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ISOMETRIES OF THE TRACE CLASS

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Let $\mathfrak{I}$ denote the Banach space of trace class operators on a complex Hilbert space H, in the norm $\|T\|_1 = \mathrm{Tr}(|T|)$. The space $\mathfrak{I}$ is a two-sided ideal in the algebra $\mathfrak{L}$ of all bounded operators on H. See [4].

THEOREM. *If Φ is a linear isometry of the Banach space $\mathfrak{I}$ onto itself, then there exists a *-automorphism or a *-antiautomorphism α of $\mathfrak{L}$ and a unitary operator U in $\mathfrak{L}$ such that* $\Phi(T) = \alpha(TU)$, (T in $\mathfrak{I}$).

REMARK 1. The theorem provides a partial answer to [3, Remark 1, p. 231].

PROOF. The adjoint Φ' is a linear isometry of $\mathfrak{L}$ onto $\mathfrak{L}$ so by results of Kadison [2, Theorem 7, Corollary 11] has the form $\Phi'(A) = U\alpha(A)$ where α and U are as described in the statement of the theorem. It is elementary that $\Phi(T) = \Psi(TU)$ where $\Psi' = \alpha$. The proof will be complete if it is shown that α is the adjoint of α^{-1} (restricted to $\mathfrak{I}$). By the folk result [1, pp. 256, 9] it is sufficient to check this in the following two cases:

(i) $\alpha(A) = VAV^{-1}$ with V a fixed unitary operator; then $\langle T, \alpha(A)\rangle = \langle T, VAV^{-1}\rangle = \langle V^{-1}TV, A\rangle = \langle \alpha^{-1}(T), A\rangle$,

(ii) after the choice of an orthonormal basis, $\alpha(A)$ is the transposed matrix of A; then $\langle T, \alpha(A)\rangle = \mathrm{Tr}(T\alpha(A)) = \mathrm{Tr}(\alpha(T)A) = \langle \alpha^{-1}(T), A\rangle$.

REMARK 2. A previous version of the above proof exploited a knowledge of the extreme points of the unit sphere of $\mathfrak{I}$. These were determined to be the partial isometries with initial (hence final) domain one-dimensional.

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