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(Ph. D. Thesis)

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LOW FREQUENCY ELECTROSTATIC INSTABILITIES IN THE MAGNETOSPHERE

Contents

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ABSTRACT

The stability of the Van Allen belt in the outer trapping zone $(2 < L < 8)$ against electrostatic low-frequency perturbations is studied. Inside the plasmapause, the energetic belt $(E > 40\ \text{keV})$ is found to be stable. Outside the plasmapause, the ring current belt is found to be unstable when the density gradient at the outer edge exceeds a certain critical value. A component of the electric field along the magnetic field line is associated with the instability. When the ring current belt is stable by itself, it supports a wave which can resonantly interact with the drift motion of the energetic particles. The condition for overstability is derived. The possible relevance of the instabilities to auroral phenomena is discussed.

I. INTRODUCTION

Two outstanding problems in current space research are the origin of the radiation belts[1] and the occurrence of the aurora and its associated phenomena. It has become increasingly clear that the two are closely related problems and must be explained simultaneously. Since the discovery of the radiation belts, there have been numerous attempts to explain the auroral production as a simple dumping of the trapped particles into the atmosphere. However, later measurements indicate that the flux of the energetic belt is insufficient to sustain an auroral precipitation[2] (O'Brien, 1964); that the flux of the trapped particles, in fact, increases as the precipitation flux increases (O'Brien, 1964). Thus the mechanisms that produce auroras must accompany the increase in the population of the radiation belts, and the "splash catcher" model replaces the leaky bucket (O'Brien, 1964).

This is also evident from the magnetic storm observations: the polar substorn -- aurora breakup and its associated bay event -- occurs intermittently during the main phase decrease, and the main phase develops more or less strongly as the substorms are more or fewer (Chapman, 1961). Because the polar substorm is due to intense precipitation of charged particles into the polar ionosphere and the main phase decrease is due to the ring current belt, this correlation again shows that the enhanced injection of particles into the closed-field-line region of the magnetosphere is accompanied by or may even be due to the intense precipitation. The ring current belt, as recently discovered by Frank (1967), is composed mainly of low-energy particles (proton energy $\approx$ 40 keV, electron energy $\approx$ 10 keV) with relatively narrow energy range.

The main source of the trapped particles in the geomagnetic field is the solar wind. There are two processes by which the solar wind particles are introduced into the earth's magnetic field: diffusion and convection (injection).

It is believed that the ring current belt, which has most energy content during the storm time, is injected from the tail of the magnetosphere (Axford, 1967). A possible injection mechanism is the reconnection of the field lines in the tail(Axford, Petchek and Siscoe, 1963). Implicit in the reconnection model is that the substorm should coincide with the relaxation of the magnetic field in the tail and with the injection of plasma into the inner magnetosphere to form the ring current belt. The question remaining is that of the sporadic nature of the polar substorm.

The more energetic particles ($E > 100$ keV for protons and $E > 40$ keV for electrons) are believed to be diffused into the outer zone ($2 < L < 8$), where L is the magnetic shell parameter (McIlwain, 1961). In analyzing Davis and Williamson's data (1963) on the outer radiation zone protons, Dungey, Hess, and Nakada (1965) found evidence that supports Kellogg's conjecture (1959) that the particles are of solar origin and are diffused into the outer zone by processes that conserve the magnetic moment and longitudinal invariant, but break the flux invariant of the particles.

Subsequently Nakada and Mead (1966) calculated the diffusion coefficient due to the magnetic fluctuations of the sudden commencements and sudden impulses. Their result, however, is smaller by a factor of ten than the observed value (Falthammer, 1966). Furthermore, in their calculation only the collisional loss is included, whereas the anomalous loss due to various plasma instabilities, such as the loss cone and whistler, are certainly important (Kennel and Petchek, 1965). Thus a diffusion rate at least a factor ten larger than that calculated by Nakada and Mead is needed to account for the observed diffusion in the outer belt. Recently Crifo and Mozer (1967), in analyzing the proton data (0.5 MeV $< E <$ 150 MeV) at $L = 1.5$, $B = 0.18$ (L, B are McIlwain coordinates), found that the inward diffusion rate driven by the magnetic disturbances associated with sudden commencements and sudden impulses fails by many orders of magnitude to explain the observed flux of protons with energy $\approx$ 500 keV at $L \approx 1.5$, $B \approx 0.18$.

In plasma physics, it is well known that plasma instabilities can cause anomalous diffusion across the magnetic field. Dungey (1965) has suggested that the polar substorm, with its time scale typically of the order of an hour, can contribute significantly to the inward diffusion of the energetic particles in the outer zone.

Because of its sporadic nature, the polar substorm is most likely to be cause by plasma instability in the magnetosphere (Akasofu, 1967; Cole, 1967; Swift, 1967; Coppi et al, 1966). In this paper we study the low-frequency electrostatic instabilities (which conserve the first two adiabatic invariants of the particles) in the outer trapping zone $(2 < L < 8)$ and their possible relation to polar substorms.

Condition of the Outer Trapping Zone $(2 < L < 8)$

In the study of electron density in the magnetosphere by means of whistlers, Carpenter (1966) found that there exists a sharp boundary - the plasmapause, separating a dense plasma $(n \approx 10^3/cm^3)$ inside and a rarified plasma $(n \approx 1/cm^3)$ outside. The location of the plasmapause depends not only on longitude but also on the disturbance condition of the magnetosphere (Carpenter, 1967). In the periods of magnetic quiescence, the average location (over longitude) of the plasmapause is at $L \approx 5$. During the storm time, the plasmapause contracts to $L \approx 3$. (Carpenter, 1967; Taylor, et al., 1968)

The region inside the plasmapause, called plasmasphere, is mainly populated by charged particles originating from the ionosphere with thermal energy $(E \lesssim 1$ eV). In addition to the thermal plasma of terrestrial origin, there is also a plasma $(E \gtrsim 1$ keV) of solar origin. The main energy reservoir inside the plasmapause are energetic protons $E \gtrsim 100$ keV (Davis, 1965; Frank, 1967a). At the equatorial plane inside the plasmapause, this energetic component has an energy density of the order of 10^{-7} ergs/cm^3, or about one tenth of that of the local magnetic field. On the other hand, the energy density of the low-energy proton (190 eV $< E <$ 50 keV), during the period of magnetic quiescence, is of the order of 10^{-9} ergs/cm^3 for $L < 5$

(Frank, 1967a). The energy density of the thermal plasma ($E \lesssim 1$ eV) is also of the order of 10^{-9} ergs/cm^3. Because the equatorial pitch angle distribution of the energetic component (protons $E > 100$ keV, electrons > 40 keV) has a maximum about the equatorial plane (Davis and Williamson, 1963), the number density of the energetic particles in a given flux tube has a maximum at the equatorial plane and falls off rapidly away from it. The region near the equatorial plane where the main body of the energetic particles resides will be called "Region I". In this region, the temperature is clearly determined by the energetic component. The energetic proton's spectra may be approximated by $\exp(-E/E_0)$ with $E_0 \propto L^{-3} \propto B$ (Davis and Williamson, 1963). This suggests that the particles are accelerated by betatron processes that diffuse particles radially inward, while conserving the particles' magnetic moments μ and longitudinal invariant J (Nakada et al., 1965). Nakada et al., (1965) further showed that the particle distribution function $f_1[E(\mu\ J\ L), \alpha_0(\mu\ J\ L), L]$, i.e. the number density of particles with energy E and EPA (equatorial pitch angle) α_0 in the equatorial plane at L, is such that $(\partial f_1/\partial L)_{\mu J} > 0$ (Fig. 1), suggesting that the source is at large L. In fact, $f_1(\mu\ J\ L)$ is factorizable in its $\mu\ J$ and L dependence: $f_1(\mu_1 J_1 L) = g(\mu, J)\ h(L)$ (Hess, 1967). Despite their large variability in flux in relatively short times ($\approx$ order of hours, Forbush et al., 1962), the outer-belt electrons (40 keV $< E <$ 5MeV) have characteristics very similar to the protons. The energy range is about the same, and the fluxes are comparable. The spectra show similar falloff with increasing energy, and similar softening with L (Hess et al., 1965). The similarities indicate a common source and that common mechanisms are operating on protons and electrons.

Outside the plasmapause, the main energy content is associated with the ring current belt of low-energy particles ($1 \text{ keV} < E < 50 \text{ keV}$) (Frank, 1967 a, b). Its flux is peaked in L-space. During the quiet times the peak is located at about $L \approx 6$, approximately the auroral zone. During the storm time, the peak moves inward to $L \approx 4$ while enhanced tenfold in magnitude, causing the main phase decrease. The inward movement of the peak is correlated with the contraction of the plasmasphere (Taylor, et al., 1968), and is also in agreement with shift toward the equator of the southernmost auroral arc (Frank, 1967 a, b). By comparing Figs. 2 and 3 for $D_{st} \approx -50\ \gamma$, we conjecture that the instantaneous L-shell of the auroral arc is about the same as the peak of the ring current belt. At a given L-shell, the differential energy spectrum of the flux is peaked around 10 keV (Frank, 1967 a).

Comment on Previous Work

Gold (1959) first suggested the possible interchange motion in the magnetosphere and derived from thermodynamic considerations the stability criteria for both the adiabatic and the isothermal processes of a tenuous plasma in a dipole field. Subsequently his work was extended by Sonnerup and Laired (1963) to include other effects such as gravity.

Chang et al. (1965, 1966) have previously considered the ionospheric effects in stabilizing the low-frequency electrostatic instabilities in the magnetosphere. But in view of the fact that

$$\left.\frac{\partial f_1}{\partial L}\right|_{\mu J} > 0$$

for the energetic belt which they considered, it is not clear whether instability exists in the first place, In fact, in the presence of the dense thermal plasma, it will be shown that the Van Allen belt is stable against μJ-conserving electrostatic perturbations. Furthermore they treated the thermal plasma ($\lesssim 1$ eV, which exists inside the plasmapause) as a dynamic component on equal footing with the energetic Van Allen component ($E \gtrsim 1$ keV). But the two components have widely different time scales as well as total energy contents. For the perturbation with period ≈ 1 hour (such as they considered) which breaks the flux invariant but conserves μJ for the energetic component, the longitudinal invariant J for the thermal protons would also be broken, as their bounce period is ≈ 1 hour. Thus one cannot use the μJ-conserving formalism for both components. Due to these inconsistencies, their resulting stability criterion is unreasonable in that it depends only on the number density of the particles (Eq. 66 of Chang et al.) but not on the energy, and the thermal plasma becomes the dominant contributor to the instability. But we know that the energy content (or energy density) is mainly associated with the energetic belt, and it is the free energy residing in the energy density gradient that drives the instability.

Recently Swift (1967) extended the work of Chang et al. to include the ring current belt, but retained the assumption that the perturbation electric field has no component along the magnetic field line, which is valid within the plasmapause (as is shown in Section II). But the ring current belt is outside the plasmapause, and therefore in a collisionless region.[4] A parallel electric field in general exists in a collisionless plasma (Alfven and Falthammer, 1963; Persson, 1966). Thus it is not justified to set $E_{\parallel} = 0$ as in Swift's treatment. There is evidence (Johansen and Omholt, 1963; O'Brien, 1964; Mozer, 1965, 1966) that the parallel electric field indeed exists in the auroral zone during the breakup phase. Furthermore Swift (1967) suggested that the interchange instability occurring at the outer edge of the ring current belt could explain the auroral breakup. But with the assumption of no parallel electric field, it is not clear how this can cause enhanced precipitation.

Chamberlain (1963) has proposed a drift wave instability (Krall and Rosenbluth, 1963) with an electric field along the magnetic field line as a mechanism for auroral precipitation. But the calculation is based on a model of slab geometry with straight magnetic field lines, and it is not clear whether it is applicable to the trapped plasma in the geomagnetic field (Dungey, 1966). Furthermore, the ring current belt is not included in his treatment.

Coppi et al. (1966) have suggested that the dynamics of the tail and hence auroral phenomena is entirely due to the sheet-pinch instability. Concerning their work, Axford (1968) has the following criticism: The nonlinear effect of such an instability would only produce a turbulent resistivity η. But the maximum merging rate as given by Petschek (1961) is inversely proportional to the logarithm of the magnetic Reynold number $R_M = V_A L/\eta$, where V_A is the Alfven speed, L the characteristic dimension of the system. Therefore the merging rate is not very sensitive to the change in resistivity, but it is mainly determined by the macroscopic conditions such as pressure difference between the tail and the ring current belt or boundary conditions.

In this work, we study the low-frequency electrostatic instabilities in the outer trapping zone $(2 < L < 8)$ in the low β approximation. (β is the ratio of plasma pressure to magnetic pressure.) The region inside the plasmapause and that outside the plasmapause are treated separately. Inside the plasmapause, where there is a dense thermal plasma in addition to the energetic particles $(E > 1$ keV), we treat the thermal plasma as a cold plasma because of its very low temperature compared with the energy of the Van Allen particles. The cold plasma then provides a large conductivity along the field line and a dielectric constant across the field line.

In Section II we discuss the ordering scheme with small parameter ϵ, the ratio of proton gyroradius to the characteristic dimension for μJ-conserving perturbations. With the conditions on μJ conservation, the Poisson equation can be ordered. Outside the plasmapause, the quasi-neutrality condition is found to be valid to $O(\epsilon^2)$. Thus we must use the quasi-neutrality condition in our lowest-order calculation while neglecting the Laplacian of potential for consistency in ordering. Moreover, the parallel electric field, of the same order as the perpendicular one, in general exists in this region. Inside the plasmapause, the parallel electric field is found to be much smaller than the perpendicular field because of the large conductivity along the field line. An ordered Poisson equation is derived. The reduced Vlasov equation (Northrop and Teller, 1960) in the ordering used in this paper is also discussed.

Section III is devoted to the stability of the outer belt inside the plasmapause. Using the variational principle, we derive a dispersion relation from which the stability condition is obtained. The outer belt with the distribution function such that $(\partial f_1/\partial L)_{\mu J} > 0$ is found to be stable against the low-frequency perturbations. There is no "resonant instability" due to the interaction of particle drifts and the wave as claimed by Chang et al. (1966).

In Section IV, we study the stability of the plasma outside the plasmapause. Since the ring current belt dominates in both energy and particle density, we neglect the energetic belt in the first approximation and study the ring current belt by itself. The ionosphere, taken to be perfectly conducting, provides the boundary condition.

A necessary and sufficient condition for stability is obtained. The inner edge of the ring current belt, where the density gradient is opposite to the magnetic field gradient, is found to be always stable. The outer edge of the ring current belt, where the density gradient is along the field gradient, is found to be stable for a weak density gradient, but becomes unstable when the density gradient reaches a certain critical value. The instability has a finite parallel electric field along the magnetic field line. The electric field, though $O(\epsilon)$ in magnitude, i.e., $cE_{\parallel}/B \approx \epsilon v$, where v is the velocity of the ring current particles, can cause a potential energy drop along the field line of the order of $Mv^2 \approx 10$ keV, for the fundamental mode.

If the ring current belt is stable by itself, as at the inner edge of the belt or during the period of magnetic quiescence when the density gradient is weak, it supports a wave. The wave can interact resonantly with the drift motion of the energetic particles ($E > 100$ keV) when the azimuthal phase velocity of the wave is equal to the drift velocity of the particles. The wave is damped if the distribution of the energetic particles at the resonant drift frequency is such that $(\partial F_0^{\text{energetic}}/\partial L)_{\mu J} > 0$, where $F_0(\mu\ J\ \psi)$ is the distribution in $\mu\ J\ \psi$ space.[3] On the other hand, if $(\partial F_0^{\text{energetic}}/\partial L)_{\mu J} < 0$, then the wave grows. This overstability has a finite parallel electric field. A physical interpretation is given, and a possible overstability in the magnetosphere is discussed in Section V.

Finally, in Section VI, we discuss the possible relation of the ring current belt instability found in Section IV to the polar substorm -- the injection and precipitation mechanisms. The inward pressure gradient of the ring current belt on its outer edge tends to stop the merging of field lines in the tail of the magnetosphere, until the instability sets in. (Axford, private communication.) Because the instability has a finite parallel electric field, it causes intense electron precipitation. Since the instability tends to relax the pressure gradient of the ring current, it allows merging to occur again. Thus the sporadic nature of precipitation and injection as evidenced by the substorms can be explained.

II. ORDERING SCHEMES

The stability of low-frequency oscillations in an inhomogeneous plasma in a nonuniform magnetic field has been extensively studied in the Finite Larmor Radius ordering scheme: $\frac{\omega}{\Omega_i} \approx \frac{r_g}{L_{\parallel}} \approx \left(\frac{r_g}{L_{\perp}}\right)^2 \equiv \epsilon^2$ (Krall and Rosenbluth, 1963), where Ω_i and r_g are ion gyrofrequency and gyroradius respectively. This scheme treats the detailed motion along the magnetic field line consistently with the drift motion across the field line in the drift time scale, thus requiring $L_{\parallel} \approx \frac{L_{\perp}}{\epsilon}$, where $L_{\parallel}$ is the characteristic dimension along B and $L_{\perp}$ is the characteristic dimension perpendicular to B, for the system is equilibrium. Thus FLR ordering is suitable only for systems with small aspect ratios (long-thin system, $L_{\parallel} \gg L_{\perp}$); and for such systems, the problem is essentially two-dimensional, and curvature effects are negligible (Kennel and Greene, 1966).

For systems with comparable characteristic dimensions (short-fat systems) such as Van Allen radiation belts, the bounce frequency ν_b along the field line is much faster than the drift frequency ω_d across the field lines: $\omega_d \approx \epsilon\, \nu_b \approx \epsilon^2\, \Omega_i$. If one is interested in the stability of such a system against low-frequency modes $\omega \approx \epsilon^2\, \Omega_i$, we can simply average the parallel motion along the field line by introducing the longitudinal invariant $J = \oint p_{\parallel}\, ds$. With perturbations conserving μ J, the Liouville equation (Northrop-Teller, 1960) for guiding center distribution function F in the Eulerian coordinate space (ψ, φ) defined by $\nabla\psi \times \nabla\varphi = \underset{\sim}{B}$ can be used for stability analysis. The treatment will be nonrelativistic.

Coordinate System

We shall use the natural coordinate system ψ, φ, χ, where χ is the magnetic potential $\chi = \int \underset{\sim}{B} \cdot d\underset{\sim}{s}$ for $\nabla \times \underset{\sim}{B} = 0$. For an axisymmetric poloidal magnetic field, we can take φ to be the azimuthal angle, and ψ would then be the flux function

$\psi(r_0) = \int_{\infty}^{r_0} dr\, r\, B(r)$, where r_0 is the equatorial distance from the axis. For a dipole field with dipole moment $\mathcal{M}$, $\psi = \frac{\mathcal{M}}{r_0}$, and essentially measures the L value ($L \equiv r_0$ in the units of earth radius; McIlwain, 1960). The elements of length along the three coordinates are

$$d\,\ell_\psi = \frac{1}{B\rho}\, d\psi \ ,$$

$$d\,\ell_\varphi = \rho\, d\,\varphi,$$

and

$$ds = \frac{1}{B}\, d\chi \ ,$$

where ρ is the perpendicular distance to the axis of symmetry from a point X of a given field line at (ψ, φ).

Adiabatic Motion of Charged Particles

Suppose a particle of charge e, mass M moves in a magnetic field $\underset{\sim}{B}$ and an electric field $\underset{\sim}{E}$ with potential Φ, with velocity components $v_{\parallel}$, $v_{\perp}$ parallel and perpendicular to the magnetic field. For fields with time variation much longer than the bounce period and spatial variation much larger than a gyroradius, the magnetic moment $\mu = \frac{1}{2} M v_{\perp}^{2}/B$ and longitudinal invariant $J = M \oint v_{\parallel}\, ds$ of the particle are conserved. Its drift motion averaged over a bounce is described by the following Hamiltonian equations (Northrop and Teller, 1960; Northrop, 1961):

$$\langle \dot{\psi} \rangle = - \frac{c}{e} \frac{\partial K}{\partial \varphi}\bigg|_{\mu J \psi} , \qquad (1)$$

$$\langle \dot{\varphi} \rangle = \frac{c}{e} \frac{\partial K}{\partial \psi}\bigg|_{\mu J \varphi} , \qquad (2)$$

where

$$K(\mu\ J\ \psi\ \varphi;\ t) = \frac{1}{2} M v_{\parallel}^{2} + \mu B + e\Phi , \qquad (3)$$

is just the lowest-order total energy of the particle. $K(\mu\ J\ \psi\ \varphi;\ t)$ is determined by the equation for the longitudinal invariant,

$$J(\mu\ K\ \psi\ \varphi;\ t) = \oint ds\ [2M(K - \mu B - e\Phi)]^{\frac{1}{2}}\ . \tag{4}$$

The bounce period is given by

$$\nu_b^{-1} = \left.\frac{\partial J}{\partial K}\right|_{\mu\psi\varphi} = \oint \frac{ds}{\left[\frac{2}{M}\ (K - \mu B - e\Phi)\right]^{\frac{1}{2}}}\ . \tag{5}$$

Kinetic Equation

Let $F(\mu\ J\ \psi\ \varphi;\ t)$ be the distribution function in $\mu\ J\ \psi\ \varphi$ space; i.e., $F(\mu\ J\ \psi\ \varphi\ t)\ d\mu\ dJ\ d\psi\ d\varphi$ gives the number of particles in the flux tube $d\psi\ d\varphi$ at (ψ, φ) with magnetic moment and longitudinal action in the intervals $d\mu$ at μ and dJ at J respectively at time t. Since $\mu\ J$ are invariants for each particle, and the motion in (ψ, φ) is described by Hamiltonian equations, there is a Liouville theorem in (ψ, φ) space:

$$\frac{\partial F}{\partial t} - \frac{c}{e}\left[\frac{\partial F}{\partial \psi}\frac{\partial K}{\partial \varphi} - \frac{\partial F}{\partial \varphi}\frac{\partial K}{\partial \psi}\right] = 0. \tag{6}$$

This equation was first derived by Northrop and Teller (1960) from a study of particle motion, and was recently derived from the Vlasov equation by Hastie et al. (1967).

The local particle spatial density is related to $F(\mu\ J\ \psi\ \varphi)$ in the lowest order (Appendix B) by

$$n[r(\psi\ \varphi\ X)] = 2B(r) \iint d\mu\ dJ\ \frac{\nu_b\ F(\mu\ J\ \psi\ \varphi)}{\left[\frac{2}{M}(K - \mu B - e\Phi)\right]^{\frac{1}{2}}}, \tag{7}$$

because the cross section of the flux tube is inversely proportional to B, and the fraction of time the particle spends in a unit segment at X on the field line during one bounce period is $\nu_b/v_{\parallel}(X)$.

Equilibrium

The long-term equilibrium of a low-β plasma in a magnetic field is defined as a steady state over a time much longer than the drift period, and is given by $F_0[\mu\ J\ K(\mu\ J\ \psi\ \varphi)]$, the steady-state distribution function, which is the general solution of Eq. (1) with $\partial F/\partial t = 0$.

Note that the dependence upon (ψ, φ) of the equilibrium distribution F_0 must be implicit through its dependence upon K. For axisymmetric systems, $K = K(\mu\ J\ \psi)$ and $F_0(\mu\ J\ \psi)$, and we have

$$\left.\frac{\partial F_0}{\partial \psi}\right|_{\mu J} = \left.\frac{\partial F_0}{\partial K}\right|_{\mu J} \left.\frac{\partial K}{\partial \psi}\right|_{\mu J} . \tag{8}$$

Perturbations

We are interested in low-frequency electrostatic perturbations that occur in a time scale long compared with the bounce period of the energetic particles (1-keV protons) but shorter than or comparable to their drift periods, so that their flux invariants are broken while their μ J are still conserved. For μ J to be conserved, three conditions on the perturbation must be satisfied (Northrop, 1963):

(i) The frequency ω and the Doppler-shifted frequency $\omega \pm m\,\omega_d$ (m is the azimuthal mode number) of the perturbation must be much smaller than the bounce frequency,

$$\omega,\ m\,\omega_d \ll \nu_b.$$

Because $\omega_d/\nu_b \approx \epsilon$, the second inequality above implies that $m \ll \epsilon^{-1}$, or the azimuthal wavelength of the perturbation must be much larger than the proton gyroradius.

(ii) The perpendicular electric field associated with the perturbation must be such that the resulting $E \times B$ drift is of the same order as or higher than v_d, the drift due to magnetic field gradient and curvature,

$$c\,\frac{E_\perp}{B} \approx \epsilon^p\, v \quad (p \geq 1),$$

where v is the velocity of the Van Allen particles ($E \gtrsim 1$ keV). Hence in one bounce period, the particles drift to a neighboring position where the magnetic field differs from that of the previous position only in ϵ order, $\nu_b^{-1}\, d\, \ell n\, B/dt \approx O(\epsilon)$, and J is thus conserved. If ExB were of the order of v, the particle could have drifted to a very different region of the magnetic field in a bounce period, and J could not be invariant.

(iii) The parallel electric field associated with the perturbation must be such that

$$c\, E_{\parallel}/B \sim \epsilon^q\, v \quad (q \geq 1) ,$$

for μ conservation.

If there is no other constraint that limits the magnitude of the electric field, then the perturbation electric field will take the lowest allowable order, $p = q = 1$.

Perturbed Number Density

The perturbed number density can be obtained by varying (7),

$$\delta n = 2B \iint d\mu\, dJ\, \frac{\nu_b}{v_{\parallel}} \left[\delta F - \frac{F_0}{M} \frac{\delta K - e\delta\Phi}{v_{\parallel}^{\,2}} + F_0\, \nu_b \oint \frac{dX}{B} \frac{(\delta K - e\delta\Phi)}{M\, v_{\parallel}^{\,3}} \right], \tag{9}$$

where the second term results from the perturbation of

$$v_{\parallel} = [2(K - \mu B - e\Phi)/M]^{\frac{1}{2}} ,$$

and the last term comes from the variation of ν_b. Noting that

$\frac{1}{M v_{\parallel}^3} = - \frac{\partial}{\partial K}\left(\frac{1}{v_{\parallel}}\right)_{\mu\psi}$, we can rewrite the last term in (9) as

$$\frac{1}{M} \oint \frac{d\chi}{B} \frac{(\delta K - e\delta\Phi)}{v_{\parallel}^3} \Big/ \oint \frac{d\chi}{B} \frac{1}{v_{\parallel}}$$

$$= - \frac{\partial}{\partial K} \oint \frac{d\chi}{B} \frac{(\delta K - e\delta\Phi)}{v_{\parallel}} \Big/ \oint \frac{d\chi}{Bv_{\parallel}} + \frac{\partial \delta K}{\partial K} = \frac{\partial \delta K}{\partial K} .$$

The vanishing of the first term in the above equation is due to J conservation:

$$\delta J = 0 = \oint \frac{d\chi}{B} \frac{\delta K - e\delta\Phi}{v_{\parallel}} . \tag{10}$$

Using (5), we can rewrite the last two terms in (9) as

$$\iint d\mu \, dJ \left(\frac{\partial K}{\partial J}\right)_{\mu\psi} \left[F_0 (\delta K - e\delta\Phi) \frac{\partial}{\partial K}\left(\frac{1}{v_{\parallel}}\right)_{\mu\psi} + \frac{F_0}{v_{\parallel}} \frac{\partial \delta K}{\partial K}\right]$$

$$= - \iint d\mu \, dK \left[\frac{1}{v_{\parallel}} \left(\frac{\partial F_0}{\partial K}\right)_{\mu\psi} (\delta K - e\delta\Phi)\right]$$

$$= - \iint d\mu \, dJ \frac{\nu_b}{v_{\parallel}} (\delta K - e\delta\Phi) \left(\frac{\partial F_0}{\partial K}\right)_{\mu\psi} . \tag{11}$$

Substituting (11) into (9), we have an expression for the perturbed density:

$$\delta n_j = 2B \iint d\mu \, dJ \, \frac{\nu_b}{v_{\parallel}} \left[\delta F - (\delta K - e_j \delta\Phi) \left. \frac{\partial F_0}{\partial K} \right|_{\mu\psi} \right] . \tag{12}$$

The perturbed distribution function δF can be solved from the linearized Eq. (6). Setting

$$\Phi(\psi, \varphi, \chi; t) = \Phi_0(\psi, \chi) + \sum_m \Phi_m(\psi, \chi) \, e^{i(m\varphi - \omega t)},$$

$$F(\mu \; J \; \psi \; \varphi; t) = F_0(\mu \; J \; \psi) + \sum_m F_m(\mu \; J \; \psi) \, e^{i(m\varphi - \omega t)} . \tag{13}$$

(Φ_0 is the equilibrium potential, m the azimuthal mode number), we have, from linearized Eqs. (6) and (2),

$$F_m(\mu \; J \; \psi) = \frac{c}{e} \, \frac{K_m (\partial F_0/\partial\psi)_{\mu J}}{\omega_d - \omega/m} , \tag{14}$$

where

$$K_m(\mu \; J \; \psi) = e\langle\Phi_m\rangle = e \, \nu_b \oint \frac{d\chi}{Bv_{\parallel}} \, \Phi_m(\chi) ,$$

by J-conservation as in Eq. (10).

Ordering Scheme

(a) Outside the plasmapause, there is no cold plasma effect, and the ordering is straightforward. With conditions (ii) and (iii), we can now order the Poisson equation,

$$\nabla \cdot \underset{\sim}{E} = 4\pi \sum_j n_j e_j . \tag{15}$$

Multiplying (15) by $c/B\,\Omega_i$ to render it dimensionless, we have the ordering:

$$\frac{c}{B\Omega_i} \nabla_{\parallel} E_{\parallel} \approx \epsilon^{q+1}(k_{\parallel} L_{\parallel}) \approx \epsilon^{q+1},$$

$$\frac{c}{B\Omega_i} \nabla_{\perp} E_{\perp} \approx k_{\perp} \epsilon^{p} v_i/\Omega_i \approx \epsilon^{p+1} (k_{\perp} L_{\perp}) \approx m\, \epsilon^{p+1} ,$$

$$\frac{c}{B\Omega_i} 4\pi n e \frac{n_i - n_e}{n} \approx \frac{4\pi n e^2}{M_i v_i^2} \frac{M_i^2 v_i^2 c^2}{e^2 B^2} \frac{n_i - n_e}{n}$$

$$\approx \left(\frac{r_g}{\lambda_D}\right)^2 \frac{n_i - n_e}{n}$$

where we have taken $k_{\parallel} L_{\parallel} \approx O(\epsilon^0)$ and $\lambda_D = [M_i v_i^2/4\pi n e^2]^{\frac{1}{2}}$.

Since there is no other constraint on the magnitude of the electric field, $p = q = 1$. The Poisson equation is thus ordered as

$$\nabla_{\parallel} E : \nabla_{\perp} E_{\perp} : 4\pi \sum_j n_j e_j ,$$

$$\epsilon^2 : m\, \epsilon^2 : \left(\frac{r_g}{\lambda_0}\right)^2 \frac{n_i - n_e}{n} . \tag{16}$$

Let us consider the following situations:

(i) High density:

$$\left(\frac{r_g}{\lambda_D}\right)^2 \equiv \frac{4\pi\, n\, M_i\, c^2}{B^2} \gg 1.$$

The charge neutrality condition is valid at least through the first order, as $m\,\epsilon \ll 1$:

$$\sum_j e_j\, n_j^{(0)} = 0. \tag{17}$$

(ii) Medium density: $4\pi\, n\, M_i\, c^2/B^2 = (r_g/\lambda_D)^2 \approx 1.$

In this case, the left-hand side (lhs) of (10) $\approx m\,\epsilon^2$ while rhs $\approx O(\epsilon^0)$. Thus the quasi-neutrality condition must again be used in the lowest-order calculation.

(iii) Low density: $4\pi\, n\, M_i\, c^2/B^2 = (r_g/\lambda_D)^2 \ll 1.$

In this case, the lhs of (10) $\approx m\,\epsilon^2$ while the rhs is of the order $m\,\epsilon^2 \dfrac{m}{(k_\perp \lambda_D)^2} \dfrac{\delta n}{n}$. For $(k_\perp \lambda_D)^2 \ll 1$, the quasi-neutrality condition may still be used in the lowest order of calculation. For a short-wavelength case $(k_\perp \lambda_D)^2 \gtrsim 1$, we must use the Poisson equation even in the lowest-order calculation. Thus it is only for the case of low density and small wavelength that the Laplacian of the potential in the Poisson equation should be kept in the lowest-order calculation.

In the region outside the plasmapause, $n \approx 1/\text{cm}^3$, $T \approx 10$ keV (Carpenter, 1965; Frank, 1967), $\lambda_D \approx 1$ km, $r_g \approx 10^2$ km. Thus we are in a high-density region $r_g \gg \lambda_D$, and the quasi-neutrality condition must be used. The characteristic parameters of the plasma in the outer zone are tabulated in Table I.

(b) Inside the plasmapause, the cold plasma with density n_c has important effects. For low-frequency perturbations under study, $\omega \approx \epsilon^2 \Omega_i$, the cold plasma contribution can be represented by a dielectric tensor $\underset{\approx}{\kappa} = \kappa_{\parallel} \hat{e}_{\parallel} \hat{e}_{\parallel} + \kappa_\perp(\underset{\approx}{I} - \hat{e}_{\parallel} \hat{e}_{\parallel})$, where $\hat{e}_{\parallel}$ is the unit vector along the field line,

$$\kappa_\perp = 1 + 4\pi n_c M_i c^2/B^2,$$

$$\kappa_{\parallel} = 1 - 4\pi n_c e^2/M_e \omega(\omega + i \nu_c), \tag{18}$$

where ν_c is the collision frequency of the electrons with protons. The Poisson equation then becomes

$$\nabla \cdot \underset{\approx}{\kappa} \cdot \underset{\sim}{E} = -4\pi \sum_j e_j n_j^h \quad , \tag{19}$$

where n_j^h is the density of the Van Allen belt particles alone. The collision frequency ν_c is given (Alfven and Falthammar, 1963) by

$$\nu_c = 0.3\, n_c\, v_e\, 10^{-5}\, T_e^{-2}\, \ell n\, \Lambda\ \sec^{-1},$$

where Λ is the plasma parameter, v_e the electron thermal velocity in cgs units and T_e the electron temperature in degrees Kelvin. The electron density inside the plasmapause is $n \approx 5 \times 10^2/\mathrm{cm}^3$ (Carpenter, 1966), and the temperature is taken to be $10^4\,{}^\circ K$ (Gringauz, 1967). With these values, the collision frequency is estimated to be $\nu_c \approx 3 \times 10^{-2}\ \sec^{-1}$, which is of the same order of magnitude as the bounce frequency of the energetic protons ν_b^h. Because of the low frequency of the perturbation $\omega \approx \epsilon\nu_b^h \approx \epsilon\nu_c \approx \epsilon^2\, \Omega_i$,

$$\kappa_{\parallel} \approx 1 - \frac{4\pi\, n_c\, e^2}{2\, M_e\, \omega\nu_c} = 1 - \frac{M_i}{M_e}\left(\frac{4\pi\, n_c\, e^2}{M_i\, v_i^2}\right)\frac{v_i^2}{\Omega_i^2}\ \frac{\Omega_i^2}{\omega\, \nu_c}$$

$$\approx 1 - \frac{M_i}{M_e}\ \frac{\Omega_i^2}{\omega\, \nu_c}\left(\frac{r_g}{\lambda'_D}\right)^2 \approx \epsilon^{-4}\, (r_g/\lambda'_D)^2 \quad , \tag{20}$$

where $\lambda'_D = [T_h/4\pi n_c e^2]^{\frac{1}{2}}$ is the Debye length obtained by using the temperature of the hot plasma and the density of the cold plasma, and we have taken $\frac{M_i}{M_e} \approx \epsilon^{-1}$. $\lambda'_D \approx 50$ m and $r_g \approx 10$ km for 100 keV protons at $L = 3$. Thus $r_g \gg \lambda'_D$ inside the plasmapause.

Now we can order Eq. (19) according to conditions (ii) and (iii): To make it dimensionless, we multiply (19) by $c/B\,\Omega_i$,

$$\frac{c}{B\,\Omega_i} \nabla_\parallel \kappa_\parallel E_\parallel \approx \kappa_\parallel k_\parallel \frac{c\,E_\parallel}{B\,\Omega_i} \approx \kappa_\parallel k_\parallel \frac{\epsilon^q v_i}{\Omega_i}$$

$$\approx \epsilon^{-4} (r_g/\lambda'_D)^2 (k_\parallel L)\, \epsilon^{q+1}$$

$$\approx \epsilon^{q-3} (r_g/\lambda'_D)^2,$$

because $\omega \approx \epsilon^2 \Omega_i$, $\nu \approx \nu_b^h \approx \epsilon\,\Omega_i$ and $\frac{M_i}{M_e} \approx \epsilon^{-1}$ and $k_\parallel L \approx O(\epsilon^0)$.

Similarly

$$\frac{c}{B\,\Omega_i} \nabla_\perp \kappa_\perp E_\perp \approx \epsilon^{p+1} (k_\perp L)\, [1 + (r_g/\lambda'_D)^2]$$

$$\approx m\, \epsilon^{p+1} (r_g/\lambda'_D)^2$$

$$\frac{c}{B\,\Omega_i} 4\pi\, \delta n_h\, e = \frac{4\pi n_c e^2}{M_i v_i^2} \frac{M_i v_i^2 c^2}{e^2 B^2} \frac{\delta n_h}{n_c}$$

$$\approx (r_g/\lambda'_D)^2 \frac{\delta n_h}{n_h} \frac{n_h}{n_i} \approx (r_g/\lambda'_D)^2\, n_h/n_c,$$

where we have put $\delta n_h/n_h \approx O(\epsilon^0)$. Thus the terms in (19) stand in ratios:

$$\nabla_{\parallel} \kappa_{\parallel} E_{\parallel} : \nabla_{\perp} \kappa_{\perp} E_{\perp} : 4\pi(n_i^h - n_e^h),$$

$$\epsilon^{q-3} : \epsilon^{p+1} : n_h/n_c . \qquad (21)$$

Since the pitch angle distribution of the energetic belt plasma is peaked around the equatorial plane (Davis and Williamson, 1963), its density variation along a given field line has a sharp maximum at the equatorial plane and falls off rapidly as one moves away from the equatorial plane. Davis and Williamson (1963) found the pitch angle distribution at $L = 3.5$ to be proportional to $\sin^3 \alpha_e$ $\left[\alpha_e = \tan^{-1}\left(\frac{v_{\parallel}}{v_{\perp}}\right)_{eq}\right.$ is the equatorial pitch angle$\left.\right]$. The density $n_h(s)$ then varies like $B^{-3/2}(s)$ (Appendix F).

In the region near the equatorial plane, called Region I, $n_h/n_c \approx 10^{-4} \approx \epsilon$, and the ratio of the terms in (21) is

$$\epsilon^{q-3} : \epsilon^{p+1} : \epsilon. \qquad (22a)$$

In the region far from the equatorial plane, called Region II, where B increases by a factor of 10^2 above its equatorial value, n_h would be a factor $10^{-3} \approx \epsilon$ smaller than the equatorial density, which in turn is ϵ smaller than the cold plasma density, and the ratio of the terms in (21) is

$$\epsilon^{q-3} \ : \ \epsilon^{p+1} \ : \ \epsilon^{2} \ . \tag{22b}$$

Since $p \geq 1$, by comparing the second and the third terms in (22a), we see that the $\nabla_\perp \ \kappa_\perp E_\perp$ is always negligible in Region I. Furthermore, by comparing the first and third terms in (22a), we see that $q = 4$ or $E_\parallel \approx O(\epsilon^4)$ in Region I.

Similarly, from (22b) $q = 5$ or $E_\parallel \approx O(\epsilon^5)$ in Region II, and $\nabla_\perp \kappa_\perp E_\perp$ is no longer negligible in Eq. (21), as there is no other constraint on $E_\perp$, and it will take the lowest-order value allowable, i.e., $p = 1$.

Thus the existence of the cold plasma effectively provides a large conductivity along the field line, which limits the magnitude of the parallel electric field to $O(\epsilon^4)$ in Region I, and to $O(\epsilon^5)$ in Region II.

Let us expand the potential $\Phi_m(X)$ formally in an asymptotic series,

$$\Phi_m(X) = \sum_{\ell} \epsilon^{\ell} \Phi_m^{(\ell)}(X) , \tag{23}$$

where $e\Phi^{(0)} \approx m v^2$, and $\epsilon^{\ell} \frac{c}{v} \frac{\partial \Phi^{(\ell)}}{\partial X} \approx O(\epsilon^{\ell+1})$. From the above discussion we have in Region I

$$\frac{\partial \Phi_m^{(0)}}{\partial X} = \frac{\partial \Phi_m^{(1)}}{\partial X} = \frac{\partial \Phi_m^{(2)}}{\partial X} = 0, \quad \frac{\partial \Phi_m^{(3)}}{\partial X} \neq 0 , \qquad (24)$$

or $\Phi_m^{(\ell)}(X)$ constant along the field line for $\ell = 0, 1, 2$. In Region II,

$$\frac{\partial \Phi_m^{(\ell)}}{\partial X} = 0 \quad \text{for } \ell = 0, 1, 2, 3. \qquad (25)$$

Since we are interested in Region I, where the main body of the Van Allen particles resides, we have, from (22a) and (24), the following ordered Poisson equation:

$$-B^2 \frac{\partial}{\partial X} \left(\kappa_{\parallel} \frac{\partial \Phi_m^{(3)}}{\partial X} \right) = 4\pi \sum_j n_j^h e_j \ , \qquad (26)$$

with boundary condition

$$E_{\parallel}^{(4)} = - \frac{\partial \Phi_m^{(3)}}{\partial X} = 0 , \qquad (27)$$

because of (25).

III. STABILITY OF VAN ALLEN BELT INSIDE PLASMAPAUSE

Let us first consider the region inside the plasmapause. Substituting Eq. (12) and (14) into Eq. (26), and using (24), we get

$$B \frac{\partial}{\partial X}\left(\kappa_{\parallel} \frac{\partial \Phi_m^{(3)}}{\partial X}\right) = 8\pi \sum_j e_j \iint d\mu\, dJ\, \frac{\nu_b}{v_{\parallel}}\left[\frac{c \langle \Phi_m^{(0)} \rangle}{\omega_d^j - \omega/m}\left(\frac{\partial F_0^j}{\partial \psi}\right)_{\mu J}\right.$$

$$\left. - e_j \left(\langle \Phi_m^{(0)} \rangle - \Phi_m^{(0)}\right)\left(\frac{\partial F_0^j}{\partial K}\right)_{\mu \psi}\right]$$

$$= 8\pi\, \Phi^{(0)} \sum_j e_j \iint d\mu\, dJ\, \frac{\nu_b}{v_{\parallel}}\, \frac{c}{\omega_d^j - \omega/m}\left(\partial F_0^j/\partial \psi\right)_{\mu J} . \tag{28}$$

Because we are primarily interested in the stability problem, the location of the eigenvalues in the complex ω plane, we shall construct a variational principle from which one can derive certain stability criteria by employing a suitable trial function, without having to obtain a complete solution to Eq. (28). Multiplying (28) by $\Phi_m^*(X)$ and integrating over the line of force $\int_{X_0}^{X_1} \frac{dX}{B}$ in Region I, and we have a variational expression:

$$\epsilon^3 \int_{X_0}^{X_1} dX\, \kappa_{\parallel} \left| \frac{\partial \Phi_m^{(3)}}{\partial X} \right|^2 = -8\pi \sum_j e_j \iint d\mu\, dJ \frac{c \left| \Phi_m^{(0)} \right|^2 \left. \frac{\partial F_0^{\,j}}{\partial \psi} \right|_{\mu J}}{\omega_d^{\,j} - \omega/m}$$

$$+ O(\epsilon) , \qquad (29)$$

where we have used (23), (24), and (27). Since the lhs of (29) is $O(\epsilon^3)$ smaller than the rhs, we have, in the lowest order,

$$\sum_j e_j \iint d\mu\, dJ \frac{\left(\partial F_0^{\,j}/\partial \psi\right)_{\mu J}}{\omega_d^{\,j} - \omega/m} = 0. \qquad (30)$$

This is the dispersion relation whence we can derive the stability condition. Note that (28) and (30) are valid for each and every ψ. Thus the stability of our system becomes local in the sense that the stability of a given shell at $(\psi, \psi + d\psi)$ depends only upon the local properties of the system at ψ.

Before considering the specific equilibrium distribution function, we shall examine the conditions for the existence of the purely growing mode and of overstability, using the Nyquist method (Appendix E).

In (30), as $\omega \to \infty$, $D\left(\frac{\omega}{m}\right) \sim \omega^{-2}$, and the number of roots in the upper half ω plane is

$$N = -1 + \frac{\Delta\theta}{2\pi} \quad \text{(Appendix E)}, \qquad (31)$$

where

$$\Delta\Theta \equiv \text{Arg } D(\omega = \infty) - \text{Arg } D(\omega = -\infty), \tag{31}$$

is the change in the argument of D as ω goes from $-\infty$ to ∞ along the real axis.

To find $\Delta\Theta$, we express the real and imaginary part of D on the real ω axis explicitly, using the Plemelj formula,

$$\frac{1}{x \pm (y + i\epsilon)} = P\frac{1}{x \pm y} \mp \pi i\, \delta(x \pm y) \,. \tag{32}$$

Inside the plasmapause, there can be no parallel electric field along the field line in equilibrium. We assume that radial electric field is also zero, neglecting the effect of the earth's rotation. Thus we can set the equilibrium potential to be zero inside the plasmapause.

Now we change variables from μJ to kinetic energy $E \equiv K - e\Phi_0$ (in this case, $\Phi_0 = 0$, $E = K$), $\lambda \equiv \mu/E = 1/B_T$, the inverse of the magnetic field strength at the turning point $v_{\parallel} = 0$. In terms of the new variables, the bounce and drift frequency can be written (Appendix F) as $\nu = \nu_0(\lambda, \psi)E^{\frac{1}{2}}$, $\omega_d = a(\lambda, \psi)E$. Then (31) becomes

$$D\left(\frac{\omega}{m},\psi\right) = \sum_j e_j^{\,2} \int_{1/B_I}^{1/B_0} d\lambda \int_{E_0}^{\infty} dE \; \frac{E^{3/2}\, \nu_0^{\,-1}\, (\partial F_0^{\,j}/\partial K)_{\mu J}}{E - \omega/m\, a_j}\,, \tag{33}$$

where E_0 is the low-energy cutoff[5] of the Van Allen particles ($E_0 \approx 1$ keV), and B_0, B_I are the magnetic field strength at ψ in the equatorial plane and the boundary of Region I respectively; we have used Eq. (8) and (2), with $\omega_d = aE$. For a dipole type field, $e_j\, a_j < 0$ $(a_e = -a_i > 0)$. The singularity in the integrand is to be handled by considering ω to have a positive imaginary part. Using (33) for ω on the real axis, we have, for $m > 0$,

$$\mathrm{Im}\, D\left(\frac{\omega}{m},\psi\right) = \begin{cases} \pi\, e^2 \displaystyle\int d\lambda \left(\frac{\omega}{m\, a_e}\right)^{3/2} \nu_0^{\,-1} \left(\frac{\partial F_0^{\,-}}{\partial K}\right)_{\mu J} \left(\lambda,\, E = \frac{\omega}{m\, a_e}\,,\, \psi\right) & \text{for } \omega > m\, E_0\, a_0, \\[2ex] -\pi\, e^2 \displaystyle\int d\lambda \left(\frac{\omega}{m\, a_i}\right)^{3/2} \nu_0^{\,-1} \left(\frac{\partial F_0^{\,+}}{\partial K}\right)_{\mu J} \left(\lambda,\, E = \frac{\omega}{m\, a_i}\,,\, \psi\right) & \text{for } \omega < -m\, E_0\, a_0, \\[2ex] 0 & \text{for } -m\, E_0\, a_0 < \omega < m\, E_0\, a_0, \end{cases} \tag{34}$$

where a_0 is the minimum value of $a_\ell(\lambda)$, and

$$\mathrm{Re}\ D\left(\frac{\omega}{m_1}\ \psi\right) = e^2 \sum_{\pm} P \int_{1/B_I}^{1/B_0} d\lambda \int_{E_0}^{\infty} dE\ \frac{v_0^{-1}\ E^{3/2}\ (\partial F^{\pm}/\partial K)_{\mu J}}{E \pm \omega/m\ a_e} . \tag{35}$$

Note that $\mathrm{Re}\ D < 0$ for $|\omega| < m\ a_0\ E_0$, and $\mathrm{Re}\ D \rightarrow -\infty$ as $\omega \rightarrow \pm\ m\ E_0\ a_0$ if $(\partial F/\partial K)_{\mu J} < 0$.

Clearly, $\mathrm{Im}\ D$ goes to zero also at $\omega = \pm\infty$. If $F_0{}^j(\mu\ J\ K)$ is a monotonic function of K for all μJ at ψ, i.e., $(\partial F_0/\partial K)_{\mu J} < 0$ for all μJ at ψ, then $\mathrm{Im}\ D$ will not go through any additional zero besides those at $\omega = \pm\infty$ and between $-m\ a_0\ E_0$ and $m\ a_0\ E_0$. The change of the argument of D as ω moves from $-\infty$ to ∞ along the real axis is 2π (Fig. 5). From (32), $N = 0$, i.e., there is no unstable mode. Therefore $(\partial F_0/\partial K)_{\mu J} < 0$ is a sufficient condition for stability against electrostatic μJ-conserving modes of an energetic plasma in an axisymmetric field in the presence of a dense, cold plasma background.

For the outer belt inside the plasmapause $(2 < L < 5)$, the distribution function $f_1[E(\mu\ J\ \psi),\ \alpha(\mu\ J\ \psi),\ \psi]$ of the energetic protons $(0.1\ \mathrm{MeV} < E < 5\ \mathrm{MeV})$ found by Davis and Williamson (1963) has the property that $(\partial f_1/\partial\psi)_{\mu J} > 0$ (Nakada et al., 1965). It can be shown (Appendix B-2) that f_1 is simply related to $F_0(\mu\ J\ \psi)$ by $F_0 = f_1/[2\ \mu\ B/M]^{\frac{1}{2}}$,

$$\left.\frac{\partial F_0}{\partial \psi}\right|_{\mu J} = \left.\frac{\partial f_1}{\partial \psi}\right|_{\mu J} \frac{1}{(2\ \mu\ B/M)^{\frac{1}{2}}} - \frac{f_1}{2[2\ \mu\ B/M]^{\frac{1}{2}}} \frac{1}{B} \frac{dB}{d\psi} . \tag{36}$$

As $dB/d\psi < 0$ for a dipole-type field, $(\partial f_1/\partial\psi)_{\mu J} > 0$ implies that $(\partial F_0/\partial\psi)_{\mu J} > 0$, i.e., $(\partial F_0/\partial K)_{\mu J} < 0$. Thus the outer energetic belt inside the plasmapause is stable against electrostatic μJ-conserving perturbation.

In order to have instability, Im D must go through zero at least twice at some ω, say ω_1, ω_2, besides $\omega = \pm\infty$ and $|\omega| < m\ a_0\ E_0$. In this case, it is not possible to have Penrose-type criteria (Penrose, 1961), i.e., necessary and sufficient conditions for stability in terms of the general properties of the distribution function. Only in a special case can simple conditions for the existence of the purely growing mode be found (Appendix G). In the following we consider a model distribution function and derive sufficient conditions for instability in terms of macroscopic parameters such as density and temperature.

As we have noted previously, the energy content of the outer zone inside the plasmapause is mainly associated with the energetic belt (40 keV < E < 5 MeV). The energy spectra of its flux are well represented by $\exp(-E/T)$, and $T \approx L^{-3} \approx B$ for the equatorial particles. For simplicity, we assume the ptich-angle distribution as a power law in the sine of the equatorial pitch angle

(EPA) α_e : $\sin^{\ell} \alpha_e = \lambda^{\ell/2}$. In general, ℓ is a function of both energy E and the magnetic shell parameter L. As computed by Hoffman and Bracken (1965), the dependence of ℓ on E is rather weak, and the dependence on L for particles with average energy is found to be $\ell = 2.84 - 0.12\,L$ for L between 3.7 and 10. For the region inside the plasmapause $(L < 5)$, we put $\ell = 2$ as a crude approximation. Thus we have a model distribution function whose dependence on energy and pitch angle is factorizable:

$$F[\mu\ J\ (\lambda\ E\ \psi)\ \psi] = \frac{3\ n(\psi)\ B_0(\psi)\ e^{-E/T(\psi)}}{4(2M)^{\frac{1}{2}}\ T(\psi)\ E^{\frac{1}{2}}} \cdot \frac{\mu}{E}\ , \qquad (37)$$

where $n(\psi)$ is the particle density in the equatorial plane at ψ as given by (7), $B_0(\psi)$ is the equatorial magnetic field strength at ψ, and $T(\psi)$ the temperature of the plasma at ψ. In view of the similarities between the characteristics of the electron and proton fluxes in the outer belt (Hess et al., 1965), we assume the same distribution function for electrons and protons for simplicity. Inside the plasmasphere, the distortion of the magnetic field by solar wind is not important, and we assume the magnetic field is a dipole field. For axisymmetric systems with electrons and protons having the same distribution, the azimuthally propagating overstability for a given mode number m must exist in pairs. For each unstable mode propagating eastward, there must also be one propagating westward. Thus if there is only one unstable mode for a given m, it must be a purely growing mode.

In a dipole magnetic field, the bounce and drift frequencies of the charged particles with energy E and $\lambda = \sin^2 \alpha_e / B_0$ are given approximately (Hamlin et al., 1961) by

$$\nu_b \equiv \nu_0(\lambda)\, E^{\frac{1}{2}} = \frac{1}{4r}\left(\frac{2E}{M}\right)^{\frac{1}{2}} \left[1.30 - 0.50\, (\lambda\, B_0)^{\frac{1}{2}}\right]^{-1}, \tag{38}$$

$$\omega_d \equiv a(\lambda)\, E = \frac{6\, c\, E}{e\, B_0\, r} \left[0.35 + 0.15\, (\lambda\, B_0)^{\frac{1}{2}}\right], \tag{39}$$

where r is the equatorial distance from the dipole axis. Because the energetic particles have their pitch angle distribution peaked about the equatorial plane, they are confined mainly in the region near the equatorial plane, $\lambda \approx B_0^{-1}$. Therefore (39) can be approximated by

$$\omega_d \approx \frac{3\, c\, E}{e\, B_0\, r} . \tag{40}$$

Using (37), (40), and (2), we have

$$\left.\frac{\partial F_0}{\partial \psi}\right|_{\mu J} = \left.\frac{\partial F_0}{\partial E}\right|_{\mu\psi} \left.\frac{\partial E}{\partial \psi}\right|_{\mu J} + \left.\frac{\partial F_0}{\partial \psi}\right|_{\mu E}$$

$$= \frac{3n\, B_0}{4(2M\, E)^{\frac{1}{2}}\, T} (G + HE)\, e^{-E/T}\, \lambda, \tag{41}$$

where

$$G(\psi) \equiv \frac{1}{B_0 r^2} \left(\frac{d \, \ell n \, n}{d \, \ell n \, r} - \frac{d \, \ell n \, T}{d \, \ell n \, r} + \frac{3}{2} \right) , \tag{42}$$

$$H(\ell) \equiv \frac{1}{B_0 r^2 T} \left(\frac{d \, \ell n \, T}{d \, \ell n \, r} + 3 \right) . \tag{43}$$

Substituting (41) into (30) and noting that for a dipole type field, $a_e(\lambda) = -a_i(\lambda) > 0$, we obtain

$$D \equiv \sum_{\pm} \int_{E_0}^{\infty} dE \, \frac{(G + HE) \, e^{-E/T}}{E \pm \omega/m \, a_e} = 0. \tag{44}$$

If $G/H < 0$, then Im D vanishes at $\omega = \pm m \, a_e \, G/H$, from (35) and (41), and there is a possible instability. Since these are the only possible zeros of Im D in addition to those at $\omega = \pm\infty$ and $|\omega| < m \, a_0 \, E_0$, the change in the argument of D from $\omega = -\infty$ to $+\infty$ can at most be 4π and there is at most one unstable mode--a purely gowing one. Setting $\omega = i\gamma$ in (41), where γ is real, gives

$$D(\gamma, \psi) = \int_{E_0}^{\infty} dE \, \frac{E(G + HE) \, e^{-E/T}}{E^2 + \gamma^2/(m \, a_e)^2} = 0, \tag{45}$$

which is always real. Its asymptotic forms are

$$\gamma \to 0 \; : \; D(\gamma \to 0) \sim GE_1(E_0/T) + HT\,\exp(-E_0/T),$$

$$\gamma \to \infty \; : \; D(\gamma \to \infty) \sim (m\,a_e)^2\,(G + 2HT)\,\exp(-E_0/T)/\gamma^2,$$

where $E_1(X)$ is the exponential integral (Abramowitz and Stegum, 1964). If these two limiting values of D are of opposite signs, then the expression $D(\gamma)$ must go through zero at some value of γ between 0 and ∞, i.e., there exists an unstable mode. Thus a sufficient condition for instability is

$$G\,E_1(x) + HT\,e^{-x} < 0,$$

$$G + 2HT > 0, \tag{46}$$

where $x \equiv E_0/T$. Or alternatively,

$$G\,E_1(x) + HT\,e^{-x} > 0,$$

$$G + 2HT < 0. \tag{47}$$

We note that

$$G + 2HT = -\iint d\mu\, dJ \left(\frac{\partial F_0}{\partial K}\right)\left(\frac{\partial K}{\partial \psi}\right)^2 = \delta^2 W > 0, \tag{48}$$

is just the necessary and sufficient condition for the interchange stability derived on the energetic grounds (Taylor, 1963) (Appendix H). Condition (47) thus corresponds to the occurrence of the interchange instability. The additional condition, the second inequality of (47), arises from the use of the kinetic equation, representing the additional constraint of the motion. Even when $\delta^2 W > 0$ is satisfied, there is another instability which will be called drift mode [as in (46)]. Using (42) and (43) to express (46) and (47) in terms of macroscopic parameters for two ratios of cutoff energy to temperature, we have, for instability in a dipole field,

Ratio (i) for:

$$x \equiv E_0/T = 0.01, \text{ i.e., } E_1(x) = 4.0, \quad e^{-x} = 0.99,$$

The condition

$$\frac{1}{4}\left(7\frac{d\,\ell n\,T}{d\,\ell n\,r} - 9\right) > \frac{d\,\ell n\,n\,T}{d\,\ell n\,r} > -7.5 \tag{49}$$

from (46) for the dirft mode, and

$$-7.5 > \frac{d\,\ell n\,n\,T}{d\,\ell n\,r} > \frac{1}{4}\left(7\frac{d\,\ell n\,T}{d\,\ln\,r} - 9\right), \tag{50}$$

from (47) for the interchange mode;

Ratio (ii) for:

$$x = 0.1, \text{ i.e., } E_1(x) = 1.7, \quad e^{-x} = 0.9,$$

The condition

$$-3 + 1.5 \frac{d \ell n T}{d \ln r} > \frac{d \ell n n T}{d \ell n r} > -7.5 , \tag{51}$$

from (46) for the drift mode, and

$$-7.5 > \frac{d \ell n n T}{d \ell n r} > -3 + 1.5 \frac{d \ell n T}{d \ell n r} , \tag{52}$$

from (47) for the interchange mode. For the energetic particles inside the plasmapause, $T \propto r^{-3}$ and $\frac{d \ell n n}{d \ell n r} > 0$. From (42) and (43) it follows that $G > 0$, $H = 0$. Hence (48) is always satisfied and the plasma is stable against the interchange. Furthermore, conditions (46) or (47) cannot be fulfilled, and the long-term equilibrium of the system is stable against the μJ-conserving perturbations. This is in fact already obvious from the analysis by Dungey et al., that $(\partial f_1/\partial L)_{\mu J} > 0$, which implies $\left.\frac{\partial F}{\partial \psi}\right|_{\mu J} > 0$ for a dipole field, i.e., $\left.\frac{\partial F}{\partial K}\right|_{\mu J} < 0$ for all μJ inside the plasmasphere, the sufficient condition for the long-term stability, as previously discussed.

IV. THE STABILITY OF THE RING CURRENT BELT

By comparing the simultaneous measurements of the location of the plasmapause (Taylor et al., 1968) and the ring current belt (Frank, 1967) with OGO satellite, we see that the peak of the ring current belt is just outside of the plasmapause (Figs. 3 and 4). This close relation suggests that the formation of the plasmapause is directly related to the ring current belt. Furthermore, the outer edge of the ring current belt is outside the plasmapause. Because the plasma outside the plasmapause is collisionless, in general there is an electric field along the magnetic field line (Alfven and Falthammar, 1963), and stability analysis must include the effect of such a parallel electric field.

The stability of a low-β plasma against μJ-conserving electrostatic perturbations with a finite parallel electric field has been examined by Rosenbluth (1967) for a plasma in a multipole field. Frieman and Rutherford (1968) derived sufficient conditions for stability from an energy principle for general geometry. In the following, we first give an alternative derivation of the Rutherford-Frieman criteria, then derive a necessary and sufficient condition for stability in a special case. The result is applied to the ring current belt with ionosphere as boundary condition.

From the ordering scheme (16), the charge neutrality condition is to be used for analyzing the low-frequency stability of the ring current belt that lies outside the plasmapause. Substituting (12) and (14) into the linearized quasi-neutrality condition, (17), leads to

$$\sum_j e_j \, \delta n_j \equiv \sum_j e_j \iint d\mu \, dJ \, \frac{v_b}{v_\parallel} \left[\frac{c \langle \Phi_m \rangle \left.\frac{\partial F_0{}^j}{\partial \psi}\right|_{\mu J}}{\omega_d{}^j - \frac{\omega}{m}} - e_j \left(\langle \Phi_m \rangle - \Phi_m \right) \left.\frac{\partial F_0{}^j}{\partial K}\right|_{\mu\psi} \right] = 0, \tag{53}$$

where $F_0\Big(\mu, J(\mu, K, \psi), \psi\Big) = F_0(\mu, K, \psi)$. Equation (53) is an eigenvalue equation for $\Phi_m(X)$, localized in ψ.

Multiplying by $\Phi_m{}^*(X)$ and integrating over the field line, $\int_{X_0}^{X_1} \frac{dX}{B}$, with the limits of integration at the ionosphere, we have a variational expression,

$$D\left(\frac{\omega}{m}, \psi\right) \equiv \sum_j e_j \iint d\mu \, dJ \left[\frac{c \, |\langle \Phi_m \rangle|^2 \left.\frac{\partial F_0{}^j}{\partial \psi}\right|_{\mu J}}{\omega_d{}^j - \frac{\omega}{m}} - e_j \left.\frac{\partial F_0{}^j}{\partial K}\right|_{\mu J} \Big(|\langle \Phi_m \rangle|^2 - \langle |\Phi_m|^2 \rangle \Big) \right] = 0. \tag{54}$$

Transforming to new variables $E = K - e\Phi^{(0)}(s)$ (the kinetic energy) and $\lambda \equiv \frac{\mu}{E}$, and setting $\nu_b = \nu_0(\lambda\psi)E^{\frac{1}{2}}$ and $\omega_d{}^j = a_j(\lambda, \psi)E$ (Φ_0 is a function of s alone assumed,[6] Appendix F), we have

$$D\left(\frac{\omega}{m}, \psi\right) = \sum_j e_j{}^2 \iint d\lambda\, dE\, E^{\frac{1}{2}}\, \nu_0{}^{-1}(\lambda) \left[\frac{|\langle\Phi_m\rangle|^2 (\partial F_0{}^j/\partial K)_{\mu J}\, E}{E - \omega/m\, a_j} - \left.\frac{\partial F_0{}^j}{\partial K}\right|_{\mu\psi} \left(|\langle\Phi_m\rangle|^2 - \langle|\Phi_m|^2\rangle\right)\right]$$

$$= 0, \qquad (55)$$

where we have used the relation

$$\left.\frac{\partial F}{\partial \psi}\right|_{\mu J} = \left.\frac{\partial F}{\partial K}\right|_{\mu J} \left.\frac{\partial K}{\partial \psi}\right|_{\mu J} = \frac{e_j}{c}\, \omega_d^j \left.\frac{\partial F}{\partial K}\right|_{\mu J} = \frac{e_j\, a_j}{c}\, E \left.\frac{\partial F}{\partial K}\right|_{\mu J} . \qquad (56)$$

At the ionosphere χ_1, χ_0, the variation of the potential $\delta\Phi_m(\psi, \chi)$, is taken to be zero due to the high conductivity there. With this boundary condition $\delta\Phi = 0$ at χ_0, χ_1, the lhs of Eq. (55) becomes a variational expression, i.e., $\delta D/\delta\Phi_m{}^* = 0$ yields (53). The eigenfunctions of (53) form a subset of the set of all trial functions of (55). If Eq. (55) has no root in the upper half ω plane, for any trial function, then Eq. (53) has no unstable solution. The sufficient condition of stability derived from (55) implies a sufficient condition of stability for (53). Such conditions can be obtained by means of Nyquist analysis.

In (56), for large ω, $D(\omega) \to \text{Const.}$ and the number of roots in the upper half ω plane is $N = \frac{\Delta\theta}{2\pi}$ (Appendix E), where

$\Delta\theta \equiv \arg D(\omega = +\infty) - \arg D(\omega = -\infty)$ is the change in the argument of D as ω goes from $-\infty$ to $+\infty$ along the real axis.

To find $\Delta\theta$, we express the real and imaginary part of D on the real ω axis explicitly, using the Plemelj formula (33),

$$\operatorname{Im} D\left(\frac{\omega}{m},\psi\right) = \pi \sum_j e_j^2 \int d\lambda \left(\frac{\omega}{m\, a_j}\right)^{3/2} v_0^{-1} \, |\langle\Phi_m\rangle|^2 \left.\frac{\partial F_0^j}{\partial K}\right|_{\mu J} \left(E = \frac{\omega}{ma_j}\right) \qquad \text{for } \omega > 0$$

$$= -\pi \sum_j e_j^2 \int d\lambda \left(\frac{\omega}{m\, a_j}\right)^{3/2} v_0^{-1} \, |\langle\Phi_m\rangle|^2 \left.\frac{\partial F_0^j}{\partial K}\right|_{\mu J} \left(E = \frac{\omega}{ma_j}\right) \qquad \text{for } \omega < 0. \quad (57)$$

Clearly $\operatorname{Im} D$ vanishes at $\omega = 0$ and $\omega = \pm\infty$. If $(\partial F_0^j/\partial K)_{\mu J} < 0$ for all μJ at ψ, then these are the only zeros of $\operatorname{Im} D$. When this is the case, the change in the argument of D as ω goes from $-\infty$ to ∞ along the real axis is at most 2π.

Now we look at the real part of D at $\omega = 0, \pm\infty$:

$$\mathrm{Re}\, D(\omega = 0, \psi) = \sum_j e_j^{\,2} \iint d\lambda\, dE\, E^{\frac{1}{2}}\, \nu_0^{-1} \left[|\langle \Phi_m \rangle|^2 \left.\frac{\partial F_0^{\,j}}{\partial K}\right|_{\mu J} + \left(\langle |\Phi_m|^2 \rangle - |\langle \Phi_m \rangle|^2 \right) \left.\frac{\partial F_0^{\,j}}{\partial K}\right|_{\mu J} \right],$$

$$\mathrm{Re}\, D(\omega \to \infty, \psi) = \sum_j e_j^{\,2} \iint d\lambda\, dE\, E^{\frac{1}{2}}\, \nu_0^{-1} \left[\langle |\Phi_m|^2 \rangle - |\langle \Phi_m \rangle|^2 \right] \times \left.\frac{\partial F_0^{\,j}}{\partial K}\right|_{\mu\psi}. \qquad (58)$$

Note that $[\langle |\Phi_m|^2 \rangle - |\langle \Phi_m \rangle|^2] \geq 0$ by Schwartz's inequality. If

$$\left.\frac{\partial F_0^{\,j}}{\partial K}\right|_{\mu J} < 0 \quad \text{and} \quad \left.\frac{\partial F_0^{\,j}}{\partial K}\right|_{\mu\psi} < 0 \text{ , then } \mathrm{Re}\, D(\omega = 0) < 0, \text{ and}$$

$\mathrm{Re}\, D(\omega \to \pm\infty) < 0$. In this case, the mapping of the real ω axis onto the D plane does not enclose the origin, and $N = \frac{\Delta \varphi}{2\pi} = 0$. There can be no unstable mode, and the plasma is thus stable. These sufficient conditions for stability were first derived by Rutherford and Frieman (1968) from an energy principle.

To derive a necessary condition for stability, we consider a special case in which the eigenfrequency as a function of temperature and density gradient is much greater than the mean drift frequency, $\omega \gg m\, \omega_d$; we follow the approach by Rosenbluth (1967). Neglecting the resonance effect, we expand the denominator in (55),

$$\left(\omega_d{}^j - \frac{\omega}{m}\right)^{-1} = -\frac{m}{\omega}\left(1 + \frac{m\, \omega_d{}^j}{\omega}\right) . \tag{59}$$

Using (59) and (8), we have the expanded variational expression of (55),

$$\xi \frac{\omega^2}{m^2} + \eta \frac{\omega}{m} + \zeta = 0, \tag{60}$$

where

$$\xi \equiv \sum_j e_j{}^2 \iint d\mu\, dJ \left(|\langle \Phi_m \rangle|^2 - \langle |\Phi_m|^2 \rangle \right) \left.\frac{\partial F_0{}^j}{\partial K}\right|_{\mu\psi} ,$$

$$\eta \equiv c \sum_j e_j \iint d\mu\, dJ \left.\frac{\partial F_0{}^j}{\partial K}\right|_{\mu J} \left.\frac{\partial K}{\partial \psi}\right|_{\mu J} |\langle \Phi_m \rangle|^2 ,$$

$$\zeta = c^2 \sum_j \iint d\mu\, dJ \left.\frac{\partial F_0{}^j}{\partial K}\right|_{\mu J} \left(\frac{\partial K}{\partial \psi}\right)^2_{\mu J} |\langle \Phi_m \rangle|^2 . \tag{61}$$

Equation (61) is not a very useful variational expression, because ω is in general complex and no minimization principle is available. (But we can obtain sufficient conditions for stability by requiring $\eta^2 - 4\xi\zeta > 0$. It is sufficient to have $\xi\zeta < 0$. This is the case if

$$\left.\frac{\partial F_0{}^j}{\partial K}\right|_{\mu\psi} < 0 \quad \text{and} \quad \left.\frac{\partial F_0{}^j}{\partial K}\right|_{\mu J} < 0,$$

which are again the Frieman-Rutherford conditions.) The variational expression (60) becomes a minimal principle when $\eta = 0$. Such is the case when the electrons and protons have the same pitch angle distribution for distribution functions factorizable in their energy and pitch angle dependence (Appendix I) (or trivially so when they have the same distribution functions). Then the minimization expression is (Appendix J)

$$\frac{\omega^2}{m^2} = \left(\frac{c}{e}\right)^2 \frac{\sum_j \iint d\mu\; dJ (\partial F_0{}^j/\partial K)_{\mu J}\, (\partial K/\partial\psi)^2_{\mu J}\; |\langle\Phi_m\rangle|^2}{\sum_j \iint d\mu\; dJ (\partial F_0{}^j/\partial K)_{\mu\psi} [\langle|\Phi_m|^2\rangle - |\langle\Phi_m\rangle|^2]} . \qquad (62)$$

The minimum value of ω^2 is just the square of the eigenfrequency of the fundamental mode, the minimum eigenfrequency. As the eigenfunction of the fundamental mode yields the minimum value of ω^2 in

(62), the necessary and sufficient condition for stability is the non-negativeness of the minimum value of ω^2. Therefore, the system is unstable if $\omega^2 < 0$ for any suitable trial function (by "suitable", we mean that it satisfies proper boundary conditions). For the system to be stable, ω^2 must be positive for all trial functions. [Note that this is a much stronger condition than the stability condition for interchange given by (48).] Even when (48) is satisfied, the system may still become unstable with respect to the low frequency modes according to (62). It has been shown (Rosenbluth, 1967) that, for $(\partial F/\partial K)_{\mu\psi} < 0$, a necessary and sufficient condition for stability is $(\partial F \, \partial K)_{\mu J} < 0$.

We use (62) to derive the stability criteria for the ring current belt.

Ring Current Belt Stability

As observed by Frank (1967), the ring current belt energy density is predominantly shared by low-energy protons (30 keV < E < 50 keV) (75%) and electrons (0.2 keV < E < 50 keV) (25%). Its flux is peaked in L space. In the region around the peak, the ring current belt dominates the energetic belt of Davis and Williamson (protons 100 keV < E < 5 MeV, electrons 50 keV < E < 5 MeV) in both the energy density and the particle density. The pitch angle distribution of the ring current belt is almost isotropic. From above information, we construct the following model distribution for the ring current belt:

$$F^{\pm}[\mu J(\mu\ E\ \psi)\ \psi] = f^{\pm}(\mu_1\ E_1\ \psi) = \frac{n_R(\psi)\ \delta(E - E_{\pm})}{2[2M_{\pm}\ E_{\pm}]^{\frac{1}{2}}}, \qquad (63)$$

where $n_R(\psi)$ is the ring current belt density at ψ in the equatorial plane. The narrowness in its energy spectrum is approximated by a δ function, with electrons and protons having different energies, and the near-isotropy is approximated by the independence of f on μ. Therefore, the region outside the plasmapause is populated with the predominating ring current belt particles as well as the energetic belt particles. Their contributions to D in Eq. (55) are separable:

$$D\left(\frac{\omega}{m_1}, \psi\right) = D_R + D_E, \qquad (64)$$

where D_R is the part due to the ring current belt and D_E is that due to the energetic belt. Because the ring current belt dominates in both energy density and particle density, we first study the stability of the ring current belt by itself, neglecting the contribution of the energetic particles in the first approximation.

When the scale length of the density gradient of the ring current belt becomes much smaller than that of the average magnetic field gradient experienced by the particles over a bounce, $\frac{1}{n}\frac{dn}{d\psi} \gg \frac{1}{B}\left\langle\frac{\partial B}{\partial \psi}\right\rangle$, the diamagnetic drift is much faster than the particle drift because of (2):

$$\omega_c \equiv c \frac{E}{e} \frac{1}{n} \frac{dn_R}{d\psi} \gg \omega_d . \tag{65}$$

In this case, $\omega \approx m\sqrt{\omega_c \, \omega_d} \gg m\,\omega_d$, as can be seen in (67). We substitute (63) and (55) and expand the denominator, using (59):

$$D_R \approx \frac{-n_R(\psi)}{2} \sum_{\pm} E_{\pm}^{-\frac{1}{2}} \int d\lambda \; \nu_0^{-1}(\lambda) \; [\langle|\Phi_m|^2\rangle - |\langle\Phi_m\rangle|^2]$$

$$- \; n_R'(\psi) \; \frac{m^2}{\omega^2} c \sum_{\pm} E_{\pm}^{3/2} \int d\lambda \; \nu_0^{-1} \; a_{\pm} \; e_{\pm} \; |\langle\Phi_m\rangle|^2$$

$$= 0, \tag{66}$$

or

$$\frac{\omega^2}{m^2} = - \frac{c^2 \frac{1}{n} \frac{dn_R}{d\psi} \sum_{\pm} E_{\pm}^{3/2} \int d\lambda \; \nu_0^{-1}(\lambda) \;\; a_{\pm} \; e_{\pm} \; |\langle\Phi_m\rangle|^2}{\sum_{\pm} E_{\pm}^{-\frac{1}{2}} \int d\lambda \; \nu_0^{-1} \; [\langle|\Phi_m|\rangle^2 - |\langle\Phi_m\rangle|^2]} . \tag{67}$$

Note that $\frac{\omega^2}{m^2} \approx \frac{c}{e} \frac{E}{n_R} \frac{dn_R}{d\psi} \cdot a\,E \approx \omega_c\,\omega_d$. For a dipole-type field, the average drift for electrons is in the positive sense (eastward), i.e.,

$$e_+ \; a_+/c = \frac{1}{E} \left.\frac{\partial E}{\partial \psi}\right|_{\mu J} \approx \left\langle \frac{\partial B}{\partial \psi} \right\rangle_{\mu J} < 0. \tag{68}$$

Thus the plasma will become locally unstable: $\omega^2 < 0$ at ψ for inward density gradient $\frac{dn}{d\psi} < 0$ at ψ. Therefore, the outer edge of the ring current belt, where $\frac{dn}{d\psi} < 0$, is likely to become unstable during the storm time when its density gradient becomes sharpened. On the other hand, the inner edge of the ring current belt, where $\frac{dn}{d\psi} > 0$, is always stable and supports a wave with angular velocity $\frac{\omega}{m} \approx \sqrt{\omega_d \omega_c}$.

The potential variation along the field line--the eigenfunction of the fundamental unstable mode--is such that is minimizes (67) or maximizes the growth rate, subject to the boundary conditions. If there were no boundary conditions, such as the case of multipole geomtry with closed field lines, then one could argue (Rosenbluth, 1967) that the fastest growing mode is $\Phi_m(s) = \text{constant}$, for which the denominator of (67) vanishes. One then recovers the hydromagnetic interchange instability which occurs in a bounce time scale, thus appearing with infinite growth rate in the present scheme. However, in the present situation, the ionosphere E layer as a conducting end imposes a boundary condition that Φ_m be zero at the ionosphere. Thus the unstable mode must have a potential variation along the field line, i.e., a finite parallel electric field. Furthermore, this instability is not the interchange in the sense of interchange two flux tubes with plasma "frozen in." The perfect conductivity of the ionosphere as we assumed here would have prevented such instability from occurring for a low-β plasma.

Marginal Stability Criterion

As the parameters of the system are continually varying, the system goes through a series of equilibrium configurations. If the system is originally in a stable configuration but its parameters are varying in such a way as to make it approach an unstable configuration, then the transition from stable to unstable configuration is characterized by a set of critical values of the parameters--the marginal stability condition. To obtain such a marginal stability condition for our present case, in which only purely growing modes or pure oscillations are considered, we let $\omega = i\gamma$ and sum over species in (55),

$$D(\gamma/m, \psi) = c^2 \sum_j \iint d\mu \, dJ \, \frac{|\langle\Phi_m\rangle|^2 \, (\partial F_0{}^j/\partial K)_{\mu J} \, (\partial K/\partial\psi)^2_{\mu J}}{\omega_d{}^2 + (\gamma/m)^2}$$

$$+ \; e^2 \sum_j \iint d\mu \, dJ \left(\frac{\partial F_0}{\partial K}\right)_{\mu\psi} [\langle|\Phi_m|^2\rangle - |\langle\Phi_m\rangle|^2] = 0. \tag{69}$$

where we have used (2), and the assumption that electrons and protons have the same pitch angle distribution. At the onset of the instability,

$\gamma = 0$. Upon extremizing $D(\gamma = 0, \psi) = 0$ with respect to Φ_m, we then obtain the critical condition at the onset of the purely growing instability.

As the density gradient of the ring current belt is being built up, it approaches an unstable configuration. To find the critical density gradient at the onset of the instability, we substitute (63) into (69) and put $\gamma = 0$:

$$D(\psi, \gamma = 0) \equiv e^2 \sum_{\pm} \frac{n_R}{(2M_\pm)^{\frac{1}{2}} E_\pm} \left[-\int d\lambda\ \nu_0^{-1} \langle |\Phi_m|^2 \rangle + \frac{1}{n_R} \frac{dn_R}{d\psi} \int d\lambda\ \nu_0^{-1} \frac{c}{ea} \ |\langle \Phi_m \rangle|^2 \right], \tag{70}$$

where $ea/c = \frac{1}{E} \left.\frac{\partial E}{\partial \psi}\right|_{\mu J} \sim \frac{1}{B} \langle \frac{\partial B}{\partial \psi} \rangle < 0$ for a dipole-type field.

We can rewrite (70) as

$$\frac{1}{n_R} \frac{dn_R}{d\psi} = \frac{1}{2} \frac{\int d\lambda\ \nu_0^{-1} \langle |\Phi_m|^2 \rangle}{\int d\lambda\ \nu_0^{-1}\ c(ea)^{-1}\ |\langle \Phi_m \rangle|^2} . \tag{71}$$

The trial function $\Phi_m(X)$ is chosen to maximize D. Substituting this extremizing trial function (eigenfunction) into (70), we then obtain the critical density gradient. This is equivalent to extremizing the density gradient in (71) (Appendix K), i.e., Φ_m is so chosen as to minimize $\left| \frac{1}{n_R} \frac{dn_R}{d\psi} \right|$. This minimum value is then the critical density gradient, and in general it depends upon the field geometry and the boundary conditions. It is important to note that the minimizing function $\Phi_m(X)$ in general depends upon X, i.e., the parallel electric field is nonvanishing at the onset of the instability. Furthermore, this marginal stability condition is different from and usually weaker than that of interchange stability in that it requires a less steep density gradient. Substituting (63) into (48), and setting $\delta^2 W = 0$, we have the marginal condition for interchange:

$$\frac{1}{n_R} \frac{dn_R}{d\psi} = \frac{5}{2} \frac{\int d\lambda \, \nu_0^{-1} (ea/c)^2}{\int d\lambda \, \nu_0^{-1} (ea/c)} .$$

For a dipole field, $ea/c \simeq \partial \ \ell n \ B/\partial\psi \approx -3$. The critical density gradient for the onset of the low-frequency instability as given by (71) is $d \ \ell n \ n_R/d\psi \approx -\frac{3}{2}$. But that for interchange is $\frac{d \ \ell n \ n_R}{d\psi} \approx -15/2$. If the density gradient is to be built up gradually, the low-frequency instability would occur first.

V. RESONANT INSTABILITY

When the ring current belt is stable by itself and supporting an oscillation, we can no longer ignore the contribution from the energetic particles in Eq. (64) (the second term), because the energetic particles can now resonate with this azimuthally propagating wave when their drift velocity equals the phase velocity of the wave. This resonant exchange of energy between particle drift and wave leads either to the damping or to the growth of the wave, depending upon the sign of $(\partial F^E/\partial\psi)_{\mu J}$ at the resonance drift frequency. Setting $\omega = \Omega + i\gamma$, where $\gamma \ll \Omega$ for the weak growth or damping, we can expand $D(\omega)$ in (64) about $\omega = \Omega$, the solution as given by $\mathrm{Re}\, D = 0$:

$$D(\omega) = \mathrm{Re}\, D(\Omega) + \left.\frac{\partial\, \mathrm{Re}\, D}{\partial\omega}\right|_{\omega=\Omega} (i\gamma) + i\, \mathrm{Im}\, D(\Omega) = 0. \tag{73}$$

Since the ring current is dominating in particle density, the real part of D is approximately D_R, the contribution of the ring current belt:

$$\mathrm{Re}\, D(\Omega) \simeq D_R(\Omega) = 0. \tag{74}$$

Equation (74) then determines the real part of the frequency. When the ring current belt has sharp density gradient, $\omega_c \gg m\,\omega_d$, Eq. (74) can be approximated by Eq. (66) and

$$\left.\frac{\partial D_R}{\partial \omega}\right|_{\omega=\Omega} = c \frac{dn_R}{d\psi} \frac{m^2}{\Omega^2} \sum_{\pm} E_{\pm}^{3/2} \int d\lambda \, \nu_0^{-1} a_{\pm} e_{\pm} |\langle \Phi_m \rangle|^2 . \tag{75}$$

Using the Plemelj formula (32) and (56), we find from (57) the contribution of the energetic belt to the imaginary part of D,

$$\mathrm{Im}\, D = +\pi c \int d\lambda \left(\frac{e_-}{a_-}\right)\left(\frac{\Omega}{m\, a_-}\right)^{\frac{1}{2}} \nu_0^{-1} |\langle \Phi_m \rangle|^2 \left.\frac{\partial F_0^-}{\partial \psi}\right|_{\mu J} \left(\lambda,\ E = \frac{\omega}{m\, a_-}\ \psi\right)$$

for $\Omega > 0$

$$= -\pi c \int d\lambda \left(\frac{e_+}{a_+}\right)\left(\frac{\Omega}{m\, a_+}\right)^{\frac{1}{2}} \nu_0^{-1} |\langle \Phi_m \rangle|^2 \left.\frac{\partial F_0^+}{\partial \psi}\right|_{\mu J} \left(\lambda,\ E = \frac{\omega}{m\, a_+},\ \psi\right)$$

for $\Omega < 0$,

where $F_0^{\pm}(\omega\ \psi)$ are the distribution functions in $\mu\ J\ \psi$ space for the energetic protons and electrons, respectively. Equating the imaginary part of (73) to zero, we have

$$\frac{\gamma}{\Omega} = -\frac{\mathrm{Im}\, D(\Omega)}{(\partial D/\partial \omega)_{\Omega}} . \tag{77}$$

Substituting (75) and (76) into (77), we obtain

$$\frac{\gamma}{\Omega} = \pm \pi \frac{\Omega^2}{m^2} (2E)^{\frac{1}{2}} \frac{\int d\lambda \left(\frac{e_\pm}{a_\pm}\right)\left(\frac{\Omega}{m\, a_\pm}\right)^{\frac{1}{2}} \nu_0^{-1} \; |\langle \Phi_m \rangle|^2 \frac{\partial F_0^{\pm}}{\partial \psi}\bigg|_{\mu J}}{\frac{dn_R}{d\psi} \sum_{\pm} E_{\pm}^{-3/2} \int d\lambda \; \nu_0^{-1} (a_\pm \, e_\pm) \; |\langle \Phi_m \rangle|^2}$$

$$\text{for } \Omega \lesseqgtr 0. \qquad (78)$$

For a dipole-type field, $e_\pm \, a_\pm < 0$, and for the ring current belt to be stable in a dipole-type field: $dn/d\psi > 0$. Therefore,

$$\gamma < 0 \quad \text{for} \quad (\partial F_0/\partial \psi)_{\mu J} > 0,$$

$$\gamma > 0 \quad \text{for} \quad (\partial F_0/\partial \psi)_{\mu J} < 0. \qquad (79)$$

Thus the wave supported by the stable ring current belt may become overstable if the distribution function for the energetic belt $F(\mu \; J \; \psi)$ decreases with ψ for fixed $\mu \; J$.

We have noted that for the energetic proton belt (100 keV < E), $(\partial f_1/\partial L)_{\mu J} > 0$ (Dungey et al., 1965) for $2 < L < 7$, implying $(\partial F_0/\partial L)_{\mu J} > 0$ for dipole-type field. Thus the westward propagating wave, capable of resonating with energetic protons, is always damped. On the other hand, the flux of the energetic electrons (40 keV < E) outside the plasmapause is highly variable (Hess et al., 1965). It has a rather well-defined trapping boundary, beyond which the flux of the energetic electrons drops sharply (Frank et al., 1964). The trapping boundary appears to be slightly outward of the peak of the ring current belt (Frank, 1967 a); its location also depends upon the storm condition, and it moves inward to a lower L shell during the magnetic storm (Williams and Ness, 1967). Hence, at the outer edge of the ring current belt the energetic electron flux decreases sharply with L, and it is possible that $(\partial F/\partial L)_{\mu J} < 0$ for some values of μ J. The eastward propagating wave supported by the stable ring current may then become unstable due to resonance with the energetic electrons.

Physical Interpretation for the Overstability

The physical mechanism for the overstability is the resonant interaction between the wave and the particle drifts. This is essentially the same mechanism for Landau damping or growth, except that the thermal velocity of the particle is replaced by the drift velocity of the guiding centers. For a low-frequency wave traveling in the direction of the particle drift, it can exchange energy with those resonant particles whose drift velocity is almost equal to the phase velocity of the wave. Those resonant particles with drift velocities slightly greater than the phase velocity of the wave will give up energy to the wave, while those with drift velocities slightly less than the phase velocity will pick up energy from the wave. If there are more resonant particles picking up energy, then the wave will be damped as the energy of the wave is positive[7] (Rutherford and Frieman, 1968). Conversely, the wave will grow. Let us consider the case of guiding centers in a dipole field; the drift speed of the guiding centers at ψ near the equatorial plane is

$$v_d(\psi, \mu\ J) = \rho\ \omega_d = \rho \frac{c}{e} \mu \left\langle \frac{\partial B}{\partial \psi} \right\rangle$$

$$\simeq \rho \frac{c}{e} \mu \frac{\partial B_0}{\partial \psi} \simeq \frac{c}{e} \mu \frac{1}{r_0} , \qquad (80)$$

where r_0 is the equatorial distance. For a given $\mu\ J$, $v_d(\psi)$ is greater for the smaller r_0 (or smaller ψ). Thus the particles resonating with the wave at ψ but lying slightly outward of ψ will be moving more slowly than the wave, and the particle resonant with the wave at ψ but lying slightly inward of ψ is moving faster than the wave. Thus for the group of resonating particles with the same $v_d(\mu\ J\ \psi)$, if there are more of them lying just inside ψ, i.e., $(\partial F/\partial \psi)_{\mu J} < 0$ for the set of $\{\mu\ J\}$ at ψ such that $\omega_d(\mu\ J\ \psi) = \omega/m$, then the wave will gain energy.

We can also see this by calculating the work done on the resonating particles by the wave. The azimuthal component of the electric field of the wave is $E_\varphi = -im\ \Phi_m/\rho$ directed along φ which is positive for eastward direction. But the current of the particle drifts is always westward, as the electrons drift to the east while the protons drift to the west. Let $\delta \underset{\sim}{j}_{res}$ be the resonant current density due to perturbation. The rate of work done on the resonating electrons with mth mode at ψ is

$$P = \int d\varphi \frac{ds}{B} \delta \underset{\sim}{j}_{res} \cdot \underset{\sim}{E}_\varphi = i \sum_m \frac{m}{c} \int \frac{ds}{B} \Phi_m^*(s) (-e) \omega_d \delta n_m^{res}(s), \tag{81}$$

where δn_m^{res}, the resonant part of the perturbed electron density associated with the mth mode, is related to δF_m, the perturbed distribution in $\mu J \psi$ space, by Eq. (12). From (12) and (14),

$$\delta n_m^{res} = i \pi B c \iint d\mu \, dJ \frac{\nu_b}{v_{\parallel}} \langle \Phi_m \rangle \left.\frac{\partial F_0}{\partial \psi}\right|_{\mu J} \delta(\omega_d - \omega/m)$$

$$\text{for } \omega/m > 0. \tag{82}$$

Substituting (76) into (75) and noting $\nu_b \oint \frac{ds}{v_{\parallel}} \Phi_m = \langle \Phi_m \rangle$, we have

$$P = \pi m e \iint d\mu \, dJ \, |\langle \Phi_m \rangle|^2 \, (\partial F_0/\partial \psi)_{\mu J} \, \delta(\omega_d - \omega/m).$$

If $\left.\frac{\partial F_0}{\partial \psi}\right|_{\mu J} (\omega_d = \omega/m) > 0$, then $P > 0$. The resonating particles gain energy and the wave will be damped. Conversely, for $\left.\frac{\partial F}{\partial \psi}\right|_{\mu J} (\omega_d = \omega/m) < 0$, $P < 0$, and the wave will gain energy and grow.

Theoretical Conclusions

To summarize, we have studied the stability of the Van Allen belt in the outer zone $(2 < L < 8)$ against electrostatic μJ-conserving perturbations in the low-β nonrelativistic approximations. Inside the plasmapause, a sufficient condition for the stability of the energetic belt is that the distribution function $F(\mu\ J\ \psi)$ be a monotonically increasing function of the magnetic shell parameter L for fixed $\mu\ J$, i.e., $(\partial F/\partial\psi)_{\mu J} > 0$. As this seems to be the case for the outer belt particles, we concluded that the outer belt inside the plasmapause is always stable (Section III). Outside the plasmapause, there is a collisionless plasma dominated by the ring current belt even during the periods of magnetic quiescence (Frank, 1967). The outer edge of the ring current belt, where the density gradient is along the magnetic field gradient, is found to be unstable when the density gradient exceeds a certain critical value. The growth rate of the instability divided by the mode number is of the order of the geometric mean of the diamagnetic drift frequency and particle drift frequency. There is in general a parallel electric field associated with the instability. When the ring current belt is stable by itself, it can support a wave which then interacts with the energetic particles drifts. If the distribution function of the energetic particles is such that $(\partial F^{ener.}/\partial\psi)_{\mu J} < 0$, then the wave becomes overstable.

VI. POSSIBLE RELEVANCE TO POLAR SUBSTORM AND AURORA PHENOMENA

Two major problems in geomagnetic storm and auroral phenomena are the injection of plasma and energy into the closed-field-line region of the magnetosphere and the precipitation of the charged particles into the ionosphere to cause the polar substorm. The injection of the enhanced plasma into the inner magnetosphere and its subsequent inflation following a polar substorm was observed by Explorer 26 (Cahill, 1966; Davis, 1966; Brown and Roberts, 1965). Thus the processes of enhanced injection and precipitation, both sporadic in nature, coincide with each other, as evidenced by the sporadic nature of the polar substorm.

Injection

It has been suggested that the reconnection of the field line can be an important injection mechanism (Axford, Petchek and Siscoe, 1963; Axford, 1968). The sporadic nature of the polar substorm suggests that it is most likely due to plasma instability in the magnetosphere (Akasofu, 1967; Cole, 1967). Axford (1968) has emphasized the importance of the boundary conditions that could constrain the fluid from moving, thereby reducing the merging rate to zero. Thus the inward pressure gradient of the ring current belt in our case tends to prevent further merging when it is sufficiently steep. But as soon as the pressure gradient reaches a certain critical value, the ring current belt becomes unstable (Sec. IV), and the instability tends to relax the pressure gradient. With the collapse of the pressure gradient, the merging of the field line is resumed, and the injection of plasma again tends to rebuild the density gradient of the ring current belt. This, then, accounts for the intermittent nature of the substorms.

Precipitation

The instability of the ring current belt has a finite parallel electric field. Although the parallel electric field is of the order of $\epsilon v B/c$ (v is the velocity of a 10-keV proton), the potential drop along the field line for the fundamental mode is of the order of Mv^2/e. The parallel electric field accelerates the charged particles along the field line with the only resistance due to magnetic inhomogeneity:

$$M \frac{dv_{\parallel}}{dt} = -\mu \frac{\partial B}{\partial s} - e \frac{\partial \phi}{\partial s} .$$

Because of the smallness of the electron-to-proton mass ratio, the parallel electric field tends to eject electrons into the ionosphere or pull electrons out of the ionosphere. This is the reason that the precipitation particles are mainly electrons.

The magnitude of the potential drop along the magnetic field line can be estimated as follows: The parallel electric field can grow to a critical value for which the particles with average energy--the ones that are responsible for the instability--are themselves being pulled out of the system. This happen when

$$-\tilde{\mu}\,(\partial B/\partial s) - e\,\partial\phi/\partial s = 0$$

or

$$e\,\Delta\phi \approx \tilde{\mu}B \approx E_{av} \approx 10 \text{ keV} .$$

When this steady-state potential difference is reached, low-energy electrons ($E < 10$ keV) will be accelerated to 10 keV while being pulled out of the magnetosphere to cause the precipitation. In this sense, the auroral electrons are freshly accelerated. Recent measurements by Albert (1967) have shown that the auroral electrons are indeed nearly monoenergetic, with fluxes peaked at about 10 keV. The conjugacy of the auroral phenomena is due to the evenness of the potential variation along the field line with respect to the equatorial plane, which results from the evenness of the ring current belt distribution with respect to the equatorial plane.

According to some observations, the polar magnetic disturbances are proportional to the maximum electron density in the auroral sporadic layers (Nagata, 1963). This suggests that the variations in the polar magnetic disturbances are produced by varying amounts of precipitation, which produce the variations in the conductivity, while the electric field associated with the current system (or potential drop across the polar cap along the dawn-dusk meridian) remains approximately constant (Bostrom, 1966). The electric field can be regarded as necessarily accompanying the injection of the ring current belt into the geomagnetic field by drift (Block, 1967; Axford, 1968). Thus the problem of the auroral electrojet simply reduces to that of intense precipitation with simultaneous injection, and can be understood from the previous discussion.

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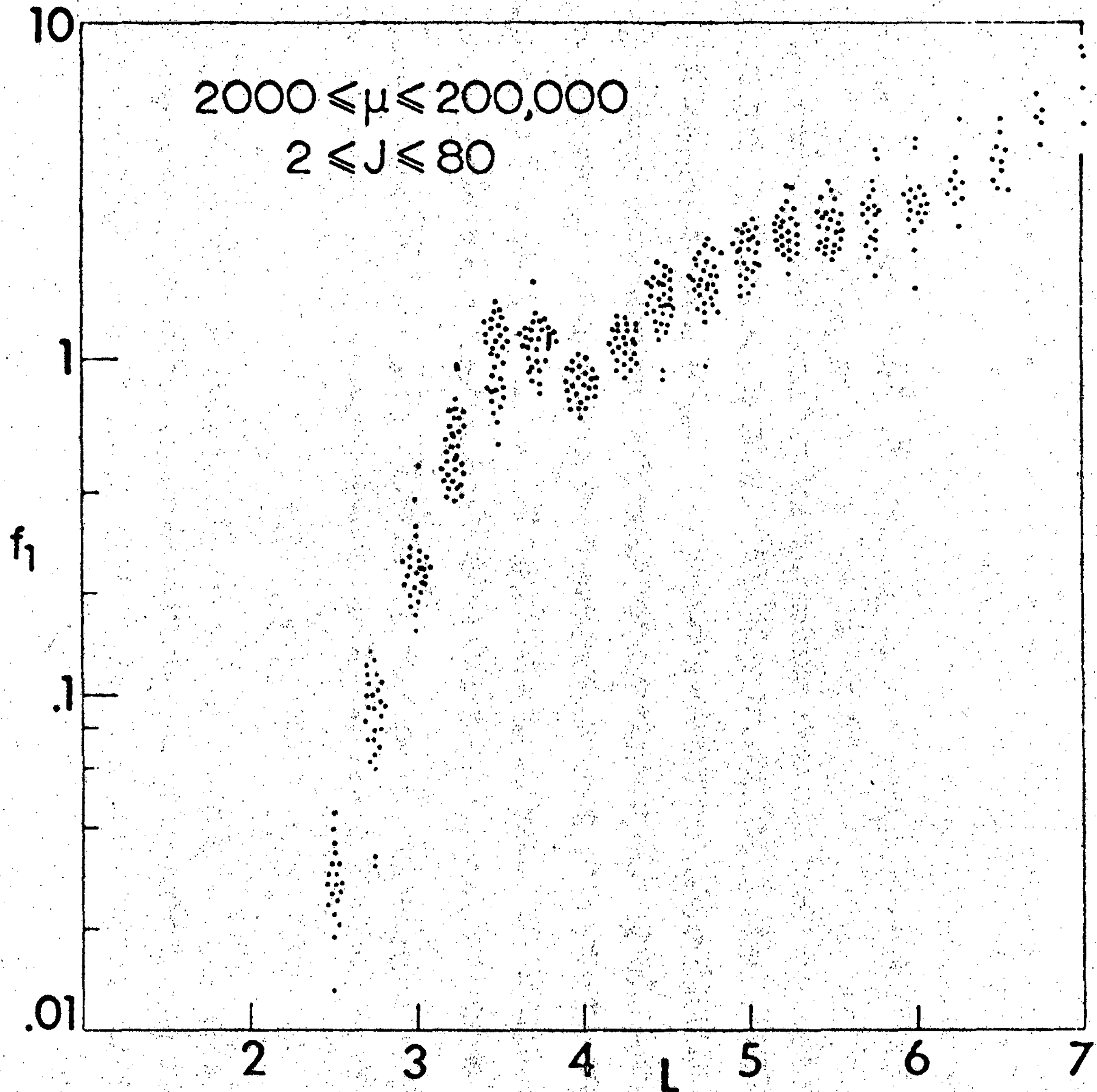

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Fig. 1. Variation of distribution function $f_1[E(\mu\ J\ L)_1\ \alpha(\mu\ J\ L)_1\ L]$ of energetic protons with L for fixed μ J (After Hess, 1967).

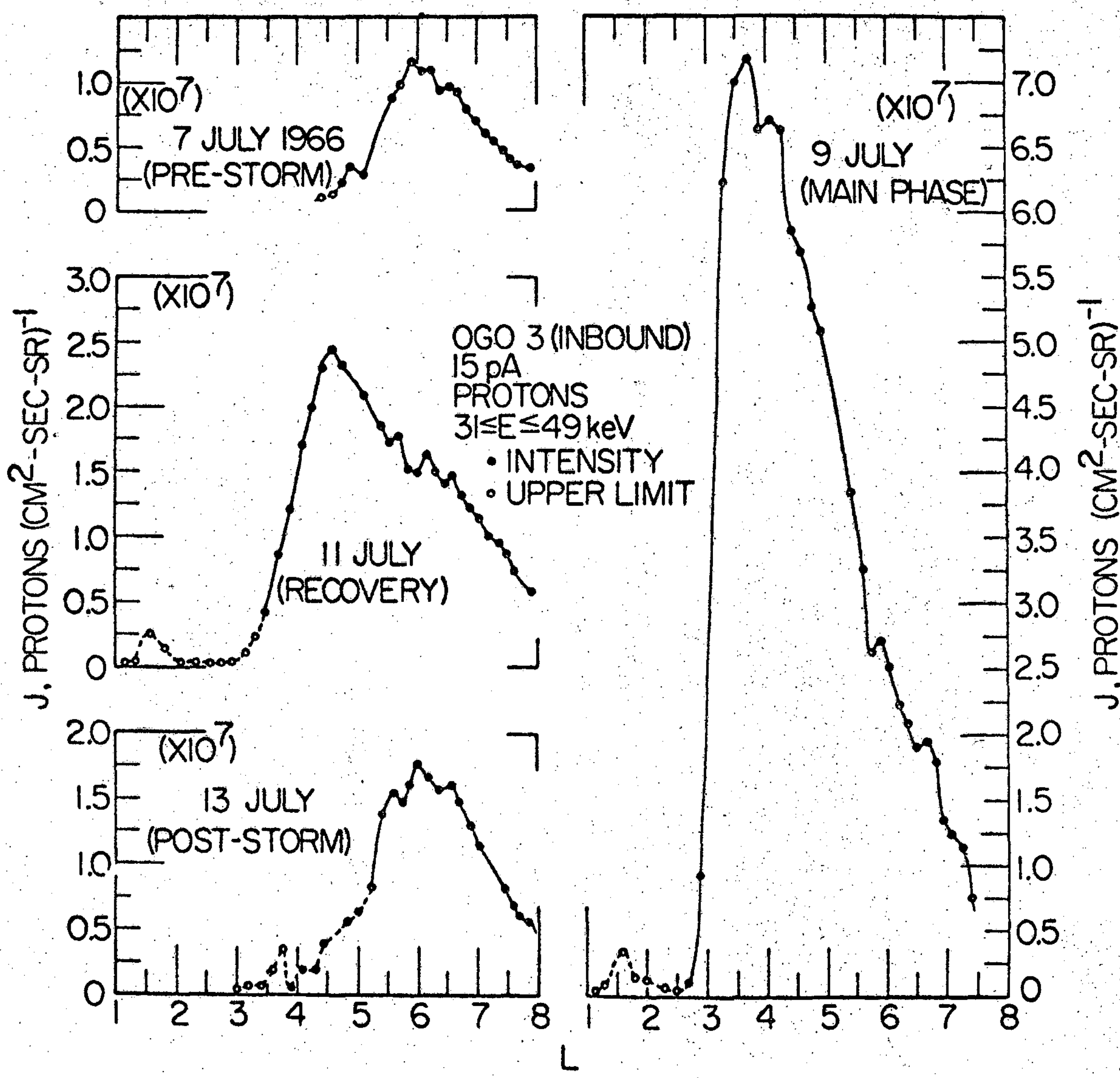

Fig. 2. Flux of ring current belt (After Frank, 1967).

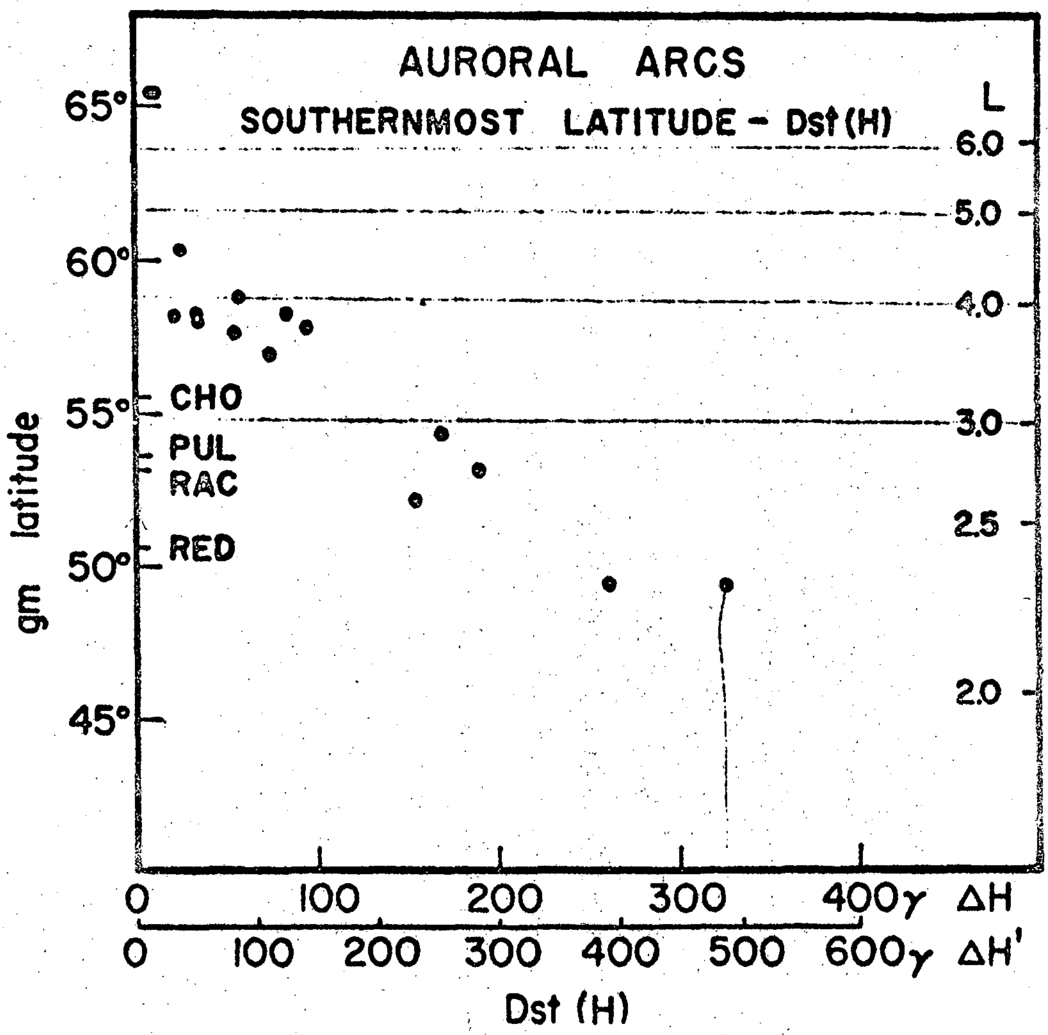

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Fig. 3. Latitude of the Southernmost auroral arc as a function of D_{st} (After Akasofn, 1967).

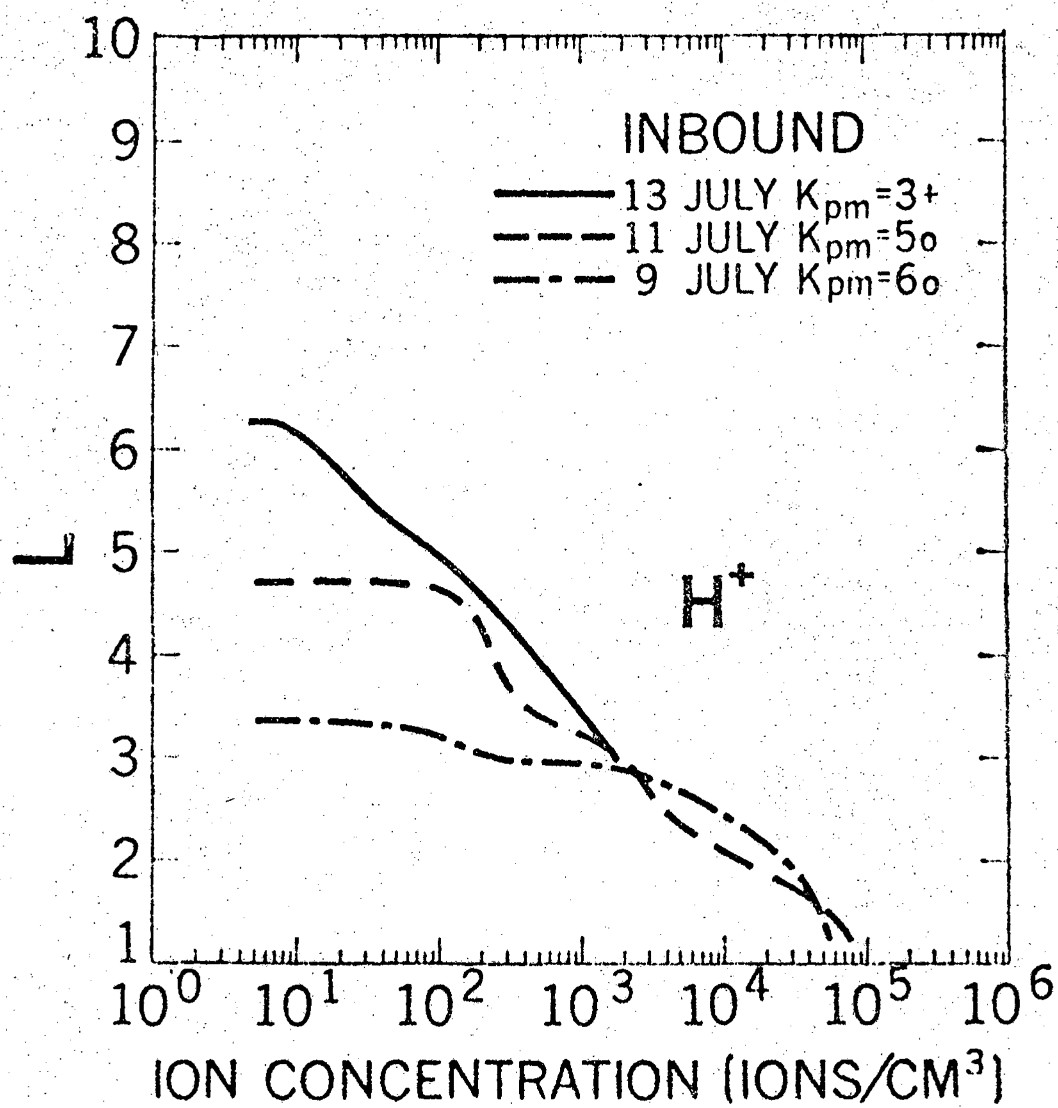

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Fig. 4. Location of the plasmapause as measured by ion concentration (After Taylor et al., 1968).

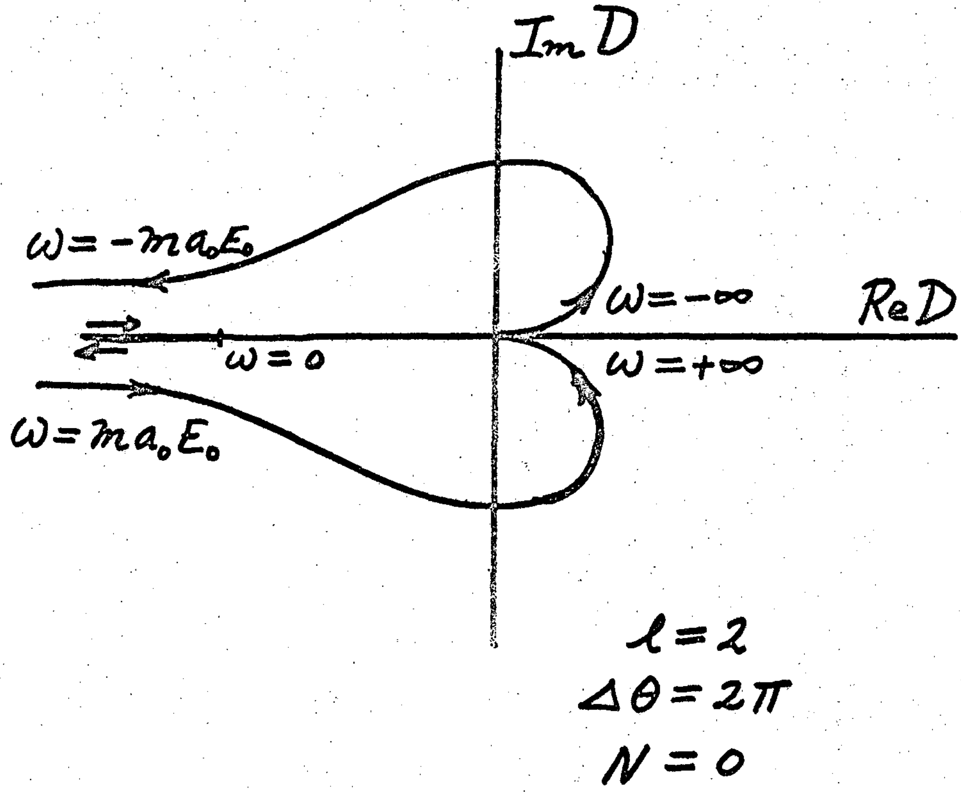

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Fig. 5. Nyquist diagram: Stable case.

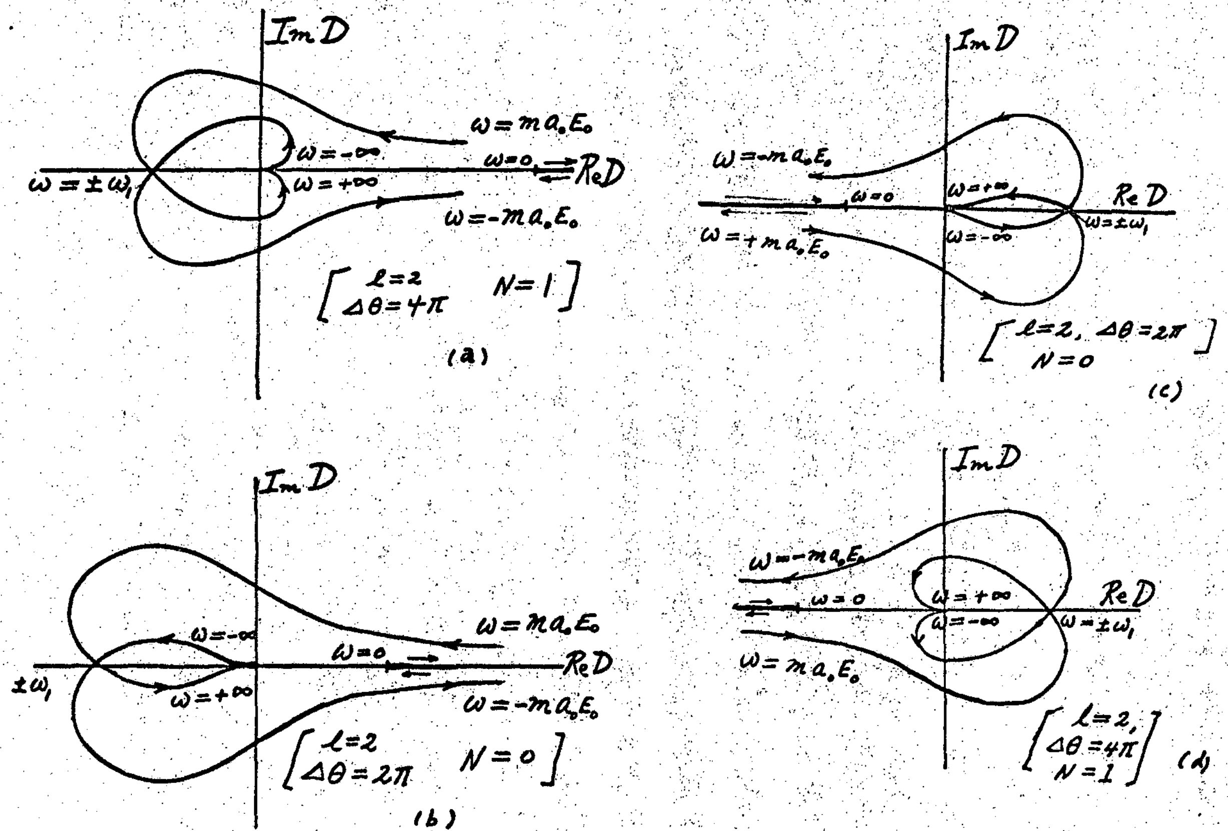

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Fig. 6. Nyquist diagram: a, b, c, d.

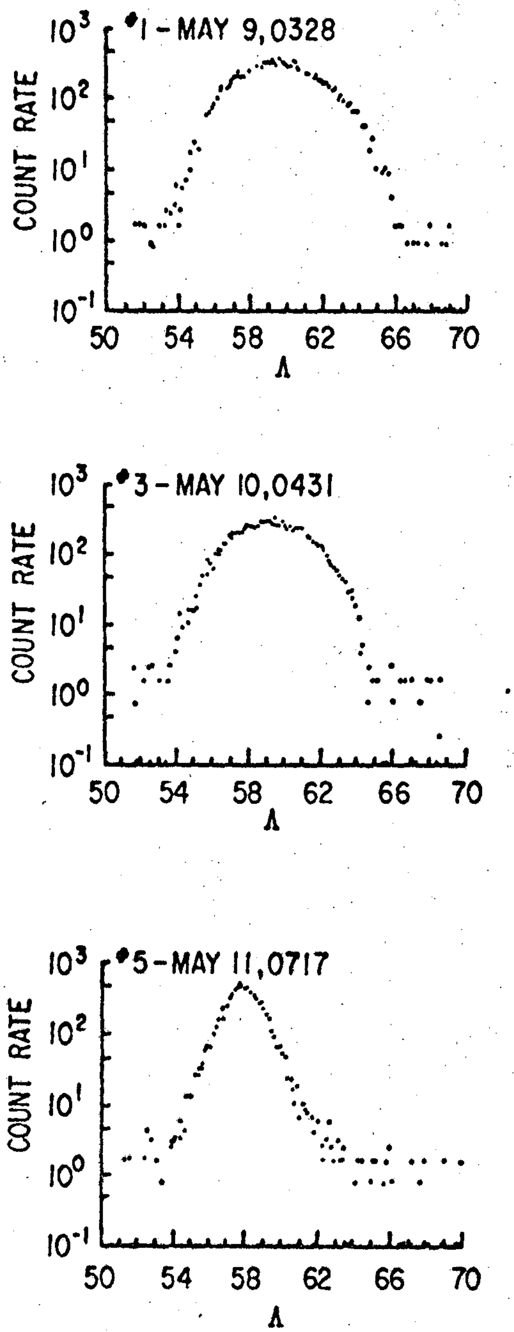

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Fig. 7. Trapping boundary of energetic electrons (After Williams and Ness, 1967).

Table I. Characteristic Parameters

Particle parameters, 10-keV particle in a dipole field

	Protons		Electrons	
	L = 3	L = 6	L = 3	L = 6
Gyrofrequency	18 cps	2 cps	33 kc	4 kc
Bounce period	1 min	2 min	1 sec	2 sec
Drift period	29 hr	14 hr	29 hr	14 hr
Gyroradius	12 km	100 km	0.5 km	2.3 km

Plasma parameters

	Inside plasmapause	Outside plasmapause
Density	$10^3/cm^3$	$1/cm^3$
Temperature	1 eV	1 keV
Debye length	20 cm	20 meters
Electron collision frequency	3×10^{-2}/sec	10^{-8}/sec
Mean free path	10^4 km	10^{13} km
Plasma frequency	2×10^6/sec	0.5×10^5/sec
β at equatorial plane	1/6	1

APPENDICES

A. Hamiltonian Equations for Drift Motion of Guiding Centers

For charged particles moving in a slowly varying and weakly inhomogeneous, curved magnetic field, the guiding-center approximation often greatly simplifies the problem whenever it is applicable and the adiabatic invariants exist. For the magnetic moment $\mu = p_\perp^2/2mB$ to be an adiabatic invariant, we require that

$$\Omega_i^{-1} \frac{1}{B}\frac{dB}{dt} = \Omega_i^{-1} \frac{\partial \ln B}{\partial t} + \Omega_i^{-1} v \cdot \nabla \ln B$$

$$\approx \mathcal{O}\left(\frac{\omega_B}{\Omega_i}\right) + \mathcal{O}\left(\frac{r_g}{L_\perp}\right) + \mathcal{O}\left(\frac{v_\parallel}{\Omega_i}\frac{1}{L_\parallel}\right) + \mathcal{O}(\epsilon) \approx \epsilon \ll 1,$$

where Ω_i, r_g are the ion gyrofrequency and gyroradius respectively; ω_B, $L_\perp$, $L_\parallel$ are the characteristic frequency of the magnetic field, and the linear dimensions perpendicular and parallel to the field line, and $\epsilon \equiv r_g/L_\perp$.

In general each of the three quantities must be small:

$$\frac{\omega_B}{\Omega_i} \approx \epsilon, \qquad \frac{r_g}{L_\perp} \approx \epsilon, \qquad \frac{v_\parallel}{L_\parallel}\frac{1}{\Omega_i} \approx \epsilon.$$

The third inequality can be written as

$$\nu_b \approx \epsilon\Omega_i.$$

This implies also a limit on the magnitude of the parallel electric field, as

$$\Delta v_{\|} \approx \frac{e}{m}\int_0^{\Omega_i^{-1}} E_{\|}dt = \frac{c}{B} E_{\|} \approx \epsilon v_{th}.$$

For J to be conserved, it is necessary that

$$\nu_b^{-1} \frac{1}{B} \frac{dB}{dt} = \nu_b^{-1} \left[\frac{1}{B} \frac{\partial B}{\partial t} + \underset{\sim}{v}_d \cdot \frac{\nabla B}{B}\right] \ll 1,$$

$\underset{\sim}{v}_d$ being the drift velocity of the guiding center across the field line, including E x B, ∇B drifts, and curvature drifts in general. This, in general, requires that $\omega_B \ll \nu_b$, and $\underset{\sim}{v}_d \cdot \nabla \ln B \ll \nu_b$. This second inequality implies the limit on the magnitude of electric field components perpendicular to the field line,

$$\frac{cE_\perp}{B} \approx \epsilon v_{th}.$$

When μ, J exist as adiabatic invariants (in fact, there are two invariant asymptotic series for which μ, J are the first terms in the expansion), the average drift motion of the guiding center can be written as the Hamiltonian equation with $\frac{e}{c}\psi$, φ as canonically conjugate variables. We shall give here an heuristic derivation following Taylor (1963) and referring to Northrop (1961) for a rigorous derivation.

The Lagrangian for a guiding center with mass M, magnetic moment μ, charge e moving in a magnetic field $B = \nabla\psi \times \nabla\varphi$ with vector potential $A = \psi\nabla\varphi$ and magnetic potential $\chi = \int \underset{\sim}{B}\cdot ds$, and electric field given by potential Φ is

$$\mathcal{L} = \frac{m v_{\parallel}^{2}}{2} + \frac{e}{c}\,\underset{\sim}{v}\cdot\underset{\sim}{A} - e\Phi - \mu B + (\epsilon^{2})$$

$$= \frac{m\dot{\chi}^{2}}{2B^{2}} + \frac{e}{c}\,\psi\dot{\varphi} - e\Phi(\psi\varphi\chi) - \mu B(\psi\varphi\chi),$$

where we have used

$$\underset{\sim}{v}_{d}\cdot\underset{\sim}{A} \approx \underset{\sim}{v}_{d}\cdot\psi\nabla\varphi \approx \psi\dot{\varphi},$$

because for a stationary magnetic field we have $\partial\varphi/\partial t = 0$.

The conjugate momenta to ψ, φ, χ are

$$p_{\psi} = \frac{\partial\mathcal{L}}{\partial\dot{\psi}} = 0, \qquad p_{\varphi} = \frac{\partial\mathcal{L}}{\partial\dot{\varphi}} = \frac{e}{c}\;, \qquad \text{and} \qquad p_{\chi} = \frac{m\dot{\chi}}{B^{2}} .$$

Note that $\frac{e}{c}\,\psi$ is the conjugate momentum to φ. The Hamiltonian is

$$H = \sum_{q_i = \psi\varphi\chi} p_i \dot{q}_i - \mathcal{L}$$

$$= B^{2}\,\frac{p_{\chi}^{2}}{2m} + \mu B + e\Phi .$$

We transform from p_{χ}, χ to action and angle variable J, θ, where

$$J(\mu H\psi\varphi) = \oint \frac{d\chi}{B}\left[2m(H - \mu B - e\Phi)\right]^{1/2}, \quad \theta = \nu_{b}^{-1}\int^{s} \frac{ds}{v_{\parallel}} .$$

This equation defines H implicitly as a function of $J\mu\psi\varphi$, which is denoted by $K(\mu J\psi\varphi)$. Recalling that $\frac{e}{c}\,\psi$, φ are canonical conjugates, we have

$$\dot{\psi} = - \frac{c}{e} \frac{\partial K}{\partial \varphi} (\mu J \psi \varphi),$$

$$\dot{\varphi} = + \frac{c}{e} \frac{\partial K}{\partial \psi} (\mu J \psi \varphi),$$

and the Liouville equation,

$$\frac{\partial F}{\partial t} (\mu J \psi \varphi) + \frac{c}{e} \left(\frac{\partial F}{\partial \varphi} \frac{\partial K}{\partial \psi} - \frac{\partial F}{\partial \psi} \frac{\partial K}{\partial \varphi} \right) = 0.$$

This equation is just the lowest order reduced Vlasov equations in the drift time scale. This equation has recently been derived from Vlasov equations by Hastie et al. (1967).

B. 1. Transformation of Phase-Space Variables

Let us begin with cylindrical coordinates in velocity space: $(v_\parallel, v_\perp, \theta)$, where $v_\parallel$, $v_\perp$ are the magnitudes of velocity components parallel and perpendicular to the magnetic field, and θ the phase angle of $\underset{\sim}{v}_\perp$. Let ψ, φ, χ, the intrinsic coordinate system for the magnetic field, be the spatial coordinates. Introducing the conjugate momentum of χ, p_χ, and the magnetic moment μ, which is proportional to the conjugate momentum of θ (θ thus becomes ignorable, as μ is invariant), we have the distribution function in μ, ψ, φ, p_χ, χ space

$$\mathcal{F}(\mu, \psi, \varphi, p_\chi, \chi) = \frac{2\pi}{m^2} f(\underset{\sim}{r}, \underset{\sim}{v});$$

$$\text{Particle density } n(\underset{\sim}{r}) = \int d^3\underset{\sim}{v}\, f(\underset{\sim}{r}, \underset{\sim}{v}) = \int B^2 \left(\frac{m^2}{2\pi}\right)^{-1} d\mu\, dp_\chi \cdot \left(\frac{2\pi}{m^2}\right)^{-1} \mathcal{F}$$

$$= B^2 \int d\mu\, dp_\chi\, \mathcal{F}.$$

As we have seen in Appendix A, p_χ, χ; $\frac{e}{c}\psi$, φ are the canonical conjugate pairs (μ is a parameter), so they satisfy the Hamiltonian equations of motion,

$$\dot{p}_\chi = -\frac{\partial H}{\partial \chi}, \qquad \dot{\chi} = \frac{\partial H}{\partial p_\chi}; \qquad \text{and} \qquad \dot{\psi} = -\frac{c}{e}\frac{\partial H}{\partial \varphi}, \quad \dot{\varphi} = \frac{c}{e}\frac{\partial H}{\partial \psi}$$

where

$$H = \frac{B^2 p_\chi^{\,2}}{2m} + \mu B + e\Phi .$$

Also there is a Liouville theorem

$$\frac{\partial \mathcal{F}}{\partial t}(\mu;\ p_\chi,\ \chi;\ \psi\varphi) + \frac{\partial H}{\partial p_\chi}\frac{\partial \mathcal{F}}{\partial \chi} - \frac{\partial H}{\partial \chi}\frac{\partial \mathcal{F}}{\partial p_\chi} + \frac{c}{e}\left[\frac{\partial H}{\partial \psi}\frac{\partial \mathcal{F}}{\partial \varphi} - \frac{\partial H}{\partial \varphi}\frac{\partial \mathcal{F}}{\partial \psi}\right] = 0. \quad \text{(B-1)}$$

This is the so-called drift kinetic equation in the Russian literature. It treats the detailed motion along the field line in a bounce time scale (hydromagnetic time scale) and is of the Chew-Goldberger-Low (1958) ordering scheme. For all terms of this equation to be of the same order, $E_\perp$ must be of zeroth order (i.e., $cE_\perp/B \approx v_{th}$) to give a zeroth-order drift velocity. On the other hand, if $cE_\perp/B \approx \epsilon v_{th}$ and thus all drifts are of $\mathcal{O}(\epsilon)$, then the last two terms can be neglected, and we have

$$\frac{\partial \mathcal{F}}{\partial t} + \frac{\partial H}{\partial p_\chi}\frac{\partial \mathcal{F}}{\partial \chi} - \frac{\partial H}{\partial \chi}\frac{\partial \mathcal{F}}{\partial p_\chi} = 0, \quad \text{(B-2)}$$

which is Grad's (1967) guiding-center equation.

If the bounce motion is nearly periodic, there exists a second adiabatic invariant J. For low-frequency perturbations conserving J, we can transform p_χ, χ to J, θ, where θ, being the angle variable conjugate to J; $\theta = \nu_b \int ds/v_\parallel$, is ignorable. Noting that $d\psi\, d\varphi = B d^2 r_\perp$, we have for the distribution function in $\mu J\psi\varphi$ space,

$$F(\mu J\psi\varphi)d\mu\ dJ\ d\theta\ B = f(\underset{\sim}{r},\ \underset{\sim}{v})2\pi v_\perp\ dv_\perp\ dv_\parallel\ ds.$$

The density in space $(\psi\varphi\chi)$ expressed in terms of F is

$$n\left[\underset{\sim}{r}(\psi\varphi\chi)\right] = \int d^3\underset{\sim}{v}\ f(\underset{\sim}{v},\ \underset{\sim}{r}) = B\int d\mu\ dJ\ F(\mu J\psi\varphi)\ \frac{\partial\theta}{\partial s}$$

$$= B\iint d\mu\ dJ\ F\ \frac{\nu_b}{\left[(2/m)(E - \mu B - e\varphi)\right]^{1/2}}\ .$$

The Jacobian of the transformation is

$$\frac{\partial(J,\ \theta)}{\partial(p_\chi X)} = \frac{\partial J}{\partial p_\parallel}\frac{\partial \theta}{\partial s} = \frac{\partial J}{\partial E}\frac{\partial}{\partial p_\parallel}\cdot\frac{\nu_b}{v_\parallel} = \nu_b^{-1} v_\parallel \frac{\nu_b}{v_\parallel} = 1,$$

which only confirms that this is a canonical transformation.

B. 2. Relation between $f_1(E,\ \alpha,\ \underset{\sim}{r})$ and $F_0(\mu,\ J,\ \psi)$

Let $f_1(E,\ \alpha,\ \underset{\sim}{r})$, $f_2(E,\ \mu,\ \underset{\sim}{r})$, $f_3(\mu,\ J,\ \underset{\sim}{r})$ be the distribution functions in $(E,\ \alpha,\ \underset{\sim}{r})$, $(E,\ \mu,\ \underset{\sim}{r})$, $(\mu,\ J,\ \underset{\sim}{r})$ spaces respectively. The number density at $\underset{\sim}{r}$ is given by

$$n(\underset{\sim}{r}) = \iint f_1(E,\ \alpha,\ \underset{\sim}{r})dE\ d\alpha = \iint f_2(E,\ u,\ \underset{\sim}{r})dE\ d\mu = \iint f_3(\mu,\ J,\ \underset{\sim}{r})du\ dJ,$$

$$f_1(E,\ \alpha,\ \underset{\sim}{r})dE\ d\alpha = f_2(E,\ \mu,\ \underset{\sim}{r})dE\ d\mu$$
$$= f_3(\mu,\ J,\ \underset{\sim}{r})d\mu\ dJ.$$

Since $\quad d\mu = 2E \sin\alpha \cos\alpha\ d\alpha/B$

$$(\partial J/\partial E)_{\mu\psi\varphi} = \nu_b^{-1}(\mu J\psi\varphi),$$

$$f_3 = \nu_b f_2 = \frac{\nu_b B f_1}{2E \sin\alpha \cos\alpha}. \tag{B-3}$$

But we also have

$$f_3 = \frac{\nu_b B}{(2E/M)^{1/2} \cos\alpha} F_0(\mu,\ J,\ \psi,\ \varphi). \tag{B-4}$$

From (1) and (2)

$$F_0(\mu,\ J,\ \psi,\ \varphi) = \frac{f_1(E,\ \alpha,\ \underset{\sim}{r})}{(2E/M)^{1/2} \sin\alpha}$$
$$= f_1(E,\ \alpha,\ \underset{\sim}{r})/(2\mu B/M)^{1/2}. \tag{B-5}$$

At the equational plane, the distribution function f_1 for an axisymmetric system is a function of E, α, ψ only. E, α, in turn, are functions of $\mu J \psi$:

$$f_1(E, \alpha, \underset{\sim}{r}) = f_1\left[E(\mu J\psi), \alpha(\mu J\psi), \psi\right].$$

From (3) we have

$$\left.\frac{\partial F_0}{\partial L}\right|_{\mu J} = \left.\frac{\partial f_1}{\partial L}\right|_{\mu J} \frac{1}{(2\mu B/m)^{1/2}} - \frac{f_1}{2(2\mu B/m)^{1/2}} \frac{1}{B} \frac{dB}{d\psi} .$$

For a dipole-type field, $dB/d\psi < 0$, and the positiveness of $(\partial f_1/\partial L)_{\mu J}$ thus implies the positiveness of $(\partial F_0/\partial L)_{\mu J}$.

C. Existence of Electric Field Component Along the Magnetic Field Line

For the very-low-frequency perturbations ω, $m\omega_d \ll \nu_b$ that we consider here, the system can be regarded as in a series of hydromagnetic equilibria, i.e., steady states over the bounce time scale, during the course of the perturbation. And for a system in a hydromagnetic equilibrium, $\frac{\partial \mathcal{F}}{\partial t}(\mu, p_{\|}s, \psi\varphi) = 0$, Eq. (B-2) becomes (with $p_{\|}$, s instead of p_{χ}, χ, where $p_{\chi} = p_{\|}/B$)

$$\frac{\partial H}{\partial p_{\|}}(\mu, p_{\|}s, \psi\varphi)\frac{\partial \mathcal{F}}{\partial s} - \frac{\partial H}{\partial s}\frac{\partial \mathcal{F}}{\partial p_{\|}} = 0. \tag{C-1}$$

The general solution is

$$\mathcal{F}(\mu;\ p_{\|};\ s;\ \psi\varphi) = \mathcal{F}\left[\mu H(\mu,\ \psi\varphi;\ p_{\|}s),\ \psi\varphi\right]. \tag{C-2}$$

The particle density in the equilibrium is given by

$$\begin{aligned} n\left[\underset{\sim}{r}(\psi\varphi s)\right] &= B(\underset{\sim}{r})\iint d\mu\ dp_{\|}\ \mathcal{F}(\mu,\ H,\ \psi\varphi) \\ &= -\ B(\underset{\sim}{r})\iint d\mu\ dH\ \frac{\partial \mathcal{F}}{\partial H}\ p_{\|}\ , \end{aligned} \tag{C-3}$$

where $\quad p_{\|} = \sqrt{2m(H - \mu B - e\Phi)}$.

The quasi-neutrality condition is

$$\sum_{\pm} e_{\pm}n_{\pm}(\underset{\sim}{r}) = 0 = \sum_{\pm} e_{\pm}B(\underset{\sim}{r})\iint d\mu\ dH\ \frac{\partial \mathcal{F}_0}{\partial H}(\mu H\psi\varphi)\left[2m(H - \mu B - e\Phi)\right]^{1/2}. \tag{C-4}$$

This equation determines the potential variation along the field line. To find $E = -\ \partial\Phi/\partial s$, we differentiate (C-4) with respect to s and obtain

$$eE_{\parallel} = - \frac{\frac{1}{B}\frac{\partial B}{\partial s}\sum_{\pm} e_{\pm}\iint d\mu \; dH \frac{\partial \mathcal{F}}{\partial H}\frac{\mu B}{v_{\parallel}}}{B\sum_{\pm} e_{\pm}\iint d\mu \; dH \frac{\partial \mathcal{F}}{\partial H}\frac{1}{v_{\parallel}}}$$

$$= + \frac{\frac{1}{B}\frac{\partial B}{\partial s}\sum_{\pm}\left\langle \frac{v_{\perp}^{2}}{v_{\parallel}^{2}} \right\rangle_{\pm} e_{\pm}}{B\sum_{\pm} e_{\pm}\iint d\mu \; dH \frac{\partial \mathcal{F}}{\partial H}\frac{1}{v_{\parallel}}} \qquad \text{(C-5)}$$

where $$\left\langle \frac{v_{\perp}^{2}}{v_{\parallel}^{2}} \right\rangle \equiv - \iint d\mu \; dH \frac{\partial \mathcal{F}}{\partial H}\frac{\mu B}{v_{\parallel}} = - \iint d\mu \; dp_{\parallel} \mathcal{F} \frac{v_{\perp}^{2}/2}{v_{\parallel}^{2}} \qquad \text{(C-6)}$$

is just a measure of pitch angle distribution. The parallel electric field can exist in the hydromagnetic steady state if and only if the magnetic field is nonuniform along the field line and the electrons and protons have different pitch angle distributions.

D. Variation of Particle Density Along the Field Line

Suppose a distribution function, factorizable in its energy E and pitch angle α dependence, is proportional to $\sin^{2\ell} \alpha \equiv \lambda^{\ell}$:

$$F\left[\mu(E, \lambda, \psi\varphi), J(E, \lambda, \psi, \varphi), \psi, \varphi\right] = f(E,\psi,\varphi)\lambda^{\ell}.$$

The number density n(s) at a distance s from the equatorial plane s = 0, along the field line at (ψ,φ), is given by (7):

$$n(\psi, \varphi s) = 2B(\psi\varphi s)\iint d\mu \; dJ \frac{v_b}{v} F$$

$$= 2B(\psi\varphi s)\int dE E^{1/2} f(E, \psi, \varphi)\int_{1/B_{max}}^{1/B} \lambda^{\ell}/(1 - \lambda B)^{1/2}$$

$$\approx \frac{1}{B^{\ell}} + \mathcal{O} \; \frac{1}{B^{\ell}} \frac{B}{B_{max}} .$$

For $B_{max} \gg B$, $\quad n(\psi, \varphi, s) \approx \dfrac{1}{B^{\ell}(\psi, \varphi, s)}$.

E. Nyquist Method

For an eigenvalue equation

$$D(\omega) = 0,$$

where $D(\omega)$ is analytic in the upper half ω plane, the number of roots in the upper half ω plane is given by

$$N = \frac{1}{2\pi i} \int_Z \frac{dD}{D} = \frac{1}{2\pi i} \int_C d\omega \, \frac{dD/d\omega}{D} \,, \tag{E-1}$$

where the contour C consists of the real axis and a semicircle enclosing the upper half ω plane in the positive sense and the contour Z is the mapping of C onto the D plane. Then by Cauchy theorem, the number of zeroes D has in the upper half ω plane is equal to the number of times the contour Z encloses the origin in the D plane in the counter-clockwise sense. Thus the necessary and sufficient condition for the existence of unstable modes is that Z enclose the origin in the counter-clockwise sense at least once.

Usually the integration of $(dD/d\omega)/D$ along the semicircle at infinity can readily be performed by knowing the asymptotic behavior of $D(\omega)$ for ω large. For example, if $D(\omega) \sim \omega^{-\ell}$ for ω large, then

$$\int_{\substack{\text{semi-}\\\text{circle}}} d\omega \, \frac{dD/d\omega}{D} = \int_0^{\pi} de^{i\theta} \, \frac{(-\ell)}{e^{i\theta}} = -\,\ell\pi i.$$

Therefore the number of roots in the upper half ω plane is

$$N = -\frac{\ell}{2} + \frac{1}{2\pi i}\int_{-\infty}^{\infty} d\omega \, \frac{dD/d\omega}{D}$$

$$= -\frac{\ell}{2} + \frac{1}{2\pi i}\left\{\ell n \frac{|D(\infty)|}{|D(-\infty)|} + i\left[\arg D(\infty) - \arg D(-\infty)\right]\right\}.$$

since $|D(\infty)| = |D(-\infty)|$,

$$N = -\frac{\ell}{2} + \frac{\Delta\theta}{2\pi} \tag{E-2}$$

$$\Delta\theta \equiv \arg D(\infty) - \arg D(-\infty)$$

is the change in the argument of D as ω goes from $-\infty$ to $+\infty$ along the real axis.

F. The Explicit Expression of Bounce and Drift Frequencies in Terms of Kinetic Energy E and $\lambda = \mu/E$

The bounce frequency as given by Eq. (6) in terms of E and λ is

$$\nu_b^{-1} = \frac{\partial J}{\partial K}\bigg|_{\mu\psi} = \oint \frac{ds}{\left[\frac{2}{M}\left(K - \mu B - e\Phi\right)\right]^{1/2}}$$

$$\frac{\partial J}{\partial K}\bigg|_{\mu\psi} = \oint ds \left[\frac{2}{M}\left(E - \mu B\right)\right]^{-1/2} = \nu_b^{-1}(x)E^{-1/2}$$

where $\quad \nu_b^{-1} = \oint ds \left[\frac{2}{M}\left(1 - \lambda B\right)\right]^{-1/2}.$

The drift frequency is given by

$$\omega_d = \frac{c}{e}\frac{\partial K}{\partial \psi}\bigg|_{\mu J} = -\frac{c}{e}\frac{(\partial J/\partial\psi)_{\mu K}}{(\partial J/\partial K)_{\mu\psi}}$$

$$= -\frac{c}{e}\nu_b \oint \frac{ds\ -\mu(\partial B/\partial\psi) - e(\partial\Phi/\partial\psi)}{\left[(2/M)(K - \mu B - e\Phi)\right]^{1/2}}$$

$$= E\frac{c}{e}\nu_0(x)\oint \frac{ds}{(1 - \lambda B)^{1/2}(2/M)^{1/2}}\left(\lambda\frac{\partial B}{\partial\psi} + \frac{e}{E}\frac{\partial\Phi}{\partial\psi}\right)$$

$$= a(\lambda, \psi, \varphi)E + b(\lambda, \psi, \varphi),$$

where

$$a(\lambda, \psi, \varphi) \equiv \frac{c}{e}(M/2)^{1/2}\nu_0(\lambda)\oint ds \frac{\lambda(\partial B/\partial\psi)}{(1 - \lambda B)^{1/2}}$$

$$b(\lambda, \psi, \varphi) \equiv c\ (M/2)^{1/2}\nu_0(\lambda)\oint ds \frac{\partial\Phi/\partial\psi}{(1 - \lambda B)^{1/2}}$$

$$= c\left\langle\frac{\partial\Phi}{\partial\psi}\right\rangle(\lambda, \psi, \varphi).$$

If the component of the equilibrium electric field vanishes in the ψ direction, then $b = 0$.

G. Sufficient Conditions for Instability

We have noted in Chapter III that for Eq. (30) to have unstable roots, it is necessary for $(\partial F/\partial K)_{\mu J}$ to change sign. In general, the necessary and sufficient conditions for instability are quite complicated. Consider a special case in which $F_0^- = F^+$ and $(\partial F/\partial K)_{\mu J} = 0$ only for one set of μJ at ψ, $\left[\mu J/\omega_d^{\mp}(\mu J\psi) = \pm\, \omega_1/m\right]$,[7] where $\omega_1 > mE_0 a_0$.

Situation 1

Suppose $(\partial F/\partial K)_{\mu J}$ changes sign in the following way:

$$\begin{aligned} (\partial F/\partial K)_{\mu J} &> 0 \qquad \text{for} \qquad |\omega_d^{\pm}| < \omega_1/m \\ &< 0 \qquad\qquad\qquad\qquad > \omega_1/m \end{aligned} \tag{G-1}$$

and ω_1 is such that

$$\iint du\; dJ\; (\partial F/\partial K)_{\mu J} > 0. \tag{G-2}$$

Then from (34) we have

$$\begin{aligned} \text{Im } D &< 0 \qquad \text{for} \qquad \omega > \omega_1 > ma_0E_0, \\ &> 0 \qquad\qquad\qquad \omega_1 > \omega > ma_0E_0, \\ &= 0 \qquad\qquad\qquad ma_0E_0 > \omega > -\, ma_0E_0, \\ &< 0 \qquad\qquad\qquad -\, ma_0E_0 > \omega > -\, \omega_1, \\ &> 0 \qquad\qquad\qquad \omega < -\, \omega_1. \end{aligned}$$

Putting $a_e = -a_i = a$, we have

$$\begin{aligned} \text{Re } D\,(\pm\,\omega_1,\ \psi) &= 2e^2 P \iint_{E_0}^{\infty} d\lambda\; dE\; \frac{\nu_0^{-1} E^{5/2} (\partial F/\partial K)_{\mu J}}{E^2 - (\omega_1/ma)^2} \\ &= 2e^2 \int d\lambda \int_{E_D}^{\infty} dE\; \frac{\nu_0^{-1} E^{1/2} \omega_d^{\,2} (\partial F/\partial K)_{\mu J}}{\omega_d^{\,2} - (\omega_1/m)^2} < 0, \end{aligned}$$

$$\text{Re } D\ (\omega \to 0) = 2e^2 \iint d\mu\ dJ\ (\partial F/\partial K)_{\mu J} > 0,$$

$$\text{Re } D\ (\omega \to \pm\, m a_- E_0) \to +\infty, \quad \text{as } (\partial F/\partial K)_{\mu J} > 0 \text{ for } \omega_d = aE_0$$

$$\text{Re } D\ (\omega \to \pm\infty) = -\frac{2m^2}{\omega^2}\int d\lambda \int dE\ v_0^{-1} E^{1/2} \omega_d\ (\partial F/\partial K)_{\mu J}$$

$$= -\frac{2m^2c^2}{\omega^2 e^2}\int d\mu\ dJ \left(\frac{\partial F}{\partial K}\right)_{\mu J}\left(\frac{\partial K}{\partial \psi}\right)^2_{\mu J}.$$

There are two subcases:

Subcase 1a. The interchange stability criterion is satisfied:

$$\iint d\mu\ dJ \left(\frac{\partial F}{\partial K}\right)_{\mu J}\left(\frac{\partial K}{\partial \psi}\right)^2_{\mu J} < 0$$

or

$$\text{Re } D\ (\omega \to \pm\infty) > 0.$$

This is possible because of (G-1).

The resulting Nyquist diagram is shown in Fig. 6a. The change in the argument of D is 4π and there is one unstable root. We will call this a "drift mode," as the plasma is energetically stable against interchange.

Subcase 1b. The interchange stability criterion is not satisfied:

$$\text{Re } D\ (\omega \to \pm\infty) = \frac{2m^2c^2}{e^2\omega^2}\iint d\mu\ dJ \left(\frac{\partial F}{\partial K}\right)_{\mu J}\left(\frac{\partial K}{\partial \psi}\right)_{\mu J} < 0.$$

The Nyquist diagram is shown in Fig. 6-b. The change in argument of D is 2π and there is no unstable root.

Situation 2

Suppose $(\partial F/\partial K)_{\mu J}$ changes sign in the following way:

$$(\partial F/\partial K)_{\mu J} < 0 \quad \text{for} \quad |\omega_d^{\pm}| < \omega_1/m, \qquad \text{(G-3)}$$
$$> 0 \qquad\qquad |\omega_d^{\pm}| > \omega_1/m,$$

and ω_1 is such that

$$\iint d\mu \; dJ \; (\partial F/\partial K)_{\mu J} < 0. \qquad \text{(G-4)}$$

Then from (35) we have

$$\begin{aligned} \text{Im } D &> 0 \quad \text{for} \quad \omega > \omega_1 > ma_0E_0, \\ &< 0 \qquad\quad \omega_1 > \omega > ma_0E_0, \\ &= 0 \qquad\quad ma_0E_0 > \omega > - ma_0E_0, \\ &> 0 \qquad\quad - ma_0E_0 > \omega > - \omega_1, \\ &< 0 \qquad\quad \omega < - \omega_1. \end{aligned}$$

From (36) and (G-3) and (G-4) we have

$$\text{Re } D \;(\pm\, \omega_1, \psi) = 2e^2 \iint d\mu \; dJ \; \frac{\omega_d^{\,2}(\partial F/\partial K)_{\mu J}}{\omega_d^{\,2} - (\omega_1/m)^2} > 0,$$

$$\text{Re } D \;(\omega \to 0) = 2e^2 \iint d\mu \; dJ \; (\partial F/\partial K)_{\mu J} < 0,$$

$$\text{Re } D \;(\omega \to \pm\, ma_0E_0) \to -\infty, \qquad \text{as } (\partial F/\partial K)_{\mu J} < 0$$
$$\text{at } \omega_d = a_0E_0 < \omega_1/m,$$

$$\text{Re } D \;(\omega \to \pm\, \infty) = - \frac{2m^2c^2}{\omega^2e^2} \iint d\mu \; dJ \left(\frac{\partial F}{\partial K}\right)_{\mu J} \left(\frac{\partial F}{\partial K}\right)^2_{\mu J}$$

There are two possibilities:

Subcase 2a.

$$\mathrm{Re}\, D\, (\omega \to \pm \infty) > 0.$$

The stability criterion for the interchange is still satisfied. The Nyquist diagram (6-c) shows there is no unstable root.

Subcase 2b.

$$\mathrm{Re}\, D\, (\omega \to \pm \infty) < 0.$$

The stability criterion for interchange is violated and the Nyquist diagram (6-d) shows there is one unstable mode.

H. Taylor's Criteria for Interchange Stability

For a low-β plasma, the criteria for the interchange stability have been derived from energy considerations (Taylor 1963) under the assumptions that

(1) particles are tied to the field line, i.e., particles initially in a given flux tube remain in that flux tube after interchange. This results from the usual hydromagnetic "frozen-in" condition $\underset{\sim}{E} + \frac{\dot{u}}{c} \times \underset{\sim}{B} = 0$, the validity of which requires that the particles' drifts can be neglected in the time scale of interest (hydromagnetic time scale ≈ average bounce period), and that $cE_\perp/B \gg v_{drift}$.

(2) J be conserved in the process. For J to be conserved the time scale of the variation must be sufficiently long compared with the bounce period, and $cE_\perp/B \ll v_{th}$, where v_{th} is the thermal velocity.

To satisfy both conditions, we would required that $cE_\perp/B \approx \epsilon^{1/2} v_{th}$, i.e., that $E_\perp$ be so large compared with the drift velocity that the frozen-in condition is still valid but so small compared with the thermal velocity that J is still conserved.

With these assumptions, we can find the variation in energy resulting from the interchange of particles on a flux tube $(\psi_1\varphi_1)$ with those on another flux tube $(\psi_2\varphi_2)$

$$\begin{aligned}\Delta W &= \iint d\mu\, dJ \Big\{ \left[F(\mu J\psi_1\varphi_1)K(\mu J\psi_1\varphi_1) + F(\mu J\psi_2\varphi_2)K(\mu J\psi_2\varphi_2)\right] \\ &\qquad - \left[F(\mu J\psi_1\varphi_1)K(\mu J\psi_2\varphi_2) + F(\mu J\psi_2\varphi_2)K(\mu J\psi_1\varphi_1)\right]\Big\} \\ &= -\iint d\mu\, dJ \Big\{\left[F(\mu J\psi_2\varphi_2) - F(\mu J\psi_1\varphi_1)\right]\left[K(\mu J\psi_2\varphi_2) - K(\mu J\psi_1\varphi_1)\right]\Big\} .\end{aligned}$$

For infinitesimal variations, we have

$$\delta^2 W = -\iint d\mu\ dJ \left[\frac{\partial F}{\partial \psi}\bigg|_{\mu J}\delta\psi + \frac{\partial F}{\partial \varphi}\bigg|_{\mu J}\delta\varphi\right]\left[\frac{\partial K}{\partial \psi}\bigg|_{\mu J}\delta\psi + \frac{\partial K}{\partial \varphi}\bigg|_{\mu J}\delta\varphi\right]$$

$$= -\iint d\mu\ dJ \left[\frac{\partial K}{\partial \psi}\bigg|_{\mu J} d\psi + \frac{\partial K}{\partial \varphi}\bigg|_{\mu J} d\varphi\right]^2 \frac{\partial F}{\partial K}\bigg|_{\mu K}. \qquad \text{(H-1)}$$

The necessary and sufficient condition for stability is

$$\delta^2 W > 0.$$

Thus for an axisymmetric system, the necessary and sufficient condition for stability is

$$\iint d\mu\ dJ \left(\frac{\partial K}{\partial \psi}\bigg|_{\mu J}\right)^2 \frac{\partial F}{\partial K}\bigg|_{\mu J} < 0. \qquad \text{(H-2)}$$

Since

$$\left(\frac{\partial F_0}{\partial \psi}\right)_{\mu J} = \left(\frac{\partial F_0}{\partial K}\right)_{\mu J}\left(\frac{\partial K}{\partial \psi}\right)_{\mu J},$$

(H-2) can be rewritten

$$\iint d\mu\ dJ \left(\frac{\partial K}{\partial \psi}\right)_{\mu J}\left(\frac{\partial F_0}{\partial \psi}\right)_{\mu J} < 0. \qquad \text{(H-3)}$$

I. Condition for the Vanishing of η in Eq. (62)

The quasi-neutrality condition is

$$\sum_j e_j \iint d\mu \; dJ \; F_0{}^j \frac{v_b}{v_\|} = 0. \tag{I-1}$$

Integrating over the line of force, we have

$$\sum_j e_j \iint d\mu \; dJ \; F_0{}^j(\mu J \psi) = 0.$$

Differentiating with respect to ψ and changing to variables λ, E gives

$$\sum_j e_j \iint d\mu \; dJ \left.\frac{\partial F_0{}^j}{\partial \psi}\right|_{\mu J} = \sum_j e_j \iint d\lambda \; dE \; v_0^{-1}(\lambda) E^{1/2} \left.\frac{\partial F_0{}^j}{\partial \psi}\right|_{\mu J} = 0. \tag{I-2}$$

In terms of λ, E, the average potential $\langle \Phi_m \rangle$ is a function of λ and only:

$$\langle \Phi_m \rangle \equiv \frac{1}{E^{1/2}} \oint \frac{dX}{B(1 - \lambda B)^{1/2}} \, v_0(\lambda, \psi) E^{1/2} = v_0(\lambda, \psi) \oint \frac{dX}{B(1 - \lambda B)} .$$

For $F_0{}^j$ factorizable in its dependence on λ and E, i.e., $F_0{}^j(\lambda, E, \psi) = g^j(\lambda) h(E, \psi)$, and if the two species have the same $g(\lambda)$, $g^e = g^i$, Eq. (I-2) becomes

$$\sum_j e_j \int dE \; E^{1/2} \left.\frac{\partial h^j}{\partial \psi}\right|_{\mu J} = 0. \tag{I-3}$$

From (62)

$$\eta \equiv c \sum_j e_j \iint d\mu \; dJ \left(\frac{\partial F_0{}^j}{\partial \psi}\right)_{\mu J} |\langle \Phi_m \rangle|^2$$

$$= c\int d\lambda\ g(\lambda)\nu_0^{-1}(\lambda)\ \langle\Phi_m\rangle(\lambda)^2 \sum_j dE\ E^{1/2} \frac{\partial h^j}{\partial\psi}\bigg|_{\mu J}$$

$$= 0,$$

because of (I-3).

J. Proof of the Minimal Principle (62)

Equation (62) can be put into the form

$$\int \frac{dX}{B}\, \Phi_m^*(X) \sum_j \iint d\mu\, dJ\, \frac{\nu_b}{v_\parallel(X)} \left\{ - c^2 \langle \Phi_m \rangle \left[\frac{\partial F_0^j}{\partial \psi}\bigg|_{\mu J} \frac{\partial K}{\partial \psi}\bigg|_{\mu J} - \frac{\partial F_0^j}{\partial K}\bigg|_{\mu\psi} \frac{\partial K}{\partial \psi}\bigg|_{\mu J}^2 \right] \frac{m^2}{\omega^2} + e_j^{\,2} \Phi_m(X) \cdot \frac{\partial F_0^j}{\partial K}\bigg|_{\mu\psi} \right\} = 0.$$

Varying this equation with respect to Φ_m^* and using,

$$\frac{\partial F}{\partial \psi}\bigg|_{\mu J} = \frac{\partial F}{\partial \psi}\bigg|_{\mu K} + \frac{\partial F}{\partial K}\bigg|_{\mu\psi} \frac{\partial K}{\partial \psi}\bigg|_{\mu J}$$

we obtain the eigenvalue equation

$$\mathcal{L}(X, X')\Phi_m(X') = \left(\frac{\omega}{m}\right)^2 \Phi_m(X), \tag{J-1}$$

where

$$\mathcal{L}(X, X') = \frac{\sum_j c^2 \int \frac{dX'}{B(X')} \frac{1}{v\,(X')} \iint d\mu\, dJ\, \frac{\nu_b^{\,2}}{v_\parallel(X)} \frac{\partial F_0^j}{\partial \psi}\Big|_{\mu K} \frac{\partial K}{\partial \psi}\Big|_{\mu J}}{\sum_j e_j^{\,2} \iint du\, dJ\, \frac{\nu_b}{v_\parallel(X)} \left(\frac{\partial F_0^j}{\partial K}\right)_{\mu\psi}} \tag{J-2}$$

is self-adjoint, as ω^2 is real in Eq. (62). Suppose the eigenfunctions $Z_n(X)$ of $\mathcal{L}$,

$$\mathcal{L}(X, X')Z_n(X') = \Omega_n^{\,2} Z_n(X), \tag{J-3}$$

form a complete orthonormal set. Then we can expand $\Phi_m(X)$

$$\Phi_m(X) = \sum_n a_n Z_n(X), \qquad \text{(J-4)}$$

where $\sum |a_n|^2 = 1$ because Φ_m is normalized to unity.

Substituting (J-4) into (J-1) gives

$$\mathcal{L}(X, X')\Phi_m(X') = \sum_n a_n \Omega_n{}^2 Z_n(X)$$

$$\int \frac{dX}{B} \Phi_m^*(X) \quad (X, X')\Phi_m(X') = \sum_n |a_n|^2 \Omega_n{}^2$$

$$= \Omega_1{}^2 \sum_n |a_n|^2 + \sum_n (\Omega_n{}^2 - \Omega_1{}^2)|a_n|^2$$

$$= \Omega_1{}^2 + \sum_n (\Omega_n{}^2 - \Omega_1{}^2)|a_n|^2,$$

where $\Omega_1{}^2$ is the smallest eigenvalue.

Because $$\Omega_n{}^2 - \Omega_1{}^2 \geq 0,$$

we have

$$\Omega_1{}^2 \leq \int \frac{dX}{B} \Phi_m^*(X) \mathcal{L}(X, X')\Phi_m(X').$$

Therefore (62) is a minimizing expression.

K. The Equivalence of the Extremization of D in (70) and the Extremization of $dn/d\psi$ in (71)

Writing (70) symbolically as

$$D = g(\Phi) + n'(\psi)h(\Phi) = 0 \tag{K-1}$$

and extremizing D: $\delta D = \delta g + n'(\psi)\delta h = 0$, we obtain

$$n' = - \delta g/\delta h. \tag{K-2}$$

From (K-1) and (K-2), the extremization of D corresponds to

$$- \frac{\delta g}{\delta h} = - \frac{g}{h} . \tag{K-3}$$

Alternatively, we can first solve (K-1)

$$n' = - g(\Phi)/h(\Phi). \tag{K-4}$$

Extremizing n' in (K-4) gives

$$\delta n' = 0 = - \frac{h\delta g - g\delta h}{h^2} ,$$

which is the same as (K-3).

FOOTNOTES

1. We shall use the terms "radiation belt" and "Van Allen belt" synonymously, meaning the energetic charged particles trapped in the earth's magnetic field. By energetic belt, we mean the trapped particles with energy $\geqslant$ 1 keV.

2. The energy flux needed for an intense auroral excitation during an auroral substorm is typically $\approx 5 \times 10^{18}$ ergs/sec. In the meantime, the rate of energy dissipation in the ionosphere due to ionospheric current is about 2×10^{18} ergs/sec. The lifetime of a substorm is of the order of an hour, $\approx 10^{4}$ sec. Thus the total energy input into the ionosphere for a substorm is 10^{23} ergs. The total kinetic energy of the trapped particles is estimated to be of the order of 10^{23} ergs (Van Allen, 1966). Therefore the trapped particles cannot supply the energy needed for an auroral substorm without fresh enhancement.

The total kinetic energy E_P of the trapped particles is related to the decrease of the geomagnetic field ΔB on earth (Sckophe, 1966) according to

$$\frac{\Delta B}{B_0} = -\frac{2}{3}\frac{E_P}{E_M},$$

where B_0 is the magnetic field on earth, ≈ 0.3 gauss, E_M is the total magnetic field energy, $\approx 10^{25}$ ergs. For a main phase decrease of $100\ \gamma = 10^{-3}$ gauss, total particle energy E_P must be of the order of 10^{23} ergs. Since the substorm occurs intermittently during the main phase, the trapped particle energy and the precipitated energy must increase simultaneously.

3. The symbols μ, J, L, F_0 are defined in Sec. II. The relation between f_1 and F_0 is discussed in Appendix B-2.

4. By collisionless plasma, we mean that the mean free time is much greater than the characteristic time scale under consideration, and the mean free path is much greater than the characteristic dimension of the system.

The electron density (for all energies) outside the plasmapause is of the order of one particle per cm^3, from the whistler measurement (Carpenter, 1964). The particle density of the ring current belt (Frank, 1966) is also about 1 per cm^3. Thus the cold plasma (1 eV) density outside the plasmapause can <u>at most</u> be of the same order as the ring current belt density. (If the ring current belt were injected into the magnetosphere, then the electric field associated with the injection would sweep away any thermal particles, and one would not expect to have any thermal particles at all.) The temperature of the plasma outside the knee is effectively that of the ring current belt, which is of the order of 1 keV. The mean free path can then be estimated (Alfven and Falthammar, 1963) as

$$\lambda \approx 10^4 \frac{T_e^2}{n_e} \approx 10^{13} \text{ km} ,$$

which is much greater than the characteristic length of the system,

$$L \approx R_e \approx 10^4 \text{ km} .$$

The collision frequency $\nu_c \approx 10^{-8} \sec^{-1}$ is much less than the characteristic frequency, $\approx 10^{-4} \sec^{-1}$. The plasma is therefore collisionless.

5. The flow energy of the solar wind is about 1 keV per particle. When the flow is stopped (for instance, at the bow shock of the earth), the flow energy is converted into the thermal energy of the particles. Thus the trapped particles of the solar origin have at least 1 keV.

6. In general $\omega_d = a(\lambda, \psi)E + b(\lambda, \psi)$ (Appendix F). When the equilibrium potential $\bar{\phi}_0(\psi, \chi)$ is independent of ψ, $b = 0$. This corresponds to the assumption of no radial electric field in equilibrium, i.e., the effect of the earth's rotation is neglected.

7. The energy of the μJ-conserving perturbation is derived by Rutherford and Trieman (1968):

$$W = W_1 + W_2,$$

where

$$W_1 = -\frac{1}{2} \sum_j \iint d\mu \, dJ \, d\psi \, d\phi (\delta F)^2 / \left. \frac{\partial F_0}{\partial K} \right|_{\mu J} ,$$

$$W_2 = - \sum_j e_j^2 \iint d\mu \, dJ \, d\psi \, d\phi \left[\langle \phi^2 \rangle - \langle \phi \rangle^2 \right] \left. \frac{\partial F_0}{\partial K} \right|_{\mu\psi} .$$

For the system to be stable, $W_1 + W_2 > 0$. Thus the stable ring current belt support a positive energy wave.

8. From the analysis by Hess (1968), the distribution function for the energetic protons is factorizable: $f_1(\mu J\psi) = g(\mu J)h(\psi)$. From Appendix B-2,

$$F(\mu,J,\psi) = g(\mu,J)h(\psi)/(2\mu B/M)^{1/2} = g(\mu J)\mathcal{H}(\psi).$$

Then $(\partial F/\partial\psi)_{\mu J} = 0$ for all μJ at some ψ requires only that $d\mathcal{H}/d\psi = 0$ at ψ.

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