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Eigenvalue Estimate, Minimal Hypersurfaces and Isoperimetric Inequalities

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Eigenvalue Estimates, Minimal Hypersurfaces and Isoperimetric Inequalities

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by

Yucheng Tu

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PUBLICATIONS

Tu, Yucheng. ”On the Lower Bound of the Principal Eigenvalue of a Nonlinear Operator.” arXiv preprint arXiv:2008.00185 (2020). *Submitted.*

Li, Xiaolong; Tu, Yucheng; and Wang, Kui. ”On a class of quasilinear operators on smooth metric measure spaces.” arXiv preprint arXiv:2009.10418 (2020). *Submitted.*

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ABSTRACT OF THE DISSERTATION

Eigenvalue Estimates, Minimal Hypersurfaces and Isoperimetric Inequalities

by

Yucheng Tu

Doctor of Philosophy in Mathematics

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Professor Lei Ni, Chair

In this thesis we study three problems in the field of geometric analysis: eigenvalue estimate of non-linear operators, existence of minimal surfaces and isoperimetric problems. These problems are more or less related to the topic of geometric calculus of variations, which is the study of extreme points of functionals defined on manifolds.

The first part is devoted to the study of lower bound of the principal eigenvalue of a family of non-linear elliptic operator L_p. Using the gradient and maximum comparison technique developed in [9] together with ideas from [14], we proved that on a compact metric measure space(possibly with convex boundary) (M, g, m) with curvature-dimension condition $BE(\kappa, N)(\kappa < 0)$, if L is a elliptic diffusion operator whose invariant measure is m, then the principal eigenvalue of L_p is bounded below by the first eigenvalue of a one-dimensional ODE with Neumann boundary condition. We showed that this is sharp result by constructing an example of metric measure space M on which the eigenvalue problem of L_p degenerates into the model equation problem. This work extends the

$\kappa = 0$ case proved in [9].

The second part is devoted to the study of existence of free boundary minimal hypersurfaces in compact manifolds, from a min-max theoretical point of view. Following the ideas from [1] and [19], we prove that in a simply connected compact manifold $(M, \partial M, g)$ under certain conditions) with its metric that is locally maximising the width of M, there is a sequence of equidistributed free boundary minimal hypersurfaces.

The third part is devoted to the study of anisotropic isoperimetric inequality for regions outside of a ball in $\mathbb{R}^n$. Based on Alexandrov-Bakelman-Pucci's method, we use the concept of generalized normal cone introduced by [17], to show that for any region outside a Euclidean ball, its isoperimetric ratio has a lower bound that equals to the case where half-Wulff shape is cut by a half-space.

Part I

Principal Eigenvalue Estimate of Nonlinear Operators

Chapter 1

Introduction to the Main Result

In the first part we prove the following result:

Theorem 1.0.1. *Let M be compact smooth manifold and L be an elliptic diffusion operator with invariant measure m. Assume that L satisfies $BE(\kappa, N)$ with $\kappa < 0$ and $N < \infty$. Let u be an eigenfunction associated with λ satisfying Neumann boundary condition if $\partial M \neq \emptyset$, where λ is the first nonzero eigenvalue of L_p. Let D be diameter defined by the intrinsic distance metric on M. Then we have a sharp comparison:*

$$\lambda \geq \lambda_D$$

where λ_D is the first nonzero eigenvalue of the Neumann eigenvalue problem on $[-D/2, D/2]$:

$$\begin{cases} \frac{d}{dt}\left[(w')^{(p-1)}\right] - (N-1)\sqrt{-\kappa}\tanh(\sqrt{-\kappa}t)(w')^{(p-1)} + \lambda w^{(p-1)} = 0. \\ w'(-\frac{D}{2}) = w'(\frac{D}{2}) = 0 \end{cases}$$

Our theorem is an extension of the Theorem 1.1 in [9] to the case $\kappa < 0$, and the proof is based on a gradient comparison method due to Bakry-Qian [2].

There has been a long history of estimate of the principal eigenvalue of elliptic partial differential operators on Euclidean space and Riemannian manifolds. In 1960, Payne and Weinberger studied the following Neumann problem for a convex domain $C \subset \mathbb{R}^n$:

$$\begin{cases} \Delta u + \mu u = 0 \quad \text{in } C \\ \partial u/\partial n = 0 \quad \text{on } \partial C \end{cases}$$

and here μ is called an eigenvalue of the Laplacian. It is easily seen that we always have $\mu \geq 0$, so despite the trivial eigenvalue 0 corresponding to constant u, we are interested in bounding the first positive eigenvalue μ_1 from below. We have the variational characterization of μ_1:

$$\mu_1 = \inf_{\int_C u=0} \frac{\int_C |\nabla u|^2}{\int_C u^2}$$

Payne and Weinberger [21] showed that $\mu_1 \geq \pi^2/D^2$, where D is the diameter of C. They also showed that this lower bound cannot be achieved by n-dimensional region, but can be approximated by choosing C degenerating into a segment $[-D/2, D/2]$. On Riemannian manifolds, the Ricci curvature plays an important role in the estimate of the principal eigenvalue. An analogous situation is when M has no boundary, where Lichnerowicz [15] proved the first sharp lower bound of λ_1 when $\mathrm{Ric} \geq (n-1)\kappa > 0$:

Theorem 1.0.2 ([15])**.** *Let M be a compact Riemannian manifold without boundary. If $Ric(M) \geq (n-1)\kappa > 0$ where $n = dim(M)$, then*

$$\lambda_1 \geq n\kappa$$

In 1970, Cheeger [4] introduced some isoperimetric constants and proved lower bound of principal eigenvalue of Laplacian in terms of these constants. Later in 1980, assuming nonnegative Ricci curvature, Li and Yau [11] used the Bochner formula and gradient estimate to prove the following result:

Theorem 1.0.3 ([11])**.** *Let M be a compact Riemannian manifold, with either empty or convex boundary ∂M. If $Ric_M \geq 0$, then*

$$\lambda_1 \geq \frac{\pi^2}{2D^2}$$

where D is the diameter of M.

This lower bound is non-sharp and is finally optimized by Zhong-Yang [25], $\lambda_1 \geq \pi^2/D^2$, and it is optimal in analogues sense as the Payne-Weinberger result: the lower bound can be approximated by a sequence of manifolds degenerating into a circle(empty boundary case) or a segment(non-empty boundary case). Proof-wise, the lower bounds obtained in the above results does not arise from a model space, however they can indeed be realized on certain model spaces. In 1992, Kroger [10] recovered Zhong-Yang's result by a comparison argument involving a one-dimensional model eigenvalue problem.

In the meantime, people turns attention to more general class of elliptic operators. In 1986, Bakry and Emery introduced the so-called Γ_2-Calculus on manifolds, which includes the notion of generalized metric Γ, a diffusion operator L over a smooth manifold M, together with generalized Ricci curvature R. The lower bound on Ricci curvature is expressed by the curvature-dimension condition $BE(\kappa, N)$, which is a condition satisfied by the operator L instead of merely a geometric assumption on M. In 2000, Bakry and Qian used a gradient comparison technique to prove sharp lower bounds of $\lambda_1(L)$ under the curvature-dimension condition $BE(\kappa, N)$ for three cases: $\kappa > 0$, $\kappa = 0$ and $\kappa < 0$. In fact, they proved the following theorem:

Theorem 1.0.4 ([2]). *Let M be a compact Riemannian manifold with either empty or convex boundary, and L be an elliptic differential operator in the form $L = \Delta + B$, where B is a smooth vector field on M. Suppose that L satisfies the $BE(\kappa, N)$ ($\kappa \in \mathbb{R}$, $N \in [1, \infty]$), and the diameter of M is bounded by d. Let λ_1 be a nonzero eigenvalue of L, then we have $\lambda_1 \geq \hat{\lambda}(N, \kappa, d)$, where $\hat{\lambda}(N, \kappa, d)$ is the first nonzero eigenvalue of the problem*

$$\begin{cases} v'' - Tv' + \lambda v = 0 \quad \text{on } [-d/2, d/2] \\ v'\big(-\frac{d}{2}\big) = v'\big(\frac{d}{2}\big) = 0 \end{cases}$$

where the function T is

$$
\begin{aligned}
&T = \sqrt{(N-1)\kappa}\tan\left(\sqrt{\frac{\kappa}{N-1}}t\right) && \textit{if } \kappa > 0 \textit{ and } n < \infty \\
&T = \sqrt{-(N-1)\kappa}\tanh\left(\sqrt{-\frac{\kappa}{N-1}}t\right) && \textit{if } \kappa < 0 \textit{ and } n < \infty \\
&T = 0 && \textit{if } \kappa = 0 \textit{ and } n < \infty \\
&T = \kappa t && \textit{if } n = \infty
\end{aligned}
$$

Recently much attention were drawn on the non-linear operator derived from the Laplacian: the p-Laplacian $\Delta_p u = \mathrm{div}(|\nabla u|^{p-2}\nabla u)$, where $p \in (1, \infty)$. The Neumann eigenvalue problem associated with Δ_p is

$$
\begin{cases}
\Delta_p u = -\lambda u^{(p-1)} := -\lambda |u|^{p-2} u & \text{in } M \\
\frac{\partial u}{\partial n} = 0 & \text{on } \partial M
\end{cases}
$$

When the Ricci curvature is assumed to be nonnegative, Kawai and Nakauchi [8] showed that $\lambda_1(\Delta_p) \geq \frac{1}{p-1}\frac{\pi_p^p}{(4D)^p}$ for $p > 2$, and was later improved by Zhang [] to $\lambda_1(\Delta_p) \geq (p-1)\frac{\pi_p^p}{(2D)^p}$ for $p > 1$ and assuming Ricci curvature is positive at a point. In 2012, Valtorta [23] considered the linearization of p-Laplacian and derived a Bochner formula, and used it to prove the sharp estimate $\lambda_1(\Delta_p) \geq (p-1)\frac{\pi_p^p}{D^p}$. Later in 2014, Naber and Valtorta extended the cases to that Ricci curvature has a negative lower bound, and proved the following comparison result:

Theorem 1.0.5 ([20])**.** *Let M be an n-dimensional complete Riemannian manifold with either empty or convex boundary, with diameter bounded by D. Suppose $\boldsymbol{Ric}_M \geq (n-1)\kappa$ for some $\kappa < 0$. Then we have the sharp lower bound*

$$\lambda_1(\Delta_p) \geq \hat{\lambda}(n, \kappa, D)$$

where $\hat{\lambda}(n,\kappa,D)$ *is the first nonzero eigenvalue of the following Neumann problem on* $[-D/2, D/2]$*:*

$$\begin{cases} \frac{d}{dt}\left[(w')^{(p-1)}\right] - (N-1)\sqrt{-\kappa}\tanh(\sqrt{-\kappa}t)(w')^{(p-1)} + \lambda w^{(p-1)} = 0. \\ w'(-\frac{D}{2}) = w'(\frac{D}{2}) = 0 \end{cases}$$

More recently there are several important extension of the above estimates to the Bakry-Emery generalized metric setting. In 2018, Koerber[9] showed that if L satisfies $BE(0,N)$ condition for $N \in [1,\infty)$, then the first nonzero eigenvalue of L_p (the p-operator of L) is bounded below by $\lambda_1(\Delta_p) \geq (p-1)\frac{\pi_p^p}{D^p}$. In an important special case($N=\infty$), Li and Wang ([13], [14]) showed that for $L = \Delta - \langle \nabla f, \cdot \rangle$, if the Bakry-Emery Ricci $\mathrm{Ric} + \nabla^2 f \geq \kappa g$, then one can also get sharp comparison result with a suitable one-dimensional Neumann problem.

Chapter 2

Γ_2 Calculus on Smooth Metric Measure Space and Curvature-Dimension Condition

2.1 Basic Settings

We first recall the notion of a smooth metric measure space.

Definition 2.1.1 (Smooth Metric Measure Space). *A smooth metric measure space is a triple* $(M, g, e^{-f} dVol_g)$*, where* M *is a Riemannian manifold,* g *is the Riemannian metric on* M *and* $f \in C^\infty(M)$*. In other words, a smooth metric measure space is a Riemannian manifold with a measure conformal to its volume induced by the metric* g*.*

In the setting of Γ_2 calculus, the working definition will be slightly more general. Starting with a smooth manifold M, we consider a second order diffusion operator L and its invariant measure m. Then we shall use the Γ_2 calculus to recover necessary geometric information on M, like metric and Ricci curvature.

Definition 2.1.2. *A linear second order operator* $L : C^\infty(M) \to C^\infty(M)$ *is called an elliptic* *diffusion operator if for any $\Phi : \mathbb{R}^n \to \mathbb{R}$, we have*

$$L(\Phi(f_1, f_2, \ldots, f_n)) = \sum_{i=1}^{n} \partial_i \Phi \cdot L(f_i) + \sum_{i,j=1}^{n} (\partial_i \partial_j \Phi) \cdot \Gamma(f_i, f_j) \tag{2.1.1}$$

where $\Gamma : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$ is defined as

$$\Gamma(f, g) = \frac{1}{2}\Big(L(fg) - fL(g) - gL(f)\Big)$$

and in addition, the Γ-operator needs to satisfy

$$\Gamma(f) := \Gamma(f, f)(x) \geq 0, \quad \forall x \in M$$

with equality if and only if $df(x) = 0$.

Definition 2.1.3 (Invariant Measure)**.** *A locally finite Borel measure m is called L-invariant if there is a generalized function $\nu : \partial M$ such that the integration-by-parts formula holds:*

$$\int_M \Gamma(f, g) dm = -\int_M fL(g) dm + \int_{\partial M} f\Gamma(g, \nu) dm$$

for all $f, g \in C^\infty(M)$. Here ν is called the outward unit normal function and is defined to be a collection of pairs (ν_i, U_i) for a covering U_i of ∂M such that $\nu_i \in C^\infty(U_i)$ and $\Gamma(\nu_i - \nu_j, \cdot)|_{U_i \cap U_j} = 0$.

Remark 2.1.4. *The relation of a diffusion operator L and its invariant measure m is analogous to the Bakry-Emery drift Laplacian $\Delta_f := \Delta - \langle \nabla f, \cdot \rangle$ with the conformal volume measure $e^{-f} dVol_g$.*

We can define the intrinsic distance on M base on L and Γ.

Definition 2.1.5 (Intrinsic Distance)**.** *The intrinsic distance $d : M \times M \to [0, \infty)$ is defined as*

$$d(x, y) := \sup\Big\{f(x) - f(y) : f \in C^\infty(M), \Gamma(f, f) \leq 1\Big\}.$$

The diameter of M is defined as $D := \sup\{d(x, y) : x, y \in M\}$.

Γ operator is analogous to the scalar product of gradient of two functions on M. To define higher order quantities like Hessian and curvature, we use Γ iteratively.

Definition 2.1.6. *For any $f, u, v \in C^{\infty}(M)$, we define the Hessian of f in direction of u, v by*

$$H_f(u,v) = \frac{1}{2}\Big(\Gamma(u, \Gamma(f,v)) + \Gamma(v, \Gamma(f,u)) - \Gamma(f, \Gamma(u,v))\Big) \tag{2.1.2}$$

and the Γ_2-operator by

$$\Gamma_2(u,v) = \frac{1}{2}\Big(L(\Gamma(u,v)) - \Gamma(u, Lv) - \Gamma(v, Lu)\Big)$$

With Γ_2 operator, we may define the N-Ricci curvature as

Definition 2.1.7. *The N-Ricci curvature is defined as*

$$Ric_N(f,f)(x) = \inf\Big\{\Gamma_2(\phi,\phi)(x) - \frac{1}{N}(L\phi)^2 : \phi \in C^{\infty}(M), \Gamma(\phi - f)(x) = 0\Big\}$$

and the ∞-Ricci curvature is $Ric_{\infty} := \lim_{N\to\infty} R_N$.

Remark 2.1.8. *The definition of N-Ricci curvature is a generalization of Bochner's Formula:*

$$Ric(\nabla u, \nabla u) = \frac{1}{2}\Delta|\nabla u|^2 - \langle \nabla u, \nabla \Delta u\rangle - |\nabla^2 u|^2.$$

If (M,g) is a Riemannian manifold and $N = dim(M)$, the N-Ricci curvature coincides with the Ricci curvature of M.

Definition 2.1.9 (Curvature-Dimension Condition)**.** *Let $\kappa \in \mathbb{R}$ and $N \in [1,\infty]$, we say that L satisfies $BE(\kappa, N)$ condition if and only if*

$$Ric_N(f,f) \geq \kappa\Gamma(f) \quad \forall f \in C^{\infty}(M). \tag{2.1.3}$$

We can also define the second fundamental form of a submanifold in M. In our situation, M may have a convex boundary, by which we define as

Definition 2.1.10. *Let ν be the outward normal of ∂M as in Definition 2.1.3. Let $U \subset M$ be an open set and $\phi, \eta \in C^{\infty}(U)$ such that $\Gamma(\nu,\phi) = \Gamma(\nu,\eta) = 0$ on $U \cap \partial M$. The second fundamental* *form on ∂M in the direction of ϕ, η is*

$$\mathrm{II}(\phi,\eta) = -H_\phi(\eta,\nu) = -\frac{1}{2}\Gamma(\nu,\Gamma(\eta,\phi)).$$

If $\mathrm{II}(\phi,\phi) \leq (<) 0$ for any $\phi \in C^\infty(M)$ such that $\Gamma(\phi) > 0$ on $U \cap \partial M$, then ∂M is called convex(strictly convex) in M.

2.2 The generalized p-Laplacian and its eigenvalue problem

Now we work on the Neumann eigenvalue problem of the non-linear operator L_p derived from the second order diffusion operator L. First we give the definitions.

Definition 2.2.1. *If L is a second order diffusion operator, and $p \geq 1$, then*

$$L_p u(x) = \begin{cases} \Gamma(u)^{\frac{p-2}{2}} \left(Lu + (p-2)\frac{H_u(u,u)}{\Gamma(u)} \right) & \textit{if } \Gamma(u)(x) \neq 0; \\ 0 & \textit{otherwise.} \end{cases}$$

defines a nonlinear operator $L_p : C^\infty(M) \to C^\infty(M)$ when $p \neq 2$, which is called the p-operator of L.

Since L_p is in general nonlinear, our gradient comparison method will base on the linearization of L_p at u, which is an important technique due to [23].

Definition 2.2.2. *We define the following $\mathcal{L}_p^u : C^\infty(M) \to C^\infty(M)$*

$$\mathcal{L}_p^u(\eta) = \begin{cases} \Gamma(u)^{\frac{p-2}{2}} \left(L\eta + (p-2)\frac{H_\eta(u,u)}{\Gamma(u)} \right) & \textit{if } \Gamma(u)(x) \neq 0; \\ 0 & \textit{otherwise.} \end{cases}$$

as the linearization of L_p at u. Notice that $\mathcal{L}_p^u$ is a second order linear operator in terms of η as u is fixed.

Now we define the eigenvalue problem of L_p. If $\lambda \in \mathbb{R}$ and $u \in C^2(M)$ satisfies the Neumann boundary problem:

$$\begin{cases} L_p u = -\lambda u|u|^{p-2} & \text{on } M^\circ \\ \Gamma(u, \tilde{\nu}) = 0 & \text{on } \partial M \end{cases}$$

Then we call λ an eigenvalue, and u an eigenfunction of L_p.However, we may not always find a classical solution. To define the eigenfunction in a weak sense, we first use the invariance of m to deduce the following integration-by-parts formula:

Lemma 2.2.1. *Let $\phi \in C^\infty(M)$ and $u \in C^2(M)$ and $\Gamma(u) > 0$ on supp(ϕ). Then we have*

$$\int_M \phi L_p u dm = -\int_M \Gamma(u)^{\frac{p-2}{2}} \Gamma(u, \phi) dm + \int_{\partial M} \Gamma(u, \tilde{\nu}) \Gamma(u)^{\frac{p-2}{2}} \phi dm'$$

So we define the eigenvalue and eigenfunction by

Definition 2.2.3. *We say that λ is an eigenvalue of L_p if there is a $u \in W^{1,p}(M)$ such that for any $\phi \in C^\infty(M)$ the following identity holds:*

$$\int_M \Gamma(u)^{\frac{p-2}{2}} \Gamma(u, \phi) dm = \lambda \int_M \phi u |u|^{p-2} dm$$

We have the following result concerning the regularity of principal eigenfunctions.

Lemma 2.2.2 (Lemma 2.2 in [9])**.** *If M is a compact smooth Riemannian manifold with an elliptic diffusion operator L and an L-invariant measure m. Then the principal eigenfunction is in $C^{1,\alpha}(M)$ for some $\alpha > 0$, and u is smooth near points $x \in M$ such that $\Gamma(u)(x) \neq 0$ and $u(x) \neq 0$; for $p < 2$, u is $C^{3,\alpha}$, and for $p > 2$, u is $C^{2,\alpha}$ near x where $\Gamma(u)(x) \neq 0$ and $u(x) = 0$.*

2.3 p-Bochner Formula and Estimate

The Bochner formula in Riemannian geometry provides a powerful tool to study manifolds with bounds on Ricci curvature. It associates the Ricci curvature with the Laplacian of norm of the gradient of functions. For manifolds with metric defined by Γ, we have the following generalized Bochner formula due to Sturm [22]:

Theorem 2.3.1 (Theorem 1.1 [22])**.** *For any $f \in C^\infty(M)$, we have*

$$\Gamma_2(f,f) = Ric_\infty(f,f) + \|H_f\|_{HS}^2 \tag{2.3.1}$$

where $\|\cdot\|_{HS}$ is the Hilbert-Schmidt norm, i.e., $\|H_f\|_{HS} = \sum_{i,j=1} H_f(u_i,u_j)^2$ where $\{u_i\}$ forms a complete orthonormal basis for Γ at x.

In this section, we will derive the Bochner formula for the linearized p-operator $\mathcal{L}_p^u$ and an estimate which is the key to prove gradient comparison theorem.

Proposition 2.3.1 (Bochner formula)**.** *Let $u \in C^3(M)$ be a first eigenfunction of L_p, and $x \in M$ be a point such that $\Gamma(u)(x) \neq 0$ and $u(x) \neq 0$. Then at x we have the following formula:*

$$\frac{1}{p}\mathcal{L}_p^u\left(\Gamma(u)^{\frac{p}{2}}\right) = \Gamma(u)^{\frac{p-2}{2}}\left(\Gamma(L_pu,u) - (p-2)L_puA_u\right) + \Gamma(u)^{p-2}\left(\Gamma_2(u,u) + p(p-2)A_u^2\right) \tag{2.3.2}$$

where $A_u := H_u(u,u)/\Gamma(u)$.

To carry out the computation we need a few properties of the Γ operator and Hessian:

Lemma 2.3.1. *For $u,v,w \in C^2(M)$, $f \in C^\infty(\mathbb{R})$, we have*

(1) $\Gamma(u, v\cdot w) = v\Gamma(u,w) + w\Gamma(u,v)$

(2) $\Gamma(f(u),v) = f'(u)\cdot\Gamma(u,v)$

(3) $H_{uv}(w,w) = uH_v(w,w) + vH_u(w,w) + 2\Gamma(u,w)\Gamma(v,w)$

(4) $H_{f(u)}(v,v) = f'(u)H_u(v,v) + f''(u)\Gamma(u,v)^2$

Proof of Lemma 2.3.1. For (1), let using equation 2.1.1 for $\Phi(f_1,f_2,f_3) = f_1f_2f_3$ we get

$$\begin{aligned}
\Gamma(u,vw) &= \frac{1}{2}\Big[L(uvw) - uL(vw) - vwLu\Big] \\
&= \frac{1}{2}\Big[\big[uvLw + 2u\Gamma(v,w)|u,v,w\big] - u\big[vLw + \Gamma(v,w)|v,w\big] - vwLu\Big] \\
&= \frac{1}{2}\Big[2w\Gamma(u,v) + 2v\Gamma(w,u)\Big] \\
&= v\Gamma(u,w) + w\Gamma(u,v)
\end{aligned}$$

Here $[f(u,v,w)|u,v,w]$ means sum over clockwise permutations, i.e, $f(u,v,w)+f(w,u,v)+f(v,w,u)$. Now for (2), let $\Phi(f_1,f_2)=f(f_1)f_2$, we have

$$\begin{aligned}
\Gamma(f(u),v) &= \frac{1}{2}\Big[L(f(u)v)-f(u)Lv-vL(f(u))\Big] \\
&= \frac{1}{2}\Big[f'(u)vLu+f(u)Lv+2f'(u)\Gamma(u,v)+f''(u)\Gamma(u)-f(u)Lv \\
&\qquad -v(f'(u)Lu+f''(u)\Gamma(u))\Big] \\
&= f'(u)\Gamma(u,v)
\end{aligned}$$

For (3), we just repeatedly use (1) to separate the product term in Γ; and for (4), we use (2) repeatedly to separate f term from Γ. $\square$

Proof of Proposition 2.3.1. We have by 2.2.2

$$\mathcal{L}_p^u\Big(\Gamma(u)^{\frac{p}{2}}\Big)=\Gamma(u)^{\frac{p-2}{2}}\left[L\big(\Gamma(u)^{\frac{p}{2}}\big)+(p-2)\frac{H_{\Gamma(u)^{\frac{p}{2}}}(u,u)}{\Gamma(u)}\right]$$

To compute the first term in the bracket, let $\Phi(x)=x^{\frac{p}{2}}$ in (2.1.1), we have

$$\begin{aligned}
\mathrm{I}=\Gamma(u)^{\frac{p-2}{2}}L\big(\Gamma(u)^{\frac{p}{2}}\big) &= \Gamma(u)^{\frac{p-2}{2}}\Big[\frac{p}{2}\Gamma(u)^{\frac{p}{2}-1}L(\Gamma(u))+\frac{p}{2}\frac{p-2}{2}\Gamma(u)^{\frac{p-4}{2}}\Gamma(\Gamma(u))\Big] \\
&= \frac{p}{2}\Gamma(u)^{p-2}L(\Gamma(u))+\frac{p(p-2)}{4}\Gamma(u)^{p-3}\Gamma(\Gamma(u)) \\
&= p\Gamma(u)^{p-2}\big[\Gamma_2(u,u)+\Gamma(u,Lu)\big]+\frac{p(p-2)}{4}\Gamma(u)^{p-3}\Gamma(\Gamma(u))
\end{aligned}$$

For the second term we have

$$\begin{aligned}
\mathrm{II}=\Gamma(u)^{\frac{p-2}{2}}(p-2)\frac{H_{\Gamma(u)^{\frac{p}{2}}}(u,u)}{\Gamma(u)} &= (p-2)\Gamma(u)^{\frac{p-4}{2}}H_{\Gamma(u)^{\frac{p}{2}}}(u,u) \\
&= (p-2)\Gamma(u)^{\frac{p-4}{2}}\big[\frac{p}{2}\Gamma(u)^{\frac{p-2}{2}}H_{\Gamma(u)}(u,u)+\frac{p(p-2)}{4}\Gamma(u)^{\frac{p-4}{2}}\Gamma(u,\Gamma(u))^2\big]
\end{aligned}$$

By the definition of H we have

$$\begin{aligned}
H_{\Gamma(u)}(u,u) &= \frac{1}{2}\big[2\Gamma(u,\Gamma(\Gamma(u),u))-\Gamma(\Gamma(u))\big] \\
&= \Gamma(u,2H_u(u,u))-\frac{1}{2}\Gamma(\Gamma(u))
\end{aligned}$$

Hence we can simplify II as

$$p(p-2)\Gamma(u)^{p-3}\big[\Gamma(u,H_u(u,u))+\frac{p-2}{4}\big]-\frac{p(p-2)}{4}\Gamma(u)^{p-3}\Gamma(\Gamma(u))$$

Hence

$$\begin{aligned}\mathrm{I}+\mathrm{II}&=p\Gamma(u)^{p-2}\big[\Gamma_2(u,u)+\Gamma(u,Lu)\big]+p(p-2)\Gamma(u)^{p-3}\big[\Gamma(u,H_u(u,u))+\frac{p-2}{4}\big]\\&=p\Gamma(u)^{\frac{p-2}{2}}\Big[\Gamma(u)^{\frac{p-2}{2}}\Gamma(u,Lu)+(p-2)\Gamma(u)^{\frac{p-4}{2}}\Gamma\big(u,H_u(u,u)\big)\Big]+p\Gamma(u)^{p-2}\Gamma_2(u,u)\\&\qquad+\frac{p(p-2)^2}{4}\Gamma(u)^{p-4}\Gamma(u,\Gamma(u))^2\end{aligned}$$

Notice that the first term above can be rewritten using the Leibniz rule by

$$\begin{aligned}&p\Gamma(u)^{\frac{p-2}{2}}\Big[\Gamma(u,L_pu)-\Gamma(u,\Gamma(u)^{\frac{p-2}{2}})Lu-(p-2)H_u(u,u)\Gamma\big(u,\Gamma(u)^{\frac{p-4}{2}}\big)\Big]\\=&p\Gamma(u)^{\frac{p-2}{2}}\Big[\Gamma(u,L_pu)-\frac{p-2}{2}\Gamma(u)^{-1}\Gamma(u,\Gamma(u))L_pu+(p-2)\Gamma(u)^{\frac{p-6}{2}}H_u(u,u)\Gamma(u,\Gamma(u))\Big]\\=&p\Gamma(u)^{\frac{p-2}{2}}\Big[\Gamma(u,L_pu)-(p-2)L_pu\frac{H_u(u,u)}{\Gamma(u)}+2(p-2)H_u(u,u)^2\Gamma(u)^{\frac{p-6}{2}}\Big]\end{aligned}$$

Then after rearranging the terms and reuse $2H_u(u,u)=\Gamma(u,\Gamma(u))$, we get the desired Bochner formula. □

By exploring the curvature dimension condition (equation (2.1.3)) we can get certain improvement on the bounds of the second term in equation (2.3.2), explicitly by the curvature lower bound κ:

Proposition 2.3.2. *Suppose L satisfies $BE(\kappa,N)$ for some $\kappa\in\mathbb{R}$ and $N\in[1,\infty]$. Then for any $n\geq N$, we have for $p\in(1,\infty)$,*

$$\Gamma(u)^{p-2}\left(\Gamma_2(u,u)+p(p-2)A_u^2\right)\geq\frac{(L_pu)^2}{n}+\frac{n}{n-1}\left(\frac{L_pu}{n}-(p-1)\Gamma(u)^{\frac{p-2}{2}}A_u\right)^2+\kappa\Gamma(u)^{p-1}$$

for $n=\infty$,

$$\Gamma(u)^{\frac{p}{2}}\left(\Gamma_2(u,u)+p(p-2)A_u^2\right)\geq(p-1)^2\Gamma(u)^{p-2}A_u^2+\kappa\Gamma(u)^{p-1},$$

for $n = 1$,

$$\Gamma(u)^{\frac{p}{2}} \left(\Gamma_2(u,u) + p(p-2)A_u^2\right) \geq (L_p u)^2 + \kappa\Gamma(u)^{p-1}$$

Proof. Following [9] Lemma 3.3, we can scale u on both sides so that $\Gamma(u)(x) = 1$. We can assume $n = N$ since $B(\kappa, N)$ implies $B(\kappa, n)$ for $n \geq N$. When $n = 1$, by the curvature-dimension inequality and $Lu = \mathrm{tr}H_u = A_u$, we get

$$\Gamma_2(u,u) + p(p-2)A_u^2 \geq \kappa + (Lu)^2 + p(p-2)A_u^2 = \kappa + (p-1)^2 A_u^2 = (L_p u)^2 + \kappa.$$

When $n = \infty$, we have $\Gamma_2(u,u) \geq \kappa + A_u^2$, therefore $\Gamma_2(u,u) + p(p-2)A_u^2 \geq \kappa + (p-1)^2 A_u^2$. Now if $1 < n < \infty$, for any $v \in C^\infty(M)$, by the curvature-dimension inequality we have

$$\Gamma_2(v,v) \geq \kappa\Gamma(v) + \frac{1}{N}(Lv)^2$$

Now we consider a quadratic form $B(v,v) = \Gamma_2(v,v) - \kappa\Gamma(v) - \frac{1}{N}(Lv)^2$, which is non-negative for any $v \in C^\infty(M)$. Let $v = \phi(u)$ where $\phi \in C^\infty(\mathbb{R})$. Then by standard computations, together with the assumption $\Gamma(u) = 1$, we have

$$\Gamma(\phi(u)) = (\phi')^2, \qquad L(\phi(u)) = \phi' Lu + \phi'',$$
$$\Gamma_2(\phi(u), \phi(u)) = (\phi')^2 \Gamma_2(u,u) + 2\phi'\phi'' A_u + (\phi'')^2.$$

Then we get

$$\begin{aligned} B(\phi(u), \phi(u)) &= \Gamma_2(\phi(u), \phi(u)) - \kappa\Gamma(\phi(u)) - \frac{1}{N}(L(\phi(u)))^2 \\ &= (\phi')^2\Gamma_2(u,u) + 2\phi'\phi'' A_u + (\phi'')^2 - \kappa(\phi')^2 - \frac{1}{N}\left[\phi' Lu + \phi''\right]^2 \\ &= (\phi')^2 B(u,u) + 2\phi'\phi''(A_u - \frac{Lu}{N}) + \frac{N-1}{N}(\phi'')^2 \end{aligned}$$

Since $B(\phi(u), \phi(u)) \geq 0$ for any ϕ, we have non-positive discriminant

$$\left(A_u - \frac{Lu}{N}\right)^2 - B(u,u)\frac{N-1}{N} \leq 0$$

Therefore we have

$$
\begin{aligned}
&\Gamma_2(u,u) + p(p-2)A_u^2 \\
=&\kappa + \frac{1}{N}(Lu)^2 + B(u,u) + p(p-2)A_u^2 \\
\geq&\kappa + \frac{1}{N}(L_p(u) + (p-2)A_u)^2 + \frac{N}{N-1}\Big(A_u - \frac{L_p(u) + (p-2)A_u}{N}\Big)^2 + p(p-2)A_u^2 \\
=&\kappa + \frac{1}{N}(L_p(u))^2 + \frac{N}{N-1}\Big(\frac{L_p(u)}{N} - (p-1)A_u\Big)^2
\end{aligned}
$$

□

Chapter 3

Gradient Comparison and the Proof of the Main reuslt

3.1 One Dimensional Comparison Model Equation

In this section we describe the one dimensional comparison model equation. The equation will be designed to include the curvature-dimension condition $BE(\kappa, N)$ and the Neumann-boundary condition. In the case $\kappa < 0$, the equation models the eigenvalue problem of Δ_p operator over a manifold which is a warped product of a ray and a space form, while in the $\kappa > 0$ case, we pick an equation which is simpler to compute. First we fix $N > 1$ and $p > 1$.

Consider three functions T_1, T_2 and T_3 as follows:

(1) $T_1(t) = -(N-1)\sqrt{-\kappa}\,\mathrm{cotanh}(\sqrt{-\kappa}t)$, defined on $I_1 = (0, \infty)$;

(2) $T_2(t) = -(N-1)\sqrt{-\kappa}$, defined on $I_2 = \mathbb{R}$;

(3) $T_3(t) = -(N-1)\sqrt{-\kappa}\tanh(\sqrt{-\kappa}t)$, defined on $I_3 = \mathbb{R}$;

Definition 3.1.1 (Model Equations). *For fixed $a \in I_i$, $i = 1, 2, 3$ and $\lambda \in \mathbb{R}$ the following initial* *value problem*

$$\begin{cases} \frac{d}{dt}\left[|w'|^{p-2}w'\right] - T_i(t)|w'|^{p-2}w' + \lambda|w|^{p-2}w = 0 \\ w(a) = -1, w'(a) = 0 \end{cases} \tag{3.1.1}$$

is called the i-th model equation. If a solution w exists, we denoted it by $w_{i,a,\lambda}$.

When $p = 2$, equation 3.1.1 is a linear second order equation, so the solution w shows oscillatory behavior under certain conditions. For general $p > 1$, we use the Prufer transform to model the amplitude and phase parts of the solution.

Definition 3.1.2 (p-sin and p-cos functions)**.** *For every $p \in (1, \infty)$, let π_p be defined by:*

$$\pi_p = \int_{-1}^{1} \frac{ds}{(1 - s^p)^{\frac{1}{p}}} = \frac{2\pi}{p\sin(\pi/p)}$$

The C^1 periodic function $\sin_p : \mathbb{R} \to [-1, 1]$ is defined via the integral on $[-\frac{\pi_p}{2}, \frac{3\pi_p}{2}]$ by

$$\begin{cases} t = \int_0^{\sin_p(t)} (1 - s^p)^{-\frac{1}{p}} ds & \text{if } t \in \left[-\frac{\pi_p}{2}, \frac{\pi_p}{2}\right] \\ \sin_p(t) = \sin_p(\pi_p - t) & \text{if } t \in \left[\frac{\pi_p}{2}, \frac{3\pi_p}{2}\right] \end{cases}$$

and we extend it to a periodic function on $\mathbb{R}$. Let $\cos_p(t) = \frac{d}{dt}\sin_p(t)$.

Remark 3.1.3. *For $\sin_p$ and $\cos_p$ functions, we have the following identity which resembles the case of usual $\sin$ and $\cos$:*

$$|\sin_p(t)|^p + |\cos_p(t)|^p = 1.$$

Let us define the Prufer transformation of equations (3.1.1) as the polar decomposition of w and w':

Definition 3.1.4 (Prufer transformation)**.** *Let $\alpha = \left(\frac{\lambda}{p-1}\right)^{\frac{1}{p}}$, then for some solution w of the equations (3.1.1), we define functions e and ϕ by*

$$\alpha w = e\sin_p(\phi), \quad w' = e\cos_p(\phi).$$

Standard calculation shows that for $i = 1, 2, 3$, ϕ and e satisfies the following first order systems:

$$\begin{cases} \phi' = \alpha - \frac{T_i}{p-1}\cos_p^{p-1}(\phi)\sin_p(\phi) \\ \phi(a) = -\frac{\pi_p}{2} \end{cases} \tag{3.1.2}$$

$$\begin{cases} \frac{d}{dt}\log(e) = \frac{T_i}{p-1}\cos_p^p(\phi) \\ e(a) = \alpha \end{cases} \tag{3.1.3}$$

Since $\sin_p$ and $\cos_p$ are both Lipshitz functions with uniformly bounded Lipshitz constants, and for $i = 1, 2, 3$, T_i is also Lipshitz, we have the existence and uniqueness of solution ϕ and e for all parameters and $t \geq a$. For the equation with T_1, if $a \in I_1 = (0, \infty)$, then we still have existence and uniqueness. The boundary case $i = 1, a = 0$ is summarized in the following proposition:

Proposition 3.1.1. *For any $i = 1, 2, 3$, $\lambda \in \mathbb{R}$ and $a \in I_i \cup \{0\}$, the initial value problems 3.1.1 has a unique solution $w_{i,a,\lambda}$.*

Proof of Proposition 3.1.1. We already demonstrated the proposition in cases except for $i = 1$ and $a = 0$. In this case let $\mu(t) = \sinh^{N-1}(\sqrt{-\kappa}t)$, we can rewrite the model equation as

$$\left[\mu \cdot (w')^{(p-1)}\right]' + \lambda\mu w^{(p-1)} = 0 \tag{3.1.4}$$

then we can integrate this equation and get for $t > 0$

$$w(t) = w(0) + \int_0^t \left[\frac{-\lambda}{\mu(s)}\int_0^s \mu(r)w^{(p-1)}(r)dr\right]^{\frac{1}{p-1}} ds$$

Considering a continuous bounded function h such that $h(w) = w$ when $|w - w(0)| \leq 1$, and let X be the subspace of $C([0, T))$ consisting o function w such that $|w + 1| \leq 1$, $S : X \to C([0, T))$ be

$$S(w)(t) = w(0) + \int_0^t \left[\frac{-\lambda}{\mu(s)}\int_0^s \mu(r)h(w)^{(p-1)}(r)dr\right]^{\frac{1}{p-1}} ds \qquad (t < T)$$

Since $\mu(r) = O(r^{N-1})$ as $r \to 0$, the integrand on the right hand side is a bounded continuous function of s, which implies the existence of a fixed point of S when $T > 0$ is chosen to be small.

The rest of the proof of global existence and uniqueness will be similar to section 3 of [24]. □

3.2 Gradient Comparison Theorem and Its Applications

In this section we prove the gradient comparison theorem of the eigenfunction with he solution to the one-dimensional model.

Theorem 3.2.1 (Gradient Comparison Theorem)**.** *Assume that L satisfies $BE(\kappa, N)$. Let u be a weak solution of*

$$L_p u = -\lambda u^{(p-1)}$$

in the sense of definition 2.2.3, satisfying Neumann boundary condition if $\partial M \neq \emptyset$, where λ is the first nonzero eigenvalue of L_p. Let $w : [a,b] \to \mathbb{R}$ be a solution of the following ODE:

$$\begin{cases} \frac{d}{dt}\left[(w')^{(p-1)}\right] - T_i(t)(w')^{(p-1)} + \lambda w^{(p-1)} = 0 \\ w(a) = -1, \quad w'(a) = 0 \end{cases} \tag{3.2.1}$$

such that w is strictly increasing on $[a,b]$ and Range$(u) \subset w([a,b])$. Then for all $x \in M$,

$$\Gamma(w^{-1}(u(x))) \leq 1.$$

Proof. By chain rule of Γ we need to show equivalently that

$$\Gamma(u)^{\frac{1}{2}}(x) \leq w'(w^{-1}(u(x))) \qquad \forall x \in M.$$

Since T_κ depends smoothly on κ, we will first prove that for any $\tilde{\kappa} < \kappa$, the gradient comparison holds when the curvature-dimension condition is $BE(\tilde{\kappa}, N)$, and then we can take $\tilde{\kappa} \to \kappa$. This will allow us to use proof by contradiction.

For $c > 0$ we denote $\phi_c = (cw' \circ w^{-1})^p$, and consider the function $Z_c : M \to \mathbb{R}$

$$Z_c(x) = \Gamma(u)^{\frac{p}{2}}(x) - \phi_c(u(x))$$

Assume for contradiction that $Z_1(x) > 0$ for some $x \in M$. Let

$$c_0 = \inf\{c : Z_c(x) > 0 \text{ for some } x \in M\}$$

By our definition of c_0, there is a $x_0 \in M$ such that Z_{c_0} takes maximum at x_0. Now we fix c_0 and denote Z_{c_0} as Z, ϕ_{c_0} as ϕ while there is no confusion. When x_0 is in the interior of M, clearly we have

$$\Gamma(Z, u)(x_0) = 0 \tag{3.2.2}$$

and second derivative test by ellipticity of $\mathcal{L}_p^u$:

$$\frac{1}{p}\mathcal{L}_p^u(Z)(x_0) \leq 0 \tag{3.2.3}$$

Boundary Case. If $x_0 \in \partial M$, since $\Gamma(u, \tilde{\nu}) = 0$ by the Neumann boundary condition, we have that $\Gamma(Z, u) = 0$ at x_0. Since Z achieves maximum at x_0 and ∂M is convex, we have

$$\begin{aligned} 0 \leq \Gamma(Z, \tilde{\nu}) = \Gamma(\Gamma(u)^{\frac{p}{2}} - \phi(u), \tilde{\nu}) = \frac{p}{2}\Gamma(u)^{\frac{p-2}{2}}\Gamma(\Gamma(u), \tilde{\nu}) - \phi'(u)\Gamma(u, \tilde{\nu}) \\ = -p\Gamma(u)^{\frac{p-2}{2}}\mathrm{II}(u, u) - \phi'(u) \cdot 0 \leq 0 \end{aligned}$$

Therefore $\Gamma(Z, \tilde{\nu})(x_0) = 0$. This implies that the second derivative of Z along the normal direction is nonpositive. On the other hand, the second derivatives along tangential directions are nonpositive, hence the ellipticity of $\mathcal{L}_p^u$ implies that $\mathcal{L}_p^u(Z)(x_0) \leq 0$. Hence we comfirmed (3.2.2) and (3.2.3) above when $x_0 \in M$ and $x_0 \in \partial M$.

From (3.2.2) we get

$$\frac{p}{2}\Gamma(u)^{\frac{p-2}{2}}\Gamma(\Gamma(u), u) - \phi'(u)\Gamma(u) = 0$$

which implies $\phi'(u) = p\Gamma(u)^{\frac{p-2}{2}}A_u$. From (3.2.3) we have

$$\frac{1}{p}\mathcal{L}_p^u(\phi(u)) = \frac{1}{p}\Big(\phi'(u)L_p u + (p-1)\phi''(u)\Gamma(u)^{\frac{p}{2}}\Big)$$

By chain rule we have $\phi' = p \cdot \big[(w')^{p-2} \cdot w''\big] \circ w^{-1}$, and $\phi'' = p\Big[(p-2)(w'')^2 + w'''w'\Big] \cdot (w')^{p-4} \circ w^{-1}$, and by differentiating the ODE satisfied by w we have

$$(p-1)(w')^{p-3}\Big[(p-2)(w'')^2 + w'''w'\Big] = T_i'(w')^{p-1} + (p-1)T_i w''(w')^{p-2} - \lambda(p-1)w'w^{p-2}$$

Therefore

$$\phi'' = p \cdot \frac{T_i'(w')^{p-1} + (p-1)T_i w''(w')^{p-2} - \lambda(p-1)w'w^{p-2}}{w'} \circ w^{-1}.$$

Now we evaluate the above expression at $u(x_0)$. Since $\phi'(u) = p \cdot \big[(w')^{p-2} \cdot w''\big] \circ w^{-1}(u) = p\Gamma(u)^{\frac{p-2}{2}} A_u$, and by (1) we have $\phi(u) = w' \circ w^{-1}(u) = \Gamma(u)^{\frac{p}{2}}$, we have

$$\frac{1}{p}\mathcal{L}_p^u(\phi(u)) = -\lambda u^{(p-1)}\Gamma(u)^{\frac{p-2}{2}} A_u + T_i'\Gamma(u)^{p-1} + (p-1)T_i\Gamma(u)^{\frac{2p-3}{2}} A_u - \lambda(p-1)u^{p-2}\Gamma(u)^{\frac{p}{2}} \quad (3.2.4)$$

Evaluating the model equation (3.2.1) at $w^{-1}(u(x_0))$, we have

$$(p-1)\Gamma(u)^{\frac{p-2}{2}} A_u - T_{\tilde{\kappa}}\Gamma(u)^{\frac{p-1}{2}} + \lambda u^{(p-1)} = 0$$

Hence

$$(p-1)T_{\tilde{\kappa}}\Gamma(u)^{\frac{2p-3}{2}} A_u = (p-1)\big[(p-1)\Gamma(u)^{\frac{p-2}{2}} A_u + \lambda u^{(p-1)}\big]\Gamma(u)^{\frac{p-2}{2}} A_u.$$

Plugging the above equation into the third term of (5.5), we have

$$\begin{aligned}\frac{1}{p}\mathcal{L}_p^u(\phi(u)) = \lambda(p-2)u^{(p-1)}\Gamma(u)^{\frac{p-2}{2}} A_u + T_{\tilde{\kappa}}'\Gamma(u)^{p-1} + (p-1)^2\Gamma(u)^{p-2}A_u^2 \\ - \lambda(p-1)u^{p-2}\Gamma(u)^{\frac{p}{2}}\end{aligned}$$

For $i = 1, 2, 3$, we have $T_i' = T_i^2/(n-1) + \tilde{\kappa}$, to rewrite the second term of equation (3.2.4):

$$\begin{aligned}\frac{1}{p}\mathcal{L}_p^u(\phi(u)) =& -\lambda u^{(p-1)}\Gamma(u)^{\frac{p-2}{2}} A_u + \frac{1}{n-1}\Big[\lambda u^{(p-1)} + (p-1)\Gamma(u)^{\frac{p-2}{2}} A_u\Big]^2 + \tilde{\kappa}\Gamma(u)^{p-1} \\ &+ (p-1)^2\Gamma(u)^{p-2}A_u^2 + (p-1)\lambda u^{(p-1)}\Gamma(u)^{\frac{p-2}{2}} A_u - \lambda(p-1)u^{p-2}\Gamma(u)^{\frac{p}{2}} \\ =&(p-2)\lambda u^{(p-1)}\Gamma(u)^{\frac{p-2}{2}} A_u - \lambda(p-1)u^{p-2}\Gamma(u)^{\frac{p}{2}} + \tilde{\kappa}\Gamma(u)^{p-1} \\ &+ \frac{n}{n-1}\Big[\frac{\lambda u^{(p-1)}}{n} + (p-1)\Gamma(u)^{\frac{p-2}{2}} A_u\Big]^2 + \frac{\lambda^2 u^{2p-2}}{n}\end{aligned}$$

By Proposition 2.3.2, 2.3.2, and $L_p u = -\lambda u^{(p-1)}$ we have

$$\begin{aligned}\frac{1}{p}\mathcal{L}_p^u\big(\Gamma(u)^{\frac{p}{2}}\big) &= \Gamma(u)^{\frac{p-2}{2}}(\Gamma(L_p u, u) - (p-2)L_p u A_u) + \Gamma(u)^{p-2}(\Gamma_2(u,u) + p(p-2)A_u^2)\\ &\geq \Gamma(u)^{\frac{p-2}{2}}\big(-\lambda(p-1)u^{(p-2)}\Gamma(u) + \lambda(p-2)u^{(p-1)}A_u\big) + \frac{\lambda^2 u^{2p-2}}{n}\\ &\quad + \frac{n}{n-1}\Big[\frac{\lambda u^{(p-1)}}{n} + (p-1)\Gamma(u)^{\frac{p-2}{2}}A_u\Big]^2 + \kappa\Gamma(u)^{p-1}\end{aligned}$$

Hence we have $\frac{1}{p}\mathcal{L}_p^u\big(\Gamma(u)^{\frac{p}{2}} - \phi(u)\big) \geq (\kappa - \tilde{\kappa})\Gamma(u)^{p-1} > 0$, which is a contradiction with the second derivative test. Therefore we conclude that $Z_1 \leq 0$ on M, which implies our gradient comparison result. □

Remark 3.2.2. *When $1 < p < 2$ we know that $u \in C^{2,\alpha}$ near x_0, hence the Bochner formula can not be directly applied to x_0. In this case notice that u does not vanish identically in a neighborhood of x_0, we can choose $x' \to x_0$ with $u(x') \neq 0$. As we apply the Bochner formula at x', The first term $\Gamma(u)^{\frac{p-2}{2}}\Gamma(L_p u, u) = -\lambda\Gamma(u)^{\frac{p-2}{2}}\Gamma(u^{(p-1)}, u)$ since u is a eigenfunction. Now this diverging term will cancel with $-\lambda(p-1)u^{p-2}\Gamma(u)^{\frac{p}{2}}$ in the expression of $\frac{1}{p}\mathcal{L}_p^u(\phi(u))$, which makes it possible to define $\frac{1}{p}\mathcal{L}_p^u\big(\Gamma(u)^{\frac{p}{2}} - \phi(u)\big)(x_0)$ to be the limit of $\frac{1}{p}\mathcal{L}_p^u(\Gamma(u)^{\frac{p}{2}} - \phi(u))(x')$ as $x' \to x_0$. Therefore the previous proof still works when $1 < p < 2$.*

3.3 Fine Analysis of Model Equation 3.2.1

The gradient comparison theorem 3.2.1 relies on the assumption that we may find a solution to the model equation w such that $w([a,b])$ contains the range of u. Potentially it might be the case that the range of u is strictly smaller that $w_{i,a,\lambda}([a,b])$ for any admissible parameters. In these cases our comparison will be non-sharp, so we need to show that for all possible choice of the principal eigenvalue $\lambda(L_p)$ and range $[-1, \max u] \subset [-1,1]$, there is always a $w_{i,a\lambda}$ such that $\max w = \max u$. In this section we carry out some finer analysis of the behavior of solution $w_{i,a,\lambda}$ to confirm the existence of a sharp comparison.

For this purpose we introduce some notations. For $a \in \mathbb{R}$, let $w_{i,a}$ be the solution to the equation (6.1) with $T = T_i$, and $b(i,a)$ be the first critical point of $w_{i,a}$ after a. If $w'_{i,a}(t) > 0$ for $t > a$, then we say $b(i,a) = \infty$. Also let $\delta_{i,a} = b(i,a) - a$ and $m(i,a) = w_{i,a}(b(i,a))$. We shall prove the following statement in the current and next section:

Proposition 3.3.1. *Under the same setting, assume that L satisfies $BE(\kappa, N)$ condition where $\kappa < 0$. Let u be an eigenfunction of L_p operator corresponding to the eigenvalue $\lambda > 0$, rescaled so that* $\min u = -1$ *and* $\max u \leq 1$*. Then there is some $a \in \mathbb{R}$, $i \in \{1, 2, 3\}$ and a solution $w_{i,a,\lambda}$ such that $m(i, a, \lambda) = \max u$.*

To prove Proposition 3.3.1, we first confirm that there is a comparison solution whose range is $[-1, 1]$.

Proposition 3.3.2. *Fix $\alpha > 0$, $n \geq 1$ and $\kappa < 0$. Then there always exists a unique $\bar{a} > 0$ such that the solution $w_{3,-\bar{a}}$ is odd, and in particular, the maximum of w restricted to $[-\bar{a}, \bar{a}]$ is 1.*

Proof. Ignoring the initial condition $w(-a) = -1$ and $w'(a) = 0$, the existence of an odd solution w is clear, hence we only need to verify that w has a critical point $a > 0$, and by rescaling can make $w(a) = 1$, hence $w_{3,-a}$ will be the desired solution to (3.2.1). By the Prufer transformation, we have

$$\begin{cases} \phi' = \alpha - \frac{T_3(t)}{p-1} \cos_p^{p-1}(\phi) \sin_p(\phi) \\ \phi(0) = 0 \end{cases}$$

Since $\phi' > \alpha$ as long as $\phi \in [-\frac{\pi_p}{2}, \frac{\pi_p}{2}]$, there exists $\bar{a} \leq \pi_p/(2\alpha)$ such that $\phi(\bar{a}) = \frac{\pi_p}{2}$. Hence the sign of w' changes at $\bar{a}$, therefore $w_{0,-\bar{a}}$ is an odd function whose range is $[-1, 1]$. $\square$

By studying the equation of ϕ one can show that there is a critical value $\alpha = \bar{\alpha}$ so that the oscillatory and asymptotic behavior of w changes. To begin with, let us point out the value of $\bar{\alpha}$ for an intuitive grasp:

Definition 3.3.1. *Denote*

$$\bar{\alpha} := \max_{\theta \in [-\frac{\pi_p}{2}, 0]} \frac{T_2}{p-1} \cos_p^{p-1}(\theta) \sin_p(\theta)$$

and

$$\bar{\theta} := \operatorname*{argmax}_{\theta \in [-\frac{\pi_p}{2}, 0]} \frac{T_2}{p-1} \cos_p^{p-1}(\theta) \sin_p(\theta)$$

Given the IVP of ϕ in T_2-model, we can see that $\phi'(-\pi_p/2) = \alpha > 0$, and ϕ' will keep positive unless $\alpha - \frac{T_2}{p-1}\cos_p^{p-1}(\phi)\sin_p(\phi) = 0$ for some value of $\phi(t)$. Hence if $\alpha < \bar{\alpha}$, ϕ' would become negative before ϕ would reach $\bar{\theta}$, which forces ϕ to be bounded. But if $\alpha > \bar{\alpha}$, then ϕ' has a positive lower bound, hence ϕ will be able to increase forever, thus making w an oscillatory solution. This trichotomy of α also change the behavior of model T_1 and T_3 as they are both asymptotic to T_2 on $[a, \infty)$ as $a \to \infty$. Now let us state these observations precisely:

Proposition 3.3.3. *For $\alpha > \bar{\alpha}$ we have $\delta(3, a) < \infty$ for every $a \in \mathbb{R}$. For $\alpha < \bar{\alpha}$, we have*

$$\lim_{t\to\infty} \phi_{3,a}(t) < \infty \qquad \textit{for all } a \in \mathbb{R}.$$

for a sufficiently large we have

$$-\frac{\pi_p}{2} < \lim_{t\to\infty} \phi_{3,a}(t) < 0 \qquad \textit{and } \delta(3, a) = \infty.$$

When $\alpha = \bar{\alpha}$, we have $\lim_{a\to\infty} \delta(3, a) = \infty$.

For model T_1 we get the following result:

Proposition 3.3.4. *For $\alpha > \bar{\alpha}$ we have $\delta(1, a) < \infty$ for all $a \in [0, \infty)$. If $\alpha \le \bar{\alpha}$ then $\phi_{1,a}$ has finite limit at infinity and $\delta(1, a) = \infty$ for all $a \in [0, \infty)$.*

To prove Proposition 3.3.1 we need to discuss two cases: $\alpha < \bar{\alpha}$ and $\alpha \ge \bar{\alpha}$. We have different situations, where in the first case we can always use model T_3 to produce the comparison solution w, and in the second case we have restriction on the maximum value that u can achieve. Namely we have

Proposition 3.3.5. *Let $\alpha \le \bar{\alpha}$. Then for each $0 < \max u \le 1$, there is an $a \in [-\bar{a}, \infty)$ such that $m(3, a) = \max u$.*

Proof. By (2.6.2) we know that if $\max u = 1$, we have $m(3, -\bar{a}) = \max u$. We can use the continuous dependence of $m(3, a)$ on a to prove the lemma if we can show

$$\lim_{a\to\infty} m(3, a) = 0.$$

Suppose first that $\alpha < \bar{\alpha}$. For any $\bar{\alpha}-\alpha > \epsilon > 0$, take A sufficiently large so that $|T_3(t)-T_2(t)| \leq \epsilon/2$ for $t > A$. Let $a > A$ and $\phi(a) = -\pi_p/2$ be the initial condition on ϕ. By our setting above, for any $t > a$, there exists $-\pi_p/2 < \theta_1 < \bar{\theta} < \theta_2 < 0$ such that

$$\alpha = \frac{T_3(t)}{p-1} \cos_p^{p-1}(\theta_i) \sin_p(\theta_i) \qquad i = 1, 2$$

and $\theta_2 - \theta_1 \leq C(\epsilon)$ as T_3 is asymptotic to T_2. By simple ODE consideration, we can conclude that $\phi(t)$ is asymptotic to θ_1 from below as $t \to \infty$. By the IVP satisfied by e:

$$\begin{cases} \frac{d}{dt} \log(e) = \frac{T_i}{p-1} \cos_p^p(\phi) \\ e(a) = \alpha \end{cases}$$

we have

$$\log(e(t)) - \log(\alpha) = \int_a^t \frac{T_3(s)}{p-1} \cos_p^p(\phi(s)) ds \leq -\frac{|T_2| - \frac{\epsilon}{2}}{p-1} \cos_p^p \left(\bar{\theta} - C(\epsilon)\right)(t-a) \to -\infty$$

as $t \to \infty$, hence $e(t) \to 0$ and so, we showed that $m(3, a) \to 0$ as $a \to \infty$. Now for $\alpha = \bar{\alpha}$, we can use the continuity of $m(3, a)$ with respect to the parameter α to prove this case. □

In the case $\alpha > \bar{\alpha}$, we have $\phi'(t) > \alpha - \bar{\alpha} > 0$ for all $t \geq a$. Hence we shall deal with generic oscillatory behavior of w and cannot hope for $\lim_{a\to\infty} m(3, a) = 0$. Noticing that for model T_2 is translation invariant, hence for all $a \in [0, \infty)$, $m(2, a) = m_2$ is a constant. By different monotonicity of T_1 and T_3, we observe that

Proposition 3.3.6. *If $\alpha > \bar{\alpha}$, then $m(3, a)$ is a decreasing function of a, while $m(1, a)$ is an increasing function of a and*

$$\lim_{a\to\infty} m(3, a) = \lim_{a\to\infty} m(1, a) = m_2.$$

Therefore we can conclude that we can find a comparison solution w from T_i ($i = 1, 2, 3$) when $\alpha > \bar{\alpha}$ and $m(1, 0) \leq \max u \leq 1$. In section 3.4 we will confirm that the maximum of u must lie in that range.

3.3.1 Diameter Comparison

In this section we discuss the relation between $\delta(i, a, \lambda)$ and choice of a and λ, for each model $i = 1, 2, 3$. It will be a key step to prove that under the same diameter assumption, the principal eigenvalue of L_p is bounded below by the eigenvalue model problem over $[-D/2, D/2]$.

Definition 3.3.2. *We define the minimum diameter of the one-dimensional model associated with λ to be*

$$\bar{\delta}(\lambda) = \min\{\delta(i, a, \lambda) | i = 1, 2, 3, a \in I_i\}$$

We can find a lower bound of $\delta(i, a)$ by convexity arguments for $i = 1, 2$:

Proposition 3.3.7 (cf.[20] Proposition 8.2)**.** *For $i = 1, 2$ and any $a \in I_i$, we have $\delta(i, a, \lambda) > \frac{\pi_p}{\alpha}$, where $\alpha = \big(\lambda/(p-1)\big)^{\frac{1}{p-1}}$.*

Model 3 needs a little bit careful attention. For this one we notice first that there is always $\bar{a} > 0$ with an odd solution for initial data at $-\bar{a}$. Namely $w_{3,\bar{a}}$ is odd function with min -1 and max 1. This is a critical situation which minimizes the diameter D given λ:

Proposition 3.3.8 (cf.[20] Proposition 8.4)**.** *For $i = 3$ and $a \in \mathbb{R}$, we have*

$$\delta(3, a, \lambda) \geq \delta(3, -\bar{a}, \lambda) = 2\bar{a}$$

and if $a \neq -\bar{a}$, the inequality is strict.

It is also easy to see from the ODE for ϕ when $i = 3$ that, $\phi' > \alpha$. Therefore $\delta(3, -\bar{a}, \lambda) < \frac{\pi_p}{\alpha}$. Also from this we have $\delta(3, -\bar{a}, \lambda)$ is strictly decreasing function of α, hence it is also a decreasing function of λ. This means that $\bar{\delta}(\lambda)$ is a strictly decreasing function. Thus if we see λ in turn as a function of δ when we fix $a = -\bar{a}$ and $i = 3$, we also have the decreasing monotonicity of λ with respect to δ: if $\delta_1 \leq \delta_2$, we have

$$\lambda(\delta_1) \geq \lambda(\delta_2).$$

3.4 Maximum of Eigenfunctions

From the fine properties of the model equation 3.2.1, we have reduced the search for the range-matching comparison solution to showing that $\max u > m(1, 0)$. In this section we are going to compare the maximum of the eigenfunction and the model functions. The idea of the maximum comparison follows from [2] for the case $p = 2$ and the extension to all $p > 1$ case in [23]. Basically we need to explore the geometric consequence of the curvature-dimension condition $BE(\kappa, N)$, namely that volume of ball of radius r has volume at least Cr^N by Bishop-Gromov comparsion theorem. We will compare the volume of a sublevel set of u with a sublevel set of w under a different measure. We define a new measure on the interval $[a, b(a)]$:

Definition 3.4.1. *Given u and w as the eigenfunctions defined in 3.2.1, let μ be a measure on $[a, b(a)]$ defined by*

$$\mu(A) = m(u^{-1} \circ w(A)). \tag{3.4.1}$$

μ is essentially the pullback of the volume measure on M by $w^{-1} \circ u$. Currently let us focus on the first model with $a = 0$, i.e. $T_1 = -(N-1)\sqrt{-\kappa}\, \operatorname{cotanh}(\sqrt{-\kappa} t)$, with initial condition $w(0) = -1$ and $w'(0) = 0$.

First we have a theorem which can be seen as a comparison between the model function and the eigenfunction.

Theorem 3.4.2 (Theorem 34 [20])**.** *Let u and w be as above and define*

$$E(s) := -\exp\left(\int_{t_0}^{s} \frac{w^{(p-1)}}{w'^{(p-1)}} dt\right) \int_a^s w^{(p-1)} d\mu(t)$$

then E is increasing on $(a, t_0]$ and decreasing on $[t_0, b)$.

The quantity $E(s)$ does not have a very clear intuitive meaning in the form above. However, we can use the model equation and the definition of μ to rewrite $E(s)$ as the ratio of integral of $u^{(p-1)}$ and $w^{(p-1)}$ on corresponding sublevel sets $\{u \le w(s)\}$ and $\{w \le w(s)\}$, with respect to certain measures:

Theorem 3.4.3 (Theorem 35,[20]). *Under the hypothesis of Theorem 6.1 the function*

$$E(s) := \frac{\int_a^s w^{(p-1)} d\mu}{\int_a^s w^{(p-1)} \sinh^{n-1}(\sqrt{-\kappa}t)dt} = \frac{\int_{u \leq w(s)} u^{(p-1)} dm}{\int_a^s w^{(p-1)} \sinh^{n-1}(\sqrt{-\kappa}t)dt}$$

is increasing on $(a, t_0]$ *and decreasing on* $[t_0, b)$*.*

Proof of the equivalent definition of $E(s)$*.* We need to show that

$$-\exp\left(\int_{t_0}^s \frac{w^{(p-1)}}{w'^{(p-1)}} dt\right) = C\left(\int_a^s w^{(p-1)}(t) \sinh^{n-1}(\sqrt{-\kappa}t)\right)^{-1} \tag{3.4.2}$$

Denoting $\eta(t) = \sinh^{n-1}(\sqrt{-\kappa}t)$, we can verify that the model equation (3.2.1) can be rewritten as

$$\frac{d}{dt}\left[\eta w'^{(p-1)}\right] + \lambda \eta w^{(p-1)} = 0$$

Dividing both sides by $w'^{(p-1)}$ we get

$$\frac{d}{dt} \log\left[\eta(w')^{(p-1)}\right] + \lambda \frac{w^{(p-1)}}{w'^{(p-1)}} = 0$$

by integrating the first equation from a to s, and the second equation from t_0 to s followed by an exponentiation, we can see that choosing

$$C = \lambda^{-1}[\eta w'^{(p-1)}]\Big|_{t_0}$$

suffices to prove the equation (3.4.2). □

The proof of Theorem 3.4.2 will be reduced to prove that a certain measure is nonnegative over $[a, b]$, for which we will use essentially the gradient comparison theorem 3.2.1 and some integration by parts technique. The following proof is adapted from [20] (proof of Theorem 7.2) and some regularity issues were taken cared of.

Proof of 3.4.2. We consider an arbitrary smooth function $H : (a, b) \to \mathbb{R}$ with compact support in

(a,b). For integration-by-part purpose, let $G:[-1,w(b)]\to\mathbb{R}$ be defined as

$$\begin{cases}\frac{d}{dt}\Big[\big(G\circ w\big)^{(p-1)}(t)\Big]=H(t)\\ G(-1)=0\end{cases}$$

Then we choose $K:[-1,w(b)]\to\mathbb{R}$ such that $(K(t))'=G(t)$. Then for $x\in M$ at which u is $C^{2,\alpha}(M)$ and $K'(u)(x)\neq 0$, we have

$$L_p(K(u))=\Gamma(K(u))^{\frac{p-2}{2}}\left[L(K(u))+(p-2)\frac{H_{K(u)}(K(u),K(u))}{\Gamma(K(u))}\right]$$

By Lemma 2.3.1, we get $\Gamma(K(u))=K'^2(u)\Gamma(u)$, $L(K(u))=K'(u)L(u)+K''(u)\Gamma(u)$ and $H_{K(u)}(K(u),K(u))=K'^3(u)H_u(u,u)+K''(u)K'(u)\Gamma(u)^2$. By our definition, $K'(u)=G(u)$ and $K''(u)=G'(u)$, therefore

$$L_p(K(u))=G(u)^{(p-1)}L_pu+(p-1)|G(u)|^{p-2}G'(u)\Gamma(u)^{\frac{p}{2}}$$

For the exceptional points $E=\{\Gamma(K(x))=0\}$, the calculation above does not hold. Using the integration by part formula, for any $\phi\in C^\infty(M)$, we have

$$\int_M\phi L_p(K(u))dm=-\int_M\Gamma(K(u))^{\frac{p-2}{2}}\Gamma(K(u),\phi)dm+\int_{\partial M}\Gamma(K(u),\nu)\Gamma(K(u))^{\frac{p-2}{2}}\phi dm$$

the second term is always 0 as $\Gamma(K(u),\nu)=K'(u)\Gamma(u,\nu)=0$ by the Neumann boundary condition on u. Given $\epsilon>0$, let $\phi_\epsilon:M\to[0,1]$ be 1 on $B_\epsilon(E)$, the ϵ-neighborhood of E, and $\phi_\epsilon=0$ on $M\setminus B_{2\epsilon}(E)$ and smooth everywhere with $\Gamma(\phi)<C/\epsilon$. Hence we have

$$\begin{aligned}0=\int_M L_p(K(u))dm&=\int_M(1-\phi_\epsilon)L_p(K(u))dm+\int_M\phi_\epsilon L_p(K(u))dm &\quad(3.4.3)\\ &=\int_M(1-\phi_\epsilon)\Big[G(u)^{(p-1)}L_pu+(p-1)|G(u)|^{p-2}G'(u)\Gamma(u)^{\frac{p}{2}}\Big]dm &\quad(3.4.4)\\ &\quad-\int_{B_{2\epsilon}(E)}\Gamma(K(u))^{\frac{p-2}{2}}\Gamma(K(u),\phi_\epsilon)dm &\quad(3.4.5)\end{aligned}$$

Suppose $E^\circ\neq\emptyset$, then $\Gamma(K(u))=0$ implies $G(u)=0$ or $\Gamma(u)=0$ on E°, and in the latter case, we also have $u=0$ by the weak definition of the eigenfunction(Definition 2.2.3). Therefore we can always extend the integrand on the left hand side by 0 to whole of E by continuity of $\Gamma(u)$. Hence we have

$$\lim_{\epsilon\to 0}\int_M (1-\phi_\epsilon)\Big[G(u)^{(p-1)}L_p u+(p-1)|G(u)|^{p-2}G'(u)\Gamma(u)^{\frac{p}{2}}\Big]dm$$
$$=\int_M \Big[G(u)^{(p-1)}L_p u+(p-1)|G(u)|^{p-2}G'(u)\Gamma(u)^{\frac{p}{2}}\Big]dm$$

For the second term we have

$$\begin{aligned}\left|\int_{B_{2\epsilon}(E)}\Gamma(K(u))^{\frac{p-2}{2}}\Gamma(K(u),\phi_\epsilon)dm\right| &\le \sup_{B_{2\epsilon\setminus\epsilon}(E)}\Big[\Gamma(K(u))^{\frac{p-1}{2}}\Gamma(\phi_\epsilon)^{\frac{1}{2}}\Big]\cdot m(B_{2\epsilon\setminus\epsilon}(E))\\ &\le \sup_{B_{2\epsilon\setminus\epsilon}(E)}\Big[\Gamma(K(u))^{\frac{p-1}{2}}\Big]\cdot\frac{C}{\epsilon}\cdot C'\epsilon\\ &\le C\sup_{B_{2\epsilon\setminus\epsilon}(E)}\Big[\Gamma(K(u))^{\frac{p-1}{2}}\Big]\end{aligned}$$

for some C independent of ϵ. By the continuity of $\Gamma(K(u))$, the right hand side converges to 0 as $\epsilon\to 0$. Hence by taking $\epsilon\to 0$ in equation (3.4.3), we have

$$\int_M \Big[-G(u)^{(p-1)}\lambda u^{(p-1)}+(p-1)|G(u)|^{p-2}G'(u)\Gamma(u)^{\frac{p}{2}}\Big]dm=0$$

Applying the gradient comparison theorem 3.2.1, we have

$$\begin{aligned}\lambda\int_M G(u)^{(p-1)}u^{(p-1)}dm &\le \int_M (p-1)|G(u)|^{p-2}G'(u)(w'\circ w^{-1}(u))^p dm\\ &=\int_a^b (p-1)|G(w)|^{p-2}G'(w)(w')^p d\mu\\ &=\int_a^b H(s)(w'(s))^{(p-1)}d\mu(s)\end{aligned}$$

Changing the left hand side integral to $d\mu$ we have

$$\begin{aligned}\lambda \int_M G(u)^{(p-1)} u^{(p-1)} dm &= \lambda \int_a^b G(w(s))^{(p-1)} w(s)^{(p-1)} d\mu(s) \\ &= \lambda \int_a^b \Big[\int_a^s H(t)dt \Big] w(s)^{(p-1)} d\mu(s) \\ &= \lambda \int_a^b \Big[\int_s^b w(t)^{(p-1)} d\mu(t) \Big] H(s) ds \\ &= \lambda \int_a^b \Big[- \int_a^s w(t)^{(p-1)} d\mu(t) \Big] H(s) ds\end{aligned}$$

Since $\int_a^b w^{(p-1)} d\mu = \int_M u^{(p-1)} = -\lambda^{-1} \int_M L_p u dm = 0$. Hence we have

$$-\lambda \Big[\int_a^s w(t)^{(p-1)} d\mu(t) \Big] ds - w'(s)^{(p-1)} d\mu(s) \text{ is a nonpositive measure on } [a, b].$$

Since $w^{(p-1)}/w'^{(p-1)}$ is nonpositive on $[a, t_0]$ and nonnegative on $[t_0, b]$, we can multiply it to the above positive measure and get

$$-\lambda \Big[\int_a^s w(t)^{(p-1)} d\mu(t) \Big] \frac{w(s)^{(p-1)}}{w'(s)^{(p-1)}} ds - w(s)^{(p-1)} d\mu(s) = \exp\Big(- \lambda \int_{t_0}^s \frac{w^{(p-1)}}{w'^{(p-1)}} dt \Big) dE(s)$$

is nonnegative measure on $[a, t_0]$ and a nonpositive measure on $[t_0, b]$. Therefore $E(s)$ is increasing on $[a, t_0]$ and decreasing on $[t_0, b]$. □

To prove the maximum comparison we study the volume of a small ball around the minimum of u. By the gradient comparison we have the following:

Lemma 3.4.1. *For ϵ sufficiently small, the set $u^{-1}[-1, -1+\epsilon)$ contains a ball of radius $w^{-1}(-1+\epsilon) - a$.*

Proof. Fix $x_0 \in M$ be such that $u(x_0) = -1$. Consider $r > 0$ and $x \in M$ such that $\text{dist}(x_0, y) = r$. By the gradient comparison theorem 3.2.1, we have $\Gamma(w^{-1} \circ u) \leq 1$ on M, hence we have

$$w^{-1}(u(x)) - w^{-1}(u(x_0)) \leq r$$

i.e., $u(x) \le w(a+r)$. If $r = w^{-1}(-1+\epsilon) - a$, we get $u(x) \le -1+\epsilon$. Hence

$$B_{w^{-1}(-1+\epsilon)-a}(x_0) \subset u^{-1}[-1, -1+\epsilon)$$

as is claimed. $\square$

Now we can prove the maximum comparison by combining Bishop-Gromov volume comparison theorem and the following estimate:

Theorem 3.4.4. *Let $n \ge N$ and $n > 1$. If u is an eigenfunction satisfying $\min u = -1 = u(x_0)$ and $\max u \le m(1,0) = w_{1,0}(b(1,0))$, then there exists a constant $c > 0$ such that for all r sufficiently small, we have*

$$m(B_{x_0}(r)) \le cr^n.$$

Proof. To keep notations short, let $w = w_{1,0}$. Let ϵ be small such that $-1+\epsilon < -2^{-p+1}$. Then we have $u^{(p-1)} < -\frac{1}{2}$ when $u < -1+\epsilon$. Let t_0 be the first zero of w, then by Theorem 6.1 we have $E(t) \le E(t_0)$. Therefore by Theorem 6.2 we get

$$m(B_{x_0}(r_\epsilon)) \le C \int_{u \le -1+\epsilon} u^{(p-1)} dm \le CE(t_0) \int_a^{w^{-1}(-1+\epsilon)} w^{(p-1)} \sinh^{n-1}(t) dt \le C' r_\epsilon^n$$

Since ϵ can be arbitrarily small, we have the claim holds for r sufficiently small. $\square$

Corollary 3.4.5. *Let $n \ge N$, $n > 1$, and $w_{(1,0)}$ be the corresponding model function. If u is an eigenfunction with $\min u = -1$, then $\max u \ge m(1,0)$.*

Proof. Suppose that $\max u < m(1,0)$, from the analysis of the model equation, $m(1,0)$ is the least possible value among $\max w$ for all model solutions w. Therefore by continuous dependence of the solution of model equation on n, we can find $n' > n$ so that $\max u$ is still less that the maximum of the correspoding model equation. Since $BE(\kappa, n')$ is still satisfied, we have by Theorem 6.3, that $m(B_{x_0}(r)) \le cr^{n'}$ for r sufficiently small. However by Bishop-Gromov volume comparison we have $m(B_{x_0}(r)) \ge Cr^N$. This is a contradiction since $n' > n \ge N$. $\square$

3.5 Proof of Main Result

Now we can combine the gradient and maximum comparison, together with properties of the model equation to show the eigenvalue comparison.

Theorem 3.5.1. *Let M be compact and connected and L be an elliptic diffusion operator with invariant measure m. Assume that L satisfies $BE(\kappa, N)$ with $\kappa < 0$ and $N < \infty$. Let u be an eigenfunction associated with λ satisfying Neumann boundary condition if $\partial M \neq \emptyset$, where λ is the first nonzero eigenvalue of L_p. Let D be diameter defined by the intrinsic distance metric on M. Then we have a sharp comparison:*

$$\lambda \geq \lambda_D$$

where λ_D is the first nonzero eigenvalue of the Neumann eigenvalue problem on $[-D/2, D/2]$:

$$\frac{d}{dt}\left[(w')^{(p-1)}\right] - (N-1)\sqrt{-\kappa}\tanh(\sqrt{-\kappa}t)(w')^{(p-1)} + \lambda w^{(p-1)} = 0.$$

Proof. We scale u so that $\min u = -1$ and $\max u \leq 1$. By Proposition 7.1 we can find a model function $w_{i,a}$ such that $\max u = \max w_{i,a}$. By the gradient comparison theorem, $\Gamma(w_{i,a}^{-1} \circ u) \leq 1$. Let x and y on M be points where u attains maximum and minimum, then we have

$$D \geq |w_{i,a}^{-1} \circ u(x) - w_{i,a}^{-1} \circ u(y)| = w_{i,a}^{-1}(m(i,a)) - w_{i,a}^{-1}(-1) = \delta(i, a, \lambda) \geq \delta(i, \bar{a})$$

Therefore by the monotonicity of eigenvalue of the model equation, we have that

$$\lambda \geq \lambda_D.$$

To check the sharpness of this result, we have the following examples: let $M_i = [-D/2, D/2] \times_{i^{-1}\tau_3} S^{n-1}$ be a warped product where S^{n-1} is the standard unit sphere, and $\tau_3(t) = \cosh(\sqrt{-\kappa}t)$. If we consider L being the classical Laplacian on M, then standard computation shows that M_i has $\mathrm{Ric} \geq -(n-1)\kappa$ and geodesically convex boundary. Hence it also satisfy the $BE(\kappa, n)$ condition. If we take $u(t, x) = w(t)$ where w is the solution to our one-dimensional model equation with $\lambda = \lambda_D$. Since the diameter of M_i tends to d as $i \to \infty$, we see that the first eigenvalue on M_i converges to

λ_d, which shows the sharpness of our lower bound.

$\square$

Part II

Min-max Theory and Existence of Minimal Hypersurfaces

Chapter 4

Introduction to the Main Results

In the second part we shall prove the following results:

Theorem 4.0.1. *Let $(M^{n+1}, \partial M, g)$ be a compact simply connected Riemannian manifold with boundary and $2 \leq n \leq 6$. If g maximizes the normalized width $W(M,g)$ in the conformal class of g, and all free boundary minimal hypersurfaces in M is properly embedded, then there is a sequence of free boundary minimal hypersurfaces $\{\Sigma_i\}$ with index $ind(\Sigma_i) = 0$ or 1 and $|\Sigma_i| \leq W(M,g)$, and the following limit holds for all $f \in C(M)$:*

$$\lim_{k\to\infty} \frac{1}{\sum_{i=1}^{k} |\Sigma_i|} \sum_{i=1}^{k} \int_{\Sigma_i} f dA_g = \frac{1}{vol(M,g)} \int_M f dV_g \tag{4.0.1}$$

Under stronger assumptions, we can prove that these free boundary minimal hypersurfaces $\{\Sigma_i\}$ can be chosen so that $\mathrm{ind}(\Sigma_i) = 1$ and $|\Sigma_i| = W(M,g)$:

Theorem 4.0.2. *Let $(M^{n+1}, \partial M, g)$ be a compact simply connected Riemannian manifold with boundary and $2 \leq n \leq 6$. If g maximizes the normalized width $W(M,g)$ in the conformal class of g, and there is no stable free boundary minimal hypersurface of area less than $W(M,g)$, then there is a sequence of free boundary minimal hypersurfaces $\{\Sigma_i\}$ with index $ind(\Sigma_i) = 1$ and $|\Sigma_i| = W(M,g)$, and the following limit holds for all $f \in C(M)$:*

$$\lim_{k\to\infty} \frac{1}{k} \sum_{i=1}^{k} \int_{\Sigma_i} f dA_g = \frac{W(M,g)}{vol(M,g)} \int_M f dV_g \tag{4.0.2}$$

Despite the similarity of statements in Theorem 4.0.1 and 4.0.2, we will use different ideas to prove them. The crucial reason is that the existence of an optimal sweep-out under the extra assumption in Theorem 4.0.2, which allows us to compute the variation of width without perturbing the metric to a better behaved family of metrics.

Equations (4.0.1) and (4.0.2) are called equidistribution properties. Intuitively, they imply that the sequence of hypersurfaces Σ_i will fill up the manifold M(possibly with repetition) in a uniform manner as Radon measures. These theorems reflects certain symmetry of extremizing metric of width functional W.

The problem of existence of minimal hypersurfaces has been studied extensively from the beginning of last century. The min-max theory of closed minimal hypersurfaces was first developed by Almgren (1960) in an effort to bring the power of Morse theory into the space of closed hypersurfaces. Pitts improved the regularity theory of the min-max hypersurfaces and proved that for $3 \leq n \leq 6$, the hypersurfaces produced by Almgren's min-max procedure is smooth. More recently, Marques-Neves-Song [19] proved that for C^∞-generic (in the Baire sense) metric g on a closed manifold M^{n+1} of dimension $2 \leq n \leq 6$, there is a sequence of smooth, embedded and closed hypersurfaces $\{\Sigma_i\}$ such that the equation (4.0.1) holds.

In 2019, Ambrozio-Montezuma [19] considered a new class of metrics on three-spheres $(\mathbb{S}^3, g)$, which are maximizer of the Simon-Smith width functional. They proved that there exists a sequence of embedded minimal $\mathbb{S}^2$ that satisfies equation (4.0.1) and if an extra assumption is satisfied, equation (4.0.2).

These works inspires us to look at the space of free boundary minimal hypersurfaces. Using the min-max theory for free boundary minimal hypersurface developed mainly by Zhou-Guang-Li-Wang [12][7], we proved Theorem 4.0.1 and 4.0.2. They can be regarded as extension of the work of Ambrozio-Montezuma [1] and Marques-Neves-Song [19] in the case of free boundary minimal hypersurfaces.

Chapter 5

Free boundary minimal hypersurfaces

In this section we introduce some definitions and preliminary results on free boundary minimal surfaces in Riemannian manifolds with boundary. Let $(M^{n+1}, \partial M, g)$ be a compact Riemannian manifold with smooth boundary ∂M. First we define the notion of a free boundary minimal hypersurface in M.

Definition 5.0.1. *A submanifold Σ is called a free boundary minimal hypersurface (hereafter FBMH) if Σ has vanishing mean curvature ($H = 0$) and $\partial\Sigma \subset M$, and the normal vector $\vec{n}$ of Σ in M is orthogonal to the conormal vector η of $\partial\Sigma$ in Σ.*

The definition above can also be derived from the fact that a free boundary minimal hypersurface is a critical point of the area functional.

Proposition 5.0.1 (First variational formula)**.** *Let $\phi : M \times (-\epsilon, \epsilon) \to M$ be smooth so that $\phi(\cdot, 0) = id_M : M \to M$, and for any $t \in (-\epsilon, \epsilon)$, $\phi_t(\partial M) := \phi(\partial M, t) \subset \partial M$. Let $A(s) := |\phi_s(\Sigma)|$. Then we have the following formula:*

$$A'(0) = \int_\Sigma -H\langle \vec{n}, \frac{\partial \phi}{\partial t}\Big|_{t=0}\rangle d\mu_\Sigma + \int_{\partial\Sigma} \phi\langle \eta, \vec{n}\rangle d\mu_{\partial\Sigma} \tag{5.0.1}$$

From the proposition above we can see that if $A'(0) = 0$ for any admissible variation ϕ, then both integral terms above should vanish, which is equivalent to $H \equiv 0$ on Σ and $\langle \eta, \vec{n}\rangle = 0$ along $\partial\Sigma$. In addition, we use the second variation formula to characterize the stability of free boundary minimal hypersurfaces:

Proposition 5.0.2 (Second variation formula)**.** *Let ϕ be defined as above, and assume that $\frac{\partial \phi}{\partial t}\big|_{t=0} = f\vec{n}$ is a normal vector field on Σ. Then we have the following formula:*

$$A''(0) = \int_{\Sigma} \left(|\nabla f|^2 - Ric_M(\vec{n},\vec{n})f^2 - |A|^2 f^2\right) d\mu_{\Sigma} - \int_{\partial\Sigma} h^{\partial M}(\eta,\eta) f^2 d\mu_{\partial\Sigma} \tag{5.0.2}$$

The right hand side of the second variation formula can be seen as a quadratic form $I : C^{\infty}(M) \times C^{\infty}(M) \to \mathbb{R}$ applied to (f, f). This symmetric quadratic form is the index form of Σ defined as

Definition 5.0.2. *For $f, g \in C^{\infty}(M)$, the quadratic form*

$$I(f,g) = \int_{\Sigma} \Big(\langle \nabla f, \nabla g\rangle - Ric_M(\vec{n},\vec{n}) fg - |A|^2 fg\Big) d\mu_{\Sigma} - \int_{\partial\Sigma} h^{\partial M}(\eta,\eta) fg d\mu_{\partial\Sigma} \tag{5.0.3}$$

is called the index form of Σ. The dimension of negative eigenspace of I is called the index of Σ. If the index is 0, then we call Σ a stable FBMH in M, i.e. the area of Σ does not decrease to the second order under any variations that preserves $\partial\Sigma \subset \partial M$.

Example 5.0.3. *Let $\mathbb{B}_1^{n+1} \subset \mathbb{R}^{n+1}$ be the unit ball centered at 0, then all equatorial disks are FBMH in $\mathbb{B}_1^{n+1}$. They are congruent to the standard unit disk $\mathbb{D}_1^n = \{(x, 0) \in \mathbb{R}^{n+1} : \|x\| \leq 1\}$. The index of these FBMHs is 1, namely pushing $\mathbb{D}_1^n$ up to a flat disk $\{(x, x_{n+1}) : \|x\| \leq \sqrt{1 - x_{n+1}^2}\}$ reduces its area, but any other infinitesimal deformation orthogonal to this one does not decrease the area.*

Next we deal with the issue of embeddedness. There is a peculiar situation that happens uniquely to free boundary hypersurfaces which needs to be taken special care of: non-properly embeddedness. We have the following definition:

Definition 5.0.4. *Let $\Sigma \subset M$ be an embedded hypersurface, with $\partial\Sigma \subset \partial M$. We say Σ is properly embedded if $\Sigma \cap \partial M = \emptyset$, i.e. the interior of Σ cannot touch the boundary of M.*

Remark 5.0.5. *If E is not properly embedded, then we should be careful with defining the variation vector field on Σ as the variation that move Σ away from ∂M at $\partial M \cap \Sigma$ can be considered.*

5.1 Min-max Construction of Minimal Hypersurfaces

In this section we introduce fundamentals of the min-max theory for free boundary minimal hypersurfaces. We will follow the formulation using integer rectifiable currents in [7]. Let (M^{n+1}, g) be a smooth manifold with nonempty boundary. We can regard M^{n+1} as a submanifold isometrically embedded in the Euclidean space $\mathbb{R}^L$ of sufficiently high dimension.

Definition 5.1.1 (k-currents and mass)**.** *Given an open set $U \subset \mathbb{R}^L$, let $D^k(U)$ denotes the space of smooth k-forms compactly supported in U. The space of k-currents in U is the space of continuous linear functionals on $D^k(U)$, which is denoted as $D_k(U)$. The mass of T, denoted as $\mathbf{M}(T)$, is defined as*

$$\mathbf{M}(T) := \sup_{\|\omega\| \leq 1, \omega \in D^k(U)} T(\omega)$$

where $\|\omega\| = \sup_{x \in U} \sqrt{\omega(x) \cdot \omega(x)}$.

A general current can be wild, so we will work on the more regular class of currents, called integer multiplicity k-rectifiable currents, namely the currents modeled on k-rectifiable submanifolds.

Definition 5.1.2 (Integer multiplicity rectifiable currents)**.** *If $T \in D_k(U)$, we say T is an integer multiplicity rectifiable k-current if it can be expressed as*

$$T(\omega) = \int_M \langle \omega(x), \xi(x) \rangle \theta(x) d\mathcal{H}^k(x), \qquad \omega \in D_k(U)$$

where M is an $\mathcal{H}^k$-measurable countbly k-rectifiable subset of U, θ is a locally $\mathcal{H}^k$ integrable positive integer valued function(called multiplicity), and $\xi : M \to \Lambda^k(\mathbb{R}^L)$ is $\mathcal{H}^k$-measurable and for $\mathcal{H}^k$-a.e. points $x \in M$, $\xi(x) = \tau_1 \wedge \tau_2 \wedge \cdots \wedge \tau_k$, where τ_i forms an orthonormal basis of $T_x M$.

In order to better characterize the convergence of integer rectifiable currents, we consider the flat metric topology:

Definition 5.1.3 (flat metric)**.** *Let $\mathcal{I}$ be the set of integer multiplicity rectifiable currents which* *satisfies* $\mathbf{M}_W(\partial T) < \infty$ *for all compact sets* W *in* U*. For all such* W*, we define a metric on* $\mathcal{I}$ *by*

$$d_W(T_1, T_2) = \inf\{\mathbf{M}_W(S) + \mathbf{M}_W(R) : T_1 - T_2 = \partial R + S,$$
$$\text{where } R \in D_{k+1}(U), S \in D_k(U) \text{ have integer multiplicity}\}$$

In min-max theory, we begin with the notion of a sweep-out of M by a family of n-dimensional currents which represents the top dimensional homology of M relative to ∂M. Let us denote $\mathcal{Z}_n(M, \partial M, g, \mathbb{Z})$ as the class of relative n-cycles in M with integer coefficients.

Definition 5.1.4 (One Sweep-out)**.** *Let* $(M^{n+1}, \partial M, g)$ *be as above. A one parameter family of maps* $\Phi : I = [-1, 1] \to \mathcal{Z}_n(M, \partial M, g, \mathbb{Z})$ *is called a 1-sweep-out if the following conditions are satisfied:*

(1) Φ *is a continuous map in flat topology;*

(2) $\sup_{t\in I} \mathbf{M}(\Phi(t)) < \infty$*;*

(3) Φ *does not have mass concentration;*

(4) $F(\Pi_\Phi)$ *represents a non-zero element in* $H_{n+1}(M, \partial M)$*.*

The maximum slice in the sweep out Φ contains the information on the width of the M, but we still need to take the infimum among all sweep-outs in order to find the best fit. Hence we have the notion of 1-width of a manifold.

Definition 5.1.5 (Normalized Width)**.** *The* 1*-width of a manifold with metric* g

$$W(M, \partial M, g) = \inf_{\Phi \in \bar{\Lambda}} \left(\max_{t\in[-1,1]} \mathbf{M}(\Phi(t), g) \right)$$

where Φ *is a sweepout of* $(M, \partial M, g)$*. The normalised* 1*-width is defined as*

$$W^*(M, \partial M, g) = \frac{W(M, \partial M, g)}{Vol(M, g)^{\frac{n}{n+1}}}.$$

Now we can define the maximizer of width in a conformal class of metrics.

Definition 5.1.6. *We say that a metric g maximizes the normalized width in its conformal class if for any $f \in C^2(M)$, $W^*(M, \partial M, e^{-f}g) \leq W^*(M, \partial M, g)$.*

Given a manifold, width can be regarded as a nonlinear functional over the space of metrics g. The following proposition is a basic property of width: locally Lipschitz.

Proposition 5.1.1. *Let g be a Riemannian metric on $(M, \partial M)$, and $0 < C_1 < C_2$ be constants. Then there exists $C = C(g, C_1, C_2) > 0$ such that whenever $C_1 g < g_1, g_2 < C_2 g$, we have*

$$|W(M, \partial M, g_1) - W(M, \partial M, g_2)| \leq C|g_1 - g_2|_{g,\infty} \tag{5.1.1}$$

Proof. Since M is a compact manifold, we have $W(M, \partial M, g) < \infty$. It is easy to verify that whenever $g' < C_2 g$, we have $W(M, \partial M, g') \leq C_2^{\frac{n}{2}} W(M, \partial M, g)$ by a scaling argument. For any admissible one parameter family Φ and $t \in [-1, 1]$, we have the point-wise difference

$$\begin{aligned}
\mathbf{M}(\Phi(t), g_1) - \mathbf{M}(\Phi(t), g_2) &\leq \left[\sup_{(x,v) \in T\Phi(t)} \left(\frac{g_1(v,v)}{g_2(v,v)}\right)^{\frac{n}{2}} - 1\right] \mathbf{M}(\Phi(t), g_2) \\
&\leq \left[\left(1 + \sup_{(x,v) \in T\Phi(t)} \frac{|g_1(v,v) - g_2(v,v)|}{g_2(v,v)}\right)^{\frac{n}{2}} - 1\right] \mathbf{M}(\Phi(t), g_2) \\
&\leq C|g_1 - g_2|_{g,\infty} \mathbf{M}(\Phi(t), g_2) \\
&\leq C C_2^{\frac{n}{2}} |g_1 - g_2|_{g,\infty} \mathbf{M}(\Phi(t), g)
\end{aligned}$$

Since we may take Φ and t so that $\mathbf{M}(\Phi(t), g_2)$ is arbitrarily close to $W(M, \partial M, g_2)$, we have

$$\begin{aligned}
W(M, \partial M, g_1) - W(M, \partial M, g_2) &\leq \mathbf{M}(\Phi(t), g_1) - W(M, \partial M, g_2) \\
&\leq C|g_1 - g_2|_{g,\infty} W(M, \partial M, g)
\end{aligned}$$

The other direction of inequality is similar, hence we proved the Lipshitz continuity of width. $\square$

5.2 Existence Theorems of Free Boundary Minimal Hypersurface

In this section we include the previous results on general existence of free boundary minimal hypersurfaces, mostly developed in [7].

Theorem 5.2.1 (cf. [7] Proposition 7.3)**.** *Suppose* $3 \leq (n+1) \leq 7$*. Then there exist a finite disjoint collection* $\{\Sigma_1, \ldots, \Sigma_N\}$ *of smooth, compact, almost properly embedded FBMHs in* $(M, \partial M, g)$*, and integers* $\{m_1, \ldots, m_N\} \subset \mathbb{N}$ *such that*

$$W(M, \partial M, g) = \sum_{j=1}^{N} m_j \cdot area_g(\Sigma_j) \quad and \quad \sum_{j=1}^{N} ind(\Sigma_j) \leq 1$$

5.3 Proof of the Main Theorems

In this section we prove Theorem 1.1 using a perturbation method originally due to Marques-Neves-Song [19], and prove Theorem 1.2 by a calculation of derivative of width inspired by Fraser-Schoen's work[6] on Steklov eigenvalues.

5.3.1 Proof of Theorem 4.0.1

In view of the abstract theorem 4.2, we can reduce the equi-distribution property to the following lemma:

Lemma 5.3.1. *Let* g *be a Riemannian metric on* M *that maximizes the normalized width in its conformal class. For every continuous function* f *satisfying*

$$\int_M f dV_g < 0,$$

there exists some integers $n_1, \cdots, n_N$*, and disjoint embedded free boundary minimal hypersurfaces* $\Sigma_1, \cdots, \Sigma_N$ *in* (M, g) *such that*

$$W(M, g) = \sum_{i=1}^{N} n_j area(\Sigma_i, g), \quad \sum_{i=1}^{N} ind_g(\Sigma_i) \leq 1$$

and

$$\sum_{i=1}^{N} n_i \int_{\Sigma_i} f dA_g \leq 0.$$

In order to associate the function f with the derivative of width under a conformal change of metric, we need to perturb the conformal family of the original metric to a new family so that the width is differentiable. The following technical lemma is crucial:

Lemma 5.3.2. *Let $q \geq 4$ be an integer, and $g : [0,1] \to \Gamma_q$ be a smooth embedding. Then there exist smooth embeddings $h : [0,1] \to \Gamma_q$ which are arbitrarily close to g in the smooth topology, and $J \subset [0,1]$ with full Lebesgue measure such that*

(1) The function $W(M, h(t))$ is differentiable at every $\tau \in J$; and

(2) For each $\tau \in J$, there exist a collection of integers $\{n_1, \cdots, n_N\}$ and a finite collection $\{\Sigma_1, \cdots, \Sigma_N\}$ of disjoint free boundary embedded minimal hypersurfaces of class C^q in $(M, h(\tau))$ such that

$$W(M, h(\tau)) = \sum_{k=1}^{N} n_k \cdot area(\Sigma_k, h(\tau)), \quad \sum_{k=1}^{N} ind_{h(\tau)}(\Sigma_k) \leq 1,$$

$$and \quad \frac{d}{dt}\Big|_{t=\tau} W(M, h(t)) = \frac{1}{2} \sum_{k=1}^{N} n_k \int_{\Sigma_k} Tr_{(\Sigma_k, h(\tau))}(\partial_t h(\tau)) dA_{h(\tau)}.$$

Proof of Lemma 5.3.2. First, due to the density of bumpy metric on M and Rademacher's theorem, we can perturb the smooth family $g : [0,1] \to \Gamma_q$ to $h : [0,1] \to \Gamma_q$ which is arbitrarily close to g in smooth topology, and a set $J \subset [0,1]$ of full measure such that $h(\tau)$ is a bumpy metric and $W(M, h(t))$ is differentiable at τ, for all $\tau \in J$.

For all $\tau \in J$, fix a sequence $t_i \to \tau$, we have

$$\frac{d}{dt} W(M, h(t))\Big|_{t=\tau} = \lim_{i \to \infty} \frac{W(M, h(t_i)) - W(M, h(\tau))}{t_i - \tau}.$$

By [7], we can find a finite disjoint collection of FBMHs $\{\Sigma_1(t_i), \cdots, \Sigma_{i_k}(t_i)\}$ and integers $\{N_1, \cdots, N_{i_k}\}$ such that

$$W(M, h(t_i)) = \sum_{j=1}^{k} N_j \text{area}(\Sigma_{i_j}(t_i)) \qquad \sum_{j=1}^{k} N_j \cdot \text{Ind}(\Sigma_{i_j}(t_i)) \leq 1$$

Now as $t_i \to \tau$, since h is a smooth family we have area$(\Sigma_{i_j}(t_i))$ uniformly bounded below and above by $W(M, h(\tau))$ as t_i is sufficiently close to τ. Therefore by the compactness theorem 5.4.1

we can extract a subsequence t_{i_j} so that $\Sigma_{i_{j_k}}$ converges in the varifold sense to Σ_k. As M is simply connected, Σ_k is two sided. Since the metric $h(\tau)$ is bumpy, we can conclude that the convergence is graphical and smooth with multiplicity one. Therefore standard calculation shows

$$\lim_{i\to\infty} \frac{\text{area}(\Sigma_{i_{j_k}}, h(t_{i_j})) - \text{area}(\Sigma_j, h(\tau))}{t_{i_j} - \tau} = \frac{1}{2}\int_{\Sigma_k} \text{Tr}_{(\Sigma_k, h(\tau))}(\partial_t h(\tau)) dA_{h(\tau)}$$

and hence we have the derivative of width formula. □

Now we can finish the proof of Theorem 4.0.1 by showing Lemma 5.3.1. For a continuous function f with $\int_M f dV_g < 0$, we can define a conformal change of metric:

$$g(t) = (1 + \frac{n+1}{n} tf)^{\frac{n}{n+1}} g \quad \text{for} \quad 0 \le t \le T.$$

We have $\partial_t g(t)\big|_{t=0} = fg$, hence for small $T > 0$ we have $\text{Vol}(M, g(t))$ less than the the volume under the original metric. Since g maximizes the normalised width, we have

$$\frac{W(M, g(t))}{\text{Vol}(M, g(t))^{\frac{n}{n+1}}} \le \frac{W(M, g(0))}{\text{Vol}(M, g(0))^{\frac{n}{n+1}}} \quad \text{for} \quad 0 \le t \le T.$$

Hence

$$W(M, g(t)) \le W(M, g(0)) \Big(\frac{\text{Vol}(M, g(t))}{\text{Vol}(M, g(0))}\Big)^{\frac{n}{n+1}} < W(M, g(0)) \quad \text{for} \quad 0 \le t \le T.$$

Fix $q \ge 4$. Now for each $i \in \mathbb{N}$ with $1/i < T$, we can find a perturbation $h_i : [0, 1/i] \to \Gamma_q$ and $J_i \subset [0, 1/i]$ with full Lebesgue measure such that

$$W(M, h_i(1/i)) < W(M, h_i(0))$$

and so there is $\tau_i \in J_i$ such that

$$\frac{d}{dt} W(M, h_i(t))\Big|_{t=\tau_i} \le 0$$

due to the first fundamental theorem of calculus. Hence by the previous lemma there are FBMHs

Σ_{i_j}, $j = 1, 2, \cdots, n_i$ and a set of integers $\{n_{i_1}, \cdots, n_{i_N}\}$ such that

$$W(M, h_i(\tau_i)) = \sum_{k=1}^{N} n_{i_k} \cdot \mathrm{area}(\Sigma_{i_k}, h_i(\tau_i)), \quad \sum_{k=1}^{N} \mathrm{ind}_{h_i(\tau_i)}(\Sigma_{i_k}) \leq 1,$$

$$\text{and} \quad \frac{d}{dt}\Big|_{t=\tau_i} W(M, h_i(t)) = \frac{1}{2}\sum_{k=1}^{N} n_{i_k} \int_{\Sigma_{i_k}} \mathrm{Tr}_{(\Sigma_{i_k}, h_i(\tau_i))}(\partial_t h_i(\tau_i)) dA_{h_i(\tau_i)} \leq 0.$$

We can relabel these Σ_{i_k} such that except for Σ_{i_1}, others have index 0. Now we can use the Compactness Theorem 5.4.1 to conclude that, by picking a subsequence $\tau_{i_j} \to 0$, the FBMHs subconverges smoothly and graphically to $\{\Sigma_1, \cdots, \Sigma_N\}$ with multiplicity 1, except for Σ_1, where the multiplicity can be 2 if Σ_1 is stable. Therefore we can pass the limit of the formula above and show

$$\frac{1}{2}\sum_{k=1} n_k \int_{\Sigma_k} f dA_g = \frac{1}{2}\sum_{k=1} n_k \int_{\Sigma_k} \mathrm{Tr}_{\Sigma_k}(\partial_t g(0)) dA_g \leq 0$$

Hence this finish the proof when f is a smooth function on $(M, g(0))$. When f is a continuous function we can use smooth functions to approximate f uniformly. Therefore we have proved Lemma 5.3.1.

We can use the implication i) to iv) in Theorem 5.4.2 and the remark after it can be applied by letting Y be the Radon measure defined by embedded FBMHs in M with index at most one, and μ_0 be the original volume measure on (M, g).

5.3.2 Proof of Theorem 4.0.2

Now we prove Theorem 4.0.2. First we need a result that guarantees the existence of optimal sweepout in Lemma 1.2.2, and then we can compute the derivative of width under a general smooth family of metrics.

Lemma 5.3.3 ([7] Proposition 5.4)**.** *Let $(\Sigma, \partial\Sigma) \subset (M, \partial M)$ be an orientable, almost properly embedded, free boundary minimal hypersurface with Area(Σ) less than the least area of the stable free boundary minimal hypersurface in M. Then there is a sweepout*

$$\Psi : [-1, 1] \to Z_n(M, \partial M),$$

such that:

(1) $\Psi(0) = \Sigma$*;*

(2) $F(\Psi) = M$*;*

(3) $\mathbf{M}(\Psi(t)) < Area(\Sigma)$ *for* $t \neq 0$*.*

Lemma 5.3.4. *Let M be a compact simply manifold Given a one parameter family of metrics $\{g(t)\}_{t\in(a,b)}$ on M varying smoothly, if $t_0 \in (a,b)$ is a point where $W(t) := W(M, \partial M, g(t))$ is differentiable, then there is an almost properly embedded free boundary minimal hypersurface Σ in $(M, \partial M, g(t_0))$ such that*

$$area(\Sigma, g(t_0)) = W(t_0) \quad and \quad \frac{d}{dt}W(M^{n+1}, \partial M, g(t))\Big|_0 = \frac{1}{2}\int_\Sigma Tr_\Sigma(\frac{\partial}{\partial t}g(t)\Big|_{t=t_0})dA_{g(t_0)}.$$

Proof of Lemma 5.3.4. By Lemma 5.3.3, there exist an optimal sweepout $\{\Sigma_s\}_{s\in[-1,1]}$ such that $\mathrm{area}(\Sigma_0) = W(M, g(t_0))$ and for all $s \neq 0$, $\mathrm{area}(\Sigma_s) < \mathrm{area}(\Sigma_0)$. Consider a smooth function $F : (a,b) \times [-1,1] \to \mathbb{R}$ defined as $F(t,s) = \mathrm{area}(\Sigma_s, g(t))$, then we have $F_s(t_0, 0) = 0$ and $F_{ss}(t_0, 0) < 0$. Now let us show that there exists $\epsilon > 0$ such that there is a differentiable function $s = s(t)$ for $t \in (t_0 - \epsilon, t_0 + \epsilon)$, such that

$$F(t, s(t)) = \max_{s\in[-1,1]} F(t,s).$$

Since $F_{ss}(t_0, 0) < 0$, the implicit function theorem guarentees that $F_s(t,s) = 0$ defines a smooth function $s = s(t)$ on $(t_0 - \epsilon, t_0 + \epsilon)$. Now there is a neighborhood of $(t_0, 0)$ such that $F_{ss} < 0$, and therefore $F(t, s(t))$ is a local maximum for each fixed $t \in (t_0 - \epsilon', t_0 + \epsilon')$. Due to the construction of sweepout(property 3) and possibly making ϵ' even smaller we can make sure $F(t, s(t))$ is a strict maximum. Hence the claim is proved. Now we define a function $h(t) = F(t, s(t)) - W(t)$ over a neighborhood of t_0. We have that $h(t) \geq 0$ due to the definition of width, and $h(t_0) = 0$ is the local minimum. Since $W(t)$ is differentiable at t_0, h is also differentiable and $h'(t_0) = 0$. Hence we have

$$W'(t_0) = \frac{\partial}{\partial t}F(t, s(t))\big|_{t=t_0} = F_s(t_0, 0)s'(t_0) + F_t(t_0, 0) = \frac{1}{2}\int_\Sigma \mathrm{Tr}_\Sigma(\frac{\partial g}{\partial t}(t_0))dA_{g(t_0)}$$

□

Similar to the proof of Theorem 4.0.1, we can define a conformal change of the metric g, now with a volume preserving factor. More precisely, for a smooth function f with $\int_M f dV_g = 0$, we fix a small $T > 0$ and let

$$g(t) = \frac{\mathrm{Vol}(M,g)^{\frac{n}{n+1}}(1+ft)}{\mathrm{Vol}(M,(1+ft)g)^{\frac{n}{n+1}}} g \quad \text{for all } t \in [0,T).$$

It is straightforward to show that $\mathrm{Vol}(M, g(t)) = \mathrm{Vol}(M, g(0))$ for all $t \in [0,T)$, and that $\partial_t g(0) = fg$.

Lemma 5.3.5. *Let $g(t)$, $t \in [0, \epsilon)$ be a smooth family of Riemannian metrics on M that contains no stable free boundary minimal surface with area greater than $W(M,g)$. If*

$$W(M, g(0)) \geq W(M, g(t))$$

then there exists a free boundary minimal hypersurface Σ such that

$$area(\Sigma, g(0)) = W(M, g(0)) \quad and \quad \int_\Sigma Tr_\Sigma(\partial_t g(0)) dA_{g(t_0)} \leq 0.$$

Proof. Take an $\epsilon > 0$. By Rademacher's Theorem, W is differentiable at almost all $t \in [0, \epsilon)$. Since W assumes local maximum at 0, There exists a sequence $t_n \in [0,\epsilon)$ converging to t_0 such that $W'(t_n) \leq 0$ for all n. Hence by the previous lemma we can find an embedded free boundary minimal hypersurface Σ_n in $(M, g(t_n))$ with $\mathrm{area}(\Sigma_n, g(t_n)) = W(t_n)$ and $\int_{\Sigma_n} \mathrm{Tr}_{\Sigma_n}(\partial_t g(t_n)) dA_{g(t_n)} \leq 0$. Now by the compactness theorem we see that Σ_n subconverges to a embedded free boundary minimal disk Σ. By the smooth convergence we have $\mathrm{area}(\Sigma, g(0)) = W(0)$ and $\int_\Sigma \mathrm{Tr}_\Sigma(\partial_t g(0)) dA_{g(0)} \leq 0$.
□

Combining Lemma 5.3.5 and the conformal change of metric $g(t)$, we can show the following statement:

Proposition 5.3.1. *Let f be a continuous function on (M, g) with zero average, and if (M, g) contains no stable free boundary minimal surface with area greater than $W(M,g)$, we can find an embedded free boundary minimal surface Σ in (M, g) such that $\int_\Sigma f dA_g \leq 0$.*

Proof. This statement follows when we approximate the function f uniformly by smooth functions, and use the previous conformal change of metric. $\square$

Then as in the proof of Theorem 4.0.1, the implication ii) to iv) in Theorem 5.4.2 will confirm the existence of equidistibuted FBMHs in M, and as Lemma 5.3.5 shows, each Σ_i has area equal to $W(M, g(0))$. This ends the proof of Theorem 4.0.2.

5.4 Compactness Theorem and Equidistribution Theorem

In this section we prove a compactness theorem of FBMHs for varying background metric and an abstract theorem on the existence of equidistributed sequence of measures.

Theorem 5.4.1 (Compactness of FBMHs with bounded index and area)**.** *Let $2 \leq n \leq 6$ and N^{n+1} be a compact manifold with boundary and $\{g_k\}_{k\in\mathbb{N}}$ a family of Riemannian metrics on N converging smoothly to some limit g. If $\{M_k^n\} \subset N$ is a sequence of connected and properly embedded free boundary minimal hypersurface in (N, g_k) with*

$$H^n(M_k) \leq \Lambda < \infty \quad and \quad index_k(M_k) \leq I,$$

for some fixed constants $\Lambda \in \mathbb{R}$, $I \in \mathbb{N}$, both independent of k. Then up to subsequence, there exists a connected and free boundary embedded minimal hypersurface $M \subset (N, g)$ where $M_k \to M$ in the varifold sense with

$$H^n(M) \leq \Lambda < \infty \quad index_k(M_k) \leq I$$

we have that the convergence is smooth and graphical for all $x \in M - Y$ where $Y = \{y_i\}_{i=1}^K \subset M$ is a finite set with $K \leq I$ and the following dichotomy holds:

- *if the number of leaves in the convergence is one then $Y = \Phi$, i.e. the convergence is smooth an graphical everywhere*
- *if the number of sheets is ≥ 2*
  -if N has $Ric_N > 0$ then M cannot be one-sided
  -if M is two-sided the M is stable.

Proof. We know by Allard's compactness theorem that there is an M such that after passing to a subsequence, $M_k \to M$ in $\mathbf{IV}_n(N)$. Let $Y \subset M$ be the singular set of M. First we show that $|Y| \leq I$. Suppose on the contrary that Y contains at least $I+1$ points $y_1, \cdots, y_{I+1}$. Then we can find $\{\epsilon_i\}_1^{I+1}$ such that $B(y_i, \epsilon_i) \cap B(y_j, \epsilon_j) = \emptyset$, and that $\sup_k \sup_{M_k \cap B(y_i, \epsilon_i)} |A|^2 = \infty$, for all $i = 1, \cdots, I+1$. Since g_k converges to g smoothly, the sectional curvature of (N^{n+1}, g_k) are uniformly bounded. Hence curvature estimate of [4] applies to this varying metric case, that is, in $\Sigma_k \cap B_r(p)$ the second fundamental form of Σ_k are bounded by a uniform constant C that depends only on N. Hence we infer that for sufficiently large k, $M_k \cap B(y_i, \epsilon_i)$ is not stable for all $i = 1, \cdots, I+1$. This implies that $\text{index}_k(M_k) \geq I+1$ which contradicts with the assumption.

To Show that $\text{Index}(M) \leq I$, we suppose that there are $u_1, u_2, \cdots, u_{I+1} \in C^\infty(M)$ that are L^2-orthogonal such that $I(u_i, u_i) < 0$ for $i = 1, 2, \cdots, I+1$. Then we extend u_i to $\tilde{u}_i \in C^1(M)$ and let $u_i^k = \tilde{u}_i\big|_{M_k}$. Since $M_k \to M$ as varifold, we have for sufficiently large k, $I_k(u_i^k, u_i^k) < 0$ for $i = 1, 2, \cdots, I+1$. Since $\text{Index}(M_k) \leq I$, $\{u_i^k\}_{i=1}^{I+1}$ must be linearly dependent. By taking a subsequence and relabeling if necessary, we can find $\{\lambda_i\}_{i=1}^I \subset \mathbb{R}$ and λ_i's not all zero such that $u_{I+1}^k = \sum_{i=1}^n \lambda_i u_i^k$. By varifold convergence we have $\langle u_i^k, u_j^k \rangle \to \langle u_i, u_j \rangle = \delta_{ij}$ for $i, j = 1, 2, \cdots, n+1$. Therefore by the varifold convergence,

$$0 = \langle u_{n+1}, u_i \rangle_M = \lim_{k \to \infty} \langle u_{n+1}^k, u_i^k \rangle_{M_k} = \lim_{k \to \infty} \lambda_i$$

This implies that $u_{n+1} = 0$ which contradicts $I(u_{n+1}, u_{n+1}) < 0$.

Now if the multiplicity of convergence is 1, then the convergence is smooth everywhere by the regularity theorem of [8]. Hence the theorem is proved. □

The following abstract theorem is an important observation that leads to the equidistribution equation 4.0.1. The proof of the main step in this theorem is a technical combinatorial argument, so we include it for the sake of completeness.

Theorem 5.4.2 (c.f. [1] Theorem B.2)**.** *Let Y be a non-empty weak-* compact subset of $M(X)$. The following assertions about a measure μ_0 in $M(X)$ are equivalent to each other:*

i) For every function $f \in C^0(X)$ such that $\int_X f d\mu_0 < 0$, there exists $\mu \in Y$ such that $\int_X f d\mu \leq$

0.

ii) For every function $f \in C^0(X)$ such that $\int_X f d\mu_0 = 0$, there exists $\mu \in Y$ such that $\int_X f d\mu \leq$ 0.

iii) μ_0 belongs to the weak- closure of the convex hull of the positive cone over Y.*

iv) There exists a sequence $\{\mu_k\}$ in Y such that

$$\lim_{k\to\infty} \frac{1}{k} \sum_{i=1}^{k} \frac{1}{\mu_i(X)} \int_X f d\mu_i = \frac{1}{\mu_0(X)} \int_X f d\mu_0 \quad \text{for all } f \in C^0(X). \tag{5.4.1}$$

Proof. i) implies ii). consider a sequence of functions $f_k = f - 1/k$. Then as $\int_X f_k d\mu_0 < \int_X f d\mu_0 =$ 0, we can find $\mu_k \in Y$ such that $\int_X f_k d\mu_k \leq 0$. By weak-* compactness of Y, there is a subsequence of $\{\mu_k\}$ (still denoted as μ_k) that converges in weak-* to some $\mu \in Y$. Since $\int_X f d\mu_k = \int_X f_k d\mu_k + \mu_k(X)/k$, and as we take $k \to \infty$, we get $\int_X f d\mu \leq 0$.
ii) implies iii): Suppose that μ_0 is not in the closure of the convex hull of the positive cone over Y, then by Hahn-Banach theorem, we can find a continuous function f such that $\int_X f d\mu_0 = 0$ and $\int_X f d\mu > 0$ for all $\mu \in Y$. This contradicts with ii).
iii) implies iv). First let us normalize a measure by $\bar{\mu} = \frac{1}{\mu(X)}\mu$. Such a normalization will not change the positive cone over Y. Now assuming that μ_0 is a normalized measure in the closure of the convex hull of the positive cone over Y, then we can find a sequence of μ_k with $\mu_k \to \mu_0$ in weak * sense, and each μ_k is a convex combination of N_k measures in the positive cone over Y:

$$\mu_k = \sum_{i=1}^{N_k} a_{k,i} \cdot (\lambda_{k,i}\mu_{k,i}) \qquad \text{with } \sum_{i=1}^{N_k} a_{k,i} = 1 \text{ and } a_{k,i}, \lambda_{k,i} > 0, \forall 1 \leq i \leq N_k$$

and $\mu_{k,i}$ are normalized measure in Y. Now we have

$$\lim_{k\to\infty} \sum_{i=1}^{N_k} a_{k,i}\lambda_{k,i} \int_X f d\mu_{k,i} = \int_X f d\mu_0 \qquad \text{for all } f \in C^0(X).$$

By setting $f = 1$ we see that $\sum_i a_{k,i}\lambda_{k,i} = 1$. By standard rational approximation, for each k, we can find $d_k \in \mathbb{N}$, and for each $i = 1, \cdots, N_k$ there is $c_{k,i}$ such that

$$\left| a_{k,i}\lambda_{k,i} - \frac{c_{k,i}}{d_k} \right| \le \frac{1}{kN_k}$$

This implies

$$\left| 1 - \sum_{i=1}^{N_k} \frac{c_{k,i}}{d_k} \right| = \left| \sum_{i=1}^{N_k} \left(a_{k,i}\lambda_{k,i} - \frac{c_{k,i}}{d_k} \right) \right| \le \sum_{i=1}^{N_k} \frac{1}{kN_k} = \frac{1}{k} \to 0$$

also for any $f \in C^0(X)$

$$\left| \sum_{i=1}^{N_k} a_{k,i}\lambda_{k,i} \int_X f d\mu_{k,i} - \sum_{i=1}^{N_k} \frac{c_{k,i}}{d_k} \int_X f d\mu_{k,i} \right| \le \|f\|_0 \cdot \frac{1}{k} \to 0.$$

Therefore we conclude that

$$\frac{\sum_{i=1}^{N_k} c_{k,i}\mu_{k,i}}{\sum_{i=1}^{N_k} c_{k,i}} \to \mu_0 \quad \text{in weak} \ * \ \text{topology} \tag{5.4.2}$$

Now we shall describe how to select the desired sequence $\{\mu_k\}$ that satisfies equation (5.4.1). The idea is to repeat the group of measures $\{\mu_{k,i}\}_{i=1}^{N_k}$ more often as $k \to \infty$.

Let $\mathcal{M}_k$ be the finite sequence of measures consisting of $\mu_{k,1}$ repeated for $c_{k,1}$ times, followed by $\mu_{k,2}$ repeated for $c_{k,2}$ times and so on until μ_{k,N_k} repeated for c_{k,N_k} times, and let $M_k = \mathrm{Card}(\mathcal{M}_k) = \sum_i c_{k,i}$. Let us relabel the measures in $\mathcal{M}_k$ by $\{\mu_{k,j}\}_1^{M_k}$, which ignores the multiplicity $c_{k,i}$.

We shall define a sequence of integers L_k by induction, where L_k can be understood as number of repetitions of $\mathcal{M}_k$ in the eventual sequence $\{\mu_k\}$. Let us fix $f \in C^0(X)$ and denote $f_{k,j} = \int_X f d\mu_{k,j}$, and $f_0 = \int_X f d\mu_0$. By 5.4.2 we have

$$\left| \frac{\sum_{j=1}^{M_k}(f_{k,j} - f_0)}{M_k} \right| \le \epsilon_k \text{ for a sequence } \epsilon_k \to 0$$

Consider a finite sequence obtained by packing $\mathcal{M}_1$ repeated L_1 times, followed by $\mathcal{M}_2$ repeated for L_2 times. For each $L_1M_1 \le N \le L_1M_1 + L_2M_2$, there is $0 \le k \le L_2$ and $0 \le l < M_2$ such that

$N = L_1M_1 + kM_2 + l$. Thus we have

$$\left| \frac{L_1 \sum_{j=1}^{M_1}(f_{1,j} - f_0) + k \sum_{i=1}^{M_2}(f_{2,j} - f_0) + \sum_{j=1}^{l}(f_{2,j} - f_0)}{L_1M_1 + kM_2 + l} \right| \leq \epsilon_1 + \epsilon_2 + \frac{2\|f\|M_2}{L_1M_1 + kM_2 + l}$$

Thus we can pick L_1 to be sufficiently large (comparing to M_2), so that the last term on the right hand side can be controlled. Notice that the choice of L_1 does not depend on L_2. Now suppose that $L_1, \cdots, L_{n-1}$ have been chosen, and we shall choose L_n. Continue to pack up the sequence by repeating $\mathcal{M}_i$ for L_i times for $i = 1, 2, \cdots n+1$, so for $N = \sum_{i=1}^{n} L_iM_i + kM_{n+1} + l$ with $0 \leq k < L_{n+1}$ and $0 \leq l < M_{n+1}$ we have

$$\begin{aligned}
&\left| \frac{\sum_{i=1}^{n} L_i \sum_{j=1}^{M_i}(f_{i,j} - f_0) + k \sum_{i=1}^{M_{n+1}}(f_{n+1,j} - f_0) + \sum_{j=1}^{l}(f_{n+1,j} - f_0)}{\sum_{i=1}^{n} L_iM_i + kM_{n+1} + l} \right| \\
\leq &\frac{\sum_{i=1}^{n} L_iM_i\epsilon_i + kM_{n+1}\epsilon_{n+1} + 2\|f\|l}{\sum_{i=1}^{n} L_iM_i + kM_{n+1} + l} \\
= &\frac{L_nM_n\epsilon_n + \sum_{i=1}^{n-1} L_iM_i\epsilon_i + kM_{n+1}\epsilon_{n+1} + 2\|f\|l}{L_nM_n + \sum_{i=1}^{n-1} L_iM_i + kM_{n+1} + l} \\
\leq &\frac{L_nM_n\epsilon_n + \sum_{i=1}^{n-1} L_iM_i\epsilon_i + kM_{n+1}\epsilon_{n+1}}{L_nM_n + \sum_{i=1}^{n-1} L_iM_i + kM_{n+1} + l} + \frac{2\|f\|M_{n+1}}{L_nM_n}
\end{aligned}$$

As before we can choose L_n to be large enough so that the first term above is dominated by $O(\epsilon_n)$, and also making $\frac{M_{n+1}}{L_nM_n} \leq 1/n \to 0$. Thus we get the sequence $\{\mu_k\}$ with the desired property: for any $f \in C^0$,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} \int_X f d\mu_k = \int_X f d\mu_0$$

This is exactly iv) since μ_k above are a normalized measures.

iv) implies i). This is trivial as the limit of average is negative implies there is at least a negative term in the summation. □

Remark 5.4.3. *In the proof of Theorem 4.0.1, the Radon measure on minimal hypersurfaces* Σ *can be written as the following form:*

$$\mu = \sum_{i=1}^{K} c_i\mu_i$$

where μ_i is the measure concentrated on each connected component of Σ. Since all nontrivial embedded FBMH in a compact manifold has an positive area lower bound, we have $\mu_i(M) \geq c > 0$ for some c. This is crucial for us to use a similar combinatorial argument as in the previous proof to extract a equidistributed sequence.

Part III

Relative Anisotropic Isoperimetric Inequality

Chapter 6

Introduction to the Main Result

In the third part we prove the following anisotropic relative isoperimetric inequality:

Theorem 6.0.1. *Let $\Phi : \mathbb{R}^n \to [0, \infty)$ be a convex, one homogeneous coercive function, and W_Φ is the Wulff shape associated with Φ. If $\Omega \subset \mathbb{R}^n \setminus B_R(0)$ is a smooth region, we have the following sharp inequality:*

$$P_\Phi(\Omega, \mathbb{R}^n \setminus B_R(0)) \geq n(\beta |W_\Phi|)^{\frac{1}{n}} |\Omega|^{\frac{n-1}{n}}$$

where

$$\beta := \inf_{v \in \mathbb{S}^{n-1}} \frac{|W \cap \{\langle x, v \rangle \geq 0\}|}{|W|}.$$

This theorem is a partial extension of the relative isoperimetric inequality outside convex regions in $\mathbb{R}^n$ by Choe, Ghomi and Ritore [5]:

Theorem 6.0.2 (Theorem 5.1 [5])**.** *Let $C \subset \mathbb{R}^n$ be a proper convex set wiwth smooth boundary. For any bounded set $D \subset \mathbb{R}^n \setminus C$ with finite perimeter,*

$$\Big(area(\partial D)_C\Big)^n \geq \frac{1}{2} n^n \omega_n \, Vol(D)^{n-1}$$

where $\omega_n = Vol(B_1)$, the volume of unit ball in $\mathbb{R}^n$.

Theorem 6.0.1 can also be seen as a relative version of the classical anisotropic isoperimetric inequality:

Theorem 6.0.3 (Anisotropic Isoperimetric Inequality in Euclidean Space [18]). *If $\Phi : \mathbb{R}^n \to \mathbb{R}$ is a one-homogeneous, convex and coercive function, W_Φ is the Wulff shape associated with Φ, then we have the following inequality:*

$$|K| \leq (n^n |W_\Phi|)^{-\frac{1}{n-1}} P_\Phi(K)^{-\frac{n}{n-1}} \tag{6.0.1}$$

The main idea of the proof of Theorem 6.0.1 is based on Alexandrov-Bakelman-Pucci Method. In the following chapter we will give an overview of several related results on isoperimetric inequalities.

Chapter 7

Isoperimetric Inequalities in Euclidean Spaces

The study of isoperimetric problem dates back to more than 2000 years ago. The classic isoperimetric problem is the following optimization problem:

Among all planar shapes with fixed perimeter, which one has the largest area?

It has been known without proof that the round circle is the only answer to the question above for a long time. Since the invention of Calculus in 17th century and in particular, the subject called Calculus of Variations, people began to ask questions that generalizes the isoperimetric problem in various perspectives. For example, in Euclidean Spaces of dimension $n \geq 2$, does the round ball $B_r(0) := \{x \in \mathbb{R}^n | |x| \leq r\}$ maximize the volume of all compact sets with same perimeter(the $(n-1)$-dim volume of the boundary)?

This turns out to be true and is often refered to as the classic isoperimetric inequality in $\mathbb{R}^n$. More precisely,

Theorem 7.0.1 (Classical Isoperimetric Inequality in Euclidean Space)**.** *Let $K \subset \mathbb{R}^n$ be a domain with smooth boundary. Then we have following inequality:*

$$Vol(K) \leq (n^n \omega_n)^{-\frac{1}{n-1}} Vol(\partial K)^{\frac{n}{n-1}}$$

and the equality case holds if and only if K is a round ball.

There are several approaches of proofs of the the classical isoperimetric inequality, the ideas among which includes Schwarz Symmetrisation, Brunn-Minkowski Inequality and Optimal Transport. In the next section we will give a proof using the Alexandrov-Bakelman-Pucci's Maximum Principle, by exploring the solution to a elliptic PDE with Neumann boundary condition, defined on the domain K.

7.1 Sets of Finite Perimeter

In the classical isoperimetric problem, we assume the boundary of K to be smooth, or at least piecewise smooth. This is indeed a necessary assumption for the boundary volume to be well defined in the classic sense. However, in a usual search of the optimal domain K, we often need to take a certain limit of domains that approaches the optimal shape. The space of domains with smooth boundary is not closed(more importantly, it is not compact), which might lead to the limiting optimal shape being non-existent. The space of sets of finite perimeter provides a sound framework so that the isoperimetric inequality admits a solution.

Definition 7.1.1. *Let $E \subset \mathbb{R}^n$ be a Lebesgue measurable set, and $A \subset \mathbb{R}^n$. The perimeter of E in A is defined as*

$$P(E;A) = \sup\Big\{ \int_E divT(x)dx : T \in C_c^\infty(A;\mathbb{R}^n), \sup_{\mathbb{R}^n} |T| \leq 1 \Big\}$$

where $C_c^\infty(A;\mathbb{R}^n)$ is the space of smooth vector fields compactly supported in A. Moreover, we call E a set of finite perimeter in A if $P(E;A) < \infty$. If for every compact set $K \in \mathbb{R}^n$, we have $P(E;K) < \infty$, then P is called a set of locally finite perimeter. In particular, when $A = \mathbb{R}^n$, we denote $P(E;\mathbb{R}^n)$ as $P(E)$.

When E is a set of finite perimeter, we can associate a vector valued radon measure with E, called the Gauss-Green measure:

Proposition 7.1.1 ([18] Proposition 12.1)**.** *Let $E \in \mathbb{R}^n$ be Lebesgue measurable. Then E is a set* *of finite perimeter in* $\mathbb{R}^n$ *if and only if there is a* $\mathbb{R}^n$ *valued Radon measure* μ_E *on* $\mathbb{R}^n$ *such that*

$$\int_E divT = \int_{\mathbb{R}^n} T \cdot d\mu_E$$

for every smooth vector field T *with compact support on* $\mathbb{R}^n$*.*

Remark 7.1.2. *When* E *has piecewise* C^1 *boundary* ∂E*,* $P(E)$ *coincide with the volume of* ∂E*, and* $\mu_E = \nu_{\partial E} \cdot H^{n-1}|_{\partial E}$*, where* $\nu_{\partial E}$ *is the outer normal vector of* ∂E*.*

We have the following fundamental theorem on the approximation of set of finite perimeter by open sets with smooth boundary. This theorem will help us resolve the regularity issue when applying PDE tools to prove the isoperimetric inequalities.

Theorem 7.1.3 ([18] Theorem 13.8)**.** *A Lebesgue measurable set* E *has finite perimeter if and only if there is a sequence of open sets* $\{E_k\}$ *with smooth boundary in* $\mathbb{R}^n$ *such that*

$$\chi_{E_k} \to \chi_E \quad \text{in } L^1(\mathbb{R}^n) \qquad \text{and } P(E_k) \to P(E)$$

The isoperimetric problem is a natural geometric variational problem, as we can consider all bounded sets with finite perimeter, so that their volumes and perimeter are well defined. In the dual sense, we can formulate the problem as follows:

$$\text{Find domain } K \in \mathbb{R}^n \text{ such that } P(K) = \min\Big\{P(E) : |E| = 1\Big\}.$$

To guarantee the existence of the minimizer, we have the following important lower semicontinuity of the perimeter and local compactness of the space of sets of finite perimeter:

Proposition 7.1.2 (Lower semicontinuity of perimeter)**.** *Let* $\{E_k\}$ *be a sequence of sets of locally finite perimeter and suppose that*

$$\chi_{E_k} \to \chi_E \quad \text{in } L^1_{loc}(A), \qquad \limsup_{k\to\infty} P(E_k, K) < \infty$$

for every compact $K \subset \mathbb{R}^n$*, then* E *is a set of locally finite perimeter and*

$$P(E; A) \leq \liminf_{k\to\infty} P(E_k; A)$$

The lower semicontinuity implies immediately that the potential limit of a minimizing sequence must also be a minimizer. The following compactness theorem further guarantees the existence of a limit:

Proposition 7.1.3 (Compactness of uniformly bounded sets of finite perimeter)**.** *Let $R > 0$ and $\{E_k\}$ be a sequence of sets of finite perimeter in $\mathbb{R}^n$, such that $\sup_k P(E_k) < \infty$ and $E_k \subset B_R(0)$ for all k. Then there is a set E of finite perimeter in $\mathbb{R}^n$ and a subsequence E_{k_j} such that*

$$\chi_{E_{k_j}} \to \chi_E \quad \text{in } L^1_{loc}(\mathbb{R}^n), \qquad \text{and } E \subset B_R(0).$$

Using the compactness result it is easy to see that the isoperimetric problem with boundedness constraint has a minimizer.

Proposition 7.1.4. *For $m < \omega_n R^n$, the set $\{P(E) : E \subset B_R(0), |E| = m\}$ admits a minimizer.*

To show the existence of a minimizer to the full problem, one considers an unbounded set E with finite perimeter $P(E)$ and $|E| = m$. Then we can approximate E by $E \cap B_R(0)$. By coarea formula, $H^{n-1}(E \cap \partial B_R(0)) \in L^1(0, \infty)$. Hence we can choose a sequence $R_k \to \infty$ so that $H^{n-1}(E \cap \partial B_{R_k}(0)) \to 0$. On the other hand $P(E; B_R(0)^c) \to 0$ as $R \to \infty$. Since

$$\begin{aligned} P(E) &= P(E; B_R(0)) + P(E; B_R(0)^c) &\quad (7.1.1)\\ &= P(E \cap B_R(0)) - H^{n-1}(E \cap \partial B_R(0)) + P(E; B_R(0)^c) &\quad (7.1.2) \end{aligned}$$

Replacing R by R_k we see that $P(E \cap B_{R_k}(0)) \to P(E)$. This approximation tells that unbounded set E obeys the same isoperimetric inequality as bounded ones, hence the minimizer is achieved by bounded sets.

As the ball is the most symmetric geometric object, the fact that it is the minimizer of the isoperimetric ratio is not surprising. It is natural to ask whether other shapes has similar minimizing property with respect to a notion of perimeter and with a volume constraint. As we will be considering a variational problem, there should be some convexity assumption for the sake of uniqueness of minimizer. Hence we turn to the relation between convex bodies and anisotropic surface energy, as generalization of the dual relation between balls and usual surface area(perimeter).

In order to define the anisotropic surface energy on sets of finite perimeterwe need some regularity result on their boundary structure. First we define the notion of reduced boundary.

Definition 7.1.4. *Let E be a set of finite perimeter. The reduced boundary of E, denoted as $\partial^* E$, is the following set*

$$\left\{ x \in supp(\mu_E) : \lim_{r \to 0^+} \frac{\mu_E(B(x,r))}{|\mu_E|(B(x,r)|} \in S^{n-1} \right\}$$

where S^{n-1} is the unit sphere in $\mathbb{R}^n$.

The reduced boundary provides us with an approximate notion of outer normal vector. The following nice structure theorem, due to De Giorgi, states that $|\mu_E|$-almost all points in supp(μ_E) are in ∂E, where the outer unit normal of the boundary can be defined.

Theorem 7.1.5 (De Giorgi's structure theorem [18] Theorem 15.9)**.** *If E is a set of finite perimeter, then we have*

$$\mu_E = \nu_{\partial^* E} \cdot H^{n-1}|_{\partial^* E}, \qquad |\mu_E| = H^{n-1}|_{\partial^* E}$$

and the generalized Gauss-Green formula holds:

$$\int_E div(T) = \int_{\partial^* E} T \cdot \nu_{\partial^* E} dH^{n-1}$$

Moreover, $\partial^* E$ is a $(n-1)$-rectifiable set.

7.2 Anisotropic Isoperimetric Problem

By the theorem 7.1.5, we can define the anisotropic surface energy for sets of finite perimeter as follows:

Definition 7.2.1. *Let $\Phi : \mathbb{R}^n \to [0, \infty)$ be a 1-homogeneous function, i.e. for all $\lambda \geq 0$ and $x \in \mathbb{R}^n$, we have $\Phi(\lambda x) = \lambda \Phi(x)$. If E is a set a finite perimeter in $A \subset \mathbb{R}^n$, then P_Φ-surface energy of E relative to A is defined as*

$$P_\Phi(E; A) = \int_{\partial^* E \cap A} \Phi\big(\nu_{\partial * E}(x)\big) dH^{n-1}(x)$$

Remark 7.2.2. *If $\Phi(x) = |x|$, we have the usual definition of perimeter of E in A.*

Likely, we can formulate the isoperimetric problem in anisotropic setting:

Problem 7.2.3. *Find the minimizer of $\{P_\Phi(E;\mathbb{R}^n) : |E| = m\}$.*

The ensure the existence of such a minimizer, we need further assumptions on the anisotropic function Φ, which includes convexity and coercivity.

Definition 7.2.4 (Convexity and Coercivity)**.**

- *A lower semi-continuous function $\Phi : \mathbb{R}^n \to \mathbb{R}$ is called convex if for any $x, y \in \mathbb{R}^n$, and any $\lambda \in [0,1]$, we have*

$$\Phi\big(\lambda x + (1-\lambda)y\big) \le \lambda\Phi(x) + (1-\lambda)\Phi(y).$$

- *A one-homogeneous function $\Phi : \mathbb{R}^n \to \mathbb{R}$ is called coercive if there is $c > 0$ such that $\Phi(x) \ge c|x|$ for all $x \in \mathbb{R}^n$.*

Remark 7.2.5. *If Φ is a one-homogeneous function, the convexity is equivalent to subadditivity: $\Phi(x+y) \le \Phi(x) + \Phi(y)$ for all $x, y \in \mathbb{R}^n$. In this case one can also view Φ as a function over the unit sphere S^{n-1} extended to $\mathbb{R}^n$ by $\Phi(x) = |x|\Phi(x/|x|)$. The coercivity of Φ would simply mean that Φ a positive lower bound c over all unit directions, and it implies that $P_\Phi(E) \ge cP(E)$.*

By proving the lower-semicontinuity and similar compactness result for anisotropic surface energy, one has the following existence theorem:

Theorem 7.2.6 (Existence of Minimizer of Anisotropic Isoperimetric Problem)**.** *If $\Phi : \mathbb{R}^n \to \mathbb{R}$ is a one-homogeneous, convex and coercive function, then the 7.2.3 has a minimizer which is a bounded set of finite perimeter.*

The shape of the minimizer to this anisotropic isopermetric problem is called the Wulff shape. It is a convex set whose support function is Φ:

$$W_\Phi = \bigcap_{\nu \in S^{n-1}} \Big\{ x : \langle x, \nu \rangle < \Phi(\nu) \Big\}$$

Then we have the following generaization of the classical isoperimetric inequality:

Theorem 7.2.7 (Theorem 6.0.3, Anisotropic Isoperimetric Inequality in Euclidean Space)**.** *If $\Phi : \mathbb{R}^n \to \mathbb{R}$ is a one-homogeneous, convex and coercive function, W_Φ is the Wulff shape associated with Φ, then we have the following inequality:*

$$|K| \le (n^n |W_\Phi|)^{-\frac{1}{n-1}} P_\Phi(K)^{-\frac{n}{n-1}} \tag{7.2.1}$$

In the next section, we will use Alexandrov-Bakelman-Pucci maximum principle to prove 6.0.3.

7.3 The ABP Method and the Proof of Anisotropic Isoperimetric Inequality

The ABP method is oringinally use by Alexandrov-Bakelman-Pucci in the sixties to study the fully nonlinear PDE. It takes different forms in various contexts, the basic idea is connecting the size(or integral over) the lower/upper contact set of the solution to a PDE, with the maximum principle. Here we apply this approach to a Neumann boundary value problem that was designed for our domain K, and prove the anisotropic isoperimetric inequality.

Proof of Theorem 6.0.3. (c.f. [3]) Due to Theorem 7.1.3, we may assume that K is a domain with smooth boundary. We consider the following linear PDE defined on K:

$$\begin{cases} \Delta u = \frac{P_\Phi(K)}{|K|} & \text{in } K \\ \frac{\partial u}{\partial \nu} = \Phi(\nu) & \text{on } \partial K \end{cases} \tag{7.3.1}$$

By the regularity theory of elliptic linear PDE, u is smooth up to the boundary of K. Consider the lower contact set of u on K as:

$$K_+ = \big\{ p \in K : u(x) - u(p) \ge \langle \nabla u(x), x - p \rangle, \forall x \in K \big\}$$

Intuitively, $p \in K_+$ if there is a hyperplane that support the graph of u from below and tangent at p. We claim that the differential map $\nabla u : K_+ \to \mathbb{R}^n$ is onto W_Φ. Take any $v \in W_\Phi$. By the definition of W_Φ, we have that for any $x \in \mathbb{R}^n$, $\langle v, x\rangle < \Phi(v)$. Consider the following function

$$u^*(x) = u(x) - \langle x, v\rangle, \qquad x \in \bar{K}$$

we claim that u^* assumes its minimum in the interior of K. This follows from that, at any $x \in \partial K$,

$$\frac{\partial u^*}{\partial \nu} = \frac{\partial u^*}{\partial \nu} - \langle \nu, v\rangle = \Phi^*(\nu) - \langle \nu, v\rangle > 0$$

where the last inequality uses the duality between Φ and Φ^*. This implies u^* is increasing near all the boundary points and therefore its minimum is in K. Let x^* be the point of minimum, we have

$$\nabla u^*(x^*) = \nabla u(x) - v = 0$$

Hence we proved our first claim that ∇u is onto W_Φ. By the change of variable formula and AM-GM inequality we have

$$|\nabla u(K_+)| = \int_{K_+} \det(J(\nabla u)) = \int_{K_+} \det(\nabla^2 u) \leq \int_{K_+} \Big(\frac{\Delta u}{n}\Big)^n$$

Hence combining with the fact $|W_\Phi| \leq |\nabla u(K_+)|$ and $\Delta u = P_\Phi(K)/|K|$ on K, we conclude that

$$|W_\Phi| \leq \frac{P_\Phi(K)^n}{n^n |K|^{n-1}}$$

which is equivalent to the anisotropic isoperimetric inequality. $\square$

Chapter 8

Relative Isoperimetric Inequalities and the Main Result

8.1 Relative Isoperimetric Inequalities

In the classical isoperimetric inequality in Euclidean spaces, there are no obstacles or fixed boundary condition on the domains of which we are trying to minimize the isoperimetric ratio. As a result, the minimizer is unique under possible translation and rescaling. When minimizing the perimeter while holding the volume to be a constant, it is also natural to consider sets that are restricted to certain subset of $\mathbb{R}^n$. For example, in Dido's Problem, we need to use thin strips of bull's hide, the total length of which is fixed, to enclose land of the maximum area by the river. In modern language, the admissible land must be a subset of the halfplane $\{(x, y) \in \mathbb{R}^2 | y > 0\}$. The equivalent dual problem is to minimize the length of boundary curve while the area of land is fixed. The original Dido's problem has an elegant answer that a semicircle whose diameter coincides with the river will be the area-maximizing land. In general situations, we can ask the following question:

Problem 8.1.1 (relative isoperimetric problem)**.** *Given an open set $A \subset \mathbb{R}^n$ and $m > 0$, find the minimizer of $P(E)$ in $\{E \subset A : |E| = m\}$.*

This problem has not been understood to its full generality in the class of sets of finite perimeter. In some cases the minimizer does not exist, for example, in our main result that follows, an optimal domain will be a limit of shrinking domains that eventually becomes empty. There have been a few results in which A is assumed to relate to certain convexity. The following theorem, is one of the earliest result in this direction:

Theorem 8.1.2 ([16] Theorem 1.1)**.** *If $C \in \mathbb{R}^n$ is a convex cone with vertex at the origin, $n \geq 2$, and $\alpha_n = P(B_1(0); C)$, then the following isoperimetric inequality holds:*

$$P(E; C) \geq n\alpha_n^{\frac{1}{n}} |E|^{\frac{n-1}{n}}$$

for any Lebesgue measurable set E with $|E| < \infty$. Moreover, if ∂C is smooth away from the origin, then the equality holds if and only if E is homothetic to $C \cap B_R$.

Remark 8.1.3. *If C is not a convex cone, then the above sharp inequality does not hold, as can be seen from the following example: in $\mathbb{R}^2$, let C be the union of the first and third quadrant. If we fix $|E| = \pi$, to minimize $P(E; C)$, it would be better to let E be the quarter-circle of radius 2 contained in the first quadrant, than to choose E as the union of two quarter-circles of radius $\sqrt{2}$ contained in the first and third quadrant.*

One can also consider $A = \mathbb{R}^n \setminus C$, where C is a convex set that has non-empty interior and non-empty boundary. We have the following theorem:

Theorem 8.1.4 ([5])**.** *Let $C \subset \mathbb{R}^n$ be a convex set with non-empty interior and smooth boundary. Then for any bounded set $D \subset \mathbb{R}^n \setminus C$ of finite perimeter, we have*

$$P(D; \mathbb{R}^n \setminus C) \geq n\Big(\frac{1}{2}\omega_n\Big)^{\frac{1}{n}} |D|^{\frac{n-1}{n}}$$

with equality if and only if D is a halfball and $\partial D \setminus C$ is a hemisphere.

8.2 Proof of the Main Result

In the isotropic case of the relative isoperimetric problem, recently Liu-Wang-Weng[17] gave a proof based on ABP method. By exploiting the relation between the convexity of C and a solution to ellptic Neumann boundary problem, they introduced the generalized normal cone of ∂C. The symmetry of Euclidean sphere is an important ingredient to get a volume estimate of the normal cone restricted to outward direction of C. So it would be interesting to ask if a similar isoperimetric inequality also holds for anisotropic case, since the Wulff shape has no symmetry. In this short note we will show that when C is an Euclidean ball, then the anisotropic problem has a similar answer. Due to the lost of rotation invariance of the Wulff shape, we need to replace the half-ball by the least volume of a Wulff shape cut by a halfspace.

We have the following theorem:

Theorem 8.2.1 (Theorem 6.0.1). *Let $\Phi : \mathbb{R}^n \to [0, \infty)$ be a convex, one homogeneous coercive function, and W_Φ is the Wulff shape associated with Φ. If $\Omega \subset \mathbb{R}^n \setminus B_R(0)$ is a smooth region, we have the following sharp inequality:*

$$P_\Phi(\Omega, \mathbb{R}^n \setminus B_R(0)) \geq n(\beta|W_\Phi|)^{\frac{1}{n}}|\Omega|^{\frac{n-1}{n}}$$

where

$$\beta := \inf_{v \in \mathbb{S}^{n-1}} \frac{|W \cap \{\langle x, v\rangle \geq 0\}|}{|W|}.$$

We denote $\Gamma = \partial\Omega \cap \partial B_R(0)$ and $\Sigma = \partial\Omega \setminus \Gamma$, by perturbing $\partial\Omega$ nearby $\partial\Gamma$ without varing the anisotropic perimeter and volume of Ω significantly, we may assume that Σ meets ∂C orthogonally, i.e. the normal vectors of Σ is orthogonal to the normal of ∂C along $\partial\Sigma \cap \partial\Gamma$. Now we consider the following Neumann problem on Ω:

$$\begin{cases} \Delta u = \frac{P_\Phi(\Sigma)}{|\Omega|} & \text{on } \Omega \\ \frac{\partial u}{\partial \nu} = \Phi(\nu) & \text{on } \Sigma - \partial\Gamma \\ \frac{\partial u}{\partial \sigma} = 0 & \text{on } \Gamma \end{cases}$$

we define the lower contact set of u in Ω by

$$\Omega_+ := \{p \in \Omega : u(x) - u(p) \geq \langle \nabla u(p), x - p\rangle, \forall x \in \Omega\}.$$

Our goal is to prove that $|\nabla u(\Omega_+)| \geq \inf_{v \in \mathbb{S}^{n-1}} |W \cap \{\langle x, v\rangle > 0\}|$. Indeed we have the following lemma.

Lemma 8.2.1. $\nabla u(\Omega_+)$ *contains* $W \cap \{\langle x, v\rangle > 0\}$ *for some* $v \in \mathbb{S}^{n-1}$.

Let $H_v := \{x \in \mathbb{R}^n | \langle x, v\rangle > 0\}$, and in order to prove lemma 8.2.1, following the notation of [17] we introduce the notion of a generalized restricted normal cone at $p \in \Gamma$. Let

$$N_p^u\Gamma := \{v \in \mathbb{R}^n | \langle x - p, v\rangle \leq u(x) - u(p), \forall x \in \Gamma\}$$

which can also be interpreted as the set of vectors v such that the function $u(\cdot) - \langle \cdot, v\rangle : \Gamma \to \mathbb{R}$ has a minimum at p. For a continuous mapping $\sigma : \Gamma \to \mathbb{S}^{n-1}$, we define the restricted normal cone at p as

$$N_p^u\Gamma/\sigma(p) := \left\{v \in N_p^u\Gamma \;\middle|\; \langle v, \sigma(p)\rangle \geq 0\right\}$$

We shall also define

$$N^u\Gamma/\sigma = \bigcup_{p\in\Gamma} N_p^u\Gamma/\sigma(p)$$

If σ maps Γ to its outer normal, we simplify the notation as $N_p^u\Gamma^+ = N_p^u\Gamma/\sigma(p)$, and $N_p^u\Gamma^- = N_p^u\Gamma/(-\sigma(p))$, and $N^u\Gamma^+$, $N^u\Gamma^-$ respectively.

Since ν is the outer normal of $\partial\Gamma$ in Γ and $\partial u/\partial\nu = \Phi(\nu) > 0$, the function $u|_\Gamma$ has a minimum inside Γ. By suitable translation and rotation, we can assume $(0, 0, \cdots, r) \in \Gamma$ is the minimum point of $u|_\Gamma$ and $\sigma(0) = e_n = (0, 0, \cdots, 1)$. We shall prove that $W \cap H_{e_n} \subset N^u\Gamma^+$, as stated in the following lemma.

Lemma 8.2.2. *If* $v \in W \cap \{\langle x, e_n\rangle > 0\}$, *then* $v \in N^u\Gamma^+$.

Proof of Lemma 8.2.2. Let $v \in W$, since $\Gamma\cup\partial\Gamma$ is compact, $u(\cdot)-\langle v, \cdot\rangle$ has a minimum at $p \in \Gamma\cup\partial\Gamma$. Notice that

$$\frac{\partial}{\partial\nu}(u(\cdot) - \langle v, \cdot\rangle) = \frac{\partial u}{\partial\nu} - \langle v, \nu\rangle = \Phi(\nu) - \langle v, \nu\rangle \geq \Phi(\nu) - \Phi^*(v)\Phi(\nu) > 0$$

since $\Phi^*(v) < 1$. Hence $p \notin \partial\Gamma$. Therefore we consider $p \in \Gamma$ as the minimum point of $u(\cdot) - \langle v, \cdot\rangle$, then

$$\langle v, p_0 - p\rangle \leq u(p_0) - u(p)$$

Using $u(p_0) - u(p) \leq 0$ and that Γ is a subset of a standard sphere, $p_0 - p = r(\sigma_0 - \sigma(p))$, we have

$$\langle v, \sigma(p)\rangle \geq \langle v, \sigma_0\rangle > 0$$

Hence $v \in N_p^u\Gamma^+$. $\square$

We now prove Lemma 8.2.1 using Lemma 8.2.2.

Proof. For any $v \in W \cap N^u\Gamma^+$, we consider the minimum point of $u(\cdot) - \langle \cdot, v\rangle$ over the whole region $\bar{\Omega}$. Let p be the minimum point. Firstly, since

$$\frac{\partial}{\partial \nu}(u(\cdot) - \langle v, \cdot\rangle) = \frac{\partial u}{\partial \nu} - \langle v, \nu\rangle = \Phi(\nu) - \langle v, \nu\rangle \geq \Phi(\nu) - \Phi^*(v)\Phi(\nu) > 0$$

we have $p \notin \partial\Sigma$. If $p \in \Omega$, by first derivative test we have $\nabla u(p) = v$, which implies $v \in \nabla u(\Omega_+)$, and we are done. If $p \in \Gamma$, then we have

$$\frac{\partial}{\partial \sigma}(u(\cdot) - \langle v, \cdot\rangle)\Big|_p \geq 0.$$

But we have

$$\frac{\partial}{\partial \sigma}(u(\cdot) - \langle v, \cdot\rangle) = \frac{\partial u}{\partial \sigma(p)} - \langle v, \sigma\rangle = 0 - \langle v, \sigma(p)\rangle < 0$$

since $v \in N_p^u\Gamma^+$. This contradiction rules out the last case $p \in \Gamma$. Therefore we have $p \in \Omega$, and $v \in \nabla u(\Omega^+)$. Combining with *Lemma 1.2* we have

$$W \cap \{\langle x, v\rangle > 0\} \subset W \cap N^u\Gamma^+ \subset \nabla u(\Omega^+)$$

$\square$

Now let us finish the proof of Theorem 6.0.1. We have

$$|W \cap \{\langle x, v\rangle > 0\}| \leq |\nabla u(\Omega^+)| = \int_{\Omega^+} |\det(\nabla^2 u)| \leq \int_{\Omega^+} \left(\frac{\Delta u}{n}\right)^n$$

where the last inequality follows from the non-negative definiteness of $\nabla^2 u$ over Ω^+. Using the original PDE that u satisfies, we have

$$\int_{\Omega^+} \left(\frac{\Delta u}{n}\right)^n = \left(\frac{P_\Phi(\Sigma)}{n|\Omega|}\right)^n |\Omega^+| \leq \frac{1}{n^n} \frac{P_\Phi(\Sigma)^n}{|\Omega|^{n-1}}.$$

Since for the standard Wulff shape $W = \{\Phi^*(v) < 1\}$ we have $P_\Phi(W) = n|W|$, we have

$$\frac{P_\Phi(\Sigma)^n}{|\Omega|^{n-1}} \geq \frac{|W \cap \{\langle x, e_n \rangle > 0\}|}{|W|} \frac{P_\Phi(\partial W)^n}{|W|^{n-1}} \geq \beta \frac{P_\Phi(\partial W)^n}{|W|^{n-1}}.$$

where β is the infimum of volume among all intersection between half-space and W over $|W|$. Hence Theorem 6.0.1 is proved.

Remark. The inequality in Theorem 6.0.1 is actually sharp. It is easy to see that if the volume $|W \cap \{\langle x, v \rangle > 0\}|$ has minimizer at v, then we can consider $\Omega_r = W - \mathbb{B}_r^n(-rv)$, as $r \to \infty$, Ω_r converges to $W \cap \{\langle x, v \rangle > 0\}$. Also we shall notice that the equality is never achieved. By a similar argument as in [17], the equality case implies that Γ is flat, but this cannot be achieved in the present case as Γ is a part of the sphere.

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