UC Santa Barbara

UC Santa Barbara Previously Published Works

Title

Knots in knots: A study of classical knot diagrams

Permalink

<https://escholarship.org/uc/item/6b74w5vv>

Journal

Journal of Knot Theory and Its Ramifications, 25(09)

ISSN

0218-2165

Author

Millett, Kenneth C

Publication Date

2016-08-01

DOI

10.1142/s0218216516410133

Peer reviewed

Knots in knots: A study of classical knot diagrams

Kenneth C Millett

*Department of Mathematics,
University of California,
Santa Barbara, CA 93106, USA
millett@math.ucsb.edu*

Received 18 August 2015

Accepted 3 September 2015

Published 7 June 2016

ABSTRACT

Looking at the structure of minimal prime knot presentations, one can notice that there are often, perhaps always, segments that present either the trefoil or the figure-eight knot. This note explores the question as to whether this is always the case, reporting on conversations with Jablan Slavik that began at a 2009 conference in Trieste, Italy and which never reached a conclusion. Evidence supporting this conjectured presence is reported and potential consequences are described.

Keywords: Prime knots; minimal diagrams; subknots.

Mathematics Subject Classification 2010: 57M25

1. Introduction

In May 2009, I presented two conjectures concerning the fine structure of prime knots, understood as either minimal crossing prime knot diagrams or ideal representations of the prime knot type, during a lecture at the Abdus Salam International Centre for Theoretical Physics Advanced School for the Conference on Knot Theory and its Applications to Physics and Biology organized by Slavik Jablan, Louis Kauffman, Sofia Lambropoulou and, Jozef Przytycki. They are:

Conjecture V. *The probability that a minimal prime knot diagram contains a trefoil segment goes to one as the crossing number goes to infinity.*

Conjecture VI. *The probability that a minimal prime knot diagram contains a trefoil slipknot segment goes to one as the crossing number goes to infinity.*

During our time in Trieste, Slavik Jablan and I began a conversation concerning just what these conjectures meant, what evidence I had that suggested they and their refinements were true and, finally, how one might imagine proving of them. As we have not yet shared the substance of our explorations, I wish to contribute this

account of our investigation in commemoration of Jablan's enormous enthusiasm for all manner of mathematics, especially those concerning the beauty of its visual expression and for matters involving knots and links. First, I will briefly describe our discussion of what these conjectures mean; next, how one is lead to believe that they might be true and some refinements that occurred during our interaction; and conclude with some comments on subsequent related explorations.

Of course, the responsibility for the accuracy of this report is solely mine, relying sole on our email exchanges, draft discussion documents and, my memory of these matters.

1.1. *Historical perspectives*

At a 2003 meeting in Les Diablerets, Switzerland, William Taylor described the presence of a deep trefoil knot in a protein, PDB 1lipa, using a random structure shrinking method that, when applied to less robust structures, could lead to different conclusions depending on the sequence of operations [7]. Wishing to avoid this problem, Dobay, Stasiak and I developed and employed a method that could be applied to open molecular chains such as proteins or, mathematically, to open arcs. The method is based upon a statistical analysis of the collection of closures to points on a very large sphere containing the open chain in a relatively small neighborhood of its center [3]. With this new method, we were able to assess the consequences of knotting of an open chain on its radius of gyration and determine how it scales with increasing length, to reassess the presence of knots in proteins, to measure the average size of knots and slipknots in random walks, and, more recently, undertake a systematic analysis of known protein structures and the biological character of these knots as well as a new analysis of the presence of knots in ideal knots. In 2006, I noticed that many of the presentations of prime knots in standard tables, e.g. Rolfsen [5], contained subsegments that were intrinsically trefoil or figure-eight knotted arcs. After a preliminary visual analysis, I proposed the project of a systematic analysis of the known prime knots to Joseph Migler, who wrote a computer code that identified all those containing trefoil knots. Supplemented with a case by case analysis of those remaining, we discovered that all contained either trefoil or figure-eight knots or both. The resulting data lead to the conjectures proposed in my 2009 lecture and reported in this note.

1.2. *Subknots of a knot*

The first question is "How one can identify a knot supported by a subsegment of a knot diagram?" Consider the case of the 8_{18} knot, shown in Fig. 1. Imagine that the entire figure lies in the plane except for small "indentations" at the crossings where the lower strand dips below the plane. Consider the short segment indicated by the arrow. One might expect that roughly half the closures would result in a trivial knot due to the almost planar character of the segment. In fact, as the initial

Fig. 1. Presentation of 8_{18} , a short segment of 8_{18} in the direction of the arrow giving an unknotted segment shown in the right.

Fig. 2. Presentation of 8_{18} , a long segment of 8_{18} in the direction of the arrow giving a trefoil subknot, the crossing removed and, the equivalent result via a crossing change.

and terminal points of the segment lie in the same complementary region of the complement of the projection of the segment, shown in the right of Fig. 1, almost all closures result in the unknot.

There is, however, a trefoil knotted subsegment in 8_{18} as shown in Fig. 2. This is achieved by “removing” a crossing as shown in the central image. Using the closure method, one sees that nearly half of the closures are a trefoil knot. These closures are equivalent to the result of having changed the crossing, as shown on the right of the figure. This segment illustrates the more complicated situation in which the initial and terminal points of the segment lie in distinct complementary regions of the projection of the segment. This complication leads one to consider alternative ways to express the character of subknots. In our discussions, Slavik and I explored several possible definitions that might better capture the intent of the conjectures and, eventually, facilitate a rigorous proof of the conjecture(s) without employing the statistics of the closures to “the sphere at infinity.” Thus, we focused on crossing changes modifying the minimal presentation of the prime knot.

Definition A. A knot, K^* , will be called a **subknot** of a knot K if it can be obtained from a minimal diagram of K by crossing changes.

While this definition does not quite capture my intent, it does allow one to prove that a minimal diagram of a non-trivial knot has a trefoil subknot.

Theorem 1.1. *Every minimal diagram of a non-trivial knot, K , contains a trefoil subknot, in the sense of Definition A.*

Sketch of Proof. Consider any presentation of the non-trivial knot, K , and any point along the curve. Proceed along the curve (you may select the direction) until reaching the first crossing of the constructed arc with itself producing a γ diagram. Continue in this direction, changing crossings to achieve a descending arc, until intersecting the loop of the γ for the first time. This must occur as the diagram for K is minimal in order to avoid a type I Reidemeister move. The increasing end of the arc now lies within the γ and, therefore, must exit at a third point. If necessary, change these three crossings to make then alternating along the ' γ '. Change the remaining crossings to the remaining arc is strictly ascending and lies "above" the created trefoil. With the exception of the three 'trefoil' crossings, the others are descending so that, the result is a very complicated diagram of a trefoil knot.

As a consequence, we concluded that Definition A missed the point of the conjectures in that one should conserve the crossings of the subsegment. Although we considered the implications of 'virtual' crossings, it was more productive to focus on the theme of 'crossing changes' to capture the original intent of the conjectures. Basically, one should not change any of the self crossings of segment under consideration but, in the spirit of the above, change all others so that the result is that the complementary segment is strictly ascending (or strictly descending). This, we felt, best expressed the essence of the closure statistical philosophy and the conjectures.

Definition B. A knot, K^* , will be called a **subknot** of a knot K if it can be obtained from a minimal diagram of K by crossing changes that preserve those within a segment of the diagram and change those outside this segment so that the complementary segment is strictly ascending.

Observe that this captures the structure in the two 8_{18} cases considered earlier and shown in Figs. 1 and 2.

2. Subknots of Minimal Prime Knot Diagrams

Following an initial visual analysis of the first minimal prime knot diagrams, through eight crossings, three phenomena are striking: (1) not all diagrams contain trefoils, e.g. the figure eight diagram does not contain a trefoil; (2) a large number of diagrams do contain trefoils; and (3) every diagram contained either a trefoil (all but 4_1 and 6_1), or a figure eight subknot (confirmed in 4_1 and 6_1). Table 1 reports the number of distinct trefoil segments and whether a figure-eight knotted segment in a visual analysis of the diagram (and confirmed by later computer analyses).

Table 1. Analysis of minimal prime knots (through 7 crossings).

Knot	3_1	4_1
3_1	1	0
4_1	0	1
5_1	2	0
5_2	5	0
6_1	0	1
6_2	1	0
6_3	2	0
7_1	7	0
7_2	2	0
7_3	4	0
7_4	1	0
7_5	2	0
7_6	2	0
7_7	2	0

2.1. Initial computer analysis

Encouraged by the visual evidence, I asked Joseph Migler to write a program to analyze minimal crossing prime knots for the presence of trefoil knots. Using Joseph Migler's Java code, we undertook an analysis of the prime knots through ten crossings [1, 8] and, later, a search for trefoil knots through sixteen crossings, looking only for those of the first type. Those that did not contain trefoils were visually analyzed to determine if they contained figure-eight knots.

Table 2. Analysis of minimal prime knots (8 crossings knots).

Knot	3_1	4_1
8_1	0	1
8_2	2	0
8_3	0	1
8_4	1	0
8_5	4	0
8_6	1	0
8_7	3	0
8_8	3	0
8_9	2	0
8_{10}	5	0
8_{11}	2	0
8_{12}	0	1
8_{13}	1	0
8_{14}	1	0
8_{15}	4	0
8_{16}	3	0
8_{17}	2	0
8_{18}	4	0
8_{19}	1	0
8_{20}	1	0
8_{21}	2	0

The eight crossing prime knots give three types, 8_1 , 8_3 , and 8_{12} , of the 21 that do not contain trefoils. Visual analysis shows that they contain figure-eight knots, Table 2. In general, we looked first for trefoils and, failing to find them, looked for a figure-eight knot. For nine crossing prime knots, 83% contained trefoils, Table 3

Table 3. Analysis of ideal knots (9 crossings knots).

Knot	3_1	4_1		Knot	3_1	4_1
9_1	9	0		9_{26}	1	0
9_2	2	0		9_{27}	1	0
9_3	6	0		9_{28}	3	0
9_4	4	0		9_{29}	3	0
9_5	1	0		9_{30}	3	0
9_6	2	0		9_{31}	2	0
9_7	2	0		9_{32}	1	0
9_8	3	0		9_{33}	1	0
9_9	4	0		9_{34}	1	0
9_{10}	3	0		9_{35}	6	0
9_{11}	2	0		9_{36}	2	0
9_{12}	2	0		9_{37}	0	1
9_{13}	1	0		9_{38}	5	0
9_{14}	2	0		9_{39}	1	0
9_{15}	2	0		9_{40}	4	0
9_{16}	6	0		9_{41}	3	0
9_{17}	1	0		9_{42}	1	0
9_{18}	2	0		9_{43}	1	0
9_{19}	3	0		9_{44}	2	0
9_{20}	2	0		9_{45}	2	0
9_{21}	2	0		9_{46}	2	0
9_{22}	2	0		9_{47}	3	0
9_{23}	2	0		9_{48}	6	0
9_{24}	3	0		9_{49}	3	0
9_{25}	3	0

Fig. 3. The proportion of prime alternating knots containing trefoil subknots through sixteen crossings.

and, for 10 crossing prime knots, 92% contained trefoil knots. In this way, one finds that, through ten crossings, every prime knot contains either a trefoil or a figure-eight knot.

For eleven through sixteen crossing prime knots, the program only identified trefoil knots of the first type so that the resulting data gives a lower bound for the fraction containing trefoils. One must expect the actual fraction to be much larger. In addition, we did not search for figure-eight knots beyond ten crossing prime knots. Figure 3 shows the fraction of trefoils contained in all minimal crossing prime knots through ten crossings and all minimal crossing alternating prime knot presentations through sixteen crossings that have been identified so far. There are many more complex trefoils that have not been identified by our simple algorithm as has proved to be the case using a partial visual inspection for the presence of trefoils and figure-eight knots in these more complex presentation.

3. Trefoils and Figure-Eights and Others

So far, every minimal diagram of a prime knot has contained either a trefoil or a figure-eight knot. Perhaps, if the conjecture is not true, there may still be a finite collection of prime knot types such that every minimal diagram contains one of these. We considered possible prime knot types that might contain neither a trefoil nor a figure-eight knot but did not identify a counter-example to the conjecture. Interesting possible cases included the Conway knots, $8^* = 8_{18}$, $9^* = 9_{40}$, and $10^* = 10_{123}$. We have already discussed 8^* above. Figure 4 shows 9^* and 10^* , both of which contain figure-eight knots. Thus, so far, all the evidence suggests that the conjecture may be true.

Question 1. Are there minimal prime knot diagrams that contain neither a trefoil nor a figure-eight subknot?

Fig. 4. Both 9_{40} and 10_{123} knots contain figure-eight knots.

4. Ideal Knots

Subsequent to these discussions, Rawdon, Stasiak and I studied the presence of subknots, using the spatial closure method discussed earlier, in ideal and random knots [4], the knotting fingerprint, in a generalization of the earlier protein structural analysis [2, 6] in which we used another version of a knotting fingerprint. Here, each cell in the circular fingerprint expresses the strength of the knotting of

Fig. 5. Knot 8_{18} contains trefoils while both 9_{40} and 10_{123} knots contain figure-eight knots.

a subknot of a length and location specified by the cells location in the circular fingerprint, see Fig. 5. The study of subknots of ideal knots provided further evidence of the conjecture in that those ideal prime knots analyzed contained trefoil or figure-eight (or both) subknots. The knotting fingerprint appears to give an even finer structural analysis than is possible from the minimal diagram. For example, the knotting finger print of 9_{40} shown in Fig. 5 indicates the presence of slipknotted trefoil knots that do not appear to be visible as subknots of the minimal diagram, at least in the sense of Definition B.

5. Concluding Thoughts

These are the first steps toward an analysis of subknots present in minimal knot diagrams of prime knots and leave many questions. For example, if one were to undertake a complete analysis of the already classified prime alternating and non-alternating knot presentations, what would one find? Would there be additional evidence to confirm these conjectures are might one find it necessary to add a new basic knot type to the trefoil and figure-eight types?

If the evidence supports the conjectures, then one must face the question of how one might go about providing a proof. This seems to be a rather complex question if only because we have not been able to envisage a strategy to test. The perspective of the 2009 Trieste conversation was the search for a small collection of irreducible knot types from which every prime knot is created, a theory of “elementary knot types.” An unexplored dimension related to the presence of these unknots concerns their topological consequences. For example, one reason that a result of this type would be interesting is that it could lead to an elementary strategy by which one could prove that certain knot invariants, e.g. the Jones polynomial, could detect non-trivial knotting. Others would be the implications for the genus of the knot or for non-trivial representations of the knot group.

We did not explore the presence of slipknotted trefoils in minimal prime knot presentations nor have we explored the asymptotic question of increasing presence as the number of crossings increases. The initial data suggests that the probability of either the trefoil or the slipknotted trefoil do not go to one as the number of crossings goes to infinity. One wonders if their probability converges and, if so, to what value as the number of crossings goes to infinity.

Unfortunately, our conversations did not lead to a definitive understanding of these questions and other related questions. It now seems time to share our quite preliminary thoughts in the hope that they will be of interest to others who can go illuminate these issues where we were unable to do so.

Acknowledgments

I would like to express my thanks to Joseph Migler for his contribution to this project, to Eric Rawdon and Andrzej Stasiak for their interest in subknots and our

subsequent study of the knotting fingerprints of ideal knots and, of course, most especially to Slavik Jablan for his conversations and collaboration in the exploration of the conjectured subknots present in minimal knot presentations.

References

- [1] J. Hoste, M. Thistlethwaite and J. Weeks, The first 1,701,936 knots, *Math. Intelligencer* **20**(4) (1998) 33–48.
- [2] M. Jamroz, W. Niemyska, E. J. Rawdon, A. Stasiak, K. C. Millett, P. Sulkowski and J. I. Sulkowska, Knotprot: A database of proteins with knots and slipknots, *Nucleic Acids Research* **43**(1) (2015) D306–D314.
- [3] K. Millett, A. Dobay and A. Stasiak, Linear random knots and their scaling behavior, *Macromolecules* **38**(2) (2005) 601–606.
- [4] E. J. Rawdon, K. C. Millett and A. Stasiak, Subknots in ideal knots, random knots, and knotted proteins, *Sci. Rep.* **5** (2015) 8928, doi:10.1038/srep08928.
- [5] D. Rolfsen, *Knots and Links* (Publish or Perish, Inc., Berkeley, CA, 1976).
- [6] J. I. Sulkowska, E. J. Rawdon, K. C. Millett, J. N. Onuchic and A. Stasiak, Conservation of complex knotting and slipknotting patterns in proteins, *Proc. Natl. Acad. Sci. USA* **109**(26) (2012) E1715–E1723.
- [7] W. R. Taylor, A deeply knotted protein and how it might fold, *Nature* **406** (2000) 916–919.
- [8] M. B. Thistlethwaite, Knot tabulations and related topics, in *Aspects of Topology*, Volume 93 of *London Math. Soc. Lecture Note Ser.* (Cambridge Univ. Press, Cambridge, 1985), pp. 1–76.