix* $\boldsymbol{R}$, since

$$\begin{aligned} \boldsymbol{S}^{-1} &:= \left(\boldsymbol{G} + \boldsymbol{B}\boldsymbol{G}^{-1}\boldsymbol{B}\right)^{-1} \\ &= \left(\boldsymbol{A}^\top\,\mathsf{diag}(\boldsymbol{g})\boldsymbol{A} + \boldsymbol{A}^\top\,\mathsf{diag}(\boldsymbol{b})\,\mathsf{diag}(\boldsymbol{g})^{-1}\,\mathsf{diag}(\boldsymbol{b})\boldsymbol{A}\right)^{-1} \\ &= \left(\boldsymbol{A}^\top\,\mathsf{diag}\left(\left[\frac{g_{ij}^2 + b_{ij}^2}{g_{ij}}\right]_{ij\in\mathcal{E}}\right)\boldsymbol{A}\right)^{-1} \\ &= \boldsymbol{A}^{-1}\,\mathsf{diag}\left(\left[\frac{g_{ij}}{g_{ij}^2 + b_{ij}^2}\right]_{ij\in\mathcal{E}}\right)\boldsymbol{A}^{-\top} \\ &= \boldsymbol{A}^{-1}\,\mathsf{diag}\left(\boldsymbol{r}\right)\boldsymbol{A}^{-\top} := \boldsymbol{R}. \end{aligned}$$

Further calculation reveals that the Schur complement of $\boldsymbol{M}$ in $\boldsymbol{G}$ is $-\boldsymbol{S}$.

Therefore, applying well-known block matrix inversion identities yields

$$\boldsymbol{F}^{-1} = \begin{bmatrix} \boldsymbol{G} & -\boldsymbol{B} \\ -\boldsymbol{B} & -\boldsymbol{G} \end{bmatrix}^{-1} \qquad (23a)$$

$$= \begin{bmatrix} \boldsymbol{S}^{-1} & \mathbf{0}_{n\times n} \\ \mathbf{0}_{n\times n} & -\boldsymbol{S}^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{I}_{n\times n} & -\boldsymbol{B}\boldsymbol{G}^{-1} \\ \boldsymbol{B}\boldsymbol{G}^{-1} & \mathbf{I}_{n\times n} \end{bmatrix} \qquad (23b)$$

$$= \begin{bmatrix} \boldsymbol{S}^{-1} & -\boldsymbol{S}^{-1}\boldsymbol{B}\boldsymbol{G}^{-1} \\ -\boldsymbol{S}^{-1}\boldsymbol{B}\boldsymbol{G}^{-1} & -\boldsymbol{S}^{-1} \end{bmatrix}. \qquad (23c)$$

Finally, we must compute the off-diagonal matrices of (23). We obtain

$$\begin{aligned} -\boldsymbol{S}^{-1}\boldsymbol{B}\boldsymbol{G}^{-1} &= -\boldsymbol{A}^{-1}\,\mathsf{diag}\left(\left[\frac{g_{ij}}{g_{ij}^2 + b_{ij}^2}\right]_{ij}\right)\mathsf{diag}(\boldsymbol{b} \oslash \boldsymbol{g})\boldsymbol{A}^{-\top} \\ &= \boldsymbol{A}^{-1}\,\mathsf{diag}\left(\left[\frac{-b_{ij}}{g_{ij}^2 + b_{ij}^2}\right]_{ij\in\mathcal{E}}\right)\boldsymbol{A}^{-\top} \\ &= \boldsymbol{A}^{-1}\,\mathsf{diag}\left(\boldsymbol{x}\right)\boldsymbol{A}^{-\top} := \boldsymbol{X}, \end{aligned}$$

where $\oslash$ denotes element-wise division. Therefore, we have that the inverse of the linear power flow matrix $\boldsymbol{M}$ is

$$\begin{bmatrix} \boldsymbol{G} & -\boldsymbol{B} \\ -\boldsymbol{B} & -\boldsymbol{G} \end{bmatrix}^{-1} = \begin{bmatrix} \boldsymbol{S}^{-1} & -\boldsymbol{S}^{-1}\boldsymbol{B}\boldsymbol{G}^{-1} \\ -\boldsymbol{S}^{-1}\boldsymbol{B}\boldsymbol{G}^{-1} & -\boldsymbol{S}^{-1} \end{bmatrix} \qquad (24a)$$

$$= \begin{bmatrix} \boldsymbol{R} & \boldsymbol{X} \\ \boldsymbol{X} & -\boldsymbol{R} \end{bmatrix}, \qquad (24b)$$

as desired. □

B. Proof of Theorem 2

Proof. First, bounding the operator norm uniformly across $\boldsymbol{M}_{ij}$, we have

$$\begin{aligned} ||\boldsymbol{M}_{ij}|| &= ||\boldsymbol{\Upsilon}_{ij}|| \cdot ||\boldsymbol{E}_{ij}|| \\ &= 2\sqrt{\lambda_{\mathsf{max}}\left(\boldsymbol{\Upsilon}_{ij}^\top\boldsymbol{\Upsilon}_{ij}\right)} = 2\sqrt{g_{ij}^2 + b_{ij}^2} \\ &\leq 2\sup_{(i,j)\in\mathcal{E}} |w_{ij}| \leq 2 := R. \end{aligned}$$

On the other hand, denoting by $\boldsymbol{Z}^*$ the conjugate transpose of a matrix $\boldsymbol{Z}$, the matrix variance statistic $\nu\left(\bar{\boldsymbol{Y}}\right) := \|\mathsf{E}[\bar{\boldsymbol{Y}}^2]\|$ (see [3]) can be expressed as follows:

$$\begin{aligned}
\nu\left(\bar{\boldsymbol{Y}}\right) &= \left\|\sum_{(i,j)\in\mathcal{E}} \mathsf{E}\left[\left(\boldsymbol{\Upsilon}_{ij}\otimes\boldsymbol{E}_{ij}\right)\left(\boldsymbol{\Upsilon}_{ij}^\top\otimes\boldsymbol{E}_{ij}^\top\right)\right]\right\| \\
&\overset{(1)}{=} \left\|\sum_{(i,j)\in\mathcal{E}} \mathsf{E}\left[\left(\boldsymbol{\Upsilon}_{ij}\boldsymbol{\Upsilon}_{ij}^\top\right)\otimes\left(\boldsymbol{E}_{ij}\boldsymbol{E}_{ij}^\top\right)\right]\right\| \\
&= 2\left\|\sum_{(i,j)\in\mathcal{E}} \begin{bmatrix}\boldsymbol{E}_{ij} & 0\\ 0 & \boldsymbol{E}_{ij}\end{bmatrix}\right\| \\
&= 2\left\|\begin{bmatrix}\boldsymbol{A}^\top\boldsymbol{A} & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{A}^\top\boldsymbol{A}\end{bmatrix}\right\| \\
&\overset{(2)}{=} 2\|\boldsymbol{A}^\top\boldsymbol{A}\| \\
&\overset{(3)}{\le} 4\Delta.
\end{aligned} \tag{25}$$

In this chain of equalities, step (1) is by the mixed-product property of the Kronecker product, namely, $(\boldsymbol{A}\otimes\boldsymbol{B})(\boldsymbol{C}\otimes\boldsymbol{D}) = (\boldsymbol{AC})\otimes(\boldsymbol{BD})$ for any matrices $\boldsymbol{A},\boldsymbol{B},\boldsymbol{C},\boldsymbol{D}$ with appropriate dimensions; step (2) follows since the operator norm of a block-diagonal matrix is the largest operator norm of any block.

For step (3), we observe that $\boldsymbol{A}^\top\boldsymbol{A}\in\mathbb{R}^{n\times n}$ is the graph Laplacian matrix of the simple undirected graph corresponding to the network topology. We use that $\boldsymbol{Y}=\boldsymbol{D}-\boldsymbol{M}$ where $\boldsymbol{M}$ is the adjacency matrix; since $\boldsymbol{Y}\succeq 0$ and $\boldsymbol{D}+\boldsymbol{M}\succeq 0$ for the "signless" Laplacian matrix, see e.g. [18], we conclude that

$$-\boldsymbol{D}\preceq\boldsymbol{M}\preceq\boldsymbol{D}, \quad \text{thus} \quad \|\boldsymbol{M}\|\le\|\boldsymbol{D}\|$$

and therefore $\|\boldsymbol{Y}\|\le\|\boldsymbol{D}\|+\|\boldsymbol{M}\|\le 2\|\boldsymbol{D}\|=2\Delta$. A direct application of matrix Bernstein gives the desired result. □

C. Proof of Theorem 3

Proof. For each line $l := (i,j)\in\mathcal{E}$, let $\boldsymbol{M}_l = \xi_l y_l \boldsymbol{a}_l\boldsymbol{a}_l^\top$ denote the summand matrices associated with each line in the matrix series (5). Observe that

$$\mathsf{E}\,\boldsymbol{M}_l = \mathfrak{p}_l y_l \boldsymbol{a}_l\boldsymbol{a}_l^\top$$

is the expectation of each elementary admittance matrix. With this, define the *centered* elementary admittance matrices as

$$\tilde{\boldsymbol{M}}_l := \boldsymbol{M}_l - \mathsf{E}\,\boldsymbol{M}_l = (\xi_l-\mathfrak{p}_l)\, y_l \boldsymbol{a}_l\boldsymbol{a}_l^\top.$$

The centered admittance matrix of the network is then

$$\tilde{\boldsymbol{Y}} = \boldsymbol{Y} - \mathsf{E}\,\boldsymbol{Y} = \sum_{l\in E}\tilde{\boldsymbol{M}}_l = \sum_{l\in E}(\xi_l-\mathfrak{p}_l)\, y_l \boldsymbol{a}_l\boldsymbol{a}_l^\top.$$

Naturally, we have that $\mathsf{E}\,\tilde{\boldsymbol{M}}_l=\boldsymbol{0}$ for any l, and $\mathsf{E}\,\tilde{\boldsymbol{Y}}=\boldsymbol{0}$.

Furthermore, for each line l, we have the upper bound

$$\begin{aligned}
\|\tilde{\boldsymbol{M}}_l\| &= \left|\left|(\xi_l-\mathfrak{p}_l)\, y_l \boldsymbol{a}_l\boldsymbol{a}_l^\top\right|\right| \\
&\overset{(1)}{=} |\xi_l-\mathfrak{p}_l|\cdot|y_l|\left|\left|\boldsymbol{a}_l\boldsymbol{a}_l^\top\right|\right|_{\mathsf{op}} \\
&\overset{(2)}{\le} 2,
\end{aligned}$$

where step (1) is by absolute homogeneity and step (2) is due to the fact that $\left|\left|\boldsymbol{a}_l\boldsymbol{a}_l^\top\right|\right| = ||\boldsymbol{a}_l||_2^2 = 2$, $|y|_l\le 1$ and

$$|\xi_l-\mathfrak{p}_l| \le \max_l |\xi_l-\mathfrak{p}_l| \le \max_l\{\max\{\mathfrak{p}_l, 1-\mathfrak{p}_l\}\}\le 1.$$

Now, we compute the matrix-valued variance of the centered admittance matrix under random contingencies. We have

$$\begin{aligned}
\boldsymbol{V} := \mathsf{E}\,\tilde{\boldsymbol{Y}}\tilde{\boldsymbol{Y}}^* &\overset{(1)}{=} \sum_{l\in\mathcal{E}}\mathsf{E}\,\tilde{\boldsymbol{M}}_l\tilde{\boldsymbol{M}}_l^* \\
&= \sum_{l\in\mathcal{E}}\mathsf{E}\left[(\xi_l-\mathfrak{p}_l)^2|y_l|^2\boldsymbol{a}_l\boldsymbol{a}_l^\top\boldsymbol{a}_l\boldsymbol{a}_l^\top\right] \\
&\overset{(2)}{=} \sum_{l\in\mathcal{E}} 2\mathfrak{p}_l(1-\mathfrak{p}_l)|y_l|^2\boldsymbol{a}_l\boldsymbol{a}_l^\top.
\end{aligned}$$

In the above display, step (1) is by independence of the summands, and step (2) is by definition of the Bernoulli variance $\mathsf{E}\,(\xi_l-\mathfrak{p}_l)^2=\mathfrak{p}_l(1-\mathfrak{p}_l)$, and $\boldsymbol{a}_l^\top\boldsymbol{a}_l=2$. Observe that the matrix-valued variance $\boldsymbol{V}$ is itself a graph Laplacian matrix that describes a graph with the same topology as the power network, with the contingency factors as line weights. This Laplacian can be written as

$$\boldsymbol{L} := \boldsymbol{A}^\top\boldsymbol{C}\boldsymbol{A}, \qquad \boldsymbol{C}=\mathsf{diag}(\boldsymbol{c}).$$

Now, we compute the intrinsic dimension of the matrix-valued variance, which is defined as follows.

Definition A.1 (Intrinsic dimension)**.** For any matrix $\boldsymbol{A}$, let

$$\mathsf{intdim}(\boldsymbol{A}) := \frac{\mathsf{tr}(\boldsymbol{A})}{||\boldsymbol{A}||}.$$

First, note that we have

$$\begin{aligned}
\mathsf{intdim}(\boldsymbol{V}) := \frac{\mathsf{tr}(\boldsymbol{V})}{||\boldsymbol{V}||} &= \frac{2\sum_{l\in\mathcal{E}}\mathfrak{p}_l(1-\mathfrak{p}_l)|y_l|^2\,\mathsf{tr}\left(\boldsymbol{a}_l\boldsymbol{a}_l^\top\right)}{||\boldsymbol{V}||} \\
&= \frac{4\sum_{l\in\mathcal{E}}\mathfrak{p}_l(1-\mathfrak{p}_l)|y_l|^2}{||\boldsymbol{V}||}
\end{aligned}$$

The second equality is due to the linearity of the trace, and the third is due to the fact that $\mathsf{tr}\,\boldsymbol{a}_l\boldsymbol{a}_l^\top=2$. From this juncture, we can now bound the operator norm of the matrix-valued variance as follows. First, note that since $\boldsymbol{V}$ is a Laplacian matrix, it can be written as $\boldsymbol{V}:=\boldsymbol{L}=\boldsymbol{D}-\boldsymbol{M}$, where $\boldsymbol{M}\in\mathbb{R}^{n\times n}$ is an adjacency matrix with $M_{ij}=-c_{ij}$ if $(i,j)\in\mathcal{E}$, and zeros along the diagonal, and $D_{ii}:=\sum_{l:l\ni i}c_l=d_i(\boldsymbol{c})$. We obtain

$$\begin{aligned}
\nu = ||\boldsymbol{V}|| = ||\boldsymbol{L}|| &= ||\boldsymbol{D}-\boldsymbol{M}|| \\
&\overset{(1)}{\le} \underbrace{||\boldsymbol{D}||}_{=\Delta_{\boldsymbol{c}}} + ||\boldsymbol{M}|| \\
&\overset{(2)}{\le} \Delta_{\boldsymbol{c}} + \sqrt{||\boldsymbol{M}||_1\,||\boldsymbol{M}||_\infty} \\
&\overset{(3)}{=} \Delta_{\boldsymbol{c}} + \sqrt{\left(\max_j\sum_i|M_{ij}|\right)\left(\max_i\sum_j|M_{ij}|\right)} \\
&= 2\Delta_{\boldsymbol{c}},
\end{aligned}$$

where step (1) is by the triangle inequality and the fact that $\boldsymbol{D}$ is diagonal, and step (2) is because for any matrix $\boldsymbol{A}$

$$\|\boldsymbol{A}\| = \sqrt{\lambda_{\max}(\boldsymbol{A}^*\boldsymbol{A})} \le \sqrt{\|\boldsymbol{A}^*\boldsymbol{A}\|_\infty} \le \sqrt{\|\boldsymbol{A}\|_1\|\boldsymbol{A}\|_\infty},$$

and step (3) is by definition of the matrix norms $||\cdot||_1, ||\cdot||_\infty$. The final equality follows by noting that

$$||\boldsymbol{M}||_1 = ||\boldsymbol{M}||_\infty = ||\boldsymbol{D}|| = \Delta_{\boldsymbol{c}}.$$

Furthermore, we can lower bound the spectral norm by considering the Rayleigh quotient; as $\boldsymbol{V}\succeq\boldsymbol{0}$, we can write $||\boldsymbol{V}||_2 := \sup_{||\boldsymbol{x}||\le 1} \boldsymbol{x}^\top \boldsymbol{V}\boldsymbol{x}$. Take $\boldsymbol{x} \leftarrow \boldsymbol{e}_i$, then we always have the lower bound

$$||\boldsymbol{V}|| = \sup_{||\boldsymbol{x}||\le 1} \boldsymbol{x}^\top \boldsymbol{V}\boldsymbol{x} \ge \sup_i \boldsymbol{e}_i^\top \boldsymbol{V}\boldsymbol{e}_i = \max_i d_i(\boldsymbol{c}) := \Delta_{\boldsymbol{c}}.$$

Thus,

$$\frac{\sum_i d_i(\boldsymbol{c})}{2\Delta(\boldsymbol{c})} \le \mathsf{intdim}(\boldsymbol{V}) \le \frac{\sum_i d_i(\boldsymbol{c})}{\Delta(\boldsymbol{c})} \le n-1,$$

since $\mathsf{rank}(\boldsymbol{V}) \le n-1$ for Laplacian matrices $\boldsymbol{V}$.

We now prepare to invoke the matrix Bernstein inequality [3]. For non-Hermitian matrices such as (5), the intrinsic dimension factor d is given as

$$d = \frac{2\,\mathsf{tr}(\boldsymbol{V})}{\|\boldsymbol{V}\|} = 2\,\mathsf{intdim}(\boldsymbol{V}) \le 2\frac{\sum_i d_i(\boldsymbol{c})}{\Delta(\boldsymbol{c})}.$$

Consequently, for all $t \ge \sqrt{\nu} + L/3 = \sqrt{2\Delta_{\boldsymbol{c}}} + 2/3$, we have

$$\begin{aligned}\Pr(\|\tilde{\boldsymbol{Y}}\| \ge t) &\le 4d\exp\left(\frac{-t^2}{2\,(\nu + Lt/3)}\right)\\ &\le 8\left(\frac{\sum_i d_i(\boldsymbol{c})}{\Delta(\boldsymbol{c})}\right)\exp\left(\frac{-t^2}{4\,(\Delta_{\boldsymbol{c}} + t/3)}\right),\end{aligned}$$

which is the desired result for the tails (7). To yield the expectation bound (8), see [3, Sec. 7.7.4]. Set $\bar{D}$ as in (6). Then, a short calculation reveals

$$\begin{aligned}\mathsf{E}\left[\|\tilde{\boldsymbol{Y}}\|\right] &\le \sqrt{2\nu\log(1+d)} + \frac{2}{3}L\log(1+d) + 4\sqrt{\nu} + \frac{8}{3}L\\ &\le C\left(\sqrt{\nu\log(1+d)} + L\log(1+d)\right)\\ &\le C\left(\sqrt{2\Delta_{\boldsymbol{c}}\log(1+2\bar{D})} + 2\log(1+2\bar{D})\right)\end{aligned}$$

for some universal constant $C > 0$, which is the desired result (8). This completes the proof of Theorem 3. $\square$

D. Proof of Lemma 1

Proof. By assumption, each $\boldsymbol{g} \overset{(\mathsf{d})}{=} \boldsymbol{Q}\boldsymbol{z}$, $\boldsymbol{b} \overset{(\mathsf{d})}{=} \boldsymbol{Q}\boldsymbol{z}$, where $\boldsymbol{z} \overset{(\mathsf{iid})}{\sim} \textsc{Normal}(0, \frac{1}{2}\mathbf{I})$ is a vector of iid Gaussians and $\boldsymbol{Q}_g, \boldsymbol{Q}_b \in \mathbb{R}^{n\times n}$ are orthonormal matrices.

First, note that

$$||\boldsymbol{M}_{ij}|| = ||\boldsymbol{\Upsilon}_{ij}||\,||\boldsymbol{E}_{ij}|| \le 4 := L.$$

The positive semidefinite upper bound for the matrix-valued variance $\mathsf{E}[\boldsymbol{F}\boldsymbol{F}^*] = \mathsf{E}[\boldsymbol{F}^*\boldsymbol{F}]$ is then

$$\begin{aligned}\mathsf{E}[\boldsymbol{F}\boldsymbol{F}^*] &= \sum_{ij\in\mathcal{E}} \mathsf{E}\left[\boldsymbol{M}_{ij}\boldsymbol{M}_{ij}^*\right]\\ &= \sum_{ij\in\mathcal{E}} \mathsf{E}\left[\begin{bmatrix} 2\boldsymbol{E}_{ij}g_{ij}^2 + 2\boldsymbol{E}_{ij}b_{ij}^2 & \mathbf{0}\\ \mathbf{0} & 2\boldsymbol{E}_{ij}b_{ij}^2 + 2\boldsymbol{E}_{ij}g_{ij}^2\end{bmatrix}\right]\\ &= 2\sum_{ij\in\mathcal{E}} \begin{bmatrix} \boldsymbol{E}_{ij}\left(\mathsf{E}\left[g_{ij}^2 + b_{ij}^2\right]\right) & \mathbf{0}\\ \mathbf{0} & \boldsymbol{E}_{ij}\left(\mathsf{E}\left[g_{ij}^2 + b_{ij}^2\right]\right)\end{bmatrix}\\ &\overset{(1)}{\preceq} \frac{2}{n}\sum_{ij\in\mathcal{E}} \begin{bmatrix} \boldsymbol{E}_{ij} & \mathbf{0}\\ \mathbf{0} & \boldsymbol{E}_{ij}\end{bmatrix}\\ &\overset{(2)}{=} \frac{2}{n}\begin{bmatrix} \boldsymbol{A}^\top\boldsymbol{A} & \mathbf{0}\\ \mathbf{0} & \boldsymbol{A}^\top\boldsymbol{A}\end{bmatrix} = \frac{2}{n}\mathbf{I}_2 \otimes \boldsymbol{A}^\top\boldsymbol{A},\end{aligned}$$

where step (1) stems from $\mathsf{E}\,g_{ij}^2 \le 1$ and $\mathsf{E}\,b_{ij}^2 \le 1$ by assumption of the uniform distribution over the unit sphere. $\square$

E. Proof of Theorem 4

Proof. Under the zero-mean hypothesis, the PSD upper bound for the matrix-valued variance $\mathsf{E}[\boldsymbol{F}\boldsymbol{F}^*] = \mathsf{E}[\boldsymbol{F}^*\boldsymbol{F}]$ is

$$\begin{aligned}\mathsf{E}[\boldsymbol{F}\boldsymbol{F}^*] &= \sum_{ij\in\mathcal{E}} \mathsf{E}\left[\boldsymbol{M}_{ij}\boldsymbol{M}_{ij}^*\right]\\ &= 2\sum_{ij\in\mathcal{E}} \begin{bmatrix} \boldsymbol{E}_{ij}\left(\mathsf{E}\left[g_{ij}^2 + b_{ij}^2\right]\right) & \mathbf{0}\\ \mathbf{0} & \boldsymbol{E}_{ij}\left(\mathsf{E}\left[g_{ij}^2 + b_{ij}^2\right]\right)\end{bmatrix}\\ &\overset{(1)}{\preceq} 4\Delta^2\sum_{ij\in\mathcal{E}} \begin{bmatrix} \boldsymbol{E}_{ij} & \mathbf{0}\\ \mathbf{0} & \boldsymbol{E}_{ij}\end{bmatrix}\\ &= 4\Delta^2\begin{bmatrix} \boldsymbol{A}^\top\boldsymbol{A} & \mathbf{0}\\ \mathbf{0} & \boldsymbol{A}^\top\boldsymbol{A}\end{bmatrix} = 4\Delta^2\mathbf{I}_2 \otimes \boldsymbol{A}^\top\boldsymbol{A},\end{aligned}$$

where step (1) stems from the fact that $\mathsf{E}\left[g_{ij}^2\right] \le \Delta^2$ and $\mathsf{E}\left[b_{ij}^2\right] \le \Delta^2$ by assumption of bounded model uncertainty. Then, the matrix variance statistic is bounded as

$$\nu \le 4\Delta^2\left|\left|\boldsymbol{A}^\top\boldsymbol{A}\right|\right| \le 4\Delta^2 n.$$

Now, the matrix Bernstein inequality completes the proof. $\square$

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I. Review #25A

A. Paper summary

This paper presents conservative probabilistic bounds for the spectrum of the admittance matrix and classical linear power flow models under uncertain network parameters. However, it isn't clear why this is being presented.

B. Comments for authors

Certainly this is a mathematically rigorous paper! As such, there will certainly be interest the some people in its contributions. However, probably most will wonder at its contribution to practical power system design and operation. I would question the assumption in the paper's first paragraph that the structure of a power system might be uncertain, estimated, controlled or optimized. I certainly could be, but the paper doesn't provide strong justification for this being at all a common situation. The Applications in Section VI are interesting, but don't really explain how the paper's results can address these issues.

C. Summarize your recommendation in one to two sentences

People who are interested in strong mathematical rigor will likely enjoy this paper. However, its practical contributions appear to be limited.

— *Anonymous reviewer*

II. Review #25B

A. Paper summary

This paper develops a theoretical framework for understanding how uncertainty in the electrical parameters of the branches of an electric power network propagates through the admittance matrix, in the context of linear approximations to the non-linear AC power flow equations The authors model branch admittances as independent, bounded random variables and leverage matrix concentration inequalities to derive probabilistic upper bounds on the spectral norm of the admittance matrix perturbation. A key contribution is the introduction of nodal criticality, a network-theoretic metric that identifies nodes most exposed to local parameter volatility. The authors prove that the spectral perturbation of the admittance matrix scales with this criticality. A limited collectio of results on IEEE benchmarks confirm that these bounds accurately capture the qualitative scaling of network uncertainty while also highlighting a significant degree of conservativeness.

B. Comments for authors

The consideration of branch admittances as random variables rather than known parameters is a very interesting starting point for innovation in the field of probabilistic methods applied to power systems. That being said, my general feeling is that the setting considered here is much more attractive from an abstract theoretical point of view rather than in relation to the practical operation of electric power systems.

A first fundamental issue is the relevance/suitability of the DC power flow and/or LinDistFlow approximations for N-1 (let alone N-k) contingency analysis. These linear approximations are already a significant simplification under normal operating conditions, ignoring reactive power entirely and assuming flat voltage profiles. These limitations become considerably more severe under N-k contingency scenarios, precisely the setting the paper targets. Especially under, rare yet challenging N-k contingency events such as the ones mentioned in the paper (e.g., natural disasters), the behaviour of the grid is often dictated by fast dynamic phenomena completely outside the scope of any static simulation. Even in the context of steady-state security assessment, any practically meaningful analysis requires to closely monitor voltage/reactive power limits as well as to explicitly account the (semi-automated) responses of the growing population of the grid's automatic controllers (from simple tap-changers to elaborate remedial action schemes). Ultimately, deriving an (admittedly conservative) mathematical bound for a model that excludes these primary drivers in the grid's post-contingency response appears to me as an interesting theoretical exercise rather than a potentially useable contribution.

It would perhaps be more physically intuitive to reconsider admittances as consisting of a fixed nominal value plus a random error term, representing parameter estimation drift or data management inaccuracies. Framing the problem this way would allow the spectral bounds to be interpreted as a measure of the network model's sensitivity to measurement/data uncertainty.

Even while adopting the linear approximations as acceptable, the translation of the presented findings/results into operationally meaningful guarantees for power system operation seems really not straightforward. In the linear approximation context, the most obvious quantity of interest is the flow through individual transmission lines, which is proportional to the angle difference across each pair of connected buses. The paper's bounds, being spectral in nature, refer to the aggregate shift of the entire voltage angle vector (given fixed power injections) rather than these individual branch quantities. While a conversion between the two is mathematically possible, to the best of my understanding, it would result in a bound that is identical for every branch in the network. This uniformity seems to be a meaningful limitation. A uniform bound effectively treats a critical, heavily loaded transmission corridor the same as a peripheral, lightly loaded branch. In practice, operators care most about the branches closest to their thermal limits, and a useful probabilistic guarantee should reflect the heterogeneity of the network.

An additional questionable point is the core assumption that line outages occur independently. This is formalized through independent Bernoulli random variables assigned to each branch. To the best of my understanding, this assumption is mathematically crucial in the sense that it is enables the application of the matrix Bernstein inequality. Unfortunately, it is clearly invalid for the types of events mentioned by the authors in framing their work (natural disasters, cascading faults, etc.). Weather-driven outages, protection system cascades, and common right-of-way failures are inherently correlated, and it is precisely these common-mode events that motivate N-k analysis in the first place. One possible way out here would be to reframe the methods and results as statements about independently sampled network configurations rather than physical outage sequences. Kindly note here that the space of topologies operators actually consider is far more structured and concentrated than an independent edge sampling model implies. Restricting the Bernoulli model to this set is both operationally natural and mathematically compatible with the existing framework, provided the events within the curated sub-set can be treated as approximately independent. This would simultaneously sharpen the bounds, by reducing the effective support of the probability measure and therefore Δc, and restore operational interpretability, by ensuring the probabilistic guarantees are statements over scenarios the operator actually considers credible. The primary remaining challenge is that the validity of the guarantee depends on the completeness of the contingency list, as any credible event outside it would fall outside the probability measure entirely.

The numerical validation presented in Section V, while demonstrating that the bounds are mathematically correct, simultaneously reveals a significant limitation of the proposed framework. The primary bound developed in Theorem 3 is shown to be approximately 2-2.5x conservative on the chosen benchmark. In many engineering contexts, a factor of two conservativeness might be acceptable. In power systems operations, however, it is particularly problematic. Transmission systems are deliberately operated close to their limits for economic reasons, and the operational margins available to accommodate uncertainty are typically kept as small as necessary. Taken together, these observations raise a fundamental question about the practical utility of the proposed framework. It is not clear what operational decision a system operator could make better, or more confidently, armed with these bounds compared to existing contingency analysis tools.

I would finally like to acknowledge the considerable care and rigor with which this paper is written. The mathematical exposition is dense but internally consistent and the proofs are thorough and well-documented. The authors have done a commendable job on this. That being said, the paper is submitted to a power systems conference and will primarily be read by a power systems audience, and in this context the presentation poses a significant accessibility challenge. The paper draws heavily on tools from applied probability and random matrix theory that are not part of the standard toolkit of most power systems researchers and practitioners. While the introductory section makes a genuine effort, the paper transitions quickly into a level of mathematical abstraction that is likely to lose a significant portion of its intended audience before the main results are reached. A more accessible presentation might include intuitive interpretations of the key quantities, particularly nodal criticality and the matrix variance statistic, alongside their formal definitions, as well as worked numerical examples that connect the theoretical bounds to recognizable power systems concepts earlier in the paper. Bridging this gap more deliberately would significantly strengthen the paper's impact and allow its genuine theoretical contributions to reach the audience that stands to benefit most from them.

C. Summarize your recommendation in one to two sentences

The paper presents a mathematically rigorous framework for analysing uncertainty in linearized power flow models using tools from random matrix theory. However, the modelling assumptions and the nature of the resulting bounds make it difficult to translate the results into operationally meaningful guarantees for practical power system security assessment.

— *Efthymios Karangelos*

III. Review #25C

A. Paper summary

This work proposes probabilistic bounds for the admittance matrix and classical linear power flow under uncertain network parameters. The work provides a theoretical framework for error bounds of common approximations of the AC power flow equations under parameter uncertainty.

B. Comments for authors

Please consider the following comments/suggestions:

- This is an excellent contribution to the field.
- "See [8] for a discussion of the conditions on the susceptances required for $B \preceq 0$ to hold." It would be helpful if a one or two sentence summary could be included in the manuscript as opposed to completely referring the reader to [8].
- A comma is missing after "e.g." in the caption for Fig. 1.

- "B. Uncertainty contingencies." Can the authors discuss how the framework could change if removing nodes (not only lines) were to be considered? This is also a practical scenario as some buses can be de-energized during extreme events. Would it be as simple as keeping the current framework and assuming all lines connected to the node in question are also disconnected? Or are there any nuances?
- Lines/legend for Fig. 2 uses two thicknesses, however they are not very noticeable. I suggest increasing more the thickness of the thick line.
- Replace "The numerical experiments confirm that our bounds" with "The numerical experiments confirm the following:"
- The work would be stronger if applied/practical cases were included to ground the contributions for the practitioner.
- The manuscript is well-written.

C. Summarize your recommendation in one to two sentences

This is very good work! It would be stronger for the power systems community if applied/practical cases were included to ground the contributions for the practitioner.

— Patricia Hidalgo-Gonzalez

IV. Brief response to reviewers (optional)

We would like to thank the reviewers for their thorough comments. A recurring concern between reviewers was a request of a more practical application of the theory in this paper. While we agree such an experiment would be helpful for the applied community, the main contribution of this work is the theory itself and its validation. Applying this theory to such problems is itself a major contribution and warrants their own paper. We clarified this in the conclusions.

We now focus on addressing reviewer-specific comments.

Response to Reviewer #1:

The main ways the theory developed in this paper can help with practical problems are by guiding the design of problem-specific fast randomized algorithms with guarantees of their performance and giving way towards a new class of probabilistic analysis of existing algorithms or entire problem structures, see updated intro and conclusions.

Response to Reviewer #2:

The applicability of these results are not limited to any linear approximation. The central object of study for this paper is the admittance matrix, thus these results are applicable to AC power flow as well. We separately showed that the admittance matrix is isomorphic to the flat start Jacobian, thus our results on the admittance matrix also carry over to these linearizations.

I would also argue against the premise that linear approximations are unsuitable for contingency analysis given the large corpus of literature that has found approximations like DC PF to be meaningful for these studies. Indeed, DC PF is widely used for electricity market clearing and security constrained OPF in practice.

The comment on treating admittances as a fixed nominal value plus a random error term is a very useful perspective, one that we have taken recently in [1] to develop error bounds on network topology learning. In the setting of bounded error, such as from quantized measurement error, the analysis developed in this paper is more general and thus applicable.

The comment on the independence assumption is also valid. There exist methods, such as Stein's Method, to extend our results to a dependent version. However, their form would be similar to the current bounds, but certain variables are replaced with their "dependent" version. A proof of this bound would essentially start from the independent case we developed, so we leave that for future papers where we specifically address faults like natural hazards and cascade failures.

Response to Reviewer #3:

Regarding the comment on node removal, these results might not directly carry over. For intuition, note that removing lines can only monotonically decrease quantities such as average nodal degree of a graph, whereas removing nodes may increase the average nodal degree. Studying the case of removing nodes would be interesting future work.

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V. Brief summary of changes made to the final paper

Minor typos, type setting, and reference formatting were corrected.

Rearranged some content in the introduction to make motivation of this work more clear (in response to reviewer #1).

Added intuition to Assumption 1 (in response to reviewer #3).

Added short paragraph in Applications and Future Work section to clarify framing of paper (in response to reviewer #1 and #2).

Removed original Figure 1 for lack of clarity, replaced original Figure 2 with updated version on a 200 bus network, and added a new figure that validates the inequality bounds on larger networks. Adjusted text accordingly.

Fixed a factor of 2 in the bound of Theorem 2 and 3 due to the lifting of admittance matrix to real domain.

Explicitly stated zero mean and independence assumptions in a few places for clarity.