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BORN EXPANSION TO ALL ORDERS FOR THE HETEROTIC STRING *

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Abstract

A perturbation expansion to all orders in loops is developed for the $E_8 \otimes E_8$ heterotic string. A light-cone gauge Lagrangian is presented and applied within the functional-integral formalism of Mandelstam to compute scattering amplitudes. Unitarity is manifest and Lorentz invariance can easily be established based on previous proofs for the superstring and bosonic string within the context of this formalism. It is shown that the determinant arising from the functional integration associated with the compactified degrees is analytic. This, along with its modular properties, gives a unique specification of it up to an overall moduli-independent factor.

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INTRODUCTION

The heterotic string [1] is a hybrid of a 26-dimensional left-moving (lm) bosonic string and a 10-dimensional right-moving (rm) superstring. Sixteen of the lm bosonic string degrees propagate on a torus whose directions are specified by the roots of the $E_8 \otimes E_8$ Lie Group, "the $E_8 \otimes E_8$ lattice". All the other degrees of this string live in a $9+1$ dimensional flat spacetime. The spectrum of the theory has a massless spin-two particle and the theory is anomaly free. These properties make this theory a hopeful contender as a theory of gravity interacting with matter.

Despite the attractive possibilities that this theory has, calculations of scattering amplitudes have been restricted to the one loop level which in fact were obtained by Gross *et al.*, [2] in their seminal work. Although in principle their operator method should work to all orders in loops, the task if formidable and has not been done.

In this paper we present a Lagrangian approach which is equivalent to the theory of Gross *et al.* [1] at least to one-loop order. Within the path integral formalism we then compute scattering amplitudes to all orders in loops. The major breakthrough that has opened the way for this calculation was made a few years back by Mandelstam [3], who obtained the complete perturbation expansion for the superstring. His important observation was that the determinant of the superstring Laplacian operator, which is needed in order to compute the path integral in the lightcone gauge, can be deduced solely by its analytic and modular properties.

For our purposes this only gives the analytic portions of the determinant for the superstring sector of the heterotic string. This still leaves the problem of determining the nonanalytic factors of the determinant, which in particular mix the left moving bosonic sector with the right moving superstring sector. We also have to adopt a suitable Lagrangian for the compactified degrees so that they can be treated within a light-cone functional integral formalism.

In section two of the paper we introduce a light cone Lagrangian for the heterotic string. Then in section three the computation of the path integral is carried out. We obtain, as our end result, the formula for the scattering amplitude in terms of known expressions such as prime forms and Green's functions. There are also three appendices which have been resersved for some technical matters which are indirectly related to our derivation.

SECTION 2

We will evaluate the following functional integral in imaginary time $t = i\tau$, where τ is Minkowski time, for the n-loop contribution to the scattering amplitude.

$$A_n = (const.) \int D[Teich]D[External]DXD\mathbf{X}e^{-(S_c+S_{nc})}\Psi_i(X,\mathbf{X})\Psi_f(X,\mathbf{X}) \quad (2.1)$$

In (2.1) the total action S is written as a sum of the compactified S_c and noncompactified, S_{nc}, degrees.

$$S = S_c + S_{nc}$$

where

$$S_c = \frac{1}{4\pi}\int d\rho d\bar{\rho}\partial_\sigma X^I\partial_\rho X^I \quad (2.2)$$

$$S_{nc} = \frac{1}{4\pi}\int d\rho d\bar{\rho}d\psi\partial_{\bar{\rho}}\mathbf{X}^I D_\psi\mathbf{X}^I \quad (2.3)$$

$$\mathbf{X}^I(\rho,\bar{\rho},\psi) = X^I(\rho,\bar{\rho}) + \psi S^I(\rho,\bar{\rho}) \quad (2.4)$$

$$\partial_{\bar{\rho}} = \frac{\partial_t + i\partial_\sigma}{2} \quad (2.5)$$

$$\partial_\rho = \frac{\partial_t - i\partial_\sigma}{2} \quad (2.6)$$

$$D_\psi = \frac{\partial}{\partial\psi} + \psi\partial_\rho \quad (2.7)$$

and $\Psi_i(\Psi_f)$ are the initial (final) wave functions of the external states. X^I and S^I in (2.4) are the I-th component of the bosonic and fermionic fields respectively. For convenience later, we also will use the following notation for the quadratic operators appearing in S.

$$\Delta_c \equiv \partial_\sigma\partial_\rho$$

$$\Delta_{nc} \equiv \partial_{\bar{\rho}}D_\psi \quad (2.8)$$

S_c is comprised of sixteen left-moving bosonic degrees of freedom which are compactified on an $E_8 \odot E_8$ lattice. S_{nc} is comprised of eight left-moving degrees of a bosonic string and eight right-moving degrees of a superstring all of which propogate in flat spacetime.

The Lagrangian in S_c was obtained from the following motivations. It gives the desired classical equations of motion for the heterotic string, as defined in the original operator formalism (1), except that it also allows σ-independent solutions. In addition, using the standard canonical prescription, it gives the desired commutation relations. Hence, upon quantization, the fields are promoted to operators. Observe that the Lagrangian is invariant to changes in X which depend only on t. We regard this as a gauge invariance. To fix the gauge, the field configurations will be restricted by the conditions,

$$\oint_{\sigma_j} d\sigma X^I(\sigma,t) = 0 \quad (2.9)$$

where the set $\{\sigma_j\}$ contains one path which rounds each string in the diagram.

In the case of S_{nc} the Lagrangian is written in terms of the superfield $\mathbf{X}$. Observe that unlike the case of the superstring, our superspace does not have a fermionic partner $\bar{\psi}$ to $\bar{\rho}$.

In our problem we define a superconformal transformation as a regular superconformal transformation on ψ and ρ but only a conformal transformation on $\bar{\rho}$. It is important to realize the point here, that we do not have to treat $\bar{\rho}$ as the complex conjugate of ρ. To understand this statement, suppose first, for orientational purposes, that $\bar{\rho}$, ρ, and ψ, define the geometrical point with $\bar{\rho}$ as the complex conjugate to ρ. Now suppose we choose to refer to (ρ, ψ) by another set of coordinates $(\hat{\rho}, \hat{\psi})$ but do not affect our reference to $\bar{\rho}$. Then, for any field configuration $\mathbf{X}(\rho, \psi)$, although its functional form in terms of $(\hat{\rho}, \hat{\psi})$ will look different,

$$\mathbf{X}(\rho,\psi) = \mathbf{X}(\rho(\hat{\rho},\hat{\psi}),\psi(\hat{\rho},\hat{\psi})) = \tilde{\mathbf{X}}(\hat{\rho},\hat{\psi}) \quad (2.10)$$

the value of the action S_{nc}, and so its weighted amplitude, remains the same.

This important property allows us to apply an analysis similar to the one in Berkovits' paper [4] to the right-moving sector in S_{nc} and immediately conclude that the present superfield formalism is equivalent to the original component formalism of Mandelstam [5]. The crucial step in such a derivation would be that one would need to make a superconformal transformation at the joining points to another set of coordinates which are defined in Berkovits' paper. Hence, having established superconformal invariance for our Lagrangian, our proof of equivalence follows directly from Berkovits' analysis.

To establish the equivalence of the superfield formalism to the component formalism is necessary since only in the latter has the proof of Lorentz invariance been established. The advantage of the superfield formalism is that the resulting equations are more convenient to handle since there is no explicit appearance of the nontrivial interaction vertex operator which is needed in the component form. For further motivation the reader is encouraged to examine this vertex operator in Mandelstam's original paper [5].

SECTION 3

In this section the amplitude is explicitly evaluated for the computation of the n-loop contribution to the scattering amplitude of N tachyons. Amplitudes involving any other types of external particles can in principle be obtained from the present one by factorization.

The first step in the calculation is to decompose the field for the compactified degrees as

$$X^I = X_p^I + F_{n-loop}^I \tag{3.1}$$

where X_p is periodic on all strings and F_{n-loop} changes by n_{ji} times the circumference of the i-th radii when rounding the j-th a-cycle. K^I is defined in terms of the radii, $\{R_i, i = 1, 16\}$, of the $E_8 \times E_8$ lattice as

$$K_j^I = \frac{1}{\sqrt{2}} \sum_i n_{ji} e_i^I R_i \tag{3.2}$$

where e_i^I is the i-th component of the projection vector for the i-th radii.

In contrast to its properties around a-cycles, F_{n-loop} is periodic around all b-cycles. The reason is so as to not overcount equivalent surfaces; our gauge constraint requires this.

We can write an expression for F_{n-loop} most simply in a mixed representation which uses both string diagram coordinates ρ and complex plane coordinates $\bar{z}$ as the sum of three terms.

$$F_{n-loop}^I = F_1^I(\bar{\rho}) + F_2^I(t) + F_3^I(t) \tag{3.3}$$

where

$$\begin{aligned} F_1^I(\bar{\rho}) &\equiv \sum_{j=1}^{n} 2\pi K_j^I \left[\frac{v_j(\bar{z}) - v_j(\bar{z}_0)}{2\pi i}\right] \\ F_2^I(t) &\equiv -\sum_{j=1}^{n} 2\pi K_j^I \sum_{i=1}^{n} \bar{\tau}_{ij} g_j(t) \\ F_3^I(t) &\equiv \sum_{i=1}^{n} C_i^I \theta(t - t_{iL})\theta(t_{iR} - t) \\ &\textit{with} \\ C_i^I &\equiv \frac{-1}{\alpha_i} \oint_{a_i-cycle} (F_1^I + F_2^I) d\sigma \\ \alpha_i &\equiv \oint_{a_i-cycle} d\sigma \end{aligned} \tag{3.4}$$

and $t_{iL}(t_{iR})$ is the left (right) t-coordinate of the joining point for the i-th loop.

The function $F_3(t)$ is needed in order that the gauge fixing condition (2.10) is satisfied for F. It is necessary that F satisfy this condition independently since the other term, X_p, already satisfies it.

The functions v_j, j=1...n, are the integrals of the one forms and $\bar{\tau}$ the period matrix on the surface. Explicit expressions for these functions and the transformation between the string diagram and the complex plane can be found for the Schottky parameterization in Ref.[6]. In (3.4) the function g_j(t) satisfies $\Delta_{nc} g_j = 0$ and it changes by 1 when rounding the j-th b-cycle. Such a function exists in all cases although its representation can be a bit cumbersome to describe.

One possible general form for such a function is,

$$g_j(t) = \begin{cases} \frac{t - t_{jL}}{t_{jR} - t_{jL}} \text{For} & \sigma_j < \sigma < \sigma_{mj} \\ & t_{jL} < t < t_{jR} \\ 0 & \text{otherwise} \end{cases} \tag{3.5}$$

where by continuation, g is defined elswehere. In particular there is no jump at t_{jR}. On the z-plane such a continuation can be seen explicitly by using, as an example, the Schottky representation. In (3.5) t_{jL} and t_{jR} are chosen so that the next boundary above σ_j is at the same value of σ, denoted σ_{mj} and so that this boundary is nonterminating between t_{jL} and t_{jR}. As an example consider the diagram in Figure 1 for the cut (loop) which is labelled 2 where in this case $\sigma_m = \sigma_3$.

Using the above decomposition of X, we now substitute into S_c. This leaves us to consider,

$$I_c \equiv \int d\sigma dt \partial_\sigma X^I \partial_\rho X^I = \int d\sigma dt \partial_\sigma \left(X_p^I + F_{n-loop}^I\right) \partial_\rho \left(X_p^I + F_{n-loop}^I\right) \tag{3.6}$$

The right-hand side can be greatly simplified by exploiting the coordinate dependences of F_1, F_2 and F_3. From this we obtain.

$$I_c = \int d\sigma dt \partial_\sigma X_p^I \partial_\rho X_p^I - i \int d\sigma dt \partial_\sigma F_1^I(\bar{\rho}) \partial_t (F_2^I + F_3^I) \tag{3.7}$$

We now observe that the term involving F_3 vanishes as can be seen by noting that,

$$\begin{gathered} \partial_t F_3^I \;\alpha\; \sum_i C_i^I \{\delta(t - t_{iL}) - \delta(t - t_{iR})\} \\ \text{and} \oint_{a_i} d\sigma \partial_\sigma F_1(\bar{\rho})|_{t_{iL}} = \oint_{a_i} d\sigma \partial_\sigma F_1(\bar{\rho})|_{t_{iR}} \end{gathered} \tag{3.8}$$

We comment that the final answer should not have depended on F_3 since it is needed due to our specific gauge choice.

Substituting for X^I in S_c, and preforming the integral over the fields and summation over the $E_8 \odot E_8$ lattice we obtain the following formula for the amplitude:

$$A_n = (\text{const.}) \int \prod_p d\tau_p \prod_q d\alpha_q \prod_r d\theta_r \prod_s d\beta_s d^{2n-2}\xi d^N Q d^N \bar{Q}$$
$$\times M_c^{-8} M_{nc}^{-4} L(\bar{\tau}, \vec{v}, 0, 0)$$
$$\exp\left[-\frac{1}{2}\sum_{\alpha>\beta} P_\alpha \cdot P_\beta \;\; N_{nc}(Q_\alpha, \bar{Q}_\alpha; Q_\beta \bar{Q}_\beta)\right] \tag{3.9}$$

where

$$\begin{aligned} &\tau_a;\, a = 1\ldots 2n-3 \\ &\theta_a;\, a = 1\ldots 2n-3 \\ &\alpha_a;\, a = 1\ldots n \\ &\beta_a;\, a = 1\ldots n \end{aligned} \tag{3.10}$$

are the set of real string diagram coordinates,

$$\xi_a;\, a = 1\ldots 2n-2$$

are the odd moduli parameters,

$$Q_a;\, a = 1\ldots N$$

are the joining points of the external states. M_c and M_{nc} are the determinants of the regularized operators: $M_c = \det \Delta_c$, $M_{nc} = \det \Delta_{nc}$ and the Green's function N_{nc} satisfies

$$\begin{aligned} \partial_{\bar{\rho}} D_\psi N_{nc}(\bar{\rho}, \rho, \psi, \bar{\rho}', \rho', \psi') = {}& 2\pi i(\psi - \psi')\delta(\rho - \rho')\delta(\bar{\rho} - \bar{\rho}') \\ & + F_{nc}(\bar{\rho}, \rho, \psi) \end{aligned} \tag{3.11}$$

We also define the Green's function. N_c associated with the compactified degrees which satisfies

$$\partial_\sigma \partial_\rho N_c(\rho, \bar{\rho}') = \pi\delta(\rho - \rho')\delta(\bar{\rho} - \bar{\rho}') + F_c(\rho) \tag{3.12}$$

Although N_c does not arise explicitly in the amplitude for Tachyons presently being computed, it will be needed later in our derivation when we compute the measure factor. In the above definitions the functions F_c and F_{nc} arise because the quadratic operators Δ_c and Δ_{nc} have zero modes and therefore have illdefined inverses.

$L(\bar{\tau}, \bar{\nu}, z_i, Q_i)$ in (3.7) is the lattice sum over the $E_8 \odot E_8$ lattice. We define it more generally as,

$$L(\tau, \vec{v}, z_i, Q_i) = \sum_{K \epsilon E_8 \otimes E_8} \exp\left[i\pi \sum_{i,j=1}^{n} K_i^I \tau_{ij} K_j^I + \sum_{r=1}^{N} \frac{Q_r^I}{\pi \alpha_r} \sum_{i=1}^{n} K_i^I v_i(z_r)\right]. \tag{3.13}$$

Where, for our case of Tachyon scattering, we set all the Q_r equal to zero.

The Modular properties of $L(\bar{\tau}, \bar{\nu}, z_i, Q_i)$ can be deduced by using the generators of Sp(2,Z) which are given in Mumford (7), page (189). Under the Modular transformation

$$\tau \to (A\tau + B)(C\tau + D)^{-1} \tag{3.14}$$

we find

$$\begin{aligned} & L\left((A\tau + B)(C\tau + D)^{-1}, (C\tau + D)^{-1}\vec{v}, z_i, Q_i\right) \\ & = \det(C\tau + D)^{1/2} \exp\left[i\pi \vec{v}^t (C\tau + D)^{-1} \vec{\tau}\right] L(\tau, \vec{v}, z_i, Q_i) \end{aligned} \tag{3.15}$$

We will now calculate the explicit expressions for the Green's functions and measure factors for the compactified and non-compactified degrees in that order.

The specifications for N_c are that it satisfies equation (3.10) and that it be periodic around a- and b-cycles. Such a function can be expressed as,

$$N_c = N^a + N^{(1)} + N^{(2)} \tag{3.16}$$

where

$$N^{(1)} = -\left[\theta(t - t')N^a(\bar{\rho}, \bar{\rho}_L) + \theta(t' - t)N^a(\bar{\rho}, \bar{\rho}_R)\right]$$

with

$$\begin{aligned} \bar{\rho}_{L(R)} &= -i\sigma_{L(R)} + t_{L(R)} \\ \sigma_L &= \sigma_R = \sigma' \\ t_L &= -\infty \\ t_R &= +\infty \end{aligned}$$

and

$$N^{(2)} = \oint d\sigma'' \frac{\partial N^a(\sigma, t, \sigma'', t')}{\partial t'} \left[N^a(\sigma'', t'; \bar{\rho}_L) - N^a(\sigma'', t'; \bar{\rho}_R) + g_{\sigma'}(\sigma'')\right] \tag{3.17}$$

and N^a is the antianalytic portion of the usual n–loop bosonic Green's function. Again, as convenience mandates, a mixture of $\bar{\rho}$ and z coordinates are used.

The function N^a alone produces the desired δ-function singularity at $\bar{\rho} = \bar{\rho}'$, however it has two undesirable features which are corrected by adding N^1 and N^2 to it. The problems with N^a is first that it produces a logarithmic jump along a line starting at $\bar{\rho} = \bar{\rho}'$ on the string diagram and second that it is not periodic around b-cycles. Both these problems are almost fully corrected by adding $N^{(1)}$ to it. It should be noted that both these problems occur with respect to either the $\bar{\rho}$ or $\bar{\rho}'$ coordinate. In the following we will carry out our discussion in terms of the $\bar{\rho}$ coordinate.

For the sum $N^a + N^{(1)}$, it is possible to define the logarithmic jump on the $\bar{\rho}$ plane so that it never lies on the string diagram. Furthermore, the sum of these two quantities, although still not periodic when $\bar{\rho}$ rounds any b-cycle, will change by a function independent of $\bar{\rho}'$. This in fact is sufficient for the defining properties for this Green's function, since we can always define these jumps at each loop to be independent of σ. Stated differently, we can rid ourselves of these jumps by adding to the Green's function (3.16) a σ independent function which jumps by appropriate compensating values at each loop to offset the jumps which are present. By adding such a function to equation (3.16), (3.12) would still be satisfied.

However, $N^a + N^{(1)}$ is not the desired Green's function because this function has a jump at $t = t'$. In particular, as you cross from $t = t' - \varepsilon$ to $t = t' + \epsilon$, there is a discontinuity arising from $N^{(1)}$. It is the purpose of $N^{(2)}$ to remove this jump so that N_c is smoothly defined on the entire string diagram as is necessary to satisfy equation (3.12). By examining $N^{(2)}$ one sees that the first two terms in the bracket $N^a(\sigma'', t'; \bar{\rho}_L) - N^a(\sigma'', t'; \bar{\rho}_R)$, correct the discontinuity produced by $N^{(1)}$. However the functions $N^a(\sigma'', t'; \bar{\rho}_L)$ and $N^a(\sigma'', t'; \bar{\rho}_R)$, in $N^{(2)}$ are not defined on the same branch as the ones in $N^{(1)}$. In our case the branch cut for $N^a(\sigma'', t'; \bar{\rho}_L) - N^a(\sigma'', t'; \bar{\rho}_R)$ in N^a runs between $\bar{\rho} = \bar{\rho}_R$ contrasting the case of N^a-$N^{(1)}$. This branch cut introduced by $N^a(\sigma'', t'; \bar{\rho}_L) - N^a(\sigma'', t'; \bar{\rho}_R)$, in $N^{(2)}$ produces another jump which $g_{\sigma'}(\sigma'')$ removes.

One important property of N_c which will be useful later we will quote here. When the two coordinates $\bar{\rho}$ and $\bar{\rho}'$ are set equal to each other, the sum of the nonsingular terms, *i.e.*, all but the terms which go as $\ln(\bar{\rho} - \bar{\rho}')$, in N_c conspire so that only N^a remains.

We also point out here that only the antianalytic part, N^a, of N_c would survive in the momentum dependent term of any scattering amplitude because of momentum conservation. However, this point is not of importance for the specific amplitude we are presently computing.

As a final remark, although our discussion has singled out the $\bar{\rho}$- coordinate, observe that the function we added to N^a, $N^{(1)} + N^{(2)}$, introduced only σ'-independent discontinuities when considered as a function of the $\bar{\rho}'$ coordinate. This means we

could add a similar function with the roles of $\bar{\rho}$ and $\bar{\rho}'$ reversed and construct a symmetric Green's function in $\bar{\rho}$ and $\bar{\rho}'$ which satisfied (3.12) with respect to both these coordinates.

Before proceeding to construct M_c, it will be necessary to transform to a "new" set of complex string diagram coordinates first introduced by Mandelstam [8]. Although the "old" string diagram coordinates $(\alpha, \beta, \tau, \theta)$ are useful for their graphical interpretation, they are not complex analytic coordinates. The definition of both types of coordinates for the bosonic string and superstring are given in Appendix A. In our case, due to the asymmetry of the right-moving superstring modes versus left-moving bosonic modes, the Jacobian between the two sets of coordinates is different. The derivation is given in Appendix A. We find

$$\prod_{r=1}^{2n-3} d\tau_r \prod_{s=1}^{n} d\alpha_s \prod_{t=1}^{3n-3} d\theta_t = \frac{1}{|T - \bar{\tau}|} d^{3n-3} Y_S d^{3n-3} \bar{Y}_B \tag{3.18}$$

where we will use the definition

$$Im'\tau \equiv \frac{T - \bar{\tau}}{2i} \tag{3.19}$$

with T and $\bar{\tau}$ being the period matrices associated with the rm superstring and lm bosonic string respectively.

We now proceed to compute the measure factor M_c. To do this we use the observation by Mandelstam (3) that it can be determined soley by its analytic and modular properties. We must therefore establish the fact that M_c is analytic with respect to the Teichmüller parameters $\{\bar{Y}_B\}$. We do this by showing that an infinitesimal change of M_c, $\delta_m M_c$, is analytic with respect to an infinitesimal change in any of the Teichmüller parameters. We use the standard formula applicable to any linear operator O for the variation of $\ln O$

$$\delta \ln O = \delta O O^{-1} \tag{3.20}$$

and

$$\delta \ln \det O = \delta(\mathrm{Tr} \ln O) = \mathrm{Tr}(\delta O O^{-1}) \tag{3.21}$$

Since the operators we are dealing with require regularization, equation (3.18) is not a complete formula for the variation, because it assumes that the trace can be defined without regularization. However, careful analysis which takes the ordering into account shows that naive use of equation (3.21) is allowed in this situation. In all subsequent discussion, it is to be understood that we are discussing the regularized operator so that implicitly, the necessary subtractions are always done where needed.

Any infinitesimal variation of the Teichmüller parameters can be affected by a suitable coordinate transformation $\rho_o \to \rho_n$. One possible choice which generates all the desired variations is

$$\rho_n = \begin{cases} \rho_o + cG(t_o) & \text{For } t_1 < t_o < t_2; \sigma_d < \sigma < \sigma_u \\ \rho_o & \text{Otherwise} \end{cases}$$

where

$$G(t_o) = \frac{t_o - t_1}{\Delta t} \quad \text{and} \quad \Delta t = t_2 - t_1 \tag{3.22}$$

where for definiteness one can imagine this as a relabeling of the coordinates of the string diagram. The limits for σ in (3.21), σ_u and σ_d, are always at two edges of a string, either cuts within the diagram or the ends of the diagram, where, in particular, these points are identified. Therefore the transformation (3.22) only changes ρ_o on some retangular region which is part of one internal or external string. Examples of such regions are shown in Fig. 2. Implicitly it is to be understood that the transformation (3.22) does not jump at t_1 and t_2 since one re-defines all functions as consisting of two parts that are identified at these boundaries.

From (3.21) we obtain for the quadratic operator Δ_c,

$$\delta \operatorname{Tr} \ln \Delta_c = \frac{1}{\pi} Tr \left[\delta\Delta_c N_c\right] \tag{3.23}$$

where

$$\delta\Delta_c = \delta\left(\partial_{\sigma_o}\partial_{\bar\rho_o}\right) \equiv \partial_{\sigma_n}\partial_{\bar\rho_n} - \partial_{\sigma_o}\partial_{\bar\rho_o} = -c(\partial_{\rho_o}G)\partial_{\sigma_o}\partial_{\rho_o} - \bar c(\partial_{\rho_o}G)\,\partial_{\sigma_o}\partial_{\bar\rho_o}$$
$$+O(c^2)$$

Substituting for $\partial_{\sigma_n}\partial_{\rho_n}$ in (3.23) we want to establish that the resulting expression is antianalytic with respect to the Teichmüller parameters $\{\bar Y_B\}$. In particular that it is only a function of $\bar c$ and not c. In fact, the c-dependent term in (3.23) will be removed upon regularization. The remaining possible trouble arises from the fact that both N_c and the term containing $G(t)$ in (3.22) involve nonanalytic quantities.

In order to show that they do not appear in the variation we first argue that N_c appearing in (3.23) can be replaced by only N^a. For this, recall that in order to take the trace the rule is to first set the prime coordinates equal to the unprimed (this is after regularizing which is essential equivalent to the prescription of ignoring the singular terms) and then integrating over the unprimed coordinate. Now, when one sets $t = t'$ we can use the property already stated above and can replace the

nonsingular part of N_c by just the nonsingular part (means all but the part behaving like $\ln(\bar\rho - \bar\rho')$) of N^a.

What then remains in total for the variation is the expression.

$$\delta \ln \det \Delta_c = \frac{\bar c}{2\Delta t}\int_{t_1}^{t_2} dt F(t', t'')|_{t'=t''=t} + (\text{analytic in Teich}) \tag{3.24}$$

where

$$F(t', t'') = \int_{\sigma_d}^{\sigma_u} d\sigma \left[\partial_\sigma \partial_{\bar\rho} N^a(\sigma, t', \sigma'', t'')\right]|_{\sigma''=\sigma}$$

Recall that in the above formulas, it is understood that we only consider the nonsingular portion of N^a. The variation of the singular part, which goes as $\ln(\bar\rho - \bar\rho')$, is absorbed in the regularization so we can ignore its effect. The only cause of nonanalyticity in this expression is the apparent dependence on t_1 and t_2. We will show that it is in fact independent of t_1 and t_2 hence establishing analyticity of $\delta \ln \Delta_c$ with respect to the Teichmüller parameters. Observe that by applying the Cauchy–Riemann equation we have

$$(\partial_t + \partial_{t'})F(t, t') = 0$$

since

$$\int_{\sigma_d}^{\sigma_u} d\sigma(\partial_t + \partial_{t'})N^a(\sigma, t, \sigma', t')|_{\sigma'=\sigma} = \int_{\sigma_d}^{\sigma_u} d\sigma(\partial_\sigma + \partial_{\sigma'})N^a(\sigma, t, \sigma', t')|_{\sigma'=\sigma}$$
$$= \int_{\sigma_d}^{\sigma_u} d\sigma \partial_\sigma N^a(\sigma, t, \sigma, t') = \Delta_{a_j} N^a = 0 \tag{3.25}$$

where ΔN_{a_j} is the change of N^a around the jth a-cycle, which is zero from the known properties of N^a. This means the integral over t in (3.19) simply gives the length of the interval, Δt, which cancels with the same quantity in the denominator leaving only $\bar c/2$, an antianalytic quantity.

We now turn to the noncompactified sector and consider the construction of the Green's function. We write the result as a sum of three terms.

$$N_{nc} = N_B + N_S + N_{non}$$

where N_B is the antianalytic portion of the usual bosonic Green's function, N_S is the superanalytic portion of the superstring Green's function, and N_{non} is a nonanalytic piece which is defined so that N_{nc} has the correct periodicity properties. An application of the Riemann Bilinear Relations can establish the behavior of $N_S + N_B$ under transformations around a- and b-cycles. This then gives the specifications needed of

N_{non}. The appropriate function one finds is:

$$N_{non} = \frac{-1}{8\pi} \sum_{r,s} \left[\left(\overline{v_r(z)} + \mu_r(z,\psi) \right) - \left(\overline{v_r(z')} + \mu_r(z',\psi') \right) \right] \times \left(\frac{T - \bar{\tau}}{2i} \right)^{-1}_{rs} \times \left[\left(\overline{v_s(z)} + \mu_s(z,\psi) \right) - \left(\overline{v_s(z')} + \mu_s(z',\psi') \right) \right] \tag{3.26}$$

where the functions $\bar{v}$ and μ are the integrals of the 1–forms for the lm bosonic sector and 1/2–forms of the rm superstring sector respectively.

We now turn to the computation of the measure factor M_{nc}. We use formula (3.21) to study the variation of $\ln \det M_{nc}$. Of course the trace operations now adhere to the standard rules of superspace. It is again possible to use just the naive expression (3.18). The other terms which would arise in the variation (not shown) do not affect the result once both the original and varied operators are appropriately regularized. We then find.

$$\delta \ln \det M_{nc} = Tr(\delta\Delta_{nc}N_B) + Tr(\delta\Delta_{nc}N_S) + Tr(\delta\Delta_{nc}N_{non}) \tag{3.27}$$

One can establish by an analysis similar to that for M_c that the first two terms above are analytic with respect to the Teichmüller parameters. Then, by Mandelstam's trick, we can write down the explicit expressions arising from these two terms which we will denote by M_{lm} and M_s where $M_{nc} = M_{lm}M_sM_{non}$. M_{lm} will be, up to a constant, the bosonic determinant and M_s will be the analytic portion (right moving part) of the superstring determinant already obtained by Mandelstam [3].

The computation of M_{non} arising from integration of the third term in (3.27) must be done explicitly.

We now turn to this calculation. After operating on N_{non} by (3.21) and setting $(\bar{\rho}', \rho', \psi') = (\bar{\rho}, \rho, \psi)$ we get

$$\delta\Delta_{nc}N_{non}(\bar{\rho},\rho,\psi;\bar{\rho},\rho,\psi) = \left[\frac{\partial\delta\bar{\rho}}{\partial\bar{\rho}}\Delta_{nc} + \frac{\partial\delta\rho}{\partial\bar{\rho}}\partial\rho D_\psi + \frac{\partial\delta\psi}{\partial\bar{\rho}}\partial_\psi D_\psi + D_\psi\delta\bar{\rho}\partial^2_{\bar{\rho}} + D_\psi\delta\rho\partial\rho\partial\bar{\rho} + D_\psi\delta\psi\partial_{\bar{\rho}}\partial_\psi - \delta\psi\partial_\rho\partial_{\bar{\rho}} \right] N_{non} \tag{3.28}$$

Recall now from (3.11) that N_{nc} is not actually the inverse of Δ_{nc} so we must be more careful in the analysis of (3.28). Inspection on N_{nc} shows that the function F_{nc} in

(3.11) arises from the N_{non} term. Specifically

$$F_{nc}(\rho,\bar{\rho},\psi) = \partial_{\bar{\rho}}D_\psi N_{non} \tag{3.29}$$

In principle, we could constrtuct the "true" Green's Function by adding to N_{nc} the term N_{fix} given as,

$$N_{fix}(\bar{\rho},\rho,\psi) = \frac{1}{2\pi}\int N_{nc}(\bar{\rho},\rho,\psi;\bar{\rho}',\rho',\psi')F_{nc}(\bar{\rho}',\rho',\psi')d\psi'd\rho'd\bar{\rho}' - \frac{1}{2\pi}F_{nc}(\bar{\rho},\rho,\psi)\int d\bar{\rho}'d\rho'd\psi'F_{nc}(\bar{\rho}',\rho',\psi') \tag{3.30}$$

This term will replace $F_{nc}(\rho)$ from the right hand side of (3.11) by a constant. At least a constant must appear in addition to the δ–function, since the operator Δ_{nc} has no inverse due to the presence of a zero mode. For purpose of computing on–shell scattering amplitudes there is never a need to explicitly write down this extra term, N_{fix}, because of the momentum conservation condition. Even though both Green's functions give the same amplitude, one must keep in mind the extra term, N_{fix}, when treating variations of the operator Δ_{nc} as we are doing.

From these considerations what emerges is that any term in (3.28) which is proportional to Δ_{nc} should be ignored, since it would not have appeared had we used the "true" Green's functions. In particular the first term on the right hand side in (3.28) is ignored. We now show that the last three terms in (3.28) are also to be ignored. If we consider these three terms in isolation, they are:

$$I \equiv \frac{-i}{2\pi}\int d\bar{\rho}d\rho d\psi \left[\left(D_\psi\delta\rho\partial_\rho\partial_{\bar{\rho}} + D_\psi\delta\psi\partial_\psi\partial_{\bar{\rho}} - \delta\psi\partial_{\bar{\rho}}\partial_{\bar{\rho}} \right) N_{non} \right]_{(\bar{\rho}',\rho',\psi')=(\bar{\rho},\rho,\psi)} \tag{3.31}$$

We will choose to make the variation of ψ, $\delta\psi$, not independent of $\delta\rho$ but rather given by,

$$\delta\psi = D_\psi\delta\rho - \frac{\psi}{2}\frac{\partial\delta\rho}{\partial\rho} + 0(\delta^2) \tag{3.32}$$

We are free to impose such a restriction as long as it can generate all the desired Teichmüller deformations. In Appendix C we show that this is possible. Observe that our choice would just correspond to the restrictions imposed by superconformal transformations of the (ρ,ψ) coordinates had it not been for the fact that $\delta\rho$ depends on both ρ and $\bar{\rho}$. In fact relation (3.32) was motivated by this correspondence and was done so as to simplify (3.28). However this should not be confused with a superconformal transformation since, in fact, performing only conformal or superconformal transformations cannot induce Teichmüller deformations. Using the above relation and substituting for $\delta\psi$ in (3.31) we are left with

$$I = \frac{-i}{2\pi}\int d\bar{\rho}d\rho d\psi \left[\left(D_\psi \delta\psi\partial_\psi\partial_{\bar{\rho}} + \frac{\psi}{2}\frac{\partial\delta\rho}{\partial_\rho}\partial_\rho\partial_{\bar{\rho}}\right) N_{non}\right]_{(\bar{\rho}',\rho',\psi')=(\bar{\rho},\rho,\psi)} \tag{3.33}$$

Again using (3.32) for $\delta\psi$ in the first term above, we get

$$I = \frac{-i}{2\pi}\int d\bar{\rho}d\rho d\psi \frac{1}{2} D_\psi \partial_{\bar{\rho}} N_{non} \tag{3.34}$$

which, from what we said earlier, must be ignored.

Hence we find for the nonanalytic portion of $\det \Delta_{nc}$ the relation,

$$\delta \ln\det\Delta_{non} = \frac{-i}{2\pi}\int d\bar{\rho}d\rho d\psi \left[\left(\frac{\partial\delta\rho}{\partial\bar{\rho}}\partial_\rho D_\psi + \frac{\partial\delta\psi}{\partial\bar{\rho}}\partial_\psi D_\psi + D_\psi\delta\bar{\rho}\partial_{\bar{\rho}}\partial_{\bar{\rho}}\right) N_{non}\right]_{(\bar{\rho}',\rho,\psi)} \tag{3.35}$$

The area integrals can be converted to contour integrals to give,

$$\delta \ln\det\Delta_{non} = \frac{+i}{2\pi}\sum_{p=1}^{n}\left\{\oint_{2p} - \oint_{1p} d\rho d\psi 2\left(\delta\rho\frac{\partial\mu_s}{\partial\rho} + \delta\psi\frac{\partial\mu_s}{\partial\psi}\right)\frac{(Im'\tau)^{-1}_{st}}{8\pi} D_\psi\mu_t \right.$$
$$\left. + \oint_{2p} - \oint_{1p} d\bar{\rho} 2\delta\bar{\rho}\frac{\partial\bar{v}_s}{\partial\bar{\rho}}\frac{(Im'\tau)^{-1}_{st}}{8\pi}\frac{\partial\bar{v}_t}{\partial\bar{\rho}}\right\} \tag{3.36}$$

The variation of the coordinates $(\delta\rho, \delta\psi)$ induces variations in the period matrix as discussed in Appendix C. Using those results we get,

$$\frac{i}{2\pi}(2\pi i)\sum_{p=1}^{n}\left\{\oint_{2p} d\rho d\psi\frac{1}{4\pi}\delta T_{ps}(Im'\tau)^{-1}_{st} D_\psi\mu_t \right.$$
$$\left. - \oint_{2p} d\bar{\rho}\frac{1}{4\pi}\delta\bar{\tau}_{ps}(Im'\tau)^{-1}_{st}\partial_{\bar{\rho}}\mu_t\right\}$$
$$= \frac{i}{8\pi^2}(2\pi i)^2(\delta T - \widetilde{\delta\tau})_{ps}(Im'\tau)_{sp}$$
$$= \delta(Im'\tau)_{ps}(Im'\tau)_{sp} \tag{3.37}$$

and therefore

$$\det\Delta_{non} = \det(Im'\tau) \tag{3.38}$$

We can now combine our results for the n-loop contribution to the scattering amplitude and write it as,

$$A_n = (\text{const.})\int \prod_{I=1}^{3n-3} dY_S^I d\bar{Y}_B^I \prod_{I=1}^{2n-2} d\xi^I \prod_{m=1}^{N} dQ_m\bar{Q}_m$$
$$M_c^{-8} M_{nc}^{-4} L(\bar{\tau}, \bar{\tau}, O, O)$$
$$\exp\left[-\frac{1}{2}\sum_{\alpha>\beta} P_\alpha\cdot P_\beta N_{nc}\left(Q_\alpha,\bar{Q}_\alpha; Q_\beta\bar{Q}_\beta\right)\right] \tag{3.39}$$

CONCLUSION

The derivation we have given is for the even spin structures; however, there are only slight modifications for the odd spin structures which we will briefly discuss now. The form of the final formula for the scattering amplitude would be similar to (3.39) except there will be additional factors in the integrand which arise from the zero modes of the external particles. This can be handled by the same procedure as Mandelstam has used in the case of the superstring [3]. Aside from this, the main problem is that there does not exist a set of holomorphic half-forms for odd spin structures. The analogue for even spin structures were essential in constructing needed expressions such as the Green's functions. This problem can be overcome by using a set of half-forms constructed by D'Hoker and Phong [9] which have one pole. Mandelstam has shown in the case of the Superstring that the final expression is independent of the position of the pole. His reasoning can be extended to the heterotic string with only slight modifications due to the differences in the nonanalytic pieces of the determinant. Hence the scattering amplitude for the odd spin structures can also be computed.

One might ask at this stage what is the use of an explicit formula for the string scattering amplitude. Clearly at present experimental energies there is no evidence indicating that strings are relevant, at a fundamental level, to the theory of elementary particles. On the other hand, the search for a consistent S-matrix, arising from a local field theory, still remains the basic theoretical problem in particle physics. In this light, string theory is a theoretical experiment based on hopes that it may resolve the shortcomings of conventional point-particle field theory and give a correct description of nature. Whether these hopes are to be realized or not, eventually rests on understanding what statements the theory has to make about the measurable quantities of particle physics, central to which is the scattering amplitude.

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APPENDIX A

In this appendix we derive the Jacobian transformation matrix between the real string diagram coordinates and the "new" complex coordinates which will be referred to as the complex moduli parameters

Let ω_B (ω_S) denote the one-(half-) form that give the real n-loop string diagram coordinates by the integrals,

$$\begin{aligned}
\alpha_{v\binom{B}{S}} &\equiv \frac{1}{2\pi i}\oint_{A_v}\omega_{\binom{B}{S}} \quad v=1\dots n\\
\beta_{v\binom{B}{S}} &\equiv \frac{1}{2\pi i}\oint_{B_v}\omega_{\binom{B}{S}} \quad v=1\dots n\\
\tau_{v\binom{B}{S}} &\equiv Re\int_{R_o}^{R_v}\omega_{\binom{B}{S}} \quad v=1\dots 2n-3\\
\theta_{v\binom{B}{S}} &\equiv \mathrm{Im}\int_{R_o}^{\bar{R}_v}\omega_{\binom{B}{S}} \quad v=1\dots 2n-3
\end{aligned} \tag{A1}$$

The complex moduli parameters $\{Y_S, \bar{Y}_B\}$ are defined through the following integrals.

$$\begin{aligned}
Y_S^q &\equiv \frac{1}{2\pi i}\oint_{B_q}\tilde{\omega}_S \quad q=1\dots n\\
\bar{Y}_B^q &\equiv \frac{-1}{2\pi i}\oint_{B_q}\bar{\tilde{\omega}}_B \quad q=1\dots n\\
Y_S^{n+r} &\equiv \int_{P_o}^{P_r}\tilde{\omega}_S \quad r=1\dots 2n-3\\
\bar{Y}_B^{n+r} &\equiv \int_{P_o}^{P_r}\bar{\tilde{\omega}}_B \quad r=1\dots 2n-3
\end{aligned} \tag{A2}$$

The definition of the complex moduli parameters in terms of the real string diagram parameters is uniquely given by requiring that the integrals of ω_B and ω_S around a-periods remain unchanged. This means,

$$\begin{aligned}
\delta_S\tilde{\omega}_S &= \delta\omega_S - \sum_q \delta\alpha_q\omega_q^S\\
\delta_B\bar{\tilde{\omega}}_B &= \delta\bar{\omega}_B - \sum_q \delta\alpha_q\bar{\omega}_q^B
\end{aligned} \tag{A3}$$

These equations give relations between the infinitesimal change of the complex parameters and the real parameters in the form.

$$\begin{aligned}
\delta Y_S^q &= \frac{1}{2\pi i}\oint_{B_q}\delta\tilde{\omega}_S = \delta\beta_q - \sum_\ell \delta\alpha_\ell T_{\ell q}\\
\delta\bar{Y}_B^q &= \frac{-1}{2\pi i}\oint_{B_q}\bar{\tilde{\omega}} = -\delta\beta_q + \sum_\ell \delta\alpha_\ell\bar{\tau}_{\ell q}\\
\delta Y_S^{n+r} &= \int_{P_o}^{P_r}\delta\tilde{\omega}_S = \delta T_r - i\delta\theta_r - \sum_j C_{rj}^S\delta\alpha_j\\
\delta\bar{Y}_B^{n+r} &= \int_{P_o}^{P_r}\delta\bar{\tilde{\omega}}_B = \delta\tau_r + i\delta\theta_r - \sum_j C_{rj}^B\delta\alpha_j
\end{aligned} \tag{A4}$$

Hence the Jacobian is.

$$\prod d\alpha\prod d\beta\prod d\tau\prod d\theta = J\prod dY_S d\bar{Y}_B$$

with

$$J \equiv \left[\det(T-\bar{\tau})\right]^{-1} \tag{A5}$$

APPENDIX B

In this appendix we will show that a change of coordinates

$$\begin{aligned} \rho_0 &\longrightarrow \rho_n = \rho_0 + \delta\rho(\bar\rho_0, \rho_0, \psi_0) \\ \bar\rho_0 &\longrightarrow \bar\rho_n = \bar\rho_0 + \delta\bar\rho(\bar\rho_0, \rho_0, \psi_0) \\ \psi_0 &\longrightarrow \psi_n = \psi_0 + \delta\psi(\bar\rho_0, \rho_0, \psi_0) \end{aligned} \tag{B1}$$

induces the following changes in the period matrices

$$\begin{aligned} \delta\bar\tau_{rs} &= \frac{1}{4\pi^2}\sum_{p=1}^{n}\oint_{2p_n} - \oint_{1p_n} d\bar\rho \frac{\partial\bar v_r}{\partial\bar\rho}\frac{\partial\bar v_s}{\partial\bar\rho}\delta\bar\rho \\ \delta T_{rs} &= \frac{-1}{4\pi^2}\sum_{p=1}^{n}\oint_{2p_n} - \oint_{1p_n} d\rho d\psi\left(\delta\rho\frac{\partial\mu_r}{\partial\rho} + \delta\psi\frac{\partial\mu_r}{\partial\psi}\right)D_\psi\mu_s \end{aligned} \tag{B2}$$

To show this, first observe that a change of coordinates implies that the function v_r and $\mu_r = 1\ldots n$, which are the integrals of the 1-forms in the bosonic sector and half-forms in the superstring sector respectively, will change. The new functions (denoted by v_r^n and μ_r^n) associated with the new coordinates are related to the old functions (denoted with no superscript) by

$$\begin{aligned} \bar v_r^n(\bar\rho) &= \bar v_r(\bar\rho - \delta\bar\rho) + \\ \frac{1}{2\pi i}\int d^2\bar\rho' \frac{\partial}{\partial\bar\rho'} N_B^a(\rho,\bar\rho;\rho',\bar\rho') &\left[\partial_\rho v_r(\bar\rho' - \delta\bar\rho(\bar\rho,\rho,\psi))|_{(\bar\rho,\rho,\psi)=(\bar\rho',\rho',\psi')}\right] \\ \mu_r^n(\rho,\psi) &= \mu_r(\rho - \delta\rho, \psi - \delta\psi) + \\ \frac{1}{2\pi i}\int d^2\rho' d\psi' D_\psi N_{nc}^a(\rho,\psi,\rho',\psi') &\left[\partial_{\bar\rho}\mu_r(\rho' - \delta\rho, \psi' - \delta\psi)|_{(\bar\rho,\rho,\psi)=(\bar\rho',\rho',\psi')}\right] \end{aligned} \tag{B3}$$

where N_{nc}^a is the Green's function defined in Section 3 equation (3.11) without the nonanalytic piece and N_B^a is the analytic plus antianalytic portion of the usual Bosonic Green's function. These relations are obtained by the conditions that $v_r^n(\mu_r^n)$ be antianalytic (superanalytic) with respect to $\bar\rho(\rho,\psi)$ and that they have the appropriate periodicity properties around all a-cycles.

The transformation properties of the function $v_r^n(\mu_r^n)$ around b-cycles will now determine the new period matrix. Note that to round the s-th b-cycle in the new coordinates means for the $\bar\rho$ coordinate for example.

$$\bar\rho_{1S} + \delta\bar\rho(\bar\rho_{1S}, \rho_{1S}, \psi_{1S}) \longrightarrow \bar\rho_{2S} + \delta\bar\rho(\bar\rho_{2S}, \rho_{2S}, \psi_{2S}) \tag{B4}$$

when

$$\bar\rho_{1S} \longrightarrow \bar\rho_{2S} \tag{B5}$$

is the corresponding transformation in the old coordinates. Hence we obtain for the change in the period matrix,

$$\begin{aligned} \delta\bar\tau_{rs} &= \frac{-i}{2\pi^2}\int d^2\bar\rho'\partial\sigma'\left[N_c^a(\bar\rho_{2S},\bar\rho') - N_c^a(\bar\rho_{1S},\bar\rho')\right]\frac{\partial v_r}{\partial\bar\rho}\frac{\partial\delta\bar\rho}{\partial\rho} \\ \delta T_{rs} &= \frac{i}{4\pi^2}\int d^2\rho' d\psi' D_{\psi'}\left[N_{nc}^a(\rho_{2S},\psi_{2S},\rho',\psi') - N_{nc}^a(\rho_{1S},\psi_{1S},\rho',\psi')\right]\left(\frac{\partial\mu_r}{\partial\rho}\frac{\partial\delta\rho}{\partial\bar\rho} + \frac{\partial\mu_r}{\partial\psi}\frac{\partial\delta\psi}{\partial\bar\rho}\right) \end{aligned} \tag{B6}$$

where we have expanded v_r and μ_r to first order in $\delta\bar\rho$ and $(\delta\rho, \delta\psi)$ respectively. Using the transformation properties of the Green's function around b-cycles, we then obtain the relations in B2.

APPENDIX C

We will show in this appendix that given any arbitrary variations of the Teichmüller parameters, we can find a suitable function $\delta\rho(\bar\rho,\rho,\psi)$, with $\delta\psi$ determined by the constraint (3.28), which will induce this desired change. The main issue concerns the joining points where an odd and even coordinate must be changed simultaneously.

We recall first the definition [3] of the odd coordinate, $\hat\psi_s$, at joining point s is related to the behavior of the string diagram coordinate ψ by,

$$\psi = \frac{\hat\psi_s}{(\rho - \hat\rho_s)^{1/4}} + \cdots \tag{C1}$$

where $\hat\rho_s$ is the value of the even coordinate at joining point s. Suppose we want to change $\hat\rho_s$ and $\hat\psi_s$ by arbitrary given values, $\delta\hat\rho_s$ and $\delta\hat\psi_s$ respectively. Our problem is to show that we can find a function $\delta\rho(\bar\rho,\rho,\psi)$ such that,

$$\begin{aligned} \delta\rho(\bar\rho,\rho,\psi)|_s &= \delta\hat\rho_s \\ (\rho - \hat\rho_s)^{1/4}\delta\psi(\bar\rho,\rho,\psi)|_s &= \delta\hat\psi_s \end{aligned} \tag{C2}$$

where the function $\delta\psi$ is determined by (3.28). Here and elsewhere in this appendix the symbol $|_s$ is to mean the limit as the arguments approach the joining point s.

To show that this is possible first write $\delta\rho$ as,

$$\delta\rho(\bar\rho,\rho,\psi) = \rho_1(\bar\rho,\rho) + \psi\rho_2(\bar\rho,\rho) \tag{C3}$$

Using (3.28) and (C1) we then have.

$$
\begin{aligned}
&a.\ \ \delta\hat{\psi}_s = \frac{1}{2}\hat{\psi}_s\frac{\partial\rho_1}{\partial\rho}\Big|_s + (\rho-\bar{\rho}_s)^{1/4}\rho_2\big|_s \\
&\qquad + \frac{1}{2}\psi'(\rho-\hat{\rho}_s)^{1/2}\frac{\partial\rho_1}{\partial\rho}\Big|_s \\
&b.\ \ \delta\hat{\rho}_s = \rho_1\big|_s + \left(\frac{\hat{\psi}_s}{(\rho-\hat{\rho}_s)^{1/4}} + \psi'(\rho-\hat{\rho})^{1/4}\right)\rho_2\big|_s
\end{aligned}
\tag{C4}
$$

We can further always write ρ_1 and ρ_2 as

$$
\begin{aligned}
&a.\ \ \rho_1(\bar{\rho},\rho) = \rho_{1e}(\bar{\rho},\rho) + \hat{\psi}_s\rho_{1o}(\bar{\rho},\rho) \\
&b.\ \ \rho_2(\bar{\rho},\rho) = \rho_{2o}(\bar{\rho},\rho) + \hat{\psi}_s\rho_{2e}(\bar{\rho},\rho) \\
&c.\ \ \delta\hat{\psi}_s = b_1 + \hat{\psi}_s b_2 \\
&d.\ \ \delta\hat{\rho}_s = c_1 + \hat{\psi}_s c_2
\end{aligned}
\tag{C5}
$$

where the above decomposition is not necessarily unique. Using the above expressions we can write (C4) as

$$
\begin{aligned}
&a.\ \ \delta\hat{\psi}_s = \hat{\psi}_s\Big[\frac{1}{2}\frac{\partial\rho_{1e}}{\partial\rho}\Big|_s + (\rho-\hat{\rho})^{1/4}\rho_{2e}\big|_s\Big] \\
&\qquad + (\rho-\hat{\rho}_s)^{1/4}\rho_{2o}\big|_s \\
&b.\ \ \delta\hat{\rho}_s = \rho_{1e}\big|_s + \psi'(\rho-\hat{\rho})^{1/4}\rho_{2o}\big|_s \\
&\qquad + \hat{\psi}_s\Big[\rho_{1o}\big|_s + \frac{\rho_{2o}}{(\rho-\hat{\rho}_s)^{1/4}}\Big|_s + \psi'(\rho-\hat{\rho}_s)^{1/4}\rho_{2e}\big|_s\Big]
\end{aligned}
\tag{C6}
$$

We now make the following specifications for $\delta\rho$. From (C6a) let

$$
\begin{aligned}
\frac{\partial\rho_{1e}}{\partial\rho}\Big|_s &= 2b_2 \\
\rho_{2e}\big|_s &= 0 \\
(\rho-\hat{\rho}_s)^{1/4}\rho_{2o}\big|_s &= b_1
\end{aligned}
\tag{C7}
$$

By this procedure we find that ρ_{2o} is singular at joining point s but that ρ_{1e} by choice is not. We now take advantage of the fact that one is still free to fix the function ρ_{1o} and ρ_{1e} at joining point s to allow us to satisfy (C6b). We make the choices.

$$
\begin{aligned}
\rho_{1e}\big|_s &= c_1 - \psi'(\rho-\bar{\rho}_s)^{1/4}\rho_{2o}\big|_s \\
\rho_{1o}\big|_s &= c_2 - \frac{\rho_{2o}}{(\rho-\hat{\rho}_s)^{1/4}}\Big|_s
\end{aligned}
\tag{C8}
$$

which means in general the ρ_{1o} is also singular at joining point s.

The above accomplishes our task of constructing $\delta\rho$ locally about all joining points. The local construction of $\delta\rho$ for the remaining even Teichmüller parameters is straightforward since $\delta\psi$ is not needed. Hence we have established our claim.

FIGURE CAPTIONS

figure 1: String diagram with three cuts (loops). In the region $\sigma_1 < \sigma < \sigma_3$, $t_{L2} < t < t_{R2}$ the function $g_2(t)$ (see text equation 3.5) is non-vanishing.

figure 2: Shaded regions a and b are examples of rectangular regions as discussed in the text regarding transformation (3.22). In region a the upper boundary is a cut and the lower boundary coincides with the boundary of the string diagram. In region b both boundaries are cuts.

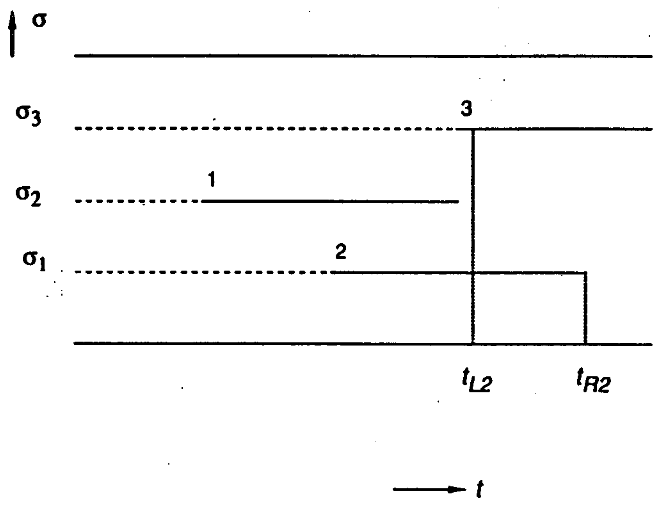

Figure 1

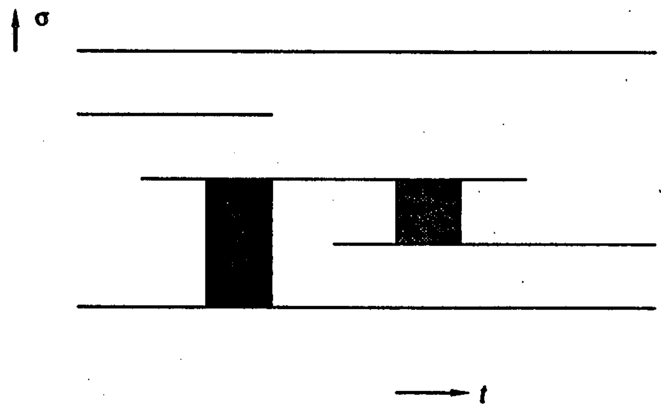

Figure 2