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Six-Dimensional Supergravity on $S^3 \times AdS_3$ and 2d Conformal Field Theory

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Abstract

In this paper we study the relation between six-dimensional supergravity compactified on $S^3 \times AdS_3$ and certain two-dimensional conformal field theories. We compute the Kaluza-Klein spectrum of supergravity using representation theory; these methods are quite general and can also be applied to other compactifications involving anti-de Sitter spaces. A detailed comparison between the spectrum of the two-dimensional conformal field theory and supergravity is made, and we find complete agreement. This applies even at the level of certain non-chiral primaries, and we propose a resolution to the puzzle of the missing states recently raised by Vafa. As a further illustration of the method the Kaluza-Klein spectra of F-theory on $M^6 \times S^3 \times AdS_3$ and of M-theory on $M^6 \times S^2 \times AdS_3$ are computed, with M^6 some Calabi-Yau manifold.

1 Introduction

One of the most interesting examples of the $\text{AdS} \leftrightarrow \text{CFT}$ conjecture proposed in [1] and refined in [2, 3] is the duality between type IIB string theory on $M^4 \times S^3 \times \text{AdS}_3$ and certain two-dimensional conformal field theories. Two-dimensional conformal field theories are very well understood, enabling us to test the conjecture in more detail than in other dimensions. Furthermore, by S-dualizing the type IIB background we obtain a string theory with only NS-NS fields turned on, which looks like the product of a $K3$ conformal field theory and an $SU(2)$ and $Sl(2, \mathbf{R})$ WZW theory. Thus we may hope to better understand the AdS side of the conjecture as well, although after the S-duality the type IIB string theory is strongly coupled and it is not clear to what extent we can trust naive conformal field theory considerations.

The main example of the conjecture arises when we consider a system of parallel D1 and D5 branes, where the D5 branes are wrapped on some four manifold M^4 which can be either T^4 or $K3$. The solitonic description of this brane configuration yields a five-dimensional black hole. The relation between the D1-D5 brane system and the five-dimensional black hole has been examined in great detail. In particular, the D1-D5 brane system correctly accounts for the entropy of the black hole [4] and various emission and absorption probabilities and greybody factors [5–8]. The D1-D5 brane system is described by a certain 1+1 dimensional gauge theory. Related gauge theories appear in the study of the M5 brane in M(atric) theory, in the M(atric) description of M-theory on T^5 , and in the description of little string theories, and were studied in [9–19]. For all these application it is important to know to which conformal field theory the gauge theory flows in the infrared. In [15] it was argued that in the infrared the Coulomb and Higgs branches of the gauge theory decouple. For the case where the D5 branes are wrapped over M^4 , the conformal field theory is a deformation of the $N = (4, 4)$ sigma model with target space $(M^4)^N/S_N$ [20, 4]. Going to the infrared for the gauge theory is the same as going to the near horizon region of the five-dimensional black hole [1], whose geometry is $M^4 \times S^3 \times \text{AdS}_3$ ¹. Thus, type IIB string theory in this background should be dual to a 2d conformal field theory associated to $(M^4)^N/S_N$. In fact, most of the D1-D5 brane calculations rely only on the IR conformal field theory description of the gauge theory on the branes, and support this duality. Further evidence and results were recently obtained in [38, 39].

One of the goals of this paper is to compare in detail the spectrum of the conformal field

¹There has been a lot of work on the relation of AdS_3 to black holes, entropy, and brane configurations; see [21–37] for an incomplete list of references.

theory and 6d (2,0) supergravity² on $S^3 \times AdS_3$. This is of particular interest in view of the recent paper [40] in which it is argued that certain states in the conformal field theory are absent in supergravity. We will see that the Kaluza-Klein spectrum of supergravity can be completely determined using only representation theory, and in particular we can find the masses and spins of all KK-fields on AdS_3 , without the need to examine the equations of motion of supergravity. A careful comparison between the multi-particle KK spectrum of supergravity and the spectrum of the conformal field theory reveals a complete agreement and automatically suggests a resolution to the puzzle of [40].

Another goal of this paper is to examine to what extent six-dimensional supergravity on $S^3 \times AdS_3$ is always dual to some conformal field theory on the boundary of AdS_3 , as arguments based on holography suggest [41]. To do this, we compute the KK spectrum of arbitrary 6d supergravities on $S^3 \times AdS^3$. This includes cases that do not obviously correspond to the near-horizon geometry of some brane configuration. For all six-dimensional supergravities with $N > 1$ supersymmetry, the KK spectrum allows for a straightforward interpretation in terms of some conformal field theory with target space $(M^4)^N/S_N$. The case with $N = 1$ supersymmetry is more mysterious. For six-dimensional supergravities obtained from F-theory on a Calabi-Yau three-manifold M^6 , the KK spectrum is organized naturally in terms of the Betti numbers of the mirror Calabi-Yau $\tilde{M}^6$. The precise meaning of this remains to be understood. However, in view of the duality between F-theory on M^6 and type I on $K3$, it is natural to identify F-theory on $M^6 \times S^3 \times AdS_3$ with the near horizon geometry of a D1-D5 brane system in type I. The (0,4) theory on the boundary of AdS_3 would then be the IR description of the gauge theory on the D1 branes.

With the same goal in mind, we will also briefly consider five-dimensional supergravity on $S^2 \times AdS_3$; the techniques in this paper are however rather general and can be extended to other compactifications involving some anti-de Sitter space AdS_p .

The outline of this paper is as follows.

In section 2 we will briefly review the D1-D5 system wrapped on M^4 , and its near horizon geometry.

In section 3, we discuss some generic features of six-dimensional supergravity on $S^3 \times AdS_3$. We will explain the relation between the global symmetries of supergravity and the chiral algebra of the conformal field theory living at the boundary of AdS_3 . We will also derive the relation between the conformal weights of fields at the boundary, and the

²We can restrict attention to six-dimensional supergravity because the radius of M^4 is much smaller than that of S^3 or AdS_3 .

masses and spins of the fields in the bulk.

In section 4, we explain the general idea how to use representation theory to determine the KK spectrum, and use the oscillator method to determine the short representations of the AdS supergroup $SU(1, 1|2)$ relevant for this problem.

In section 5 we discuss the spectrum of the conformal field theory at the boundary, by considering the cohomology and elliptic genus of a symmetric product.

In section 6, we compute the KK spectrum for the chiral $N = (2, 0)$ $d = 6$ supergravity which describes type IIB supergravity compactified on $K3$. The multiparticle spectrum of states in supergravity contains all chiral primaries of the conformal field theory, but also various non-chiral primaries. We check in an example that these non-chiral primaries occur with the right multiplicity as predicted by the elliptic genus, and propose that they constitute the missing states of [40].

In section 7, other six-dimensional supergravities compactified on $S^3 \times AdS_3$ are considered, and a preliminary discussion of five-dimensional supergravity on $S^2 \times AdS^3$ is given.

Finally, we give some conclusions in section 8.

Several of the results in this paper were announced in [42]. While this work was nearing completion, a paper [43] appeared that also studies KK spectra using representation theory.

2 The D1-D5 Brane System

The metric for the extremal system of Q_1 D1 branes and Q_5 D5 branes wrapped on $K3$ is given by [8, 38]

$$\frac{ds^2}{\alpha'} = \frac{U^2}{\ell^2}(-dt^2 + (dx^5)^2) + \frac{\ell^2}{U^2}dU^2 + \ell^2 d\Omega_3^2 \quad (2.1)$$

$$+ \sqrt{\frac{Q_1}{vQ_5}} ds_{K3}^2 \quad (2.2)$$

where $\ell^2 = g_6 \sqrt{N}$, $N = Q_1 Q_5$, and ds_{K3}^2 is the metric on $K3$ with volume $16\pi^4 v \alpha'^2$. The dilaton is given by $\exp(-2\phi) = Q_5/g_6^2 Q_1$, with g_6 the six-dimensional and $g = g_6 \sqrt{v}$ the ten-dimensional string coupling constant. The RR three-form field strength H satisfies $\int_{S^3} H = \int_{S^3} *_6 H = 4\pi^2 \alpha' Q_5$.

The limit in which we trust the supergravity approximation is the one where we keep

$g_6 Q_i$ large and fixed, and send g_6 to zero. Because $\text{vol}(K3)/\alpha'^2 \ell^4 \sim 1/(g_6 Q_5)^2$ is very small for these values of the parameters, only the massless modes of supergravity on $K3$ will be important, giving rise to a six-dimensional supergravity theory with $(2, 0)$ supersymmetry and 21 tensor multiplets, compactified on $S^3 \times AdS_3$.

Several of the moduli of $K3$ in the near horizon limit are independent of their values of infinity in the original type IIB setup. The values of these fixed scalars, such as the volume of $K3$, are a function of the charges only and determined by minimizing a suitable central charge as in [44, 45]. Since the central charge is linear in Q_1 and Q_5 , rescaling both by a fixed factor will rescale the central charge by the same factor and therefore not affect the moduli of the $K3$. Hence, the $K3$ will generically be a smooth $K3$, and the sizes of the two-cycles in $K3$ will be of order $\sim \alpha' f(Q_1/Q_5)$ for certain functions f . Altogether this shows that we can neglect all branes wrapped on various cycles in $K3$. The KK fields from S^3 will have masses of order $m^2 \alpha' \sim 1/\ell^2$, whereas massive string states will have masses of order $m^2 \alpha' \sim 1$. KK states from $K3$ will have masses of order $m^2 \alpha' \sim \sqrt{Q_1/Q_5}$ and of order $m^2 \alpha' \sim f(Q_1/Q_5)$.

S-duality of type IIB string theory sends $\exp(\phi) \rightarrow \exp(-\phi)$, and $g_{\mu\nu} \rightarrow \exp(-\phi) g_{\mu\nu}$. After a trivial change of the U -coordinate the metric becomes

$$\frac{ds^2}{\alpha'} = \frac{U^2}{Q_5} (-dt^2 + (dx^5)^2) + \frac{Q_5}{U^2} dU^2 + Q_5 d\Omega_3^2 + \frac{1}{g_6 \sqrt{v}} ds_{K3}^2 \quad (2.3)$$

This metric corresponds to an exact string background, which consists of a $K3$ piece, a level Q_5 $SU(2)$ WZW theory and a level Q_5 $Sl(2, \mathbf{R})$ piece. The dilaton $\exp(2\phi) = Q_5/g_6^2 Q_1$ blows up in the limit we are interested in, so we expect that only certain BPS-protected quantities can be meaningfully compared. In particular, one should be able to recover the chiral primaries of the conformal field theory from this string background. A recent result in this direction is [36] where the stringy exclusion principle¹ of [38] is related to a unitary truncation of the $Sl(2, \mathbf{R})$ WZW spectrum. It would be interesting to explore the relation between the conformal field theory and this exact string background in more detail.

¹The stringy exclusion principle can also be understood classically in terms of large gauge transformations [46, 47].

3 Six-Dimensional Supergravity

3.1 Symmetries and the anti-de Sitter supergroup

As is clear from the discussion in the previous section, the low-energy excitations of the conformal field theory of the D1-D5 brane system should be compared to the KK spectrum of six-dimensional supergravity compactified on $\text{AdS}^3 \times S^3$. Although initially our focus will be on the chiral six-dimensional theory obtained by putting type IIB supergravity on $K3$, later we want consider various six-dimensional theories, with various numbers of matter multiplets. The supersymmetry generators in six dimensions are Weyl spinors of a fixed chirality, and the total number of chiral (antichiral) spinors will be denoted by $2n_L$ ($2n_R$). Six-dimensional theories exist for various values of n_L and n_R . The minimal theory has $n_L = 1, n_R = 0$, and appears for instance in heterotic and type I compactifications on $K3$. Type IIA on $K3$ yields a 6d theory with $(n_L, n_R) = (1, 1)$, type IIB on $K3$ yields a 6d theory with $(n_L, n_R) = (2, 0)$, and type IIA or IIB on T^4 yields a theory with $(n_L, n_R) = (2, 2)$. Although other values of (n_L, n_R) are possible as far as the existence of a supersymmetry algebra is concerned, there is only one other algebra whose representations include a graviton, namely $(n_L, n_R) = (2, 1)$. The corresponding supergravity theory is presumably anomalous [48], but we can still consider its KK spectrum, as that requires only classical supergravity. A convenient table of the various superalgebras and their multiplets is given in [49].

Six-dimensional supergravity has an $Sp(n_L) \times Sp(n_R)$ global symmetry. These symmetries do not play an important role in the rest of this paper, and will therefore be ignored. In any case, it is straightforward to keep track of the global symmetries at each step and determine how various quantities transform under them. Besides these global symmetries, 6d supergravity on $S^3 \times \text{AdS}_3$ has an $SO(4) \times SO(2, 2) \simeq SU(2) \times SU(2) \times Sl(2, \mathbf{R}) \times Sl(2, \mathbf{R})$ group of isometries. These are part of an anti-de Sitter supergroup $G = G_L \times G_R$, where both G_L and G_R contain $SU(2) \times Sl(2, \mathbf{R})$. G_L and G_R are global symmetries of the left and right-moving degrees of freedom of the two-dimensional conformal field theory. The simple supergroups that contain $Sl(2, \mathbf{R})$ were classified in [50]. The only simple supergroups whose bosonic part is $SU(2) \times Sl(2, \mathbf{R})$ are $Osp(3|2, \mathbf{R})$ and $SU(1, 1|2)$, but the first one (that actually contains an $O(3)$ subgroup) can be easily ruled out by looking at the transformation properties of the supercharges. All supercharges transform in the spinor representation of $SO(4)$ and should therefore be in the spin-1/2 representation of $SU(2)$. However, the fermionic generators of $Osp(3|2, \mathbf{R})$ are in the vector representation of $O(3)$. Thus the only possible supergroup is $SU(1, 1|2)$.

The $S^3 \times AdS_3$ background is maximally supersymmetric. For $n_L + n_R > 1$, the supercharges transform in the $2(n_L + n_R)((\mathbf{2}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{2}))$ representation of $SO(4)$, and for $n_L = 1, n_R = 0$ as $4(\mathbf{2}, \mathbf{1})$. In general, only a subset of the supercharges will close into the $SO(4) \times SO(2, 2)$ generators, and these will form the AdS supergroup $G_L \times G_R = SU(1, 1|2) \times SU(1, 1|2)$ for $n_L + n_R > 1$ and $G_L \times G_R = SU(1, 1|2) \times Sl(2, \mathbf{R}) \times SU(2)$ for $n_L = 1, n_R = 0$.

It is very useful to know the anti-de Sitter supergroups. It helps organize the KK spectrum of supergravity, which should fall in representations of the supergroup, and it also tells us something about the chiral algebra of the conformal field theory living at the boundary.

3.2 The chiral algebra at the boundary

The relation between the chiral algebra at the boundary and G_L, G_R arises as follows. At low energies the supergravity theory on AdS_3 is described by the difference between two Chern-Simons theories [51–53],

$$S = \frac{k}{4\pi} \int (A \wedge dA + \frac{2}{3} A \wedge A \wedge A) - \frac{k}{4\pi} \int (\bar{A} \wedge d\bar{A} + \frac{2}{3} \bar{A} \wedge \bar{A} \wedge \bar{A}) \quad (3.1)$$

where A is a connection for G_L and $\bar{A}$ a connection for G_R . Here, $k = Q_1 Q_5$ for the D1-D5 brane system. More generally, $k = \ell^4 / G_6$, where G_6 is a dimensionless six-dimensional Newton's constant that appears in the six-dimensional action as $S = \frac{1}{8\pi^3 \alpha'^2 G_6} \int d^6 x \sqrt{-g} R$. For the D1-D5 brane system, $G_6 = g^2 / v = g_6^2$.

To see the relation between this Chern-Simons theory and the chiral algebra on the boundary, we generalize the arguments of [54–56, 39, 27], and in particular of [57, 58]. Since the connections $A, \bar{A}$ contain the three-dimensional spin-connection and vielbein, we have to impose a suitable boundary condition on them in order to describe AdS_3 . The vielbein and spin-connection live in the $Sl(2, \mathbf{R})$ subgroup of G_L . Denote by $Sl(2, \mathbf{R}) \xrightarrow{j} G_L$ the corresponding embedding. Then the boundary conditions are that to leading order in $r = U/\ell^2$

$$A_i \sim b^{-1} \partial_i b + b^{-1} A_i^{(0)} b \quad (3.2)$$

where

$$b = j \begin{pmatrix} \sqrt{r} & 0 \\ 0 & 1/\sqrt{r} \end{pmatrix} \quad (3.3)$$

and $A_r^{(0)} = A_-^{(0)} = 0$, $x^\pm = t \pm x^5$, and

$$A_+^{(0)} = j(T^-) + T(x^+, x^-). \quad (3.4)$$

Here, T^+, T^0, T^- is a basis for $sl(2, \mathbf{R})$ and $T(x^+, x^-)$ is an arbitrary x^+, x^- -dependent element of the Lie algebra of G_L subject to the condition $[j(T^+), T(x^+, x^-)] = 0$. The component of $T(x^+, x^-)$ proportional to $j(T^+)$ is the stress energy tensor. For instance, for $Sl(2, \mathbf{R})$ $A_+^{(0)} = \begin{pmatrix} 0 & T(x^+, x^-) \\ 1 & 0 \end{pmatrix}$. The constraints imposed on $A_+^{(0)}$ are exactly the same constraints one imposes on the currents of WZW theory when one performs Hamiltonian reduction based on embeddings of sl_2 [59–62]. The relation between constrained currents and 2d gravity [63, 64] and Chern-Simons theory [65, 66] is well-known. The generators of the chiral algebra of the boundary theory are in one-to-one correspondence with the components of $T(x^+, x^-)$. The symmetries they generate are precisely the gauge transformations that preserve the form of $A_+^{(0)}$, and from the form of the gauge transformations one can immediately read off their Poisson brackets. In particular, the central charge is equal to $c = 6k$.

Thus, to summarize, the left-moving chiral algebra of the conformal field theory at the boundary is given by the Hamiltonian reduction of the current algebra based on G_L , and similarly for the right-movers. Conversely, G_L is obtained by keeping only the modes A_m with $|m| < \Delta$ for each spin- Δ generator of the chiral algebra. Below we list the supergroups listed in [50] and the chiral algebras one obtains from them by Hamiltonian reduction.

Table 1: 2d anti-de Sitter supergroups and their chiral algebras. See [67–72] and references therein for more details.

$Os p(N 2, \mathbf{R})$	$O(N)$ extended superconformal algebras; $N = 1$ and $N = 2$ are the usual $N = 1, 2$ superconformal algebras
$SU(N 1, 1)$	$U(N)$ extended superconformal algebras
$SU(2 1, 1)$	“small” $N = 4$ algebra
$Os p(4^* 2N)$	$Sp(N)$ extended superconformal algebras
$G(3)$	octonionic $N = 7$ algebra
$F(4)$	octonionic $N = 8$ algebra
$D(2, 1, \alpha)$	“large” $N = 4$ algebra $\mathcal{A}_\gamma$

As we already pointed out, the algebra of interest for our case is $SU(2, 1|1)$, in agreement with the fact that the boundary CFT is a $N = (4, 4)$ or $N = (4, 0)$ conformal field theory.

3.3 AdS masses and conformal weights

The relation between the mass in AdS and the conformal weights at boundary was already discussed for scalars in section 4.1 of [38]. Here we extend this discussion to include fields of arbitrary spin s on AdS_3 . The first step is to map the plane at $U = \infty$ to a cylinder, after which the AdS_3 part is described by the metric

$$\frac{ds^2}{\ell^2} = -\cosh^2 \rho d\tau^2 + \sinh^2 \rho d\phi^2 + d\rho^2. \quad (3.5)$$

Including the “isospin part”, the Virasoro generators are ($u = \tau + \phi$, $v = \tau - \phi$)

$$\begin{aligned} L_0 &= i\partial_u \\ L_{-1} &= ie^{-iu} \left(\coth 2\rho \partial_u - \frac{1}{\sinh 2\rho} \partial_v + \frac{i}{2} \partial_\rho - \frac{i}{2} s \coth \rho \right) \\ L_1 &= ie^{iu} \left(\coth 2\rho \partial_u - \frac{1}{\sinh 2\rho} \partial_v - \frac{i}{2} \partial_\rho + \frac{i}{2} s \coth \rho \right) \end{aligned} \quad (3.6)$$

and similarly for $\bar{L}_{0,\pm 1}$ with $u \leftrightarrow v$ and $s \rightarrow -s$. Primary fields satisfying $L_1 \psi = \bar{L}_1 \psi = 0$, $L_0 \psi = h\psi$ and $\bar{L}_0 \psi = \bar{h}\psi$ exist only if $s = h - \bar{h}$ and are then given by

$$\psi \sim \frac{e^{-ihu - i\bar{h}v}}{(\cosh \rho)^{h+\bar{h}}} \quad (3.7)$$

By evaluating the sum of the Casimirs of the two $Sl(2, \mathbf{R})$'s using the explicit expressions (3.6) we find that for the primary field ψ

$$(2h(h-1) + 2\bar{h}(\bar{h}-1))\psi = \ell^2 \square \psi + s^2 \coth^2 \rho \psi \quad (3.8)$$

where $\square$ is the Laplacian on scalar fields. This can be rewritten as

$$\left(\square + \frac{s^2}{\ell^2 \sinh^2 \rho} \right) \psi = m^2 \psi \quad (3.9)$$

with

$$\ell^2 m^2 = 2h(h-1) + 2\bar{h}(\bar{h}-1) - s^2 = (h+\bar{h})(h+\bar{h}-2). \quad (3.10)$$

For large ρ , the angular momentum part in (3.9) decouples, and m is what we would like to call the AdS mass. Equation (3.10), together with $s = h - \bar{h}$, completely determine the spin and mass on AdS_3 in terms of the conformal weights at the boundary, and vice versa.

4 The Kaluza-Klein Spectrum

4.1 The spectrum of harmonics

To determine the KK spectrum of some 6d supergravity compactified on $S^3 \times AdS^3$, we need to expand every field in harmonics on S^3 , insert this into the linearized field equations, and diagonalize the remaining equations on AdS^3 . From these field equations one can read off the masses of the various KK excitations on AdS^3 . This is a rather complicated procedure, which was done for various ten and eleven-dimensional cases in [73–78], and recently for 6d (2,0) supergravity in [79]. To determine precisely the AdS mass corresponding to each harmonic requires a rather precise knowledge of the field equations, but to determine which harmonics appear requires only a knowledge of the field content of the theory.

In fact, it is rather easy to find the complete spectrum of harmonics in the theory. The sphere S^3 is the homogeneous space $SO(4)/SO(3)$, and the harmonics that appear on homogeneous spaces were discussed in [80]. Any field on the theory can be decomposed as the sum of products of fields living on AdS^3 and S^3 . Each field on S^3 transforms in some representation R_4 of the isometry group $SO(4)$, but in addition transforms in some representation R_3 of the local Lorentz group $SO(3)$ of S^3 . According to [80], the only representations R_4 of $SO(4)$ that appear in the harmonic expansion are those that contain R_3 in the decomposition of R_4 in $SO(3)$ representations. For example, a vector field on S^3 transforms in the $\mathbf{3}$ of $SO(3)$. This $SO(3) = SU(2)$ is the diagonal $SU(2)$ subgroup of $SO(4) = SU(2) \times SU(2)$. If we decompose a representation $(2\mathbf{j}_1 + \mathbf{1}, 2\mathbf{j}_2 + \mathbf{1})$ of $SO(4)$ in terms of $SO(3)$ representations and require that the result contains $\mathbf{3}$, we find that $|j_1 - j_2|$ has to be zero or one. Thus, a vector field on S^3 gives rise to the harmonics that transform as $(\mathbf{m}, \mathbf{m} - \mathbf{2})$, $(\mathbf{m}, \mathbf{m})$ and $(\mathbf{m}, \mathbf{m} + \mathbf{2})$ under $SO(4)$.

The various $SO(3)$ representations that appear when decomposing a field in pieces living on AdS^3 and S^3 can be found as follows. Since ultimately we are interested in the solutions of the field equations modulo gauge invariance, the degrees of freedom contained in some six-dimensional field are labeled by a representation of the little group $SO(4)$. The fields can depend arbitrarily on the coordinates of S^3 , and the local Lorentz group $SO(3)$ of the three-sphere should therefore be identified with an $SO(3)$ subgroup of the little group $SO(4)$. Thus, all $SO(3)$ representations are found by decomposing the representation of the little group in representations of $SO(3)$ ¹.

¹More generally, consider some $p + q$ -dimensional theory compactified on $S^p \times AdS_q$. Any field ϕ transforming in some representation R of the little group $SO(p+q-2)$ can be decomposed as $\oplus_i n_i S_i \otimes T_i$,

As an example, consider the metric in six dimensions. The graviton transforms in the $(\mathbf{3}, \mathbf{3})$ of the little group $SO(4)$, so that the relevant $SO(3)$ representations are $\mathbf{1} + \mathbf{3} + \mathbf{5}$. For each of these we find the corresponding harmonics as above, and altogether we find that the total set of harmonics coming from the graviton transforms as

$$\oplus_m ((\mathbf{m}, \mathbf{m} - \mathbf{4}) + 2(\mathbf{m}, \mathbf{m} - \mathbf{2}) + 3(\mathbf{m}, \mathbf{m}) + 2(\mathbf{m}, \mathbf{m} + \mathbf{2}) + (\mathbf{m}, \mathbf{m} + \mathbf{4})) \quad (4.1)$$

As another example, consider $(n_L, n_R) = (2, 0)$ supergravity with n_T tensor multiplets. The theory is anomalous unless $n_T = 21$ which is the theory obtained from type IIB on K3, but for the purpose of counting harmonics we can take n_T arbitrary. The gravity multiplet consists of a graviton, five self-dual two-forms and four gravitinos, and the tensor multiplet of one anti self-dual two-form, four fermions and five scalars. The total representation of the little group is [49]

$$(\mathbf{3}, \mathbf{3}) + 4(\mathbf{2}, \mathbf{3}) + 5(\mathbf{1}, \mathbf{3}) + n_T(\mathbf{3}, \mathbf{1}) + 4n_T(\mathbf{2}, \mathbf{1}) + 5n_T(\mathbf{1}, \mathbf{1}) \quad (4.2)$$

which yields for the total set of harmonics

$$\begin{aligned} \oplus_m ((\mathbf{m}, \mathbf{m} \pm \mathbf{4}) + 4(\mathbf{m}, \mathbf{m} \pm \mathbf{3}) + (n_T + 7)(\mathbf{m}, \mathbf{m} \pm \mathbf{2}) \\ + 4(n_T + 2)(\mathbf{m}, \mathbf{m} \pm \mathbf{1}) + (6n_T + 8)(\mathbf{m}, \mathbf{m})) . \end{aligned} \quad (4.3)$$

4.2 Representations of the AdS supergroup

Having determined the total set of harmonics, we would like to extract from them the spectrum of the fields of the SCFT living at the boundary of AdS^3 . In principle we could first determine the masses and spins of the various KK modes on AdS^3 from the field equations, and then relate these masses to the scaling dimensions using (3.10). However, it is also possible to avoid this and to obtain the scaling dimensions and $U(1)$ weights at the boundary using representation theory, if there are sufficiently many supersymmetries. The point is that the complete KK spectrum should fall into representations of the anti-de Sitter supergroup $G_L \times G_R$, whose generators include the generators L_0 and J_0 of the boundary (super)conformal algebra. If our knowledge of the harmonics is enough to completely determine the set of representations of the anti-de Sitter supergroup that appear, we can simply read off the various values of L_0 and J_0 that appear in the boundary CFT, and in particular we can determine the set of chiral primaries. In [74] this strategy

where S_i and T_i are irreducible representations of $SO(p)$ and $SO(q-2)$. If $U_{i,r}$, $r = 1, \dots$ is the set representations of $SO(p+1)$ that contain S_i when decomposing it into $SO(p)$ representations, then the set of harmonics for ϕ is $\oplus_{i,r} n_i C_{i,r}$

was used to determine the KK spectrum of IIB supergravity on $S^5 \times AdS^5$. In that case, the only representations of the relevant anti-de Sitter supergroup $U(2, 2|4)$ that appear are in so-called short representations. This follows from the fact that short representations are the only ones having spins of at most two as required for a theory of supergravity. Although this argument is restricted to theories with 32 supersymmetries, we will find that also for the six-dimensional theories with 16 and 8 supersymmetries the only multiplets appearing in the KK reduction are short multiplets.

In the case of AdS^3 , the relevant anti-de Sitter supergroup is not simple but the product of two groups $G_L \times G_R$. The possible simple supergroups for $G_{L,R}$ are reproduced in Table 1. For a discussion of these algebras and their representations, see [50]. The representations are constructed in [50] using the oscillator method [81], which means that one expresses the generators of the algebra in terms of a set of free bosonic and harmonic oscillators and then uses the Fock space of these oscillators to construct representations of the algebra. Short representations are obtained by acting with the generators on the Fock vacuum, and are labeled by the number of oscillators in terms of which the generators are constructed.

In our case, we are mainly interested in the representations of $SU(2|1, 1)$. Although the structure of short representations of this superalgebra is well-known (it parallels the representation theory of the $N = 4$ superconformal algebra), we will rederive it here using the oscillator method because that method generalizes to other algebras and other dimensions.

For $SU(2|1, 1)$, we introduce $\{\vec{a}_i, \vec{a}_i^\dagger, \vec{\psi}_i, \vec{\psi}_i^\dagger\}$, where $i = 1, 2$, $\vec{a}_i$ are k -component vectors of bosonic creation and annihilation operators and $\vec{\psi}_i$ are k -component vectors of fermionic creation and annihilation operators. The brackets are

$$[a_i(p), a_j(q)^\dagger] = \delta_{p,q} \delta_{i,j}, \quad \{\psi(p)_i, \psi(q)_j^\dagger\} = \delta_{p,q} \delta_{i,j} \quad (4.4)$$

The lowering operators of $SU(2|1, 1)$ are

$$L_- = \{\vec{a}_1 \cdot \vec{a}_2, \vec{a}_1 \cdot \vec{\psi}_1 \pm \vec{a}_2 \cdot \vec{\psi}_2, \vec{\psi}_1 \cdot \vec{\psi}_2\}, \quad (4.5)$$

the generators of the maximal compact subsuperalgebra are

$$L_0 = \{\vec{a}_1 \cdot \vec{a}_1^\dagger + \vec{a}_2 \cdot \vec{a}_2^\dagger, \vec{\psi}_1^\dagger \cdot \vec{\psi}_1 - \vec{\psi}_2^\dagger \cdot \vec{\psi}_2, \vec{a}_1 \cdot \vec{\psi}_2^\dagger \pm \vec{a}_2 \cdot \vec{\psi}_1^\dagger, \vec{a}_1^\dagger \cdot \vec{\psi}_2 \pm \vec{a}_2^\dagger \cdot \vec{\psi}_1\}, \quad (4.6)$$

and the raising operators are

$$L_+ = \{\vec{a}_1^\dagger \cdot \vec{a}_2^\dagger, \vec{a}_1^\dagger \cdot \vec{\psi}_1^\dagger \pm \vec{a}_2^\dagger \cdot \vec{\psi}_2^\dagger, \vec{\psi}_1^\dagger \cdot \vec{\psi}_2^\dagger\}. \quad (4.7)$$

A single irrep of $SU(2|1,1)$ contains several different irreps of $Sl(2, \mathbf{R}) \times SU(2)$. The fermionic operators $Q^\pm = \vec{a}_1^\dagger \cdot \vec{\psi}_1^\dagger \pm \vec{a}_2^\dagger \cdot \vec{\psi}_2^\dagger$ map between the various irreps of $Sl(2, \mathbf{R}) \times SU(2)$. The algebra $Sl(2, \mathbf{R})$ consists of the Virasoro generators L_{-1}, L_0, L_{+1} . These generators are given by

$$L_{-1} = \vec{a}_1^\dagger \cdot \vec{a}_2^\dagger \quad (4.8)$$

$$L_0 = \frac{1}{2}(\vec{a}_1 \cdot \vec{a}_1^\dagger + \vec{a}_2 \cdot \vec{a}_2^\dagger) \quad (4.9)$$

$$L_{+1} = \vec{a}_1 \cdot \vec{a}_2. \quad (4.10)$$

It will also be convenient to identify the $SU(2)$ subgroup

$$J^- = \vec{\psi}_2 \cdot \vec{\psi}_1 \quad (4.11)$$

$$J^0 = \vec{\psi}_1^\dagger \cdot \vec{\psi}_1 - \vec{\psi}_2 \cdot \vec{\psi}_2^\dagger \quad (4.12)$$

$$J^+ = \vec{\psi}_1^\dagger \cdot \vec{\psi}_2^\dagger. \quad (4.13)$$

Here, J^0 is normalized so that it agrees with the usual definition of J^0 in an $N = 2$ superconformal algebra.

As in [50], we find that the short multiplets consist of

states	j	j'	L_0
$ 0\rangle$	$k/2$	0	$k/2$
$Q^\pm 0\rangle$	$(k-1)/2$	1/2	$(k+1)/2$
$Q^\pm Q^\pm 0\rangle$	$(k-2)/2$	0	$(k+2)/2$

(4.14)

where j is the spin of $SU(2)$, j' is the spin with respect to the global $SU(2)$ automorphism group under which $Q^\pm$ is a doublet, and L_0 is the conformal weight of the ground state. Thus a short multiplet of $SU(2|1,1)$ consists of four representations of $Sl(2, \mathbf{R}) \times SU(2)$. Not surprisingly, the structure is exactly the same as that of a representation of the $N = 4$ superconformal algebra whose highest weight state is a chiral primary. The representation (4.14) contains indeed precisely one chiral primary, namely the vacuum $|0\rangle$ which has J^0 eigenvalue k and L_0 eigenvalue $k/2$. It is easy to see that only short multiplets contain a chiral primary field, as one would expect.

An example of a long multiplet is the one built out of the ground states $\psi_1^\dagger(1)|0\rangle$,

$a_2^\dagger(1)|0\rangle$. The content of this multiplet reads

states	j	j'	L_0
$a_2^\dagger(1) 0\rangle$	$k/2$	0	$(k+1)/2$
$\psi_1^\dagger(1) 0\rangle$	$(k-1)/2$	0	$k/2$
$Q^\pm a_2^\dagger(1) 0\rangle$	$(k-1)/2$	1/2	$(k+2)/2$
$Q^\pm \psi_1^\dagger(1) 0\rangle$	$(k-2)/2$	1/2	$(k+1)/2$
$Q^\pm Q^\pm a_2^\dagger(1) 0\rangle$	$(k-2)/2$	0	$(k+3)/2$
$Q^\pm Q^\pm \psi_1^\dagger(1) 0\rangle$	$(k-3)/2$	0	$(k+2)/2$

(4.15)

Let us now come back to the compactifications of 6d supergravities on $S^3 \times AdS^3$ with $n_L + n_R > 1$. The complete KK spectrum should organize itself as representations of $G_L \times G_R = SU(2|1, 1) \times SU(2|1, 1)$. We will assume that all KK states will fall into short representations of $SU(2|1, 1) \times SU(2|1, 1)$. If we denote the short representation given in (4.14) by $\mathbf{k} + \mathbf{1}$, then any short representation of the product group is of the form $(\mathbf{k} + \mathbf{1}, \mathbf{k}' + \mathbf{1})_S$. The subscript S has been included in order to avoid confusion with representations $(\mathbf{m}, \mathbf{m}')$ of $SO(4)$, the group of rotations of the three-sphere. The idea is now to take a set of short multiplets, decompose these into $SO(4)$ representations, and compare the result to the set of $SO(4)$ representations obtained from the KK analysis of supergravity described in the previous section. Requiring that the two agree will give us the set of short multiplets, and from (4.14) we then obtain the spectrum of highest weight states of the CFT, and in particular the set of chiral primaries of the CFT. Supergravity yields only “single particle” states of the conformal field theory at the boundary. The full spectrum is obtained by taking arbitrary products of these single particle states.

It turns out that in all cases with $n_L + n_R > 1$ the spectrum of short multiplets is of the form

$$\oplus_m (t_2(\mathbf{m}, \mathbf{m} \pm \mathbf{2})_S + t_1(\mathbf{m}, \mathbf{m} \pm \mathbf{1})_S + t_0(\mathbf{m}, \mathbf{m})_S) \quad (4.16)$$

The reason that the difference between the two integers is at most two is because supergravity has fields of spin at most two. A larger difference would correspond to higher spin fields. Using the structure of the short multiplet (4.14), we can decompose each $(\mathbf{m}, \mathbf{m}')_S$ in representations of $SO(4)$,

$$(\mathbf{m}, \mathbf{m}')_S = \oplus_{i=0}^2 \oplus_{j=0}^2 \begin{pmatrix} 2 \\ i \end{pmatrix} \begin{pmatrix} 2 \\ j \end{pmatrix} (\mathbf{m} - \mathbf{i}, \mathbf{m}' - \mathbf{j}) \quad (4.17)$$

which shows that (4.16) is equivalent to the following spectrum of $SO(4)$ representations

$$\begin{aligned} & \oplus_m (t_2(\mathbf{m}, \mathbf{m} \pm \mathbf{4}) + (4t_2 + t_1)(\mathbf{m}, \mathbf{m} \pm \mathbf{3}) + (6t_2 + 4t_1 + t_0)(\mathbf{m}, \mathbf{m} \pm \mathbf{2}) \\ & + (4t_2 + 7t_1 + 4t_0)(\mathbf{m}, \mathbf{m} \pm \mathbf{1}) + (2t_2 + 8t_1 + 6t_0)(\mathbf{m}, \mathbf{m})) \end{aligned} \quad (4.18)$$

Before proceeding, we will first discuss the spectrum of the conformal field theory in some more detail, and then compare (4.18) to the KK spectrum of various supergravities.

5 The $K3^N/S_N$ Conformal Field Theory

The conformal field theory of the D1-D5 system with the D5 branes wrapped on $K3$ has been conjectured to be described by a deformation of the supersymmetric sigma model whose target space is the orbifold $K3^N/S_N$ [20, 4]. In order to compare the spectrum obtained from the KK reduction to that of the conformal field theory, we will need to know in some detail the spectrum of this conformal field theory.

The most robust set of states in the conformal field theory are the states which are chiral primary both for the left and right movers. Since these states are in ultrashort multiplets and their conformal weights satisfy a BPS bound, their spectrum is independent of any perturbation of the conformal field theory, and can be conveniently encoded in terms of the generalized Poincaré polynomial

$$P_{t,\bar{t}} = \text{Tr}(t^{J_0} \bar{t}^{\bar{J}_0}) \quad (5.1)$$

where the trace is taken over the space of chiral primaries only. In case the superconformal field theory is a supersymmetric sigma model with target space M , the Poincaré polynomial equals

$$P_{t,\bar{t}} = \sum_{p,q} h_{p,q} t^p \bar{t}^q \quad (5.2)$$

where $h^{p,q}$ are the Betti numbers of M [82]. The Poincaré polynomial of a resolution of $K3^N/S_N$ called the Hilbert scheme of N points on $K3$ was computed in [83] and has generating function

$$\sum_{N \geq 0} Q^N P_{t,\bar{t}}(K3^N/S_N) = \prod_{m=1}^{\infty} \prod_{p,q} \left(1 + (-1)^{p+q+1} Q^m t^{p+m-1} \bar{t}^{q+m-1}\right)^{(-1)^{p+q+1} h^{p,q}}. \quad (5.3)$$

An alternative derivation of this result uses standard orbifold conformal field theory. The Hilbert space of S_N orbifolds can be decomposed in terms of Hilbert spaces of $\mathbf{Z}_n$ orbifolds as in [84]. According to the discussion of $\mathbf{Z}_n$ orbifolds in [85], a state with conformal weight h and $U(1)$ weight q in the original CFT M gives rise to various states in the orbifold CFT $M^n/\mathbf{Z}_n$. In the untwisted sector we get a state with $h' = nh$ and $q' = nq$, and in the twisted sector states with $q' = q$ and $h' = \frac{h+m}{n} + \frac{c}{24} \frac{n^2-1}{n}$, where m is some nonnegative integer.

Consider now a supersymmetric sigma model with target space a complex Kähler manifold M of complex dimension d , and central charge $c = 3d$. A chiral primary with $h = p/2$ and $q = p$ in the left-moving NS sector corresponds via spectral flow to a Ramond ground state with $h = d/8$ and $q = p - d/2$. According to the discussion above, the untwisted sector of $M^n/\mathbf{Z}_n$ contains a Ramond ground state with $h = nd/8$ and $q = n(p - d/2)$, whereas the twisted sector contains a Ramond ground state with $h = nd/8$ and $q = p - d/2$ (the case $m = 0$). This corresponds to chiral primaries with $(h, q) = (np/2, np)$ and $(h, q) = ((p + \frac{d}{2}(n-1))/2, p + \frac{d}{2}(n-1))$ in the NS sector. The complete Poincare polynomial is now determined by combining these latter states with their right moving counterparts, and by subsequently writing down a second quantized partition function for these generators [84]. This leads to

$$\sum_{N \geq 0} Q^N P_{t, \bar{t}}(M^N/S_N) = \prod_{m=1}^{\infty} \prod_{p,q} \left(1 + (-1)^{p+q+1} Q^m t^{p+\frac{d}{2}(m-1)} \bar{t}^{q+\frac{d}{2}(m-1)} \right)^{(-1)^{p+q+1} h^{p,q}}. \quad (5.4)$$

For $d = 2$ we indeed recover (5.3).

Interestingly, $P_{t, \bar{t}}(M^N/S_N)$ does have a well-defined $N \rightarrow \infty$ limit. Because it has a single factor of $(1 - Q)^{-1}$, (5.4) is of the form

$$a_0 + (a_0 + a_1)Q + (a_0 + a_1 + a_2)Q^2 + \dots = (1 - Q)^{-1}(a_0 + a_1Q + a_2Q^2 + \dots) \quad (5.5)$$

Thus the $N \rightarrow \infty$ limit is obtained by extracting the factor of $(1 - Q)^{-1}$ and taking $Q \rightarrow 1$,

$$P_{t, \bar{t}}(M^\infty/S_\infty) = \lim_{Q \rightarrow 1} (1 - Q) \sum_{N \geq 0} Q^N P_{t, \bar{t}}(M^N/S_N). \quad (5.6)$$

For $d = 2$, this is the partition function for a set of unconstrained bosonic and fermionic oscillators. If we define $n_\Delta = \sum_p h^{p+\Delta, p}$, then for sufficiently large m there will be n_Δ oscillators of degree $(m + \Delta, m)$. The oscillators are bosonic (fermionic) depending on whether Δ is even (odd). The only exception is that there are only $h^{1,0}$ generators of degree $(1, 0)$, $h^{0,1}$ of degree $(0, 1)$, and $h^{0,0} + h^{1,1}$ of degree $(1, 1)$. In particular, for K3 there are only bosonic generators, one of degree $(m, m + 2)$ and one of degree $(m + 2, m)$ for $m \geq 0$, 22 of degree (m, m) for $m > 1$ and 21 of degree $(1, 1)$.

The spectrum of left and right-moving chiral primaries is not the only part of the spectrum which is independent of marginal deformations of the theory. A more general object with this property is the elliptic genus, which can only change if a phase transition occurs. The elliptic genus is defined by

$$Z(\tau, z) = \text{Tr}_{RR}(-1)^F q^{L_0 - c/24} \bar{q}^{\bar{L}_0 - c/24} y^{J_0} \quad (5.7)$$

with $q = e^{2\pi i\tau}$ and $y = e^{2\pi iz}$, and the trace is over the Ramond sector of the Hilbert space [86–88]. The elliptic genus for $K3$ was considered in [89], and its explicit form is [90]

$$Z(\tau, z) \equiv \sum_{m,l} c(m, l) q^m y^l = 24 \left(\frac{\theta_3(\tau, z)}{\theta_3(\tau, 0)} \right)^2 - 2 \frac{\theta_4(\tau, 0)^4 - \theta_2(\tau, 0)^4}{\eta(\tau)^4} \left(\frac{\theta_1(\tau, z)}{\eta(\tau)} \right)^2. \quad (5.8)$$

With this definition of $c(m, l)$, the elliptic genus of $K3^N/S_N$ has generating function [84]

$$\sum_{N \geq 0} p^N Z(K3^N/S_N; \tau, z) = \prod_{n > 0, m \geq 0, l} \frac{1}{(1 - p^n q^m y^l)^{c(nm, l)}}. \quad (5.9)$$

The first few terms in the elliptic genus of $K3^N/S_N$ for $N > 6$ read¹

$$\begin{aligned} Z(K3^N/S_N; \tau, z) = & ((N+1)y^{-N} + \dots) \\ & + q((22N-2)y^{-N-1} + (464N-592)y^{-N} + \dots) \\ & + q^2((277N-323)y^{-N-2} + (5652N-13716)y^{-N-1} \\ & + (67131N-244053)y^{-N} + \dots) \\ & + \mathcal{O}(q^3) \end{aligned} \quad (5.10)$$

Translating (5.10) back to the NS-NS sector, we find for example that the orbifold CFT has $67131N - 244053$ states of the form $|q, h\rangle_L \otimes |q', h'\rangle_R = |0, 2\rangle_L \otimes |q', q'/2\rangle_R$ where we should count the states weighted with the sign $(-1)^{q'}$. To count how many states of this form are descendants of chiral primaries, we need to know the first few terms in the Poincaré polynomial $P_{t,-1}(K3^N/S_N)$. The coefficient in front of t^q counts the number of chiral primaries of the form $|q, q/2\rangle_L \otimes |q', q'/2\rangle_R$, weighted with $(-1)^{q'}$. For sufficiently large N we find

$$\begin{aligned} P_{t,-1}(K3^N/S_N) = & (N+1) - (22N-2)t + (277N-323)t^2 - (2576N-5752)t^3 \\ & + (19574N-64474)t^4 + \mathcal{O}(t^5). \end{aligned} \quad (5.11)$$

Since we know the number of descendants with $|q, h\rangle_L = |0, 2\rangle_L$ of the chiral primary $|q, q/2\rangle_L$ (namely 4, 8, 5, 2, 1 for $q = 0, 1, 2, 3, 4$ respectively) we find that the descendants of the chiral primaries contribute $15397N - 54521$ to the $67131N - 244053$ states of the form $|0, 2\rangle_L \otimes |q', q'/2\rangle_R$. Therefore, there are many states which are not descendants of chiral primaries but which do survive the marginal perturbation of the orbifold. The first such states are of the form $|0, 1\rangle_L \otimes |q', q'/2\rangle_R$. The elliptic genus shows that there are $464N - 592$ such states, but only $233N - 319$ are descendants of chiral primaries. How to account for the missing states in the supergravity description was the puzzle raised in [40]. We will explicitly identify the missing states of type $|0, 1\rangle_L \otimes |q', q'/2\rangle_R$ in section 6.2.

¹Notice that the elliptic genus diverges for $N \rightarrow \infty$, so that we cannot use it to count states in the strict supergravity limit.

6 The CFT Spectrum versus the KK Spectrum: $(n_L, n_R) = (2, 0)$

6.1 Single particle states

The set of $SO(4)$ representations that appears in the compactification of $(n_L, n_R) = (2, 0)$ supergravity on $S^3 \times AdS_3$ was already computed in (4.3). We can compare this to (4.18) and find that the two precisely agree when $t_2 = 1$, $t_1 = 0$ and $t_0 = n_T + 1$. Thus, the set of short multiplets is of the form

$$\oplus_m ((\mathbf{m}, \mathbf{m} \pm \mathbf{2})_S + (n_T + 1)(\mathbf{m}, \mathbf{m})_S) \quad (6.1)$$

Since each short multiplet contains precisely one chiral primary, we find that there are $n_T + 1$ chiral primaries of degree (m, m) , one of degree $(m, m + 2)$ and one of degree $(m + 2, m)$. These are all single particle states from the point of supergravity. To construct an arbitrary state in the boundary conformal field theory we should consider arbitrary products of these single particle states. Therefore, the complete set of chiral primaries is obtained by taking arbitrary products of the single particle chiral primaries. A convenient way to write down the spectrum is to associate a bosonic creation operator to each single particle chiral primary and then to look at the Fock space they create. For each state in the Fock space there is exactly one chiral primary. This is precisely the same picture (for $n_T = 21$) as we found in conformal field theory below equation (5.6). Thus, it seems that there is perfect agreement as far as the chiral primaries are concerned.

This analysis is, however, restricted to the higher harmonics, as (4.3) is only valid for sufficiently large m . For example, the $SO(3)$ representation $\mathbf{5}$ is only contained in $(\mathbf{m}, \mathbf{m})$ if $m \geq 3$. The multiplicities of $(\mathbf{m}, \mathbf{n})$ in (4.3) are correct for $m + n > 4$, for $m + n \leq 4$ they are given by

$SO(4)$ representation	$(\mathbf{1}, \mathbf{1})$	$(\mathbf{1}, \mathbf{2})$	$(\mathbf{1}, \mathbf{3})$	$(\mathbf{2}, \mathbf{1})$	$(\mathbf{2}, \mathbf{2})$	$(\mathbf{3}, \mathbf{1})$
multiplicity	$5n_T + 1$	$4n_T + 4$	$n_T + 6$	$4n_T + 4$	$6n_T + 7$	$n_T + 6$

(6.2)

If we carefully take these multiplicities into account and redo the decomposition into short multiplets, we find that there is one chiral primary of degree $(m, m + 2)$ and one of degree $(m + 2, m)$ for $m > 0$ only, that there are n_T chiral primaries of degree $(1, 1)$ and $n_T + 1$ of degree (m, m) for $m > 1$. This is exactly the same as what we found below (5.6), except that the chiral primaries of degrees $(2, 0)$ and degrees $(0, 2)$ are absent. This is perhaps not too surprising, as these chiral primaries correspond in conformal field theory to the descendants $J_{-1}^+|0\rangle$ and $\bar{J}_{-1}^+|0\rangle$ of the identity operator.

The way to account for these states is similar to what happens in four dimensions, where $N = 4$ super Yang-Mills is conjectured to be dual to type IIB supergravity on

$S^5 \times AdS_5$. In [73] it was shown that there are AdS_5 degrees of freedom that are pure gauge in the bulk and can be gauged away completely except at the boundary. These degrees of freedom form a so-called singleton representation of the relevant AdS supergroup $SU(2, 2|4)$, and the field content of the singleton representation is precisely that of an $U(1)$ $N = 4$ super Yang-Mills multiplet. This $U(1)$ is naturally identified with the decoupled $U(1)$ of the $U(N)$ $N = 4$ super Yang-Mills theory.

The group $SU(1, 1|2)$ does not have singleton representations, but nevertheless something similar happens in our case. The $SU(1, 1|2) \times SU(1, 1|2)$ Chern-Simons theory does not have any propagating degrees of freedom in $2 + 1$ dimensions. However, as discussed in section 3.2, the gauge field is subject to a boundary condition that contains fields living at the boundary. These fields are the generators of the left and right-moving chiral algebra, and their positive frequency modes make up the representations $(\mathbf{1}, \mathbf{3})_S$ and $(\mathbf{3}, \mathbf{1})_S$ of $SU(1, 1|2) \times SU(1, 1|2)$. Thus, although these are not singleton representations, they do correspond to pure gauge degrees of freedom in the bulk. Including them in the list of short multiplets obtained from supergravity provides us with the missing chiral primaries of degrees $(2, 0)$ and $(0, 2)$. We now have a complete and detailed agreement between the chiral primaries in the orbifold conformal field theory and those obtained from supergravity.

Although we have no rigorous argument why all KK states should fall into short representations, it is impossible to organize the KK spectrum differently. It would be interesting to have a more fundamental understanding of this fact.

6.2 Multi-particle states

At this stage one might argue that since the multiparticle states contain all chiral primaries of the orbifold conformal field theory, and we also have all generators of the $N = 4$ algebra at our disposal, the complete KK spectrum is exactly equivalent to the set of chiral primaries and their descendants. If true, this would lead to a discrepancy between the states obtained from supergravity and the states that should be present in conformal field theory according to the elliptic genus calculation of section 5. This puzzle, raised in [40], can have several solutions, each of which are somewhat problematic.

(i) There could be a phase transition as one deforms the conformal field theory from the orbifold point to the actual conformal field theory which is living at the boundary. The latter has been argued to be a strongly coupled conformal field theory at zero world-sheet theta angle [15], so a priori nothing prevents such a phase transition point at which the elliptic genus would jump. However, it would be quite awkward to have so many

states appear/disappear at the singularity and it is not clear how that could be compatible with unitarity. Such a drastic jump is also not something that occurs when we consider type IIA on a singular K3 that gives rise to enhanced gauge symmetry. The corresponding conformal field theory is also strongly coupled, but we expect only some additional nonperturbative massless states to appear in the spectrum [91].

(ii) Since states with arbitrary large $\bar{L}_0$ contribute to each term in the elliptic genus, it could potentially receive contributions from stringy states. This would require stringy states whose spin s and mass m satisfy $s \sim m\ell\sqrt{\alpha'}$. It is hard to imagine such states with an enormous spin arising from string theory.

(iii) All the additional states could live purely on the boundary of AdS_3 and therefore correspond to “singletons”, in the same way as the stress-tensor of the boundary theory corresponds to a singleton. Singletons correspond to pure gauge degrees of freedom in the bulk, and there does not seem to be room for such a large number of additional gauge fields in six-dimensional supergravity on $S^3 \times AdS_3$. Alternatively, the duality involving supergravity could be incomplete and the proper duality would require us to add the additional degrees of freedom by hand. In that case, there would be something crucial missing from the otherwise quite successful solitonic description of brane configurations.

From all these points of view it would be much nicer if all states would already be present in the supergravity description and we will now argue that in fact they are.

The main point is the fact that although the product of two chiral primaries is again a chiral primary, this is not true for their operator product expansion. The leading term is a chiral primary, but the subleading regular terms contain non-chiral primaries as well. One easy way to see this is by looking at the transformation properties of characters of $N = 4$ representations under modular transformations [92]. An equivalent way to put this is that correlation functions of chiral primaries contain a sum over all fields in the intermediate channels, not just over the chiral primary ones. Now according to the proposal of [2, 3], we can compute arbitrary correlators in the boundary theory by taking suitable boundary conditions for the fields in the bulk and computing the partition function for the bulk theory. It is then straightforward to compute correlators of non-chiral primaries: one simply considers correlators of chiral primaries, lets the arguments of the chiral primaries approach each other and subtracts out the leading singularities from the correlation function. For example, suppose the operator product expansion of the chiral primaries A and B contains the chiral primary C and the non-chiral primary D as

$$A(z)B(w) \sim C(w) + (z - w)D(w) + \dots \quad (6.3)$$

The two-point function of D can then be computed as

$$\langle D(x)D(y) \rangle = \lim_{z \rightarrow x} \lim_{w \rightarrow y} \left\langle \frac{A(z)B(x) - C(x)}{z - x} \frac{A(w)B(y) - C(y)}{w - y} \right\rangle \quad (6.4)$$

The right hand side can be obtained directly from the prescription of [2, 3]. Although this may seem somewhat indirect, all information about the conformal field theory is in this way present in the supergravity description.

There is a different way to state the above which is closer to the supergravity description. Recall that in the supergravity description the various single particle fields were organized in representations of $SU(1, 1|2) \times SU(1, 1|2)$. Multiparticle states are products of single particle states, transforming in the tensor product representation of $SU(1, 1|2) \times SU(1, 1|2)$. The tensor product of two short representations of $SU(1, 1|2)$ does not contain just a single short representation of $SU(1, 1|2)$, but various longer ones as well. These longer ones correspond to the non-chiral primaries. In other words, the product of two descendants of chiral primaries is not necessarily the descendant of a chiral primary. Indeed, in (6.3) the field $D(z)$ is nothing but $(\partial A(z))B(z)$, and we could also have computed the D -two point function via

$$\langle D(x)D(y) \rangle = \lim_{z \rightarrow x} \lim_{w \rightarrow y} \langle \partial A(z)B(x) \partial A(w)B(y) \rangle \quad (6.5)$$

As further evidence that this is the correct interpretation of multi-particle states, we will show in an example that this procedure accounts for all additional states predicted by the elliptic genus. Consider states of the form $|0, 1\rangle_L \otimes |q', q'/2\rangle_R$. The elliptic genus shows that there are $464N - 592$ such states, but only $233N - 319$ are descendants of chiral primaries, as we mentioned at the end of section 5. The only way to make states with $q = 0$ and $h = 1$ as multiparticle states is to take the product of two states with $h = 1/2$, which can only appear as descendants of chiral primaries with $q = 1$ and q' arbitrary. The cohomology classes of weight $(1, q')$ are: (i) for each i , $0 \leq i \leq N - 1$, there are 20 forms of degree $(1, 1) \otimes (0, 2)^i \equiv (1, 1 + 2i)$ ($(0, 2)$ represents the anti-holomorphic two-form), (ii) for each i , $0 \leq i \leq N - 2$, there is one form of degree $(1, 1) \otimes (0, 2)^i \equiv (1, 1 + 2i)$ and one of degree $(1, 3) \otimes (0, 2)^i \equiv (1, 3 + 2i)$. The total number of forms of degree $(1, q')$ is $20N + 2(N - 1) = 22N - 2$, which is responsible for the appearance of this factor in (5.10) and (5.11). Each of the $SU(1, 1|2)$ representations corresponding to a form of degree $(1, q')$ contains a spin-1/2 doublet of states with $h = 1/2$, namely $|1, 1/2\rangle_L \otimes |q', q'/2\rangle_R$ and $|-1, 1/2\rangle_L \otimes |q', q'/2\rangle_R$. The tensor product of two of such spin-1/2 representations contains a spin-1 representation with descendants of a chiral primary, and a singlet, which is a non-chiral primary of weight $|0, 1\rangle_L \otimes |q', q'/2\rangle_R$.

To count the number of non-chiral primaries, denote the forms under (i) above by $\beta_k \bar{\Omega}^i$, with β_k , $k = 1, \dots, 20$ the 20 $(1, 1)$ forms and $\bar{\Omega}$ the anti-holomorphic two-form. The forms under (ii) will be denoted by $\alpha_l \bar{\Omega}^i$, $l = 1, 2$. The forms β_k originate from forms on $K3$ and can therefore be multiplied by at most the $(N - 1)$ th power of $\bar{\Omega}$, whereas α_l originate from forms on $K3^2/S_2$ and can therefore be multiplied by at most the $(N - 2)$ th power of $\bar{\Omega}$. From the product of $\beta_k \beta_l \bar{\Omega}^i$ we get one non-chiral primary for each $k > l$, and since $\beta_k \beta_l$ should be thought of as living on $K3^2/S_2$, it can be multiplied by at most the $(N - 2)$ th power of $\bar{\Omega}$. Therefore, the total number of non-chiral primaries obtained this way is $\frac{19 \cdot 20}{2}(N - 1)$. Similarly, the forms $\beta_k \alpha_l$ live on $K3^3/S_3$ and give total of $20 \cdot 2 \cdot (N - 2)$ non-chiral primaries. Finally, $\alpha_1 \alpha_2$ lives on $K3^4/S_4$ and gives upon multiplication with $\bar{\Omega}^i$ a total of $N - 3$ non-chiral primaries. Altogether the total number of non-chiral primaries of the form $|0, 1\rangle_L \otimes |q', q'/2\rangle_R$ obtained this way is

$$190(N - 1) + 40(N - 2) + (N - 3) = 231N - 273 \quad (6.6)$$

Together with the $233N - 319$ descendants of chiral primaries we find a total number of $464N - 592$ states of the form $|0, 1\rangle_L \otimes |q', q'/2\rangle_R$, which is precisely the number predicted by the elliptic genus. We consider this as strong evidence in favor of our proposal.

7 Other 6d and 5d Supergravities and their KK Spectrum

7.1 6d supergravity with $n_L + n_R > 1$

In this section we briefly redo the analysis for the other 6d supergravities compactified on $S^3 \times AdS_3$. In all cases we find that the KK spectrum agrees with that of a conformal field theory whose target space is the symmetric product of some four-manifold. Furthermore, the KK spectrum allows us to read off the cohomology of the four-manifold. One new feature is the appearance of short multiplets $(\mathbf{m}, \mathbf{m}')_S$ with $m + m'$ odd. The chiral primaries in these short multiplets come from fermionic fields in the supergravity, and correspond to odd-degree forms in the target space of the non-linear sigma model. From either point of view it is clear that we should associate fermionic creation rather than bosonic creation operators to these chiral primaries, in complete agreement with (5.3) and the discussion below (5.6). A second new feature is the appearance of additional singletons, related to the additional supersymmetries. The singletons give rise to additional short multiplets of the form $(\mathbf{1}, \mathbf{2})_S$ and $(\mathbf{2}, \mathbf{1})_S$ on the boundary.

The general result is that if the KK spectrum, when decomposed into short multiplets,

is for sufficiently high harmonics given by (4.16)

$$\oplus_m(t_2(\mathbf{m}, \mathbf{m} \pm \mathbf{2}) + t_1(\mathbf{m}, \mathbf{m} \pm \mathbf{1})_S + t_0(\mathbf{m}, \mathbf{m})_S) \quad (7.1)$$

then the hodge diamond of the four-manifold looks like

$$\begin{array}{ccccc} & & 1 & & \\ & t_1/2 & & t_1/2 & \\ t_2 & & t_0 - t_2 - 1 & & t_2 \\ & t_1/2 & & t_1/2 & \\ & & t_2 & & \end{array} \quad (7.2)$$

In certain cases the Hodge diamond will not make any sense, but only in cases in which the corresponding supergravity does not exist, so this is not something to worry about. We now give the values of t_0, t_1, t_2 for the various supergravities.

$(n_L, n_R) = (2, 2)$. This is the case with the maximal amount of supersymmetry, which is obtained for example by compactifying type IIA or IIB supergravity on T^4 . The field content consists of a graviton, eight gravitino's, 5 self-dual and 5 anti self-dual two forms, 16 gauge fields, 40 fermions and 25 scalars. The KK spectrum of $SO(4)$ representations is found to be

$$\oplus_m((\mathbf{m}, \mathbf{m} \pm \mathbf{4}) + 8(\mathbf{m}, \mathbf{m} \pm \mathbf{3}) + 28(\mathbf{m}, \mathbf{m} \pm \mathbf{2}) + 56(n_T + 2)(\mathbf{m}, \mathbf{m} \pm \mathbf{1}) + 70(\mathbf{m}, \mathbf{m})). \quad (7.3)$$

After organizing this in terms of short multiplets we find $t_0 = 6$, $t_1 = 4$ and $t_2 = 1$. The hodge diamond (7.2) is that of T^4 , showing that the conformal field theory is a sigma model with target space a symmetric product of T^4 , as expected. This theory was recently studied in [39], and is an example where we have fermionic short multiplets and additional singletons.

$(n_L, n_R) = (1, 1)$. This theory, with n_V vector multiplets and n_U $USp(2)$ vector multiplets (whose field content is given in [49]) yields $t_0 = n_V + 3n_U + 2$, $t_1 = 0$ and $t_2 = 1$. An example of such a theory is obtained by putting type IIA on $K3$, which is a theory with 20 vector multiples and $n_U = 0$. We find that the cohomology is precisely that of $K3$. Thus, type IIA supergravity on $K3 \times S^3 \times AdS_3$ seems to be dual to type IIB on a different $K3 \times S^3 \times AdS_3$.

$(n_L, n_R) = (3, 0)$. Two multiplets are listed in [49]. Taking n_1 of the first and n_2 of the second yields $t_1 = n_1 + 2n_2$, $t_2 = n_2$ and $t_0 = 2(n_1 + n_2)$.

$(n_L, n_R) = (2, 1)$. Except for the gravity multiplet the theory has one more multiplet that includes a spin-3/2 state. Taking n of these multiplets, we get $t_1 = n + 2$, $t_2 = 1$ and $t_0 = 2n + 2$.

$(n_L, n_R) = (3, 1)$ or $(4, 0)$. These supersymmetry algebras have only a single acceptable representation, of which we take n . The cohomologies come out as (7.2) with $t_1 = 4n$, $t_2 = n$ and $t_0 = 6n$.

7.2 6d supergravity with $n_L + n_R = 1$

The most interesting case is the 6d supergravity with the smallest number of supersymmetries, i.e. the theory with $(n_L, n_R) = (1, 0)$, on $S^3 \times AdS^3$. Such six-dimensional supergravities are obtained either from F-theory on a Calabi-Yau manifold, or from heterotic or type I string theory on $K3$. Although we have not verified this, this six-dimensional supergravity on $S^3 \times AdS_3$ could for instance describe the near-horizon geometry of a $D1$ - $D5$ system in type I theory on $K3$, with the $D5$ branes wrapping the $K3$. On the $D1$ branes we expect to find a sigma model with $(4, 0)$ supersymmetry of the type discussed in [93, 94]. Sigma models with $(4, 0)$ supersymmetry are similar to those with $(4, 4)$ supersymmetry, except that the right moving fermions live in some vector bundle which is not the tangent bundle, and couple to a self-dual gauge field for this vector bundle in the world-sheet lagrangian. In our case we expect to get a sigma model whose target space is the moduli space of $Sp(N)$ instantons on $K3$, with the right-moving fermions coupling to some vector bundle over this moduli space. The spectrum of this theory does not only involve the cohomology of the instanton moduli space, but also various vector bundle cohomologies as in [95]. We do not know explicit results for these cohomologies, making it difficult to determine the CFT spectrum. What is even more problematic is the lack of a precise definition of a chiral primary. The left-movers have an $N = 4$ superconformal algebra, and we can certainly take the usual definition of a chiral primary for the left-movers, but the right-movers have only an $SU(2)$ current algebra and a Virasoro algebra. Nevertheless we can go ahead and compute the KK spectrum of the supergravity theory, which will give us a prediction for the spectrum of the conformal field theory. Using the techniques used so far, we can no longer determine the $\bar{L}_0$ -eigenvalue of the various KK-fields. However, the spin of fields on AdS_3 is equal to the difference of the L_0 and $\bar{L}_0$ eigenvalue, and the spins can be determined as follows. Given a field transforming in some representation $(\mathbf{n}_1, \mathbf{n}_2)$ of the little group $SO(4)$, we decomposed it in $SO(3) \subset SO(4)$ representations, and subsequently found the $SO(4)$ representations $(\mathbf{m}, \mathbf{m} + \mathbf{d})$ that yield the same $SO(3)$ representations for $SO(3) \subset SO(4)$. Notice that these two $SO(4)$'s are different; one is the little group in six dimensions, the other one is the isometry group of the sphere. The representation $(\mathbf{n}_1, \mathbf{n}_2)$ contains various $U(1) \times U(1) \subset SU(2) \times SU(2) = SO(4)$ representations, with $U(1) \times U(1)$ eigenvalues y_1, y_2 . The $U(1)$'s are normalized so that the eigenvalues are half-integer. Then the spins of the various fields associated to $(\mathbf{m}, \mathbf{m} + \mathbf{d})$

are the possible values of $y_1 - y_2$ given that $y_1 + y_2 = d$. Incorporating these spins in the determination of the short multiplets enables us to determine the various $\bar{L}_0$ eigenvalues.

Six dimensional $(1, 0)$ supergravity theory can in general have n_H hyper multiplets, n_V vector multiplets and n_T tensor multiplets. For a discussion of the equations of motion, see [96]. Anomaly cancellation implies $29n_T + n_H - n_V = 273$, but this will not be relevant as we are only interested in classical supergravity. We will assume that the gauge group is a product of $U(1)$'s, with respect to which the hypermultiplets are neutral. When we compactify the theory on $S^3 \times AdS^3$, the KK spectrum of $SO(4)$ representations reads for sufficiently large m

$$\begin{aligned} \oplus_m ((\mathbf{m}, \mathbf{m} \pm \mathbf{4}) + 2(\mathbf{m}, \mathbf{m} \pm \mathbf{3}) + (n_T + n_V + 3)(\mathbf{m}, \mathbf{m} \pm \mathbf{2}) \\ + (2(n_T + n_H + n_V) + 4)(\mathbf{m}, \mathbf{m} \pm \mathbf{1}) + 2(n_T + 2n_H + n_V + 2)(\mathbf{m}, \mathbf{m})) . \end{aligned} \quad (7.4)$$

We want to organize this in representations of $SU(1, 1|2) \times Sl(2, \mathbf{R}) \times SU(2)$. We will denote by $(\mathbf{m}, \mathbf{m}'; s)_S$ the tensor product of the short representation (4.14) with $k = m - 1$ of $SU(1, 1|2)$, the $\mathbf{m}'$ -dimensional representation of $SU(2)$ and a highest weight representation of $Sl(2, \mathbf{R})$ with highest weight $\frac{m-1}{2} - s$. Thus s is the difference between the conformal weight of the left-moving chiral primary and the right-moving primary. Decomposing (7.4) in these short representations yields

$$\begin{aligned} \oplus_m ((\mathbf{m}, \mathbf{m} + \mathbf{2}; -1)_S + n_T(\mathbf{m}, \mathbf{m}; 0)_S + n_V(\mathbf{m}, \mathbf{m}; -1)_S \\ + (\mathbf{m}, \mathbf{m}; 0)_S + (\mathbf{m}, \mathbf{m}; -2)_S + 2n_H(\mathbf{m}, \mathbf{m} - \mathbf{1}; -1/2)_S \\ + n_T(\mathbf{m}, \mathbf{m} - \mathbf{2}; -1)_S + n_V(\mathbf{m}, \mathbf{m} - \mathbf{2}; 0)_S + (\mathbf{m}, \mathbf{m} - \mathbf{2}; -1)_S \\ + (\mathbf{m}, \mathbf{m} - \mathbf{2}; 1)_S + (\mathbf{m}, \mathbf{m} - \mathbf{4}; 0)_S) \end{aligned} \quad (7.5)$$

This result is not very transparent as it stands, but becomes very suggestive when we consider six-dimensional supergravity obtained from F-theory on a Calabi-Yau threefold M , and forget the spin dependence in (7.5). The number of multiplets is expressed in terms of the cohomology of M and the cohomology of the base B over which M is elliptically fibered, namely $n_T = h^{1,1}(B) - 1$, $n_V = h^{1,1}(M) - h^{1,1}(B) - 1$, and $n_H = h^{2,1}(M) + 1$ [97], if we assume the gauge group is a product of $U(1)$'s and all matter is neutral. The representations (7.5) can very simply be written in terms of the hodge numbers $h^{p,q}(\tilde{M})$ of the mirror Calabi-Yau

$$\oplus_m \oplus_{i=-3}^3 \left(\left(\sum_k h^{k+i,k}(\tilde{M}) \right) (\mathbf{m}, \mathbf{m} + \mathbf{i} - \mathbf{1}) \right) \quad (7.6)$$

In fact, this result turns out also to be correct in case $M = K3 \times T^2$ and $M = T^6$. It seems that the $(0, 4)$ conformal field theory knows about the duality between type I and

F-theory, and one can associate in a natural way a $(0, 4)$ conformal field theory to any Calabi-Yau. It would be interesting to understand this at a more fundamental level, and whether in other cases of string-duality the conformal field theory of the dual theory can be constructed by putting the original theory on a certain AdS space. It would also be interesting to examine the $(0, 4)$ theories themselves in some more detail, perhaps along the lines of [98].

7.3 5d supergravity on $S^2 \times AdS_3$

According to one of the conjectures of [1], M-theory compactified on $M \times S^2 \times AdS_3$ with M some Calabi-Yau manifold is dual to a $(0, 4)$ superconformal field theory on the boundary of AdS_3 . The central charge of the $(0, 4)$ theory can be used to compute the entropy of the corresponding 4d black hole [99, 100]. The $(0, 4)$ theory lives on an M-theory fivebrane wrapping some holomorphic four-cycle C in M , and the left-movers have central charge $b^{\text{even}}(C) + b^{\text{odd}}(C)$.

In order to see to what extent M-theory on $M \times S^2 \times AdS_3$ knows about this, we compute its KK spectrum using the techniques explained in section 4 and section 7.2, but now applied to five dimensions. The anti-de Sitter supergroup is $SU(1, 1|2) \times Sl(2, \mathbf{R})$, and by $(\mathbf{m}; s)_S$ we denote the tensor product of a short representation (4.14) with $k = m - 1$ of $SU(1, 1|2)$ and a highest weight representation of $Sl(2, \mathbf{R})$ with highest weight $\frac{m-1}{2} - s$. As before, s is the difference between the conformal weight of the left-moving chiral primary and the right-moving primary. For five-dimensional $N = 1$ supergravity with n_H hypermultiplets and n_V vectormultiplets compactified on $S^2 \times AdS_3$ we get for the KK spectrum (ignoring singletons)

$$\begin{aligned} & n_H(\mathbf{2}; -1/2)_S + n_V(\mathbf{3}; -1)_S + n_V(\mathbf{3}; 0)_S + (\mathbf{3}; -2)_S + (\mathbf{3}; 1)_S \\ & + \oplus_{m>1} (n_H(\mathbf{2m}; -1/2)_S + n_V(\mathbf{2m} + \mathbf{1}; -1)_S + n_V(\mathbf{2m} + \mathbf{1}; 0)_S + (\mathbf{2m} + \mathbf{1}; -2)_S \\ & + (\mathbf{2m} + \mathbf{1}; -1)_S + (\mathbf{2m} + \mathbf{1}; 0)_S + (\mathbf{2m} + \mathbf{1}; 1)_S). \end{aligned} \quad (7.7)$$

The number of vector multiplets equals $h^{1,1}(M) - 1$, and the number of hyper multiplets is $2(h^{1,2}(M) + 1)$ [101]. Thus, dropping the spin dependence (7.7) can be rewritten as

$$b^{\text{odd}}(M)(\mathbf{2})_S + (b^{\text{even}}(M) - 2)(\mathbf{3})_S + \oplus_{m>1} (b^{\text{odd}}(M)(\mathbf{2m})_S + b^{\text{even}}(M)(\mathbf{2m} + \mathbf{1})_S). \quad (7.8)$$

The KK spectrum (7.7) and its simplified version (7.8) do not depend on the cohomology of the holomorphic four-cycle, but only on the cohomology of the Calabi-Yau manifold M . The information about the four-cycle is therefore not encoded in the KK spectrum of supergravity, but should manifest itself in the interactions and correlation functions

of the conformal field theory. For instance, the central charge can be read off from the two-point function of the stress-energy tensor¹. It would be interesting to understand this in more detail.

8 Conclusions

In this paper we have shown that there is detailed agreement between the Kaluza-Klein spectrum of supergravity and the spectrum of certain conformal field theories. To account for all the states in conformal field theory we had to consider products of descendants of chiral primaries. It is an interesting question what the role of the analog fields in four-dimensional $N = 4$ $d = 4$ super Yang-Mills theory is. The relevant supergroup in four dimensions is $SU(2, 2|4)$, and there are many fields that can be constructed as products of descendants of chiral primaries. If their dimensions are still protected, we can perhaps learn something about the singularity structure of the 4d correlation functions.

In two dimensions there are many things that deserve a better understanding. In particular, the cases with 8 supersymmetries in section 7.2 and 7.3 are still somewhat mysterious. We would also like to know the implications of our results for the entropy and various emission and absorption probabilities of 4d and 5d black holes. An issue we have not discussed is the RR sector of the conformal field theories. To study those the AdS_3 part has to be replaced by a certain three-dimensional black hole [104]. It would be interesting to redo the KK analysis in that background to have an independent test of the conjectured duality. Finally, the techniques can easily be generalized to study other theories involving AdS_3 or other AdS spaces. In the examples studied in this paper, the KK spectrum turned out to be almost completely determined by the field content and symmetries of the theory. If true in general, this would imply that KK spectra are not a very deep test of the $AdS \leftrightarrow$ CFT conjecture, and that most of the interesting information is hidden in the interactions of the theory.

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¹For a recent discussion of the central charge, see [102, 103].

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