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Author
Spicer, Calum

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Higher dimensional foliated Mori theory

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Calum Spicer

Committee in charge:

Professor James McKernan, Chair
Professor Ken Intrilligator
Professor Elham Izadi
Professor Aneesh Manohar
Professor Dragos Oprea

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DEDICATION

To Rachel.

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The dissertation author was the primary investigator and author of this material.

VITA

2012	B. A. in Mathematics, Philosophy, Johns Hopkins University
2012	M. A. in Mathematics, Johns Hopkins University
2017	Ph. D. in Mathematics, University of California, San Diego

ABSTRACT OF THE DISSERTATION

Higher dimensional foliated Mori theory

by

Calum Spicer
Doctor of Philosophy in Mathematics

University of California San Diego, 2017

Professor James McKernan, Chair

We develop some foundational results in a higher dimensional foliated Mori theory, and show how these results can be used to prove a structure theorem for the Kleiman-Mori cone of curves in terms of the numerical properties of $K_{\mathcal{F}}$ for rank 2 foliations on threefolds. We also make progress toward realizing a minimal model program for rank 2 foliations on threefolds.

Chapter 1

Introduction

1.1 Statement of main results

We first state the main results to be proven here, a more complete introduction follows in the next section.

We will always work over $\mathbb{C}$. By a foliation on a normal variety X we mean a saturated subsheaf $\mathcal{F} \subset T_X$ closed under Lie bracket. Given such a pair $X, \mathcal{F}$ we define $-c_1(\mathcal{F}) = K_{\mathcal{F}}$ to be the canonical divisor of the foliation. In recent years much work has been done understanding the birational geometry of the foliation in terms of $K_{\mathcal{F}}$ when the rank of $\mathcal{F}$ is 1, especially in the case of rank 1 foliations on surfaces.

The goal of this paper is to extend this work to the case of co-rank 1 foliations, especially in the case of threefolds. An essential first step in understanding the birational geometry of a variety or foliation is a structure theorem on the closed cone of curves $\overline{NE}(X)$. Here we prove the following foliated cone theorem:

Theorem 1.1.1. *Let X be a klt, $\mathbb{Q}$-factorial threefold and $\mathcal{F}$ a co-rank 1 foliation with canonical and non-dicritical foliation singularities. Then*

$$\overline{NE}(X) = \overline{NE}(X)_{K_{\mathcal{F}} \geq 0} + \sum \mathbb{R}_+[L_i]$$

where L_i are curves.

Furthermore either L_i is contained in sing(X) or L_i may be taken to be a rational curve with $K_{\mathcal{F}} \cdot L_i \geq -6$.

In particular, the $K_{\mathcal{F}}$-negative extremal rays are locally discrete in the $K_{\mathcal{F}} < 0$ portion of the cone.

One can always find a resolution of singularities $\pi : X' \to X$ such that X' is smooth and the transformed foliation has canonical singularities. In this case all the hypotheses of the theorem are satisfied, and so each foliation-negative extremal ray in X' is spanned by the class of a rational curve.

With the cone theorem in hand we then turn to the question of constructing minimal models. That is, given a pair $(X, \mathcal{F})$ is there a sequence of birational modifications that can be performed resulting in a model $(Y, \mathcal{G})$ with $K_{\mathcal{G}}$ nef?

In general we are unable to prove the existence of minimal models, however we are able to provide partial results which may warrant optimism as well as pinpointing the difficulties in establishing existence.

In particular we are able to prove the existence of minimal models in two classes of foliations:

Theorem 1.1.2. *A minimal model for $(X, \mathcal{F})$ exists if either*

1) $\mathcal{F}$ is a smooth rank 2 foliation on a smooth 3-fold X.

2) $\mathcal{F}$ is a co-rank 1 toric foliation on a toric variety (no restriction on the dimension of the ambient variety).

We are also able to prove the existence of flips for a natural class of singularities:

Theorem 1.1.3. *Terminal foliation flips exist.*

In contrast to the classical situation where one can always arrange for a variety to be terminal after a resolution, this is not possible in the foliated situation, indeed the best one can hope for is canonical singularities. Thus, in many questions of practical interest in the study of foliations it would be preferable to know the existence of canonical flips. We are able to construct canonical flips in several cases and hope that the construction of a general flip can be reduced to one of these cases.

As we will see, there is an analogue in the classical setting for this argument whereby a "special" termination of flips allows one to reduce the existence of general flips to a "special" flipping case. We will show that this "special" foliated termination also holds.

We sketch the proofs of our main results: First, we explain the proof of the cone theorem:

For simplicity, let us assume that X is smooth. Let R be an extremal ray of the cone of curve $\overline{NE}(X)$ with $R \cdot K_{\mathcal{F}} < 0$. Suppose that H_R is a supporting hyperplane of R.

H_R is a nef divisor on X. If $H_R^k = 0$ for some $k \leq n$ then we can show by a foliated bend and break result that through a general point of X there is a rational curve tangent to the foliation spanning R.

If $H_R^n \neq 0$, then we may take H_R to be effective, and so we see that R actually comes from a lower dimensional subvariety S of X. The idea here is to proceed by induction on dimension. Unfortunately, a priori the singularities of S could be much worse than the singularities of X. Indeed, as we will see the singularities in our induction are sometimes worse than log canonical. The bulk of our work is therefore to work around these difficulties. In short, our two main techniques are an extension of an adjunction type result of Kawamata to the foliated setting and an algebraicity criterion whereby one can deduce the compactness of leaves of the foliation from numerical data about $K_{\mathcal{F}}$. These results seem to be useful outside of their place in the proof of the cone theorem.

Our results on the foliated MMP and existence and termination of flips all rely on the following observation: If S is a smooth leaf of a smooth foliation, then

$$K_{\mathcal{F}}|_S = (K_X + S)|_S = K_S.$$

This suggests that it might be possible to run a foliated MMP as a well chosen log-MMP. There are many difficulties with this approach since in general none of $X, S, \mathcal{F}$ will be smooth and it is unclear if the singularities which arise in the course of the foliated MMP will even allow a log MMP to be run. There are also additional complications arising from that fact that in general we cannot assume that our leaf S is algebraic. Nevertheless we are able to handle many of these issues and realize the foliated MMP as an appropriate log-MMP.

Finally, we present some applications of our methods to some classification problems in foliation theory.

1.2 Introduction

A major guiding philosophy for higher dimensional geometry is that the geometry of a complex projective manifold X is reflected in properties of the canonical bundle of X, $\omega_X = \det(\Omega^1_X)$.

In the case of smooth projective algebraic curves, there is a natural trichotomy namely, those curves which are isomorphic to $\mathbb{P}^1$, curves of genus $= 1$ and curves of genus > 2. This can be rephrased as a classification in terms of curves by the positivity of their canonical bundle, i.e., $\mathbb{P}^1$ has an anti-ample canonical bundle, genus 1 curves have a trivial canonical bundle and genus > 2 curves have an ample canonical bundle.

In higher dimensions such a neat classification fails, in part because the canonical bundle of higher dimensional varieties can exhibit a range of behaviors beyond anti-ample, trivial, ample. Nevertheless, one might hope that there is some way to "construct" a general variety using simpler varieties whose canonical bundles are either anti-ample, trivial or ample.

The goal behind the minimal model program (MMP) is to provide an algorithmic method of realizing such a decomposition. With that in mind one can phrase the guiding question of the MMP as follows:

Question 1. *Given a complex projective manifold X can we perform a sequence of (birational) surgery operations on X in order to simplify the global geometry of X?*

By simplify we mean either find a birational model X' such that $K_{X'}$ is nef or such that $X' \to Z$ fibres over a lower dimensional space, and such that the fibres are $K_{X'}$-negative.

Recall that K_X is any divisor such that $\mathcal{O}(K_X) = \omega_X$.

The first step to realizing such a sequence of operations is the celebrated cone and contraction theorem.

Definition 1. *Let X be a normal variety. The* **Kleiman-Mori Cone** $\overline{NE}(X)$ *is the closure of the cone generated by classes of effective curves modulo numerical equivalence.*

Theorem 1.2.1. *Suppose that X is smooth. Then we can write*

$$\overline{NE}(X) = \overline{NE}(X)_{K_X \geq 0} + \sum \mathbb{R}_{\geq 0}[L_i]$$

where the L_i are rational curves and the rays $\mathbb{R}_{\geq 0}[L_i]$ are locally discrete.

Furthermore, if R is any K_X-negative extremal ray, there exists a morphism $c_R : X \to Z$ such that $c_R(C)$ is a point if and only if $[C] \in R$. This is called the contraction of the extremal ray R.

We would like to realize our minimal model as a sequence of contractions of K_X-negative extremal rays. Indeed, in the case where X is a surface this strategy works:

Theorem 1.2.2. *Let X be a smooth surface. Then there exists a sequence of contractions of extremal rays $X \to X_1 \to \ldots \to X_n$ such that X_n is smooth and either K_{X_n} is nef or X_n fibres over Z where $dim(Z) = 0, 1$ and the fibres of the map are K_{X_n}-negative.*

Notice that any sequence of contractions must terminate at some point because each contraction drops the rank of $H^2(X, \mathbb{Z})$ by 1.

However, for threefolds, this approach fails. To explain we classify extremal contractions into 3 types:

(i) Fibre type contractions: Here the contraction realizes X as a fibre space where each fibre is K_X-negative.

(ii) Divisorial contractions: Here the contraction contracts a divisor to a point or curve.

(iii) Small contractions: Here the contraction contracts a subvariety of dimension $< n - 1$.

Contractions of type (i) are a stopping point for the minimal model program, and do not cause difficulties in higher dimensions.

Contractions of type (ii), in contrast to the surface case, can produce singular varieties. However, these singularities are relatively mild and the cone and contraction theorem is still true under the assumption that X has singularities in this class.

Contractions of type (iii) present serious difficulties. Let $f : X \to Z$ be such a contraction. We claim that K_Z is not $\mathbb{Q}$-Cartier. Assume otherwise, then because f is an isomorphism in codimension 2 we see $f^*K_Z = K_X$. But this implies that K_X is trivial on the exceptional locus of f, a contradiction of the definition of f as a K_X-negative contraction.

If K_Z is not $\mathbb{Q}$-Cartier there is no obvious way to proceed. One cannot in general define the intersection pairing of curves with K_Z and so the question of whether or not K_Z is nef does not even make sense.

The solution to this problem is the flip. Given a small contraction $f : X \to Z$, we want to produce another variety and morphism $f^+ : X^+ \to Z$ such that X, X^+ are isomorphic in codimension 1 and such that $-K_{X^+}$ is negative on the exceptional locus of f^+- notice the "flip" in the sign!

Unfortunately, from the definition of the flip it isn't immediately obvious that it should even exist. Even if we can show that the flip exists we run into the subtle problem of termination. After a flip both the dimension and Picard number are unchanged, and so one might reasonably wonder if it is possible that there is an infinite sequence of flips.

There has been much work done on this problem, and in the case of threefolds there is a definite answer:

Theorem 1.2.3. *Let X be a smooth threefold. Then flips exist and an MMP starting from X will have only finitely many flips. In particular, the MMP exists and terminates.*

For projective varieties over $\mathbb{C}$ much more than the above is known, and in recent years there has active work on extending the MMP to both the Kahler setting and the positive characteristic setting:

Theorem 1.2.4. *Let X be a smooth 3-fold and suppose either*

(i) X is Kähler

(ii) X is projective over a field of characteristic $p > 5$.

Then there is an MMP starting from X.

For (i) see [HP], for (ii) see [HX].

Unlike the complex projective case, it does not seem that much is known about the existence of the MMP beyond the 3-fold case in these settings.

In this paper we take a third approach to generalizing the classical MMP: a foliated MMP.

Recall that a foliation is a coherent subsheaf $\mathcal{F} \subset T_X$ of the tangent sheaf such that

1) $T_X/\mathcal{F}$ is torsion free.

2) $\mathcal{F}$ is closed under Lie bracket. This is sometimes stated as $\mathcal{F}$ is integrable.

Before getting to our main questions we present a range of examples of foliations to highlight the ubiquity of foliations in geometry, as well as to demonstrate their wide range of intriguing behaviors:

Example 1. *Let $\pi : X \to B$ be a fibration. Then the relative tangent bundle $T_{X/B} \subset T_X$ defines a foliation.*

Example 2. *On $\mathbb{C}^2$ an algebraic foliation is defined by a vector field $\partial = a(x,y)\partial_x + b(x,y)\partial_y$ where $a, b \in \mathbb{C}[x,y]$. Even if a, b are both polynomials the (local) solutions to the equation $\partial f = 0$ (which will be leaves of the corresponding foliation) may be transcendental.*

It is an open question whether or not there is an algorithmic method to decide whether or not a foliation on $\mathbb{C}^2$ has algebraic leaves.

More generally, given a complex manifold X, a foliation can be defined by an open cover $\{U_i\}$ such that for each U_i we have a collection of vector fields $\{v^i_1, ..., v^i_r\}$ which Lie commute, i.e., $[v^i_k, v^i_j] = \delta_{jk}$, and such that the $\{v^i_j\}$ satisfy the obvious compatability conditions.

Where the vector fields vanish or become tangent the foliation acquires singularities. If $p \in X$ is not a singular point of the foliation a classical theorem of

Frobenius states that there is a small analytic neighborhood $p \in U_p$ such that the foliation is induced by a holomorphic fibration $U_p \to B_p$, i.e. $\mathcal{F} = T_{U_p/B_p}$.

Example 3. *Let $\Lambda \subset \mathbb{C}^n$ be a co-compact lattice, i.e., $\mathbb{C}^n/\Lambda = X$ is a complex torus.*

Consider the foliation $\partial_{x_1}, ..., \partial_{x_{n-1}}$ on $\mathbb{C}^n$. This foliation is invariant underneath the action of Λ, and so descends to a "linear" foliation on X.

The leaves of the foliation are given by the quotient of $\{x_n = c\}$ for $c \in \mathbb{C}$ by Λ. Therefore the leaves are isomorphic to $\mathbb{C}^k \times T$ where T is a complex torus. If the leaves are are isomorphic to $\mathbb{C}^{n-1}$ they are not compact, in this case one can visualize the leaf as "spiraling" around the torus X. In fact such a leaf will be dense in X.

Example 4. *Let X be a projective manifold with an ample or movable class α. Suppose that T_X is slope unstable with respect to α, and let $\mathcal{F}$ be a maximal destabilizing subsheaf. Then $\mathcal{F}$ defines a foliation.*

The study of foliations arising in this way is intimately related with questions relating to the uniruledness of X.

Example 5. *Let $\mathcal{F}$ be a rank 1 foliation on a smooth complex projective surface X. Pick some point $p \in X$ and let $p \in T$ be a germ of a disc transverse to the foliation at p. If $\gamma \in \pi_1(X)$, then transport of T along γ induces a biholomorphism $f_\gamma : T \to T$, which in turn gives a representation $\pi_1(X) \to SL_2(\mathbb{R})$, the holonomy representation.*

Results of Corlette and Simpson, [CS08], imply that essentially all representations $\pi_1(X) \to SL_2(\mathbb{R})$ with Zariski dense image arise in this way.

In analogy with the classical setting, we define $K_{\mathcal{F}} = -c_1(\mathcal{F})$.

Question 2. *Supposing that we constrain the singularities of a foliation $\mathcal{F}$ in an appropriate way, can we perform a sequence of birational modifications to $\mathcal{F}$ which "simplify" its global geometry, i.e., result in a model $(Y, \mathcal{G})$ with $K_{\mathcal{G}}$-nef, or $\mathcal{G}$ fibred over a lower dimensional foliation?*

Observe that the integrability condition is necessary:

Example 6. *[Dru, Example 1.3] points out that the null correlation bundle $\mathcal{N}$ on $\mathbb{P}^{2n+1}$ for $n \geq 1$ gives rise to a corank 1 subbundle $\mathcal{D} \subset T_{\mathbb{P}^{2n+1}}$ with $-det(\mathcal{D})$ being ample. $\mathcal{D}$ is smooth distribution, but not closed under Lie bracket.*

We can embed $\mathbb{P}^3 \cong E \subset X \xrightarrow{f} C^4$ as the exceptional divisor of the blow up of $\mathbb{C}^4$ at a point. There exists a lift of $\mathcal{D}$ to a smooth distribution $\mathcal{D}'$ on all of X. If $C \subset E$ we still have $det(\mathcal{D}) \cdot C < 0$ and so there is no way to construct a nef model without contracting curves transverse to the distribution.

As we will see later there are important technical reasons for insisitng that only curves tangent to the foliation are contracted. However, morally speaking one can think of a foliation as being a "fibration" $X \to [X/\mathcal{F}]$ where $[X/\mathcal{F}]$ is the leaf space of the foliation. In this case the foliated MMP should be thought of as a relative MMP for this fibration, and so the MMP should only modify X "fibre-wise".

Work by Bogomolov, McQuillan, Brunella, Mendes and others has realized the foliated MMP in the case of rank 1 foliations on surfaces. For rank 1 foliations of higher dimensional varieties the MMP is known thanks to [McQ].

In higher dimensions and ranks there has also been recent work in classifying the "building block" foliations, see for example, [AD13], [PT13], [LPT].

However, little to no work has previously been done on the MMP for higher rank foliations. The goal of what follows is to address the case of rank 2 foliations on 3-folds. In particular we will prove a foliated cone and contraction theorem, and provide some partial results toward the existence of the MMP, as well as exploring the possibility of extending these methods to co-rank 1 foliations in general.

Before proceeding it is worth stopping to consider why the foliated MMP (and indeed why the Kähler and positive characteristic MMP) is hard.

In the classical setting the cone and contraction theorem, as well as the existence of the MMP are proven by way of theorems relating the canonical bundle to the vanishing and non-vanishing of certain cohomology groups, e.g., Kawamata-Viehweg vanishing, the basepoint free theorem, etc.. Unfortunately, this cohomological approach seems to have no hope of generalizing to the foliated/Kähler/positive characteristic case since the relevant (non-)vanishing are simply false in these contexts.

In the positive characteristic case there are results due to Keel, Hacon-Xu and Schwede which can sometimes take the place of the usual theorems in the classical MMP. In the Kähler setting the proof of the cone and contraction theorem relies on several deformation theory techniques and classification results only available in the three dimensional case, and so does not obviously (to me) generalize to higher dimensions.

Simply put, it does not seem possible to run the foliated MMP by taking a standard text on the MMP and replace X by $\mathcal{F}$. New methods need to be developed.

Finally, given the recent successes in the foliated MMP for rank 1 foliations one might reasonably wonder what makes rank 2 harder than rank 1? Morally it seems that there are two reasons.

First, the deformation theory of rank 1 foliations is simpler than rank 2 foliations. Indeed, the starting point for the rank 1 MMP is a strong foliated bend and break result which does not hold for rank 2 foliations.

Secondly, (at the cost of working on Deligne-Mumford stacks) the ambient space X remains smooth throughout the MMP. In the rank 2 case (as in the classical case), even working in the greater generality of Deligne-Mumford stacks, smoothness is not preserved.

Chapter 2

A proof of the cone theorem

2.1 Set up and basic results

Definition 2. *Given a normal variety X a foliation $\mathcal{F}$ is a coherent saturated subsheaf of the tangent sheaf of X which is closed under the Lie bracket.*

The rank of the foliation, $rk(\mathcal{F})$, is its rank as a sheaf and its co-rank is $dim(X) - rk(\mathcal{F})$.

The singular locus of the foliation is the locus where $\mathcal{F}$ fails to be a sub-bundle of T_X. Note that $sing(\mathcal{F})$ has codimension at least 2

The canonical divisor plays a central role in the birational geometry of foliations, we define it as follows:

Definition 3. *Let U be the locus where X and $\mathcal{F}$ are smooth. We can associate a divisor to $det(\mathcal{F}|_U)^*$, which gives a Weil divisor on all of X, denoted $K_\mathcal{F}$*

For the rest of this paper we will take $\mathcal{F}$ to be a co-rank 1 foliation over $\mathbb{C}$.

Definition 4. *We say $W \subset X$ is tangent to $\mathcal{F}$ if the tangent space of W factors through the tangent space of $\mathcal{F}$ along $X - sing(\mathcal{F})$. Otherwise, we say that W is transverse to the foliation.*

If $\mathcal{F}$ factors through the tangent space of W, $\mathcal{F}|_W \to T_W \to T_X|_W$, we say that W is invariant.

2.1.1 1-forms and pulled back foliations

Definition 5. *Let ω be a rational 1-form with $\omega \wedge d\omega = 0$. Then we can define a foliation by contraction. Namely, we take $\mathcal{F}$ to be the kernel of the pairing of ω with T_X. Thus, given a rank 1 coherent subsheaf of Ω_X, we can define a foliation by contraction.*

On the other hand, given a foliation $\mathcal{F}$ we can define a subsheaf of Ω_X by taking the kernel of $\Omega_X \to \mathcal{F}^*$.

Let $\mathcal{F}$ be a co-rank 1 foliation on X, and suppose that it is defined by the rank 1 subsheaf of the cotangent sheaf $0 \to \mathcal{L} \to \Omega_X$.

Definition 6. *Let $f : W \to X$. We have a morphism $df : f^*\Omega_X \to \Omega_W$. Assume that $f(W)$ is is not tangent to $\mathcal{F}$ and that $f(W)$ is not contained in the singularities of $\mathcal{L}$ or sing(X). Then $df(f^*\mathcal{L})$ is a rank 1 coherent subsheaf of Ω_W. Observe that if $\omega \in \mathcal{L}$ is an integrable 1 form that $df(\omega)$ is still integrable.*

This gives a foliation $\mathcal{F}_W$, called the pulled back foliation.

When f is a closed immersion we will sometimes refer to it as the restricted foliation.

In general, even if $\mathcal{L}$ is a saturated subsheaf, $f^*\mathcal{L}$ might not be saturated.

Definition 7. *Let $0 \to \mathcal{L} \to \Omega_X$ define a foliation. We call the saturation of $\mathcal{L}$ in Ω_X, $N^*_{\mathcal{F}}$, the conormal sheaf. On the smooth locus of X, $(N^*_{\mathcal{F}})|_{X^{sm}}$ is a line bundle represented by 1-forms with zero loci of codimension at least 2. Thus, we can associate to $N^*_{\mathcal{F}}$ a well defined Weil divisor. We will denote this divisor by $[N^*_{\mathcal{F}}]$.*

Lemma 2.1.1. *Let $f : W \to X$ be a morphism such that $f(W)$ is not tangent to $\mathcal{F}$ and not contained in sing(X). Assume that $N^*_{\mathcal{F}}$ is a line bundle. Then $[N^*_{\mathcal{F}_W}] - \Theta = f^*(N^*_{\mathcal{F}})$, where Θ is an effective divisor.*

Proof. By assumption of non-tangency, $df : f^*(N^*_{\mathcal{F}}) \to \Omega_W$ is nonzero.

The result then follows since $N^*_{\mathcal{F}_W}$ is the saturation of the image of $f^*(N^*_{\mathcal{F}})$ in Ω_W, i.e., on the smooth locus of W, W_{sm}, we have a nonzero map of line bundles $f^*N^*_{\mathcal{F}} \to \mathcal{O}_{W_{sm}}([N^*_{\mathcal{F}_W}])$. □

Remark 1. *If W is not smooth, then $N^*_{\mathcal{F}_W}$ may not be Cartier, however $[N^*_{\mathcal{F}_W}]-\Theta$ is Cartier.*

Lemma 2.1.2. *We have the following equivalence of Weil divisors: $K_X = K_{\mathcal{F}} + [N^*_{\mathcal{F}}]$*

Proof. Let $U = X - (\text{sing}(X) \cup \text{sing}(\mathcal{F}))$, note that the singularities of X and $\mathcal{F}$ are in codimension 2. On U we actually have the following equality of line bundles, $\mathcal{O}(K_X) = \mathcal{O}(K_{\mathcal{F}}) \otimes N^*_{\mathcal{F}}$, which in turn gives the result over X. □

2.1.2 Foliated Pairs and Foliation singularities

Frequently in birational geometry it is useful to consider pairs (X, Δ) where X is a normal variety, and Δ is a $\mathbb{Q}$-Weil divisor such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier. By analogy we define

Definition 8. *A foliated pair $(\mathcal{F}, \Delta)$ is a pair of a foliation and a $\mathbb{Q}$-Weil ($\mathbb{R}$-Weil) divisor such that $K_{\mathcal{F}} + \Delta$ is $\mathbb{Q}$-Cartier ($\mathbb{R}$-Cartier).*

Foliated pairs show up in [?] and [McQ] where Δ is assumed to have no components which are invariant under the foliation and the coefficients of Δ lie in the set $\{\frac{n-1}{n} | n = 1, 2, ...\} \cup \{1\}$. We make no such requirements on Δ, but we will see that these restrictions can be phrased in terms of restrictions on the singularities of the pair $(\mathcal{F}, \Delta)$. Note also that we are typically interested only in the cases when $\Delta \geq 0$, although it simplifies some computations to allow Δ to have negative coefficients.

Given any birational morphism $\pi : \widetilde{X} \to X$, we get an induced foliation $\widetilde{\mathcal{F}}$ on $\tilde{X}$. Thus, we can write, $K_{\widetilde{\mathcal{F}}} + \pi_*^{-1}\Delta = \pi^*(K_{\mathcal{F}} + \Delta) + \sum a(E_i, \mathcal{F}, \Delta)E_i$,

Definition 9. *We say that the foliation is terminal, canonical, log terminal, log canonical if $a(E_i, \mathcal{F}, \Delta) > 0$, ≥ 0, $> -\epsilon(E_i)$, $\geq -\epsilon(E_i)$, respectively, where $\epsilon(D) = 0$ if D is invariant and 1 otherwise and where π varies across all birational morphisms.*

If $(\mathcal{F}, \Delta)$ is log terminal and $\lfloor \Delta \rfloor = 0$ we say that $(\mathcal{F}, \Delta)$ is klt.

Notice that these notions are well defined, i.e., $\epsilon(E)$ and $a(E, \mathcal{F}, \Delta)$ are independent of π.

Observe that in the case where $\mathcal{F} = T_X$ no exceptional divisor is invariant, i.e., $\epsilon(E) = 1$, and so this definition recovers the usual definitions of (log) terminal, (log) canonical.

While (log) terminal and (log) canonical are natural definitions from the perspective of birational geometry, a priori it is unclear what the singularities themselves "look like". However, in the case of terminal foliation singularities on surfaces we have the following neat characterization due to [McQ08, Corollary I.2.2.]

Proposition 2.1.3. *Let $(X, \mathcal{F}, 0)$ be the germ of a terminal foliation singularity on a surface. Then there exists a smooth foliation on a smooth surface $(Y, \mathcal{G})$ and a cyclic quotient $Y \to X$ such that $\mathcal{F}$ is the quotient of $\mathcal{G}$ by this action.*

We also make note of the following easy fact:

Proposition 2.1.4. *Let $\pi : (Y, \mathcal{G}) \to (X, \mathcal{F})$ be a birational morphism. Write $\pi^*(K_{\mathcal{F}} + \Delta) = K_{\mathcal{G}} + \Gamma$. Then $a(E, \mathcal{F}, \Delta) = a(E, \mathcal{G}, \Gamma)$ for all E.*

Remark 2. *Observe that if any subvariety of supp(Δ) is foliation invariant, then $(\mathcal{F}, \Delta)$ is not log canonical. This suggests that one could think of invariant varieties as being log canonical centres for the foliation.*

We will also make use of the class of simple foliation singularities:

Definition 10. *We say that $p \in X$ with X smooth is a simple singularity for $\mathcal{F}$ provided in formal coordinates around p we can write the defining 1-form for $\mathcal{F}$ in one of the following two forms, where $1 \leq r \leq n$:*

(i) There are $\lambda_i \in \mathbb{C}^*$ such that

$$\omega = (x_1 ... x_r)(\sum_{i=1}^{r} \lambda_i \frac{dx_i}{x_i})$$

and if $\sum a_i \lambda_i = 0$ for some non-negative integers a_i then $a_i = 0$ for all i.

(ii) There is an integer $k \leq r$ such that

$$\omega = (x_1 ... x_r)(\sum_{i=1}^{k} p_i \frac{dx_i}{x_i} + \psi(x_1^{p_1} ... x_k^{p_k}) \sum_{i=2}^{r} \lambda_i \frac{dx_i}{x_i})$$

where p_i are positive integers, without a common factor, $\psi(s)$ is a series which is not a unit, and $\lambda_i \in \mathbb{C}$ and if $\sum a_i \lambda_i = 0$ for some non-negative integers a_i then $a_i = 0$ for all i.

We say the integer r is the dimension-type of the singularity.

Remark 3. *A general hyperplane section of a simple singularity is again a simple singularity.*

By Cano, [Can04], every foliation on a smooth threefold admits a resolution by blow ups centred in the singular locus of the foliation such that the transformed foliation has only simple singularities.

Lemma 2.1.5. *Let X be smooth, and $\mathcal{F}$ a co-rank 1 foliation on X. Let $x \in X$ be a smooth point of $\mathcal{F}$. Then $\mathcal{F}$ is canonical at x.*

This is proven in [AD13], we include a proof for completeness. Observe that this is perhaps not obvious since even blowing up along smooth centres can transform a smooth foliation into a non-smooth one.

Proof. Let $\mathcal{F}$ be generated at x by $n-1$ commuting vector fields $\partial_1, ..., \partial_{n-1}$. Suppose for contradiction that x is not canonical. Then we can find some discrete valuation ring R in $K(X)$ such that $\mathcal{O}_{X,x} \hookrightarrow R$ and $\partial_1 \wedge ... \wedge \partial_{n-1} = \pi^d \Theta$ where $\Theta \in \bigwedge^{n-1} T_R$ and π is a generator of the maximal ideal of R.

Thus, for any $f_1, ..., f_{n-1} \in \mathcal{O}_{X,x}$ we have

$$\partial_1 \wedge ... \wedge \partial_{n-1}(df_1 \wedge ... \wedge df_{n-1}) \in \mathfrak{m}_R \cap \mathcal{O}_{X,x} \subset \mathfrak{m}_{X,x}$$

However, by assumption of smoothness, there exists $x_1, ..., x_{n-1}$ all distinct with $\partial_i(dx_i)$ a unit, and hence $\partial_1 \wedge ... \wedge \partial_{n-1}(dx_1 \wedge ... \wedge dx_{n-1})$ is a unit □

Canonical singularities in codimension 2 have been described in [LPT, Proposition 3.4], in particular, they are simple at their generic points.

Lemma 2.1.6. *Suppose $\mathcal{F}$ has simple singularities. Any blow up along a strata of sing($\mathcal{F}$) has discrepancy 0. Furthermore, any blow up of a point of sing($\mathcal{F}$) has discrepancy at least 0.*

Proof. Working in local coordinates, let $Z = \{x_1 = ... = x_k = 0\}$ be a codimension k strata. Let ω be a defining 1-form around Z, then write $\omega = (x_1...x_k)\sum_{i=1}^{k} \frac{dx_i}{x_i} +$ *h.o.t.*.

Pulling back ω along the blow up of Z gives a 1-form which vanishes to order $k-1$ on the exceptional divisor, and the first claim follows. The second claim follows by an identical computation. □

Corollary 2.1.7. *Simple singularities are canonical.*

Proof. For divisors E whose centres are generically contained in the smooth locus, by our above lemma $a(E, \mathcal{F}) \geq 0$.

For divisors centred over the singular locus, notice that the blow up of a simple singularity along some point in the singular locus preserves the property of being a simple singularity we are done by induction. □

The converse of this statement is false:

Example 7. *Consider the germ of the foliation $(0 \in X, \mathcal{F})$ given by the degeneration of smooth surfaces to the cone over an elliptic curve. Consider the blow up π at the point 0 with exceptional divisor E and let $\mathcal{F}'$ be the transformed foliation. Observe that $\mathcal{F}'$ has simple singularities, and that E is invariant.*

Write $K_{\mathcal{F}'} = \pi^* K_{\mathcal{F}} + aE$.

Denote by L the closure of a leaf in X passing through 0, and L' its strict transform. $K_{\mathcal{F}}|_L = K_L$, $K_{\mathcal{F}'}|_{L'} = K_{L'} + E|_{L'}$ and $K_{L'} = \pi^* K_L - E|_{L'}$.

From this we see that $K_{L'} + E|_{L'} = \pi^* K_L + aE|_{L'}$ and so $a = 0$, hence $(\mathcal{F}, 0)$ is canonical. However, $(\mathcal{F}, 0)$ is not simple since simple singularities are never isolated.

We will need to define one final type of foliation singularity:

Definition 11. *Given a foliated pair $(X, \mathcal{F})$ we say that $\mathcal{F}$ has non-dicritical singularities if for any sequence of blow ups $\pi : (X', \mathcal{F}') \to (X, \mathcal{F})$ and any $q \in X$ we have $\pi^{-1}(q)$ is tangent to the foliation.*

Remark 4. *Observe that this implies that if W is $\mathcal{F}$ invariant, then $\pi^{-1}(W)$ is $\mathcal{F}'$ invariant.*

Definition 12. *Given a germ $(X, 0)$ with a foliation $\mathcal{F}$ such that 0 is a singular point for $\mathcal{F}$ we call a (formal) hypersurface germ $(S, 0)$ a (formal) separatrix if it is invariant under $\mathcal{F}$.*

Note that away from the singular locus of $\mathcal{F}$ a separatrix is in fact a leaf. Furthermore being non-dicritical implies that there are only finitely many separatrices through a singular point.

Example 8. *Let $\lambda \in \mathbb{R}$. Consider the foliation $\mathcal{F}_\lambda$ on $\mathbb{C}^2$ generated by $x\partial_x + \lambda y\partial_y$. For $\lambda \in \mathbb{Q}_{\geq 0}$ we can see that $\mathcal{F}_\lambda$ is dicritical, and otheriwse is non-dicrtical.*

Indeed, more generally on smooth surfaces dicriticality is equivalent to having infinitely many separatrices passing through a singular point. In higher dimensions this characterization is false since a dicritical singularity may fail to have any separatrices. However if we consider the set of curves γ_t tangent to the foliation and passing through the singularity non-dicriticality is equivalent to $\bigcup_t \gamma_t$ being contained in a germ of a proper closed analytic subset.

Example 9. *Simple singularities are non-dicritical.*

Even for simple foliation singularities it is possible that there are separatrices which do not converge. However, as the following definition/result of [CC92] shows there is always at least 1 convergent separatrix along a simple foliation singularity of codimension 2.

Definition 13. *For a simple singularity of type (i), all separatrices are convergent.*

For a simple singularity of type (ii), around a general point of the singularity we can write $\omega = pydx + qxdy + x\psi(x^p y^q)\lambda dy$. $x = 0$ is a convergent separatrix, called the strong separatrix.

2.2 Foliated MMP for surfaces

McQuillan in [McQ08] proves the existence of a foliated MMP, namely:

Theorem 2.2.1. *Let X be a smooth surface and $\mathcal{F}$ a foliation with canonical foliation singularities. Then, there is an MMP starting with X, namely a sequence of contractions of curves $\pi : X \to Y$ and a foliation $\mathcal{G}$ on Y, birationally equivalent to $\mathcal{F}$ such that either $K_{\mathcal{G}}$ is nef, or it is a $\mathbb{P}^1$-bundle over a curve. Furthermore, Y has rational singularities, and π can be realized as a contraction of invariant curves C with intersect the canonical divisor of the foliation negatively.*

Observe that we can make the following modifications, implicit in [McQ08]:

Corollary 2.2.2. *Let $f : X \to U$ be a birational morphism of surfaces, and let $\mathcal{F}_X, \mathcal{F}_U$ be foliations birationally equivalent by f. Suppose X is smooth and $\mathcal{F}_X$ has canonical singularities. Let Δ be a divisor not containing any fibres of f. Then we can run the relative MMP, i.e., there is a birational map $g : X \to Y$ and $h : Y \to U$ and a foliation $\mathcal{G}$ on Y such that $K_{\mathcal{G}} + g_*\Delta$ is $h-$nef.*

Proof. First, assume that $\Delta = 0$.

By the cone theorem for surface foliations we see that if C is a K_X-negative curve contracted by f that C is an invariant rational curve, and following [McQ08] we can contract it to a point, notice that the contracted space still maps down to U. Continuing inductively, and letting $(Y, \mathcal{G})$ be the output of this MMP we see that $K_{\mathcal{G}}$ is nef over U.

If $\Delta \geq 0$ and if C is contracted by f, then $\Delta \cdot C \geq 0$, and so $(K_{\mathcal{F}} + \Delta) \cdot C < 0$ implies that $K_{\mathcal{F}} \cdot C < 0$. Thus the $K_{\mathcal{F}} + \Delta$-MMP can be realized as some subset of contractions in the $K_{\mathcal{F}}$-MMP. □

2.3 Some adjunction results for foliations

We begin with a simple lemma:

Lemma 2.3.1. *Let $f : Y \to X$ be a morphism of normal varieties. Let $\mathcal{F}$ be a foliation on X. Suppose that $f(Y)$ is not tangent to $\mathcal{F}$ and that $f(Y)$ is not contained in sing(X). Let $\mathcal{F}_Y$ be the pulled back foliation. Suppose $K_X + \Delta_X$ is $\mathbb{R}$-Cartier and either*

(i) $N^*_{\mathcal{F}}$ is a line bundle (e.g. X is smooth) or,

(ii) we have a morphism $f^*\Omega_X^{[1]} \to \Omega_Y^{[1]}$ *between sheaves of reflexive differentials, and* $(N^*_{\mathcal{F}})^{**}$ *is a line bundle. Here* $\Omega_X^{[1]}$ *means* $(\Omega^1_X)^{**}$*.*

Then

$$f^*(K_{\mathcal{F}} + \Delta_X) - K_{\mathcal{F}_Y} = f^*(K_X + \Delta_X) - K_Y + \Theta$$

where $\Theta \geq 0$*.*

Proof. Write $K_{\mathcal{F}} = K_X - [N^*_{\mathcal{F}}]$, and $K_{\mathcal{F}_Y} = K_Y - [N^*_{\mathcal{F}_Y}]$.

In case (i) as noted earlier, $f^*N^*_{\mathcal{F}} = \mathcal{O}(N^*_{\mathcal{F}_Y} - \Theta)$ where Θ is a effective.

In case (ii) we have a morphism $f^*((N^*_{\mathcal{F}})^{**}) \to \Omega_Y^{[1]}$ and $(N^*_{\mathcal{F}_Y})^{**}$ is the saturation of the image of this morphism. On the smooth locus of Y this gives a morphsim $f^*((N^*_{\mathcal{F}})^{**}) \to N^*_{\mathcal{F}_Y}$, and hence an equality of divisors $f^*[N^*_{\mathcal{F}}] = [N^*_{\mathcal{F}_Y}] - \Theta$ where $\Theta \geq 0$.

In either case, the result follows. □

Remark 5. *Observe that if* X *is not klt the morphism* $f^*\Omega_X^{[1]} \to \Omega_Y^{[1]}$ *does not always exist.*

Of particular interest are the cases where f is a closed immersion, f is a blow up or f is a fibration. In these cases, we get

Corollary 2.3.2. *Let* X *be smooth.*

(1) Let $\nu : D^\nu \to D \subset X$ *be the normalization of a divisor transverse to the foliation, then* $\nu^*(K_{\mathcal{F}} + D) = K_{\mathcal{F}_D} + \Theta$*. Furthermore,* $\nu(\Theta)$ *is either contained in sing*(D) *or is tangent to* $\mathcal{F}$*.*

(2) The foliation discrepancy is less than or equal to the usual discrepancy.

(3) If the fibres of $f : Y \to X$ *are all reduced, then* $f^*K_{\mathcal{F}} - K_{\mathcal{F}_Y} = f^*K_X - K_Y$*.*

Proof. The only thing that doesn't follow immediately from the previous lemma is the claim that if B is a component of Θ not contained in $\nu^{-1}(\text{sing}(D))$ then it is tangent to the foliation. However, observe that if ω is a 1-form locally generating $N^*_{\mathcal{F}}$, then, the pull back of ω to B vanishes. This implies that B is tangent to the foliation. □

Remark 6. *In case (1), if* D *is smooth then* Θ *is supported on the tangency locus of* D *and* $\mathcal{F}$*.*

The following is a more general version of foliation adjunction that we will need. The proof mirrors the proof of the general adjunction formula in the case of varieties. We follow the presentation in [Fuj11].

Proposition 2.3.3. *Let $\mathcal{F}$ be a co-rank 1 foliation, let S be a prime divisor transverse to the foliation, with normalization S^ν, and let $\mathcal{F}_{S^\nu}$ the foliation restricted to S^ν. Then, if $K_\mathcal{F} + \Delta + S$ is an $\mathbb{R}$-Cartier divisor,*

$$\nu^*(K_\mathcal{F} + \Delta + S) = K_{\mathcal{F}_{S^\nu}} + \Delta_{S^\nu}$$

where $\Delta_{S^\nu} \geq 0$.

Proof. The case where X and S are smooth is proven above. So, assume that X is normal and $K_\mathcal{F} + \Delta + S$ is $\mathbb{R}$-Cartier.

Let $g : Y \to X$ be a resolution such that Y and the strict transform of S, call it S_Y, are smooth. Write

$$K_{\mathcal{F}_Y} + S_Y + \Gamma_Y = g^*(K_\mathcal{F} + \Delta + S)$$

Thus, $(K_{\mathcal{F}_Y} + S_Y + \Gamma_Y)|_{S_Y} = K_{\mathcal{F}_{S_Y}} + \Gamma_Y|_{S_Y} + \Theta$ where Θ is effective. Write $\Gamma_Y|_{S_Y} = \Gamma_{S_Y}$.

Let $\nu : S^\nu \to S$ be the normalization of S. We have a factorization $S_Y \xrightarrow{f} S^\nu \xrightarrow{\nu} S$. Let $\Delta_{S^\nu} = f_*(\Gamma_{S_Y} + \Theta)$. Then $K_{\mathcal{F}_{S^\nu}} + \Delta_{S^\nu} = \nu^*(K_\mathcal{F} + \Delta + S)$.

What remains to show is that Δ_{S^ν} is effective. Since $\Theta \geq 0$, it suffices to show that $f_*\Gamma_{S_Y}$ is effective.

By taking hyperplane cuts of X, we may assume that X is a surface, $\mathcal{F}$ is a foliation by curves, and S is a curve transverse to the foliation. We can run the foliated log MMP over X with respect to $K_{\mathcal{F}_Y} + S_Y$.

Replacing $\mathcal{F}_Y, S_Y$ by the output of the MMP we get that $K_{\mathcal{F}_Y} + S_Y$ is f-nef. Note that each step of the relative MMP will contract a curve with $0 > K_{\mathcal{F}_Y} \cdot E \geq -1$, and so we see that S_Y is smooth since if E meets S_Y we would have that $(K_{\mathcal{F}_Y} + S_Y) \cdot E \geq 0$ and so E is not contracted. Thus, we still have a factorization $S_Y \to S^\nu \to S$.

We have that $\Gamma_Y = -(K_{\mathcal{F}_Y} + S_Y) + f^*(K_{\mathcal{F}} + \Delta + S)$ is f-anti nef. By the negativity lemma, Γ_Y is effective, hence its restriction and pushforward to S^ν is effective. □

Definition 14. *We will refer to Δ_{S^ν} as the foliated different.*

Corollary 2.3.4. *Notation as above. Let D be a prime component of Δ_{S^ν}, then either D is supported on $\nu^{-1}\Delta$, $\nu^{-1}(sing(S)\cup sing(X))$ or is tangent to the foliation.*

Proof. As above, $\Delta_S = f_*(\Gamma_{S_Y} + \Theta)$ where Θ is tangent to the foliation and where Γ_{S_Y} is supported on the strict transform of Δ and on exceptional divisors centred above the singular loci S and X. Since $\nu(f(\Gamma_{S_Y})) \subset \Delta \cup \mathrm{sing}(S) \cup \mathrm{sing}(X)$, we have our result. □

We also have a foliated Riemann-Hurwitz formula:

Proposition 2.3.5. *Let $\pi : Y \to X$ be a surjective, finite morphism of normal varieties. Let $\mathcal{F}$ be a co-rank 1 foliation on X, with $K_{\mathcal{F}}$ $\mathbb{Q}$-Cartier. Then*

$$K_{\mathcal{F}_Y} = \pi^* K_{\mathcal{F}} + \sum \epsilon(D)(r_D - 1)D$$

where the sum is over divisors with ramification index r_D,

Proof. First, observe that $\pi^{-1}(\mathrm{sing}(X))$ is of codimension at least 2 in Y, thus, to prove our result, it suffices to restrict to $\pi : Y - \pi^{-1}(\mathrm{sing}(X)) \to X - \mathrm{sing}(X)$, and thus we may assume that X is smooth.

Pick a neighborhood of a general point of the branch divisor such that $\mathcal{F}$ is smooth, and the branch divisor consists of a single, smooth component. Let $\mathcal{F}$ be locally defined by dz. If the branch locus is not invariant, then we see that $\pi^* dz$ is a non-vanishing holomorphic form.

If the branch locus is invariant we have $\pi^* dz = kw^{k-1}dw$ where k is the ramification index. In this case the zero divisor $\pi^* dz$ agrees with the ramification divisor. Notice that the ramification divisor is also foliation invariant.

Thus $N^*_{\mathcal{F}} = \pi^* N^*_{\mathcal{F}} \otimes \mathcal{O}(R')$ where R' is the invariant part of the usual ramification divisor. Using $K_X = K_{\mathcal{F}} + N^*_{\mathcal{F}}$ and the usual Riemann-Hurwitz, we get our result. □

Remark 7. *If the ramification of $\pi : (Y, \mathcal{G}) \to (X, \mathcal{F})$ is foliation invariant then $K_{\mathcal{G}} = \pi^* K_{\mathcal{F}}$.*

Later on we will need to compute the discrepancies of pairs $(\mathcal{F}, \Delta)$. The following two results will be useful in this regard.

Corollary 2.3.6. *Suppose X is klt and $\mathbb{Q}$-factorial. Let Δ be an effective divisor. Let $\pi : Y \to X$ be a birational morphism which extracts divisors of usual discrepancy with respect to $(X, \Delta) \leq -1$. Then if π extracts E, the discrepancy of E with respect to $(\mathcal{F}, \Delta)$ is $\leq -\epsilon(E)$ with strict inequality if $\epsilon(E) = 0$. In particular, π only extracts divisors of foliation discrepancy < 0.*

Remark 8. *This result can be phrased as saying that the non-klt places of (X, Δ) are non-klt places of $(\mathcal{F}, \Delta)$. Observe that the converse of this statement is false since smooth varieties can admit foliations with log canonical singularities.*

Proof. The statement can be checked locally on X, so consider the following diagram:

$$\begin{array}{ccc} Y' & \xrightarrow{g} & Y \\ \downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \pi} \\ X' & \xrightarrow{f} & X \end{array}$$

Here $f : X' \to X$ is the index 1 cover associated to $N^*_{\mathcal{F}}$, note f is etale in codimension 2. Denote by $\mathcal{F}'$ the foliation on X'.

Y' is the normalization of $X' \times_X Y$. Observe that g is finite.

Next, note $f^* K_X = K_{X'}$ and $f^* K_{\mathcal{F}} = K_{\mathcal{F}'}$. Write $\Delta' = f^* \Delta$

Let E be a divisor contracted by π and let E' a divisor contracted by π' such that $g(E') = E$, let r be the ramification index.

Next, write

$$K_Y + \pi_*^{-1} \Delta = \pi^*(K_X + \Delta) + aE + D_1$$
$$K_{\mathcal{F}_Y} + \pi_*^{-1} \Delta = \pi^*(K_{\mathcal{F}} + \Delta) + bE + D_2$$
$$K_{Y'} + \pi'^{-1}_* \Delta' = \pi'^*(K_{X'} + \Delta') + a'E' + D_3$$
$$K_{\mathcal{F}_{Y'}} + \pi'^{-1}_* \Delta' = \pi'^*(K_{\mathcal{F}'} + \Delta') + b'E' + D_4$$

where D_i are divisors not involving E, E'.

We have $(N^*_{\mathcal{F}'})^{**}$ is a line bundle sub-sheaf of $\Omega^{[1]}_{X'}$. Next, by [GKKP11, Theorem 4.3] we have that reflexive 1-forms on X' pull back to reflexive 1-forms on Y' since X' is klt, which gives a morphism $d\pi' : \pi'^{[*]}(N^*_{\mathcal{F}'})^{**} \to \Omega^{[1]}_{Y'}$. $(N^*\mathcal{F}_{Y'})^{**}$ is the saturation of the image of $d\pi'$, which implies that the foliated discrepancy is less than the usual discrepancy, cf. Lemma 2.3.1, so $b' \leq a'$.

Next, by Riemann-Hurwitz,

$$\begin{aligned} K_{Y'} + \pi'^{-1}_* \Delta' = g^*(K_Y + \pi^{-1}_* \Delta) + (r-1)E' + F = \\ g^*(\pi^*(K_X + \Delta) + aE + D_1) + (r-1)E' + F \end{aligned}$$

where F is a divisor not involving E'. Pulling back the other way around the diagram shows that

$$a' = ra + (r-1).$$

Likewise, foliated Riemann-Hurwitz tells us that

$$\begin{aligned} K_{\mathcal{F}_{Y'}} + \pi'^{-1}_* \Delta' = g^*(K_{\mathcal{F}_Y} + \pi^{-1}_*) + \epsilon(r-1)E' + G = \\ g^*(\pi^*(K_{\mathcal{F}} + \Delta) + bE + D_2) + \epsilon(r-1)E' + G \end{aligned}$$

where $\epsilon = 0$ if E' is invariant and $= 1$ otherwie. Again, pulling back the other way around the diagram gives

$$b' = rb + \epsilon(E)(r-1).$$

Since $a \leq -1$, we get that $a' \leq -1$. And so $rb + \epsilon(E)(r-1) = b' \leq a' \leq -1$.

This gives that $b \leq \frac{-\epsilon(E)(r-1)-1}{r} \leq -\epsilon(E)$ with strict inequality if $\epsilon(E) = 0$. $\square$

Corollary 2.3.7. *Let $f : (X', \mathcal{F}') \to (X, \mathcal{F})$ be finite. Write $K_{\mathcal{F}'} + \Delta' = f^*(K_{\mathcal{F}} + \Delta)$. Suppose $(\mathcal{F}, \Delta)$ is terminal (canonical) then $(\mathcal{F}', \Delta')$ is terminal (canonical).*

Proof. Let $\pi : Y \to X$ be birational. Consider the following diagram:

$$\begin{array}{ccc} Y' & \xrightarrow{g} & Y \\ \downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \pi} \\ X' & \xrightarrow{f} & X \end{array}$$

Let $E \subset Y$ and let $E' \subset Y'$ map to E. Exactly as above if a is the discrepancy of E and if a' is the discrepancy of E' we have that $a' = ra + \epsilon(E)(r-1)$. If $a > 0$ ($a \geq 0$) then $a' > 0$ ($a' \geq 0$). □

This next result seems to be standard in the literature, but for lack of a good reference we include it and a proof here.

Lemma 2.3.8. *Let $\mathcal{F}$ be a codimension 1 foliation on a X. Let $\pi : X' \to X$ be a birational morphism. Suppose Z is a centre transverse to the foliation and Z is generically contained in the smooth locus of X. Then the foliation discrepancy of a divisor E centred over Z is equal to the usual discrepancy.*

Furthermore, E is transverse to the foliation, and the $\mathbb{P}^r$-fibration structure of E over Z is tangent to the foliation restricted to E.

Proof. Perhaps passing to a resolution $X'' \to X'$, observe that any such exceptional divisor can be reached by a sequence of blow ups along centres transverse to the foliation. Thus by induction we may assume that π is a blow up with centre Z, where Z is transverse to the foliation.

Write (analytic locally) $Z = \{x_1 = ... = x_s = 0\}$ and let one patch of our blow up π be given by $x_1 = y_1, x_2 = y_1y_2, ..., x_s = y_1y_s, x_{s+1} = y_{s+1}, ..., x_n = y_n$

Suppose that $\mathcal{F}$ is locally determined by

$$\omega = f(x_{s+1}, ..., x_n)dx_n + \text{terms featuring } x_1, ..., x_s$$

An easy computation shows that $\text{ord}_{y_1}(\pi^*\omega) = 0$ where y_1 is the equation of the exceptional divisor. Thus, $\pi^*N^*_{\mathcal{F}} = N^*_{\mathcal{F}'}$ and so by 2.3.1 the foliation disrepancy is equal to the usual discrepancy.

To see our final claim, the exceptional divisor, E is given by $y_1 = 0$, and the foliation restricted to E is given by $\omega_E = f(y_{s+1}, ...y_n)dy_n \neq 0$, which vanishes when pulled back (as a 1-form) to the fibres of $E \to Z$. In particular we see that E is transverse to the foliation. □

2.4 A foliated bend and break result

We recall the following theorem due to [Miy87], [SB92, Theorem 9.0.2] or [BM01]

Theorem 2.4.1. *Let $(X, \mathcal{F})$ be a normal foliated variety of dimension n, and let $H_1, ..., H_{n-1}$ be ample divisors. Let C be a general intersection of elements $D_i \in \mid m_i H_i \mid$ where $m_i \gg 0$. Suppose that $C \cdot K_{\mathcal{F}} < 0$ Then if A is an ample divisor through a general point of C there is a rationl curve Σ with*

$$A \cdot \Sigma \leq 2n \frac{A \cdot C}{-K_{\mathcal{F}} \cdot C}.$$

Strictly speaking the proof in [SB92] does not exactly show that the curves are tangent to $\mathcal{F}$, however a slight modification to the argument gives this conclusion. We explain it here:

Lemma 2.4.2. *Let E be a semi-stable vector bundle on a smooth curve C. Suppose that $det(E)$ is ample. Then E is ample.*

Proof. It suffices to show that every quotient $E \to Q \to 0$ has $\deg(\det(Q)) > 0$. So suppose that Q does not have positive degree, and let K be the kernel of the quotient. Then $\deg(\det(K)) \geq \deg(\det(E))$ and $\operatorname{rank}(K) < \operatorname{rank}(E)$. In particular K is a destabilizing subbundle of E, a contradiction. □

Proof. If $\mathcal{F}|_C$ is semi-stable, then it is ample and we apply [BM01]. In fact here we get the better bound on the degree of $2\operatorname{rank}(\mathcal{F})\frac{A \cdot C}{-K_{\mathcal{F}} \cdot C}$.

Otherwise, there exists a maximal destabilizing subsheaf $0 \to \mathcal{F}' \to \mathcal{F}$, recalling that a torsion free coherent sheaf is semi-stable if and only if its restriction to a general complete intersection variety is semi-stable (in our case C). Notice that $\mathcal{F}'$ is closed under Lie bracket because $\mathcal{F}$ is closed under lie bracket, and $\operatorname{Hom}(\bigwedge^2 \mathcal{F}', \mathcal{F}/\mathcal{F}') = 0$.

We have that $K_{\mathcal{F}'} \cdot C < 0$, and in fact since $\mathcal{F}|_C$ is not ample $K_{\mathcal{F}'} \cdot C \leq K_{\mathcal{F}} \cdot C$. Thus, we are done by induction on the rank of $\mathcal{F}$. □

We make a minor modification of a lemma due to [KMM94].

Corollary 2.4.3 (Bend and Break)**.** *Let X be a normal projective variety of dimension n. Let $\mathcal{F}$ be a foliation of rank r on X, and $\Delta \geq 0$. Let M be any nef divisor. Suppose that there are nef $\mathbb{R}$-divisors $D_1, ..., D_n$ such that*

(1) $D_1 \cdot D_2 \cdot ... \cdot D_n = 0$

(2) $-(K_{\mathcal{F}} + \Delta) \cdot D_2 \cdot ... \cdot D_n > 0$

Then, through a general point of X there is a rational curve Σ with $D_1 \cdot \Sigma = 0$ and

$$M \cdot \Sigma \leq 2n \frac{M \cdot D_2 \cdot ... \cdot D_n}{-K_{\mathcal{F}} \cdot D_2 \cdot ... \cdot D_n}$$

and Σ is tangent to $\mathcal{F}$

Proof. We can pick ample $\mathbb{Q}$-divisors $H_2, ...H_n$ sufficiently close to $D_2, ..., D_n$ so that

$$-K_{\mathcal{F}} \cdot H_2 \cdot ... \cdot H_n > \Delta \cdot H_2 \cdot ... \cdot H_n > 0$$

Pick $m_i \gg 0$ such that $m_i H_i$ is very ample, and let C be an intersection of general elements in $| \ m_i H_i \ |$. Then, we may take C to be contained in the smooth locus of both X and $\mathcal{F}$.

Then, apply the above theorem to give rational curves Σ_k tangent to the foliation with

$$\begin{aligned}(kD_1 + H) \cdot \Sigma_k \leq 2n \frac{(kD_1 + H) \cdot m_2 H_2 \cdot ... \cdot m_n H_n}{-K_{\mathcal{F}} \cdot m_2 H_2 \cdot ... \cdot m_n H_n} \\ = 2n \frac{(kD_1 + H) \cdot H_2 \cdot ... \cdot H_n}{-K_{\mathcal{F}} \cdot H_2 \cdot ... \cdot H_n}\end{aligned}$$

As H_i approaches D_i, the left hand side of the inequality approaches a bounded constant. Thus, as k varies, $\Sigma_k = \Sigma$ belongs to a bounded family, so for $k \gg 0$ we may take Σ to be fixed. Letting H approach M and letting k go to infinity gives our result. ◻

Remark 9. *Observe that this result is totally independent of either the rank of the foliation or the dimension of the ambient variety. We recover the usual form of bend and break when we take the rank of the foliation $r = dim(X)$.*

2.5 The cone theorem for surfaces

We will need the following minor variant of the foliated cone theorem, which is implicit in the existing literature. In proving it we use the following definition and result from convex geometry:

Definition 15. *Let K be a convex cone containing no lines. A ray R of K is called exposed if there is a hyperplane meeting K exactly along R.*

Lemma 2.5.1. *If K is a closed convex cone containing no lines, then K is the closure of the subcone generated by the exposed rays.*

Proof. See [Roc70, Corollary 18.7.1]. □

Theorem 2.5.2. *Let $(S, \mathcal{F}, \Delta = \sum a_i D_i)$ be a triple of a normal surface, a foliation, and an effective divisor. Assume that sing($\mathcal{F}$) $\cap$ sing(S) consists of non-dicritical foliation singularities. Then*

$$\overline{NE}(X) = \overline{NE}(X)_{K_{\mathcal{F}}+\Delta\geq 0} + Z_{-\infty} + \sum \mathbb{R}^+[L_i]$$

where L_i are invariant curves, and $Z_{-\infty}$ is spanned by those D_i in supp(Δ) with $a_i > \epsilon(D_i)$. Furthermore, if L_i is not contained in Δ and is disjoint from the singularities of X, then $L_i \cong \mathbb{P}^1$, and $K_{\mathcal{F}} \cdot L_i \geq -2$. In particular, if H is ample, there are only finitely many curves with extremal rays R with $(K_{\mathcal{F}}+\Delta+H)\cdot R < 0$

Proof. Let W denote the closure of the right hand side of the desired equality.

Assume that W is strictly smaller than $\overline{NE}(X)$. Then, by Lemma 5.1, if H is a sufficiently general ample divisor, if we choose t so that $H_R = K_{\mathcal{F}} + \Delta + tH$ is nef, it is zero precisely on one exposed extremal ray R, not contained in W.

We argue depending on $\nu(H_R)$. If $\nu(H_R) \leq 1$, then, as in our foliated bend and break lemma, we set $D_i = H_R$ for $i \leq \nu(H_R) + 1$ and $D_i = H$ otherwise. Then, $D_1 \cdot D_2 = 0$ and $(K_{\mathcal{F}} + \Delta) \cdot D_2 = -tH \cdot D_2 < 0$, thus we may apply our foliated bend and break lemma.

Thus, through a general point of S there is a rational curve Σ with $D_1 \cdot \Sigma = H_R \cdot \Sigma = 0$, and bounded degree. Thus, the extremal ray is spanned by the class $[\Sigma]$.

If $\nu(H_R) = 2$, then, writing $H_R = A + E$ where A is ample and E is effective we see that $E \cdot R < 0$, and hence R is spanned by some component of E, call it C.

Write $E = rC + E'$. If Δ is a boundary along C, we see that there exists some $\alpha \geq 0$ such that $K_{\mathcal{F}} + \Delta + \alpha E = K_{\mathcal{F}} + \Delta' + C$ where $\Delta' \geq 0$. However, we have $(K_{\mathcal{F}} + \Delta' + C) \cdot C < 0$ which is a contradiction of adjunction if C is not invariant. Thus C must be invariant and so R is spanned by an invariant curve, C.

If C is not contained in Δ and disjoint from $\mathrm{sing}(X)$, then $K_{\mathcal{F}} \cdot C \geq 2g(C) - 2$, which if C is $K_{\mathcal{F}}$-negative implies that $C \cong \mathbb{P}^1$.

Thus, we see that W and $\overline{NE}(X)$ coincide.

Standard arguments then apply to show that the right hand side of our equality is already closed, and that the extremal rays are locally discrete. □

Remark 10. *Observe that $Z_{-\infty}$ is in fact the contribution to the cone coming from the non-log canonical locus of (X, Δ).*

2.6 A foliation sub-adjunction result

[Kaw98] proves the following sub-adjunction result:

Theorem 2.6.1. *Let (X, D) be klt for some D. Suppose that W is a center of log canonical singularities for (X, Δ) and let $\nu : W^\nu \to W$ be its normalization. Then if H is ample, there exists an effective divisor Δ_{W^ν} such that $\nu^*(K_X + \Delta + H) = K_{W^\nu} + \Delta_{W^\nu}$.*

In this section we prove a foliated version of this sub-adjunction. We will need the following result due to Hacon on the existence of dlt models, see for example [Fuj11, Theorem 10.4]:

Theorem 2.6.2. *Let X be a quasi projective variety, and B a boundary such that $K_X + B$ is $\mathbb{R}$-Cartier. One can construct a projective birational morphism $f : Y \to X$ where Y is normal and $\mathbb{Q}$-factorial. Furthermore, f only extracts divisors of discrepancy ≤ -1, and if we set $B_Y = f^{-1}_* B + \sum_{f-exceptional} E$, then (Y, B_Y) is dlt.*

We briefly recall the definition of dlt and some related results:

Definition 16. *A pair* $(X, \Delta = \sum a_i \Delta_i)$ *is called dlt (divisorial log terminal) if* $0 \leq a_i \leq 1$ *and there exists a log resolution* $\pi : (Y, \Gamma) \to (X, \Delta)$ *such that* π *only extracts divisors of discrepancy* > -1.

Example 10. $(\mathbb{C}^2, H_1 + H_2)$ *where* H_1, H_2 *are two lines meeting at the origin is dlt, however* $(\mathbb{C}^2, D)$ *where* D *is a nodal cubic is not dlt. Thus being dlt is not a local analytic property.*

Lemma 2.6.3. *Let* (X, Δ) *be dlt and let* $S_1, ..., S_k$ *be the irreducible components of* $\lfloor \Delta \rfloor$.

(1) (X, Δ) *is log canonical.*
(2) S_i *is normal and if we write* $(K_X + \Delta)|_{S_i} = K_{S_i} + \Delta_i$ *then* (S_i, Δ_i) *is dlt.*
(3) *If* $\lfloor \Delta \rfloor = 0$ *then* (X, Δ) *is klt*

Proof. Standard, see for example [KM98]. □

Notice that if (X, B) is log canonical along W, then writing $K_Y + B' = f^*(K_X + B)$ we have that (Y, B') is dlt above the generic point of W.

We will also need the following canonical bundle formula due to [Kaw98], the formulation here is found in [Kol07, Theorem 8.5.1].

Definition 17. *Let* (F, R) *be a sub log canonical pair (i.e.,* R *not necessarily effective) with* $K_F + R = 0$. *Write* $R = R_+ - R_-$ *where* $R_+, R_- \geq 0$ *and have no components in common. Then we define*

$$p_g^+ = h^0(F, \mathcal{O}(\lceil R_- \rceil)).$$

Theorem 2.6.4. *Let* $f : X \to Y$ *be a fibration of normal varieties with general fibre* F. *Suppose that there is a divisor* R *such that*

(1) $K_X + R = f^*D$ *for some* $\mathbb{Q}$*-Cartier divisor on* Y.
(2) (X, R) *is log canonical over the generic point of* Y.
(3) $p_g^+(F, R|_F) = 1$.

Then one can write

$$K_X + R = f^*(K_Y + J + B_R)$$

where

(i) J is the pushforward of a nef divisor on some model of Y, and depends only on Y and $(F, R|_F)$.

(ii) B_R is a $\mathbb{Q}$-divisor such that for B_i a divisor on Y if λ is supremum of $\{t : K_X + R + tf^*B_i$ is log canoncical over the generic point of $B_i\}$ (the log canonical threshold) then the coefficient of B_i in B_R is $1 - \lambda$.

Remark 11. *In [Kol07] (2) is phrased as "slc fibres in codimension 1 over an open subset of Y". This condition is implied by our condition (2).*

Definition 18. *Given a fibration $f : (X, \mathcal{F}) \to (Y, \mathcal{G})$ where $\mathcal{F} = f^*\mathcal{G}$ define the ramification divisor of f as follows: for any divisor Q on Y write $f^*Q = \sum a_i P_i$. Let the ramification be the sum $\sum(a_i - 1)P_i$ as Q runs over all $\mathcal{G}$ invariant divisors on Y.*

Definition 19. *Given a pair (X, Δ) or $(\mathcal{F}, \Delta)$ we say that W is a log canonical centre of $(X/\mathcal{F}, \Delta)$ if $(X/\mathcal{F}, \Delta)$ is log canonical above the generic point of W, and there is a divisor D of discrepancy $= -\epsilon(D)1$ dominating W.*

Lemma 2.6.5. *Let $f : (X, \mathcal{F}) \to (Y, \mathcal{G})$ be a morphism with X normal and Y smooth. Suppose that f has connected fibres, and $\mathcal{F} = f^*\mathcal{G}$. Then $(K_\mathcal{F} - f^*K_\mathcal{G}) = (K_X - f^*K_Y) - R$ where R is the ramification divisor.*

Proof. Since Y is smooth, if ω is a 1-form which determines $\mathcal{G}$, then $df(\omega)$ is a 1-form which determines $\mathcal{F}$ and has zero divisor $= R$. □

Theorem 2.6.6. *Let (X, D) be klt for some $D \geq 0$ and $\mathbb{Q}$-factorial. Suppose that W is a log canonical centre of $(\mathcal{F}, \Delta)$. Furthermore, suppose that W is transverse to the foliation and generically contained in the smooth loci of X and $\mathcal{G}$. Let $\nu : W^\nu \to W$ be the normalization and $\mathcal{G}$ the induced foliation on W^ν. Let H be an ample divisor. Then $\nu^*(K_\mathcal{F} + \Delta + H) = K_\mathcal{G} + \Delta_{W^\nu}$ where $\Delta_{W^\nu} \geq 0$.*

Remark 12. *Observe that when X is smooth, this result follows immediately from Kawamata's subadjunction.*

Proof. First, notice that since W is transverse to the foliation and generically contained in the smooth loci of X and $\mathcal{F}$, that if E is a divisor such that the centre of E on X is W we have that $a(E, \mathcal{F}, \Delta) = a(E, X, \Delta)$, and therefore W is a log canonical centre of (X, Δ).

Let $f : (Y, \mathcal{H}) \to (X, \mathcal{F})$ be a dlt modification and write $f^*(K_X + \Delta) = K_Y + \Gamma'$ and $f^*(K_{\mathcal{F}} + \Delta) = K_{\mathcal{H}} + \Gamma$. Since f only extracts divisors of usual discrepancy ≤ -1, by Corollary 2.3.6 it only extracts divisors of foliation discrepancy ≤ 0, and so $\Gamma \geq 0$. Furthermore, we know that Γ, Γ' agree on divisors dominating W.

Let $E \to W$ be a divisor dominating W which has coefficient 1 in Γ, Γ'. E is transverse to $\mathcal{H}$ and if we write $\mathcal{H}_E$ for the foliation restricted to E we have that $\mathcal{H}_E$ is the pullback of the foliation on W^ν. Let $\sigma : E \to W^\nu$ be induced map.

Write $(K_X + \Gamma')|_E = K_E + \Theta'$ and $(K_{\mathcal{H}} + \Gamma)|_E = K_{\mathcal{H}_E} + \Theta$. Note that $\Theta, \Theta' \geq 0$. By construction (E, Θ') is dlt above the generic point of W^ν (and in particular is log canonical above the generic point of W^ν).

Let $E \xrightarrow{\alpha} U \xrightarrow{\beta} W^\nu$ be the Stein factorization of $E \to W^\nu$. Let $\mu : U' \to U$ be a resolution of singularities of U, let $\mu' : E' \to E$ be a resolution of singularities of the main component of $E \times_U U'$, and let $\tau : E' \to U'$ be the induced map. Let $\mathcal{G}_{U'}, \mathcal{G}_U$ be the pulled back foliations on U', U respectively and let $\mathcal{H}_{E'}$ be the pulled back foliation on E'.

Our picture is as follows:

$$\begin{array}{ccccc}
(E', \mathcal{H}_{E'}) & \xrightarrow{\mu'} & (E, \mathcal{H}_E) & & \\
\downarrow{\scriptstyle \tau} & & \downarrow{\scriptstyle \beta} & \searrow^{\sigma} & \\
(U', \mathcal{G}_{U'}) & \xrightarrow{\mu} & (U, \mathcal{G}_U) & \xrightarrow{\alpha} & (W^\nu, \mathcal{G})
\end{array}$$

Next, let $K_{\mathcal{H}_{E'}} + \Psi = \mu'^*(K_{\mathcal{H}_E} + \Theta)$. We have $K_{\mathcal{H}_{E'}} + \Psi = \tau^*\mu^*\beta^*M = \tau^*N$ where $N = \mu^*\beta^*M$, and $\mu'_*\Psi = \Theta$.

Write $K_{E'} + \Psi' = \mu_E^*(K_E + \Theta')$ and let D be a divisor on E' dominating W. Then the coefficient of D in Ψ is the same as the coefficient of D in Ψ'. This follows by realizing $\mu' : E' \to E$ as an embedded resolution and applying adjunction:

Let $g : (Z, \mathcal{H}_Z) \to (Y, \mathcal{H})$ be a log resolution and write $K_Z + \Gamma'_Z = g^*(K_Y + \Gamma')$ and $K_{\mathcal{H}_Z} + \Gamma_Z = g^*(K_{\mathcal{H}} + \Gamma)$. Then $(K_Z + \Gamma'_Z)|_{E'} = K_{E'} + \Psi'$ and $(K_{\mathcal{H}_Z} + \Gamma_Z)|_{E'} = K_{\mathcal{H}_{E'}} + \Psi$. Again, Γ_Z, Γ'_Z agree on divisors dominating W, and since D is transverse to the foliation, the foliated different along D is just the coefficient of D in $\Gamma_Z|_{E'}$ which agrees with the usual different, i.e., the coefficient of D in $\Gamma'_Z|_{E'}$.

In particular, we see that (E', Ψ') being log canonical above the generic point of U' implies that (E', Ψ) is log canonical over the generic point of U'.

By Lemma 2.6.5 we know

$$(K_{\mathcal{H}_{E'}} - \tau^* K_{\mathcal{G}_{U'}}) = (K_{E'} - \tau^* K_{U'}) - R$$

where $R \geq 0$. Thus, if we write $N = K_{\mathcal{G}_{U'}} + N'$ we have that $K_{\mathcal{H}_{E'}} + \Psi = \tau^*(K_{\mathcal{G}_{U'}} + N')$ and so

$$K_{E'} + \Psi - R = \tau^*(K_U + N').$$

Notice that no component of R dominates W, and so $(E', \Psi - R)$ is log canonical above the generic point of U'.

Let $F' = \tau^{-1}(x)$ be a general fibre and let $\mu_F : F' \to F = \alpha^{-1}(\mu(x))$ be the restricted map. For general x this will be birational with exceptional locus $\mathrm{exc}(\mu') \cap F'$.

If we write $\Psi = \Psi_+ - \Psi_-$ where $\Psi_+, \Psi_- \geq 0$ and have no components in common, then we see that Ψ_- is μ' exceptional, and so $\Psi_-|_{F'}$ is μ_F-exceptional. Choosing F' to be disjoint from $-R$ we see that

$$p_g^+(F', (\Psi - R)|_{F'}) = h^0(F', \mathcal{O}(\lceil \Psi_- \rceil)) = h^0(F, \mu_{F*}\mathcal{O}(\lceil \Psi_- \rceil)) = 1.$$

Thus we can apply the canonical bundle formula to write $N' = J + B_{\Psi - R}$ where J is the pushforward of a nef divisor and so $\mu^* \beta^* M = K_{\mathcal{G}_{U'}} + J + B_{\Psi - R}$. We now show that $\mu_* B_{\Psi - R}$ is effective.

Let B be a divisor on U' such that $\mu_* B \neq 0$. Write $\tau^* B = \sum w_j Q_j$ so that $R = \sum (w_j - 1) Q_j + R'$ where $Q_j \not\subset \mathrm{supp}(R')$. Since μ is an isomorphism along the generic point of B and since $\mu'_* \Psi \geq 0$, if we write $\Psi = \sum a_j Q_j + \Psi'$ we see that if $\mu'_* Q_j \neq 0$ then $a_j \geq 0$, and so $a_j \geq 0$ for some j.

Thus, the log canonical threshold of

$$K_{E'} + \Psi - R + t\tau^* B = K_{E'} + \sum (a_j + 1 - w_j + t w_j) Q_j + \Psi' + R'$$

is ≤ 1, and so the coefficient of B in $B_{\Psi-R}$ is non-negative and so $\mu_* B_{\Psi-R} \geq 0$.

Pushing forward along μ gives us that $\beta^* M = K_{\mathcal{G}_U} + \mu_* J + B$ where $B \geq 0$. By foliated Riemann-Hurwitz we know that $K_{\mathcal{G}_U} = \beta^* K_{\mathcal{G}} + \tilde{R}$ where $\tilde{R} \geq 0$, and so $\beta^* M = \beta^* K_{\mathcal{G}} + \mu_* J + B'$ where $B' \geq 0$.

Pushing forward by β and then dividing by $\deg(\beta)$ gives $M = K_{\mathcal{G}} + \overline{J} + \overline{B}$ where $\overline{J}$ is the pushforward of a nef divisor and $\overline{B}$ is effective.

If H is ample on X, then $\nu^* H$ is ample, and so $\mu_* J' + \nu^* H$ is $\mathbb{Q}$-equivalent to an effective divisor, and our result follows. □

We will only need the above result in the case where $\dim(X) = 3$ and $\dim(W) = 1$. In this case $\mathcal{G}$ is just a foliation by points and so we have $(K_{\mathcal{F}} + \Delta) \cdot W \geq 0$.

In many ways it would be preferable to have a truly "foliated" proof of the above result, i.e., without making reference to Kawamata's canonical bundle formula. We are able to do this in the case where $\dim(X) = 3$ (which as mentioned suffices for our purposes), but are unable to prove the result in the generality above. For the interested reader we explain the proof here:

The argument up until the use of Kawamata's canonical bundle formula is exactly the same. We replace the use of that canonical bundle formula with the following foliated version, whose proof makes no use of Kawamata's result:

Proposition 2.6.7. *Let $\pi : (X, \mathcal{F}) \to (Y, \mathcal{G})$ a fibration where $\pi^* \mathcal{G} = \mathcal{F}$. Suppose that dim$(X) = 2$ and that we have a pair $(\mathcal{F}, \Delta)$ with $\Delta \geq 0$ such that*

1) $K_{\mathcal{F}} + \Delta = \pi^* M$

2) $(\mathcal{F}, \Delta)$ is log canonical above the generic point of Y.

Then $M = J + B$ where

(i) J is nef and

(ii) The coefficient of B_i in B is $\epsilon(B_i) - \lambda$ where λ is the log canonical threshold of $K_{\mathcal{F}} + \Delta + t\pi^* B_i$.

Proof. Notice that since $\mathcal{G}$ is a foliation by points on Y $\epsilon(B_i) = 0$.

Replacing Δ by $\Delta - \pi^*_B$ we may assume that $B = 0$. We still have that $\Delta \geq 0$. Suppose for sake contradiction that M is not nef.

Then $K_{\mathcal{F}} + \Delta$ is negative on some extremal ray R and we can apply the cone theorem for foliations to deduce that R is spanned by some curve tangent to $\mathcal{F}$ or contained in $\text{supp}(\lfloor \Delta \rfloor)$.

$(K_{\mathcal{F}}+\Delta)\cdot C = 0$ for any C tangent to the foliation, and so C must be contained in a component of $\lfloor \Delta \rfloor$ dominating Y. If we write $\Delta = C + \Delta'$ we see that foliation adjunction tells us that that $(K_{\mathcal{F}} + C + \Delta') \cdot C \geq 0$. Thus M must be nef. □

Remark 13. *The formulation above is meant to suggest the generalization of this result to the setting where X is not necessarily a surface and Y is not necessarily a curve.*

2.6.1 Some applications of foliation subadjunction

In this section we present a simple application of foliation subadjunction which will nevertheless be very useful later on.

Lemma 2.6.8. *Let X be a $\mathbb{Q}$-factorial threefold and suppose X is klt. Let $\mathcal{F}$ be a co-rank 1 foliation, and let C be a curve transverse to the foliation not contained in sing(X) with $K_{\mathcal{F}} \cdot C < 0$. Let S be a reduced divisor such that $S \cdot C < 0$. Then S is smooth at the generic point of C. In particular if $\nu^*(K_{\mathcal{F}} + S) = K_{\mathcal{F}_{S^\nu}} + \Theta$, then C is not contained in supp(Θ).*

Proof. Suppose the contrary holds. Perhaps replacing S by $S + \epsilon H$ where ϵ is sufficiently small and $C \subset H$, we may assume that $S \cdot C < 0$, and that the log canonical threshold of S along C is $\lambda < 1$, so C is a log canonical centre of $(\mathcal{F}, \lambda S)$. Notice that the foliated log canonical threshold along C is the same as the usual log canonical threshold.

Since C is transverse to the foliation we have by Theorem 2.6.6 $(K_{\mathcal{F}} + \lambda S + A) \cdot C \geq 0$ where A is any ample $\mathbb{Q}$-divisor. However, this is a contradiction of the negativity of $K_{\mathcal{F}}$ and S along C. □

Corollary 2.6.9. *Let $(X, \mathcal{F}, S)$ be a triple of a $\mathbb{Q}$-factorial threefold a co-rank 1 foliation, and a surface transverse to the foliation. Suppose that (X, D) is klt for some D. Let R be an extremal ray of X such that $K_{\mathcal{F}}$ and S are negative on R.*

Then R is spanned by the class of a curve C which is either tangent to the foliation, or contained in sing$(\mathcal{F}) \cup$ sing(X).

Proof. Let $\nu : S^\nu \to S$ be the normalization of S Write $\nu^*(K_{\mathcal{F}} + S) = K_{\mathcal{F}_{S^\nu}} + \Delta$.

Since $R \cdot S < 0$, there exists an extremal ray R' in $\overline{NE}(S)$ such that $\nu_* R' = R$ in $\overline{NE}(X)$.

By the cone theorem for surface foliations, R' is spanned by a curve C and so R is spanned by $\nu(C)$. Furthermore either C is contained in the support of Δ, or R is spanned by a curve tangent to the foliation.

We know that Δ is supported on $\nu^{-1}(\text{sing}(S) \cup \text{sing}(X) \cup \text{sing}(\mathcal{F}))$ and on curves invariant by the foliation. Suppose that $\nu(C)$ is transverse to the foliation. Then C must be contained in $\nu^{-1}(\text{sing}(S) \cup \text{sing}(X))$, however by the above lemma S is smooth at the generic point of $\nu(C)$. So C is contained in $\nu^{-1}(\text{sing}(X))$ or is tangent to the foliation, and the result follows. □

2.7 On the convergence of separatrices and germs of leaves

As noted it is a subtle question whether or not a separatrix is convergent on a smooth variety, it is perhaps a bit more subtle in the case of a singular variety. The goal of this section is to understand the convergence of separatrices of the foliation, and in particular understand the convergence of separatrices along points of a curve C tangent to the foliation.

Cano and Cerveau in [CC92] prove the following:

Theorem 2.7.1. *Let $(X, \mathcal{F}, 0)$ be the germ of a 3-dimensional complex manifold with a co-rank 1 foliation. Suppose that $\mathcal{F}$ has non-dicritical singularities. Let γ be a curve tangent to the foliation and not contained in sing$(\mathcal{F})$. Then γ is contained in a unique convergent separatrix.*

In what follows we adapt their techniques and ideas to work in the setting where X is singular.

We will use the following fact about simple foliation singularities found in [CC92, Proposition II.5.5]:

Lemma 2.7.2. *Let $(X, 0)$ be a foliated germ with simple foliation singularities. Let $Q_i \to 0$. Suppose that at each Q_i there is a germ of a separatrix S_{Q_i} such that the S_{Q_i} agree on overlaps. Then there is a convergent germ of a separatrix at 0 which agrees with with the S_{Q_i}.*

Lemma 2.7.3. *Let X be smooth, $\mathcal{F}$ a foliation with simple singularities and E a compact invariant divisor. Let γ be a germ of a curve tangent to $\mathcal{F}$ meeting E but not contained in E. Then there exists a neighborhood U of E and a closed $\mathcal{F}$-invariant hypersurface $S \subset U$ such that $\gamma \subset S$.*

Proof. Without loss of generality we may assume that $\gamma \cap E = Q$, a single point.

Furthermore, by passing to a resolution, $\pi : X' \to X$, and letting $E' = \pi^{-1}E$, we may assume that each point of $\text{sing}(\mathcal{F}) \cap E$ has at most 1 (formal) separatrix not contained in $exc(\pi)$.

Let $W = \text{sing}(\mathcal{F}') \cap E'$, Let V be those components of W with exactly one separatrix not contained in $exc(\pi)$, let γ' be the strict transform of γ and let V_0 be the connected component of V containing $\gamma' \cap E'$.

We let $\mathcal{A}$ be the locus of points $P \in V_0$ such that

(a) there exists an open set $P \in U_P$ and separatrix $P \in S'_P$

(b) for every $R \in E' - V_0$ there exists an open set containing R that is disjoint from S'_P.

This set is open, and by the above lemma, we see that it is closed, and therefore is all of V_0.

There exist finitely many P_i such that U_{P_i} cover V_0. For all $R \in E' - V_0$ there exists an open W_R disjoint from all the S'_{P_i}. The W_R, U_{P_i} form an open cover of a neighborhood of E', call it U. Let $S' = \cup S'_{P_i}$, noting that these separatrices agree on overlaps.

Finally, there exists a $V \subset X$ such that $\pi^{-1}(V) \subset U$. Since π is proper, by the proper mapping theorem we have that $S = \pi(S') \subset V$ is still an (invariant) hypersurface. □

Corollary 2.7.4. *Let C be a compact curve tangent to a foliation with non-dicritical singularities on X such that C is not contained in sing$(\mathcal{F}) \cup$ sing(X). Then there is a germ of an analytic surface containing C, call it S, such that S is tangent to the foliation.*

Proof. If $P \in C$ is disjoint from $\text{sing}(\mathcal{F}) \cup \text{sing}(X)$, then there is a (unique) germ of a leaf containing P and C. Thus, it suffices to construct around the points where C meets the singular loci of X and $\mathcal{F}$ a germ of a separatrix containing C. Let Q be one such point. Without loss of generality, we may assume that C is irreducible near Q.

Perhaps replacing X by a germ around Q, we can find a resolution of singularities of both X and the foliation, $\pi : (X', \mathcal{F}') \to (X, \mathcal{F})$ such that X' is smooth, $\mathcal{F}'$ has simple foliation singularities. Furthermore, we may assume that $\pi^{-1}(Q)$ is an invariant divisor, call it E.

Let C' be the strict transfrom of C. Then C' meets E, and by our above extension lemma, there is an open subset containing E call it U and analytic foliation invariant hypersurface $S' \subset U$ containing C'. By the proper mapping theorem we have that $S_Q = \pi(S')$ is our desired germ of an invariant hypersurface.

There exist finitely many Q_i such that $C \subset \cup S_{Q_i}$, and thus there is an open set containing C such that $S = \cup S_{Q_i}$ is an analytic hypersurface in this open subset containing C. □

Remark 14. *Observe that in contrast to the smooth case where every non-dicritical singularity admits at least one convergent separatrix, if X is singular it possible for there to be no separatrices (formal or otherwise) through a particular point $x \in X$. In the case of surfaces an example is given by considering the contraction of an elliptic Gorenstein leaf. In these cases, however, there are no germs of curves tangent to $\mathcal{F}$ passing through x.*

Corollary 2.7.5. *Suppose $\mathcal{F}$ has canonical and non-dicritical singularites and suppose that $C \subset$ sing$(\mathcal{F})$ but not contained in sing(X). Then there exists a germ of a separatrix S containing C such that S agrees with the strong separatrix along C.*

Proof. Through a general point of C there is a germ of a curve γ tangent to $\mathcal{F}$, meeting C at a point and such that γ is contained in a strong separatrix along C. As above, let $\pi : (X', \mathcal{F}') \to (X, \mathcal{F})$ be a resolution of singularities such that $\pi^{-1}(C)$ is a divisor, and in particular, since C is contained in the singular locus, this divisor must be invariant. By our extension lemma there exists an extension of γ to a separatrix S'. Taking $\pi(S')$ gives our desired separatrix. □

Corollary 2.7.6. *Suppose that $\mathcal{F}$ has canonical and non-dicritical singularities and supose that $C \subset sing(X)$. Suppose that $\mathcal{F}$ is terminal at the generic point of C. Then there exists a germ of a separatrix S containing C.*

Proof. Since $\mathcal{F}$ is terminal, taking a general hyperplane section meeting C we see that through a general point of C there exists a germ of a curve γ tangent to $\mathcal{F}$, meeting C transversely. As above let π be a resolution, and since C is both terminal for $\mathcal{F}$ and contained in $\mathrm{sing}(X)$, $\pi^{-1}(C)$ is invariant, and our result follows. □

Corollary 2.7.7. *Let D be a divisor transverse to $\mathcal{F}$ such that $\mathcal{F}_D$ is induced by a fibration $D \to Z$. Let C be a curve tangent to the foliation and S a separatrix around C. Then S has an extension to a neighborhood of D.*

Proof. Let $f_1, ..., f_\ell$ be the fibres of $D \to Z$ meeting C. Perhaps shrinking S a bit, we may assume that $D \cap S \subset f_1 \cup ... \cup f_\ell$.

Observe that if we can extend S to a small neighbhorhood of f_i, call it V_i for all i, then the extension to neighborhood of all of D exists. So, we claim that such an extension exists.

Let $\pi : Y \to X$ be a resolution of singularities so that $\pi^{-1}(f_i) = E_i$ is a divisor. Notice that E_i is invariant. If we let S' be the strict transform of S, then by our extension lemma there exists an extension of S' to a neighborhood of E_i for all i. Pushing forward this extension and these open sets gives our claim. □

2.8 Foliation negative curves contained in leaves of dimension 2

We begin with an algebraicity criterion:

Lemma 2.8.1. *Let C be a compact curve, and S an analytic surface germ, sitting inside a projective variety X and C is not contained in sing(S). Assume that $K_{S^\nu} + \Delta$ is $\mathbb{Q}$-Cartier and $(K_{S^\nu} + \Delta) \cdot C < 0$, and that Δ is a boundary along C. Then either C is rational and $(K_{S^\nu} + \Delta) \cdot C \geq -2$ or S is algebraic, i.e., the Zariski closure of S is an algebraic surface.*

Proof. Let Y be the Zariski closure of S with $K(Y)$ the field of rational functions on Y. By observations due to [Bos01], [BM01] we see that the algebraicity of S follows if the transcendence degree of $K(Y)$ over $\mathbb{C}$ is 2.

Let $T \xrightarrow{f} S^\nu \xrightarrow{g} S$ be the minimal resolution of the normalization S (perhaps after restricting S to a smaller neighborhood of C) and let C' be the strict transform of C. We have $K_T + \Delta_T = f^*(K_{S^\nu} + \Delta)$ where $\Delta_T \geq 0$.

Let $\mathfrak{T}$ denote the formal scheme given by the completion of T along C' and let $K(\mathfrak{T})$ be the field of formal meromorphic functions on $\mathfrak{T}$. Notice that $K(\mathfrak{T})$ is a field extension of $K(Y)$, and so it suffices to bound the trascendence degree of $K(\mathfrak{T})$.

Next, since T is smooth $\mathcal{O}_T(C')$ is Cartier, in particular, C' is a local complete intersection in $\mathfrak{T}$. Let $\nu : C^\nu \to C'$ be the normalization.

By assumption there exists a $t \geq 0$ such that $K_T + \Delta_T + tC' = K_T + \Theta + C'$ where $\Theta \geq 0$. By adjunction $(K_T + \Theta + C') \cdot C' = 2g(C^\nu) - 2 + d_{C'}$, where $d_{C'} \geq 0$.

If $\nu^*\mathcal{O}(C')$ is not ample, then the left hand side of the equation is negative, hence C' is rational, and $(K_T + \Delta_T) \cdot C' \geq (K_T + \Theta + C') \cdot C' \geq -2$.

On the other hand, if $\nu^*\mathcal{O}(C')$ is ample, the normal bundle of C' in $\mathfrak{T}$ is ample, which by a result of Hartshorne, [Har68, Theorem 6.7], implies that $K(\mathfrak{T})$ has transcendence degree at most 2 over $\mathbb{C}$, and our result follows.

We briefly explain the idea behind the proof of Hartshorne's result in our case. If L is a line bundle on $\mathfrak{T}$ we want to bound the growth rate $h^0(\mathfrak{T}, nL) \leq Dn^2$. However, we also know that $h^0(\mathfrak{T}, nL) \leq \sum_{m=0}^{\infty} h^0(C', -mC' + nL)$. Since C' is ample, we know that $H^0(C', -mC' + nL) = 0$ for $m \geq D'n$ for some D' depending only on C' and L, and $h^0(C', -mC' + nL) \leq D''n$ where D'' depends only on L. Putting these together, we get a bound on the above sum by $\sum_{m=0}^{D'n} D''n = D'D''n^2$. $\square$

We can prove the above algebraicity criterion in a different way which suggests an algebraicity criterion in higher dimensions. First, recall the following fact about the deformation theory of curves [Kol96]

Theorem 2.8.2. *Let X be a l.c.i. n-dimensional (analytic) variety and $C \subset X$ a projective curve. Then the dimension of the component of the chow variety containing $[C]$ is at least $-K_X \cdot C + (n-3)(1-g(C^\nu))$ where $C^\nu \to C$ is the normalization.*

Corollary 2.8.3. *Let $C \subset S \subset X$ where X is a projective 3-fold, and S is a complex analytic surface with $(K_S + \Delta) \cdot C < 0$. Suppose that C is not contained in sing(S) or supp(Δ). Then one of the following holds:*

1) $C \cong \mathbb{P}^1$ and $(K_S + \Delta) \cdot C \geq -1$

2) S is algebraic

3) C moves in a family covering X.

Proof. Let $\pi : T \to S$ be the minimal resolution and write $K_T + \Gamma = \pi^*(K_S + \Delta)$ where $\Gamma \geq 0$. Let C' be the strict transform of C By the above theorem, the dimension of the deformation space of $C' \subset T$, and hence the deformation space of $C \subset S$, is positive dimensional if either C is not rational, or C is rational and $(K_S + \Delta) < -1$. If the deformation space of C in S is positive dimensional, the same is true of the deformation space of C in X.

Let Z denote the subvariety of X swept out by the images of the deformations of C, notice that $\dim(Z) \geq 2$ and that $S \subset Z$. If $Z = X$ then we are in case 3. If $\dim(Z) = 2$, then since $S \subset Z$, we have that S must be a component of Z, and is therefore algebraic. □

Corollary 2.8.4. *Let $C \subset Y \subset X$ where X is a projective 4-fold and Y is an (analytic) 3-fold with $K_Y \cdot C < 0$. Then either*

1) Y contains an algebraic surface,

2) Y is algebraic or,

3) C moves in a family covering X.

In the following proof we will make use of the following definition:

Definition 20. *Given a reflexive sheaf L and a positive integer $q \leq dim(X)$ a Pfaff field of rank q is a non-zero morphism $\Omega^q_X \to L$. Given foliation of $\mathcal{F}$ of rank q, by taking the q-th wedge power of $\Omega^1_X \to \mathcal{F}^*$ we get a Pfaff field $\Omega^q_X \to \mathcal{O}(K_{\mathcal{F}})$ of rank q.*

Lemma 2.8.5. *Let X be 3-fold. Suppose that $K_{\mathcal{F}}$ is $\mathbb{Q}$-Cartier and $\mathcal{F}$ has only non-dicritical singularities. Let C be a compact curve tangent to the foliation such that C is not contained in $sing(X) \cup sing(\mathcal{F})$.*

Then there exists a germ of an analytic surface S such that C is contained in S, and S is foliation invariant.

If $\nu : S^\nu \to S$ is the normalization, then $\nu^* K_{\mathcal{F}} = K_{S^\nu} + \Delta$ where $\Delta \geq 0$.

Proof. By our extension lemmas we get the existence of the germ S containing C.

To prove our last statement, if $\Omega^2_X \to \mathcal{O}(K_{\mathcal{F}})$ is the Pfaff field associated to our foliation, since S is foliation invariant we have a morphism $(\Omega^2_S)^{\otimes m} \to \mathcal{O}_S(mK_{\mathcal{F}})$, where m is the Cartier index of $K_{\mathcal{F}}$. We can apply [AD14, Lemma 3.7] to see that this lifts to a map $(\Omega^2_{S^\nu})^{\otimes m} \to \nu^* O_S(mK_{\mathcal{F}})$. Observe that the lemma is proven in the case where S is a variety, however the proof works just as well in the case where S is an analytic variety. For the reader's convenience we explain the proof below.

Thus, we have a nonzero map $\mathcal{O}(mK_{S^\nu}) \to \mathcal{O}(m\nu^* K_{\mathcal{F}})$ and our result follows. Observe that Δ is supported on the locus where this map fails to be surjective, which is contained within $\mathrm{sing}(X) \cup \mathrm{sing}(\mathcal{F})$.

□

Lemma 2.8.6. *Let $K_{\mathcal{F}}$ be a $\mathbb{Q}$-Cartier divisor of Cartier index m and let S be a germ of an analytic space tangent to the foliation, and hence invariant. Then there is a lift of $(\Omega^2_X)^{\otimes m} \to \mathcal{O}(mK_{\mathcal{F}})$ to $(\Omega^2_Z)^{\otimes m} \to \nu^* \mathcal{O}(mK_{\mathcal{F}})$ where $\nu : Z \to S$ is the normalization.*

Proof. This is proven in [AD14] in the case where S is algebraic, although the argument works in the analytic case. We sketch the argument here.

First assume that $K_{\mathcal{F}}$ is Cartier. Notice that the lift can be constructed locally so we may assume that $K_{\mathcal{F}} = \mathcal{O}_X$. Furthermore, it suffices construct the lift on each irreducible component of S, so we may assume that S is irreducible.

Since S is invariant, and ν is bimeromorphic, we see that the Pfaff field on S lifts to a meromorphic Pfaff field on Z. Thus to prove our result it suffices to show that the lift $\phi : \Omega^2_Z \to \mathcal{M}_Z$ is in fact holomorphic.

This can be checked on stalks. So let $x \in Z$, and our Pfaff field is equivalent ot the data of a 2-derivation of $\mathcal{O}_{Z,x}$ into $M_{Z,x}$. However, by [ADK08, Proposition 4.5] , we see that this is 2-derivation actually takes values in $\mathcal{O}_{Z,x}$ which is what we wanted to show.

Now, let $\pi : X' \to X$ be the index 1 cover associated to $K_{\mathcal{F}}$, i.e., we have a foliation $\mathcal{F}'$ such that $K_{\mathcal{F}'}$ is Cartier. Let $S' = \pi^{-1}(S)$ and Z' be the normalization of S'. Let G be the Galois group of the cover. Let π_S, π_Z be the induced maps from $S' \to S$, $Z' \to Z$

Observe that the morphsim

$$\pi^*((\Omega^2_Z)^{\otimes m}) = (\pi^*\Omega^2_Z)^{\otimes m} \to (\Omega^2_{Z'})^{\otimes m} \to \nu'^*\mathcal{O}(mK_{\mathcal{F}'}) = \\ \nu'^*\pi^*O(mK_{\mathcal{F}}) = \pi_Z^*\nu^*\mathcal{O}(mK_{\mathcal{F}})$$

is G-linear and therefore descends to a morphism $(\Omega^2_Z)^{\otimes m} \to \nu^*\mathcal{O}(mK_{\mathcal{F}})$.

□

Example 11. *In the case that X is smooth with simple singularities, the computation of Δ is easy. Δ is supported on $\nu^{-1}(sing(\mathcal{F}))$ and if Z is a component of $sing(\mathcal{F})$ and S is a strong separatrix along Z, the coefficient of Z in Δ is exactly 1. Otherwise the coefficient of Z is some positive integer k which depends on the analytic type of the singularity.*

Remark 15. *There is the following alternative construction of Δ. Let $f : (Y, \mathcal{G}) \to (X, \mathcal{F})$ be a resolution of singularities and write $K_{\mathcal{G}} + \Gamma = f^*K_{\mathcal{F}}$. Let S' be the strict transform of S, and notice that we have a moprhism $\sigma : S' \to S^\nu$.*

Write $(K_{\mathcal{G}} + \Gamma)|_{S'} = K_{S'} + \Delta'$ and take $\Delta = \sigma_*\Delta'$. Then $K_{\mathcal{F}}|_{S^\nu} = K_{S^\nu} + \Delta$. It is not hard to check that this construction of Δ agrees with our previous one.

One can think of the above result together with Lemma 2.3.3 as being a complete version of foliation adjunction, i.e.,

Proposition 2.8.7. *Let $\mathcal{F}$ be a co-rank 1 foliation. Suppose that $K_{\mathcal{F}}+\Delta+\epsilon(S)S$ is $\mathbb{Q}$-Cartier and log canonical. Let $\nu : S^\nu \to S$ be the normalization. Call the induced foliation $\mathcal{G}$, so $\mathcal{G}$ is co-rank 1 if S is transverse to the foliation and it T_{S^ν} is S is invariant. Then $\nu^*(K_{\mathcal{F}}+\Delta+\epsilon(S)S) = K_{\mathcal{G}}+\Delta$ where $\Delta \geq 0$.*

Lemma 2.8.8. *Suppose that $(\mathcal{F},\Delta)$ is log canonical, and let S be an invariant divisor. Write $(K_{\mathcal{F}}+\Delta)|_{S^\nu} = K_{S^\nu}+\Delta_{S^\nu}$. Then (S^ν,Δ_{S^ν}) is log canonical away from sing$(\mathcal{F}) \cup$ sing$(X) \cup$ sing(S).*

Proof. Let $\pi : (Y,\mathcal{G}) \to (X,\mathcal{F})$ be a log resolution of both X and $\mathcal{F}$, i.e., $\mathcal{F}$ has simple singularities and $S' \cup \mathrm{supp}(\Delta) \cup \mathrm{exc}(\pi)$ is snc, where S' is the strict transform of S.

Write $K_{\mathcal{G}}+\Gamma = \pi^*(K_X+\Delta)$ and $(K_{\mathcal{G}}+\Gamma)|_{S'} = K_{S'}+\Delta_{S'}$ so that if $\sigma : S' \to S^\nu$ is the induced map $\sigma_*\Delta_{S'} = \Delta_{S^\nu}$.

Let $B \subset \mathrm{supp}(\Delta_{S'})$ be such that $\sigma_* B \neq 0$ and $\pi(B)$ is not contained in sing$(\mathcal{F})\cup$ sing(X). If we write $K_{\mathcal{G}}|_{S'} = K_{S'}+\Theta$ notice that (S',Θ) is log canonical at the generic point of B. If $E \subset Y$ is any divisor dominating $\pi(B)$ then E must be invariant, and so the discrepancy of $(\mathcal{F},\Delta)$ along E is at least 0. Combining these two observations gives our result. □

We finish the section with our characterization of $(K_{\mathcal{F}}+\Delta)$-negative curves tangent to a foliation.

Lemma 2.8.9. *Let C be a curve tangent to $\mathcal{F}$, not contained in sing(X), with $(K_{\mathcal{F}}+\Delta)\cdot C < 0$. Suppose that $\mathcal{F}$ has canonical non-dicritical singularities and that $(\mathcal{F},\Delta)$ is log canonical. Then, $[C] = \sum a_i[M_i]+\beta$ where $(K_{\mathcal{F}}+\Delta)\cdot\beta \geq 0$ and the M_i are either*

(i) rational curves tangent to the foliation $0 > K_{\mathcal{F}}\cdot M_i \geq -4$

(ii) $M_i \subset$ sing(X).

Proof. If C is contained in the singular locus of the foliation, we argue as below.

Otherwise, let S be the surface germ tangent to the foliation which contains C. Write $(K_{\mathcal{F}}+\Delta) = K_{S^\nu}+\Theta$. Since Θ is a boundary along C, by our algebraicity criterion, we see that either C is rational, or S is algebraic, in which case we can apply the usual cone theorem for surfaces.

Notice that the non-log canonical locus of (S^ν, Θ) is supported on the singular loci of X and $\mathcal{F}$. The cone theorem for surfaces tells us that in $\overline{NE}(S^\nu)$ we can write $[C] = \sum a_i[L_i] + \beta$ where the L_i are curves contained in the non-log canonical locus of (S^ν, Θ) or are rational curves with $(K_{S^\nu} + \Theta) \cdot L_i = (K_\mathcal{F} + \Delta) \cdot L_i \geq -4$, and $(K_\mathcal{F} + \Delta) \cdot \beta \geq 0$. Pushing forward to X gives our result. □

Lemma 2.8.10. *Suppose that $\mathcal{F}$ has canonical and non-dicritical singularities. Let $C \subset sing(\mathcal{F})$ with $(K_\mathcal{F} + \Delta) \cdot C < 0$. Suppose that $(\mathcal{F}, \Delta)$ is log canonical, then C is rational, and $K_\mathcal{F} \cdot C \geq -2$.*

Proof. First, since $(\mathcal{F}, \Delta)$ is log canonical, C cannot be contained in the support of Δ. Next, since $\mathcal{F}$ is canonical along C, there is a strong separatrix for $\mathcal{F}$ around a general point of C. By our extension lemmas, this strong separatrix extends to an analytic divisor containing C, call it S. $(K_\mathcal{F} + \Delta)|_S = K_S + aC + \Theta$ where Θ is effective and does not contain C. Since S is a strong separatrix we see that $a = 1$. Adjunction and the inequality $(K_S + C + \Theta) \cdot C < 0$ imply that C is rational and $(K_S + C + \Theta) \cdot C \geq -2$. □

2.9 The cone theorem

With the work of the previous sections in hand, we are now in a position to give a proof of the foliated cone theorem. The argument is similar to the one used to prove the cone theorem for surfaces in section 5.

Theorem 2.9.1. *Let X be a klt, $\mathbb{Q}$-factorial threefold and $\mathcal{F}$ a co-rank 1 foliation with canonical and non-dicritical foliation singularities. Then*

$$\overline{NE}(X) = \overline{NE}(X)_{K_\mathcal{F} \geq 0} + \sum \mathbb{R}_+[L_i]$$

where L_i are curves.

Furthermore either L_i is contained in $sing(X)$, or L_i is a rational curve with $K_\mathcal{F} \cdot L_i \geq -6$

In particular, the $K_\mathcal{F}$-negative extremal rays are locally discrete in the $K_\mathcal{F} < 0$ portion of the cone.

Proof. Choose H and ample divisor and $t \in \mathbb{R}$ such that $H_R = K_{\mathcal{F}} + tH$ is nef, and such that H_R is zero on precisely one extremal ray, $R = \mathbb{R}\alpha$. We argue based on the numerical dimension of D.

If $\nu = \nu(H_R) < 3$ there exists a k depending on ν such that $H_R^k H^{3-k} = 0$. As in our bend and break lemma, take $D_i = H_R$ for $i \leq 3-k$ and $D_i = H$ otherwise. Observe that

$$K_{\mathcal{F}} \cdot D_2 \cdot D_3 = D_1 \cdot D_2 \cdot D_3 - tH \cdot D_2 \cdot D_3 < 0$$

Thus, our bend and break lemma applies to produce rational curves Σ with $\Sigma \cdot D_1 = \Sigma \cdot H_R = 0$.

Take $M = mH_R - K_{\mathcal{F}}$ where $m \gg 0$ so that M is ample. Our bend and break lemma tells us that

$$M \cdot \Sigma \leq 2(3) \frac{M \cdot D_2 \cdot ... \cdot D_n}{-K_{\mathcal{F}} \cdot D_2 \cdot ... \cdot D_n}$$

Noting that $M \cdot \Sigma = -K_{\mathcal{F}} \cdot \Sigma$ and $M \cdot D_2 \cdot ... \cdot D_n = -K_{\mathcal{F}} \cdot D_2 \cdot ... \cdot D_n$ gives our desired bound on the degree of Σ.

So, suppose that $\nu(D) = 3$. Since D is nef, it is also big. Then, perturbing by some $\epsilon > 0$ sufficiently small, we may take $K_{\mathcal{F}} + (t-\epsilon)H$ to still be big, and negative on R.

Thus, there exists some effective prime divisor D such that $D \cdot \alpha < 0$. Note that D is $\mathbb{Q}$-Cartier.

For any $\beta \in NE(X)$ close enough to α we can write $\beta = \sum a_i[C_i]$ where $C_i \cdot D < 0$. Letting β approach α, we see R comes from an extremal ray in $\overline{NE}(D)$

Either D is invariant, in which case Lemma 7.4 applies or D is generically transverse to the foliation, in which case, since R is $K_{\mathcal{F}}$ and D-negative Lemma 6.4 applies to show that R is spanned by the class of a curve, and furthermore we can take this curve to be contained in $\text{sing}(X)$ or tangent to the foliation, in which Lemma 7.4 applies again.

In any case, either R comes from $\text{sing}(X)$, comes from a component of $\text{sing}(\mathcal{F})$ meeting a non-simple singularity or is spanned by the class of a rational curve C tangent to the foliation with $K_{\mathcal{F}} \cdot C \geq -4$.

Our result then follows by standard arguments to show that the cone of curves indeed has the claimed structure. □

Portions of the work in the above chapter is being prepared for submission for publication.

Spicer, Calum "Higher dimensional foliated Mori Theory".

The dissertation author was the primary investigator and author of this material.

Chapter 3

Some classification results

This chapter has two main goals: describing the geometry of the extremal rays in the cone of curves and explaining how to contract these in some special cases.

We will return to question of contractions of extremal rays in the next chapter, where many of the results in this chapter will be proven in greater generality using results from the classical MMP. Nevertheless, the explicit classifications in this chapter are of some interest since they allow us to contstruct the MMP for smooth foliations, see Theorem 3.2.2.

3.1 Classifying extremal rays

In this section we provide some results classifying the structure of $K_{\mathcal{F}}$-negative extremal rays.

Definition 21. *Given an extremal ray $R \subset \overline{NE}(X)$ we define loc(R) to be all those points x such that there exists a curve C with $x \in C$ and $[C] \in R$.*

Lemma 3.1.1. *Let R be a $K_{\mathcal{F}}$-negative extremal ray. Then loc(R) is closed.*

Proof. Let H_R be a supporting hyperplane to R.

If $\nu(H_R) < 3$, then as in the proof of the cone theorem, X is covered by rational curves which span R and so $\text{loc}(R) = X$.

Otherwise H_R is big and nef, and so there exists an irreducible effective divisor S with $R \cdot S < 0$ then $\mathrm{loc}(R) \subset S$ and so it remains to show that either $\mathrm{loc}(R)$ is a finite collection of curves or is all of S.

We argue depending on whether S is (i) invariant or (ii) not invariant. Let $S^\nu \to S$ be the normalization.

In case (i), write $K_{\mathcal{F}}|_{S^\nu} = K_{S^\nu} + \Delta$ and $H_R = K_{\mathcal{F}} + A$ for some ample A. Thus $(K_{\mathcal{F}} + A)|_{S^\nu} = (K_{S^\nu} + \Delta + A|_{S^\nu})$ and we argue based on $\nu(H_R|_{S^\nu})$. If $\nu = 2$ then it is big, and therefore zero on only finitely many curves. Otherwise, we can apply bend and break to produce rational curves M through a general point of S with $M \cdot H_R = 0$.

In case (ii), let $\mathcal{G}$ be the foliation restricted to S^ν and $(K_{\mathcal{F}} + S)|_{S^\nu} = K_{\mathcal{G}} + \Delta$. Observe that we can write $H_R = (K_{\mathcal{F}} + S) + A$ for some ample divisor A, and so $H_R|_S = (K_{\mathcal{G}} + \Delta) + A|_S$ where $A|_S$ is ample. If $\nu(H_R|_{S^\nu}) = 2$ then H_R is only zero on finitely many curves. Otherwise we can apply bend and break to produce rational curves M through a general point of S which has $M \cdot H_R = 0$.

□

Remark 16. *Observe that these arguments prove that for any extremal ray R, if $loc(R) = S$ where S is a surface, then S is covered by a family of rational curves tangent to the foliation, each of which spans R.*

Remark 17. *In the case where $K_{\mathcal{F}}$ is pseudoeffective this also follows from the following general fact about big and nef divisors on threefolds: If $D = A + E$ is big and nef then the union of curves C with $D \cdot C = 0$ is a closed subset. Indeed, any such $C \subset E$, and if E_i is a component of E since $L|_{E_i}$ is a nef divisor on a surface, it is either 0 or finitely many curves, or is zero on a family of curves covering E_i.*

In rest of this section we assume that X is smooth and $\mathcal{F}$ has simple singularities.

Definition 22. *We say a foliation on a variety X is algebraically integrable if the closure of a general leaf is a closed subvariety of X.*

We make note of the following result, proved in [AD13, Theorem 5.1].

Proposition 3.1.2. *An algebraically integrable foliation with non-dicritical foliation singularities cannot have an anti-ample canonical divisor.*

Corollary 3.1.3. *Let S be a normal surface, and $\mathcal{F}$ a foliation with non-dicritical singularities. Then $-K_{\mathcal{F}}$ is not ample.*

Proof. If $-K_{\mathcal{F}}$ we ample, then by [BM01] it would be algebraically integrable, a contradiction. □

Let R be an extremal ray. There are three possibilities, either $\mathrm{loc}(R)$ is (i) all of X, (ii) a surface or (iii) 1-dimensional.

3.1.1 loc$(R) = X$

Here our arguments follow [Kol91, Section 4]

We have a diagram

$$\begin{array}{ccc} U & \xrightarrow{F} & X \\ \downarrow{\scriptstyle p} & & \\ Z & & \end{array}$$

where U is a family of rational curves over Z, Z is normal and F is dominant.

If C_z for $z \in Z$ is a fibre of p and D_z is the image of C_z under F and D_{gen} is a general curve, then either:

(1) D_{gen} intersects infinitely many other D_z

(2) D_{gen} does not intersect any other D_z

We tackle the case of (1) first.

Theorem 3.1.4. *Assume the situation is as in (1). Either*

(i) dim$(N(X)) = 1$; in particular $\mathcal{F}$ is Fano if $D_{gen}.K_{\mathcal{F}} < 0$ or,

(ii) dim$(N(X)) = 2$ and there is a map $q : X \to E$ to a smooth curve E such that every D_z is contained in a a fibre of q. The fibres of q are all irreducible. If $D_z \cdot K_{\mathcal{F}} < 0$ and $\mathcal{F}$ has non-dicritical foliation singularities, $\mathcal{F}$ is the foliation induced by q.

Proof. Except for the last claim, the proof is as in [Kol91]. For the last claim, a priori $\mathcal{F}$ might not agree with the fibration, in which case we get an induced foliation $\mathcal{F}_e$ on X_e where $X_e = q^{-1}(e)$ for $e \in E$. Observe that $\mathcal{F}_e$ has non-dicritical foliation singularities.

If we have that $K_{\mathcal{F}} \cdot D_z < 0$, then $\mathcal{F}_e$ would be rank 1 and Fano, but this is a contradiction. □

Theorem 3.1.5. *Assume that situation is as in case (2). Then, there is a morphism $g : X \to Y$ where Y is a normal surface, and the fibres of g are the D_z In particular $dg(\mathcal{F}) = \mathcal{G}$ is a foliation on Y, and $\mathcal{F}$ is the pull back of $\mathcal{G}$*

Proof. Except for the last claim, this is as in [Kol91]. The last claim follows from the observation that the fibres of g are all foliation invariant. □

3.1.2 loc(R) is 2 dimensional

We make an easy observation:

Lemma 3.1.6. *Let L be a an algebraic component of a leaf of $\mathcal{F}$, and assume that $\mathcal{F}$ has simple singularities. Then the closure of L has at worst normal crossings.*

We follow the ideas in [Kol91, Theorem 2.1].

Lemma 3.1.7. *Let X be a three dimensional smooth algebraic space. Let $S \to B$ be a proper smooth minimal ruled surface with typical fibre F. Let $f : S \to X$ be a morphism, and E the image of S, and C the image of F. Assume dim$(E) = 2, C \cdot K_{\mathcal{F}} < 0, C \cdot E < 0$. Then, we are in one of the following situations:*

(i) E is a smooth minimal ruled surface with typical fibres C. If E is not foliation invariant, the foliation restricted to E agrees with the ruling.

(ii) $E \cong \mathbb{P}^2$, and E is foliation invariant.

Proof. Either the image of S is invariant or it is not.

We first address the non-invariant case. By adjunction we get that $K_{\mathcal{F}_S} = f^*(K_{\mathcal{F}} + E) - \Delta$. Now, since $F^2 = 0$ and $K_{\mathcal{F}_S} \cdot F \leq -2 < 0$, by foliation adjunction we see that F must be invariant. For F tangent to a foliation on a surface we always have $K_{\mathcal{F}_S} \cdot F \geq -2$ so we in fact get $K_{\mathcal{F}_S} \cdot F = -2$. Thus Δ is wholly contained in the fibres of $S \to B$, and $K_{\mathcal{F}} \cdot C, E \cdot C = -1$, and the foliation agrees with the ruling on S. Notice that by non-dicriticallity of our foliation singularities f cannot contract any section of S.

Next, notice that $(K_X + E) \cdot C = K_E \cdot C = -2$. Since $E \cdot C = -1$, we get that $K_X \cdot C = -1$. Since C is K_X-negative and the classification of K_X-negative extremal rays implies that E is smooth along the image of any fibre of $S \to B$.

Observe also that in this case not every curve in E spans the same extremal ray in X- otherwise the foliation restricted to E would be an algebraically integrable Fano foliation, a contradiction.

Now we handle the invariant case. First, observe by our previous lemma that E is normal crossings. We argue depending on whether $C^2 = f(F)^2 = 0$ or $C^2 > 0$.

Let $g : T \to E$ be the minimal desingularization. Since the image of T is normal crossings, we see that g is just the normalization map. Observe that $g^*K_{\mathcal{F}} = K_T + \Delta$ where $\Delta \geq 0$.

Suppose that $C^2 > 0$. Following [Kol91] we see that $T \cong \mathbb{P}^2$ or a minimal ruled surface.

Suppose that $T \cong \mathbb{P}^2$, and that E is not normal.

Since $K_T + \Delta = \mathcal{O}(-1)$ we see that Δ is a (possibly singular) conic.

Let $\ell \subset E$ be a component of the non-normal locus of E. Note that $\ell \subset \text{sing}(\mathcal{F})$, but we also have that $K_{\mathcal{F}}.\ell < 0$ since ℓ spans R. However, by Lemma 7.5 this implies that ℓ is a smooth rational curve.

Consider the blow up $\pi : \tilde{X} \to X$ at ℓ with exceptional divisor D. Observe that D is a $\mathbb{P}^1$-bundle over ℓ, and D meets $\tilde{E}$, the strict transform, along 2 disjoint sections of D. In particular, since we have a factorization $\mathbb{P}^2 \to \tilde{E} \to E$ this implies the existence of two disjoint curves in $\mathbb{P}^2$- a contradiction.

Thus, we see that E is normal, and hence isomorphic to $\mathbb{P}^2$.

Now, suppose that T is a minimal ruled surface over B. Let $\Delta = aF + b\sigma$ where F is a fibre and σ is a section. Since $K_{\mathcal{F}} \cdot C < 0$, we must have $b \leq 1$. Thus, we see that the non-normal locus of E is entirely contained in the fibres, otherwise, we would have $b \geq 2$.

In the $f(F)^2 = 0$ case, we get that T is a ruled surface and so, as before we have $\Delta = aF + b\sigma$, and $b \leq 1$, hence the non-normal locus of E is contained in the fibres.

It remains to show that the non-normal locus is in fact empty. Let B be contained in the non-normal locus, notice that B is the image of a fibre of S. Observe that E is at worst normal crossings, so denote by D_1, D_2 the two branches of E containing B. Passing to a small neighborhood of B we see that $D_1|_{D_2} = B$, and so $D_1 \cdot B = D_1|_{D_2} \cdot B = B^2$ where the last intersection is taken in D_2. Switching the roles of D_1, D_2 shows that $D_1 \cdot B = D_2 \cdot B$. There is an exact sequence

$$0 \to \mathcal{O}_B(D_1) \to N_{B/X} \to \mathcal{O}_B(D_2) \to 0$$

.

If $D_i \cdot B \geq 0$ then in fact $K_X \cdot B < 0$ and the result follows by the classification of K_X-negative extremal rays. Otherwise the normal bundle of B is anti-ample, in which case B cannot move in a family, a contradiction.

□

Theorem 3.1.8. *Let $(X, \mathcal{F})$ be a co-rank 1 foliation on a smooth threefold X, an extremal ray R an irreducible surface E such that $R \cdot E < 0$. Assume for $[C] \in R$ that C moves in a family and that $K_{\mathcal{F}} \cdot C$ is maximal, then E is one of the surfaces described above.*

Proof. By supposition C has a non-trivial deformation. Thus, we get a morphism from a (not necessarily minimal) ruled surface $g : S \to X$.

Suppose that S is not minimal. Thus, it contains some reducible fibre $F_0 = \sum a_k f_k$ with $a_k \geq 0$ and components f_k. Let F_1 denote the fibre which gets sent to C under g. Then $g(F_1) \cdot K_{\mathcal{F}} = g(F_0) \cdot K_{\mathcal{F}} = (\sum a_k g(f_k)) \cdot K_{\mathcal{F}}$. But, in this case $g(f_k)$ is a rational curve with $K_{\mathcal{F}} \cdot g(f_k) > K_{\mathcal{F}} \cdot g(F_1) = K_{\mathcal{F}} \cdot C$, a contradiction of the maximality of the intersection of C with $K_{\mathcal{F}}$.

Thus, S is minimal, and we have reduced to our previous lemma. □

3.1.3 loc(R) is 1 dimensional

We begin with two examples showing that this case can really happen, see also [BP11] for some similar examples:

Example 12. *Let $\phi : X_1 \dashrightarrow X_2$ be the threefold toric flop. We can realize X_i as an $\mathbb{A}^1$-bundle over $\mathbb{A}^2$ blown up at a point, with exceptional curve C_i. Let $\mathcal{G}$ be a foliation on $\mathbb{A}^2$ blown up at a point so that the exceptional curve is invariant, meets exactly two other invariant curves and $\mathcal{G}$ has canonical singularities.*

Let $\mathcal{F}_2$ be the pull back of this foliation to X_2. Let $\mathcal{F}_1$ be the strict transform of $\mathcal{F}_2$ under ϕ^{-1}. C_1 is the flop of C_2, and observe that $C_1 \subset sing(\mathcal{F}_1)$, and $\mathcal{F}_1$ has canonical singularities along C_1

Let X_0 be the blow up of X_i along C_i, with exceptional divisor E. and $\mathcal{F}_0$ the transformed foliation on X_0. Let $\pi_i : X_0 \to X_i$. Let $\tilde{C}_1$ be a $\mathbb{P}^1$ sitting above C_1.

Observe that $\mathcal{F}_0|_E = K_E + \Delta$ where Δ consists of three of the four torus invariant divisors on E. Thus, $K_{\mathcal{F}_0} \cdot \tilde{C}_1 = -1$ and since $\pi^* K_{\mathcal{F}_1} = K_{\mathcal{F}_0}$ we get that $K_{\mathcal{F}_1} \cdot C_1 = -1$.

Furthermore, we can check that $K_{\mathcal{F}_2} \cdot C_2 = 1$.

Thus, we see that C_1 is an isolated $K_{\mathcal{F}_1}$-negative extremal ray, and the flip of C_1 exists.

Example 13. *Let X_0, X_1, X_2 be as above. Observe that X_0 is the blow up at the vertex of the cone over a quadric, Q, in $\mathbb{P}^3$. Let $\mathcal{F}_2$ be the foliation coming from one of the projection $\pi_2 : Q \to \mathbb{P}^1$.*

We can lift this foliation to all of X_0 in a torus invariant way, giving us a torus invariant foliation, $\widetilde{\mathcal{F}_2}$.

Let $f_i : X_0 \to X_i$ be the contraction lifting the contraction $\pi_i : Q \to \mathbb{P}^1$. Let $\mathcal{G}$ be the pushforward of $\mathcal{F}_2$ along f_1. Let $C = \pi_1(Q)$. One can compute that $K_{\mathcal{G}} \cdot C = -2$, however C does not move in a two dimensional family.

If we let $\mathcal{G}^+$ be the pushforward of $\widetilde{\mathcal{F}_2}$ under f_2 and $C^+ = f_2(Q)$ we have that $K_{\mathcal{G}^+} \cdot C^+ = 2$. Thus, $\phi : X_1 \dashrightarrow X_2$ is a foliation flip.

Observe, however, that C is a log canonical singularity of $\mathcal{G}$.

We also have following local version of Reeb stability:

Lemma 3.1.9. *Let L be a leaf of a foliation $\mathcal{F}$ on X and $K \subset L$ a compact subset. Suppose that K is simply connected. Then there is an open subset of X, $K \subset W \subset X$ and a holomorphic submersion $W \to U$ such that the leaves of $\mathcal{F}$ are given by the fibres of this map.*

Proof. The usual proof of Reeb stability, see for example [MM03, Theorem 2.9], works in this case. □

As a corollary of this we get the following description of isolated extremal rays:

Corollary 3.1.10. *Let C span an isolated $K_{\mathcal{F}}$-negative extremal ray on a smooth 3-fold. Then C is contained in the singular locus.*

Proof. Suppose the contrary. Then, we claim that C is actually disjoint from the singular locus. Let S be a germ of a leaf containing C. Then since C is isolated, $C^2 < 0$ in S.

Adjunction then implies that $K_S \cdot C = -1$, and thus if C is to be $K_{\mathcal{F}}$-negative it cannot meet the singular locus. Otherwise $K_{\mathcal{F}}|_S = K_S + \Delta$ and $\Delta \cdot C > 0$.

Now apply local Reeb stability as above to see that C actually moves to near by leaves, giving our contradiction.

J. V. Periera has given the following alternative proof: Suppose as above that C is not contained in the singular locus. By restricting Bott's partial connection on the leaf to C and noting that $N_{C/S} = \mathcal{O}(-1)$ we see that $N_{C/X} = \mathcal{O}(-1) \oplus \mathcal{O}$ and so C moves in X, a contradiction. □

Corollary 3.1.11. *Let X be a smooth 3-fold and $\mathcal{F}$ be a smooth rank 2 foliation on X and suppose that $\mathcal{F}$ is not uniruled. If R is a $K_{\mathcal{F}}$-negative extremal ray, then* loc(R) *is a divisor transverse to the foliation.*

Proof. By 3.1.10 we know that $\text{loc}(R) = D$ must be divisor. Suppose for sake of contradiction that D is foliation invariant, then D is covered by rational curves which by Reeb stability can be moved into nearby leaves and therefore $\mathcal{F}$ is uniruled. □

As noted earlier it is a somewhat subtle question of when a separatrix converges. The next proposition suggests the possibility of some general statement relating the existence of flipping curves and convergence of separatrices:

Proposition 3.1.12. *Suppose that $C \subset$ sing($\mathcal{F}$) is a flipping curve and that it meets no other components of sing($\mathcal{F}$), i.e., C is a smooth connected component of sing($\mathcal{F}$). Let S_1, S_2 be the formal separatrices along C, then S_1, S_2 are convergent.*

Proof. We already know one of these separatrices, say S_1, is a strong separatrix. So consider suppose for sake of contradiction that S_2 is not convergent. In this case $\mathcal{F}$ must have a saddle node type singularity along $K_{\mathcal{F}}|_{S_2}$, in fact, at a generic point of of C in appropriate (formal) coordinates if ω is the 1-form defining $\mathcal{F}$ we can write $\omega = y(k + \lambda x^k)dx + x^{k+1}dy$ where $k \geq 1$ where $x = 0$ is a local equation for S_1 and $y = 0$ is a local equation for S_2.

Replacing X by the formal completion of X along C we see that $N_{S_2/X}((k+1)C) = N\mathcal{F}|_{S_2}$, or $K_{\mathcal{F}}|_{S_2} = K_{S_2}((k+1)C)$, where for a formal scheme we define $K_{S_2} = K_X \otimes \mathcal{O}(S_2)|_{S_2}$

We know that $K_{\mathcal{F}}|_C = K_C$, since C is a smooth rational curve, and since $K_{S_2}(C)|_C = K_C$ we see N_{C/S_2} must be trivial. In this case we know that C has a deformation over $\mathbb{C}[\epsilon]/(\epsilon^m)$ for any m in S_2, and thus a deformation in X. But this implies that C moves, a contradiction of the fact that it is a flipping curve. □

Remark 18. *This result is a bit strange. Being a flipping curve is a global condition, yet convergence of separatrices is a fundamentally a local condition. We will come back to this point later, but a general picture is still missing for us.*

3.2 Divisorial contractions

Here we will give a concrete method description $K_{\mathcal{F}}$-negative divisorial contractions in the case where X is smooth and $\mathcal{F}$ has simple singularities. Later on we will develop other techniques to realize these contractions when X is not necessarily smooth (which will be necessary in order to run the MMP), therefore the reader interested only in running the MMP may skip this section.

Let R be an extremal ray such that loc(R) is a divisor, D.

We use a version of Castelnuovo's criterion to contract D. To this end, we want to find a very ample divisor L such that

a) $H^1(X, L) = 0$

b) $H^1(D, \mathcal{O}_D \otimes L(jD)) = 0$ for $1 \leq j \leq k-1$

c) $\mathcal{O}_D \otimes L(kD)$ is globally generated, with corresponding morphism $cont : D \to D_{cont}$

In this case, $L(kD)$ is globally generated and defines a morphism to a projective variety which contracts all those curves spanning R.

We will proceed on a case by case basis. Let H_R be a supporting hyperplane of R.

3.2.1 D is a $\mathbb{P}^1$-bundle transverse to the foliation

Let H_R be a sufficiently large multiple so that $A = H_R - K_{\mathcal{F}}$ is ample.

First, since D is a $\mathbb{P}^1$-bundle over a curve, $\pi : D \to B$, we have that $H_R|_D$ is semi-ample, i.e., since $H_R \cdot F = 0$ where F is any fibre and $H_R \cdot B_0 > 0$ where B_0 is a section, we have that $H_R|_D$ is the pullback of an ample divisor on B.

Next, $\mathcal{F}_D$ agrees with the bundle structure. For any fibre F we know that $K_{\mathcal{F}} \cdot F = D \cdot F = -1$. Furthermore, by foliation adjunction, we have $\mathcal{O}_D(D) = K_{\mathcal{F}_D} - K_{\mathcal{F}} + \pi^*\Theta$ where Θ is effective. Next, observe that $K_{\mathcal{F}_D} - 2K_{\mathcal{F}}|_D = \pi^* T$, and so perhaps replacing H_R be a sufficiently large multiple, we may assume that $H_R + K_{\mathcal{F}_D} - 2K_{\mathcal{F}}$ is globally generated.

Let $L = m(A + qH_R)$, where $m, q > 0$ to be chosen later. First, observe that $\mathcal{O}_D \otimes L(mD)$ is isomorphic to

$$m(H_R + K_{\mathcal{F}_D} - 2K_{\mathcal{F}}|_D) + mqH_R + m\pi^*\Theta$$

which is globally generated for $m \gg 0$

Second, $\mathcal{O}_D \otimes L(jD)$ for $1 \leq j \leq m-1$ is isomorphic to

$$(m-j)(A + qH_R) + j(H_R + K_{\mathcal{F}_D} - 2K_{\mathcal{F}}|_D) + j\pi^*\Theta + jqH_R$$

choosing $q \gg 0$ we see that $\mathcal{O}_D \otimes L(jD - K_D)$ is ample, so by Kodaira vanishing the first cohomology of $\mathcal{O}_D \otimes L(jD)$ is trivial.

Finally, choosing $m \gg 0$ we have that $H^1(X, L) = 0$ by Serre vanishing and that L is very ample.

We make the following useful observation:

Lemma 3.2.1. *Suppose C spans the extremal ray R and that loc$(R) = D$ is transverse to the foliation. Then C is K_X-negative. In particular, we can contract D to a smooth 3-fold.*

Proof. Following the computation in the proof of Lemma 9.7, we have that $K_{\mathcal{F}} \cdot C, C \cdot D = -1$ and that D is a $\mathbb{P}^1$-fibration such that D is normal along centres transverse to the foliation. But, we also have $(K_X + D) \cdot C = K_D \cdot C = -2$, and so $K_X \cdot C = -1$. □

Theorem 3.2.2. *Let $X, \mathcal{F}$ both be smooth. Then there is a foliated MMP for $(X, \mathcal{F})$.*

Proof. If R is any $K_{\mathcal{F}}$-negative extremal ray, then by our classification above we have that loc(R) is a $\mathbb{P}^1$-bundle, E trasnverse to the foliation and in fact R is K_X-negative. Contract E to a smooth curve $\pi : (X, \mathcal{F}) \to (Y, \mathcal{G})$. We claim that $\mathcal{G}$ is smooth.

Away from $\pi(E)$ this immediate, and since E is transverse to the foliation we must have that $\mathcal{G}$ is smooth at the generic point of E. Thus $\mathcal{G}$ has at worst isolated singularities along E. By Malgrange's theorem, see [CLN08] for example, around any such point, call it Q, $\mathcal{G}$ is induced by holomorphic fibration, thus to prove smoothness of $\mathcal{G}$ it suffices to show that the leaf passing through Q is smooth.

Let $C = \pi^{-1}(Q)$, and let S_Q be a germ of an analytic variety in a neighborhood of E containing C such that S_Q is foliation invariant. Thus $\pi(S_Q) = T$ is the leaf passing through Q. However, notice that C is a K_{S_Q}-negative rational curve, thus we also have $C^2 = -1$ in S_Q.

Next, observe that S_Q is transverse to E along C. Indeed, $-1 = E \cdot C = E|_S \cdot C = nC^2$ where $E|_{S_Q} = nC$, and so $n = 1$.

This gives us that $\pi : S \to T$ is the contraction of a (-1)-curve and so T is smooth.

Thus, we can perform the contraction in the category of smooth foliations on smooth varieties, allowing us to proceed with the MMP. Since each contraction drops the picard number by 1, this process must eventually terminate. □

Remark 19. *In fact since the map $X \to Y$ is exactly the blow up along a smooth curve $\pi(E) = B$ we see that B must be everywhere transverse to $\mathcal{G}$. This follows since the blow up of a foliation along a centre generically transverse to the foliation will acquire singularities over those points of B where B is tangent to $\mathcal{G}$, but since $\mathcal{F}$ is smooth this cannot happen.*

Remark 20. *It is perhaps a bit amusing to note that when $\mathcal{F}$ is rank 1 and smooth, the MMP for $\mathcal{F}$ is completely trivial: either $\mathcal{F}$ is a fibration in rational curves or $K_{\mathcal{F}}$ is already nef.*

As the above result shows if $\mathcal{F}$ is rank 2 and smooth the MMP takes a bit more work, but not much.

If $\mathcal{F}$ is rank 3, then we are in the classical case and even starting with a smooth $\mathcal{F}$ (i.e. X is smooth) the problem is hard!

3.2.2 $D \cong \mathbb{P}^2$

In this case, we know $H_R|_D = \mathcal{O}_D$, $K_{\mathcal{F}}|_D = \mathcal{O}(-a)$ where $a \geq 1$ and $D|_D = \mathcal{O}(-b)$ where $b \geq 1$ Let $A = H_R - K_{\mathcal{F}}$ and H_R chosen to be a sufficiently large multiple so that A is ample. Let $L = m(A + qH_R)$ $m, q > 0$ to be chosen later.

Choose k so that $ma = kb$. Then $\mathcal{O}_D \otimes L(kD)$ is isomorphic to $\mathcal{O}_D$, which is globally generated.

For $1 \leq j \leq k$ we have $\mathcal{O}_D \otimes L(jD)$ is ample, and hence has vanishing first cohomology.

Finally, choosing $m \gg 0$ we have that $H^1(X, L) = 0$

3.2.3 D is an invariant ruled surface with a section not in R

Let $\pi : D \to B$, and Σ be a section of π, with $[\Sigma] \notin R$. Note that if $g(B) \geq 1$ and D is invariant that we must be in this situation.

As above, we know that $H_R|_D$ is semi-ample.

Suppose first that F meets the singular locus of $\mathcal{F}$, and so $K_{\mathcal{F}} \cdot F = -1$. In particular, $K_{\mathcal{F}}|_D = \mathcal{O}(-1) + \pi^*\Theta$. Let $D|_D = \mathcal{O}(-c) + \pi^*N$. Perhaps replacing

H_R by a sufficiently large multiple, we may take $H_R - cK_{\mathcal{F}} + D|_D$ to be globally generated.

Next, let $A = H_R - cK_{\mathcal{F}}$ and $L = m(A + qH_R)$ $m, q > 0$ to be chosen later.

$\mathcal{O}_D \otimes L(mD)$ is isomorphic to $m(H_R - cK_{\mathcal{F}} + D|_D) + mqH_R + m\pi^*\Theta$ which is globally generated for $m \gg 0$.

For $1 \leq j \leq m$ we have that $\mathcal{O}_D \otimes L(jD)$ is isomorphic to

$$(m - j)(A + qH_R) + j(H_R - cK_{\mathcal{F}} + D|_D) + jqH_R + j\pi^*\Theta$$

again, $\mathcal{O}_D \otimes L(jD - K_D)$ is ample for $q \gg 0$ and so $\mathcal{O}_D \otimes L(jD)$ has vanishing first cohomology.

Finally, choosing $m \gg 0$ we have that $H^1(X, L) = 0$

If F does not meet sing($\mathcal{F}$), then $F \cdot K_{\mathcal{F}} = F \cdot (K_X + D)$ and so the contraction exists in this case.

Observe furthermore that these computations hold even in the case that B is singular.

3.2.4 D is a $\mathbb{P}^1$-bundle and every curve in D spans R

This implies that K_D is ample , in particular either $D \cong \mathbb{P}^1 \times \mathbb{P}^1$ or it is the blow up of $\mathbb{P}^2$ at a point.

First, assume D is the blow up of $\mathbb{P}^2$ at a point, and let be E the exceptional curve. By Reeb stability E must meet the singular locus of $\mathcal{F}$. However, if E is not contained in the singular locus, we have $K_{\mathcal{F}} \cdot E \geq 0$ a contradiction.

Consider the case where $D \cong \mathbb{P}^1 \times \mathbb{P}^1$. Let F_1, F_2 be the classes of fibres. Again, by Reeb stability, we have that $K_{\mathcal{F}} \cdot F_i = -1$, and so $K_{\mathcal{F}}|_D = \mathcal{O}(-1, -1)$. Let $D|_D = \mathcal{O}(-c, -c)$. Note that $H_R|_D = \mathcal{O}_D$

So, take $A = H_R - cK_{\mathcal{F}}$, and $L = m(A + qH_R)$.

First, $\mathcal{O}_D \otimes L(mD)$ is isomorphic to O_D, hence is globally generated.

For $1 \leq j \leq D$ $\mathcal{O}_D \otimes L(jD)$ is ample, hence has vanishing first cohomology.

Taking $m \gg 0$ we get that $H^1(X, L) = 0$.

Consider the case where D is $\mathbb{P}^2$ blown up at a point. Let F be the class of a line passing through the exceptional divisor. Then $K_{\mathcal{F}} \cdot F = -1$. Let $D \cdot F = -c$. In particular, $(-cK_{\mathcal{F}} + D)|_D = \mathcal{O}_D$. Taking $A = H_R - cK_{\mathcal{F}}$ the same arguments as above work to give our contraction.

Lemma 3.2.3. *Let M be a line bundle such that M restricted to loc(R) is trivial. Then, for $t \gg 0$ $tL+M$ where L is as in the cases above, also satisfies Casteluovo's criteria to define a contraction.*

Proof. a) By Serre's criterion, choosing $t \gg 0$ ensures that $H^1(X, tL + M) = 0$.

b) Replacing the k used above with tk the arguments used above still show that $H^1(D, \mathcal{O}_D \otimes \mathcal{O}(tL + M + jD)) = 0$ for $1 \leq j \leq tk - 1$.

c) $\mathcal{O}_D \otimes \mathcal{O}(tL + M + tkD) \cong L(kD)^{\otimes t}$ is globally generated because $L(kD)$ is. □

Corollary 3.2.4. *If M is a Cartier divisor trivial on loc(R), then it is pulled back from the contracted space.*

Proof. By our previous lemma, large enough multiples of $tL(tkD) + M$ are pulled back from the contracted space, say $r(tL(tkD) + M), (r + 1)(tL(tkD) + M)$ for $r \gg 0$, hence in fact $tL(tkD) + M$ is pulled back from the contracted space.

But, we also have $L(kD)$ is pulled back from the contracted space, so in fact M is as well.

Alternatively, we see that each of the contractions above is a $K_X + \Delta$-negative extremal contraction for some klt pair (X, Δ), and so the result follows from the usual arguments of the MMP. □

3.3 Contractions when loc(R) is 1-dimensional

Assume that X is $\mathbb{Q}$-factorial and normal, in particular we are no longer assuming that X is smooth or that $\mathcal{F}$ has simple singularities. Let R be an extremal ray where loc(R) is 1-dimensional.

For a Cartier divisor D let $\mathrm{Null}(D) = \{P : P \in V, V \cdot D^{\dim(V)} = 0\}$, and $BS(D)$ denotes the stable base locus of D, i.e., $\cap_{m \geq 0} bs(mD)$ where $bs(mD)$ is the base locus of mD. It is easy to see that $BS(D) = bs(mD)$ for m sufficiently large and divisible.

We will make use of the following result [CL14, Corollary 1]

Lemma 3.3.1. *Let X be normal threefold and let L be big and nef. Let A be an ample divisor. Then for all $\epsilon > 0$ sufficiently small $Null(L) = BS(L - \epsilon A)$.*

Let H_R be a supporting hyperplane to R. As we have seen H_R is big and nef. By our next lemma $\text{Null}(H_R)$ is a finite collection of curves.

Lemma 3.3.2. *Let $S \subset X$ be a surface. Then $H_R^2 \cdot S > 0$.*

Proof. Suppose for sake of contradiction that there is some surface S such that $H_R^2 \cdot S = 0$.

Let $f : S^\nu \to S$ be the normalization of S.

$H_R|_S$ is nef and so it is pseudoeffective, and we proceed case by case on the numerical dimension of f^*H_R.

If $\nu(f^*H_R) = 0$ then H_R is zero on a moving curve, hence is zero on infinitely many curves, a contradiction.

If $\nu(f^*H_R) = 1$ then write $H_R = K_\mathcal{F} + A$ where A is ample. We have that $f^*(H_R)^2 = 0$, and that $f^*(H_R)$ has positive intersection with any ample divisor on S^ν (otherwise H_R would be zero on a moving curve).

Thus

$$f^*K_\mathcal{F} \cdot f^*H_R = -f^*A \cdot f^*H_R < 0.$$

Perhaps rescaling H_R by a positive constant we may write $H_R = A' + D + S$ where A' is ample, and D is effective, and the support of D does not contain S. Then

$$f^*H_R \cdot f^*S = -f^*H_R \cdot f^*(A' + D) \leq -f^*H_R \cdot f^*A' < 0.$$

If S is $\mathcal{F}$ invariant, then $f^*K_\mathcal{F} = K_{S^\nu} + \Delta$ where $\Delta \geq 0$.

We apply our bend and break result to $D_1 = D_2 = f^*H_R$. $D_1 \cdot D_2 = 0$ by supposition, and by our above computation $(K_{S^\nu} + \Delta) \cdot D_1 = f^*K_\mathcal{F} \cdot f^*H_R < 0$. Thus, we get through a general point of X a rational curve Σ with $0 = D_2 \cdot \Sigma = H_R \cdot \Sigma$ a contradiction.

If S is not $\mathcal{F}$ invariant then by foliation adjunction we see that $f^*(K_{\mathcal{F}} + S) = K_{\mathcal{F}_{S^\nu}} + \Delta$, $\Delta \geq 0$. Again, by our above computations we have that $(K_{\mathcal{F}_{S^\nu}} + \Delta) \cdot f^*H_R = f^*(K_{\mathcal{F}} + S) \cdot f^*H_R < 0$

Again we apply bend and break with $D_1 = D_2 = f^*H_R$. $D_1 \cdot D_2 = 0$ by assumption, and $D_1 \cdot (K_{\mathcal{F}_{S^\nu}} + \Delta) < 0$. Again, through a general point of X we get a rational curve Σ tangent to the foliation with $0 = D_2 \cdot \Sigma = H_R \cdot \Sigma$ a contradiction.

If $\nu(f^*H_R) = 2$ then $f^*H_R^2 = H_R^2 \cdot S > 0$ and we are done. □

[Art70] proves the following result on the algebraicity of certain birational modifications:

Theorem 3.3.3. *Let X be a proper variety and let $Y \subset X$ be a closed subscheme. Let U be a some analytic neighborhood of Y. Let M be a complex analytic space and a proper modification $\hat{f} : U \to M$ or $\hat{g} : M \to U$. Then there exists an algebraic space and a morphism $f : X \to Z$ or $g : Z \to X$ extending $\hat{f}, \hat{g}$ respectively.*

Remark 21. *Artin's result actually works even when $\hat{f}, \hat{g}$ are only assumed to be formal modifications.*

Lemma 3.3.4. *loc(R) can be contracted in the category of algebraic spaces.*

Proof. By our previous lemma $\mathrm{Null}(H_R)$ is a finite collection of curves, each of which span R. Let A be an ample divisor and choose ϵ sufficiently small and m sufficiently large so that $\mathrm{Null}(H_R) = bs(m(H_R - \epsilon A)) = B$.

Let $g : Y \to X$ be a resolution of the base locus of $m(H_R - \epsilon A)$ so that we have $g^*(m(H_R - \epsilon A)) = M + F$ where M is semi-ample, F is effective and $g(F) = B$ and $g(\mathrm{exc}(g)) = B$.

If $C \subset F$ is a curve with $g(C)$ not a point, then $C \cdot (M + F) = g(C) \cdot m(H_R - \epsilon A) < 0$. Thus we have $F \cdot \alpha \leq -m\epsilon A \cdot \alpha$ for any $\alpha \in \overline{NE}(S)$ with.

Let G be an effective divisor $\mathbb{Q}$-Cartier divisor supported on $\mathrm{exc}(g)$ such that $-G$ is g-ample (such a G exists because X is $\mathbb{Q}$-factorial). Then for $1 \gg \delta > 0$ we see that $(F + \delta G)|_{\mathrm{exc}(g)}$ is anti-ample.

Let N be such that $N(F + \delta G) = D$ is an integral Cartier divisor. In this case we see that D is a subscheme with an anti-ample normal bundle, and so it may be contracted to a point, [Art70, Theorem 6.2]. Indeed, taking $f : D \to p$ where p is a point we have that Condition (i) of the theorem is satisfied by the ampleness of the co-normal bundle, and Condition (ii) of the theorem is immediate because we are contracting D to a point.

This contraction factors through g and gives a contraction $X \to Z$. By [Art70] this contraction may be taken in the category of algebraic spaces. □

Note that we have not proven that Z is projective.

Lemma 3.3.5. *Let H_R be the supporting hyperplane to R. Assume that H_R descends to a $\mathbb{Q}$-Cartier divisor on Z, then Z is projective.*

Proof. By assumption, if f is the contraction, then $H_R = f^*M$. We claim that M is in fact ample. First M is nef and $M^3 > 0$. If C is any curve in Z then we also have $M \cdot C > 0$. By our above lemma if S is any surface then $M^2 \cdot S > 0$. Thus the Nakai-Moishezon criterion for ampleness applies to show that M is ample, and so Z is projective. □

The question of whether H_R descends to a $\mathbb{Q}$-Cartier divisor on Z, or equivalently, when $\rho(X/Z) = 1$, seems to be a difficult one which we will take up in the next few sections.

As a point of comparison, in [HP, Theorem 7.12] the contraction of an isolated extremal ray is constructed in the same way as above, i.e., by realizing it first as a contraction in the category of analytic spaces. However, in the Kähler case one can make use of the relative analytic base point free theorem to show that $\rho(X/Z) = 1$.

Portions of the work in the above chapter is being prepared for submission for publication.

Spicer, Calum "Higher dimensional foliated Mori Theory".

The dissertation author was the primary investigator and author of this material.

Chapter 4

Toward running the MMP

As seen earlier if X is smooth and R is a $K_{\mathcal{F}}$-negative extremal ray with $\text{loc}(R)$ a divisor E then R is in fact $K_X + \epsilon(E)E$-negative, and thus the contraction exists by standard theorems in the classical log MMP. This is the first indication that it might be possible to run a foliated MMP by running an appropriate log MMP. Unfortunately, in the case of a flipping curve it is unclear if there is a good choice of an algebraic divisor Δ which realizes that foliated flip as a $K_X + \Delta$ flip.

Perhaps as expected the two main challenges in developing the foliated MMP are:

Question 3. *Do foliation flips exist?*

and

Question 4. *Do foliation flips terminate?*

In the classical case, one approach to the construction of flips is to show that existence of flips follows from the existence of "special" flips and termination of "special" flips. This is the philosophy that we adopt here. In the first section we will prove the existence of a "special" MMP. Then, we will show how that existence of the "special" MMP proves the existence of flips in a wider range of cases. Unfortunately, we are unable to prove the unqualified existence of flips.

We will also discuss some results related to an unqualified termination of flips.

4.1 Some results from the classical situation

We will need the following results from the classical (log) MMP, these are proven in [KM98], for example.

Theorem 4.1.1. *Let $f : X \to W$ be a projective morphism of analytic varieties with dim$(X) \leq 3$ (we include the case where W is a point). Suppose that (X, Δ) is klt and $\mathbb{Q}$-factorial. Then*

$$\overline{NE}(X/W) = \overline{NE}(X/W)_{K_X+\Delta\geq 0} + \sum \mathbb{R}_{\geq 0}[L_i]$$

where the L_i are rational curves and are locally discrete. Furthermore, if R is a $(K_X+\Delta)$-negative extremal ray, then there exists a morphism over W $c_R : X \to Z$ such that

(a) $c_R(C)$ is a point if and only if $[C] \in R$

(b) Let M be a $\mathbb{Q}$-Cartier divisor with $M \cdot R = 0$, then there exists M' on Z such that $c_R^* M' = M$.

(c) if c_R is a flipping contraction, then the flip exists.

Theorem 4.1.2. *Let $f : X \to W$ be a projective morphism of analytic varieties. Suppose that D is a f-nef divisor and that $D - (K_X + \Delta)$ is f-big and nef where (X, Δ) is klt. Then D is f-semi-ample.*

Remark 22. *Recall that D being f-semi-ample means that some multiple of D defines a morphism $\phi_{mD} : X \to \mathbb{P}_Y(f_*\mathcal{O}(mD))$. In particular, ϕ contracts all those curves C with $D \cdot C = 0$.*

Remark 23. *A brief comment on $\mathbb{Q}$-factoriality: Above we only require that X is globally $\mathbb{Q}$-factorial, i.e., every globally defined Weil divisor has some multiple which is Cartier. This does not imply that X is local analytically $\mathbb{Q}$-factorial. Indeed, it is possible to have $\mathbb{Q}$-factorial varieties which have singularities analytically isomorphic to the cone over a quadric, which can easily be shown to not be $\mathbb{Q}$-factorial.*

We also recall the following very useful result, the so-called "negativity lemma":

Lemma 4.1.3. *Let $f : X \to Y$ be a proper morphism of normal varieties. Let D be a $\mathbb{R}$-Cartier divisor such that $-D$ if f-nef. Then D is effective if and only if h_*D is effective.*

Proof. Standard, see for example [KM98, Lemma 3.39] □

This has the following handy corollary:

Corollary 4.1.4. *Consider the following morphisms:*

$$\begin{array}{ccc} (X_1, \mathcal{F}_1) & & (X_2, \mathcal{F}_2) \\ & \searrow^{f_1} \quad {}_{f_2}\swarrow & \\ & Y & \end{array}$$

where f_i is projective and birational and X_i, Y are normal varieties. Suppose that $K_{\mathcal{F}_i} + \Delta_i$ is $\mathbb{Q}$-Cartier, and that $f_{1*}\Delta_1 = f_{2*}\Delta_2$. Suppose that $-(K_{\mathcal{F}_1} + \Delta)$ is f_1-nef and $K_{\mathcal{F}_2} + \Delta_2$ is f_2-nef.

Then for any exceptional divisor E

$$a(E, \mathcal{F}_1, \Delta_1) \leq a(E, \mathcal{F}_2, \Delta_2)$$

. Furthermore, we have strict inequality if either

(i)$-(K_{\mathcal{F}_1} + \Delta_1)$ is f_1-ample and the centre of E is contained in $exc(f_1)$ or,

(ii) $K_{\mathcal{F}_2} + \Delta_2$ is f_2-ample, and the centre of E is contained in $exc(f_2)$.

Proof. The proof is the same as the proof in [KM98, Lemma 3.38]. The point is that if $g_i : Z \to X_i$ extracts E then $g_2^*(K_{\mathcal{F}_2} + \Delta_2) - g^*(K_{\mathcal{F}_1} + \Delta_1)$ is nef over Y and we apply the negativity lemma. □

We now give the precise definition of a foliation flip:

Definition 23. *Given a small contraction $f : X \to Z$ with $\rho(X/Z) = 1$ where $-K_{\mathcal{F}}$ is f-ample the flip is birational map to a projective variety $\phi : X \dashrightarrow X^+$ together with a morphism $f^+ : X^+ \to Z$ such that $K_{\mathcal{F}^+}$ is f^+-ample and such that $\rho(X^+/Z) = 1$.*

4.2 Preliminary computations, I

For completeness we collect here a suite of relatively easy results which we repeatedly use.

Lemma 4.2.1. *Let $(X, \mathcal{F})$ be a 3-fold with non-dicritical terminal foliation singularities and suppose that sing(X) is tangent to $\mathcal{F}$. Let H be a general hyperplane. Then $(H, \mathcal{F}_H)$ has terminal foliation singularities.*

Proof. Let $\pi : (X', \mathcal{F}') \to (X, \mathcal{F})$ be a resolution of singularities. Observe that we may assume $\text{exc}(\pi) = E$ is foliation invariant (otherwise $(X, \mathcal{F})$ would both be smooth along $\pi(E)$).

Write $\pi^*(K_{\mathcal{F}}) = K_{\mathcal{F}'} - \sum a_i E_i$ where $a_i > 0$

Choose H general enough so that $\pi^* H = \pi_*^{-1} H = H'$, and that $H \cup E$ is snc.

Observe that $\mathcal{F}_{H'}$ has simple singularities. Furthermore, restricting to H, by foliation adjunction, we have that $K_{\mathcal{F}_{H'}} + \pi_*^{-1}\Delta - \sum a_i E_i|_H + \sum d_i E_i|_H = \pi^*(K_{\mathcal{F}_H} + \Delta)$ where $\Delta \geq 0$ and $\pi_*^{-1}\Delta + \sum d_i E_i|_H \geq 0$ are the foliation differents. Notice that since H is transverse to E_i it is also transverse to $\mathcal{F}$ along E_i (since E_i is invariant) and so $d_i = 0$. Subtracting $\pi^*\Delta = \pi_*^{-1}\Delta + b_i E_i$ where $b_i \geq 0$ from both sides gives our result. □

Corollary 4.2.2. *Let $C \subset$ sing(X), let $\mathcal{F}$ be terminal at the generic point of C and suppose that C is tangent to $\mathcal{F}$. Then there is a unique analytic space S containing C which is foliation invariant.*

Proof. By our extension lemmas it suffices to find a germ of curve γ not contained in sing(X) with $\gamma \cap C \neq \emptyset$ to produce the desired S.

Take a general hyperplane cut H passing through C so that $(H, \mathcal{F}_H)$ is terminal at $p = C \cap H$. Since terminal foliation singularities on surfaces are quotients of smooth foliations $q : (Y, \mathcal{G}) \to (H, \mathcal{F}_H)$, letting γ be the pushforward of a germ of a leaf through $q^{-1}(p)$ gives the desired curve germ. Furthermore, γ is smooth and is the unique germ passing through p. □

Lemma 4.2.3. *Let $(X, \mathcal{F})$ be a foliated pair, and let S be an invariant (analytic) divisor. Suppose that $\mathcal{F}$ is terminal along the codimension 2 singularities of X.*

Then writing $K_{\mathcal{F}}|_S = K_S + \Delta$ and $(K_X + S)|_S = K_S + \Theta$ we have that Θ, Δ have the same coefficient along divisors contained in sing(X).

Proof. Cutting by hyperplanes, we may assume that X is a surface and S is a curve. Since $\mathcal{F}$ is terminal, this implies that X has cyclic quotient singularities.

Write $K_{\mathcal{G}} = \pi^* K_{\mathcal{F}} + A$ and $K_Y + E + S' = \pi^*(K_X + S) + B$ where π is a resolution, E is the sum of the exceptoinal divisors with coefficient 1 and S' is the strict transform of S. Notice that $K_{\mathcal{G}}$ and $K_Y + E + S'$ have the same intersection value with exceptional curves. Thus $A = B$. If we write $K_{\mathcal{G}}|_S = K_S + aP$ where P is the singular point of $\mathcal{G}$ along P, since $\mathcal{F}$ is the quotient of a smooth foliation, we must have $a = 1$. This gives us $K_{\mathcal{G}}|_S = (K_Y + E + S)|_S$ and the result follows. □

Remark 24. *In fact the above result gives us that the coefficient of a component $C \subset$ sing(X) in Δ is $\frac{n-1}{n}$ where $(Y, \mathcal{G})/(\mathbb{Z}/n\mathbb{Z}) = (X, \mathcal{F})$, i.e., n is the order of the local fundamental group around the singularity.*

If one is willing to use stacks the above computation can be done much more simply by results due to McQuillan:

Let $\pi : (\mathcal{X}, \mathfrak{F}) \to (X, \mathcal{F})$ be the index 1 covering stack so that $K_{\mathfrak{F}}$ is Cartier. Suppose that $\mathcal{F}$ is terminal along the singularities of X. Let $S \subset X$ be normal and $\mathcal{F}$-invariant and let $f : \mathcal{S} \to \mathcal{X}$ be the pull back of S to $\mathcal{X}$. McQuillan's stacky adjunction formula, [McQ, I.8.7] tells us

$$f^* K_{\mathfrak{F}} = K_{\mathcal{S}} + s_Z(f) - Ram_f$$

where $K_{\mathcal{S}}$ is the orbifold canonical bundle, $s_Z(f)$ is a contribution form the singular locus of $\mathfrak{F}$ and $Ram_f = 0$ since S is normal.

Since $\mathfrak{F}$ is terminal (hence smooth) at the stacky points of X we see that no stacky point is contained in supp$(s_Z(f))$. By Riemann-Hurwitz we know that $K_{\mathcal{S}} = \pi^* K_S + \sum \frac{n_i-1}{n_i} D_i$ where D_i runs over sing$(X) \cap S$, and so $K_{\mathcal{F}}|_S = K_S + \sum \frac{n_i-1}{n_i} D_i + \sum k_i E_i$ where $E_i \subset$ sing$(\mathcal{F}) \cap S$.

Lemma 4.2.4. *Suppose that $\mathcal{F}$ has canonical singularities and $\mathcal{F}$ is terminal along sing(X) and sing(X) is tangent to $\mathcal{F}$. Let $D, D_1, ..., D_n$ be a collection of $\mathcal{F}$-*

invariant divisors. Suppose that $D, D_1, ..., D_n$ are $\mathbb{Q}$-Cartier. Let $D^\nu \to D$ be the normalization.

Write $K_{\mathcal{F}}|_{D^\nu} = K_{D^\nu}+\Theta$ and $(K_X+D+\sum D_i)|_{D^\nu} = K_{D^\nu}+\Delta$. Then $\Theta \geq \Delta \geq 0$ with equality along those centres contained in sing(X).

Thus, if $C \subset D$ is not contained in sing$(\mathcal{F})$, then $K_{\mathcal{F}} \cdot C < 0$ implies $(K_X + D + \sum D_i) \cdot C < 0$.

Suppose that D is a strong separatrix along those components of sing$(\mathcal{F})$ which are contained in D, then if $C \subset$ sing$(\mathcal{F})$ and D_{i_0} is the other separatrix along C for some i_0 then $K_{\mathcal{F}} \cdot C < 0$ implies $(K_X + D + \sum D_i) \cdot C < 0$.

Proof. We can write $\Theta = \sum a_i T_i + \sum b_i S_i$ where $T_i \subset \text{sing}(\mathcal{F})$ and $S_i \subset \text{sing}(X)$.

Notice that since $\mathcal{F}$ has canonical singularities, and canonical singularities are simple in codimension 2, we see that $D \cup D_1 \cup ... \cup D_n$ is normal crossings in codimension 2. This gives us $D_i|_D \subset \sum T_k$ and that for $T_k \subset \text{supp}(\Delta)$ the coefficient of T_k is 1.

Furthermore, $a_i \geq 1$, with equality if D is a strong separatrix along T_i.

Observe that if D, D' are two $\mathcal{F}$ invariant divisors then they cannot intersect along sing(X) and so by 4.2.3, we see that Θ, Δ agree along the S_i. Thus, for $C \neq T_i$ for all i, we have that $0 > (K_{D^\nu} + \Theta) \cdot C \geq (K_{D^\nu} + \Delta) \cdot C$.

If D is a strong separatrix, then we have that $a_i = 1$ for all i, and so $\Theta = \Delta + \sum E_i$ where $E_i \subset \text{sing}(\mathcal{F})$ are such that the other separatrix along E_i is not contained in $\sum D_i$, and so by supposition $C \neq E_i$ for any i. Thus $0 > (K_{D^\nu} + \Theta) \cdot C \geq (K_{D^\nu} + \Delta) \cdot C$. □

The following corollary will be used extensively:

Corollary 4.2.5. *Hypotheses as above. Let C be a $K_{\mathcal{F}}$-negative curve tangent to the foliation. Suppose that all the separatrices in a neighborhood of C are convergent, call them D_i. Suppose that $\sum D_i$ is $\mathbb{Q}$-Cartier. Then C is $K_X + \sum D_i$-negative.*

Proof. If C is not contained in singular locus of $\mathcal{F}$, this follows from the first part of 4.2.4. Otherwise, one of the separatrices call it D_0 is the strong separatrix along C, and the other separatrix is D_i for some i, since by assumption all separatrices are convergent. Thus, writing $K_X + \sum D_i = K_X + D_0 + \sum_{i \geq 1} D_i$ we can apply the second part of 4.2.4 to get our conclusion. □

Under additional hypotheses on the singularities of X (which we will see are preserved by running the MMP), we have that the cone theorem holds without any qualifications.

Proposition 4.2.6. *Suppose (X, D) is klt for some D and that X is $\mathbb{Q}$-factorial. Suppose that $\mathcal{F}$ has non-dicritical canonical singularities, sing(X) is tangent to $\mathcal{F}$ and $\mathcal{F}$ is terminal along the 1-dimensional components of sing(X). Then, every $K_{\mathcal{F}}$-negative extremal ray is spanned by a rational curve tangent to the foliation.*

Proof. The only extremal rays which are not guaranteed to be rational are those spanned by curves in sing(X), which are by supposition tangent to the foliation.

Furthermore, we may assume that each such curve is in fact a flipping curve, otherwise, if loc$(R) = E$ is a divisor, we know that either K_E or $K_{\mathcal{F}_E}$ is not pseudoeffective (depending on whether or not E is invariant or not) and thus R contains many rational curves.

Let S be the germ of a foliation invariant analytic divisor containing C. By 4.2.3 we know that $K_{\mathcal{F}}|_S = K_S + \frac{n-1}{n}C + \Delta$ where Δ is effective and does not contain C and $n \geq 2$.

Let $f : T \to S$ be the minimal resolution of S and let C' be the strict transform of C. We can write $f^*(K_S + \frac{n-1}{n}C + \Delta) = K_T + \frac{n-1}{n}C' + \Gamma$ where $\Gamma \geq 0$.

Observe that $(C')^2 \leq 0$, otherwise some multiple of C' would move, implying that there are many curves in R, a contradiction. Thus $(K_T + C' + \Gamma) \cdot C' < 0$, which by adjunction implies that C is rational. □

The following lemma will imply that only curves tangent to the foliation will be contracted in the course of the MMP.

Lemma 4.2.7. *Suppose that (X, D) is klt for some D. Let X be $\mathbb{Q}$-factorial and $\mathcal{F}$ be non-dicritical, and suppose furthermore that the singularities of X are tangent to the foliation.*

Let R be a $K_{\mathcal{F}}$-negative extremal ray. Suppose that $[C] \in R$. Then C is tangent to the foliation.

Proof. Suppose not. Let E be an effective divisor such that $E \cdot R < 0$. Thus $E \cdot C < 0$, and therefore E is transverse to the foliation. By assumption X is smooth at the generic point of C, and by Lemma 2.6.8 E is smooth at the generic point of C. Writing $\nu : E' \to E$ for the normalization map and $K_{\mathcal{G}}$ the foliation induced on E' we have that $K_{\mathcal{G}} + \Delta = \nu^*(K_{\mathcal{F}} + E)$. Letting $K_{\mathcal{F}} + E + A = H_R$ be a supporting hyperplane for R, with A ample, we have that $K_{\mathcal{G}} + \Delta + \nu^* A$ is a nef divisor, and $(K_{\mathcal{G}} + \Delta + (1-\epsilon)\nu^* A) \cdot C < 0$.

By foliation adjunction we see that $K_{\mathcal{G}} + \Delta + \nu^* A$ cannot be big. Thus $(K_{\mathcal{G}} + \Delta + \nu^* A)^2 = 0$ and so foliated bend and break applies to produce rational curves tangent to the foliation which span the ray R.

By non-dicriticality of $\mathcal{F}$, we see that $\mathcal{G}$ is the foliation induced by a fibration in rational curves $E' \to B$, in particular if f is a general fibre then $[f] \in R$.

(In fact, we can show that R is K_X-negative. On one hand since f can be taken to be disjoint from the singularities of X and since $f \cdot E, f \cdot K_{\mathcal{F}} < 0$ we have that both are ≤ -1. If we write $(K_{\mathcal{G}} + \Delta) = \nu^*(K_{\mathcal{F}} + E)$ then $-2 \leq (K_{\mathcal{G}} + \Delta) \cdot f \leq -2$. This tells us that supp(Δ) is tangent to the foliation, and so E is generically smooth along centres transverse to the foliation. Thus if we write $K_{E'} + \Delta' = \nu^*(K_X + E)$ we have $f \cdot \Delta' = 0$, and so $(K_{E'} + \Delta') \cdot f = -2$. Thus $(K_X + E) \cdot f = (K_{\mathcal{F}} + E) \cdot f$ which implies that $K_X \cdot f = K_{\mathcal{F}} \cdot f < 0$.)

$[f] \in R$ implies that every component in any fibre is also in R, and so $\overline{NE}(E')$ maps entirely into R and so $K_{\mathcal{G}} + \Delta$ is anti-ample. The foliated cone theorem for surfaces applies to show that every extremal ray in $\overline{NE}(E')$ is spanned by a curve tangent to the foliation. But in this case, every curve class in E' has zero intersection with f, a contradiction of the projectivity of E'. □

Remark 25. *As mentioned (much) earlier, this result is necessary if we wish think of the foliated MMP as being a relative MMP over the leaf space.*

In more practical terms, this condition guarantees that our foliation singularities stay non-dicritical.

Corollary 4.2.8. *Let R be an extremal ray with loc$(R) = D$ let $f : X \to Y$ be a contraction of R. Suppose that D is transverse to the foliation. Then D is contracted to a curve Z. Furthermore the contraction is K_X-negative, and so the con-*

tracted space and foliation will be smooth at the generic point of Z. Additionally, if $E \geq 0$ is any foliation invariant divisor, then the contraction is $K_X + E$-negative.

Proof. By the proof above we see that the foliation on D is a $\mathbb{P}^1$-fibration and that the extremal ray contracted is spanned by a general fibre f, and that f is K_X-negative. Thus Y is terminal at generic point of Z. Furthermore, observe that if E is invariant, then a general choice of f will be disjoint from E and so $(K_X + E) \cdot f = K_X \cdot f < 0$. □

4.3 Running a special MMP

The goal of this section is to show how under some additional assumptions it is possible to run the foliated MMP, what we will call a "special" MMP.

4.3.1 The induction set up and hypotheses

The going induction assumption for this section is that we have already constructed the first n steps of the MMP

$$(X_1, \mathcal{F}_1) \overset{f_1}{\dashrightarrow} (X_2, \mathcal{F}_2) \overset{f_2}{\dashrightarrow} ...(X_n, \mathcal{F}_n)$$

where $(X_1, \mathcal{F}_1)$ is a smooth 3-fold and $\mathcal{F}_1$ is a foliation with simple singularities, and each f_i contracts a $K_{\mathcal{F}_i}$-negative extremal ray.

Each f_i contracts a divisor or it is a flip. If f_i is a divisorial contraction, only curves tangent to the foliation are contracted. If f_i is a flip, then the flipping (and hence flipped) curves are tangent to the foliation.

We will make the following three (strong!) hypotheses, what we will call the $\mathbb{Q}$-factoriality hypothesis, the convergence hypothesis and the irreducibility hypothesis.

Hypothesis 4.3.1. *Let $p \in X_i$ and let S_p be a germ of a separatrix at p, then S_p is $\mathbb{Q}$-Cartier.*

Hypothesis 4.3.2. *If C is a flipping curve contained in $sing(\mathcal{F}_i)$ then all the separatrices along C are convergent.*

Hypothesis 4.3.3. *If C is connected component of the flipping locus, then C is irreducible.*

These three hypotheses are what make our MMP "special".

By induction we may suppose

(1) there exists an analytic open set $U_1 \subset X_1$ such that U_1 contains every algebraic divisor contracted by the MMP and U_1 contains $\mathrm{exc}(X_1 \dashrightarrow X_i)$ for all i.

Set $U_{i+1} = f_i(U_i)$. If f_i is a contraction, then since U_i contains the exceptional divisor, U_{i+1} is still open, and $f_i|_{U_i}$ is still projective. If f_i is a flip, then writing

$$\begin{array}{ccc} X_i & & X_{i+1} \\ & \searrow^{g_i} \quad \swarrow_{g_{i+1}} & \\ & W_i & \end{array}$$

for the base of the flip, we see that since U_i contains the exceptional locus of g_i, $g_i(U_i)$ is open, $U_{i+1} = g_{i+1}^{-1}(g_i(U_i))$ and the flip induces a rational map $f_i : U_i \dashrightarrow U_{i+1}$. Again, $g_i|_{U_i}, g_{i+1}|_{U_{i+1}}$ are both projective.

(2) (X_i, Δ_i) has klt singularities for some Δ_i and $\mathcal{F}_i$ has canonical non-dicritical singularities. $\mathrm{sing}(X_i)$ is tangent to $\mathcal{F}_i$ and $\mathcal{F}_i$ is terminal along the generic points of 1-dimensional components of $\mathrm{sing}(X_i)$.

(3) we have constructed reduced (analytic) divisor $D_1 + T_1$ on U_1, invariant under $\mathcal{F}_1$ such that if the strict transforms of D_1, T_1 on X_i are D_i, T_i, then f_i is either a $K_{U_i} + D_i + T_i$-negative contraction or a $K_{U_i} + D_i + T_i$-flip. Here D_1 consists of algebraic divisors contracted by the MMP and T_1 consists of germs of separatrices around flipping curves.

This extra inductive data in (3) is perhaps unusual so we explain its construction. The idea is to take D_1 to be the sum of all the invariant divisors contracted by the MMP, together with the strict transforms of the germs of invariant divisors around flipping curves, T_1. By our earlier computation we know that a $K_{\mathcal{F}_i}$-negative contraction will be $K_{U_i} + D_i + T_i$-negative.

Since such germs might only be analytic, we can only define this divisor on an analytic open set. This passage to an analytic open set introduces a new subtlety which one must be very careful about: when we contract an $K_{\mathcal{F}}$-negative extremal divisor, the resulting space is guaranteed to have globally $\mathbb{Q}$-factorial singularities, however these are in general not analytically $\mathbb{Q}$-factorial. In particular, the analytic $\mathcal{F}$-invariant divisor we get by taking the germ of an analytic space around a flipping curve is not guaranteed to be $\mathbb{Q}$-Cartier in general.

In this set up, our aim is construct the next step of the MMP $(X_n, \mathcal{F}_n) \overset{f_n}{\dashrightarrow} (X_{n+1}, \mathcal{F}_{n+1})$.

4.3.2 Preliminary computations, II

Proposition 4.3.4. *$(U_i, (1-\epsilon)(D_i + T_i))$ is klt for all $1 \gg \epsilon > 0$, $i \leq n$.*

Proof. Write $\Delta_i^\epsilon = (D_i+T_i)-\epsilon(D_i+T_i)$ Each step of the MMP is $K_{U_i}+\Delta_i^\epsilon$-negative for ϵ sufficiently small. Let E be divisor sitting over U_i. Suppose that f_i is a flip. Let Y be a log resolution of both U_i, U_{i+1}. Then by the negativity lemma for any divisor E on Y, we know that $a(U_i, D_i + T_i, E) \leq a(U_{i+1}, D_{i+1} + T_{i+1}, E)$. Thus $(U_i, \Delta_{i+1}^\epsilon)$ is klt.

If f_i is a divisorial contraction, then since $f_{i*}(D_i + T_i) = D_{i+1} + T_{i+1}$, then $(K_{X_i} + D_i + T_i) - f_i^*(K_{X_{i+1}} + D_{i+1} + T_{i+1}) \geq 0$ by the negativity lemma. Thus for any E sitting over U_{i+1} we have $a(U_i, D_i + T_i, E) \leq a(U_{i+1}, D_{i+1} + T_{i+1}, E)$.

Finally, since $D_1 + T_1$ is a normal crossings divisor (because $\mathcal{F}_1$ has simple singularities) $(U_1, (D+T)_1^\epsilon)$ is klt for all ϵ sufficiently small. The previous two computations and induction imply the claim. □

Corollary 4.3.5. *$(X_i, (1-\epsilon)D_i)$ is klt.*

Proof. Follows from the fact that the MMP is an isomorphism outside of U_i and the general fact that if $(X, D+E)$ is klt, then (X, D) is klt. □

Lemma 4.3.6. *There exists a resolution of singularities of $(X_n, \mathcal{F}_n)$ $g : Y \to X_n$ such that $a(E, \mathcal{F}_n) > 0$ for every extracted divisor.*

Proof. By induction we may assume that such a resolution exists for X_{n-1}, call it $h : Y' \to X_{n-1}$.

If f_n contracts a divisor then taking $Y = Y'$ and $g = f_n \circ h$ gives our result.

If f_n is a flip, let Y be a resolution of the induced map $Y' \dashrightarrow X_n$. Y might extract a divisor E of discreapncy ≥ 0 over X_{n-1}, but the image of E on X_n will be contained in the flipping locus so by the negativity lemma we have $a(E, \mathcal{F}_n) > a(E, \mathcal{F}_{n-1}) \geq 0$. □

4.3.3 Constructing $(X_n, \mathcal{F}_n) \dashrightarrow (X_{n+1}, \mathcal{F}_{n+1})$

Proposition 4.3.7. *Assume hypotheses 4.3.1, 4.3.2, 4.3.3. Then the next step of the MMP* $(X_n, \mathcal{F}_n) \dashrightarrow (X_{n+1}, \mathcal{F}_{n+1})$ *exists.*

Proof. There are two cases, either our contraction of an extremal ray is a divisorial contraction, or it is a flipping contraction.

We first consider the case of a divisorial contraction. Suppose that we contract a divisor E.

First assume that E is contained in the support of D_n if E is invariant, and is not contained in the support of D_n, but contained in U_n. By our above computations we see that this contraction is in fact $K_{X_n} + D_n$-negative, in particular we can realize it as a klt contraction.

Furthermore, our above considerations show that only curves tangent to the foliation are contracted to points.

In this case we see that X_{n+1} is $\mathbb{Q}$-factorial, and by our $\mathbb{Q}$-factoriality hypothesis we see that if we let $U_{n+1} = \pi(U_n)$ that $D_n + T_n$ is $\mathbb{Q}$-Cartier on U_{n+1}.

Otherwise, let E_1 be the strict transform of E on X_1 and $D'_1 = D_1 + E_1$ if E_1 is invariant, and $= D_1$ otherwise. Let $U'_1 = U_1 \cup V$ where V is a small neighborhood of E_1. By our extension results we see that we can extend every separatrix in T_1 to U'_1

By our $\mathbb{Q}$-factoriality hypothesis this extension will stay $\mathbb{Q}$-Cartier, and furthermore by our above computations, each step in the MMP is $K_{X_i} + D'_i + T_i$-negative so our inductive hypotheses are satisfied and thus we may freely assume (replacing D_1 by D'_1 that $E \subset U_n$ and E is a component of D_n if it is invariant and so we have reduced to our previous assumption.

Next, we consider the case where our contraction is a flipping contraction, $X_n \xrightarrow{g_n} W$. Let C be a connected component of the flipping locus.

If C is contained in D_n or T_n then the flipping contraction is $K_{U_n} + D_n + T_n$-negative and so we can realize the contraction $g|_{U_n} : U_n \to g(U_n)$ as a klt contraction. In particular, by the base point free theorem if M is a supporting hyperplane of the extremal ray we see that $M = g_n^* M'$ for some $\mathbb{Q}$-Cartier divisor on W. This gives us that $\rho(X_n/W) = 1$.

We will now realize the foliation flip as the output of a log MMP:

It suffices to construct the flip for each connected component of the flipping locus, so we may assume that $g_n(\mathrm{exc}(g_n)) = p$ is a point. Let $p \in V \subset W$ be a small enough neighborhood so that $g_n^{-1}(W) \subset U_n$. Restricting to $g_n : U_n \to V$ we would like to run a $K_{\mathcal{F}_n}$-MMP over V.

An important issues arises here. In general, while every compact curve in $g_n^{-1}(V)$ spans the same extremal ray in $\overline{NE}(X)$, it might not be the case that $\overline{NE}(g_n^{-1}(V)/V)$ is one dimensional.

Nevertheless, we still have each $K_{\mathcal{F}_n}$-negative ray R' in $\overline{NE}(g^{-1}(V)/V)$ is spanned by some curve C, furthermore, each such curve $K_{U_n} + D_n + T_n$-negative, and thus we can construct a $K_{U_n} + D_n + T_n$-flip of this extremal ray. But, since $K_{\mathcal{F}_n} \cdot R' = t(K_{U_n} + D_n + T_n) \cdot R$ where $t > 0$, we see that the log flip is also the foliated flip. Continuing to run this MMP we see that each step is a step in the $K_{U_n} + D_n + T_n$-MMP. Since the $K_{U_n} + D_n + T_n$-MMP terminates, eventually this $K_{\mathcal{F}_n}$-MMP must terminate with $g_n^+ : U_n^+ \to V$ and $K_{\mathcal{F}_n^+}$ is g_n^+-nef.

As before, by Artin's theorem on the existence of modifications, we see that there exists an algebraic space and a birational modification $g_n^+ : X_{n+1} \to W$ extending $U_n^+ \to V$. Observe that X_{n+1} is $\mathbb{Q}$-factorial, and that we have the strict transform of T_n is $\mathbb{Q}$-Cartier on U_{n+1} (here we do not need our $\mathbb{Q}$-factoriality hypothesis).

Next, since $X_n \dashrightarrow X_{n+1}$ is a sequence of flips we know that $\rho(X_{n+1}) = \rho(X_n)$ and so $\rho(X_{n+1}/W) = 1$, furthermore, since all the flipping curves are irreducible, we see that $X_{n+1} \to W$ is projective. Finally, because $K_{\mathcal{F}_{n+1}}$ is nef over W this implies that it is $K_{\mathcal{F}_{n+1}}$ is in fact g_n^+-ample.

Since $K_{\mathcal{F}_{n+1}}$ is g^+-ample, for $n \gg 0$ we see that $K_{\mathcal{F}_{n+1}} + ng^{+*}M'$ is ample on X_{n+1}, and so X_{n+1} is in fact projective.

If C is not contained in D_n or T_n, then we claim there exists an analytic open U_1' containig U_1 and C and an extension of T_1 to U_1' call it T_n' such that replacing T_1, U_1 by T_1', U_1' we have that C is now contained in T_n.

Supposing the claim, by our $\mathbb{Q}$-factoriality hypothesis and the above computations we see that each step of the MMP is still $K_{U_i} + D_i + T_i$-negative, and thus we have reduced to our previous case.

To prove the claim let $C \subset V$ be a small analytic neighborhood, and let $S \subset V$ be the union of the separatrices containing C. Let $S_1 \subset V_1$ be strict transform of S, V back on X_1.

If S_1 meets some component of D_1, then it must meet along $\mathrm{sing}(\mathcal{F}_1)$, and therefore it must agree with some components of T_1, and thus has an extension to a neighborhood of D_1.

If S_1 meets some non-invariant divisor F contained in U_1, then the same argument as before implies that S_1 has an extension to a neighborhood of F. Thus taking $U_1' = U_1 \cup V_1$ and taking $T_1' = T_1 \cup S_1$ gives our desired extensions. □

Lemma 4.3.8. *X_{n+1}, as constructed above, is $\mathbb{Q}$-factorial.*

Proof. First, assume that f_n is a divisorial contraction. Since it is a klt contraction, $\rho(X_n/X_{n+1}) = 1$. Let D be a divisor on X_{n+1}, and let D' be its strict transform on X_n. Since $-(K_{X_n} + D_n)$ is f_n-ample there exists some c such that $D' + c(K_{X_n} + D_n)$ is f_n-trivial, and therefore there is some $\mathbb{Q}$-Cartire divisr on X_{n+1} such that $f_n^*M = D' + c(K_{X_n} + D_n)$. Thus, we see that $D = M - c(K_{X_{n+1}} + D_{n+1})$ is $\mathbb{Q}$-Cartier.

Otherwise f_n is a flip. Let D' be a divisor X_n. We will show that the strict transform of D' on X_{n+1}, call it D is still $\mathbb{Q}$-Cartier. It suffices to check that this is so on a small neighborhood around the flipping/flipped locus, therefore we may assume our foliation flip is given by a sequence of log flips, $X_n = Y_0 \dashrightarrow Y_1 ... \dashrightarrow Y_\ell = X_{n+1}$. Let D_1 be the strict transform of D' on Y_1, and let W_1 be the base of the flip, and $h_i : Y_i \to W_1$ the corresponding contractions, both of which are of relative picard number 1. Take $-(K_{Y_0} + \Delta_0$ is h_0-ample, and so $D' + c(K_{Y_0} + \Delta_0$ is h_0-trivial for some c, and so there exists M on W_1 such that $h_0^*M = D' + c(K_{Y_0} + \Delta_0)$. Then $h_1^*M - c(K_{Y_1} + \Delta_1)$ is $\mathbb{Q}$-Cartier and equivalent to D_1. We are therefore done by induction.

Observe that this also proves that if T is a $\mathbb{Q}$-Cartier divisor only defined in a neighborhood of the flipping locus, that its strict transform in a neighborhood of the flipped locus is still $\mathbb{Q}$-Cartier. In particular, if a germ of a separatrix is $\mathbb{Q}$-Cartier, the flipped germ is still $\mathbb{Q}$-Cartier. □

Lemma 4.3.9. *(X_{n+1}, Δ_{n+1}) has klt singularities for some Δ_{n+1}, $\mathcal{F}_{n+1}$ has canonical non-dicritical singularities. sing(X_{n+1}) is tangent to $\mathcal{F}_{n+1}$ and $\mathcal{F}_{n+1}$ is terminal along the generic points of 1-dimensional components of sing(X_{n+1}).*

Proof. The first claim follows from the fact tha $(X_{n+1}, D_{n+1} - \epsilon D_{n+1})$ is klt.

The fact that $\mathcal{F}_{n+1}$ has canonical singularities is immediate because (log) terminal/(log) canonical is preserved by steps of the MMP. Non-dicriticality is preserved because if f_n is a divisorial contraction then we only contracted curves tangent to the foliation. If f_n is a flip, and g_n is the flipping contraction with base W_n then $\mathcal{F}_{W_n}$ has non-diciritical singularities because the flipping curve is tangent to the foliation, but this immediately implies that $\mathcal{F}_{n+1}$ has non-dicritical singularities.

Next suppose that C is a 1-dimensional singularity of X_{n+1}. Either C is a flipped curve, in which case the result follows from the fact that flipping curves are tangent to the foliation, or it is in the image of a divisorial contraction, $E \to C$. If E is transverse to the foliation then X_{n+1} must be terminal at the generic point of C, hence smooth at the generic point of C. Thus E is foliation invariant, and so C is tangent to the foliation.

For the last claim, suppose that f_n is a flip and let C be contained in the flipped locus. If E is a divisor dominating C then by the negativity lemma $a(E, \mathcal{F}_{n+1}) > a(E, \mathcal{F}_n) \geq 0$

Otherwise C is in the image of the contraction of foliation negative divisor E, and $K_{\mathcal{F}_n} = f_n^* K_{\mathcal{F}_{n+1}} + aE$ where $a > 0$ and the result follows by induction. □

The above lemmas, and the construction of f_n imply that our inductive hypotheses (1)-(3) are all still satisfied, and thus we can always produce the next step in the MMP. Unfortunately this is not enough to show that the MMP exists since we do not know that this process will terminate. But we have shown:

Proposition 4.3.10. *Suppose hypotheses 4.3.1, 4.3.2, 4.3.3 and termination of flips. Then the foliated MMP exists.*

4.3.4 Invariant termination

In this section we prove a termination result for foliation flips:

Proposition 4.3.11. *Suppose hypotheses 4.3.1, 4.3.2, 4.3.3. Let $(X, \mathcal{F})$ be a smooth variety and $\mathcal{F}$ have simple foliation singularities. Let $D = \sum D_i$ be a collection of $\mathcal{F}$-invariant divisors. Suppose that D is snc. Let $(X, \mathcal{F}) = (X_0, \mathcal{F}_0) \dashrightarrow (X_1, \mathcal{F}_1)$... be any sequence of steps of the $\mathcal{F}$-MMP. Let D^j denote the strict transform of D. Then, eventually the flipping locus is disjoint from sing(F_j) and D^j*

The proof proceeds in several steps.

Step 1:

Claim. *After finitely many flips no 1-dimensional components of the singular locus are contained in the flipping locus. No intersection of two 1-dimensional components of the singular locus is contained in the flipping locus.*

Proof. By the work in the previous section we can realize the foliated MMP as a log MMP for some dlt pair $(X, \sum D_i + \sum T_j)$, where by abuse of notation, T_j are analytic divisors on some open subset of X.

If S is some invariant divisor then we can write $K_{\mathcal{F}}|_S = K_S + \sum k_i C_i + \sum \frac{n_j - 1}{n_j} B_j$ where $k_i \geq 1$ are integers and $C_i \subset \text{sing}(\mathcal{F}_k)$

Without loss of generality we may assume that S is one of the D_i, T_j and that the separatrices around each component of the singular locus are contained in D_i, T_j, so by dlt adjunction, we can write $(K_X + \sum D_i + \sum T_j) = K_S + \sum C_i + \sum \frac{n_j - 1}{n_j} B_j$ and $(S, \sum C_i + \sum \frac{n_j - 1}{n_j} B_j)$ is dlt.

In particular, if $C_i, C_{i'}$ intersect, they must intersect at a smooth point of S. Let Σ be a flipping curve, and let S be a germ of a separatrix around Σ. If Σ meets the intersection of $C_i, C_{i'}$, then we have that $\Sigma \cdot (\sum k_i C_i + \sum \frac{n_j - 1}{n_j} B_j) \geq 2$. However, $K_S \cdot \Sigma \geq -2$ since Σ is a flipping curve. But this is a contradiction of the $K_{\mathcal{F}}$-negativity of Σ.

Every time a curve in the singular locus is flipped, we know that $\mathcal{F}$ is terminal along the flipped curve. In particular, the number of components of $\mathrm{sing}(\mathcal{F})$ drops. This cannot hapenn infinitely often. □

Step 2:

Claim. *After finitely many flips, the flipping locus is disjoint from the singular locus of $\mathcal{F}$.*

Proof. Let $(X, \mathcal{F}) \dashrightarrow (X', \mathcal{F}')$ be the flip. Let C be a components of the singular locus. As above, by dlt adjunction we can write $K_{\mathcal{F}}|_C = K_C + \sum k_i P_i + \sum \frac{n_j - 1}{n_j} Q_j$ where k_i are integers, and P_i is supported on the intersection of C with other components of the singular locus.

Suppose that C meets the flipping locus, and write C' for the strict transform of C. Again, we can write $K_{\mathcal{F}'}|_{C'} = K_{C'} + \sum k'_i P_i + \sum \frac{n'_j - 1}{n'_j} Q_j$. By the negativity lemma we know that $\sum k'_i P_i + \sum \frac{n'_j - 1}{n'_j} Q_j \leq \sum k_i P_i + \sum \frac{n_j - 1}{n_j} Q_j$ with strict inequality for some coefficient. By our previous step we know that $k'_i = k_i$ and since $\{\frac{n-1}{n}\}$ has no infinite strictly decreasing sequence we see that the flipping locus cannot meet C infinitely often. □

Step 3:

Claim. *After finitely many flips, the flipping locus is disjoint from D_i.*

Proof. Write $S = D_i$. We first show that we cannot flip a curve into S infinitely often. Let $(X, \mathcal{F}) \dashrightarrow (X', \mathcal{F}')$ be a flip, let S' be the strict transform and let C' be the flipped curve. Suppose that $C' \subset S'$.

Writing $K_{\mathcal{F}}|_S = K_S + \Delta$ and $K_{\mathcal{F}'}|_{S'} = K_{S'} + \Delta'$ by our previous step we know that C' is disjoint from $\lfloor \Delta' \rfloor$. Notice that the coefficient of C' in Δ' is non-negative and so the rational map $S' \to S$ extracts a divisor with non-positive discrepancy with respect to $K_S + \Delta$. However, the centre of C' on S is always disjoint from $\lfloor \Delta \rfloor$, and $K_S + \Delta$ is klt away from $\lfloor \Delta \rfloor$. There are only finitely many divisors of non-positive discrepancies centred over the klt locus of (S, Δ), and so we can only flip a curve into S finitely many times.

If we flip a curve out of S, then the picard rank of S drops, which can only happen finitely many times.

Finally, if a flipping curve C intersects S, but is not conatined in S, then $C \cdot S > 0$ and so $C' \cdot S' < 0$, which implies that $C' \subset S'$.

This implies that the flipping locus must eventually be disjoint from S. $\square$

Applying the claim to all components of D we get the proof of the proposition.

4.4 Comments on the hypotheses in the special MMP

While ultimately we hope to be able to deduce the existence of the MMP from the special MMP and special termination, it will be useful to consider under which circumstances the hypotheses used are true or false.

4.4.1 The irreducibility hypothesis

In the case of foliations by curves, McQuillan [McQ] has noticed that the existence of a connected component of the flipping locus which is not smooth is an obstruction to the projectivity of the flip, and that such an obstruction really does occur. However, in the case of foliations with canonical singularities he is able to classify those foliations where this obstruction occurs.

One can view this obstruction as the failure for canonical models of foliations to exist in general. Indeed, in the classical case one can reduce to the case where each flipping curve is irreducible by running a relative MMP over a small neighborhood of the base of the flip. This MMP results in a minimal model over the base, and by taking the canonical model of this minimal model over the base we get the flip. However, in our situation such a reduction is not possible. Later on we will see how to address this problem in some situations.

4.4.2 The $\mathbb{Q}$-factoriality hypothesis

In the case where $\mathcal{F}$ has terminal singularities, the $\mathbb{Q}$-factoriality hypothesis is not necessary.

We will need to make use of the following generalization of Malgrange's theorem due to Cerveau and Lins-Neto [CLN08, Corollary 1]

Lemma 4.4.1. *Let X be a germ of an analytic variety at $0 \in \mathbb{C}^N$ of dimension n, and let $\mathcal{F}$ be a holomorphic foliation on $X^* = X - sing(X)$. Suppose that:*

1) X is a complete intersection,

2) $dim(sing(X)) \leq n-3$,

3) $\mathcal{F}$ is defined by a holomorphic 1 form ω such taht $dim(sing(\omega)) \leq n-3$.

Then $\mathcal{F}$ has a holomorphic first integral.

Proposition 4.4.2. *Suppose (X, Δ) has klt singularities for some divisor Δ, and that $\mathcal{F}$ has terminal singularities. Let C be curve tangent to the foliation and S a germ of a leaf containing C. Then S is $\mathbb{Q}$-Cartier.*

Proof. The statement can be checked analytically locally around the singular points of X along C, so replace everything by its germ around a singular point.

Next, if after passing to a finite cover $\pi : Y \to X$ we have that $\pi^{-1}(S)$ is $\mathbb{Q}$-Cartier, then S is $\mathbb{Q}$-Cartier.

So, take an index 1 cover (ramified only over sing(X)) so that $K_X, K_{\mathcal{F}}$ are both Cartier. Call this new cover $(Y, \mathcal{G})$. $K_Y = \pi^* K_X$ and $K_{\mathcal{G}} = \pi^* K_{\mathcal{F}}$, since Y is log terminal and K_Y is Cartier, this implies that Y is canonical.

By Lemma 2.3.7 we see that $\mathcal{G}$ is terminal.

Next we claim that Y is actually terminal. Notice that $(N^*_{\mathcal{G}})^{[1]}$ is a line bundle being the difference of 2 Cartier divisors, and since Y is log terminal, we have that the foliation discrepancies are less than or equal to the usual discrepacies, 2.3.1. Thus, since $\mathcal{G}$ is terminal, this immediately implies that Y is terminal.

Y is terminal and index 1, which implies by [KM98] that it is a cDV hypersurface singularity, in particular Y is a complete intersection and sing(Y) is isolated.

Notice also that $\mathcal{G}$ is smooth away from sing(Y). We claim that $\mathcal{G}$ has a holomorphic first integral.

Observe that for any $0 \in \mathrm{sing}(Y)$, if we write $Y^* = Y - 0$ that $\mathcal{G}$ is defined by global a 1-form on Y^* near 0. Indeed, take any generator ω of $(N^*_{\mathcal{G}})^{[1]}$ around 0. Observe that since $\mathcal{G}$ is smooth away from 0 we have that $\mathrm{sing}(\omega) \subset \{0\}$.

Thus, 4.4.1 applies to show that $\mathcal{G}$ has a holomorphic first integral, i.e., there is a holomorphic $f : Y \to \mathbb{C}$ whose fibres determine $\mathcal{G}$. In particular $\pi^{-1}(S)$ is exactly $(f = 0)$, which implies that $\pi^{-1}(S)$ is Cartier.

□

Remark 26. *In fact, the above proof shows that if $(X, \mathcal{F}, 0)$ is a singularity arising in the course of the foliated MMP, then $(X, 0)$ is a quotient of a terminal Gorenstein singularity $(Y, 0')$, in particular, Y is cDV hypersurface. Thus the singularities arising in the course of the foliated MMP are at worst LCIQ.*

In contrast to the terminal case, consider the following example which shows that the $\mathbb{Q}$-factoriality hypothesis is not always satisfied:

Example 14. *Let $(X, \mathcal{F})$ be a foliated $\mathbb{Q}$-factorial threefold with a singularity 0 isomoprhic (in the analytic topology) to the cone over a quadric. Let X_0 be the formal completion of X along this singularity. Let $Y \to X_0$ be the resolution of the vertex of the cone with exceptional divisor Q. Consider any foliation on Y such that Q is invariant, and Q meets exactly two other invariant divisors each one along a fibre in each ruling on Q.*

This induces a foliation on X_0 which can be chosen to extend to X. In this case, each separatrix around 0 is not $\mathbb{Q}$-Cartier, however their sum is Cartier.

4.4.3 The convergence hypothesis

The convergence hypothesis is perhaps the most mysterious. As we saw the hypothesis is satisfied when X is smooth and the flipping curve is contained in a smooth component of $\mathrm{sing}(\mathcal{F})$.

Nevertheless, it seems that one could do away with the convergence hypothesis. We sketch the (potential) argument here:

Let $\mathfrak{Y} \to \mathfrak{Z}$ be the formal completions of Y, Z along $f^{-1}(0), 0$ respectively. Here our formal separatrices now become divisors on $\mathfrak{Y}$.

If we could run a formal log MMP over $\mathfrak{Z}$ then by using [Art70] we could realize this formal log MMP as an actual MMP over Z which would in turn be the $\mathcal{F}$-MMP, without needing our convergence hypothesis.

The existence of such an MMP would follow if we could deduce relative formal versions of the theorems in the log MMP from the relative analytic versions of these theorems. Unfortunately, the literature seems to be missing such statements.

4.5 A special relative MMP and an existence of flips result

In this section we will show that invariant termination proves the existence of the special MMP in a relative analytic situation, what we will call the "special relative MMP". Here our set up is a projective morphism of complex analytic varieties $\pi : Y \to (Z, 0)$ where Z is the germ around a point 0. Let $\mathcal{F}$ be a foliation on Y with canonical singularities. Suppose furthermore that $\text{exc}(\pi)$ is foliation invariant.

We will continue to need our convergence hypothesis, 4.3.2.

4.5.1 Existence of special relative MMP

The goal here is to prove the following:

Proposition 4.5.1. *Suppose that Y is smooth and $\mathcal{F}$ has simple singularities. Then there is a nef model of $K_{\mathcal{F}}$ over Z, i.e., a pair $(Y', \mathcal{F}')$ birational to $(Y, \mathcal{F})$ with $K_{\mathcal{F}'}$ π-nef.*

Before proceeding we check the (easy) fact that the cone theorem holds in our relative analytic situation:

Lemma 4.5.2. *Suppose that $\pi : Y \to (Z, 0)$ is a projective morphism with $dim(Y) = dim(Z) = 3$. Let $\mathcal{F}$ be a foliation on Y with canonical singularities. Suppose that $exc(\pi)$ is foliation invariant. Then*

$$\overline{NE}(Y/Z) = \overline{NE}(Y/Z)_{K_{\mathcal{F}} \geq 0} + \sum \mathbb{R}_{\geq 0}[L_i]$$

where the L_i are either rational curves tangent to the foliation with $K_{\mathcal{F}} \cdot L_i \geq -4$ or are contained in sing(Y).

Proof. Since the argument is in many ways simpler than the complete cone theorem, we only sketch it here.

Notice first that if an extremal ray is spanned by a curve, then by assumption this curve must be tangent to the foliation. Furthermore, if $C \subset S$ where S is a an invariant divisor containing C, either $S \to \pi(S)$ is bimeromorphic, $\pi(S)$ is a curve or $\pi(S)$ is a point (in which case S is algebraic). In either case writing $K_{\mathcal{F}}|_S = K_S + \Delta$ and applying the (relative) cone theorem for surfaces gives our result.

Let R be a $K_{\mathcal{F}}$-negative extremal ray, we see that there exists an effective divisor S with $S \cdot R < 0$.

If S is invariant, then we see that R comes from a $K_S + \Delta$-negative extremal ray, in which case we are done by the (usual) relative cone theorem.

If S is not invariant, let $\mathcal{G}$ be the induced foliation on S, and let $T = \pi(S)$. Notice that $S \to T$ must be birational and so we see that R comes from a $K_{\mathcal{G}} + \Delta$-negative extremal ray contracted by π, in particular it is spanned by a curve. □

Now we prove the proposition:

Proof. Suppose now that Y is smooth, and $\mathcal{F}$ has simple singularities. Let $D = \pi^{-1}(0)$ and let $W = \text{sing}(\mathcal{F}) \cap S$. By assumption for each component W_i of W (perhaps shrinking $(Z, 0)$) there exists an analytic divisor T_i such that T_i is a separatrix along W_i. Write $T = \sum T_i$. Perhaps passing to a higher model we may assume that $D + T$ is a snc divisor (not just normal crossings), so in particular $(Y, D + T)$ is dlt.

We claim that running a $K_Y + D + T$-MMP over Z will be the $K_{\mathcal{F}}$-MMP over Z. In fact, since the separatrices around flipping curves are global divisors on Y, we see that the steps of the MMP will always keep T $\mathbb{Q}$-Cartier, in particular, for this MMP the $\mathbb{Q}$-factoriality hypothesis is always satisfied. Furthermore, by shrinking Z further, if needed, we may assume that the irreducibility hypothesis is satisfied.

Thus we can always realize each step of the $K_{\mathcal{F}}$-MMP as a step in the $K_Y + D + T$-MMP over Z.

It is clear that there cannot be an infinite sequence of divisorial contractions, and since any sequence of steps of the K_Y+D+T-MMP must eventually terminate, any sequence of foliation flips must termiante. The output of this MMP is our desired nef model. □

For later reference we briefly summarize some of the properties of $(Y', \mathcal{F}')$:

Corollary 4.5.3. *1) The singularities are Y' are tangent to $\mathcal{F}'$ and $\mathcal{F}'$ is terminal along the 1-dimensional singularities of Y'*

2) $\mathcal{F}'$ has canonical foliation singularities.

3) If D is a collection of $\mathcal{F}'$-invariant divisiors on Y', then (Y', D) is dlt.

4) Y' is projective over Z

5) Y' is $\mathbb{Q}$-factorial.

In light of point 4, we make the following warning/definition:

Definition 24. *By [Art70] we can use the above proposition to produce a "weak flip". Given $f : (X, \mathcal{F}) \to Z$ a flipping contraction, there exists a morphism $f' : (X', \mathcal{F}') \to Z$ where $K_{\mathcal{F}'}$ is f'-nef. However, X' may only be an algebraic space (not projective) and $K_{\mathcal{F}'}$ might fail to be f'-ample.*

4.5.2 Special relative MMP implies the existence of some flips

Proposition 4.5.4. *Let $\pi : X \to Z$ be a flipping contraction. Suppose that $\rho(X/Z) = 1$. If the "weak flip" $X' \to Z$ as constructed above is a projective morphism, then the flip exists.*

Proof. If H_R is a supporting hyperplane in $\overline{NE}(X)$ to the ray corresponding to the contraction, since $\rho(X/Z) = 1$ we see that $H_R = \pi^*M$ for some $\mathbb{Q}$-Cartier divisor M. M is therefore an ample divisor, and so Z (which is a priori just an algebraic space) is in fact projective.

Let $\pi(\text{exc}(\pi)) = p \in Z$. Let $g : Y \to X$ be a resolution of singularities, furthermore we can choose this resolution so that $a(E, \mathcal{F}) > 0$ for every divisor extracted by g.

As above, perhaps replacing Z by a small neighborhood of p in Z, call it U, we can run a MMP to find a nef model $(W, K_{\mathcal{F}_W}) \to U$. As before, by Artin we can find an algebraic space $X^+ \to Z$ extending $W \to U$, and a foliation $K_{\mathcal{F}^+}$ which is nef over Z.

Next, notice that X^+ is $\mathbb{Q}$-factorial, and so by the negativity lemma every divisor extracted by g must be contracted in X^+.

Thus we have $\rho(X^+) = \rho(X)$, and so $\rho(X^+/Z) = 1$. But this implies that $K_{\mathcal{F}^+}$ must be ample over Z, and not just nef. Furthermore if $\pi^+ : X^+ \to Z$ is the induced map then $K_{\mathcal{F}^+} + m\pi^{+*}M$ is ample for $m \gg 0$, and so X^+ is projective, and is therefore the flip. $\square$

The above lemma can be seen as showing that the challenge in constructing the flip is to pass from a model over Z where $K_{\mathcal{F}}$ is nef, the minimal model, to a model over Z where $K_{\mathcal{F}}$ is ample, the canonical model. As the next example, due to McQuillan, shows this is not always possible.

Example 15. *Let $(X, \mathcal{F})$ be a surface foliation which has an elliptic Gorenstein leaf (e.g.l.). $\mathcal{F}$ may be chosen so that $K_{\mathcal{F}}$ is big and nef. However, $K_{\mathcal{F}}$ is not semi-ample. In particular, $\mathbb{Q}$-Gorenstein canonical models of foliations do not always exist.*

As noted earlier, in the case where every connected component of the flipping locus is irreducible, the "weak flip" is projective, and therefore $\rho(X/Z) = 1$ is enough to guarantee the existence of the flip.

Denoting $(Y', D' + T')$ the output of this MMP, we see $(Y', D' + T')$ is dlt and that for any C mapping to a point in Z we still have that $K_{\mathcal{F}'} \cdot C \geq (K_{Y'} + D' + T') \cdot C$, in particular $K_{\mathcal{F}'} - (KY' + D' + T')$ is big and nef over Z. Thus, if the base point free theorem held for the dlt pair $(Y', D' + T')$ the $\mathcal{F}$-canonical model would exist, for example, if $D' + T'$ is LSEPD.

Corollary 4.5.5. *Suppose that* $(X, \mathcal{F})$ *is a step of the MMP, and that* $\mathcal{F}$ *has terminal singularities. Let* $f : X \to Z$ *be a flipping contraction. Then the flip exists.*

Proof. As before, we may assume that we are working in the neighborhood of a connected component of $\mathrm{exc}(f)$. Let C be the connected component.

We have the following sequence

$$0 \to \mathrm{Pic}(Z) \otimes \mathbb{Q} \xrightarrow{f^*} \mathrm{Pic}(X) \otimes \mathbb{Q} \to \mathbb{Q}$$

where the last arrow is given by intersecting with C. Thus, to prove $\rho(X/Z) = 1$ it suffices to show that if $M \cdot C = 0$ then $M = f^*M'$ for some M'.

Let S be the germ of a separatrix around C, and let $p = f(C)$. Let U be a small neighborhood around p so that S is defined on $W = f^{-1}(U)$. As above we know that $(W, S - \epsilon S)$ is klt for ϵ sufficiently small, and that $-(K_W + S)$ is f-ample. For $n \gg 0$ we have that $nM|_W - (K_W + (1 - \epsilon)S)$ is ample, and so the relative analytic base point free theorem applies to show that $M|_W$ is semi-ample over W, hence for some sufficiently large n, nM is pulled back from a Cartier divisor nM' on U. Since f is an isomorphism away from C, it is easy to extend nM' to a Cartier divisor on all of Z, and the result follows. In particular, Z is projective.

Now let $(W', \mathcal{F}') \xrightarrow{f'} V$ be the weak flip as constructed above, i.e. $K_{\mathcal{F}'}$ is nef over V, but not necessarily ample. Notice that since $\mathcal{F}$ is terminal every separatrix in a neighborhood of C is convergent, and so the convergence hypothesis is satisfied. Furthermore, we know that $W \dashrightarrow W'$ is an isomorphism in codimension 1.

We claim that we can construct a canonical model of $K_{\mathcal{F}'}$ over V. Again, since $\mathcal{F}$ is terminal we see that $K_{\mathcal{F}} \cdot \Sigma = (K_X + S) \cdot \Sigma$ for all Σ contracted by f. Indeed, since $\mathcal{F}$ is terminal this implies that if we write $K_{\mathcal{F}}|_S = K_S + \Delta$ and if we write $(K_X + S)|_S = K_S + \Theta$ then $\Delta = \Theta$.

Letting S' be the strict transform of S we see that $K_{\mathcal{F}'} - (K_{X'} + S')$ is numerically trivial over V, hence it is big and nef over V. We have that (W', S') is dlt, and our goal is to be able apply the base point free theorem to $K_{\mathcal{F}'}$.

Write $S = K_{\mathcal{F}} - K_X$, since f is a contraction of Picard rank 1, this implies that

$S =_{num} \lambda K_{\mathcal{F}}$ for some $\lambda \in \mathbb{Q}$, and so $S' =_{num} \lambda K_{\mathcal{F}'}$. In particular, either S' is nef or $-S'$ is nef over V, and if Σ is contracted by f' we have $K_{\mathcal{F}'} \cdot \Sigma = 0$ if and only if $S' \cdot \Sigma = 0$. In particular for $1 \gg \delta > 0$ and $m \gg 0$ we know that $mK_{\mathcal{F}'} + \delta S'$ is nef over V.

Thus for large enough m, and small enough δ, $mK_{\mathcal{F}'} - (K_{W'} + (1-\delta)S')$ is big and nef over V, and $(X', (1-\delta)S')$ is klt. Thus, we may apply the base point free theorem to conclude that $K_{\mathcal{F}'}$ is semi-ample.

Let $\phi : W' \to W^+$ be the induced map over Z such that there is an ample divisor A over Z such that $K_{\mathcal{F}'} = \phi^* A$.

Since $W' \to Z$ is small, we see that ϕ is small, and since $K_{\mathcal{F}^+} = \phi_* K_{\mathcal{F}}$ we get that $K_{\mathcal{F}^+}$ is ample over V.

As usual, we can realize the flip $X \dashrightarrow X^+$ in the category of algebraic spaces, and since $K_{\mathcal{F}^+}$ is ample over Z we get that X^+ is in fact projective. □

Remark 27. *Notice that the above result does not assume any of our earlier hypotheses, and therefore gives an unqualified existence of flips statement.*

The argument is a special case of the more general statement that if $K_{\mathcal{F}}$ is numerically equivalent over Z to $K_X + D$ where D is the sum of our separatrices, then the flip exists. It is unclear what sort of conditions could guarantee this numerical equivalence statement, since it is not true in general.

4.6 MMP under fewer hypotheses

Suppose that $(X, \mathcal{F})$ is a smooth 3-fold and $\mathcal{F}$ a foliation with simple singularities. Let $D = \sum D_i$ be a collection of $\mathcal{F}$-invariant divisors. Denote We know that every divisorial contraction in the MMP is $K_{X_n} + D_n$-negative. If there are no flips in the MMP, then our earlier computations show that each of these contractions can be realized as a klt contraction.

However, a $K_{\mathcal{F}_n}$-flip is not always a $K_{X_n} + D_n$-flip, and so we cannot see immediately that $(X_{n+1}, D_{n+1} - \epsilon D_{n+1})$ is klt.

Lemma 4.6.1. *Let $(X_n, \mathcal{F}_n) \dashrightarrow (X_{n+1}, \mathcal{F}_{n+1})$ be a foliation flip. Then $(X_{n+1}, D_{n+1} - \epsilon D_{n+1})$ is klt.*

Proof. Away from the flipping locus the result is clear. In a small neighborhood of the flipped locus, by 4.5.3 item 4 we see that $(X_{n+1}, D - \epsilon D)$ is klt for any collection of invariant divisors, in particular, $(X_{n+1}, D_{n+1} - \epsilon D_{n+1})$ is klt. □

We can therefore partly summarize the work of the previous sections by

Proposition 4.6.2. *Suppose X is smooth and $\mathcal{F}$ has simple singularities. Assume*

(i) termination of flips

(ii) all flips encountered in running the MMP for $(X, \mathcal{F})$ are terminal.

Then the MMP for $(X, \mathcal{F})$ exists.

Remark 28. *While we could have phrased this as requiring that $\mathcal{F}$ be terminal, this would would have implied that $\mathcal{F}$ is in fact smooth, hence the reason for the formulation of (ii).*

4.6.1 Termination of flips

As noted any sequence of divisorial contractions must terminate, thus the challenge is therefore to show that flips terminate.

In the classical case, the termination of threefold flips turns out to be easy, we sketch the argument here:

Given X a canonical threefold define $d(X)$, the difficulty, to be the number of divisors above X with discrepancy < 1. It is easy to see that the difficulty is finite, and one can show that the difficulty never increases under flips, and after enough flips must eventually go down. This implies immediately that there can be no infinite sequence of flips.

In the case of a foliation with canonical singularities, the situation is more subtle. If we try to define the foliated difficulty, i.e., the number of divisors above X with discrepancy < 1, we see that any canonical non-terminal singularity with have infinitely many divisors with discrepancy $= 0$. Indeed, blowing up a surface singularity repeatedly along its singular locus furnishes such an example. One can therefore view the problem as the fact that there is in general no way to resolve a singular foliation into a smooth one.

Alternatively, the challenge can be viewed as comparable to showing in the classical case that a sequence of (X, Δ) flips terminates where Δ is log canonical, but not klt.

In the case of rank 1 foliations, a flip must always decrease the number of components of the singular locus, and thus any sequence of flips must terminate. However, this is not true in general for co-rank 1 foliations. At any rate, it does not seem like there is an easy numerical reason for termination of foliation flips.

4.6.2 Some foliated conjectures of Shokurov and termination

We make the following foliated versions of two conjectures due to Shokurov:

Conjecture 4.6.3. *Let $\mathcal{F}$ be a co-rank 1 foliation. The function $mld(x, \mathcal{F}, \Delta)$ which assigns to a point the minimal foliated log discrepancy at that point is lower semicontinuous, i.e., the function only "jumps down".*

Conjecture 4.6.4. *Denote by L_n the set of all $a \in \mathbb{Q}$ such that a is the log discrepancy of some co-rank 1 foliation on some variety of dimension n. Then L_n satisfies the ACC.*

Observe that as stated the first conjecture is false for rank 1 foliations on 3-folds (even smooth threefolds):

Example 16. *Consider the foliation $\mathcal{L}$ on $\mathbb{C}^3$ given by $\partial_x + z\partial_y$. For $x \in \mathbb{R} \leq 0$, $mld(x, \mathcal{L}, \emptyset) = -1$ if x is rational and 0 otherwise.*

Proposition 4.6.5. *These conjectures are true for toric foliations.*

Proof. This follows from the fact that the classical versions of conjectures are true for toric varieties, and from our results on toric foliations. □

Proposition 4.6.6. *These conjectures are true for $\mathbb{Q}$-Gorenstein surface foliations with canonical singularities.*

Proof. This follows directly from the classification of $\mathbb{Q}$-Gorenstein canonical singularities due to [McQ08]. □

Observe that lower semicontinuity of foliated log discrepancies immediately implies that the minimal log discrepancy at any point is n. To see this, observe that any point is a limit of smooth points, which have m.l.d. exactly n and the result follows. The same result applies to generic points.

In the classical setting, these conjectures imply termination of flips. Thus, one may wonder if the foliated versions of these conjectures imply foliated termination of flips.

We will argue that this is true for the threefold case:

Proof. Let $\phi_i : X_i \dashrightarrow X_{i+1}$ be an infinite sequence of flips, with exceptional locus E_i. Notice that E_i is a curve. Let a_i be the log discrepancy along E_i. Let $\alpha_i = inf\{a_j : j \geq i\}$, then α_i is increasing, and by conjecture is eventually constant, call this value a.

Let $W_i = \{x \in X : x \in V, \dim(V) = 1, mld(V, \mathcal{F}, \Delta) \leq a\}$

Observe that the log discrepancy along any curve in W_i is at most a, with equality for a general curve in W_i. If W_i is one dimensional for all i sufficiently large, then in fact eventually the flipping locus cannot be contained in W_i

Thus, for i sufficiently large, we may assume that $W_i \dashrightarrow W_{i+1}$ is birational. Let $Z_i \subset W_i$ be those curves with minimal log discrepancy strictly less than a, which is equal to the set of those curves with log discrepancy less than or equal to $a - \epsilon$ for some ϵ. In particular Z_i is closed. Let z_i be the number of irreducible components of Z_i.

Thus $z_i \geq z_{i+1}$, with equality if the flipping locus is not contained in Z_i, and for a sequence of flips whose flipping/flipped loci are all contained in Z_i, eventually strict inequality.

Indeed, the log discrepancy along the flipped locus is greater than the log discrepancy along the flipping locus unless there is a point along the flipping locus with mld less than the mld at the generic point of the flipping locus.

But, observing that the mld of any point along the flipped locus is greater than the minimum mld of points on the flipping locus, we see that this issue can only happen finitely many times, since eventually the flipping locus is disjoint from those points with mld $< a$, or the number of points with mld less than $a - \epsilon$ for some ϵ decreases.

Thus we see that the pairs $(\rho(W_i), z_i)$ are strictly decreasing under lexicographic order under flips. In particular, any sequence of flips must terminate. □

Unfortunately, these conjectures are probably very hard. Indeed, very little is known about them in the classical case and the cases which are known largely proceed by explicit classification of singularities.

4.7 Toric foliated MMP

Definition 25. *Let X be a toric variety. Let $\mathcal{F}$ be a foliation on X. We say that $\mathcal{F}$ is toric provided that it is invariant under the torus action on X.*

Lemma 4.7.1. *Let $\mathcal{F}$ be a co-rank 1 toric foliation. Then $K_{\mathcal{F}} = -\sum D_\tau$ where the sum is over all the torus invariant and non-$\mathcal{F}$-invariant divisors.*

Proof. By passing to a toric resolution $\pi : (X', \mathcal{F}') \to (X, \mathcal{F})$, and noting that the strict transform of a divisor D is torus and $\mathcal{F}'$-invariant if and only if it is torus and $\mathcal{F}$-invariant, we see that it suffices to prove the result on a resolution of X. Thus we may assume that X is smooth.

Observe that $\mathcal{F}$ is defined by a rational torus invariant 1-form ω. Working in torus coordinates $x_1, ..., x_n$, we see that $\omega = \sum \lambda_i \frac{dx_i}{x_i}$. Where $\lambda_i \in \mathbb{C}$. $\lambda_i \neq 0$ if and only if the divisor associated to $\{x_i = 0\}$ is foliation invariant. In particular ω has a pole of order 1 along each torus and $\mathcal{F}$-invariant divisor. Thus $N^{\mathcal{F}}$ is equivalent to the sum of the torus and foliation invariant divisors, and the result follows. □

Definition 26. *Let σ be a cone in a fan Δ defining a toric variety. Let $D(\sigma)$ denote the closed subvariety corresponding to σ.*

Remark 29. *We note that if $\tau = \langle v_1, ..., v_n \rangle$ is a full dimensional cone in the fan defining X, then this argument in fact shows $D(v_i)$ is $\mathcal{F}$-invariant for some i.*

Furthermore, if $w = \langle v_1, ..., v_{n-1} \rangle$ is a codimension 1 cone in the fan, then $D(w)$ is tangent to $\mathcal{F}$ if and only if $D(v_i)$ is invariant for some i.

Remark 30. *Let $p \in sing(\mathcal{F})$. Observe that every separatrix at p is convergent.*

We also make the following simple observation:

Proposition 4.7.2. *Suppose that $\mathcal{F}$ is defined by $\omega = \sum_{i=1}^{n} \lambda_i \frac{dx_i}{x_i}$. If $\lambda_i = 0$ for some i, then ω, and hence $\mathcal{F}$, is pulled back along some dominant rational map $f : X \dashrightarrow Y$. In particular, if $K_{\mathcal{F}}$ is not nef, then $\mathcal{F}$ is a pull back.*

Proof. Suppose for sake of contradiction that $\mathcal{F}$ is not a pull back. This remains true after passing to a resolution of singularities of X. Let $\mathcal{F}'$ be the transformed foliation. Since $\mathcal{F}'$ is not a pull back we have that every torus invariant divisor is also $\mathcal{F}'$ invariant, thus $K_{\mathcal{F}'}$, and hence $K_{\mathcal{F}}$ is trivial. □

We show that the cone theorem holds for co-rank 1 toric foliations in all dimensions- first we have the following result:

Theorem 4.7.3. *Let $\mathcal{F}$ be a toric foliation with canonical singularities. Let C be a curve in X, and $K_{\mathcal{F}}.C < 0$, then $[C] = [M] + \alpha$ where M is a torus invariant curve tangent to the foliation, and α is a pseudo-effective class.*

Proof. By [Mat02], we can write

$$C = \sum_{\text{tangent to } \mathcal{F}} a_u D(u) + \sum_{\text{not tangent to } \mathcal{F}} b_w D(w)$$

and where u, w run over the codimension 1 subcones of the fan.

We show that some a_u can be taken to be non-zero. Assume the contrary, that $a_u = 0$ for all u.

Since $D(w)$ is not tangent to the foliation, we have that if $w = \langle v_1, ..., v_{n-1} \rangle$, then all the $D(v_i)$ are not foliation invariant.

In order to have $K_{\mathcal{F}} \cdot C < 0$, we must have $D(w) \cdot D(v_i) > 0$ for some w, v_i. Let τ, τ' be the two full dimensional cones which are spanned by w and v_n, v_{n+1} respectively. Then $\tau \cup \tau'$ must be concave along $\langle v_1, ..., \hat{v}_i, ..., v_{n-1} \rangle$.

Thus, there must be $\sigma_1, ..., \sigma_r$ cones in our fan such that $\tau \cup \tau' \bigcup_{i=1}^{r} \sigma_i$ is a convex subcone of our fan. Furthermore, we know that both $D(v_n)$ and $D(v_{n+1})$ are foliation invariant. By [Mat02] $\langle v_1, ..., \hat{v}_i, ..., v_{n-1}, v_n \rangle$ or $\langle v_1, ..., \hat{v}_i, ..., v_{n-1}, v_{n+1} \rangle$ are in the same extremal ray as $D(w)$, and both correspond to torus invariant curve tangent to the foliation.

Thus, in the extremal ray spanned by $D(w)$ there is a curve tangent to the foliation. □

Thus, we have

Corollary 4.7.4. *Let $\mathcal{F}$ be a co-rank 1, toric foliation with non-dicritical singularities. Then, $\overline{NE}(X)_{K_{\mathcal{F}}<0} = \sum \mathbb{R}^+[M_i]$ where the M_i are torus invariant rational curves tangent to the foliation.*

Lemma 4.7.5. *Let R be a $K_{\mathcal{F}}$-negative ray. Then there is a contraction corresponding to this extremal ray, and falls into one of the following types:*

(i) Fibre type contractions.

(ii) Divisorial contractions.

(iii) Small contractions.

Furthermore, in cases (i) and (ii) if a curve is contracted, it is tangent to the foliation. In particular, after a contraction of type (i) or (ii) if $\mathcal{F}$ has non-dicritical singularities, the resulting foliation will still have non-dicritical singularities.

Proof. We know that the contraction exists, what is unclear if the curves being contracted are tangent to the foliation. By our cone theorem for toric foliations we know that some curve contracted is tangent to the foliation, however, it might be the case that there is a contracted curve transverse to the foliation.

In case (i) suppose that $\pi : X \to Z$ is the contraction and Y is a general fibre. Suppose for sake of contradiction that Y is not tangent to the foliation. Then there is an induced foliation on Y, call it $K_{\mathcal{G}}$ and $K_{\mathcal{G}}$ is negative on every curve in Y and furthermore $\rho(Y) = 1$. However, this implies that $\mathcal{G}$ must have dicritical singularities, implying that $\mathcal{F}$ does as well- a contradiction. Finally, if a general fibre is tangent to the foliation, then every fibre is.

In case (ii), if D is the divisor contracted by π, and if D is invariant, the result is immediate. Otherwise D is transverse to the foliation, and thus we have an induced foliation on D, keeping in mind that $D \cdot R < 0$ we see that the induced foliation is still negative on R. Thus, π induces fibre type contraction $D \to \pi(D)$, and the result follows by case (i). □

We now handle the flipping case:

Lemma 4.7.6. *In the case of a small contraction, the flip exists and no infinite sequence of flips exsits. Furthermore, if $\mathcal{F}$ has canonical and non-dicritical singularities, then the flipped foliation, $\mathcal{F}^+$ does as well.*

Proof. The existence and termination of the flip can be seen by the fact that toric log flips exist and terminate. Lemma 10.12 implies that $\mathcal{F}^+$ has canonical singularities.

What remains to be shown is the claim about the non-dicriticalness of $\mathcal{F}^+$. Let S be the flipping locus of $\mathcal{F}$.

Non-dicriticalness of $\mathcal{F}^+$ follows easily if either S is contained in sing$(\mathcal{F})$ or if it is tangent to the foliation.

So, suppose that S is transverse to the foliation. Let $f : X \to Z$ be the contraction, and let T be a fibre of f. Suppose for sake of contradiction that T is transverse to the foliation. Notice that the Picard number of T is 1. Furthermore, observe that the foliation restricted to T, $\mathcal{F}_T$, has $-K_{\mathcal{F}_T}$ ample. However, this implies that $\mathcal{F}_T$ is pulled back along a rational map $T \dashrightarrow W$, a contradiction of the non-dicritcality of $\mathcal{F}$.

Thus, the foliation restricted to S is the fibration induced by f. If T is a general fibre, then there exists an analytic germ of a hypersurface containing T which is foliation invariant. Call this hypersurface Y_T. If $\mathcal{F}^+$ failed to be non-dicritical, then for infinitely many fibres T, the strict transforms of Y_T would intersect. However, this implies that $\mathcal{F}^+$ is dicritical along a codimension 2 singularity, which contradicts the fact that $\mathcal{F}^+$ has canonical singularities.

□

Putting all this together:

Theorem 4.7.7. *The foliated toric MMP exists, and ends either with a foliation where $K_{\mathcal{F}}$ is nef, or with a fibration $\pi : X \to Z$ and $\mathcal{F}$ is pulled back from a foliation on Z.*

Proof. If $K_{\mathcal{F}}$ is not nef, there is an extremal ray on which $K_{\mathcal{F}}$ is negative. We can contract this ray resulting in a either:

(i) a fibration, in which case we stop.

(ii) a divisorial contraction, in which case we repeat with the new variety.

(iii) a flipping contraction, in which case we perform the flip.

Each of these steps can happen only finitely many times.

□

Portions of the work in the above chapter is being prepared for submission for publication.

Spicer, Calum "Higher dimensional foliated Mori Theory".

The dissertation author was the primary investigator and author of this material.

Chapter 5

Applications to classification problems

In the Kodaira-Enriques classification, smooth projective surfaces are classied in terms of their Kodaira dimension. For foliations on smooth surfaces Brunella, McQuillan and Mendes have performed a similar classification in terms of the Kodaira dimension and numerical dimension. We recall the definitions of these quantities:

Definition 27. *Let D be a $\mathbb{R}$-Cartier divisor on a normal variety X. We define the* **Kodaira dimension** *by*

$$\kappa(D) = limsup_{m \to \infty} \frac{log(h^0(X, \mathcal{O}(mD)))}{log(m)}$$

If D is nef we define the **numerical dimension** *by*

$$\nu(D) = max\{k : D^k \neq 0\}$$

. If D is not pseudo-effective we set $\nu(D) = -\infty$.

If D is not nef it is still possible to define $\nu(D)$ in a way which agrees with the definition here, but we will not need this.

We collect the following standard facts

Lemma 5.0.1. *Let $dim(X) = n$. (i) $\nu(D) \geq \kappa(D)$*

(ii) If $\nu(D) = n$ then $\kappa(D) = n$.

(iii) If $h^0(X, \mathcal{O}(mD)) = 0$ for all m, then $\kappa(D) = -\infty$.

Definition 28. *Given a variety X or a foliation $\mathcal{F}$ we define $\kappa(X) = \kappa(K_X)$ and $\kappa(\mathcal{F}) = \kappa(K_{\mathcal{F}})$, and likewise for ν.*

In the course of the Kodaira-Enriques classification it turns out that $\kappa(X) = \nu(X)$. Intrestingly this fails for $\mathcal{F}$, in fact the subtlest point of the Brunella-McQuillan-Mendes classification is classifying those foliations with $\kappa(\mathcal{F}) \neq \nu(\mathcal{F})$.

In these next few subsections we will perform a partial classification of smooth foliations in terms of κ, ν.

5.1 Smooth foliations with $\nu = 0$

Smooth foliations with $c_1(K_{\mathcal{F}}) = 0$ have been classified by Touzet [Tou08]

Theorem 5.1.1. *Let X be a complex projective manifold and let $\mathcal{F}$ be a smooth co-rank 1 foliation with $K_{\mathcal{F}} =_{num} 0$. Then $\mathcal{F}$ fits into one of the following categories:*

A) X is a $\mathbb{P}^1$ bundle over a manifold Y with $K_Y = 0$ and $\mathcal{F}$ induces a flat connection on the bundle.

B) There is an etale cover $\pi : A \times Y \to X$ where A is an abelian variety and $\pi^*\mathcal{F}$ is the pull back of a co rank 1 linear foliation on A.

C) There exists a curve B with $g(B) \geq 2$, a manifold Y with $K_Y = 0$ and an etale cover $\pi : Y \times B \to X$ such that $\pi^*\mathcal{F}$ is induced by the fibration over B.

Corollary 5.1.2. *Let $(X, \mathcal{F})$ be a smooth rank 2 foliation on a smooth threefold. Suppose that $\nu(\mathcal{F}) = 0$. Then $\mathcal{F}$ is birational to one of the foliations above.*

Proof. Run the foliated MMP starting from $(X, \mathcal{F})$. This exists and terminates because $\mathcal{F}$ is smooth. Let $(Y, \mathcal{G})$ be the output of the MMP. Then $\nu(\mathcal{G}) = 0$ and $\mathcal{G}$ is nef which implies that $c_1(K_{\mathcal{G}}) = 0$. We are therefore done by Touzet's classification. □

Corollary 5.1.3. *Let $(X, \mathcal{F})$ be a smooth rank 2 foliation on a smooth threefold. If $\nu(\mathcal{F}) = 0$ then $\kappa(\mathcal{F}) = 0$.*

Proof. The first claim follows from the above result and Touzet's classification. □

5.2 Smooth foliations with $\nu = \kappa = 1$

We need the following simple fact:

Lemma 5.2.1. *Let D be a nef divisor with $\nu(D) = \kappa(D) = 1$. Then D is semi-ample.*

Proposition 5.2.2. *Let $(X, \mathcal{F})$ be a smooth foliation with $\nu(\mathcal{F}) = 1$ and $\kappa(\mathcal{F}) = 1$. Then $(X, \mathcal{F})$ is birational to one of the following:*

1) A non-isotrivial fibration of Calabi-Yau surfaces.

2) There is a fibration $f : X \to B$ with general fibre transverse to the foliation. Furthermore, the foliation induced on a general fibre is of type A, B or C in Touzet's classification.

Proof. Run the foliated MMP to get $(Y, \mathcal{G})$ with $K_{\mathcal{G}}$ nef and $\nu(\mathcal{G}) = \kappa(\mathcal{G}) = 1$. By our above lemma this implies that $K_{\mathcal{G}}$ is semi-ample. Let $f : X \to B$ be the fibration coming from $K_{\mathcal{G}}$ where B is a curve. We have that $K_{\mathcal{G}} = f^*M$ for some divisor M on B.

We have two cases, either (i) $\mathcal{G}$ is tangent to f or (ii) it is transverse to f.

Case (i): If $\mathcal{G}$ is tangent to the fibration, then the two must agree. Even though $\mathcal{G}$ is smooth, the fibration is not necessarily smooth due to the possibility of non-reduced fibres. However, up to a finite cover we know that the fibration is smooth.

If S is a general fibre, then $0 = K_{\mathcal{G}}|_S = K_S$, and so $\mathcal{G}$ is a fibration in surfaces with trivial canonical bundle. Notice that f cannot be isotrivial, since otherwise $c_1(K_{\mathcal{G}}) = 0$, contrary to supposition.

Case (ii): If $\mathcal{G}$ is transverse to the fibration, let F be a general fibre. Then there is an induced foliation $\mathcal{H}$ on F. By foliation adjunction we get that $K_{\mathcal{H}} + \Delta = 0$ where $\Delta \geq 0$.

If Δ is non-zero, then $K_{\mathcal{H}}$ is not pseudoeffective, and so $\mathcal{H}$ is a rational fibration. However, Δ must also be tangent to the fibration, and this is a contradiction. Indeed, if L is a general leaf of $\mathcal{H}$ then $L \cdot \Delta = 0$, and so $L \cdot (K_{\mathcal{H}} + \Delta) = -2 \neq 0$. Thus, we see that for a general fibre Δ must be zero.

If $\Delta = 0$, then $K_{\mathcal{H}} = 0$. In this case, we see that either a fibre is everywhere transverse to the foliation or it is tangent to the foliation. This implies that for a general fibre $\mathcal{H}$ will be smooth. By the classification of smooth foliations with $K_{\mathcal{H}} = 0$. We apply Touzet's classifcation to conclude. □

It is possible that we have some fibres tangent to the foliation, even if the general fibre is transverse to the foliation. In this case we know that $K_{\mathcal{G}}|_S = K_S$ and so S must be a Calabi-Yau surface.

Remark 31. *The only essential use of smoothness of $\mathcal{F}$ was in using the MMP. Indeed, assuming the complete MMP, the classification in this section still holds true.*

Remark 32. *There are examples of foliations with $\nu = 1$ and $\kappa = -\infty$, it is unclear if there are examples with $\nu = 1$ and $\kappa = 0$.*

5.3 Foliations with $\nu = 3$

Proposition 5.3.1. *Let $\mathcal{F}$ be a foliation with canonical singularities on a smooth threefold X such that $K_{\mathcal{F}}$ is big and nef. Suppose that there are no nonconstant morphisms $C \to X$ where $g(C) \leq 1$, then $K_{\mathcal{F}}$ is ample.*

Proof. Write $K_{\mathcal{F}} = A + E$ where A is ample and E is effective. We have that $K_{\mathcal{F}}^3 > 0$.

Suppose that $K_{\mathcal{F}} \cdot C = 0$ for some curve C. Then $S \cdot C < 0$ for some component of E. There are three cases, either (i) $C \not\subset \operatorname{sing}(\mathcal{F})$ is tangent to the foliation, (ii) it is transverse or (iii) $C \subset \operatorname{sing}(\mathcal{F})$.

In case (i) let T be the germ of a leaf containing C. Then $(K_X + T) \cdot C = 0$ and $(K_X + T + tS) \cdot C < 0$ for all $t > 0$. Since (X, T) is log terminal along C, let t_0 be the log canonical threshold of $(K_X + T + tS)$ along C, then $(K_X + T + t_0 S) \cdot$

$C < 0$. However, by Kawamata subadjunction this implies that C is rational, a contradiction.

In case (ii) S must also be transverse to the foliation. As above, we can use Lemma 2.6.8 to see that S is smooth at the generic point of C. Thus write $(K_\mathcal{F} + S)|_{S^\nu} = K_\mathcal{G} + \Delta$ and notice that $(K_\mathcal{G} + \Delta) \cdot C < 0$.

Let $(T, \mathcal{H}) \xrightarrow{f} (S^\nu, \mathcal{G})$ be the foliated terminalization, i.e., a resolution followed by an application of the relative foliated MMP for surfaces, then writing $K_\mathcal{H} + \Gamma = f^*(K_\mathcal{G} + \Delta)$ we have that T is $\mathbb{Q}$-factorial and $(K_\mathcal{H} + \Gamma) \cdot f_*^{-1}C < 0$. This implies that $f_*^{-1}C$ is a nef divisor, and so we can apply bend and break to produce rational curves, a contradiction.

In case (iii) by our earlier computations we see that $K_\mathcal{F}|_C = K_C + \Delta$ where $\Delta \geq 0$, in which case C is either rational or of genus 1, in either case this is a contradiction.

Suppose S is a surface with $K_\mathcal{F}^2 \cdot S = 0$. We can therefore write $E = tS + E'$ where $t > 0$. Again, either (i) S is invariant or (ii) it is not.

In case (i) $K_\mathcal{F}|_S = K_S$. Thus $K_S^2 = 0$. By the Kodaira-Enriques classification of surfaces, we see that S is covered by elliptic curves, a contradiction.

In case (ii) we have $K_\mathcal{F} \cdot (A + E) \cdot S = -t(K_\mathcal{F} \cdot S^2$, but since $K_\mathcal{F}|_S \cdot A|_S > 0$ the right hand side must be strictly positive and so $K_\mathcal{F}|_S \cdot S|_S < 0$.

Thus we can write $K_\mathcal{F}|_S \cdot (K_\mathcal{F} + S)|_S < 0$. But if we write $(K_\mathcal{F} + S)|_{S^\nu} = K_\mathcal{G} + \Delta$ we get that $K_\mathcal{G} \cdot M < 0$ for a nef divisor M, and so we can apply bend and break to produce rational curves, a contradiction. □

As noted earlier, in general, canonical $\mathbb{Q}$-Gorenstein models of foliations do not exist, which is to say if $K_\mathcal{F}$ is big and nef in general there is no birational morphism $(X, \mathcal{F}) \to (Y, \mathcal{G})$ such that $K_\mathcal{G}$ is $\mathbb{Q}$-Cartier and ample. Among other things, the above result tells us (perhaps unsurprisingly) that rational and genus 1 curves are the obstructions to a minimal model being canonical.

Portions of the work in the above chapter is being prepared for submission for publication.

Spicer, Calum ”Higher dimensional foliated Mori Theory”.

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