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CIRCULARLY SYMMETRIC DISKS WHOSE TRANSPORT TWISTOR SPACES ADMIT GLOBAL BLOW-DOWN MAPS

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Zhengyi Qi

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The Dissertation of Zhengyi Qi
is approved:

Professor François Monard, Chair

Professor Jie Qing

Professor Torsten Ehrhardt

Donald Smith
Interim Vice Provost and
Dean of Graduate Studies

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Abstract

Circularly symmetric disks whose transport twistor spaces admit global blow-down maps

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In the literature on X-ray transform and Transport Twistor (TT) space, blow-down maps describe the degeneracy of the TT space of an oriented Riemannian surface, while collapsing (yet separating) geodesics of the unit tangent bundle of that surface. In the literature [3], such maps have been constructed for simple surfaces of near-constant curvature (see Definition 1.1), showing that the interior of their TT space is biholomorphic to an open set in standard $\mathbb{C}^2$. The construction there relied on a microlocal argument leveraging the absence of conjugate points.

In this dissertation, we construct explicit circularly symmetric examples, including examples with conjugate points, whose transport twistor spaces admit a global blow-down map.

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1 Background

1.1 Geometric Inverse Problem

Geometric inverse problems study the recovery of unknown geometric or analytic structures by indirect measurements. In many situations, the quantity of interest cannot be observed directly, and one instead attempts to find hidden information from boundary data, scattering measurements, travel time, or integral transforms along geodesics. Such problems arise naturally in a wide range of applications, including medical imaging, seismic tomography, ultrasound imaging, radar and remote sensing, and non-destructive material testing. Beyond their practical significance, geometric inverse problems also exhibit deep mathematical structures connecting differential geometry, partial differential equations, dynamical systems, and integral geometry.

One of the central topics in geometric inverse problems is the interaction between the geometry of a manifold and the behavior of geodesic vector flows. Given a Riemannian manifold, geodesics describe the propagation of particles, waves, or rays in the specific geometry, and measurements taken along geodesics often encode substantial information about the metric structure of the manifold. This leads naturally to the study of geodesic X-ray transforms, which integrate functions or tensor fields along geodesics. The inversion and analysis of such transforms form the foundation of many geometric tomography problems.

Another fundamental problem in geometric inverse problems concerns the geodesic X-ray transform. Given a symmetric tensor field f on a Riemannian manifold (M, g), the geodesic X-ray transform is defined by integrating f along geodesics. In the scalar case, it takes the form

$$If(\gamma) = \int_\gamma f\, ds,$$

where γ is a geodesic in M. A central question is whether one can recover f from the collection of its integrals over all geodesics. Closely related to this is the range characterization problem, which aims to describe the set of all functions on the boundary that arise as X-ray data of tensor fields.

A fundamental example is the boundary rigidity problem, which asks whether a compact two-dimensional Riemannian manifold can be uniquely determined, up to a boundary fixing isometry, from the knowledge of boundary distance functions. More precisely, given a compact Riemannian manifold (M, g) with boundary, one seeks to determine whether the metric g can be recovered from the boundary distance function

$$d_g|_{\partial M\times\partial M}.$$

This problem originates from travel time tomography in geophysics, where one attempts to recover the internal structure of the Earth from travel times of seismic waves measured at the boundary[24].

Closely related is the lens rigidity problem, where one studies whether the scattering relation and travel times of geodesics determine the geometry of the manifold. These problems are deeply connected to the global dynamics of the geodesic flow, including trapping phenomena or the case of existence of conjugate points, and the geometry of the unit tangent bundle.[10]

Another big problem is tensor tomography, which concerns the injectivity and reconstruction properties of the geodesic X-ray transform acting on symmetric tensor fields. Given a symmetric m-tensor field f, one studies whether the integrals of f along geodesics uniquely determine the solenoidal part of the tensor field. Tensor tomography appears naturally as the linearization of the boundary rigidity problem and plays a fundamental role in the analysis of geometric inverse problems.[25]

The study of these questions is particularly well developed on simple manifolds, namely compact Riemannian manifolds with strictly convex boundary, no conjugate points, and non-trapping geodesic flow. In this setting, the geodesic X-ray transform exhibits strong injectivity and stability properties, and many geometric inverse problems admit positive uniqueness results. Besides the simplicity assumption, circular symmetry provides another geometric setting in which injectivity of the geodesic X-ray transform can be established.[21]

A fundamental analytical viewpoint comes from transport equations associated with the geodesic vector field on the unit tangent bundle. Many inverse problems can be reformulated as transport-type equations, where the unknown function evolves along geodesics[21]. This dynamical formulation provides a powerful framework for studying injectivity, reconstruction formulas, and range characterizations for X-ray transforms. On surfaces, the circle structure of the fibres of the unit tangent bundle further introduces rich Fourier-analytic techniques in the fibre variable, leading to fibrewise Fourier analysis[11] and various holomorphic structures naturally associated with the transport equation.

Another fundamental problem is the range characterization of the geodesic X-ray transform. While injectivity asks whether an unknown function or tensor field can be uniquely recovered from its integrals over geodesics, the range problem asks which functions on the space of geodesics actually arise as such integrals. In the Euclidean case, this question is answered by the classical Helgason–Ludwig range conditions for the Radon transform. For the geodesic X-ray transform on a Riemannian manifold, however, the geometry of the manifold and the dynamics of the geodesic flow make the range problem substantially more subtle.

A useful way to formulate the range problem is through the transport equation on the unit tangent bundle. If X denotes the geodesic vector field on SM, then the geodesic X-ray transform of a function f can be encoded by solving

$$Xu = -f$$

with suitable boundary conditions. Thus, range characterization is closely related to understanding boundary values of solutions to transport equations and invariant functions of the geodesic flow.

On two-dimensional constant-curvature circularly symmetric disks, fibrewise Fourier analysis and singular value decompositions provide powerful tools for describing these boundary values and for deriving explicit range conditions.[16]

Motivated by the study of invariant distributions and fibrewise holomorphic structures on the unit tangent bundle, Gabriel P. Paternain, Jan Bohr, and their collaborators introduced geometric frameworks that encode transport-theoretic structures through complex-analytic methods [3, 4, 2].

In particular, by complexifying the geometry of the unit tangent bundle and studying holomorphic structures naturally associated with the geodesic flow, one is led to the notion of the transport twistor space. This space captures fibrewise holomorphicity from a global geometric perspective and provides a natural setting for studying invariant distributions, transport equations, and X-ray transforms.The research presented in this thesis is closely related to these developments. In particular, this work studies global geometric properties of transport twistor spaces and the construction of blow-down structures associated with rotationally symmetric non-simple surfaces

1.2 Geometric Preliminary

As discussed previously, two-dimensional manifolds play a significant role in geometric inverse problems. Many classical questions, including tensor tomography, boundary rigidity, and the study of geodesic X-ray transforms, have achieved their most complete understanding in dimension two. This is largely due to the additional geometric and analytic structures available on surfaces. We therefore begin by reviewing the basic setting of two-dimensional Riemannian geometry that underlies the subsequent study of transport equations and related inverse problems.

An important example of the Riemannian surface is that we can deduce the Jacobi field equation [21].

Theorem 1.1 (Jacobi Field Equation)**.** *Let (M, g) be a Riemannian surface and let $\gamma(t)$ be a unit-speed geodesic. Then any orthogonal Jacobi field along γ can be written as*

$$J(t) = b(t)\dot{\gamma}(t)^{\perp},$$

where the scalar function $b(t)$ satisfies the Jacobi equation

$$\ddot{b}(t) + K(\gamma(t))\, b(t) = 0. \tag{1}$$

Here K denotes the Gaussian curvature of (M, g).

For a fixed geodesic γ, we will use the distinguished Jacobi function b_γ determined by

$$b_\gamma(0) = 0, \qquad \dot{b}_\gamma(0) = 1.$$

This normalization is useful for detecting conjugate points: for $t \neq 0$, the point $\gamma(t)$ is conjugate to $\gamma(0)$ along γ if and only if $b_\gamma(t) = 0$. Thus, the absence of conjugate points along γ is measured by the nonvanishing of this normalized Jacobi function for $t \neq 0$.

We now introduce one of the basic geometric settings for inverse problems. Let (M, g) be a compact oriented Riemannian surface, possibly with boundary. The metric g determines the geometry of M and gives rise to a family of geodesics describing locally length-minimizing curves. The collection of all unit tangent vectors forms the unit sphere bundle

$$SM = \{(x, v) \in TM : \ |v|_g = 1\},$$

where the transport equation is formulated. The geodesic flow

$$\varphi_t : SM \to SM$$

is generated by the geodesic vector field X and describes the evolution of unit-speed geodesics on the manifold. Many geometric inverse problems can be naturally lifted from M to SM, allowing questions concerning geodesics to be reformulated as problems on the unit sphere bundle.

After introducing simple manifolds, we obtain a setting where geodesics behave in a particularly well-controlled manner: every geodesic reaches the boundary in finite time and contains no conjugate points. Consequently, geodesics establish a correspondence between incoming and outgoing boundary data. This correspondence is encoded by the scattering relation on ∂SM, which plays a central role in geometric inverse problems such as lens rigidity and boundary rigidity.

Definition 1.1. *Let (M, g) be a compact Riemannian manifold with smooth boundary. We say that (M, g) is* simple *if it satisfies the following three conditions:*

1. ***Strictly convex boundary.***

The boundary ∂M is strictly convex, meaning that its second fundamental form is positive definite at every boundary point.

2. ***Non-trapping property.***

   The manifold is non-trapping, namely every maximal geodesic exits the manifold in finite time. In other words, there are no geodesics that remain entirely inside M for all time.

3. ***Absence of conjugate points.***

   The manifold does not contain conjugate points. Equivalently, for every geodesic γ, there is no nontrivial Jacobi field along γ that vanishes at two distinct points. Geometrically, this means that geodesics do not develop conjugate points and the exponential map remains locally injective.

Simple manifolds provide the classical geometric setting for many inverse problems. The assumptions of strict boundary convexity, non-trapping, and absence of conjugate points ensure that geodesics exhibit well-behaved dynamics and are uniquely determined by their boundary data. We will introduce the scattering relation to describe those properties.

Definition 1.2. *To describe geodesics entering and leaving the manifold through the boundary, we first introduce the incoming and outgoing boundaries of the unit sphere bundle:*

$$\partial_\pm SM = \{(x, v) \in SM : \ x \in \partial M, \ \pm\langle v, \nu(x)\rangle \geq 0\},$$

where $\nu(x)$ denotes the inward unit normal vector at $x \in \partial M$. The set $\partial_- SM$ consists of outward-pointing unit vectors, while $\partial_+ SM$ consists of inward-pointing unit vectors.

Since (M, g) is a non-trapping manifold, every geodesic issued from $(x, v) \in$

$\partial_+ SM$ exits the manifold in finite time. The corresponding exit time

$$\tau(x, v) = \inf\{t > 0 : \gamma_{x,v}(t) \in \partial M\}$$

is called the travel time (or exit time) function.

The geodesic flow therefore induces a map from incoming to outgoing boundary data. The scattering relation is defined as

$$\alpha : \partial SM \to \partial SM,$$

$$\alpha(x, v) := \varphi_{\tau(x,v)}(x, v),$$

where φ_t denotes the geodesic flow. In other words, for a geodesic entering the manifold at (x, v), the scattering relation records the boundary point and direction at which the geodesic exits the manifold.

Two-dimensional simple manifolds have been a central object of study in geometric inverse problems, particularly in the investigation of the range characterization of the X-ray transform [23] and the boundary rigidity problem[24].

The class of simple manifolds provides a natural setting for tensor tomography. The first injectivity result in this direction was obtained by Mukhometov [19], who established uniqueness of the X-ray transform on functions in two dimensions. More generally, his result applies to a class of simple families of curves and is not restricted to geodesics. This work provided the first evidence that appropriate geometric control of a family of curves is sufficient for unique recovery from integral measurements and laid the foundation for the subsequent development of tensor tomography.

Subsequently, Anikonov and Romanov [1] established the corresponding uniqueness result for 1-forms, while Sharafutdinov [26] proved the s-injectivity of the geodesic ray transform on symmetric 2-tensor fields. These results suggested that simplicity provides sufficient geometric control for recovering tensor fields from their geodesic integrals. This program was completed by Paternain, Salo, and Uhlmann [20], who proved that the geodesic ray transform is s-injective on symmetric tensor fields of arbitrary order on two-dimensional simple manifolds. More precisely, if a symmetric m-tensor field f satisfies

$$I_m f = 0,$$

then there exists a symmetric $(m-1)$-tensor field h vanishing on the boundary such that

$$f = dh.$$

Here

$$d = \mathrm{Sym} \circ \nabla : C^\infty(S^{m-1}T^*M) \longrightarrow C^\infty(S^m T^*M)$$

denotes the symmetrized covariant derivative. In local coordinates,

$$(dh)_{i_1 \cdots i_m} = \frac{1}{m} \sum_{j=1}^{m} \nabla_{i_j} h_{i_1 \cdots \widehat{i_j} \cdots i_m},$$

where the hat means that the corresponding index is omitted.

Here, s-injectivity means injectivity on the solenoidal part of a tensor field. Equivalently, a tensor field f is solenoidal when $\delta f = 0$, where $\delta = -d^*$ denotes the divergence operator.

Equivalently, the geodesic ray transform is injective on the space of solenoidal tensor fields. This theorem completely resolves the tensor tomography problem in two dimensions and constitutes one of the fundamental achievements in geometric inverse problems.

1.3 Vertical Fourier Analysis

After introducing some basic geometric background, we now turn to an important analytic tool. Fiberwise Fourier analysis has become one of the most powerful tools in the study of two-dimensional geometric inverse problems. An important application of this method arises in the study of the attenuated X-ray transform[22]. By decomposing functions on the unit tangent bundle into their Fourier modes along the fibers, one can transform geometric and dynamical questions into algebraic relations between different Fourier components. This approach provides a natural framework for analyzing transport equations, geodesic X-ray transforms, and their range characterization, and has led to several fundamental developments in tensor tomography and integral geometry.

We now introduce the vector fields used in vertical Fourier analysis.

The geodesics on the unit sphere bundle can be described by three natural vector fields: the geodesic vector field X, the vertical vector field V, and the vector field $X_\perp$. Let $\varphi_t : SM \to SM$ denote the geodesic flow,

$$\varphi_t(x, v) = (\gamma_{x,v}(t), \dot{\gamma}_{x,v}(t)),$$

where $\gamma_{x,v}$ is the unit speed geodesic with initial data

$$\gamma_{x,v}(0) = x, \qquad \dot{\gamma}_{x,v}(0) = v.$$

The geodesic vector field X is the infinitesimal generator of this flow, namely

$$Xu = \frac{d}{dt}\bigg|_{t=0} u(\varphi_t(x, v)), \qquad u \in C^\infty(SM).$$

The orientation of M induces a natural circle action on the fibers of SM. For

$\theta \in \mathbb{S}^1$, let

$$\rho_\theta : SM \to SM$$

be defined by

$$\rho_\theta(x, v) = (x, R_\theta v), \qquad R_\theta v := \cos\theta\, v + \sin\theta\, v_\perp,$$

where $v_\perp \in T_x M$ is the unique vector satisfying

$$|v_\perp|_g = |v|_g, \qquad \langle v, v_\perp \rangle_g = 0,$$

such that the ordered pair $(v, v_\perp)$ is positively oriented with respect to the orientation of M. Thus, R_θ denotes the rotation by angle θ in the tangent space $T_x M$.

The infinitesimal generator of this circle action is the vertical vector field

$$V = \frac{d}{d\theta}\rho_\theta\bigg|_{\theta=0}.$$

The third vector field is defined by

$$X_\perp = [X, V].$$

Together, $(X, X_\perp, V)$ form a global frame on SM. Their commutation relations are given by

$$[X, V] = X_\perp, \qquad [V, X_\perp] = X, \qquad [X, X_\perp] = -KV,$$

where K denotes the Gaussian curvature of the surface. These structure equations encode the geometry of the manifold and play a fundamental role in the study of transport equations, Pestov identities, and the geodesic X-ray transform.

A remarkable feature of the two-dimensional setting is that the geodesic vector field admits the decomposition[11]

$$X = \eta_+ + \eta_-,$$

where

$$\eta_\pm = \frac{1}{2}(X \pm iX_\perp).$$

For $k \in \mathbb{Z}$, let

$$\Omega_k := \{u \in C^\infty(SM) : Vu = iku\}$$

denote the k-th fibrewise Fourier mode. The operators $\eta_\pm$ satisfy

$$\eta_+ : \Omega_k \to \Omega_{k+1}, \qquad \eta_- : \Omega_k \to \Omega_{k-1},$$

so that the geodesic flow only couples neighboring Fourier modes. Indeed, if

$$u = \sum_{k=-\infty}^{\infty} u_k,$$

then

$$(Xu)_k = \eta_+ u_{k-1} + \eta_- u_{k+1}.$$

This decomposition transforms transport equations into an infinite system relating adjacent Fourier modes and provides the analytical framework for tensor tomography and the geodesic X-ray transform.

Although the fibrewise Hilbert transform is not explicitly used in every argument below, it provides the natural analytical framework for the study of transport equations on SM and underlies many of the fundamental identities appearing in two-dimensional geometric inverse problems. For this reason, we briefly review its definition and basic properties. One important application is the study of tensor tomography mentioned in the previous section[20].

The fibrewise Hilbert transform is defined by its action on the Fourier components. For

$$u = \sum_{k=-\infty}^{\infty} u_k,$$

where the k-th fibrewise Fourier component is

$$u_k(x,v) = \frac{1}{2\pi}\int_0^{2\pi} u\big(\rho_\theta(x,v)\big)e^{-ik\theta}\,d\theta, \qquad Vu_k = iku_k.$$

we define

$$Hu = -i\sum_{k=-\infty}^{\infty} \operatorname{sgn}(k)u_k,$$

where

$$\operatorname{sgn}(k) = \begin{cases} 1, & k>0,\\ 0, & k=0,\\ -1, & k<0. \end{cases}$$

Equivalently,

$$Hu_k = -i\ \operatorname{sgn}(k)u_k.$$

Hence the Hilbert transform multiplies positive Fourier modes by $-i$, negative modes by i, and annihilates the zeroth Fourier mode.

Moreover,

$$H^2u = -u + u_0,$$

where u_0 denotes the zeroth Fourier component. A function $u = \sum_{k\in\mathbb{Z}} u_k$ is called fibrewise holomorphic if $u_k = 0$ for all $k < 0$, and fibrewise antiholomorphic if $u_k = 0$ for all $k > 0$. The Hilbert transform separates these fibrewise holomorphic and antiholomorphic parts of functions on SM and plays a central role in the construction of invariant distributions and holomorphic integrating factors.

Its interaction with the geodesic vector field is described by the fundamental commutator identity[11]

$$[H, X]u = X_\perp u_0 + (X_\perp u)_0,$$

where $(\cdot)_0$ denotes the projection onto the zeroth Fourier mode. This identity is one of the key analytical ingredients in the study of transport equations and the geodesic X-ray transform on surfaces.

1.4 Transport equation

The transport equation provides a partial differential equation formulation of the geodesic X-ray transform and is one of the fundamental analytical tools in geometric inverse problems. Instead of studying integrals along geodesics directly, one considers a first-order equation on the unit tangent bundle whose characteristics coincide with the geodesic flow. This point of view has proved to be particularly effective in the study of injectivity, stability, tensor tomography, and attenuated X-ray transforms [21].

For a smooth function $f \in C^\infty(SM)$, consider the transport equation

$$Xu = -f, \tag{2}$$

together with the outgoing boundary condition

$$u|_{\partial_- SM} = 0. \tag{3}$$

Since the characteristics of the geodesic vector field are precisely the geodesics of (M, g), the solution is obtained by integrating along the geodesic flow. More precisely, let $\tau(x, v)$ denote the exit time of the geodesic issued from (x, v). Then the solution admits the explicit representation

$$u(x, v) = \int_0^{\tau(x,v)} f(\varphi_t(x, v))\, dt,$$

where φ_t denotes the geodesic flow. Consequently, on non-trapping manifolds the transport equation admits a unique solution determined by the incoming boundary condition.

The geodesic X-ray transform naturally arises as the boundary trace of this solution. Indeed, for $(x, v) \in \partial_+ SM$, the geodesic X-ray transform is defined by

$$If(x, v) = \int_0^{\tau(x,v)} f(\varphi_t(x, v))\, dt.$$

Therefore,

$$If = u|_{\partial_+ SM}.$$

More generally, given an attenuation function $a \in C^\infty(M)$, one considers the attenuated transport equation

$$(X + a)u = -f,$$

whose boundary value determines the attenuated geodesic X-ray transform. This formulation extends naturally to matrix-valued connections and Higgs fields, leading to the non-Abelian X-ray transform.

Although the transport equation admits an explicit solution, its regularity is considerably more subtle. The principal difficulty comes from the exit time function τ, whose smoothness depends on the geometry of the boundary. When (M, g) has strictly convex boundary and is non-trapping, the exit time function is smooth away from the glancing set

$$\partial_0 SM = \{(x, v) \in \partial SM : \langle v, \nu_x \rangle_g = 0\},$$

consisting of unit vectors tangent to ∂M, and hence smooth source terms give rise to smooth solutions in the interior of SM. Near the glancing boundary, however, the regularity of the solution is closely related to the behaviour of the exit time function and the scattering relation.

The regularity theory developed for transport equations shows that, under suitable geometric assumptions, solutions inherit the regularity of the source term and extend smoothly on the regions where the exit time and the scattering relation are smooth. These regularity results play a fundamental role in the analysis of the geodesic X-ray transform, since they justify the application of vertical Fourier analysis, fibrewise Hilbert transforms, and Pestov-type identities. Throughout this paper, the transport equation serves as the main analytical bridge between the geometry of the unit tangent bundle and the transport twistor space introduced in the subsequent sections.

1.5 Circularly Symmetric Disks of "Herglotz" type

After introducing the geometric preliminaries, we turn our attention to a particular class of geometries exhibiting rotational symmetry. The study of rotationally symmetric metrics has a long history dating back to the work of Herglotz, who investigated the propagation of seismic waves in spherically symmetric models of the Earth [12, 27]. Under the assumption of spherical symmetry, the travel times of seismic rays can be related explicitly to the underlying wave speed through integral formulas, leading to some of the earliest successful inverse problems in geophysics. Inspired by these classical developments, rotationally symmetric manifolds have become an important model in integral geometry and geometric inverse problems.

Rotationally symmetric disks provide a rich family of examples in which many geometric quantities can be computed explicitly. The presence of rotational symmetry yields additional conserved quantities along geodesics, most notably the Clairaut relation, which significantly simplifies the analysis of geodesic flows. As a consequence, scattering relations, travel time functions, Jacobi fields, and geodesic X-ray transforms can often be reduced to one-dimensional integral equations. This additional structure has led to important advances in tensor tomography, boundary rigidity, and range characterization problems. Moreover, rotational symmetry frequently allows positive results to be established under substantially weaker regularity assumptions than those required in the general setting. For these reasons, rotationally symmetric disks serve as an ideal framework for both theoretical investigations and explicit computations in geometric inverse problems.

First, let us define the circular symmetry disk.

Definition 1.3. *Let* $R > 0$ *and*

$$\mathbb{D}_R = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq R^2\}.$$

Assume that $c : \mathbb{D}_R \to (0, \infty)$ *is a smooth radial function, that is,*

$$c(x, y) = c(r), \qquad r = \sqrt{x^2 + y^2},$$

for some smooth function $c : [0, R] \to (0, \infty)$.

We equip $\mathbb{D}_R$ with the Riemannian metric

$$g = \frac{1}{c^2(x, y)}(dx^2 + dy^2). \tag{4}$$

Passing to polar coordinates

$$x = r\cos\theta, \qquad y = r\sin\theta,$$

the metric takes the form

$$g = \frac{1}{c^2(r)}\left(dr^2 + r^2 d\theta^2\right).$$

Since the metric is originally defined in the smooth Cartesian coordinates (x, y)*, it extends smoothly across the origin. Although polar coordinates are singular at* $r = 0$*, the metric tensor itself is smooth and well-defined at* $(0, 0)$*. If, in addition,*

$$\left(\frac{r}{c(r)}\right)' > 0,$$

then $(\mathbb{D}_R, g)$ *is called a* Herglotz disk.

The first property of Herglotz disks is the following; see [21] for a proof.

Lemma 1.1. *Herglotz disks are nontrapping and have strictly convex boundary.*

Another important property of Herglotz disks is the kernel characterization of the geodesic X-ray transform.

Lemma 1.2. *[21] The geodesic X-ray transform is injective on Herglotz disks.*

One advantage of a circularly symmetric disk is that fan-beam coordinates can be used to parametrize ∂SM. We use θ to represent the angular coordinate of the boundary point in ∂M and α to represent the angle between the corresponding unit tangent vector and the inward unit normal at the boundary. Recalling the scattering relation from Definition 1.2, rotational invariance allows us to write it in the form [16]

$$S(\theta, \alpha) = (\theta + \pi + 2\mathfrak{s}(\alpha), \pi - \alpha). \tag{5}$$

Here the scattering signature $\mathfrak{s}$ satisfies

$$\mathfrak{s}(\alpha + \pi) = \mathfrak{s}(\alpha) + \pi, \qquad \mathfrak{s}(\alpha) = -\mathfrak{s}(-\alpha).$$

Thus, in the rotationally invariant setting, the scattering relation is completely determined by the function $\mathfrak{s}(\alpha)$.

The scattering relation on a Herglotz disk admits a particularly simple description due to the rotational symmetry of the metric [18]. Let $\mathfrak{s}(\alpha)$ denote the scattering function, where α is the angle between the incoming geodesic and the inward unit normal at the boundary. The variation of the scattering function is closely related to the behavior of Jacobi fields along geodesics. More precisely, if $b(\alpha, t)$ denotes the scalar Jacobi field satisfying

$$b(\alpha, 0) = 0, \qquad \dot{b}(\alpha, 0) = 1,$$

then the derivative of the scattering function is given by

$$\mathfrak{s}'(\alpha) = \frac{b(\alpha, \tau(\alpha))}{2R\cos\alpha},$$

where $\tau(\alpha)$ is the travel time of the geodesic. Consequently, the monotonicity of the scattering relation is equivalent to the positivity of the endpoint value of the Jacobi field. In particular, we establish with Monard in [17] the following result:

Theorem 1.2. *Let* $(\mathbb{D}_R, g)$ *be a Herglotz disk with* g *of the form* (4). *Then* $(\mathbb{D}_R, g)$ *is* *simple if and only if the scattering function $\mathfrak{s}(\alpha)$ satisfies $\mathfrak{s}'(\alpha) > 0$ for all $\alpha \in [-\frac{\pi}{2}, \frac{\pi}{2}]$.*

Proof of Theorem 1.2. ($\Longrightarrow$) Suppose there exists α_0 such that $\mathfrak{s}'(\alpha_0) \leq 0$. Since $\mathfrak{s}'(\alpha)$ is continuous in α and we know that $b(\alpha, \tau(\alpha)) > 0$ for α close to $\pm\frac{\pi}{2}$, by the intermediate value theorem, there must exist some $\alpha_0 \in (-\frac{\pi}{2}, \frac{\pi}{2})$ such that $\mathfrak{s}'(\alpha_0) = 0$, or equivalently, $b(\alpha_0, \tau(\alpha_0)) = 0$, and hence $p = Re^{i0}$ and $q = Re^{i(\pi+2\mathfrak{s}(\alpha_0))}$ are conjugate.

($\Longleftarrow$) Suppose there exists a pair of conjugate points (p, q) in M. By rotation invariance, we may choose p, q to be conjugate along a geodesic $\gamma(t)$ such that $\gamma(0) = Re^{i0}$, $\gamma(t_0) = p$ and $\gamma(t_1) = q$ for some t_0, t_1 satisfying $0 \leq t_0 < t_1$, $\dot{\gamma}(0) = v(\alpha_0)$ for $\alpha_0 \in (-\pi/2, \pi/2)$. Then there is a nontrivial solution $j(t)$ of (1) vanishing at t_0, t_1. If $t_0 = 0$, then $b(t_1, \alpha_0) = 0$ and we set $t^* = t_1$. If $t_0 \neq 0$, since $j(t)$ and $b(t, \alpha_0)$ both solve the same Jacobi equation, by Sturm's Separation Theorem, there exists $t^* \in (t_0, t_1)$ such that $b(\alpha_0, t^*) = 0$.

At this point, we have proved that there exists $\alpha_0 \in (-\pi/2, \pi/2)$ and $0 < t^* \leq \tau(\alpha_0)$ such that $b(\alpha_0, t^*) = 0$. Now if $t^* = \tau(\alpha_0)$, then $\mathfrak{s}'(\alpha_0) = 0$ and we are done. If $t^* < \tau(\alpha_0)$, we now analyze the structure of the zero set of $b(\alpha, t)$ to prove that there is α_1 such that $b(\alpha_1, \tau(\alpha_1)) = 0$ so that $\mathfrak{s}'(\alpha_1) = 0$. Since $b(\alpha, \cdot)$ is a non-trivial solution of a second-order linear homogeneous ODE in t, one must have $\partial b/\partial t \neq 0$ whenever $b(\alpha, t) = 0$. Hence by the implicit function theorem, there is a function $t(\alpha)$ defined in a neighborhood of α_0 such that $t(\alpha_0) = t^*$ and $b(\alpha, t(\alpha)) = 0$. One must also satisfy that $t(\alpha)$ must be greater than the radius of injectivity of (M, g). Since $\lim_{\alpha \to \pm\pi/2} \tau(\alpha) = 0$, the maximal extension of $t(\alpha)$ must encounter a point of the form $(\alpha_1, \tau(\alpha_1))$ with $\tau(\alpha_1) > 0$. Hence we have found α_1 with $\tau(\alpha_1) > 0$ such that $b(\alpha_1, \tau(\alpha_1)) = 0$ and hence $\mathfrak{s}'(\alpha_1) = 0$. Theorem 1.2 is proved. □

Theorem 1.2 is precisely the condition that no conjugate points occur along the corresponding geodesic. Therefore, on a Herglotz disk, the scattering relation provides a direct characterization of the conjugate point structure through the associated Jacobi fields.

The assumption of circular symmetry occupies a good stage in the study of geometric inverse problems. This symmetry not only considerably simplifies the mathematical structure but, more importantly, it permits positive and often definitive answers to questions that remain intractable for generic metrics, particularly in the setting of low regularity.

In parallel, the theory of tensor tomography has benefitted immensely from the radial assumption. On circularly symmetric manifolds, the geodesic X-ray transform of symmetric tensors can be fully characterized. Krishnan and Sharafutdinov, employing the Reshetnyak formula [13], provided a complete range characterization in Sobolev spaces, thereby resolving the fundamental question of which data sets are admissible. Such range characterizations are not merely of theoretical interest; they are essential for designing stable inversion algorithms and establishing uniqueness results for inverse problems involving tensor fields.

The power of spherical symmetry is perhaps most strikingly demonstrated by the recent spectral rigidity result of de Hoop, Ilmavirta, and Katsnelson, which establishes that spherically symmetric terrestrial planets with metric discontinuities are spectrally rigid [7]. Crucially, this result permits the metric to have jump discontinuities along finitely many interfaces—a feature that models physically realistic phase transitions in planetary interiors, such as the core-mantle boundary. To our knowledge, this is the first spectral rigidity theorem for a manifold with boundary and a piecewise continuous metric [7]. The authors work under the smooth Herglotz condition on each layer, but allow the metric to be only piecewise $C^{1,1}$, with jump discontinuities at the interfaces. This represents a significant relaxation of regularity requirements compared to the general theory, where C^∞ smoothness is typically assumed. The proof is based on a novel trace formula, and the result directly underlies the validity of standard reference Earth models such as PREM, which remains widely used in linearized tomography [7]. Moreover, the framework accommodates two distinct wave types (P- and S-polarizations) across different layers, modeling the toroidal and spheroidal modes observed in isotropic elastic solids. This demonstrates that spherical symmetry provides a mathematically tractable setting that captures the essential features of real-world planetary structure, including discontinuities, while still permitting definitive positive results [7].

Therefore, circular symmetry provides a fertile proving ground for geometric inverse problems. It is a setting where deep and practically relevant results—including spectral rigidity in the presence of discontinuities, range characterizations for tensor tomography, and boundary rigidity—can be established. The extension of the transport twistor space framework to Herglotz disks, which are precisely the rotationally symmetric surfaces satisfying the Herglotz condition, thus represents a natural and important step, bridging the sophisticated complex-geometric machinery of twistor theory with the rich and well-understood class of radially symmetric geometries.

Also for the explicit range characterization of the geodesic X-ray transform can be written on Herglotz disk with constant curvature [16].

Although the radial sound-speed representation

$$g = \frac{1}{c^2(r)} \left(dr^2 + r^2 d\theta^2 \right)$$

is natural from the viewpoint of isotropic wave propagation, it is less convenient for the geometric computations that follow. In particular, the conformal factor $c^{-2}(r)$ appears in both the radial and angular components of the metric. We therefore introduce a suitable radial change of coordinates. This coordinate transformation induces an isometry between the original metric and a metric of the form (6).

Lemma 1.3. *Let* $(\mathbb{D}_R, g)$ *be a Herglotz disk in the radial sound-speed parameterization* (4). *Then there exist*

$$R' = \frac{R}{c(R)} > 0$$

and a positive smooth radial function a such that $(\mathbb{D}_R, g)$ *is isometric to*

$$\left(\mathbb{D}_{R'},\ a^2(\rho)\, d\rho^2 + \rho^2\, d\theta^2\right).$$

More precisely, the isometry is induced by the radial change of variables

$$\rho = \frac{r}{c(r)}.$$

Proof. Assume that the Herglotz condition

$$\frac{d}{dr}\left(\frac{r}{c(r)}\right) > 0$$

holds on the interval $[0, R]$. Define

$$\rho = \frac{r}{c(r)}.$$

Since

$$\frac{d\rho}{dr} = \left(\frac{r}{c(r)}\right)' > 0,$$

the map

$$r \longmapsto \rho(r)$$

is strictly monotone. By the Inverse Function Theorem, it is a local diffeomorphism. Since it is also injective on $[0, R]$, it follows that $\rho(r)$ defines a global change of radial coordinate and admits a smooth inverse

$$r = r(\rho).$$

Differentiating gives

$$d\rho = \left(\frac{r}{c(r)}\right)' dr, \qquad dr = \frac{d\rho}{\left(\frac{r}{c(r)}\right)'}.$$

Substituting into the metric yields

$$g = \frac{1}{c^2(r)} \frac{d\rho^2}{\left(\frac{r}{c(r)}\right)'^2} + \frac{r^2}{c^2(r)}\, d\theta^2.$$

Using

$$\rho = \frac{r}{c(r)},$$

we obtain

$$g = a^2(\rho)\, d\rho^2 + \rho^2 d\theta^2.$$

where

$$a(\rho) = \frac{1}{c(r)\left(\frac{r}{c(r)}\right)'}.$$

Therefore,

$$g = a^2(\rho)\, d\rho^2 + \rho^2 d\theta^2.$$

Moreover, if $\Phi : (r, \theta) \mapsto (\rho, \theta)$ denotes the coordinate change, then

$$g = \Phi^* \left(a^2(\rho)\, d\rho^2 + \rho^2 d\theta^2 \right),$$

which shows that Φ is an isometry. Hence the transformation preserves the length of every curve, and consequently preserves the geodesic structure of the metric. Since the coordinate transformation

$$\Phi : (r, \theta) \mapsto (\rho, \theta)$$

is an isometry, all intrinsic geometric quantities are preserved under this change of coordinates. In particular, geodesics, conjugate points, Jacobi fields, travel times, scattering relations, and the geodesic X-ray transform are unchanged after passing from the metric

$$g = \frac{1}{c^2(r)} \left(dr^2 + r^2 d\theta^2 \right)$$

to

$$g = a^2(\rho)\, d\rho^2 + \rho^2 d\theta^2. \tag{6}$$

This proves the claimed isometric representation. Since $\rho(R) = R/c(R)$, the new radial coordinate ranges over $[0, R']$ with $R' = R/c(R)$. $\square$

Therefore, without loss of generality, we may perform all subsequent calculations in the coordinates (ρ, θ) and assume throughout the remainder of this article that the metric is written in the form

$$g = a^2(\rho)\, d\rho^2 + \rho^2 d\theta^2.$$

This representation is technically more convenient while remaining geometrically equivalent to the original metric. To explore this metric, we will first talk about some properties of $a(r)$.

Remark 1.1. *Since the metric is required to be smooth in Cartesian coordinates* (x, y)*, the radial coefficient* $a(r)$ *cannot be an arbitrary smooth function of* r*. Indeed, writing*

$$r = \sqrt{x^2 + y^2},$$

smoothness of

$$g = a^2(r)(dr^2 + r^2 d\theta^2)$$

at the origin requires that there exists a smooth function $A : [0, 1] \to \mathbb{R}$ *such that*

$$a(r) = A(r^2).$$

Equivalently, a *extends to a smooth even function near* $r = 0$ *and admits an expansion*

$$a(r) = a_0 + a_2 r^2 + a_4 r^4 + \cdots .$$

In particular,

$$a^{(2k+1)}(0) = 0, \qquad k \geq 0.$$

This characterization of smooth radial functions is classical; see, for example, Grafakos and Teschl [8, Lemma 2].

Thus, we can write

$$a(r) = 1 + r^2 \tilde{a}(r). \tag{7}$$

Then, we can talk about the following properties of the $a(r)$.

Lemma 1.4. *Suppose*

$$g = a^2(r)\, dr^2 + r^2 d\theta^2$$

extends smoothly to the origin. Then

$$a(0) = 1.$$

Proof. Using

$$dr^2 + r^2 d\theta^2 = dx^2 + dy^2,$$

the metric can be written as

$$g = dx^2 + dy^2 + (a^2(r) - 1)\frac{(x\, dx + y\, dy)^2}{r^2}.$$

Hence

$$g_{xx} = 1 + (a^2(r) - 1)\frac{x^2}{r^2}.$$

For g to be continuous at the origin, the limit of g_{xx} as $(x, y) \to (0, 0)$ must be independent of the direction. Taking $y = kx$ gives

$$\lim_{(x,y)\to(0,0)} g_{xx} = 1 + \frac{a^2(0) - 1}{1 + k^2}.$$

This is independent of k if and only if

$$a^2(0) = 1.$$

Since $a(r) > 0$, we conclude that

$$a(0) = 1.$$

□

1.6 Transport Twistor Space

For finer details of the construction, including a coordinate-invariant description of the transport twistor structure, see [2, Section 2.4.1].

Transport twistor spaces were first introduced in [5], and they are a powerful tool for using knowledge from complex geometry to study geometric inverse problems and dynamical systems on the unit tangent bundle of a given Riemannian surface. Any oriented Riemannian surface (M, g) with a tangent bundle SM gives rise to a transport twistor space, i.e. an involutive manifold $(Z, \mathscr{D})$. The rank 2 involutive structure (that is, a subbundle of $T_{\mathbb{C}}Z = TZ \otimes \mathbb{C}$ that is closed under commutators) is uniquely defined in the following that satisfies properties [4].

$$\mathscr{D} \cap \overline{\mathscr{D}} = 0 \quad \text{on } SM, \tag{8}$$

$$\mathscr{D} \cap \overline{\mathscr{D}} = \mathbb{C}X \quad \text{on } SM, \tag{9}$$

$$\mathscr{D} \cap TZ_x = T^{0,1}Z_x \quad \text{for all } x \in M. \tag{10}$$

Here $Z_x \subset Z$ is the fiber over x and $T^{0,1}Z_x$ is the anti-holomorphic tangent bundle with respect to the complex structure that g and orientation induce on T_xM.

1.6.1 Transport twistor correspondence

One of the remarkable features of transport twistor spaces is that they translate analytic problems into questions from complex geometry. For instance, invariant distributions satisfying

$$Xu = 0$$

correspond to CR-holomorphic functions on $(Z, \mathscr{D})$, while the regularity of these invariant distributions is reflected in the regularity of the corresponding holomorphic objects. This correspondence provides a powerful bridge between dynamical systems, integral geometry and several complex variables. The following theorem shows the connection between the transport space and the holomorphic function on them.

Theorem 1.3. *[2] Let (M, g) be a closed oriented Riemannian surface, and let Z denote its transport twistor space. Then the restriction map*

$$f \longmapsto f|_{SM}$$

induces an isomorphism

$$\mathcal{O}(Z) \cong \{u \in C^{\infty}(SM) : Xu = 0,\ u \text{ is fibrewise holomorphic}\},$$

where $\mathcal{O}(Z)$ denotes the algebra of holomorphic functions on Z.

1.6.2 Holomorphic blowdown maps

The construction of a blow-down structure is closely related to the existence of a suitable β-map. Indeed, the β-map provides explicit holomorphic functions on the transport twistor space whose components serve as blow-down coordinates. Consequently, much of the analysis in the sequel is devoted to constructing such a map and establishing its regularity properties. Once the β-map is available, one obtains an explicit realization of the transport twistor space together with its holomorphic structure.

An important question is whether the transport twistor space admits a holomorphic blow-down structure. Roughly speaking, this asks whether one can find globally defined holomorphic coordinates that embed the transport twistor space into a complex surface while respecting the involutive structure. When such a structure exists, the complicated geometry of the transport twistor space can be studied through explicit holomorphic coordinates, allowing one to construct invariant distributions and analyze the transport equation using methods from complex analysis. Here $\Omega_{\mathbb{C}^2}$ denotes the standard Hermitian $(1,1)$-form on the target,

$$\Omega_{\mathbb{C}^2} := i\big(dw \wedge d\bar{w} + d\xi \wedge d\bar{\xi}\big).$$

The notation C^∞_α refers to the glancing-adapted smooth structure on $\partial_+ SM$ used in the blow-down construction; see [2, Section 2.3]. We also use the reference Hermitian form $\underline{\Omega}$ on Z. Locally, η_1, η_2 denote a $(1,0)$-coframe normalized by

$$\underline{\Omega} = i\big(\eta_1 \wedge \bar{\eta}_1 + \eta_2 \wedge \bar{\eta}_2\big).$$

Thus, if

$$\beta^* \Omega_{\mathbb{C}^2} = i \sum_{j,k=1}^{2} H_{jk} \eta_j \wedge \bar{\eta}_k,$$

the uniform positivity condition below is equivalently $\beta^*\Omega_{\mathbb{C}^2} \geq c\,\underline{\Omega}$. In the coordinates used later in the thesis, the coframe η_1, η_2 is characterized explicitly in (77).

Definition 1.4. *A smooth map $\beta\colon Z \to \mathbb{C}^2$ has* **holomorphic blow-down structure** *if it has the following properties:*

(a) The restriction $\beta|_{\partial_+ SM} : \partial_+ SM \to \mathbb{C}^2$ is a totally real C^∞_α−embedding.

(b) The restriction $\beta|_{Z^\circ}$ is a biholomorphism onto its image.

(c) Writing $\beta^*\Omega_{\mathbb{C}^2} = i\sum_{j,k=1}^2 H_{jk}\eta_j \wedge \bar{\eta}_k$ for some Hermitian matrix field $H = \{H_{jk}\}_{j,k=1}^2$, there exists $c > 0$ such that $H - cId$ is positive semidefinite.

The terminology and the preceding definition are motivated by the Euclidean model and by the blow-down construction of LeBrun and Mason for projective twistor spaces associated with Zoll surfaces [14, 15]. In the Euclidean case, in the coordinates (z, μ) the involutive distribution is

$$\mathscr{D}_0 = \mathbb{C}\,\partial_{\bar{\mu}} \oplus \mathbb{C}\big(\partial_{\bar{z}} + \mu^2\partial_z\big).$$

The map

$$\beta_0(z, \mu) = \big(z - \mu^2\bar{z}, \mu\big)$$

identifies the interior of the transport twistor space biholomorphically with an open subset of $\mathbb{C}^2$. In the corresponding coordinates $(w, \xi) = \beta_0(z, \mu)$, the involutive structure becomes the standard complex structure,

$$\mathscr{D} = \operatorname{span}_{\mathbb{C}}\left\{\partial_{\overline{w}}, \partial_{\overline{\xi}}\right\}.$$

Thus, the map β_0 resolves the degeneracy of the complex structure on the interior of the transport twistor space. Definition 1.4 is designed to retain the essential geometric properties of this model in a general Riemannian setting [5].

Property (a) concerns the behaviour of the map on the boundary. Since (M, g) is assumed to be non-trapping with strictly convex boundary, the influx boundary $\partial_+ SM$

parametrizes the oriented geodesics of M. Moreover, the holomorphicity of β implies

$$X\beta = 0 \qquad \text{on } SM,$$

so that the restriction of β to SM is constant along the geodesic flow. Consequently, the injectivity of

$$\beta|_{\partial_+ SM}$$

means precisely that distinct oriented geodesics are separated by the components of β. This is the strongest natural separation property that can be required at the boundary.

There is, however, an additional subtlety near the glancing set $\partial_0 SM$. Because $X\beta = 0$, the differential of $\beta|_{\partial_+ SM}$, when computed using the ordinary smooth structure on $\partial_+ SM$, necessarily loses rank at tangential vectors. The modified C^∞_α-structure compensates for this fold-type degeneracy of the scattering parametrization. It is therefore natural to require $\beta|_{\partial_+ SM}$ to be a C^∞_α-embedding rather than an embedding with respect to the ordinary smooth structure. The additional requirement that its image be totally real,

$$d\beta_p(T_p\partial_+ SM) \cap J_{\mathbb{C}^2} d\beta_p(T_p\partial_+ SM) = \{0\},$$

ensures that the image of the space of geodesics contains no nontrivial complex tangent directions. This parallels the LeBrun–Mason construction, where the space of geodesics is realized as a totally real submanifold of $\mathbb{CP}^2$.

Property (b) requires that

$$\beta|_{Z^\circ} \colon Z^\circ \longrightarrow \mathbb{C}^2$$

be a biholomorphism onto its image. It follows that $\beta(Z^\circ)$ is open and that the components of β form global holomorphic coordinates on Z°. In particular, the possibly complicated integrable complex structure determined by $\mathscr{D}$ is represented, through β, by the standard complex structure of $\mathbb{C}^2$. Thus, this condition expresses the desingularizing role of the blow-down map in the interior. Equivalently, since β is already holomorphic, property (b) amounts to requiring that β be injective on Z° and have everywhere nondegenerate complex differential.

Finally, property (c) is a quantitative nondegeneracy condition. Writing the distinguished Hermitian form on Z° in the frame $\{\eta_1, \eta_2\}$, the inequality

$$\beta^*\Omega_{\mathbb{C}^2} \geq c\Omega$$

is equivalent to

$$H \geq c\,\mathrm{Id}.$$

It gives a uniform positive lower bound for the singular values of $d\beta$, measured relative to the distinguished Hermitian metric Ω. In particular, it implies

$$d\beta_1 \wedge d\beta_2 \neq 0$$

pointwise, and hence rules out any local collapse in the interior. Merely requiring the nonvanishing of $d\beta_1 \wedge d\beta_2$ would guarantee only that β is an immersion; it would not provide uniform control of the differential near the degenerate boundary of Z°. The stronger Hermitian estimate is precisely the condition needed to ensure that the blow-down property is stable under sufficiently small perturbations of both the map and the underlying metric.

In summary, the three conditions encode complementary features of a blow-down map: property (a) separates the geodesics and realizes their parameter space as a totally real surface, property (b) provides genuine holomorphic coordinates in the interior, and property (c) supplies the uniform metric control needed for stability under perturbations. Hence the definition both captures the characteristic behaviour of the explicit model blow-down maps and gives an open condition suitable for perturbation arguments [5]. In a recent work, Bohr and Paternain [6] applied the transport twistor space to the study of Zoll magnetic structures. The main idea is to exploit the correspondence between transport equations and holomorphic geometry established by the transport twistor space, thereby translating the classification and construction of Zoll magnetic structures into problems in complex geometry. More precisely, the authors investigate fibrewise holomorphic blow-down maps from the transport twistor space onto complex surfaces, where the boundary is collapsed onto a totally real submanifold while each fibre disk is mapped biholomorphically onto a holomorphic disk. Consequently, the dynamics of the thermostat flow are encoded by a family of holomorphic disks attached to the totally real submanifold.

Based on this correspondence, the authors prove that suitable holomorphic blow-down structures determine Zoll magnetic structures. Conversely, every Zoll magnetic structure naturally gives rise to such a holomorphic blow-down map. This establishes a geometric equivalence between Zoll magnetic systems and certain ruled complex surfaces equipped with families of embedded holomorphic disks. Furthermore, by deforming the totally real submanifold and applying the deformation theory of holomorphic disks, they construct new families of Zoll magnetic structures. Their work demonstrates that transport twistor spaces provide not only a geometric interpretation of transport equations but also an effective framework for reconstructing and deforming geometric structures through complex-analytic techniques.

2 Main Results

2.1 Statement of the main results

Consider, for $R > 0$, the surface $M = \mathbb{D}_R := \{(x,y) \in \mathbb{R}^2, x^2 + y^2 \le R^2\}$, equipped with the metric g_κ, expressed in polar coordinates $(x,y) = P(r,\theta) := (r\cos\theta, r\sin\theta)$,

$$P^* g_\kappa = (1+\kappa r^2)^2 \, \mathrm{d}r^2 + r^2 \mathrm{d}\theta^2. \tag{11}$$

After introducing the manifolds, we can construct the $\beta-$map explicitly on them and prove the β-map that satisfies the blow-down structure.

Theorem 2.1. *Consider $(M,g) = (\mathbb{D}_R, g_\kappa)$ defined by (11) where $R > 0$ and $\kappa > -\frac{1}{R^2}$. On its transport twistor space $(Z, \mathscr{D})$, in the coordinates (z,ν), and the involutive structure is*

$$\mathscr{D} = \{\partial_{\bar{\nu}}, \Xi\}, \tag{12}$$

$$\Xi := \left((2+\kappa z\bar{z})\nu^2 - \kappa z^2\right)\partial_z \tag{13}$$

$$+ \left(2 + \kappa z\bar{z} - \kappa \nu^2 \bar{z}^2\right)\partial_{\bar{z}} \tag{14}$$

$$+ \kappa(z - \nu^2\bar{z})\left(\bar{\nu}\partial_{\bar{\nu}} - \nu\partial_\nu\right), \tag{15}$$

define

$$w(z,\nu) := (z - \bar{z}\nu^2)e^{\frac{\kappa(z\bar{z} - \bar{z}^2\nu^2)}{2}}, \qquad \xi(z,\nu) := \nu e^{\frac{\kappa(z\bar{z} - \bar{z}^2\nu^2)}{2}}, \qquad (z,\nu) \in \mathbb{D}_R \times \mathbb{D}. \tag{16}$$

Then the map $\beta := (w, \xi)$ has a holomorphic blow-down structure.

Then, we can extend $a(r)$ to smooth function.

Theorem 2.2. *For $a(r) \in C^\infty[0,R]$, consider $(M,g) = (\mathbb{D}_R, g_\kappa)$ defined by* (26) *below where $R > 0$ and $a(r) > 0$ when $r \in [0,R]$. On its transport twistor space $(Z, \mathscr{D})$, in the coordinate (z,ν), and the involutive structure is $\mathscr{D} := \mathbb{C}\partial_{\bar\nu} \oplus \mathbb{C}\Xi$, where*

$$\Xi := -\tilde{a}z^2\partial_z + (1+a)\partial_{\bar z} + \nu^2((1+a)\partial_z - \tilde{a}\bar{z}^2\partial_{\bar z}) + (z - \nu^2\bar{z})\tilde{a}(\bar\nu\partial_{\bar\nu} - \nu\partial_\nu)). \tag{17}$$

Define

$$w(z,\nu) := (z - \bar{z}\nu^2)e^G, \qquad \xi(z,\nu) := \nu e^G, \qquad (z,\nu) \in \mathbb{D}_R \times \mathbb{D}, \tag{18}$$

where the function $G \in C^\infty(\mathbb{D}_R \times \mathbb{D})$ will be constructed in Section 2.5.1. Then the map $\beta := (w, \xi)$ has a holomorphic blow-down structure.

As an application of the holomorphic blow-down structure, one can also obtain geodesically invariant distributions with prescribed fiberwise Fourier modes. More precisely, given a bottom mode $\varphi \in \Omega_m \cap \ker \eta_-$, one is interested in constructing $u \in C^\infty(SM)$ such that

$$Xu = 0, \qquad u_m = \varphi, \qquad u_k = 0 \quad \text{for } k < m.$$

In our previous work [17, Corollary 2], we showed that for the family (D_R, g_κ) such invariant distributions can be constructed under a suitable holomorphic extension assumption on the corresponding holomorphic differential. This illustrates that the blow-down structure is not only a complex-geometric property of the transport twistor space, but also has consequences for inverse problems on the underlying surface.

Ouline.

The remainder of this chapter is devoted to the construction of the β-map and the verification of its blow-down structure. We first study the geometry of circularly symmetric disks and derive the corresponding scattering relation in Section 2.2, which provides the boundary information needed for the construction. We then introduce the transport twistor space in Section 2.4 and describe it in both polar coordinates (r, θ, μ) in Section 2.4.2 and the smooth coordinates (z, ν) in Section 2.4.3. These two coordinate systems play different roles: the polar coordinates are convenient for solving the transport equation and constructing the components of the β-map, while the (z, ν)-coordinates are better suited for studying their regularity and holomorphic properties on the transport twistor space. We next construct the second component ξ in Section 2.5.1 by solving an appropriate transport equation and then use the Clairaut integral to obtain the first component w in Section 2.6. After rewriting both components in the (z, ν)-coordinates, the resulting map takes the form

$$\beta(z, \nu) = \left((z - \bar{z}\nu^2)e^{G(z,\nu)}, \nu e^{G(z,\nu)}\right).$$

Finally, in Section 2.7, we verify that this map satisfies the conditions required for a holomorphic blow-down structure.

2.2 Simplicity of the Circular Symmetry Disk

We first compute the scattering function associated with the rotationally symmetric metric

$$g = a^2(r)\, dr^2 + r^2\, d\theta^2.$$

No analyticity assumption on $a(r)$ is required for this computation. By the symmetries of the scattering relation, it is enough to consider

$$\alpha \in \left(-\frac{\pi}{2}, 0\right).$$

Along every geodesic, Clairaut's first integral gives

$$r \sin\alpha = \text{constant}.$$

For a geodesic entering from $r = R$ with initial angle α, its minimal radial coordinate is therefore

$$\rho = -R\sin\alpha > 0.$$

Let (ρ, θ_v) denote the turning point of the geodesic. The angular coordinate along the descending and ascending branches is given by[21]

$$\theta(t) = \theta_v \mp \rho \int_{\rho}^{r(t)} \frac{a(r)}{r\sqrt{r^2 - \rho^2}}\, dr.$$

Hence the total angular displacement between the two boundary endpoints is

$$\theta(\tau) - \theta(0) = 2\rho \int_{\rho}^{R} \frac{a(r)}{r\sqrt{r^2 - \rho^2}}\, dr.$$

On the other hand, the scattering relation is written in fan-beam coordinates as

$$S(\theta, \alpha) = \big(\theta + \pi + 2\mathfrak{s}_a(\alpha), \pi - \alpha\big).$$

Thus

$$\theta(\tau) - \theta(0) = \pi + 2\mathfrak{s}_a(\alpha),$$

and consequently

$$\mathfrak{s}_a(\alpha) = \rho \int_\rho^R \frac{a(r)}{r\sqrt{r^2-\rho^2}}\, dr - \frac{\pi}{2}, \qquad \rho = -R\sin\alpha. \tag{19}$$

Equivalently,

$$\mathfrak{s}_a(\alpha) = -R\sin\alpha \int_{-R\sin\alpha}^R \frac{a(r)}{r\sqrt{r^2-R^2\sin^2\alpha}}\, dr - \frac{\pi}{2}.$$

The Euclidean metric corresponds to $a(r) = 1$. In this case,

$$\mathfrak{s}_1(\alpha) = \rho \int_\rho^R \frac{1}{r\sqrt{r^2-\rho^2}}\, dr - \frac{\pi}{2}.$$

Using the substitution

$$r = \rho\sec\varphi,$$

we obtain

$$\frac{dr}{r\sqrt{r^2-\rho^2}} = \frac{1}{\rho}\, d\varphi.$$

Since $r = \rho$ corresponds to $\varphi = 0$, while $r = R$ corresponds to

$$\varphi = \arccos\left(\frac{\rho}{R}\right),$$

it follows that

$$\rho \int_\rho^R \frac{1}{r\sqrt{r^2-\rho^2}}\, dr = \arccos\left(\frac{\rho}{R}\right).$$

Because

$$\frac{\rho}{R} = -\sin\alpha = \cos\left(\frac{\pi}{2}+\alpha\right),$$

we obtain

$$\arccos\left(\frac{\rho}{R}\right) = \frac{\pi}{2} + \alpha.$$

Therefore,

$$\boxed{\mathfrak{s}_1(\alpha) = \alpha.}$$

It is useful to rewrite (19) relative to the Euclidean case. Since

$$\rho \int_\rho^R \frac{1}{r\sqrt{r^2 - \rho^2}}\, dr = \frac{\pi}{2} + \alpha,$$

we have the exact identity

$$\mathfrak{s}_a(\alpha) = \alpha + \rho \int_\rho^R \frac{a(r) - 1}{r\sqrt{r^2 - \rho^2}}\, dr. \tag{20}$$

This formula shows that the difference

$$\mathfrak{s}_a(\alpha) - \alpha$$

depends linearly on $a(r) - 1$.

We now compute the contribution of the even monomial r^{2k}. Define

$$\mathcal{I}_k(\alpha) := \rho \int_\rho^R \frac{r^{2k}}{r\sqrt{r^2 - \rho^2}}\, dr = \rho \int_\rho^R \frac{r^{2k-1}}{\sqrt{r^2 - \rho^2}}\, dr, \qquad k \geq 1.$$

Set

$$u = \sqrt{r^2 - \rho^2}.$$

Then

$$r^2 = u^2 + \rho^2, \qquad r\, dr = u\, du,$$

so that

$$\frac{r^{2k-1}}{\sqrt{r^2-\rho^2}}\,dr = (u^2+\rho^2)^{k-1}\,du.$$

Therefore,

$$\mathcal{I}_k(\alpha) = \rho \int_0^{\sqrt{R^2-\rho^2}} (u^2+\rho^2)^{k-1}\,du.$$

Expanding the integrand by the binomial theorem gives

$$(u^2+\rho^2)^{k-1} = \sum_{j=0}^{k-1} \binom{k-1}{j} \rho^{2(k-1-j)} u^{2j}.$$

Hence

$$\begin{aligned}
\mathcal{I}_k(\alpha) &= \rho \sum_{j=0}^{k-1} \binom{k-1}{j} \rho^{2(k-1-j)} \int_0^{\sqrt{R^2-\rho^2}} u^{2j}\,du \\
&= \rho \sum_{j=0}^{k-1} \binom{k-1}{j} \rho^{2(k-1-j)} \frac{(R^2-\rho^2)^{j+\frac{1}{2}}}{2j+1}.
\end{aligned} \tag{21}$$

Substituting

$$\rho = -R\sin\alpha, \qquad \sqrt{R^2-\rho^2} = R\cos\alpha,$$

we obtain

$$\boxed{\mathcal{I}_k(\alpha) = -R^{2k}\sin\alpha \sum_{j=0}^{k-1} \binom{k-1}{j} \frac{\sin^{2(k-1-j)}\alpha\,\cos^{2j+1}\alpha}{2j+1}.} \tag{22}$$

Consequently, for any finite even polynomial

$$a(r) = 1 + \sum_{k=1}^{N} a_{2k} r^{2k},$$

the scattering function is exactly

$$\boxed{\mathfrak{s}_a(\alpha) = \alpha + \sum_{k=1}^{N} a_{2k}\mathcal{I}_k(\alpha).} \tag{23}$$

More generally, whenever $a(r) - 1$ is represented by a convergent even power series on $[0, R]$, the same formula holds with the corresponding infinite sum.

We now return to the particular choice

$$a(r) = 1 + \kappa r^2.$$

Since the scattering formula is affine linear in $a(r)$, the Euclidean contribution and the quadratic contribution can be computed separately. Using

$$\mathfrak{s}_1(\alpha) = \alpha$$

and

$$\mathcal{I}_1(\alpha) = -R^2 \sin\alpha\cos\alpha = -\frac{R^2}{2}\sin(2\alpha),$$

we recover

$$\mathfrak{s}_\kappa(\alpha) = \alpha - \frac{\kappa R^2}{2}\sin(2\alpha). \tag{24}$$

Differentiating gives

$$\mathfrak{s}_\kappa'(\alpha) = 1 - \kappa R^2\cos(2\alpha). \tag{25}$$

Thus, the behavior of the scattering relation is completely determined by the dimensionless parameter

$$\lambda := \kappa R^2.$$

First, the condition

$$\lambda > -1$$

is necessary for $a(r) = 1 + \kappa r^2$ to remain positive on $[0, R]$, and hence for g_κ to define a nondegenerate Riemannian metric on the closed disk.

When

$$-1 < \lambda < 1,$$

we have

$$\mathfrak{s}_\kappa'(\alpha) > 0$$

for every admissible α. Therefore, the scattering function is strictly increasing, the disk has no conjugate points, and the metric is simple.

The value

$$\lambda = 1$$

is the critical case. Indeed,

$$\mathfrak{s}_\kappa'(\alpha) = 1 - \cos(2\alpha) = 2\sin^2\alpha,$$

which vanishes at $\alpha = 0$. Thus the strict monotonicity of the scattering relation is lost, and the metric is no longer simple.

When

$$\lambda > 1,$$

the derivative becomes negative for geodesics sufficiently close to the radial geodesic. More precisely, if

$$\alpha_c = -\frac{1}{2}\arccos\left(\frac{1}{\lambda}\right),$$

then

$$\mathfrak{s}_\kappa'(\alpha_c) = 0$$

and

$$\mathfrak{s}_\kappa'(\alpha) < 0, \qquad \alpha \in (\alpha_c, 0).$$

Consequently, the scattering function is not monotone, conjugate points occur, and the disk is non-simple.

In particular, within the range where the metric is well defined,(i.e. $\kappa R^2 > -1$),

$$g_\kappa \text{ is simple} \quad \Longleftrightarrow \quad |\kappa| R^2 < 1.$$

2.3 Constructing the β-map on SM

By Theorem1.3, the construction of holomorphic functions on the transport twistor space can be reduced to the construction of smooth fibrewise holomorphic functions on SM that are invariant under the geodesic flow. Therefore, as a first step, we derive explicit coordinate expressions for the geodesic vector field X and the perpendicular vector field $X_\perp$.

Consider the circularly symmetric metric

$$g = a^2(r)\, dr^2 + r^2\, d\theta^2, \qquad (r, \theta) \in (0, R) \times \mathbb{S}^1, \tag{26}$$

where $a(r) > 0$ is smooth. The inverse metric is

$$g^{-1} = \frac{1}{a^2(r)} \partial_r \otimes \partial_r + \frac{1}{r^2} \partial_\theta \otimes \partial_\theta.$$

We derive the geodesic equations using the Hamiltonian formalism. Let

$$(r, \theta, p_r, p_\theta)$$

denote the canonical coordinates on T^*M, where p_r and p_θ are the momenta dual to r and θ, respectively. The kinetic energy Hamiltonian is

$$H(r, \theta, p_r, p_\theta) = \frac{1}{2} g^{-1}(p, p) = \frac{1}{2} \left(\frac{p_r^2}{a^2(r)} + \frac{p_\theta^2}{r^2} \right). \tag{27}$$

The canonical symplectic form on T^*M is

$$\omega = dr \wedge dp_r + d\theta \wedge dp_\theta.$$

The Hamiltonian vector field X_H is determined by

$$\iota_{X_H} \omega = dH.$$

Equivalently,

$$X_H = \frac{\partial H}{\partial p_r} \partial_r + \frac{\partial H}{\partial p_\theta} \partial_\theta - \frac{\partial H}{\partial r} \partial_{p_r} - \frac{\partial H}{\partial \theta} \partial_{p_\theta}. \tag{28}$$

We now compute each derivative separately. First,

$$\frac{\partial H}{\partial p_r} = \frac{p_r}{a^2(r)},$$

and

$$\frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{r^2}.$$

Since the metric coefficients do not depend on θ, we also have

$$\frac{\partial H}{\partial \theta} = 0.$$

For the radial derivative, we compute

$$\begin{aligned}
\frac{\partial H}{\partial r} &= \frac{1}{2}\frac{\partial}{\partial r}\left(\frac{p_r^2}{a^2(r)} + \frac{p_\theta^2}{r^2}\right) \\
&= \frac{1}{2}\left(-\frac{2a'(r)}{a^3(r)}p_r^2 - \frac{2}{r^3}p_\theta^2\right) \\
&= -\frac{a'(r)}{a^3(r)}p_r^2 - \frac{p_\theta^2}{r^3}.
\end{aligned}$$

Substituting these derivatives into (28), we obtain

$$X_H = \frac{p_r}{a^2(r)}\partial_r + \frac{p_\theta}{r^2}\partial_\theta + \left(\frac{a'(r)}{a^3(r)}p_r^2 + \frac{p_\theta^2}{r^3}\right)\partial_{p_r}. \tag{29}$$

There is no ∂_{p_θ}-component because the metric is rotationally invariant.

Thus, Hamilton's equations are

$$\dot{r} = \frac{p_r}{a^2(r)}, \tag{30}$$

$$\dot{\theta} = \frac{p_\theta}{r^2}, \tag{31}$$

$$\dot{p}_r = \frac{a'(r)}{a^3(r)}p_r^2 + \frac{p_\theta^2}{r^3}, \tag{32}$$

$$\dot{p}_\theta = 0. \tag{33}$$

We now restrict the Hamiltonian flow to the unit tangent bundle. The natural orthonormal frame associated with the metric (26) is

$$e_r = \frac{1}{a(r)}\partial_r, \qquad e_\theta = \frac{1}{r}\partial_\theta.$$

A unit tangent vector $v \in S_{(r,\theta)}M$ can therefore be written as

$$v = \cos\alpha\, e_r + \sin\alpha\, e_\theta = \frac{\cos\alpha}{a(r)}\partial_r + \frac{\sin\alpha}{r}\partial_\theta, \tag{34}$$

where $\alpha \in \mathbb{S}^1$ is the fibre angle measured from the radial unit vector e_r.

Hence, away from the origin, the coordinates on SM are

$$(r, \theta, \alpha).$$

The velocity components are

$$\dot{r} = \frac{\cos\alpha}{a(r)}, \qquad \dot{\theta} = \frac{\sin\alpha}{r}.$$

To express the corresponding momenta, we use the metric identification between TM and T^*M. Since

$$p_r = g_{rr}\dot{r} \qquad \text{and} \qquad p_\theta = g_{\theta\theta}\dot{\theta},$$

we obtain

$$p_r = a^2(r)\frac{\cos\alpha}{a(r)} = a(r)\cos\alpha, \tag{35}$$

$$p_\theta = r^2\frac{\sin\alpha}{r} = r\sin\alpha. \tag{36}$$

These formulas indeed satisfy the unit-energy condition:

$$H = \frac{1}{2}\left(\frac{a^2(r)\cos^2\alpha}{a^2(r)} + \frac{r^2\sin^2\alpha}{r^2}\right) = \frac{1}{2}.$$

Substituting (35) and (36) into Hamilton's equations gives

$$\dot{r} = \frac{\cos\alpha}{a(r)} \tag{37}$$

and

$$\dot{\theta} = \frac{\sin\alpha}{r}. \tag{38}$$

It remains to compute $\dot{\alpha}$. From (33), the angular momentum p_θ is constant along every geodesic. Since

$$p_\theta = r\sin\alpha,$$

we have

$$\frac{d}{dt}\big(r\sin\alpha\big) = 0.$$

Expanding this derivative gives

$$\dot{r}\sin\alpha + r\cos\alpha\,\dot{\alpha} = 0.$$

Using (37), we obtain

$$\frac{\cos\alpha}{a(r)}\sin\alpha + r\cos\alpha\,\dot{\alpha} = 0.$$

Therefore,

$$\dot{\alpha} = -\frac{\sin\alpha}{a(r)r}. \tag{39}$$

The computation above is initially performed when $\cos\alpha \neq 0$, but the resulting formula extends to all α by continuity.

Combining (37), (38), and (39), the geodesic vector field on SM is

$$\boxed{X = \frac{\cos\alpha}{a(r)}\partial_r + \frac{\sin\alpha}{r}\partial_\theta - \frac{\sin\alpha}{a(r)r}\partial_\alpha.} \tag{40}$$

We next compute the perpendicular vector field originally defined in Section 1.3. The vertical vector field is

$$V = \partial_\alpha.$$

Following the convention used in this thesis, we define

$$X_\perp = [X, V].$$

Write

$$X = A(r,\alpha)\partial_r + B(r,\alpha)\partial_\theta + C(r,\alpha)\partial_\alpha,$$

where

$$A(r,\alpha) = \frac{\cos\alpha}{a(r)}, \qquad B(r,\alpha) = \frac{\sin\alpha}{r}, \qquad C(r,\alpha) = -\frac{\sin\alpha}{a(r)r}.$$

Since $V = \partial_\alpha$ has constant coefficients, we have

$$[X, V] = -\partial_\alpha A\,\partial_r - \partial_\alpha B\,\partial_\theta - \partial_\alpha C\,\partial_\alpha.$$

We compute the three derivatives separately:

$$\partial_\alpha A = -\frac{\sin\alpha}{a(r)},$$

$$\partial_\alpha B = \frac{\cos\alpha}{r},$$

and

$$\partial_\alpha C = -\frac{\cos\alpha}{a(r)r}.$$

Therefore,

$$\begin{aligned} X_\perp &= -\left(-\frac{\sin\alpha}{a(r)}\right)\partial_r - \frac{\cos\alpha}{r}\partial_\theta - \left(-\frac{\cos\alpha}{a(r)r}\right)\partial_\alpha \\ &= \frac{\sin\alpha}{a(r)}\partial_r - \frac{\cos\alpha}{r}\partial_\theta + \frac{\cos\alpha}{a(r)r}\partial_\alpha. \end{aligned}$$

Thus,

$$\boxed{X_\perp = \frac{\sin\alpha}{a(r)}\partial_r - \frac{\cos\alpha}{r}\partial_\theta + \frac{\cos\alpha}{a(r)r}\partial_\alpha.} \tag{41}$$

We now explain the separated form used below. A function on SM, written in the coordinates (r, θ, α), depends on two angular variables. The variable θ describes the position on the rotationally symmetric surface, whereas α describes the direction of the unit tangent vector inside the fibre $S_{(r,\theta)}M$.

We first decompose a smooth function $u \in C^\infty(SM)$ into its fibrewise Fourier series with respect to α:

$$u(r, \theta, \alpha) = \sum_{p\in\mathbb{Z}} u_p(r, \theta) e^{ip\alpha}.$$

Indeed, since the vertical vector field is

$$V = \partial_\alpha,$$

the functions $e^{ip\alpha}$ are precisely its eigenfunctions:

$$V(e^{ip\alpha}) = ip\, e^{ip\alpha}.$$

Thus, the index p records the fibrewise Fourier degree, and

$$u_p(r,\theta)e^{ip\alpha} \in \Omega_p.$$

The rotational symmetry of the metric gives a second Fourier decomposition. Since

$$g = a^2(r)\,dr^2 + r^2\,d\theta^2$$

is independent of θ, the geodesic vector field

$$X = \frac{\cos\alpha}{a(r)}\partial_r + \frac{\sin\alpha}{r}\partial_\theta - \frac{\sin\alpha}{a(r)r}\partial_\alpha$$

also has coefficients independent of θ. Consequently,

$$[X, \partial_\theta] = 0.$$

Therefore, the transport equation $Xu = 0$ preserves each eigenspace of ∂_θ. We may hence decompose

$$u_p(r,\theta) = \sum_{\ell\in\mathbb{Z}} f_{\ell,p}(r)e^{i\ell\theta},$$

where

$$\partial_\theta\left(e^{i\ell\theta}f_{\ell,p}(r)\right) = i\ell e^{i\ell\theta}f_{\ell,p}(r).$$

Combining the Fourier decompositions in the base angle θ and the fibre angle α, a general smooth function can formally be written as

$$u(r,\theta,\alpha) = \sum_{\ell\in\mathbb{Z}}\sum_{p\in\mathbb{Z}} e^{i\ell\theta}e^{ip\alpha}f_{\ell,p}(r).$$

Because the equation is linear and different values of ℓ do not interact, we may fix one base angular frequency ℓ and study the corresponding Fourier components separately. We therefore use the ansatz

$$\boxed{\omega_p(r,\theta,\alpha) = e^{i\ell\theta} e^{ip\alpha} f_p(r),} \tag{42}$$

where the dependence of $f_{\ell,p}$ on the fixed index ℓ has been suppressed from the notation.

Thus, the factor $e^{i\ell\theta}$ is imposed by the rotational symmetry of the metric, the factor $e^{ip\alpha}$ represents the p-th fibrewise Fourier mode, and the radial coefficient $f_p(r)$ remains to be determined from the invariant equation.

This parity decomposition is reflected directly in the two components of the previously constructed β-map. On SM, these components are

$$w = re^{i\theta}(1 - e^{2i\alpha}) \exp\left(\frac{\kappa r^2}{2}(1 - e^{2i\alpha})\right)$$

and

$$\xi = e^{i\theta} e^{i\alpha} \exp\left(\frac{\kappa r^2}{2}(1 - e^{2i\alpha})\right).$$

To see their fibrewise Fourier contents, write

$$A(r) = \frac{\kappa r^2}{2}.$$

Then

$$\exp\left(A(r)(1 - e^{2i\alpha})\right) = e^{A(r)} \exp\left(-A(r)e^{2i\alpha}\right).$$

Expanding the second exponential gives

$$\exp\left(-A(r)e^{2i\alpha}\right) = \sum_{j=0}^{\infty} \frac{(-A(r))^j}{j!} e^{2ij\alpha}.$$

This expansion contains only nonnegative even powers of $e^{i\alpha}$.

It follows that

$$w = re^{i\theta}e^{A(r)}(1 - e^{2i\alpha}) \sum_{j=0}^{\infty} \frac{(-A(r))^j}{j!} e^{2ij\alpha}$$

contains only the even fibrewise Fourier modes

$$p = 0, 2, 4, \ldots .$$

Therefore,

$$w \in \bigoplus_{j\geq 0} \Omega_{2j}.$$

Similarly,

$$\xi = e^{i\theta}e^{A(r)} \sum_{j=0}^{\infty} \frac{(-A(r))^j}{j!} e^{i(2j+1)\alpha},$$

so ξ contains only the odd fibrewise Fourier modes

$$p = 1, 3, 5, \ldots .$$

Thus,

$$\xi \in \bigoplus_{j\geq 0} \Omega_{2j+1}.$$

Hence the two components of the β-map correspond precisely to the two independent parity classes of the invariant transport equation:

$$w \quad \text{represents the even Fourier chain,}$$

whereas

$$\xi \quad \text{represents the odd Fourier chain.}$$

Moreover, both functions contain only nonnegative fibrewise Fourier modes, and are therefore fibrewise holomorphic. Since

$$Xw = 0, \qquad X\xi = 0,$$

Theorem 1.3 identifies them with the restrictions of holomorphic functions on the transport twistor space. This explains why the β-map naturally has two components,

$$\beta|_{SM} = (w, \xi),$$

one supported on the even fibrewise Fourier modes and the other on the odd fibrewise Fourier modes. The parity decomposition above is valid for any rotationally symmetric metric of the form

$$g = a^2(r)\, dr^2 + r^2\, d\theta^2,$$

since it follows only from the fact that η_+ and η_- shift the fibrewise Fourier degree by one. Thus, the transport equation preserves the decomposition into even and odd fibrewise Fourier modes.

However, the explicit closed-form expressions obtained above are special to the choice

$$a(r) = 1 + \kappa r^2.$$

For this quadratic radial coefficient, the mode equations for the radial Fourier coefficients can be solved explicitly, and the resulting series can be summed into the exponential expressions

$$w = re^{i\theta}(1 - e^{2i\alpha}) \exp\left(\frac{\kappa r^2}{2}(1 - e^{2i\alpha})\right), \tag{43}$$

$$\xi = e^{i\theta} e^{i\alpha} \exp\left(\frac{\kappa r^2}{2}(1 - e^{2i\alpha})\right). \tag{44}$$

For a general smooth radial function $a(r)$, the even and odd modes still decouple, but the corresponding radial coefficients satisfy a more complicated hierarchy of first-order differential equations. In general, the coefficients must be constructed recursively, and the resulting Fourier series need not admit a simple closed-form summation. Therefore, the functions w and ξ written above should be viewed as explicit solutions adapted to the special metric $a(r) = 1 + \kappa r^2$, rather than as universal formulas for arbitrary rotationally symmetric metrics.

2.3.1 Reconstruct the scattering relation

Fix a Herglotz disk $\mathbb{D}_R$ with a metric g of the form (26). Parameterize $\partial SM = (\mathbb{R}/2\pi\mathbb{Z})_\theta \times (\mathbb{R}/2\pi\mathbb{Z})_\alpha$, where θ parameterizes the boundary point $Re^{i\theta}$ and α parameterizes the tangent vector $e^{i\alpha} \cdot (-\mathbf{e}_r)_{Re^{i\theta}}$. Recall that the rotation-invariance of the metric gives that the scattering relation $S\colon \partial SM \to \partial SM$ can be written as

$$S(\theta, \alpha) = (\theta + \pi + 2\mathfrak{s}(\alpha), \pi - \alpha), \tag{45}$$

for some function $\mathfrak{s}(\alpha)$ which we call the scattering signature.

We first establish the following

Lemma 2.1. *If $(\mathbb{D}_R, g)$ is a smooth Herglotz disk, then its scattering signature takes the* *form*

$$\mathfrak{s}(\alpha) = \alpha + \tilde{\mathfrak{s}}(\alpha), \qquad \tilde{\mathfrak{s}}(\alpha) = \sum_{p=1}^{\infty} s_{2p} \sin(2p\alpha), \tag{46}$$

for some rapidly-decaying coefficients $\{s_{2p}\}_{p\geq 0}$.

For example, the representation (46) of the Euclidean metric is given by $\tilde{\mathfrak{s}} \equiv 0$.

Proof. The non-trapping condition implies that $\mathfrak{s}(\alpha)$ is smooth, and the convexity condition implies that $\mathfrak{s}$ fixes $\pm\pi/2$. By symmetry we also have $\mathfrak{s}(-\alpha) = -\mathfrak{s}(\alpha)$. Thus, writing $\tilde{\mathfrak{s}}(\alpha) := \mathfrak{s}(\alpha) - \alpha$, we find that f satisfies

$$\tilde{\mathfrak{s}}(-\alpha) = -\tilde{\mathfrak{s}}(\alpha), \qquad \tilde{\mathfrak{s}}(\alpha+\pi) = \tilde{\mathfrak{s}}(\alpha), \qquad \tilde{\mathfrak{s}}(\alpha) \in \mathbb{R}.$$

In particular, $\tilde{\mathfrak{s}}(\alpha)$ is fiberwise even and we can expand it as $\tilde{\mathfrak{s}}(\alpha) = \sum_{k\in\mathbb{Z}} s_{2k} e^{2ik\alpha}$, and the other two conditions imply $s_{2p} = -s_{-2p} = \overline{s_{-2p}}$ for all p, imposing the sine series form (46). The rapid decay of the coefficients follows from smoothness. □

Remark 2.1. *Note that simplicity, equivalent to* $\inf_\alpha |\mathfrak{s}'(\alpha)| > 0$ *as was proved in [17, Prop. 3.1], is thus expressed as*

$$\inf_\alpha \left(1 + \sum_{p=1}^{\infty} 2p\ s_{2p} \cos(2p\alpha)\right) > 0. \tag{47}$$

By Theorem1.2, if $(\mathbb{D}_R, g)$ *is simple, then for every* $r \in (0,1)$*,* $(\mathbb{D}_{Rr}, g)$ *is simple.*

In terms of Fourier series, this means

$$\inf_\alpha \left(1+\sum_{p=1}^\infty 2ps_{2p}(1)\cos(2p\alpha)\right) > 0$$
$$\implies \inf_\alpha \left(1+\sum_{p=1}^\infty 2ps_{2p}(r)\cos(2p\alpha)\right) > 0, \quad r\in(0,1).$$

Then, we construct an operator which maps $\tilde{a}(r) := (a(r)-1)/r^2$ to $\tilde{\mathfrak{s}}(\alpha) := \mathfrak{s}(\alpha)-\alpha$, notably it has *linear* behavior, in spite of not being defined on a vector space. Using [21, Eq. (2.30)] and writing $a = 1 + r^2\tilde{a}(r)$, we have

$$\mathfrak{s}(\alpha) = \alpha - R\sin\alpha \int_{-R\sin\alpha}^{R} \frac{\tilde{a}(r)r\mathrm{d}r}{\sqrt{r^2-R^2\sin^2\alpha}} = \alpha + A_R\tilde{a}(\alpha),$$

where we have defined

$$\begin{aligned} A_R\tilde{a}(\alpha) &:= -R\sin\alpha \int_{-R\sin\alpha}^{R} \frac{\tilde{a}(r)r\mathrm{d}r}{\sqrt{r^2-R^2\sin^2\alpha}} && (48)\\ &= -R\sin\alpha \int_0^{R\cos\alpha} \tilde{a}(\sqrt{u^2+R^2\sin^2\alpha})\,\mathrm{d}u && (u=\sqrt{r^2-R^2\sin^2\alpha})\\ &= -\frac{R}{2}\sin\alpha\, I_0^{\mathbb{R}_R}\tilde{a}(0,\alpha), \end{aligned}$$

where $I_0^{\mathbb{D}_R} f(\beta,\alpha) = \int_0^{2R\cos\alpha} f(Re^{i\beta}+te^{i(\beta+\pi+\alpha)})\,\mathrm{d}t$ is the Euclidean X-ray transform on the disk $\mathbb{D}_R$.

Obvious symmetries include:

$$A_R\tilde{a}(-\alpha) = -A_R\tilde{a}(\alpha), \quad A_R\tilde{a}(\alpha+\pi) = A_R\tilde{a}(\alpha), \qquad \alpha\in\mathbb{R}/2\pi\mathbb{Z}.$$

Since we notice that for $a(r) = 1+\kappa r^2$, the ξ restricted to ∂SM is related to the scattering relation, and for the formula of the scattering relation is related the metric

$a(r)$. That is why we define the operator in this chapter and we will use it to construct ξ for general $a(r)$ in Sec. 2.5.1.

2.4 Constructing the β-map on Z

Having constructed the β-map on SM, we now turn to its counterpart on the transport twistor space Z. Recall that Z is obtained by attaching a holomorphic disk to each geodesic and therefore provides a natural complexification of the unit tangent bundle. Since the transport twistor space is the natural setting for studying holomorphic structures associated with the geodesic flow, it is natural to seek a map

$$\beta : Z \longrightarrow \mathbb{C}^2$$

whose restriction to SM agrees with the β-map constructed in the previous subsection. The remainder of this section is devoted to constructing such a map in the coordinates introduced in 2.2.

2.4.1 Coordinate-invariant construction of Z

We now recall the construction of the TT space of an oriented Riemannian surface (M, g).

The metric and orientation give rise to a unique almost-complex structure, i.e. a bundle map $\iota\colon TM \to TM$ satisfying $\iota^2 = -id$. For $v \in TM$, ιv is the unique tangent vector such that $g(\iota v, \iota v) = g(v, v)$, $g(\iota v, v) = 0$ and $(v, \iota v)$ is oriented. The almost-complex structure allows us to define a complex multiplication

$$(x, \omega \cdot v) := (x, \mathrm{Re}(\omega)v + \mathrm{Im}(\omega)\iota v), \qquad (x, v) \in TM, \quad \omega \in \mathbb{C}, \tag{49}$$

which also gives rise to the usual circle action on the fibers of SM when $|\omega| = 1$. Recalling that $Z = BM$, a way to construct the involutive structure $\mathscr{D} \subset T_{\mathbb{C}}Z$ in a coordinate-invariant way, is as the pushdown $p_\star\widehat{\mathscr{D}}$ of a distribution defined on the total space $SM \times \mathbb{D}$ of the principal $\mathbb{S}^1$-bundle

$$\mathsf{p}\colon SM \times \mathbb{D} \to BM, \qquad (x, v, \omega) \mapsto (x, \omega \cdot v), \tag{50}$$

with fibers given by the orbits of the circle action

$$\psi_t(x, v, \omega) = (x, e^{it} \cdot v, e^{-it}\omega), \quad (x, v) \in SM, \quad t \in \mathbb{R}/2\pi\mathbb{Z}. \tag{51}$$

If X denotes the generator of the geodesic flow on SM, and V denotes the generator of the circle action on the fibers of SM, we recall the Guillemin-Kazhdan operators $\eta_\pm := \frac{1}{2}(X \pm i[X, V]) \in C^\infty(M; T_{\mathbb{C}}(SM))$. Then the distribution $\widehat{\mathscr{D}}$ is given by

$$\widehat{\mathscr{D}} = \mathrm{span}\{\eta_- + \omega^2\eta_+,\ \partial_{\overline{\omega}}\}. \tag{52}$$

It is then proved in [5, Lemma 4.1] that $\widehat{\mathscr{D}}$ is $\mathbb{S}^1$-invariant and as such descends to a distribution $\mathscr{D} = \mathsf{p}^*\widehat{\mathscr{D}}$ on Z, satisfying the properties (a)-(b)-(c) of Definition 1.4.

2.4.2 The distribution in polar coordinates

The polar coordinate chart $(x, y) = P(r, \theta) = (r\cos\theta, r\sin\theta)$ is a diffeomorphism $\widetilde{M} := (0, R]_r \times (\mathbb{R}/2\pi\mathbb{Z})_\theta \to M\backslash\{0\}$. On $(\widetilde{M}, P^*g)$, an oriented orthonormal frame of $T\widetilde{M}$ is given by $(\mathbf{e}_r, \mathbf{e}_\theta)$, where $\mathbf{e}_r = \frac{1}{a(r)}\partial_r$ and $\mathbf{e}_\theta = \frac{1}{r}\mathbf{e}_\theta$, and the circle action applied to $\mathbf{e}_r$ gives rise to the global chart

$$\widetilde{M} \times \mathbb{R}/2\pi\mathbb{Z} \ni (r, \theta, \alpha) \mapsto (r, \theta, \cos\alpha\ \mathbf{e}_r + \sin\alpha\ \mathbf{e}_\theta) \in S\widetilde{M},$$

where $V = \partial_\alpha$. In these coordinates, the almost-complex structure of M is given by the a and $\tilde{a}$. in 7

$$\mathbb{C}(\mathbf{e}_r + i\mathbf{e}_\theta) = \mathbb{C}\Big(\partial_{\bar{z}} - \frac{z^2\tilde{a}(z\bar{z})}{1 + a(z\bar{z})}\partial_z\Big) \qquad (z = re^{i\theta}), \tag{53}$$

and the geodesic vector field takes the expression (40), from which we can derive an expression for the Guillemin-Kazhdan operators:

$$\eta_\pm = \frac{1}{2}(X \pm i[X, V]) = \frac{e^{\pm i\alpha}}{2}\left(\mathbf{e}_r \mp i\mathbf{e}_\theta \pm \frac{i}{ra(r)}\partial_\alpha\right).$$

The distribution $\widehat{\mathscr{D}}$ on $S\widetilde{M}_{(r,\theta,\alpha)} \times \mathbb{D}_\omega$ is thus given by $\widehat{\mathscr{D}} = \operatorname{span}\{\partial_{\bar{\omega}}, \widehat{\Xi}\}$, where

$$\widehat{\Xi} = \eta_- + \omega^2\eta_+ = \frac{1}{2}\left((e^{-i\alpha} + \omega^2 e^{i\alpha})\mathbf{e}_r + i(e^{-i\alpha} - \omega^2 e^{i\alpha})\mathbf{e}_\theta + \frac{i(-e^{-i\alpha} + \omega^2 e^{i\alpha})}{r(a(r))}\partial_\alpha\right).$$

To push $\widehat{\mathscr{D}}$ down to $B\widetilde{M}$, we use the chart

$$\widetilde{\Phi}\colon \widetilde{M} \times \mathbb{D} \ni (r, \theta, \mu) \mapsto (r, \theta, \operatorname{Re}(\mu)\mathbf{e}_r + \operatorname{Im}(\mu)\mathbf{e}_\theta), \tag{54}$$

and notice that $\widetilde{\Phi}(r, \theta, \mu) = \mathsf{p}(r, \theta, 0, \mu)$ where p is defined in (50). In particular, we have that $\mathsf{p}_*\partial_r = \partial_r$, $\mathsf{p}_*\partial_\theta = \partial_\theta$, $\mathsf{p}_*(\partial_\omega) = \partial_\mu$ and $\mathsf{p}_*(\partial_{\bar{\omega}}) = \partial_{\bar{\mu}}$ above a point $(r, \theta, 0, \omega)$. Finally, to compute $\mathsf{p}_*\partial_\alpha$, we use that p_* annihilates the infinitesimal generator of (51) given by $\partial_\alpha - i(\omega\partial_\omega - \bar{\omega}\partial_{\bar{\omega}})$, hence $\mathsf{p}_*(\partial_\alpha) = \mathsf{p}_*(i(\omega\partial_\omega - \bar{\omega}\partial_{\bar{\omega}})) = i(\mu\partial_\mu - \bar{\mu}\partial_{\bar{\mu}})$.

As a conclusion, the involutive structure of $B\widetilde{M}$ looks like $\widetilde{\mathscr{D}} = \mathbb{C}\partial_{\bar{\mu}} \oplus \mathbb{C}\widetilde{\Xi}$, where

$$\widetilde{\Xi} := \mathsf{p}_*(\widehat{\Xi}|_{(r,\theta,0,\mu)}) = \frac{1}{2}\left(\frac{(\mu^2+1)}{a(r)}\partial_r + \frac{i(1-\mu^2)}{r}\partial_\theta + \frac{1-\mu^2}{ra(r)}\left(\mu\partial_\mu - \bar{\mu}\partial_{\bar{\mu}}\right)\right). \quad (55)$$

2.4.3 Smooth coordinates on Z down to the origin

The discussion in this section applies to any disk $\mathbb{D}_R$ equipped with a rotation-invariant metric of the form 6 for some positive function $a(r)$ satisfying 6. We denote by $(\mathbf{e}_r, \mathbf{e}_\theta) = (\frac{1}{a}\partial_r, \frac{1}{r}\partial_\theta)$ an oriented orthonormal frame of P^*g. Although $(P_*\mathbf{e}_r, P_*\mathbf{e}_\theta)$ does not extend to the origin of M, we have the following.

Lemma 2.2. *The section*

$$\mathbf{e} := P_*(\cos\theta\mathbf{e}_r - \sin\theta\mathbf{e}_\theta),$$

originally defined and smooth on $M\backslash\{0\}$, extends to a smooth section of SM.

Proof of Lemma 2.2. For $(x, y) \neq (0, 0)$, we compute directly that

$$\begin{aligned}\mathbf{e} &= \frac{1}{a}\left((1+\cos^2\theta(1-a))\partial_x + \cos\theta\sin\theta(1-a)\partial_y\right)\\ &= \frac{1}{a(x^2+y^2)}\left(1 - x^2\tilde{a}(x^2+y^2))\partial_x - xy\tilde{a}(x^2+y^2)\partial_y\right),\end{aligned}$$

hence the result. □

Lemma 2.2 justifies the validity of the chart defined in polar coordinate. Moreover, relative to the chart (54), this corresponds to the change of variable

$$(r, \theta, \mu, \bar{\mu}) \to (z, \bar{z}, \nu, \bar{\nu}), \quad \text{where} \quad z = re^{i\theta}, \quad \nu = \mu e^{i\theta}. \quad (56)$$

Pushing forward the distribution $\widetilde{\mathscr{D}} = \mathbb{C}\partial_{\bar{\mu}} \oplus \mathbb{C}\widetilde{\Xi}$, we find that $\partial_{\bar{\mu}} = e^{-i\theta}\partial_{\bar{\nu}} \propto \partial_{\bar{\nu}}$, and a straightforward calculation shows that the vector field $\widetilde{\Xi}$ defined in (55) becomes $\frac{e^{-i\theta}}{2(1+\kappa r^2)}\Xi$, where Ξ is given in

$$\Xi := \left((2+\kappa z\bar{z})\nu^2 - \kappa z^2\right)\partial_z + \left(2+\kappa z\bar{z} - \kappa\nu^2\bar{z}^2\right)\partial_{\bar{z}} + \kappa\left(z - \nu^2\bar{z}\right)\left(\bar{\nu}\partial_{\bar{\nu}} - \nu\partial_\nu\right). \quad (57)$$

Remark 2.2. *As for the general $a(r)$, since the Taylor expansion of $a(r)$ contains only even terms, we can know the Lemma 2.2 still works for it, and we will construct*

$$\Xi := -(a(z\bar{z})-1)z^2\partial_z + (1+a(z\bar{z}))\partial_{\bar{z}} \quad (58)$$
$$+\nu^2\left((1+a(z\bar{z}))\partial_z - (a(z\bar{z})-1)\bar{z}^2\partial_{\bar{z}}\right) \quad (59)$$
$$+(z-\nu^2\bar{z})(a(z\bar{z})-1)\left(\bar{\nu}\partial_{\bar{\nu}} - \nu\partial_\nu\right). \quad (60)$$

2.4.4 Construction of the β map

Consider the $\beta-$map when $a(r) = 1+\kappa r^2$ for the special case, we know that the $\beta-$map restricted on SM is 43. We can compute that the $\beta-$map on Z is

$$\beta(w,\xi) = (re^{i\theta}(1-\mu^2)e^{\frac{\kappa(r^2-r^2\mu^2)}{2}}, ue^{i\theta}e^{\frac{\kappa(r^2-r^2\mu^2)}{2}}) \quad (61)$$

Since we have proved in Sec. 2.4.3 it is smooth at the origin, we will have that the $\beta-$map is

$$w = (z-\bar{z}\nu^2)\exp\left(\frac{\kappa(z\bar{z}-\bar{z}^2\nu^2)}{2}\right), \quad (62)$$
$$\xi = \nu\exp\left(\frac{\kappa(z\bar{z}-\bar{z}^2\nu^2)}{2}\right). \quad (63)$$

whose restriction to SM is geodesically invariant, i.e. $X\beta_{SM} = 0$.

Extending these explicit formulas from the special case $a(r) = 1+\kappa r^2$ to a general radial function $a(r)$ is difficult. In the next section, we therefore employ a different approach directly on the function $\mathfrak{s}(\alpha)$.

2.5 Construction on ∂SM for general $a(r)$

We explain how to construct two functions on ∂SM which are 1-equivariant, fiberwise holomorphic and scattering-invariant.

Recall the definition of the circle Hilbert transform $H\colon L^2(\mathbb{S}^1) \to L^2(\mathbb{S}^1)$ such that $H(e^{ik\alpha}) = -i\operatorname{sgn}(k)e^{ik\alpha}$ for $k \in \mathbb{Z}$.

Starting from the scattering relation (45), we have

$$Hf(\alpha) = -\sum_{p=1}^{\infty} f_{2p}\cos(2p\alpha),$$

and hence

$$f + iHf = -i\sum_{p=1}^{\infty} f_{2p}e^{2ip\alpha},$$

is a fiberwise holomorphic function. To construct the two components of the beta map on ∂SM, which behave like $e^{i\theta}$, note that $e^{i(\theta+\mathfrak{s}(\alpha))}$ is invariant under scattering relation but need not be fiberwise holomorphic. However, we can prove the following theorem.

Proposition 2.1. *Let $\mathfrak{s}(\alpha) = \alpha + \sum_{p=1}^{\infty} f_{2p}\sin(2p\alpha)$ be the scattering signature of a Herglotz disk. Then the functions (w, ξ) defined as*

$$\begin{aligned} \xi &:= e^{i(\theta+\mathfrak{s}(\alpha)+iHf(\alpha))} = e^{i(\theta+\alpha+f(\alpha)+iHf(\alpha))}, \\ w &:= \sin(\alpha)\xi = e^{i\theta}\frac{e^{2i\alpha}-1}{2i}e^{i(f+iHf)}, \end{aligned} \tag{64}$$

are fiberwise holomorphic, S-invariant.

Since ξ is fiberwise odd, w is fiberwise even, and they are 1-equivariant in θ, they make a good candidate to be the restriction to ∂SM of a beta map.

Proof. Fiberwise holomorphy can be seen directly from their dependence in $e^{i\alpha}$, and S-invariance follows from the fact that $\sin(\alpha)$ is S-invariant, and ξ if S-invariant thanks to the symmetries of Hf. □

Lemma 2.3. *The map $\partial_+ SM \ni (\theta, \alpha) \mapsto (w, \xi)$ is injective.*

Proof. Indeed, $\frac{w}{\xi} = \sin\alpha$ uniquely determines $\alpha \in [-\pi/2, \pi/2]$, after which w or ξ determines β. □

The above lemma is saying that the function (w, ξ) separates geodesics, which partially fulfills item (i) of having a holomorphic blowdown structure.

Due to their equivariance in β, note that (w, ξ) defined as such are uniquely defined up to scalar constant. We may then extend them to transport twistor space in a unique fashion, and the question is then whether they produce a map with holomorphic blowdown structure. The goal of the next sections is to construct (w, ξ) on the interior of TT space, and to show that the functions constructed, when restricted to ∂SM, recover those of Proposition 2.1.

2.5.1 Construction of ξ

In solving for $\widetilde{\Xi}\xi = 0$, an important reduction is as follows: write

$$\xi = \mu \, e^{g(r^2)} e^{i\theta} e^{-2F(r,\mu)},$$

where $g(r^2)$ solves $g'(r^2) = \frac{a-1}{2r^2}$ (see above paragraph for a polynomial expansion form of such a solution), then "$\widetilde{\Xi}\xi = 0$" is equivalent to

$$LF = \mu^2(a-1), \qquad \text{where} \quad L := (1+\mu^2)r\partial_r + (1-\mu^2)\mu\partial_\mu. \tag{65}$$

The theorem below shows how to explicitly construct F to solve (65). Below, recall that the operator A_r is defined in (48).

Theorem 2.3. *Fix $R > 0$ and consider a radial function $f \in C^\infty(\mathbb{D}_R)$ (i.e., of the form $g(r^2)$ for some $g \in C^\infty([0, R^{1/2}])$. Define*

$$F(r,\mu) := \frac{1}{i}\frac{1}{2\pi i}\int_{\mathbb{S}^1} \frac{A_r[f](z)}{z-\mu}\mathrm{d}z, \quad r \in (0,R], \quad \mu \in \mathbb{D}^\circ. \tag{66}$$

Then $F \in C^\infty(\mathbb{D}_R \times \mathbb{D})$ and we have

$$LF(r,\mu) = \mu^2 r^2 f(r). \tag{67}$$

In particular, the function F solving (65) *can be constructed by applying Theorem 2.3 to the function $f(r) = \frac{a-1}{r^2}$.*

Proof of Theorem 2.3. In what follows, we write

$$F(r,\mu) = \frac{1}{i}\frac{1}{2\pi i}\int_{\mathbb{S}^1} \frac{h(r,z)}{z-\mu}\mathrm{d}z = \frac{1}{2\pi i}\int_{\mathbb{S}^1} \frac{h(r,\alpha)}{1-\mu e^{-i\alpha}}\,\mathrm{d}\alpha, \tag{68}$$

where we have defined, for $r \in (0,R]$ and $\alpha \in [0,2\pi]$,

$$h(r,\alpha) := A_r[f](e^{i\alpha}) \overset{(48)}{=} -r\sin\alpha \int_0^{r\cos\alpha} f(\sqrt{u^2 + r^2\sin^2\alpha})\,\mathrm{d}u.$$

We first show that h is a smooth function of r^2 and α. To this end, from the hypothesis, we write $f(r) = g(r^2)$ for $g \in C^\infty([0, R^{1/2}])$, so that

$$\begin{aligned} h(r,\alpha) &= -r\sin\alpha \int_0^{r\cos\alpha} g(u^2 + r^2\sin^2\alpha)\,\mathrm{d}u \\ &= -r^2\sin\alpha\cos\alpha \int_0^1 g\Big(r^2(v^2\cos^2\alpha + \sin^2\alpha)\Big)\,\mathrm{d}v, \end{aligned}$$

upon changing variable $u(v) = vr\cos\alpha$. This last expression is clearly smooth in r^2 and α. Then smoothness of F follows from (68) and the mapping properties of the Cauchy kernel.

On to the derivation of (67), the function h satisfies the symmetry properties $h(r, \alpha + \pi) = h(r, \alpha)$ and $h(r, -\alpha) = -h(r, \alpha)$. In particular, we have

$$\int_{\mathbb{S}^1} h(r,\alpha)\,\mathrm{d}\alpha = 0, \qquad r \in (0, R]. \tag{69}$$

With the observation that

$$(\sin\alpha\partial_\alpha - \cos\alpha r\partial_r)(r\cos\alpha) = -r, \qquad (\sin\alpha\partial_\alpha - \cos\alpha r\partial_r)(r\sin\alpha) = 0,$$

it follows that

$$(\sin\alpha\partial_\alpha - \cos\alpha r\partial_r)h = r^2\sin\alpha f(r),$$

which we write as

$$\partial_\alpha h = \frac{\cos\alpha}{\sin\alpha} r\partial_r h + r^2 f(r). \tag{70}$$

Integrating this relation gives

$$\frac{1}{2\pi}\int_{\mathbb{S}^1}\frac{\cos\alpha}{\sin\alpha}r\partial_r h\,\mathrm{d}\alpha = -r^2 f(r). \tag{71}$$

By integration by parts on (68), we have

$$\begin{aligned}
\mu\partial_\mu F &= \frac{1}{2\pi i}\int_{\mathbb{S}^1}\frac{h(r,\alpha)\mu e^{-i\alpha}}{(1-\mu e^{-i\alpha})^2}\,\mathrm{d}\alpha &(72)\\
&= \frac{1}{2\pi}\int_{\mathbb{S}^1} h(r,\alpha)\partial_\alpha\left(\frac{1}{1-\mu e^{-i\alpha}}\right)\mathrm{d}\alpha &(73)\\
&= -\frac{1}{2\pi}\int_{\mathbb{S}^1}\frac{\partial_\alpha h}{1-\mu e^{-i\alpha}}\,\mathrm{d}\alpha. &(74)
\end{aligned}$$

Hence, by direct computation,

$$\begin{aligned}
LF &= \frac{\mu^2+1}{2\pi i}\int_{\mathbb{S}^1}\frac{r\partial_r h(r,\alpha)}{1-\mu e^{-i\alpha}}\,\mathrm{d}\alpha - \frac{1-\mu^2}{2\pi}\int_{\mathbb{S}^1}\frac{\partial_\alpha h(r,\alpha)}{1-\mu e^{-i\alpha}}\,\mathrm{d}\alpha\\
&\overset{(70)}{=} \frac{\mu^2+1}{2\pi i}\int_{\mathbb{S}^1}\frac{r\partial_r h(r,\alpha)}{1-\mu e^{-i\alpha}}\,\mathrm{d}\alpha - \frac{1-\mu^2}{2\pi}\int_{\mathbb{S}^1}\left[\left(\frac{1}{1-\mu e^{-i\alpha}}\right)\frac{\cos\alpha}{\sin\alpha}r\partial_r h + r^2 f(r)\right]\mathrm{d}\alpha\\
&= \frac{1}{2\pi i}\int_{\mathbb{S}^1}\left(\mu^2+1+(1-\mu^2)\frac{e^{i\alpha}+e^{-i\alpha}}{e^{i\alpha}-e^{-i\alpha}}\right)\frac{r\partial_r h(r,\alpha)}{1-\mu e^{-i\alpha}}\,d\alpha - (1-\mu^2)r^2 f(r)\\
&= -\frac{1}{2\pi}\int_{\mathbb{S}^1}\frac{(e^{i\alpha}+\mu)}{\sin\alpha}r\partial_r h(r,\alpha)\,\mathrm{d}\alpha - (1-\mu^2)r^2 f(r)\\
&\overset{(71)}{=} -\frac{1}{2\pi}\int_{\mathbb{S}^1}\left(i+\frac{\mu}{\sin\alpha}\right)r\partial_r h(r,\alpha)\,\mathrm{d}\alpha + \mu^2 r^2 f(r).
\end{aligned}$$

The term proportional to μ vanishes because the integrand is odd w.r.t. $\alpha \mapsto \alpha+\pi$, and the first term vanishes due to (69). Hence $LF = \mu^2 r^2 f(r)$. □

We now rewrite the construction in the smooth coordinates (z,ν) introduced in Section 2.4.3 Recall that

$$z = re^{i\theta}, \qquad \nu = \mu e^{i\theta}.$$

Hence

$$r^2 = z\bar{z}, \qquad \mu = \nu e^{-i\theta}.$$

Since the function constructed above is

$$\xi(r,\theta,\mu) = \mu e^{i\theta} e^{g(r^2)-2F(r,\mu)},$$

we obtain

$$\xi(z,\nu) = \nu e^{G(z,\nu)},$$

where we define

$$G(z,\nu) := g(z\bar{z}) - 2\mathcal{F}(z,\nu),$$

and

$$\mathcal{F}(z,\nu) := F(r,\mu), \qquad z = re^{i\theta}, \quad \nu = \mu e^{i\theta}.$$

Although the expression

$$\mu = \nu e^{-i\theta}$$

is written in polar coordinates, $\mathcal{F}$ extends smoothly across $z = 0$. Indeed, by the symmetry properties of $h(r,\alpha)$ and the Cauchy representation of F, the function F contains only even positive powers of μ, and its Fourier coefficients have the corresponding vanishing order in r. More precisely, locally one may write

$$F(r,\mu) = \sum_{k\geq 1} r^{2k} F_k(r^2)\mu^{2k},$$

with F_k smooth. Therefore,

$$\mathcal{F}(z,\nu) = \sum_{k\geq 1} F_k(z\bar{z})(\bar{z}\nu)^{2k},$$

which is smooth in (z, ν) down to $z = 0$. Consequently,

$$G \in C^\infty(D_R \times \mathbb{D}),$$

and the second component of the β-map takes the form

$$\boxed{\xi(z, \nu) = \nu e^{G(z,\nu)}}.$$

2.6 Construction of w

Notice that the function ξ vanishes simply at the zero section $\{\mu = 0\}$. The Clairaut first integral $c = r \sin \alpha = \frac{r}{2i}(e^{i\alpha} - e^{-i\alpha})$ naturally extends to a meromorphic function on Z with a simple pole at the zero section, namely we write $c = \frac{r}{2i}(\mu - 1/\mu)$ satisfying $\widetilde{\Xi} c = \partial_{\bar{\mu}} c = 0$ for $\mu \neq 0$. Then, with ξ defined in the previous section, one may define

$$w = -2ic\xi = r(1 - \mu^2) e^{g(r^2)} e^{i\theta} e^{F(r,\mu)}, \tag{75}$$

holomorphic on Z down to the zero section, and satisfying $\widetilde{\Xi} w = 0$ by the product rule. Moreover, we indeed observe that, using that $F(r, 0) = 0$ for all r,

$$w(r, \theta, 0) = r e^{g(r^2)} e^{i\theta},$$

for the (r, θ) coordinate. Based on the method in Section 2.5.1 to construct ξ, we can have that $w(z, \nu) = (z - \bar{z}\nu^2) e^G$.

2.7 Proofs of Blow-Down Structure - Theorems 2.1 and 2.2

Before proving Theorems 2.1 and 2.2, we first recall the definition of *holomorphic blow-down structure*, originally defined in [2] as a way to make conditions (1)-(2) of Section 1.6 open, and hence amenable to perturbation theory. First, to make (1) quantitative, one must account for a systematic degeneracy at the glancing $\partial_0 SM$, which is embodied by the C^∞_α-structure on $\partial_+ SM$ defined in [2, Section 2.3]. Since checking condition (a) below boils down to an explicit computation outlined in the proof of Theorem 2.1, we skip the specific definition. Second, as points of $Z\backslash SM$ approach SM, the blow-down behavior of a holomorphic map $\beta\colon Z \to \mathbb{C}^2$ satisfying (1)-(2) can be captured using a special Hermitian metric $\underline{\Omega}$ on Z, and the standard $\Omega_{\mathbb{C}^2} := i(\mathrm{d}w \wedge \mathrm{d}\bar{w} + \mathrm{d}\xi \wedge \mathrm{d}\bar{\xi})$ on $\mathbb{C}^2_{w,\xi}$. We define $\underline{\Omega}$ in terms of the coordinates (z, v) in (56) (see also [2, Sec. 2.4.1] for a coordinate invariant description): write $\mathscr{D} = \mathbb{C}\partial_{\bar{v}} \oplus \mathbb{C}\Xi$ with Ξ defined in (57), then

$$\underline{\Omega} := i(\eta_1 \wedge \bar{\eta}_1 + \eta_2 \wedge \bar{\eta}_2), \tag{76}$$

where η_1, η_2 are the unique one-forms on Z satisfying

$$\begin{aligned} \eta_1(\partial_v) = \eta_1(\partial_{\bar{v}}) = \eta_1(\Xi) = 0, \qquad & \eta_1(\overline{\Xi}) = 1 - |v|^4, \\ \eta_2(\overline{\Xi}) = \eta_2(\partial_{\bar{v}}) = \eta_2(\Xi) = 0, \qquad & \eta_2(\partial_v) = 1. \end{aligned} \tag{77}$$

Definition 2.1. *A smooth map $\beta\colon Z \to \mathbb{C}^2$ has* **holomorphic blow-down structure** *if it has the following properties:*

(a) The restriction $\beta|_{\partial_+ SM} : \partial_+ SM \to \mathbb{C}^2$ is a totally real $C^\infty_\alpha-$embedding.

(b) The restriction $\beta|_{Z^\circ}$ is a biholomorphism onto its image.

(c) Writing $\beta^*\Omega_{\mathbb{C}^2} = i\sum_{j,k=1}^2 H_{jk}\eta_j \wedge \bar{\eta}_k$ for some Hermitian matrix field $H =$

$\{H_{jk}\}_{j,k=1}^2$, *there exists* $c > 0$ *such that* $H - cId$ *is positive semidefinite*[1].

Proof of Theorem 2.1. **Proof of (a).** Following [4], in order to show the totally real C_α^∞−embedding, we need first show that $\beta|_{\partial_+ SM}$ is injective. Evaluating β at $(z, v) = (Re^{i\theta}, e^{i(\theta+\pi+\alpha)})$ with $\alpha \in [\frac{-\pi}{2}, \frac{\pi}{2}]$, we write

$$(w, \xi)(Re^{i\theta}, e^{i(\theta+\pi+\alpha)}) = ((1 - e^{2i\alpha})Re^{i\theta}e^{\frac{\kappa R^2(1-e^{2i\alpha})}{2}}, -e^{i\alpha}e^{i\theta}e^{\frac{\kappa R^2(1-e^{2i\alpha})}{2}}). \tag{78}$$

Notice that $w/\xi = 2iR\sin\alpha$ uniquely determines $\alpha \in [-\pi/2, \pi/2]$, after which $e^{i\theta}$ is uniquely determined by either w or ξ. By [4, Lemma 4.1], the C_α^∞-embedding property boils down to verifying that the rescaled Jacobian

$$\frac{1}{\cos^2(\alpha)}\left|\frac{\partial w}{\partial\theta}\frac{\partial\xi}{\partial\alpha} - \frac{\partial w}{\partial\alpha}\frac{\partial\xi}{\partial\theta}\right|^2$$

is nowhere-vanishing. We thus compute

$$\begin{bmatrix} \frac{\partial w}{\partial\theta} & \frac{\partial w}{\partial\alpha} \\ \frac{\partial\xi}{\partial\theta} & \frac{\partial\xi}{\partial\alpha} \end{bmatrix} = ie^{i\theta}e^{\frac{\kappa R^2}{2}(1-e^{2i\alpha})}\begin{bmatrix} R(1 - e^{2i\alpha}) & Re^{2i\alpha}(-2 - \kappa R^2 + \kappa R^2 e^{2i\alpha}) \\ -e^{i\alpha} & -e^{i\alpha}(1 - \kappa R^2 e^{2i\alpha}) \end{bmatrix}.$$

Substituting, we find:

$$\frac{1}{\cos^2(\alpha)}\left|\frac{\partial w}{\partial\theta}\frac{\partial\xi}{\partial\alpha} - \frac{\partial w}{\partial\alpha}\frac{\partial\xi}{\partial\theta}\right|^2 = 4R^2|e^{\kappa R^2(1-e^{2i\alpha})}| \neq 0.$$

Proof of (b). By [4], it suffices to prove that the Cauchy-Riemann equations are satisfied and that β is bijective onto its image. By the inverse function theorem, injectivity and holomorphicity will show that the inverse (defined on $\beta(\mathbb{D}_R \times \mathbb{D}^\circ)$) will be automatically holomorphic. The computation that $\mathrm{d}w(\mathscr{D}) = \mathrm{d}\xi(\mathscr{D}) = 0$ is a direct

[1] In [4] such a condition is written concisely as $\beta^*\Omega_{\mathbb{C}^2} \geq c\underline{\Omega}$.

calculation best done in the polar chart (54), where

$$w = re^{i\theta}(1-\mu^2)e^{\kappa r^2(1-\mu^2)/2}, \qquad \xi = \mu e^{i\theta}e^{\kappa r^2(1-\mu^2)/2}, \tag{79}$$

and $\mathscr{D}$ is given in (55).

Next, we check injectivity on Z°, by recovering *at most* one pair $(z,\nu) \in \mathbb{D}_R \times \mathbb{D}^\circ$ from the relations

$$w = (z - \nu^2\bar{z})e^{\kappa(z\bar{z}-\nu^2\bar{z}^2)/2}, \qquad \xi = \nu e^{\kappa(z\bar{z}-\nu^2\bar{z}^2)/2}.$$

First observe that if $\xi = 0$, then necessarily $\nu = 0$ and we are left with the relation $w = ze^{\kappa z\bar{z}/2}$. Writing $u := |z|^2$, we derive $|w|^2 = ue^{\kappa u} =: f(u)$. Noticing that $f'(u) = (1+\kappa u)e^{\kappa u}$, the hypothesis $1+\kappa R^2 > 0$ implies that $f'(u) > 0$ on $[0, R^2]$. Hence equation $f(u) = |w|^2$ determines a unique $u = |z|^2$, then z is determined via $z = we^{-\kappa|z|^2}$.

Now suppose $\xi \neq 0$. Then $\nu \neq 0$, and we can define the quantity

$$c := \frac{w}{\xi} = \frac{z - \nu^2\bar{z}}{\nu} = r\frac{1-\mu^2}{\mu}. \tag{80}$$

With c known from data, the above relation helps recover z when ν is determined, via

$$z = \frac{\nu}{1-|\nu|^4}(c + \bar{c}|\nu|^2). \tag{81}$$

Substituting (81) into ξ and $|\xi|^2$, we obtain

$$\xi = \nu e^{\frac{\kappa|\nu|^2}{2(1-|\nu|^4)}(|c|^2+|\nu|^2c^2)}, \tag{82}$$

$$|\xi|^2 = h_c(|\nu|^2), \qquad h_c(u) := ue^{\frac{\kappa u}{1-u^2}(|c|^2+u\mathrm{Re}(c^2))}. \tag{83}$$

Writing $c = c_R + ic_I$ with $c_R, c_I \in \mathbb{R}$, notice that $h_c(u) = ue^{\kappa u\left(\frac{c_R^2}{1-u}+\frac{c_I^2}{1+u}\right)}$. If equation (83) determines a unique $|\nu|^2 \in [0,1)$, then ν is uniquely determined from (82). Hence we now study the solvability of (83). Differentiating, we find

$$h_c'(u) = \left(1 + \kappa u\left(\frac{c_R^2}{(1-u)^2} + \frac{c_I^2}{(1+u)^2}\right)\right)\exp\left(\kappa u\left(\frac{c_R^2}{1-u} + \frac{c_I^2}{1+u}\right)\right). \tag{84}$$

In addition, observe that for any solution $u = |\nu|^2$ of (83), the reconstructed z from (81) satisfies

$$z\bar{z} = u\left(\frac{c_R^2}{(1-u)^2} + \frac{c_I^2}{(1+u)^2}\right)$$

Comparing to (84), we thus find that $\operatorname{sgn}(h_c'(u)) = \operatorname{sgn}(1+\kappa z\bar{z})$. Since we seek solutions satisfying $|z| \le R$ with $\kappa R^2 > -1$, we must have $1 + \kappa z\bar{z} > 0$, and thus any admissible solution u to (83) must also satisfy $h_c'(u) > 0$. We will conclude by showing that the region $\{h_c' > 0\}$ consists of at most a contiguous interval of the form $[0, u_c) \subseteq [0,1)$, and where h_c is injective so that (83) can have at most one admissible solution. To prove this last statement: if $\kappa \ge 0$, it's immediately clear from (84) that $h_c'(u) > 0$ on $[0,1)$; if $\kappa < 0$, notice that for all $u \in [0,1)$

$$\frac{\mathrm{d}}{\mathrm{d}u}\left(1 + \kappa u\left(\frac{c_R^2}{(1-u)^2} + \frac{c_I^2}{(1+u)^2}\right)\right) = \kappa\left(c_R^2\frac{1+u}{(1-u)^3} + c_I^2\frac{1-u}{(1+u)^3}\right).$$

If $c = 0$, then $h_c(u) = u$ is clearly injective. If $c \neq 0$, then the right side in the above display is negative on $[0,1)$, and since $h_c'(0) = 1 > 0$, the sign of $h_c'(u)$ can only go from positive to negative once as u increases. The conclusion is reached.

Proof of (c). If $(\Xi^\vee, \overline{\Xi}^\vee, \partial_\nu^\vee, \partial_{\bar{\nu}}^\vee)$ are the dual one-forms to $(\Xi, \overline{\Xi}, \partial_\nu, \partial_{\bar{\nu}})$ on Z°, we have

$$\beta^* \mathrm{d}f = \overline{\Xi} f \, \overline{\Xi}^\vee + \cancel{\Xi f} \, \Xi^\vee + \partial_\nu f \, \partial_\nu^\vee + \cancel{\partial_{\bar\nu} f} \, \partial_{\bar\nu}^\vee = \frac{\overline{\Xi} f}{1 - |\nu|^4} \, \eta_1 + \partial_\nu f \, \eta_2, \qquad f \in \{w, \xi\},$$

where η_1, η_2 are defined in (77). Denoting $A := e^{\kappa(z\bar z - \bar z^2 \nu^2)/2}$, a direct computation gives

$$\beta^* \mathrm{d}w = A(a\eta_1 + b\eta_2), \qquad \beta^* \mathrm{d}\xi = A(c\eta_1 + d\eta_2), \quad \text{where}$$
$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} := \begin{bmatrix} 2(1 + \kappa \bar z z - \kappa \nu^2 \bar z^2) & -2\nu \bar z - \kappa \nu \bar z^2 z + \kappa \nu^3 \bar z^3 \\ 2\kappa \bar z \nu & 1 - \kappa \bar z^2 \nu^2 \end{bmatrix}. \tag{85}$$

Thus the matrix H from Definition 2.1 has components

$$H_{11} = |A|^2 (|a|^2 + |c|^2), \quad H_{12} = \overline{H}_{21} = |A|^2 (a\bar b + c\bar d), \quad H_{22} = |A|^2 (|d|^2 + |b|^2).$$

To estimate the minimal eigenvalue $\lambda_{\min}$ of H, following [4, Theorem 4.1],

$$\lambda_{\min} \geq \frac{\det(H)}{\operatorname{tr}(H)} = |A|^2 \frac{|ad - bc|^2}{|a|^2 + |b|^2 + |c|^2 + |d|^2}.$$

When $1 + \kappa R^2 > 0$, we will have that $|A|^2 \geq e^{-2|\kappa| R^2}$. Given that $\det(H) = 2 + 2\kappa z \bar z$, then $\det(H) \geq 2 - 2|\kappa| R^2$, whenever $\kappa R^2 \in (-1, 1)$. When $\kappa R^2 \geq 1$, we have $2 + 2\kappa z \bar z \geq 2$. In either case, $\det(H)$ is bounded below by some positive constant. To bound $\operatorname{tr}(H)$ from above, we find uniform bounds:

$$|a| \leq 2(1 + 2|\kappa| R^2), \quad |b| \leq 2R(1 + |\kappa| R^2), \quad |c| \leq 2|\kappa| R, \quad |d| \leq 1 + |\kappa| R^2.$$

Combining these estimates gives a uniform positive lower bound for $\lambda_{\min}$, and fulfills (c). □

We now prove Theorem 2.2 along the same lines, modifying the arguments to account for generalizations.

Proof of Theorem 2.2. (1) **Separation of geodesics and C^∞_α embedding**. In order to prove the totally real C^∞_α-embedding, we need first show that $\beta|_{\partial_+ SM}$ is injective. Evaluating β at $(z, v) = (Re^{i\theta}, e^{i(\theta+\pi+\alpha)})$ when $\alpha \in [\frac{-\pi}{2}, \frac{\pi}{2}]$, we write

$$(w, \xi)(Re^{i\theta}, e^{i(\theta+\pi+\alpha)}) = ((1 - e^{2i\alpha})Re^{i\theta}e^{G(e^{i\alpha})}, -e^{i\alpha}e^{i\theta}e^{G(e^{i\alpha})}). \tag{86}$$

Since G satisfies $\partial_{\bar{\mu}} G = 0$, it is holomorphic with respect to the variable μ. Restricting to $\partial_+ SM$, where $r = R$ is fixed and $\mu = e^{i\alpha}$, we may regard

$$G(R, e^{i\alpha})$$

simply as a holomorphic function of the boundary variable $e^{i\alpha}$, and write it as $G(e^{i\alpha})$. Noticing that $\frac{w}{\xi} = 2iR\sin\alpha$ uniquely determines $\alpha \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, after which $e^{i\theta}$ is uniquely determined by either w or ξ. By [4, Lemma 4.1], the C^∞_α−embedding property can be written as to verify that the rescaled Jacobian

$$\frac{1}{\cos^2(\alpha)} \left| \frac{\partial w}{\partial \theta}\frac{\partial \xi}{\partial \alpha} - \frac{\partial w}{\partial \alpha}\frac{\partial \xi}{\partial \theta} \right|^2 \tag{87}$$

is nowhere vanishing. We can compute that

$$J = \begin{pmatrix} \dfrac{\partial w}{\partial \alpha} & \dfrac{\partial w}{\partial \theta} \\ \dfrac{\partial \xi}{\partial \alpha} & \dfrac{\partial \xi}{\partial \theta} \end{pmatrix} = e^{i\theta}e^{G(e^{i\alpha})} \begin{pmatrix} R\left[-2ie^{2i\alpha} + (1 - e^{2i\alpha})\dfrac{\partial}{\partial \alpha}G(e^{i\alpha})\right] & iR(1 - e^{2i\alpha}) \\ -e^{i\alpha}\left[i + \dfrac{\partial}{\partial \alpha}G(e^{i\alpha})\right] & -ie^{i\alpha} \end{pmatrix}.$$

Then, we will have that

$$\frac{1}{\cos^2\alpha}|J|^2 = 4R^2|e^{2G}|^2 \neq 0. \tag{88}$$

(2) **Biholomorphicity.** That the map β is holomorphic relies on its very construction. If it's invertible, then its inverse is holomorphic, so it suffice to prove it's injective. We do this by recovering (z,ν) from

$$w = (z - \nu^2\bar{z})e^G, \qquad \xi = \nu e^G. \tag{89}$$

If $\xi = 0$, then $\nu = 0$ and we thus need to reconstruct z from $w(z,0) = ze^{g(|z|^2)}$ which is the holomorphic coordinate. In this case, $|w|^2 = |z|^2e^{2g(|z|^2)}$, and thus

$$\frac{\partial|w|^2}{\partial|z|^2} = (1 + 2|z|^2g'(|z|^2))e^{2g(|z|^2)} = ae^{2g(|z|^2)} > 0,$$

thus $|w|^2$ uniquely determines $|z|^2$, and $z = we^{-g(|z|^2)}$ is uniquely determined.

If $\xi \neq 0$, then $\nu \neq 0$, and we let $C(z,\nu) := \frac{z}{\nu} - \nu\bar{z}$. The surface

$$S_c = \{(z,\nu) \in \mathbb{D}_R^\circ \times \mathbb{D}^\circ, \quad C(z,\nu) = c\}$$

is a graph over $\mathbb{D}_\nu^\circ$ given by

$$z = z_c(\nu) = \frac{1}{1 - |\nu|^4}(c\nu + \nu^2\bar{c}\bar{\nu}). \tag{90}$$

In particular if c and ν are known, then z is uniquely determined.

Notably, we also have $\frac{w}{\xi} = c(z,\nu)$, hence c is known, and it will remain to show that the map $\nu \mapsto \xi(z_c(\nu),\nu)$ is injective. This is done by first noticing that $\xi(z_c(\nu),\nu) = \nu e^{H_c(|\nu|^2)}$ for some function $H_c\colon \mathbb{R} \to \mathbb{C}$. We then show that $|\xi(z_c(\nu),\nu)|^2 = |\nu|^2 e^{2\mathrm{Re}(H_c(|\nu|^2))}$ determines $|\nu|^2$ in particular by noticing the key equality

$$\frac{\partial}{\partial|\nu|^2}\left(|\xi|^2(z_c(\nu),\nu)\right) = a(|z_c(\nu)|^2)\frac{e^{G+\overline{G}}}{2|\nu|^2} > 0, \qquad \frac{\partial}{\partial|\nu|^2} := \frac{1}{2|\nu|^2}(\nu\partial_\nu + \bar{\nu}\partial_{\bar{\nu}}), \tag{91}$$

hence $|\nu|^2$ is uniquely determined. Finally, we recover ν from the relation $\nu = \xi(z_c(\nu,\nu))e^{-H_c(|\nu|^2)}$.

It thus remains to prove (91). Writing $\xi = \nu e^G$, when $\nu \neq 0$, the relations $\Xi\xi = \partial_{\bar{\nu}}\xi = 0$ and $D_\theta\xi = \xi$ become

$$\Xi G = (z - \nu^2\bar{z})\tilde{a}, \qquad \partial_{\bar{\nu}}G = 0, \qquad D_\theta G = 0. \tag{92}$$

In particular, we obtain

$$2(\partial_{\bar{z}} + \nu^2\partial_z)G{=}\Xi G = (z - \nu^2\bar{z})\tilde{a}. \tag{93}$$

On to the computation of the left side of (91), by chain rule we first compute the real vector field Y such that

$$(\nu\partial_\nu + \bar{\nu}\partial_{\bar{\nu}})\left(|\xi|^2(z_c(\nu),\nu)\right) = (Y|\xi|^2)(z_c(\nu),\nu).$$

To do this, fix c and consider z as an implicit functions of ν via (90). Differentiating with respect to ν and $\bar{\nu}$ the equation $\frac{z}{\nu} - \nu\bar{z} = c$, we find

$$\begin{aligned}(\nu\partial_\nu z) - \nu^2(\nu\partial_\nu\bar{z}) &= z + \nu^2\bar{z}\\ -\bar{\nu}^2(\nu\partial_\nu z) + (\nu\partial_\nu\bar{z}) &= 0,\end{aligned}$$

hence inverting this 2×2 system for $\nu\partial_\nu z$ and $\nu\partial_\nu \bar{z}$, we find

$$\nu\partial_\nu z = \frac{z + \nu^2\bar{z}}{1 - |\nu|^4}, \qquad \nu\partial_\nu \bar{z} = \frac{\bar{\nu}^2}{1 - |\nu|^4}(z + \nu^2\bar{z}),$$

and $\bar{\nu}\partial_{\bar{\nu}} z$, $\bar{\nu}\partial_{\bar{\nu}}\bar{z}$ can be derived by complex conjugation. Using this, we find that

$$\begin{aligned} Y &:= \nu\partial_\nu + \bar{\nu}\partial_{\bar{\nu}} \\ &= \frac{((1 + |\nu|^4)z + 2\nu^2\bar{z})}{1 - |\nu|^4}\partial_z \\ &\quad + \frac{((1 + |\nu|^4)\bar{z} + 2\bar{\nu}^2 z)}{1 - |\nu|^4}\partial_{\bar{z}} + \nu\partial_\nu + \bar{\nu}\partial_{\bar{\nu}}. \end{aligned}$$

Finally, we compute

$$2|\nu|^2\frac{\partial}{\partial|\nu|^2}\left(|\xi|^2(z(c,\nu),\nu)\right) = \left(1 + \frac{1}{2}(YG + Y\overline{G})\right)e^{G+\overline{G}} \tag{94}$$

$$\overset{(Y=\overline{Y})}{=} \left(1 + \frac{1}{2}(YG + \overline{YG})\right)e^{G+\overline{G}}. \tag{95}$$

Writing $D_\theta G = 0$ as $\nu\partial_\nu G = \bar{z}\partial_{\bar{z}}G - z\partial_z G$ and using $\partial_{\bar{\nu}}G = 0$, we find that

$$YG = \frac{2(\bar{z} + \nu^2 z)}{1 - |\nu|^4}(\partial_{\bar{z}} + \nu^2\partial_z)G \overset{(93)}{=} |z|^2\tilde{a} + \frac{\bar{\nu}^2 z^2 - \nu^2\bar{z}^2}{1 - |\nu|^4}\tilde{a}.$$

In particular, we find that

$$1 + \frac{1}{2}(YG + Y\overline{G}) = 1 + \mathrm{Re}(YG) = 1 + |z|^2\tilde{a} = a(|z(c,\nu)|^2). \tag{96}$$

Combining (95)-(96) yields (91).

(3) **Desingularized Jacobian estimate.** If $(\Xi^\vee, \bar{\Xi}^\vee, \partial_\nu^\vee, \partial_{\bar{\nu}}^\vee)$ are the dual one-

forms to $(\Xi, \bar{\Xi}, \partial_\nu, \partial_{\bar{\nu}})$ on Z°, we have

$$\beta^* df = \Xi f\, \Xi^\vee + \bar{\Xi} f\, \bar{\Xi}^\vee + \partial_\nu f\, \partial_\nu^\vee + \partial_{\bar{\nu}} f\, \partial_{\bar{\nu}}^\vee = \frac{\bar{\Xi} f}{1 - |\nu|^4} \eta_1 + \partial_\nu f\, \eta_2, \qquad f \in \{w, \xi\},$$

where η_1, η_2 are defined as above.

We write

$$\Xi = \alpha \partial_z + \beta \partial_{\bar{z}} + \gamma \left(\bar{\nu} \partial_{\bar{\nu}} - \nu \partial_\nu \right),$$

where

$$\alpha := -\widetilde{a} z^2 + \nu^2 (1 + a), \qquad \beta := 1 + a - \widetilde{a} \bar{z}^2 \nu^2, \qquad \gamma := (z - \nu^2 \bar{z}) \widetilde{a}.$$

Then, for f holomorphic, we obtain the relation

$$\beta \bar{\Xi} f = \left(|\beta|^2 - |\alpha|^2 \right) \partial_z f + \left(\bar{\alpha} \gamma + \beta \bar{\gamma} \right) \nu \partial_\nu f.$$

By direct calculation,

$$|\beta|^2 - |\alpha|^2 = 4a(1 - |\nu|^4),$$

and

$$\bar{\alpha}\gamma + \beta\bar{\gamma} = 2\widetilde{a}\bar{z}(1 - |\nu|^4).$$

Hence

$$\frac{\bar{\Xi} f}{1 - |\nu|^4} = \frac{4a}{\beta} \partial_z f + \frac{2\widetilde{a} \nu \bar{z}}{\beta} \partial_\nu f, \qquad f \in \{w, \xi\}. \tag{97}$$

Recall that

$$w = (z - \bar{z}\nu^2) e^G, \qquad \xi = \nu e^G.$$

Writing

$$A = e^G,$$

we have

$$w = (z - \bar{z}\nu^2)A, \qquad \xi = \nu A.$$

Therefore,

$$\partial_z w = A\left(1 + (z - \bar{z}\nu^2)\partial_z G\right), \tag{98}$$

$$\partial_\nu w = A\left(-2\bar{z}\nu + (z - \bar{z}\nu^2)\partial_\nu G\right), \tag{99}$$

$$\partial_z \xi = A\nu\partial_z G, \tag{100}$$

$$\partial_\nu \xi = A\left(1 + \nu\partial_\nu G\right). \tag{101}$$

Substituting these identities into (97), we may write

$$\beta^* dw = A(h_{11}\eta_1 + h_{12}\eta_2), \qquad \beta^* d\xi = A(h_{21}\eta_1 + h_{22}\eta_2),$$

where

$$h_{11} = \frac{4a}{\beta}\left(1 + (z - \bar{z}\nu^2)\partial_z G\right) + \frac{2\widetilde{a}\nu\bar{z}}{\beta}\left(-2\bar{z}\nu + (z - \bar{z}\nu^2)\partial_\nu G\right), \tag{102}$$

$$h_{12} = -2\bar{z}\nu + (z - \bar{z}\nu^2)\partial_\nu G, \tag{103}$$

$$h_{21} = \frac{4a\nu}{\beta}\partial_z G + \frac{2\widetilde{a}\nu\bar{z}}{\beta}\left(1 + \nu\partial_\nu G\right), \tag{104}$$

$$h_{22} = 1 + \nu\partial_\nu G. \tag{105}$$

We first compute the determinant of the matrix

$$h = \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix}.$$

Using the expressions above, we obtain

$$\det h = \frac{4a}{\beta} \det \begin{pmatrix} 1 + (z - \bar{z}\nu^2)\partial_z G & -2\bar{z}\nu + (z - \bar{z}\nu^2)\partial_\nu G \\ \nu\partial_z G & 1 + \nu\partial_\nu G \end{pmatrix}. \tag{106}$$

Indeed, the terms containing $\widetilde{a}\nu\bar{z}/\beta$ in the first column are scalar multiples of the second column and therefore do not contribute to the determinant.

Expanding the determinant in (106) gives

$$\begin{aligned} &\left(1 + (z - \bar{z}\nu^2)\partial_z G\right)(1 + \nu\partial_\nu G) \\ &\qquad - \left(-2\bar{z}\nu + (z - \bar{z}\nu^2)\partial_\nu G\right)\nu\partial_z G \\ &= 1 + \nu\partial_\nu G + (z - \bar{z}\nu^2)\partial_z G + 2\bar{z}\nu^2\partial_z G \\ &= 1 + \nu\partial_\nu G + (z + \bar{z}\nu^2)\partial_z G. \end{aligned} \tag{107}$$

Since $D_\theta G = 0$, we have

$$\nu\partial_\nu G = \bar{z}\partial_{\bar{z}} G - z\partial_z G.$$

Consequently,

$$\begin{aligned} 1 + \nu\partial_\nu G + (z + \bar{z}\nu^2)\partial_z G &= 1 + \bar{z}\partial_{\bar{z}} G + \bar{z}\nu^2\partial_z G \\ &= 1 + \bar{z}\left(\partial_{\bar{z}} + \nu^2\partial_z\right)G. \end{aligned} \tag{108}$$

From the relation obtained above,

$$2\left(\partial_{\bar{z}} + \nu^2\partial_z\right) G = (z - \nu^2\bar{z})\widetilde{a},$$

we therefore obtain

$$\begin{aligned} 1 + \bar{z}\left(\partial_{\bar{z}} + \nu^2\partial_z\right) G &= 1 + \frac{\widetilde{a}}{2}\bar{z}(z - \nu^2\bar{z}) \\ &= 1 + \frac{\widetilde{a}}{2}\left(|z|^2 - \bar{z}^2\nu^2\right). \end{aligned} \tag{109}$$

Since

$$a = 1 + \widetilde{a}|z|^2,$$

we have

$$\beta = 1 + a - \widetilde{a}\bar{z}^2\nu^2 = 2 + \widetilde{a}\left(|z|^2 - \bar{z}^2\nu^2\right).$$

Hence

$$1 + \frac{\widetilde{a}}{2}\left(|z|^2 - \bar{z}^2\nu^2\right) = \frac{\beta}{2}.$$

Combining this identity with (106) gives

$$\boxed{\det h = 2a.} \tag{110}$$

We now compute the pullback of the standard Hermitian form. Recall that

$$\Omega_{\mathbb{C}^2} = i\left(dw \wedge d\bar{w} + d\xi \wedge d\bar{\xi}\right).$$

Therefore,

$$\beta^*\Omega_{\mathbb{C}^2} = i\sum_{j,k=1}^{2} H_{jk}\eta_j \wedge \bar{\eta}_k,$$

where

$$H_{11} = |A|^2 \left(|h_{11}|^2 + |h_{21}|^2\right), \tag{111}$$

$$H_{12} = |A|^2 \left(h_{11}\overline{h_{12}} + h_{21}\overline{h_{22}}\right), \tag{112}$$

$$H_{21} = |A|^2 \left(h_{12}\overline{h_{11}} + h_{22}\overline{h_{21}}\right) = \overline{H_{12}}, \tag{113}$$

$$H_{22} = |A|^2 \left(|h_{12}|^2 + |h_{22}|^2\right). \tag{114}$$

Thus $H = (H_{jk})$ is a positive Hermitian matrix. Moreover, using (110),

$$\begin{aligned} \det H &= |A|^4 |\det h|^2 \\ &= 4e^{2(G+\bar{G})} a^2. \end{aligned} \tag{115}$$

It remains to show that H is uniformly positive definite. For this purpose, we first obtain a uniform lower bound for β. Since

$$\beta = 1 + a - \widetilde{a}\bar{z}^2 \nu^2$$

and

$$\widetilde{a}|z|^2 = a - 1,$$

the reverse triangle inequality gives

$$\begin{aligned} |\beta| &\geq 1 + a - |\widetilde{a}|\, |z|^2 |\nu|^2 \\ &= 1 + a - |a-1||\nu|^2. \end{aligned} \tag{116}$$

If $a \geq 1$, then

$$|\beta| \geq 1 + a - (a-1)|\nu|^2 \geq 2.$$

If $0 < a < 1$, then

$$|\beta| \geq 1 + a - (1-a)|\nu|^2 \geq 2a.$$

Therefore,

$$|\beta| \geq 2\min\{1, a\}. \tag{117}$$

Since a is positive and smooth on the compact disk $\overline{D}_R$, there exists

$$a_0 := \min_{\overline{D}_R} a > 0.$$

Hence

$$|\beta| \geq 2\min\{1, a_0\} > 0 \tag{118}$$

uniformly on Z.

Since G is smooth on the compact set $\overline{D}_R \times \overline{\mathbb{D}}$, the functions

$$G, \qquad \partial_z G, \qquad \partial_\nu G$$

are uniformly bounded. Together with (118) and the explicit expressions (102)–(105), this implies that all four coefficients h_{ij} are uniformly bounded. Consequently, there exists a constant $C_1 > 0$ such that

$$\operatorname{tr} H = |A|^2 \left(|h_{11}|^2 + |h_{12}|^2 + |h_{21}|^2 + |h_{22}|^2\right) \leq C_1. \tag{119}$$

On the other hand, since G is smooth on a compact set, there exists a constant $C_0 > 0$ such that

$$e^{G+\bar{G}} \geq C_0.$$

Together with $a \geq a_0 > 0$, equation (115) yields

$$\det H = 4e^{2(G+\bar{G})}a^2 \geq 4C_0^2 a_0^2 > 0. \tag{120}$$

Let $\lambda_{\min}(H)$ and $\lambda_{\max}(H)$ denote the two eigenvalues of H. Since H is positive Hermitian,

$$\det H = \lambda_{\min}(H)\lambda_{\max}(H),$$

and

$$\lambda_{\max}(H) \leq \operatorname{tr} H.$$

It follows from (119) and (120) that

$$\begin{aligned} \lambda_{\min}(H) &= \frac{\det H}{\lambda_{\max}(H)} \\ &\geq \frac{\det H}{\operatorname{tr} H} \\ &\geq \frac{4C_0^2 a_0^2}{C_1} =: c > 0. \end{aligned} \tag{121}$$

Therefore,

$$H \geq c \,\mathrm{Id},$$

or equivalently,

$$H - c \,\mathrm{Id}$$

is positive semidefinite on Z. This proves the desingularized Jacobian estimate and hence condition (3) of the holomorphic blow-down structure. □

3 Future Work

We have proved that, when the metric function $a(r)$ is smooth, one can construct the associated β-map on the transport twistor space and show that it admits a blow-down structure. If instead

$$a(r) \in C^k([0, R]),$$

for $k \in \mathbb{N}$, the β-map can still be constructed in the same form as in the C^∞ case. The main difficulty is that the regularity of the resulting β-map is not immediately clear, since it depends on the regularity of the corresponding Cauchy transform. Such regularity is essential for determining whether the β-map admits a blow-down structure.

Another natural direction for future work is to determine whether the interior of the transport twistor space is a Stein manifold. The Stein property is important because it guarantees the existence of sufficiently many global holomorphic functions and provides strong analytic tools, such as holomorphic convexity, the vanishing of higher coherent sheaf cohomology, and the Oka–Grauert principle [9]. In the setting of transport twistor spaces, such properties may be useful for studying global holomorphic objects and, in particular, for constructing holomorphic integrating factors associated with transport equations.

In the Euclidean case, the Stein property can be established by using the global holomorphic coordinates

$$w = z - \bar{z}v^2, \qquad \xi = v.$$

These coordinates immediately give holomorphic separability. To prove holomorphic convexity (see the proof the detail in [5]), one constructs, for each boundary point p, a holomorphic function f_p that vanishes at p but has no zeros in the interior. The construction depends on the boundary hypersurface containing p. If $p \in SM$, one may use a function of the form

$$f_p = \xi - \xi(p).$$

If $p \in \partial M \times \mathbb{D}$, one may instead use

$$f_p = w - z_p + \overline{z_p}\,\xi^2,$$

where $z_p \in \partial M$ is the base point of p. Since f_p does not vanish in the interior, $1/f_p$ is holomorphic there and becomes unbounded as one approaches p. This prevents the holomorphic hull of a compact set from escaping through the boundary and therefore yields holomorphic convexity.

For the non-Euclidean transport twistor spaces considered in this dissertation, the blow-down map takes the form

$$w = (z - \overline{z}\,\nu^2)e^F, \qquad \xi = \nu e^F.$$

Although these functions provide global holomorphic coordinates in the interior, the additional factor e^F couples the base and fiber variables and makes the geometry of the boundary image substantially more complicated. Consequently, the elementary barrier functions used in the Euclidean case no longer have an obvious analogue: it is not clear how to construct a holomorphic function that vanishes at a prescribed boundary point while remaining nonvanishing throughout the interior. Constructing such boundary functions, or alternatively finding a strictly plurisubharmonic exhaustion function, remains an interesting problem for future work.

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