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Input-Output Triggered Control via the Small Gain Theorem and Switched Systems Modeling

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SUMMARY

This paper investigates stability of nonlinear control systems under intermittent information. Following recent results in the literature, we replace the traditional periodic paradigm, where the up-to-date information is transmitted and control laws are executed in a periodic fashion, with the event-triggered paradigm. Building on the small gain theorem, we develop input-output triggered control algorithms yielding stable closed-loop systems. In other words, based on the currently available (but outdated) measurements of the outputs and external inputs of a plant, a mechanism triggering when to obtain new measurements and update the control inputs is provided. Depending on the noise environment, the developed algorithm yields stable, asymptotically stable, and $\mathcal{L}_p$-stable (with bias) closed-loop systems. Control loops are modeled as interconnections of hybrid systems for which novel results on $\mathcal{L}_p$-stability are presented. The prediction of a triggering event is achieved by employing $\mathcal{L}_p$-gains over a finite horizon. By resorting to convex programming, a method to compute $\mathcal{L}_p$-gains over a finite horizon is devised. Finally, our approach is successfully applied to a trajectory tracking problem for unicycles.

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KEY WORDS: intermittent information; self-triggering; small gain theorem; $\mathcal{L}_p$-stability; hybrid systems

1. INTRODUCTION

In order to address demands of the modern world, the control community has recently put under scrutiny its fundamental concept – feedback. These efforts tackle the question: "How often should information between systems be exchanged in order to meet a desired performance?" The desired performance can be estimation quality or stability. This paper is concerned with stability of nonlinear control systems under *intermittent information*. Under the term intermittent information, we refer to both *intermittent feedback* (a user-designed property as in [1], [2], [3] and [4]) and *intrinsic properties* of control systems such as packet collisions, sampling period, processing time, network throughput, scheduling protocols, lossy communication channels, occlusions of sensors or a

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limited communication/sensing range (see [5] and the references in [1]). User-designed intermittent feedback is motivated by rational use of expensive resources at hand in an effort to decrease energy consumption as well as processing and sensing requirements. In addition, intermittent feedback allows *multitasking* by not utilizing resources all the time for a sole task.

1.1. Background

The traditional digital control theory provides rules of thumb to determine stabilizing sampling periods for linear systems (e.g., 20 times the time constant of the dominant pole [6]). When it comes to nonlinear systems, approximate discrete-time models are derived and analyzed because nonlinear systems, in general, cannot be discretized in closed form [7]. These traditional approaches are characterized by periodic sampling.

Several authors have recently devised aperiodic (or intermittent) sampling policies that provably stabilize control systems. Typically, one first designs a controller without taking into account a communication network and then, in the second step, one determines how often control and sensor signals have to be transmitted over the network so that the closed-loop system remains stable. We classify the recent approaches for nonlinear control systems as follows:

(i) Small gain theorem approaches [8], [9];
(ii) Dissipativity and passivity-based approaches [10], [11], [12];
(iii) Input-to-State Stability (ISS) approaches [4], [13], [14], [15]; and
(iv) Other approaches [2], [3], [16], [17].

Event-triggered and *self-triggered* realizations of intermittent feedback are proposed in [4], [13], [14], [15] and [10]. In these event-driven approaches, one defines a desired performance, and sampling (i.e., transmission of up-to-date information) is triggered when an event representing the unwanted performance occurs. The work in [15] applies event-triggering to control, estimation and optimization tasks. The work in [10] utilizes the dissipative formalism of nonlinear systems, and employs passivity properties of feedback interconnected systems in order to reach an event-triggered control strategy for stabilization of passive and output passive systems. In [12], the authors propose an event-triggered output feedback control approach for time-invariant input-feedforward output-feedback passive plants and controllers (in addition to some other technical conditions imposed on plants and controllers). A particular limitation of the results in [12] is that the number of inputs and outputs of the plant and controller must be equal (due to this, this approach cannot be applied to the systems in the examples in Section 6). The event-triggered approach of [17] converts a trajectory tracking control problem into stabilization problem for an autonomous system and then applies an invariance principle for hybrid systems in order to solve the tracking problem. It is worth mentioning that [17] is tailored for a trajectory tracking problem of unicycles which is quite similar to our case study in Section 6. However, while the results of [17] hold for a certain class of trajectories (i.e., trajectories generated with a non-constant linear velocity and angular velocity which does not change the sign nor converges to zero), we do not impose such requirements on the trajectories. In addition, [17] assumes that the trajectory is known a priori by the controller while we make no such assumption herein. Lastly, notice that the self-triggered counterparts of the event-triggered approaches from [12] and [17] are yet to be devised.

In self-triggered approaches, the current sample is used to determine the next sampling instant, i.e., to predict the occurrence of the triggering event. In comparison with event-triggering, where sensor readings are constantly obtained and analyzed in order to detect events (even though the control signals are updated only upon event detection), self-triggering decreases requirements posed on sensors and processors in embedded systems. The pioneering work on self-triggering, intended to maintain $\mathcal{L}_2$-gains of linear control systems below a certain value, is found in [18] and [19]. A comprehensive comparison of our work with [18] and [19] is provided in Subsection 6.1. The authors in [14] extend event-triggering presented in [4] and develop *state-triggering*: self-triggering based on the value of the system state in the last feedback transmission. The work in [16] utilizes Lyapunov theory and develops event-triggered trajectory tracking for nonlinear systems affine in controls.

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1.2. Contributions

The approach in [8] and [9] is developed for general nonlinear controllers and plants that render closed-loop systems stable, asymptotically stable or exponentially stable in the absence of a communication network. The generality of nonlinear models considered in [8] and [9] allows for analysis of time-varying closed-loop systems with external inputs/disturbances, output feedback and dynamic controllers. In addition, the methodology from [8] and [9] yields stable, asymptotically stable, exponentially stable or $\mathcal{L}_p$-stable control systems under intermittent information. The principal requirement in [8] and [9] is $\mathcal{L}_p$-stability of the closed-loop system. In other words, if a certain controller does not yield the closed-loop system $\mathcal{L}_p$-stable, one can seek for another controller. Hence, unlike related works, our requirements are on the closed-loop systems and not on the plant and controller per se. On the other hand, the work in [2] does not consider external inputs, and the results for nonlinear systems are provided for a class of exponentially stable closed-loop systems in the absence of communication networks. The authors in [3] do not consider external inputs and exponential stability of systems with specific nonlinearities is analyzed. The work in [16] investigates state feedback for nonlinear systems affine in controls and static controllers. A comparison of our approach and the approach from [16] can be found in [20]. The ISS approaches assume state feedback, static controllers, and do not consider external inputs. In addition, the results of [13] and [14] are applicable to state-dependent homogeneous systems and polynomial systems. The work in [10] analyzes passive plants, proportional controllers and does not take into account external inputs.

The main limitation of the approach in [8] and [9] is periodicity of the transmission instants inherited from the standard definition of $\mathcal{L}_p$-gains. Recall that the standard $\mathcal{L}_p$-gain is not a function of time (i.e., there is no prediction of when some event might happen) nor state. To circumvent this limitation (while retaining the generality of [8] and [9]), we devise an input-output triggered approach employing $\mathcal{L}_p$-gains over a finite horizon in the small gain theorem. Under the term *input-output triggering*, we refer to self-triggering based on the values of the plant's external input and output in the last feedback transmission. The triggering event, that is to be precluded in our approach, is the violation of the small gain condition.

The main contributions of this paper are:

a) The design of an input-output triggered sampling policy yielding stability of nonlinear systems employing the small gain theorem;
b) The consideration of realistic communication channels and sensors in the stability analysis;
c) The formulation of novel conditions for $\mathcal{L}_p$-stability (over a finite horizon) of hybrid systems; and
d) The design of a novel method for computing $\mathcal{L}_p$-gains over a finite horizon by resorting to convex programming.

In addition, our approach does not require construction of storage nor Lyapunov functions which can be a difficult task for a given problem.

1.3. Paper Organization

In Section 2, we provide an example that motivates the results and approach proposed in this paper. Section 3 formulates the problem of intermittent feedback under various assumptions. Section 4 presents the notation and definitions utilized in this paper. The methodology brought together to solve the problem via input-output triggering is presented in Section 5. The proposed input-output triggered sampling policy is verified on a trajectory tracking problem in Section 6 and compared with a related work. Conclusions and future challenges are in Section 7. Proofs and several technical results are included in the Appendix.

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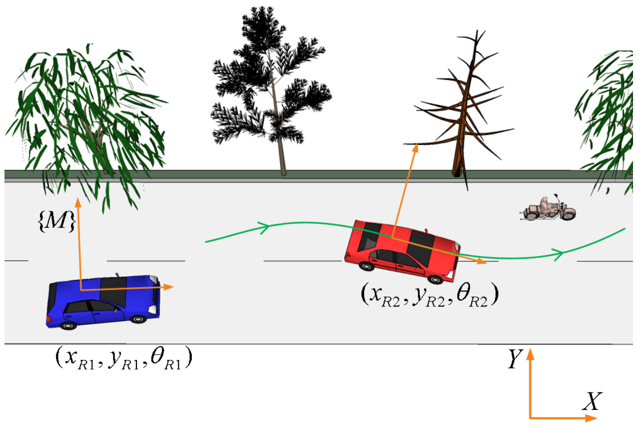

Figure 1. An illustration of the trajectory tracking problem considered in this paper.

2. MOTIVATIONAL EXAMPLE

Using laser-based or radar-based sensors, Autonomous Cruise Control (ACC) technology allows a vehicle to slow down when approaching another vehicle and accelerate to the desired speed when traffic allows. Besides reducing driver fatigue, improving comfort and fuel economy, ACC is also intended to keep cars from crashing [21]. The sampling periods of ACC loops are typically fixed and designed for the worst case scenario (e.g., fast and heavy traffic). Furthermore, these fixed sampling periods are often determined experimentally and are based on the traditional rules of thumb (e.g., 20 times the time constant of the dominant pole). Intuitively, the sampling periods of ACC loops should not remain constant as the desired speed, distance between the cars, the environment (urban on non-urban), and paths (straight or turns) change. The work presented herein quantifies this intuition.

Consider the *trajectory tracking* controller in [22] as an example of a simple ACC. In [22], a velocity-controlled unicycle robot R_1 given by

$$\dot{x}_{R1} = v_{R1}\cos\theta_{R1},\ \dot{y}_{R1} = v_{R1}\sin\theta_{R1},\ \dot{\theta}_{R1} = \omega_{R1} \tag{1}$$

tracks a trajectory generated by a virtual velocity-controlled unicycle robot R_2 with states x_{R2}, y_{R2} and θ_{R2}, and linear and angular velocities v_{R2} and ω_{R2}, respectively. See Figure 1 for an illustration. The tracking error x_p in the coordinate frame $\{M\}$ of robot R_1 is

$$x_p = \begin{bmatrix} x_{p1} \\ x_{p2} \\ x_{p3} \end{bmatrix} = \begin{bmatrix} \cos\theta_{R1} & \sin\theta_{R1} & 0 \\ -\sin\theta_{R1} & \cos\theta_{R1} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_{R2} - x_{R1} \\ y_{R2} - y_{R1} \\ \theta_{R2} - \theta_{R1} \end{bmatrix}. \tag{2}$$

After differentiating (2), we obtain:

$$\dot{x}_p = \begin{bmatrix} \omega_{R1}x_{p2} - v_{R1} + v_{R2}\cos x_{p3} \\ -\omega_{R1}x_{p1} + v_{R2}\sin x_{p3} \\ \omega_{R2} - \omega_{R1} \end{bmatrix}. \tag{3}$$

System (3) can be interpreted as a plant with state x_p and external inputs v_{R2} and ω_{R2}. Take the output of the plant to be $y = x_p$ and introduce $\omega_p := [v_{R2}\ \ \omega_{R2}]^\top$. The plant is controlled through control signals v_{R1} and ω_{R1}. In order to compute v_{R1} and ω_{R1}, and track an unknown trajectory (a trajectory is given by $v_{R2}(t)$, $\omega_{R2}(t)$ and initial conditions of x_{R2}, y_{R2} and θ_{R2}), robot R_1 needs to know the state of the plant x_p and the inputs to R_2, i.e., v_{R2} and ω_{R2}. Following [22], choose the

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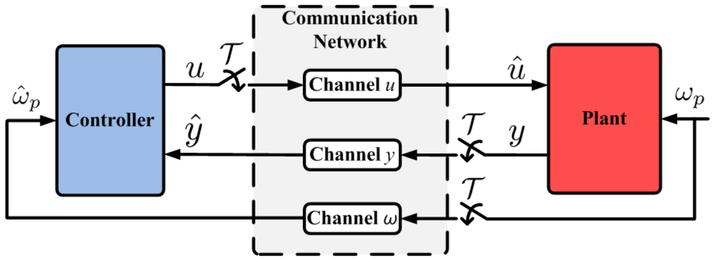

Figure 2. A diagram of a control system with the plant and controller interacting over a communication network with intermittent information updates. The three switches indicate that the information between the plant and controller are exchanged at discrete time instants belonging to a set $\mathcal{T}$.

following control law:

$$\begin{aligned} v_{R1} &= v_{R2}\cos x_{p3} + k_1 x_{p1}, \\ \omega_{R1} &= \omega_{R2} + k_2 v_{R2}\frac{\sin x_{p3}}{x_{p3}} x_{p2} + k_3 x_{p3}, \end{aligned} \tag{4}$$

where k_1, k_2 and k_3 are positive control gains. Let us introduce $u := [v_{R1} \;\; \omega_{R1}]^\top$. Proposition 3.1 in [22] shows that the control law (4) makes the origin $x_p = [0\ 0\ 0]^\top$ of the plant (3) globally asymptotically stable provided that $v_{R2}(t)$, $\omega_{R2}(t)$ and their derivatives are bounded for all times $t \geq 0$ and $\lim_{t\to\infty} v_{R2}(t) \neq 0$ or $\lim_{t\to\infty} \omega_{R2}(t) \neq 0$.

The above asymptotic stability result is obtained assuming instantaneous and continuous information. In real-life applications, continuous access to the values of y and ω_p is rarely achievable. In other words, the control signal u is typically computed using intermittent measurements corrupted by noise. The measurements of the outputs and external inputs of the plant are denoted $\hat{y}$ and $\hat{\omega}_p$, respectively. In general, as new up-to-date values of $\hat{y}$ and $\hat{\omega}_p$ arrive, the control signal may change abruptly. Afterward, the newly computed values u are sent to actuators. These values might be noisy and intermittently updated as well. Hence, the plant is not controlled by u but instead by $\hat{u}$. An illustration of such a control system is provided in Figure 2.

A goal of this paper is to take advantage of the available information from the plant, i.e., of $\hat{y}$ and $\hat{\omega}_p$, and design sampling/control update instants $\mathcal{T} = \{t_1, t_2, \dots\}$ such that stability of the control system is preserved. As our intuition suggests, different $\hat{y}$ and $\hat{\omega}_p$ may yield different time instants in $\mathcal{T}$. In fact, the intersampling intervals $\tau_1 = t_2 - t_1, \tau_2 = t_3 - t_2, \dots$, for R_1 are determined based on the distance from the desired trajectory (i.e., $\hat{y}$) and the nature of trajectory (i.e., $\hat{\omega}_p$). Driven by the desire to obtain intersampling instants τ_i's as large as possible, we adopt a hybrid systems modeling formalism and analysis in this paper. Hybrid modeling captures state jumps and permits the use of multiple models (i.e., switched systems) which in turn can be exploited to maximize intersampling intervals τ_i's. Figure 3 contrasts different methods for computing τ_i's. The solid blue line in Figure 3 represents τ_i's computed via the methodology devised in this paper. Apparently, the use of finite horizon $\mathcal{L}_p$-gains (this notion somewhat corresponds to the notion of individual $\mathcal{L}_p$-gains considered in [23]) produces larger τ_i's in comparison with the use of unified gains. Unified gains are simply the maximum of all individual gains of a switched system. As discussed in [23], unified gains are a valid (although quite conservative) choice for the $\mathcal{L}_p$-gain of a switched system. However, even such conservative $\mathcal{L}_p$-gains of interconnected switched systems, when used in the small gain theorem, do not suffice to conclude stability of the closed-loop system [23]. Essentially, one should use the finite horizon $\mathcal{L}_p$-gains of interconnected switched systems in order to decrease conservativeness, i.e., maximize τ_i's, when applying the small gain theorem.

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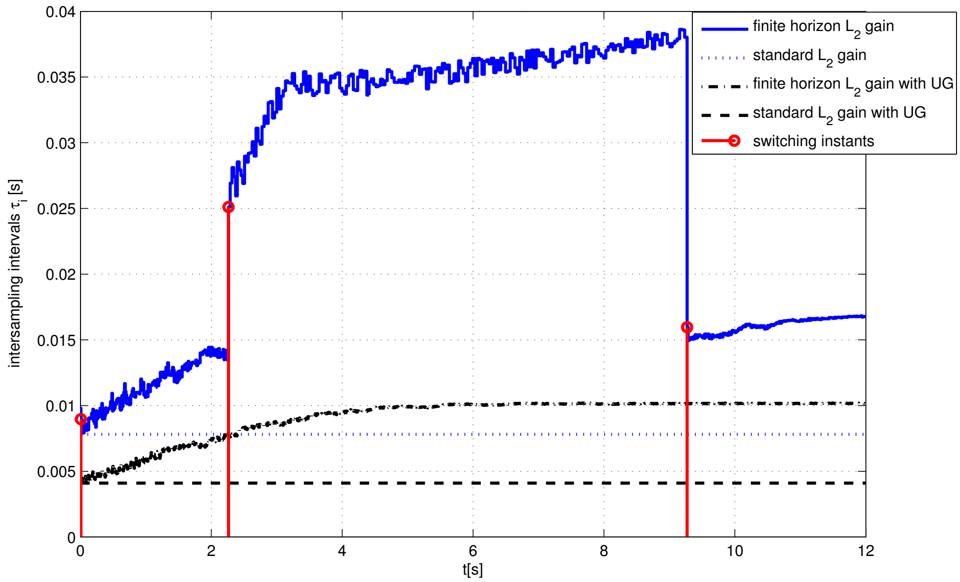

Figure 3. A comparison of the intersampling intervals τ_i's obtained for different notions of $\mathcal{L}_p$-gains. The abbreviation UG stands for 'Unified Gain'. Red stems indicate time instants when changes in $\hat{\omega}_p$ occur. The solid blue line indicates τ_i's generated via the methodology devised in this paper. Definitions of the said notions appear in Sections 4 and 5.

3. PROBLEM FORMULATION

Consider a nonlinear feedback control system consisting of a plant

$$\begin{aligned} \dot{x}_p &= f_p(t, x_p, u, \omega_p), \\ y &= g_p(t, x_p), \end{aligned} \tag{5}$$

and a controller

$$\begin{aligned} \dot{x}_c &= f_c(t, x_c, u_c, \omega_c), \\ y_c &= g_c(t, x_c), \end{aligned} \tag{6}$$

interconnected via the assignment

$$u = y_c, \qquad u_c = y, \qquad \omega_c = \omega_p, \tag{7}$$

where $x_p \in \mathbb{R}^{n_p}$ and $x_c \in \mathbb{R}^{n_c}$ are the states, $y \in \mathbb{R}^{n_y}$ and $y_c \in \mathbb{R}^{n_u}$ are the outputs, and $(u, \omega_p) \in \mathbb{R}^{n_u} \times \mathbb{R}^{n_\omega}$ and $(u_c, \omega_c) \in \mathbb{R}^{n_y} \times \mathbb{R}^{n_\omega}$ are the inputs of the plant and controller, respectively, where ω_p is an exogenous input to the plant. Following the assignment (7), we model the connections (or links) between the plant and the controller as communication networks over which intermittent exchange of information due to sampling takes place. Figure 2 depicts this setting, where the value of u computed by the controller that arrives to the plant is denoted $\hat{u}$. Similarly, the values of y and ω_p that the controller actually receives are denoted $\hat{y}$ and $\hat{\omega}_p$, respectively. In this setting, the quantity $\hat{u}$ is the input fed to the plant (5) while the quantities $\hat{y}$ and $\hat{\omega}_p$ are the measurement of y and ω_p received by the controller (6).

To study the properties of the feedback control system in Figure 2, define

$$e = \begin{bmatrix} e_y \\ e_u \end{bmatrix} := \begin{bmatrix} \hat{y} - y \\ \hat{u} - u \end{bmatrix} \tag{8}$$

and

$$e_\omega := \hat{\omega}_p - \omega_p. \tag{9}$$

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To model intermittent transmission (or sampling) of the values of y and u, the quantities $\hat{y}$ and $\hat{u}$ are updated at time instances $t_1, t_2, \ldots, t_i, \ldots$ in $\mathcal{T}$, i.e.,†

$$\left.\begin{aligned} \hat{y}(t_i^+) &= y(t_i) + h_y(t_i) \\ \hat{u}(t_i^+) &= u(t_i) + h_u(t_i) \end{aligned}\right\} \quad t_i \in \mathcal{T}, \tag{10}$$

where $h_y : \mathbb{R} \to \mathbb{R}^{n_y}$ and $h_u : \mathbb{R} \to \mathbb{R}^{n_u}$. Similarly, the quantity $\hat{\omega}_p$ may change discretely reflecting changes in the plant exogenous input or in the level of plant disturbances. The time instants at which jumps of $\hat{\omega}_p$ occur are denoted t_i^δ and belong to the set $\mathcal{T}^\delta$, which is a subset of $\mathcal{T}$. We assume that the received values of y, u, and ω_p given by $\hat{y}$, $\hat{u}$, and $\hat{\omega}_p$, respectively, remain constant in between updates, i.e., for each $t \in [t_0, \infty) \setminus \mathcal{T}$,

$$\dot{\hat{y}} = 0, \qquad \dot{\hat{u}} = 0, \qquad \dot{\hat{\omega}}_p = 0, \tag{11}$$

which is known as the zero-order hold strategy [24].

The following standing assumption summarizes the properties imposed to the feedback control system in Figure 2 throughout this paper.

Assumption 1 (standing assumption)
The jump times at the controller and plant end coincide. The set of sampling instants $\mathcal{T}^\delta := \{t_1^\delta, t_2^\delta, \ldots, t_i^\delta, \ldots\}$ at which $\hat{\omega}_p$ changes its value satisfies $\mathcal{T}^\delta \subset \mathcal{T}$, where $\mathcal{T} := \{t_1, t_2, \ldots, t_i, \ldots\}$, $t_{i+1} > t_i$ for each t_{i+1}, t_i in $\mathcal{T}$.

We are now ready to state the problem studied in this paper.

Problem 1
Determine the set of sampling instants $\mathcal{T}$ and $\mathcal{T}^\delta$ to update $(\hat{y},\, \hat{u})$ and $\hat{\omega}_p$, respectively, such that the closed-loop system (5)-(6) is stable in the $\mathcal{L}_p$ sense.

The following specific scenarios are investigated:

Case 1
The signals $\hat{u}$, $\hat{y}$, and $\hat{\omega}_p$ are not corrupted by noise, and ω_p is constant between consecutive t_i^δ's.

Case 2
The signals $\hat{u}$ and $\hat{y}$ are not corrupted by noise while $\hat{\omega}_p$ is corrupted by noise. In addition, ω_p is arbitrary between consecutive t_i^δ's.

Case 3
The signals $\hat{u}$, $\hat{y}$, and $\hat{\omega}_p$ are corrupted by noise. In addition, ω_p is arbitrary between two consecutive t_i^δ's.

4. PRELIMINARIES

4.1. Notation

To simplify notation, at times we use $(x, y) := [x^\top \quad y^\top]^\top$. The dimension of a vector x is denoted n_x. Next, let $f : \mathbb{R} \to \mathbb{R}^n$ be a Lebesgue measurable function on $[a, b] \subset \mathbb{R}$. We use

$$\|f[a,b]\|_p := \left(\int_{[a,b]} \|f(s)\|^p \mathrm{d}s \right)^{\frac{1}{p}}$$

to denote the $\mathcal{L}_p$ norm of f when restricted to the interval $[a, b]$. If the corresponding norm is finite, we write $f \in \mathcal{L}_p[a, b]$. In the above expression, $\|\cdot\|$ refers to the Euclidean norm of a vector. If the

†The formulation of the update law in (10) implies that the jump times at the controller and plant end coincide.

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argument of $\|\cdot\|$ is a matrix A, then it denotes the induced 2-norm of A. Eigenvalues and singular values of a matrix A are denoted $\lambda_i(A)$ and $\sigma_i(A)$, respectively. Given $x \in \mathbb{R}^n$, we define

$$\bar{x} = (|x_1|, |x_2|, \ldots, |x_n|),$$

where $|\cdot|$ denotes the (scalar) absolute value function. Given $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$, the partial order $\preceq$ is defined as

$$x \preceq y \iff x_i \leq y_i \quad \forall i \in \{1, \cdots, n\}.$$

The set $\mathcal{A}_n$ denotes the set of all $n \times n$ matrices and $\mathcal{A}_n^+$ denotes the subset of all matrices that are symmetric and have nonnegative entries. $\mathbb{R}_+^n$ denotes the nonnegative orthant. The natural numbers are denoted $\mathbb{N}$ or $\mathbb{N}_0$ when zero is included.

4.2. *Hybrid Systems*

Given a (finite or infinite) sequence of time instances $t_1, t_2, \ldots, t_i, \ldots$ with $t_{i+1} > t_i$, defining the set $\mathcal{T}$ as in Assumption 1, and an initial time $t_0 < t_1$. In this paper we consider hybrid systems written as

$$\Sigma^\delta \begin{cases} \chi(t^+) = h_\chi^\delta(t, \chi(t)) & t \in \mathcal{T} \\ \left.\begin{aligned} \dot{\chi} &= f_\chi^\delta(t, \chi, \omega) \\ y &= \ell_\chi^\delta(t, \chi, \omega) \end{aligned}\right\} & \text{otherwise}, \end{cases} \tag{12}$$

where χ is the state, (ω, δ) is the input, and y is the output. The input δ is given by a piecewise constant, right-continuous function of time $\delta : [t_0, \infty) \to \mathcal{P}$, which we refer to as the *switching signal*, with $\mathcal{P}$ an index set (not necessarily a finite set). The functions f_χ^δ and h_χ^δ are regular enough to guarantee existence of solutions, which, given initial state χ_0, initial time t_0, and a switching signal $\delta : [t_0, \infty) \to \mathcal{P}$, are given by right-continuous functions $t \mapsto \chi(t)$. Jumps of the state χ occur at each $t \in \mathcal{T}$. The value of the state after a jump is given by $\chi(t^+) = \lim_{t' \searrow t} \chi(t')$ for each $t \in \mathcal{T}$. The switching signal δ changes at time instances t_i^δ, defining the set $\mathcal{T}^\delta$ as in Assumption 1, which is a subset of $\mathcal{T}$.

For notational convenience, we define $t_0^\delta := t_0$, which is not a switching time, the intersampling intervals $\tau_i = t_{i+1} - t_i$ for each t_{i+1}, t_i in $\mathcal{T} \cup \{t_0\} =: \mathcal{T}_0$, and the interswitching intervals $\tau_i^\delta := t_{i+1}^\delta - t_i^\delta$ for each $t_{i+1}^\delta, t_i^\delta$ in $\mathcal{T}^\delta \cup \{t_0^\delta\} =: \mathcal{T}_0^\delta$.

The following stability notions for hybrid systems Σ^δ as in (12) are employed in this paper.

Definition 1 ($\mathcal{L}_p$-stability with bias b)
Let $p \in [1, \infty]$. Given a switching signal $t \mapsto \delta(t)$, the hybrid system Σ^δ is $\mathcal{L}_p$-stable with bias $b(t) \equiv b \geq 0$ from ω to y with (linear) gain $\gamma \geq 0$ if there exists $K \geq 0$ such that, for each $t_0 \in \mathbb{R}$ and each $\chi_0 \in \mathbb{R}^{n_\chi}$, each solution to Σ^δ from χ_0 at $t = t_0$ we have that $\|y[t_0, t]\|_p \leq K\|\chi_0\| + \gamma\|\omega[t_0, t]\|_p + \|b[t_0, t]\|_p$ for each $t \geq t_0$.

Definition 2 ($\mathcal{L}_p$-stability with bias b over a finite horizon τ)
Let $p \in [1, \infty]$. Given a switching signal $t \mapsto \delta(t)$ and $\tau \geq 0$, the hybrid system Σ^δ is $\mathcal{L}_p$-stable over a finite horizon of length τ with bias $b(t) \equiv b \geq 0$ from ω to y with (linear) constant gain $\widetilde{\gamma}(\tau) \geq 0$ if there exists a constant[‡] $\widetilde{K}(\tau) \geq 0$ such that, for each $t_0 \in \mathbb{R}$ and each $\chi_0 \in \mathbb{R}^{n_\chi}$, each solution to Σ^δ from χ_0 at $t = t_0$ satisfies $\|y[t_0, t]\|_p \leq \widetilde{K}(\tau)\|\chi_0\| + \widetilde{\gamma}(\tau)\|\omega[t_0, t]\|_p + \|b[t_0, t]\|_p$ for each $t \in [t_0, t_0 + \tau)$.

Definition 3 (detectability)
Let $p, q \in [1, \infty]$. Given a switching signal $t \mapsto \delta(t)$, the state χ of Σ^δ is $\mathcal{L}_p$ to $\mathcal{L}_q$ detectable from (y, ω) to χ with (linear) gain $\gamma \geq 0$ if there exists $K \geq 0$ such that, for each $t_0 \in \mathbb{R}$ and each $\chi_0 \in \mathbb{R}^{n_\chi}$, each solution to Σ^δ from χ_0 at $t = t_0$ satisfies $\|\chi[t_0, t]\|_q \leq K\|\chi_0\| + \gamma\|y[t_0, t]\|_p + \gamma\|\omega[t_0, t]\|_p$ for each $t \geq t_0$.

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‡The parenthesis in $\widetilde{K}$ and $\widetilde{\gamma}$ denote explicitly the dependency of these constants on the already chosen τ.

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Definition 3 is taken from [9] while Definition 2 is motivated by [25] and [26]. When $b = 0$, we say "$\mathcal{L}_p$-stability" instead of "$\mathcal{L}_p$-stability with bias 0".

Proposition 1
Given a switching signal $t \mapsto \delta(t)$, if Σ^δ is $\mathcal{L}_p$-stable with bias $b(t) \equiv b \geq 0$ from ω to y with gain $\gamma \geq 0$ and $\mathcal{L}_p$ to $\mathcal{L}_p$ detectable from (y, ω) to χ with gain $\gamma' \geq 0$ then Σ^δ is $\mathcal{L}_p$-stable with bias $\gamma' b$ from ω to state χ for the given switching signal.

Proof
From the $\mathcal{L}_p$-stability with bias assumption, Definition 1 implies that there exists $K \geq 0$ such that for each t_0 and each χ_0 we have

$$\|y[t_0, t]\|_p \leq K\|\chi_0\| + \gamma\|\omega[t_0, t]\|_p + \|b[t_0, t]\|_p \qquad \forall t \geq t_0,$$

while $\mathcal{L}_p$ to $\mathcal{L}_p$ detectability from (y, ω) to χ with gain γ' implies that there exists $K' \geq 0$ such that

$$\|\chi[t_0, t]\|_p \leq K'\|\chi_0\| + \gamma'\|y[t_0, t]\|_p + \gamma'\|\omega[t_0, t]\|_p$$

for all $t \geq t_0$. Then, we obtain

$$\|\chi[t_0, t]\|_p \leq (K\gamma' + K')\|\chi_0\| + (\gamma\gamma' + \gamma')\|\omega[t_0, t]\|_p + \gamma'\|b[t_0, t]\|_p \tag{13}$$

for all $t \geq t_0$. This proves the claim since (13) corresponds to $\mathcal{L}_p$-stability with bias $\gamma' b$, gain $\gamma\gamma' + \gamma'$, and constant $K\gamma' + K'$. □

For a given switching signal δ, the following result provides a set of sufficient conditions for $\mathcal{L}_p$-stability of a hybrid system Σ^δ with $\mathcal{L}_p$-stable (over a finite horizon) subsystems.

Theorem 1
Given a switching signal $t \mapsto \delta(t)$, consider the hybrid system Σ^δ given by (12). Let $\overline{K} \geq 0$ and $p \in [1, \infty)$. Suppose the following properties hold:

(i) For each $t_i^\delta \in \mathcal{T}_0^\delta$, there exist constants $\widetilde{K}(\tau_i^\delta)$ and $\widetilde{\gamma}(\tau_i^\delta)$ such that[§]

$$\|y[t_i^\delta, t']\|_p \leq \widetilde{K}(\tau_i^\delta)\|\chi(t_i^{\delta+})\| + \widetilde{\gamma}(\tau_i^\delta)\|\omega[t_i^\delta, t']\|_p \tag{14}$$

for all $t' \in [t_i^\delta, t_{i+1}^\delta)$ if t_i^δ is not the largest switching time in $\mathcal{T}^\delta$ (in which case $\tau_i^\delta = t_{i+1}^\delta - t_i^\delta$) or for all $t' \in [t_i^\delta, \infty)$ if t_i^δ is the largest switching time in $\mathcal{T}^\delta$ (in which case it corresponds to $\mathcal{L}_p$-stability and one can write $\widetilde{K}$ and $\widetilde{\gamma}$), and such that

$$K_M := \sup_i \widetilde{K}(\tau_i^\delta), \tag{15}$$

$$\gamma_M := \sup_i \widetilde{\gamma}(\tau_i^\delta), \tag{16}$$

exist.

(ii) The condition

$$\sum_i \|\chi(t_i^{\delta+})\| \leq \overline{K}\|\chi(t_0)\|, \tag{17}$$

holds.

Then, Σ^δ is $\mathcal{L}_p$-stable from ω to y with constant $K_M\overline{K}$ and gain γ_M for the given δ. For $p = \infty$, the same result holds with the constant $K_M\overline{K}$ and gain γ_M when (17) is replaced with $\sup_i \|\chi(t_i^{\delta+})\| \leq \overline{K}\|\chi(t_0)\|$.

[§]For $i = 0$, we have $\chi(t_0^{\delta+}) = \chi(t_0^\delta)$, which is equal to $\chi(t_0)$.

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See Appendix 7.2 for a proof.

Remark 1
The sampling policy designed herein (along with Assumption 2 provided below) ensures that hypothesis (i) of Theorem 1 always holds. Note that it is not straightforward to verify (17) beforehand. In fact, the control policy needs to make decisions "on the fly", based on the available information regarding previous switching instants, in order to enforce (17) (refer to Section 6 for further details). For example, condition (17) can be enforced by requiring $\|\chi(t_{i+1}^{\delta+})\| \leq \lambda\|\chi(t_i^{\delta+})\|$, where $\lambda \in [0,1)$, which is similar to the property exploited in the design of uniformly globally exponentially stable protocols in [8]. Notice that decision making "on the fly" by exploiting previously received information is a salient feature of self-triggering.

Building from ideas in [9], the next result proposes an expression of the $\mathcal{L}_p$-gain over a finite horizon for a generic nonlinear system $\dot\chi = \widetilde{g}(t,\chi,v)$ with state χ and input v.

Theorem 2
Given $\tau \geq 0$ and $t_0 \in \mathbb{R}$, suppose that there exist $A \in \mathcal{A}^+_{n_\chi}$ with $\|A\| < \infty$, a continuous function $\tilde{y} : \mathbb{R} \times \mathbb{R}^{n_\chi} \times \mathbb{R}^{n_v} \to \mathbb{R}^{n_\chi}_+$ such that

$$\overline{\dot\chi} = \overline{\widetilde{g}(t,\chi,v)} \preceq A\overline{\chi} + \tilde{y}(t,\chi,v), \quad \forall (t,\chi,v) \in [t_0, t_0+\tau] \times \mathbb{R}^{n_\chi} \times \mathbb{R}^{n_v}. \tag{18}$$

Then, for each solution to $\dot\chi = \widetilde{g}(t,\chi,v)$ we have

$$\|\chi[t_0, t_0+\tau]\|_p \leq \widetilde{K}(\tau)\|\chi(t_0)\| + \widetilde{\gamma}(\tau)\|\tilde{y}[t_0, t_0+\tau]\|_p, \tag{19}$$

for all $t \in [t_0, t_0+\tau)$, where

$$\widetilde{K}(\tau) = \left(\frac{\exp(\|A\|p\tau) - 1}{p\|A\|}\right)^{\frac{1}{p}}, \tag{20}$$

$$\widetilde{\gamma}(\tau) = \frac{\exp(\|A\|\tau) - 1}{\|A\|}. \tag{21}$$

Theorem 2 proposes conditions under which $\dot\chi = \widetilde{g}(t,\chi,v)$ is $\mathcal{L}_p$-stable from $\tilde{y}$ to χ over the finite horizon τ for any $p \in [1,\infty]$. Its proof is in Appendix 7.3.

5. INPUT-OUTPUT TRIGGERED CONTROL USING FINITE HORIZON $\mathcal{L}_p$-STABILITY

Inspired by the approach in [8], our solution to Problem 1 determines the sampling instants $\mathcal{T}$ using input-output information of the system resulting from the interconnection between (5) and (6), namely

$$\left.\begin{aligned} x(t^+) &= x(t) \\ e(t^+) &= h(t) \end{aligned}\right\} \quad t \in \mathcal{T} \tag{22a}$$

$$\left.\begin{aligned} \dot{x} &= f(t,x,e,\hat\omega_p,e_\omega) \\ \dot{e} &= g(t,x,e,\hat\omega_p,e_\omega) \end{aligned}\right\} \quad \text{otherwise}, \tag{22b}$$

where $x := (x_p, x_c)$, and functions f, g and h are given by

$$f(t,x,e,\hat\omega_p,e_\omega) := \begin{bmatrix} f_p(t,x_p,g_c(t,x_c)+e_u,\hat\omega_p - e_\omega) \\ f_c(t,x_c,g_p(t,x_p)+e_y,\hat\omega_p) \end{bmatrix}, \qquad h(t_i) := \begin{bmatrix} h_y(t_i) \\ h_u(t_i) \end{bmatrix}, \tag{23}$$

$$g(t,x,e,\hat\omega_p,e_\omega) :=$$

$$\begin{bmatrix} \underbrace{\hat{f}_p(t,x_p,x_c,g_p(t,x_p)+e_y,g_c(t,x_c)+e_u,\hat\omega_p-e_\omega)}_{\equiv 0 \text{ for zero-order-hold estimation strategy}} - \frac{\partial g_p}{\partial t}(t,x_p) - \frac{\partial g_p}{\partial x_p}(t,x_p) f_p(t,x_p,g_c(t,x_c)+e_u,\hat\omega_p-e_\omega) \\ \overbrace{\hat{f}_c(t,x_p,x_c,g_p(t,x_p)+e_y,g_c(t,x_c)+e_u,\hat\omega_p)} - \frac{\partial g_c}{\partial t}(t,x_c) - \frac{\partial g_c}{\partial x_c}(t,x_c) f_c(t,x_c,g_p(t,x_p)+e_y,\hat\omega_p) \end{bmatrix}, \tag{24}$$

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where h_y and h_y are introduced in (10).

By identifying the switching signal δ in (12) with $\hat{\omega}_p$[¶], we write (22) in the form of a hybrid system Σ^δ as in (12) as follows:

$$\left.\begin{aligned} x(t^+) &= x(t) \\ e(t^+) &= h(t) \end{aligned}\right\} \quad t \in \mathcal{T} \tag{25a}$$

$$\left.\begin{aligned} \dot{x} &= f^\delta(t, x, e, e_\omega) \\ \dot{e} &= g^\delta(t, x, e, e_\omega) \end{aligned}\right\} \quad \text{otherwise} . \tag{25b}$$

In this paper, we are interested in changes in $\mathcal{L}_p$-gains (over a finite horizon) of (25) for different values of the switching signal $\delta = \hat{\omega}_p$. Note that $f^\delta(t, x, e, e_\omega)$ and $g^\delta(t, x, e, e_\omega)$ are alternative (but equivalent) labels for $f(t, x, e, \hat{\omega}_p, e_\omega)$ and $g(t, x, e, \hat{\omega}_p, e_\omega)$. (For convenience, we use the former when the switching component of our model is explicitly utilized.)

5.1. Why $\mathcal{L}_p$-gains Over a Finite Horizon?

$\mathcal{L}_p$-gains over a finite horizon allow prediction of the triggering event in this paper. In addition, as suggested by the example in Section 2, they produce less conservative intertransmission intervals τ_i's than classical $\mathcal{L}_p$-gains when used in the small gain theorem. This is due to the fact that $\mathcal{L}_p$-gains over a finite horizon are monotonically nondecreasing in τ. To show this fact, we use the following characterization for $p \in [1, \infty)$ taken from [25], [27] and [28]:

$$[\widetilde{\gamma}(\tau)]^p := \sup_{\omega \in \mathcal{L}_p[t_0, t_0+\tau]} \left\{ \frac{\int_{t_0}^{t_0+\tau} \|y(t)\|^p \mathrm{d}t}{\int_{t_0}^{t_0+\tau} \|\omega(t)\|^p \mathrm{d}t} \right\}, \tag{26}$$

where $\|x(t_0)\| = 0$, $\|\omega[t_0, t_0+\tau]\|_p \neq 0$, $b = 0$ and δ is fixed to be constant to generate an output y and solution x of Σ^δ. The case $p = \infty$ is similar.

Proposition 2
The function $\tau \mapsto \widetilde{\gamma}(\tau)$ is monotonically nondecreasing.

Proof
Take $\tau > 0$ and choose any τ' such that $\tau' > \tau$. According to (26), for the horizon $[t_0, t_0 + \tau']$ we can write

$$[\widetilde{\gamma}(\tau')]^p = \sup_{\omega \in \mathcal{L}_p[t_0, t_0+\tau']} \left\{ \frac{\int_{t_0}^{t_0+\tau} \|y(t)\|^p \mathrm{d}t + \int_{t_0+\tau}^{t_0+\tau'} \|y(t)\|^p \mathrm{d}t}{\int_{t_0}^{t_0+\tau} \|\omega(t)\|^p \mathrm{d}t + \int_{t_0+\tau}^{t_0+\tau'} \|\omega(t)\|^p \mathrm{d}t} \right\}.$$

Now, choose $\omega \in \mathcal{L}_p[t_0, t_0 + \tau']$ such that $\omega(t) = 0$ for $t \in (t_0 + \tau, t_0 + \tau']$. This yields

$$\begin{aligned} [\widetilde{\gamma}(\tau')]^p &= \sup_{\omega \in \mathcal{L}_p[t_0, t_0+\tau']} \left\{ \frac{\int_{t_0}^{t_0+\tau} \|y(t)\|^p \mathrm{d}t + \int_{t_0+\tau}^{t_0+\tau'} \|y(t)\|^p \mathrm{d}t}{\int_{t_0}^{t_0+\tau} \|\omega(t)\|^p \mathrm{d}t} \right\} \geq \\ &\geq \sup_{\omega \in \mathcal{L}_p[t_0, t_0+\tau]} \left\{ \frac{\int_{t_0}^{t_0+\tau} \|y(t)\|^p \mathrm{d}t}{\int_{t_0}^{t_0+\tau} \|\omega(t)\|^p \mathrm{d}t} \right\} = [\widetilde{\gamma}(\tau)]^p. \end{aligned}$$

Taking the p^{th} root of the above inequality shows the claim. □

Since a standard (i.e., infinite horizon or classical) $\mathcal{L}_p$-gain γ can be defined as

$$\gamma := \sup_{\tau \geq 0} \widetilde{\gamma}(\tau), \tag{27}$$

we conclude that $\widetilde{\gamma}(\tau) \leq \gamma$ for all $\tau \geq 0$. Lastly, notice that some systems are $\mathcal{L}_p$-stable only over a finite horizon.

[¶]This assignment requires the space of δ and $\hat{\omega}_p$ to match, i.e., $\mathcal{P} \subseteq \mathbb{R}^{n_\omega}$.

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5.2. *Proposed Approach*

The approach proposed to provide a solution to Problem 1 is as follows. Suppose that t_i^δ and t_{i+1}^δ are two consecutive switching instants of the switching signal $t \mapsto \delta(t)$. Then, the switching signal $t \mapsto \delta(t)$ remains constant over $[t_i^\delta, t_{i+1}^\delta)$, i.e., $\delta(t) = r$ for all $t \in [t_i^\delta, t_{i+1}^\delta)$ for some $r \in \mathcal{P}$. To determine if a sample should be taken within $(t_i^\delta, t_{i+1}^\delta)$, i.e., to determine τ_i with $\tau_i \leq t_{i+1}^\delta - t_i^\delta$, suppose that for each $t \in [t_i^\delta, t_i^\delta + \tau_i)$, and for some $p \in [1, \infty]$, the solution $t \mapsto (x(t), e(t))$ to (25) resulting from an input $t \mapsto e_\omega(t)$ satisfies

$$\overline{\dot{e}(t)} \preceq A^r \overline{e(t)} + \tilde{y}^r(t, x(t), e_\omega(t)), \tag{28}$$

$$\|\tilde{y}^r[t_i^\delta, t]\|_p \leq K_n^r \|x(t_i^\delta)\| + \gamma_n^r \|(e, e_\omega)[t_i^\delta, t]\|_p, \tag{29}$$

where $A^r \in \mathcal{A}_{n_e}^+$ with $\|A^r\| < \infty$, $\tilde{y}^r : \mathbb{R} \times \mathbb{R}^{n_x} \times \mathbb{R}^{n_\omega} \to \mathbb{R}_+^{n_e}$ is continuous, and K_n^r and γ_n^r are positive constants. From (28), it follows that, for each $t \in [t_i^\delta, t_i^\delta + \tau_i)$, we have

$$\|e[t_i^\delta, t]\|_p \leq \widetilde{K}_e^r(\tau_i)\|e(t_i^\delta)\| + \widetilde{\gamma}_e^r(\tau_i)\|\tilde{y}^r[t_i^\delta, t]\|_p,$$

with $\tau \mapsto \widetilde{K}_e^r(\tau)$ and $\tau \mapsto \widetilde{\gamma}_e^r(\tau)$ given as

$$\widetilde{K}_e^r(\tau) = \left(\frac{\exp(\|A^r\|p\tau) - 1}{p\|A^r\|}\right)^{\frac{1}{p}}, \qquad \widetilde{\gamma}_e^r(\tau) = \frac{\exp(\|A^r\|\tau) - 1}{\|A^r\|} \tag{30}$$

(see (20) and (21), respectively). Then, interpreting system (25) as the interconnection shown in Figure 4 between the *nominal system* Σ_n^δ given by

$$x(t^+) = x(t) \quad \} \quad t \in \mathcal{T} \tag{31a}$$

$$\dot{x} = f^\delta(t, x, e, e_\omega) \quad \} \quad \text{otherwise}, \tag{31b}$$

with input (e, e_ω) and output $\tilde{y}^r$, and the *error system* Σ_e^δ given by

$$e(t^+) = h(t) \quad \} \quad t \in \mathcal{T} \tag{32a}$$

$$\dot{e} = g^\delta(t, x, e, e_\omega) \quad \} \quad \text{otherwise}, \tag{32b}$$

with input $\tilde{y}^r$ and output e, and using a small gain argument, we propose to choose $\tau > 0$ such that $\widetilde{\gamma}_e^r(\tau) \leq \kappa/\gamma_n^r$, where $\kappa \in (0, 1)$. In this way, the small gain condition $\gamma_n^r \widetilde{\gamma}_e^r(\tau) < 1$ is satisfied and a stabilizing sampling policy is given by $\tau^r \in (0, \tau^{r,*}]$, where[‖]

$$\tau^{r,*} = \frac{1}{\|A^r\|} \ln\left(\kappa \frac{\|A^r\|}{\gamma_n^r} + 1\right). \tag{34}$$

This policy yields the closed-loop system (25) $\mathcal{L}_p$-stable over a horizon τ when, over this horizon, the value of the switching signal is equal to r. We emphasize that the policy (34) is utilized in all subsequent results. However, different settings (i.e., Cases 1, 2 and 3) lead to different stability properties of closed-loop systems (refer to the following subsection).

[‖] Using the expression for $\widetilde{\gamma}_e^r$ in (30), we obtain

$$\frac{\exp(\|A^r\|\tau) - 1}{\|A^r\|} \leq \frac{\kappa}{\gamma_n^r}. \tag{33}$$

Then, solving the above inequality for τ and taking the largest possible value, denoted $\tau^{r,*}$, yields

$$\tau^{r,*} = \frac{1}{\|A^r\|} \ln\left(\kappa \frac{\|A^r\|}{\gamma_n^r} + 1\right).$$

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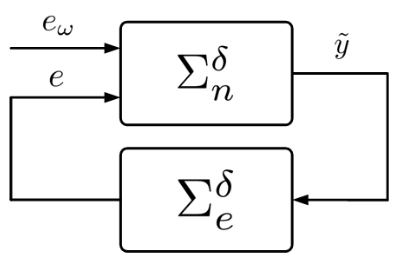

Figure 4. Interconnection of the nominal hybrid system Σ_n^δ and the output error hybrid system Σ_e^δ.

5.3. Design of Input-Output Triggering

In this section, we provide a solution to Problem 1 by designing intersampling intervals using input-output information. The following assumption is imposed in the results to follow.

Assumption 2
For each $r \in \mathcal{P}$, there exists a function $\widetilde{y}^r$ such that the following hold:

(i) There exists A^r such that

$$\overline{g^r(t,x,e,e_\omega)} \preceq A^r\overline{e} + \widetilde{y}^r(t,x,e_\omega) \qquad \forall (t,x,e,e_\omega) \in [t_0,\infty) \times \mathbb{R}^{n_x} \times \mathbb{R}^{n_e} \times \mathbb{R}^{n_\omega}.$$

Furthermore, there exists $\overline{\eta}$ such that $\sup_{r\in\mathcal{P}} \|A^r\| \leq \overline{\eta}$.

(ii) The system Σ_n^r is $\mathcal{L}_p$-stable from (e_ω, e) to $\tilde{y}^r$ with a constant K_n^r and gain γ_n^r. Furthermore, there exist constants $\overline{K_n}$ and $\overline{\gamma_n}$ such that $\sup_{r\in\mathcal{P}} K_n^r \leq \overline{K_n}$ and $\sup_{r\in\mathcal{P}} \gamma_n^r \leq \overline{\gamma_n}$.

(iii) The state x of Σ_n^r is $\mathcal{L}_p$ to $\mathcal{L}_p$ detectable from $(\tilde{y}, e_\omega, e)$ with a constant K_d^r and gain γ_d^r. Furthermore, there exist constants K and γ such that $\sup_{r\in\mathcal{P}} K_d^r \leq K$ and $\sup_{r\in\mathcal{P}} \gamma_d^r \leq \gamma$.

5.3.1. Cases 1 and 2: In each of these cases, $\hat{u}$ and $\hat{y}$ are transmitted without distortions (and without transmission delays). Then, the plant and the controller receive precise values of u and y at transmission instants; namely, h_y and h_u in (10) are equal to zero. Therefore, the error state e is reset to zero at every sampling event, in which case, the hybrid system (25) becomes

$$\left.\begin{aligned} x(t^+) &= x(t) \\ e(t^+) &= 0 \end{aligned}\right\} \quad t \in \mathcal{T} \tag{35a}$$

$$\left.\begin{aligned} \dot{x} &= f^\delta(t,x,e,e_\omega) \\ \dot{e} &= g^\delta(t,x,e,e_\omega) \end{aligned}\right\} \quad \text{otherwise}\ . \tag{35b}$$

In particular, for Case 1, the signal e_ω is also reset to zero at transmission instants, and since ω_p is constant between two consecutive transmission instants, we have that $e_\omega \equiv 0$ if $\hat{\omega}_p(0) = \omega_p(0)$. The following $\mathcal{L}_p$-stability properties from e_ω to (x, e) are guaranteed by the proposed policies.

Theorem 3
Given $p \in [1,\infty]$, suppose that Assumption 2 holds. Let the sampling instants in $\mathcal{T}$ be given by (34), computed for given values of r, that are constant on each intersampling interval, and define the switching signal $\delta : [t_0,\infty) \to \mathcal{P}$ with $\mathcal{T}^\delta \subset \mathcal{T}$. Suppose there exists $\overline{K}$ such that, for the given switching signal δ, for each $(x(t_0), e(t_0), t_0)$ and each $t \mapsto e_\omega(t)$, each solution to (35) is such that its x component satisfies

$$\sum_{t_i^\delta \in \mathcal{T}_0^\delta} \|x(t_i^{\delta+})\| \leq \overline{K}\|x(t_0)\|. \tag{36}$$

Then, there exists $\tau^*_{\min} > 0$ such that $\tau^{r,*} \geq \tau^*_{\min}$ for all $r \in \mathcal{P}$ and, for each (x_0, t_0), each solution to (35) satisfies

$$\|(x,e)[t_0,t]\|_p \leq \overline{K}\widehat{K}\|(x,e)(t_0)\| + \widehat{\gamma}\|e_\omega[t_0,t]\|_p, \qquad \forall t \geq t_0, \tag{37}$$

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for the switching signal δ. (Constants $\widehat{K}$ and $\widehat{\gamma}$ are defined in Appendix 7.4.)

Remark 2
Property (37) corresponds to $\mathcal{L}_p$ stability from e_ω to (x, e). The existence of $\tau^*_{\min}$ follows from the upper bound on $\|A^r\|$ and on γ_n^r in items (ii) and (iii) of Assumption 2, respectively. From (34) we obtain

$$\tau^{r,*} = \frac{1}{\|A^r\|} \ln\left(\kappa \frac{\|A^r\|}{\gamma_n^r} + 1\right) \geq \frac{1}{\overline{\eta}} \ln\left(\kappa \frac{\overline{\eta}}{\underline{\gamma_n}} + 1\right) > 0.$$

Note that since, without loss of generality, there exists $\underline{\gamma_n} > 0$ such that $\inf_{r\in\mathcal{P}} \gamma_n^r \geq \underline{\gamma_n}$, then we have that $\tau^{r,*}$ is upper bounded by $\tau^*_{\max} = \lim_{\|A^r\|\searrow 0} \frac{1}{\|A^r\|} \ln\left(\kappa \frac{\|A^r\|}{\underline{\gamma_n}} + 1\right) = \frac{\kappa}{\underline{\gamma_n}}$.

Remark 3
For Case 1, inequality (37) becomes $\|(x, e)[t_0, t]\|_p \leq \overline{K}\widehat{K}\|(x, e)(t_0)\|$. In addition, notice that $\|(x, e)(t_i^+)\| \leq \|(x, e)(t_i)\|$, $t_i \in \mathcal{T}$. Consequently, one can infer stability, asymptotic and exponential stability after imposing additional structure on $f^\delta(t, x, e, e_\omega)$ and $g^\delta(t, x, e, e_\omega)$ in (35) (refer to [8, 9, Section II], [29] and [30]).

Remark 4
Let us consider the case where u or y (or both) is the output of a state observer. In other words, the plant or controller (or both) is fed with an estimate provided by an observer. Consequently, in (35a) we have $e(t^+) = h(t)$ for all $t \in \mathcal{T}$, where $h(t)$ is the observer error. For $p \in [1, \infty)$, if the observer error satisfies the condition that there exists $\overline{K}_h \geq 0$ such that $\sum_{t_i^\delta \in \mathcal{T}_0^\delta} \|h(t_i^\delta)\| \leq \overline{K}\|e(t_0)\|$ (e.g., exponentially converging observers), then Theorem 3 holds with

$$\|(x, e)[t_0, t]\|_p \leq K_1\widehat{K}\|(x, e)(t_0)\| + \widehat{\gamma}\|e_\omega[t_i, t]\|_p, \qquad \forall t \geq t_0,$$

where $K_1 = \sqrt{n_x + n_e} \max\left\{\overline{K}, \overline{K}_h\right\}$. For $p = \infty$, the observer error has to be bounded, i.e., there exists $\overline{K}_h \geq 0$ such that $\|h(t)\| \leq \overline{K}_h\|e(t_0)\|$ for every $t \geq t_0$, in order for Theorem 3 to hold.

5.3.2. Case 3: This is the most general case, for which (25) is rewritten as

$$\left.\begin{aligned} x(t^+) &= x(t) \\ e(t^+) &= h(t) \end{aligned}\right\} \; t \in \mathcal{T} \tag{38a}$$

$$\left.\begin{aligned} \dot{x} &= f^\delta(t, x, e, e_\omega) \\ \dot{e} &= g^\delta(t, x, e, e_\omega) \end{aligned}\right\} \; \text{otherwise} \tag{38b}$$

where $t \mapsto h(t)$ models measurement noise, quantization error, and related perturbations. For this system, we have the following result.

Theorem 4
Given $p \in [1, \infty]$, suppose that Assumption 2 holds and that there exists $\overline{K}_h \geq 0$ such that $\|h(t)\| \leq \overline{K}_h$ for all $t \geq t_0$. Let the sampling instants in $\mathcal{T}$ be given by (34), computed for given values of r that, at least, are constant on each interval and define the switching signal $\delta : [t_0, \infty) \to \mathcal{P}$ with $\mathcal{T}^\delta \subset \mathcal{T}$. Suppose there exists $\overline{K}$ such that, for the given switching signal δ, for each $(x(t_0), e(t_0), t_0)$ and each $t \mapsto e_\omega(t)$, each solution to (38) is such that its x component satisfies (36). Then, there exists $\tau^*_{\min} > 0$ such that $\tau^{r,*} \geq \tau^*_{\min}$ for all $r \in \mathcal{P}$ and, for each $(x(t_0), e(t_0), t_0)$, each solution to (38) satisfies

$$\|(x, e)[t_0, t]\|_p \leq \overline{K}\widehat{K}\|(x, e)(t_0)\| + \widehat{\gamma}\|e_\omega[t_0, t]\|_p + \|\widehat{b}[t_0, t]\|, \qquad \forall t \geq t_0 \tag{39}$$

for the switching signal δ. (Expressions for $\widehat{K}$, $\widehat{\gamma}$ and $\widehat{b}(t)$ are provided in Appendix 7.4.)

Property (39) corresponds to $\mathcal{L}_p$ stability from e_ω to (x, e) with bias $\widehat{b}(t) \equiv \widehat{b} \geq 0$. See Appendix 7.4 for a proof.

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Remark 5
Noisy measurements can be a consequence of quantization errors. According to [31], interconnections of systems with linear $\mathcal{L}_p$-gains prone to quantization errors do not yield closed-loop systems with linear $\mathcal{L}_p$-gains. Hence, $\mathcal{L}_p$-stability with bias in Theorem 4 cannot be relaxed without contradicting the points in [31].

Remark 6
Let us consider the case of lossy communication channels. If there is an upper bound on the maximum number of successive dropouts, say $N^D \in \mathbb{N}$, simply use $\tau^{r,*}/N^D$ as intertransmission intervals in order for Theorem 4 to hold.

Remark 7
In order to account for possible delays introduced by the communication networks in Figure 2, one can use scattering transformation for the small gain theorem [32]. Provided that Σ_n^δ and Σ_e^δ are input-feedforward output-feedback passive systems satisfying certain conic relations, the work in [32] makes stability properties of (25) independent of constant time delays and Theorem 4 is applicable again. In light of [11] and [12], these constant time delays are allowed to be larger than the intersampling intervals $\tau^{r,*}$'s.

5.4. *Implementation of Input-Output Triggering*

Note that condition (28) is only needed over a horizon $[t_i^\delta, t_i^\delta + \tau_i)$, where τ_i is yet to be determined. Asking for this condition to hold for every (x, e_ω) would be too restrictive or lead to conservative sampling times. Since $x(t_i^\delta)$ and $e(t_i^\delta)$ are known after every sampling event, then, for the current constant value of $t \mapsto \delta(t)$ and the input $t \mapsto e_\omega(t)$ over $[t_i^\delta, t_i^\delta + \tau_i)$, the required property is guaranteed when

$$\overline{g^\delta(t, x, e, e_\omega)} \preceq A^\delta \bar{e} + \tilde{y}^\delta(t, x, e_\omega(t))$$

holds for each

$$t \in [t_i^\delta, t_i^\delta + \tau_i], \qquad (x, e) \in \mathcal{S}_i := \text{Reach}_{t_i^\delta, t - t_i^\delta}((x(t_i^\delta), e(t_i^\delta))),$$

where $\text{Reach}_{t_i^\delta, t - t_i^\delta}((x(t_i^\delta), e(t_i^\delta)))$ is the reachable set of

$$\dot{x} = f^\delta(t, x, e, e_\omega), \qquad \dot{e} = g^\delta(t, x, e, e_\omega) \tag{40}$$

from $(x(t_i^\delta), e(t_i^\delta))$ at t_i^δ after $t - t_i^\delta$ units of time, namely

$$\text{Reach}_{\underline{t}, \tau}(z_0) \quad = \quad \{z(t') \ : \ z \text{ is a solution to (40) from } z_0 \text{ at } \underline{t}, t' \in [\underline{t}, \underline{t} + \tau]\}.$$

Then, exploiting the reachability ideas outlined above, the following algorithm can be used at each time instant t_i, $i = 0, 1, 2, \ldots$:

Step 1. – Obtain measurements $\hat{y}(t_i)$ and $\hat{\omega}_p(t_i)$.
Step 2. – Extract state estimate $\hat{x}(t_i)$ from the measurements.
Step 3. – Update the control law (6) with $\hat{y}(t_i)$ and, if (36) is not compromised, with $\hat{\omega}_p(t_i)$.
Step 4. – Actuate the plant with $\hat{u}(t_i)$.
Step 5. – Estimate $\mathcal{S}_i$ from (5)-(6) using reachability analysis.
Step 6. – Compute (34) and pick τ_i.

With the objective of obtaining values of $\widetilde{\gamma}_e(\tau)$ in (21) as small as possible, we propose to minimize $\|A\|$. This leads to less conservative τ_i's in a solution to Problem 1. Following the statement of Theorem 2, we consider the following optimization problem: given $r \in \mathcal{P}$, $\underline{t}$, $\tau > 0$, and $e_\omega : [\underline{t}, \underline{t} + \tau] \to \mathbb{R}^{n_\omega}$,

$$\begin{aligned} \text{minimize} \quad & \|A\| && \text{(41a)} \\ \text{subject to:} \quad & A \in \mathcal{A}_{n_e}^+ && \text{(41b)} \\ & \overline{g^r(t, x, e, e_\omega)} \preceq A\bar{e} + \tilde{y}^r(t, x, e_\omega) && \text{(41c)} \end{aligned}$$

for all $t \in [\underline{t}, \underline{t} + \tau]$, $(x, e) \in \mathcal{S} := \text{Reach}_{\underline{t}, \tau}((x(\underline{t}), e(\underline{t})))$.

Proposition 3
The optimization problem (41) is convex.

Proof
It is well known that $\|A\|$ is a convex function of A (see [33], Chapter 3). Now, let us prove that constraints (41b) and (41c) yield a convex set. First, a convex combination of two matrices in $\mathcal{A}^+_{n_e}$ is again in $\mathcal{A}^+_{n_e}$. This is due to the fact that symmetric matrices with nonnegative elements remain symmetric with nonnegative elements when multiplied with nonnegative scalars and when added together. Let us now show that inequality (41c) yields a convex set in A. For any $t' \in [\underline{t}, \underline{t}+\tau]$, pick any $(x', e') \in \mathcal{S}$ and $e'_\omega \in \mathbb{R}^{n_\omega}$. Now, let us introduce substitutions $E(\bar{e}) = \overline{g^r(t', x', e', e'_\omega)}$ and $F = \tilde{y}^r(t', x', e'_\omega)$. Our goal is to show that if

$$E(\bar{e'}) \preceq A_1\bar{e'} + F, \tag{42}$$

$$E(\bar{e'}) \preceq A_2\bar{e'} + F, \tag{43}$$

then

$$E(\bar{e'}) \preceq [(1-\alpha)A_1 + \alpha A_2]\bar{e'} + F \tag{44}$$

where $\alpha \in [0, 1]$. Using (42) and (43), we obtain

$$(1-\alpha)A_1\bar{e'} + \alpha A_2\bar{e'} \succeq (1-\alpha)(E(\bar{e'}) - F) + \alpha(E(\bar{e'}) - F) = E(\bar{e'}) - F$$

which is equivalent to (44). Since t', (x', e') and e'_ω were picked arbitrarily from $[\underline{t}, \underline{t}+\tau]$, $\mathcal{S}$ and $\mathbb{R}^{n_\omega}$, respectively, therefore (44) holds for all $t \in [\underline{t}, \underline{t}+\tau]$ and all $(x, e) \in \mathcal{S}$. The fact that the intersection of a family of convex sets is a convex set concludes the proof. □

6. CASE STUDY - TRAJECTORY TRACKING

In the first part of this section, we apply the input-output triggered update policy (34) to the *trajectory tracking* controller presented in Section 2 and exploit the ideas from Section 5.4. Subsection 6.1 is reserved for a comparison with a related work.

Since the controller (4) is not a dynamic controller, we have $f_c \equiv 0$. Next, we take $x_c = (v_{R1}, \omega_{R1})$ and

$$u = g_c(t, x_c) = x_c. \tag{45}$$

Recall that the states of the plant (3) are measured directly, i.e.,

$$y = g_p(t, x_p) = x_p, \tag{46}$$

and assume that the communication network for transmitting the control input (v_{R1}, ω_{R1}) to the actuators of R_1 can be neglected due to on-board controllers. Because of the absence of the communication network for transmitting u and $f_c \equiv 0$, we have that $e_u \equiv 0$. Consequently, we can exclude x_c from x and take $x = x_p$. Notice that f_p is given by (3). Recall that the external input is $\omega_p = (v_{R2}, \omega_{R2})$. Now we have

$$e = \hat{x} - x = [e_1 \;\; e_2 \;\; e_3]^\top, \tag{47}$$

and

$$e_\omega = \hat{\omega}_p - \omega_p = [e_{\omega,1} \;\; e_{\omega,2}]^\top. \tag{48}$$

After substituting (3), (4), (45), (46), (47) and (48) into (23) and (24), we obtain (compare with expression (25))

$$\dot{e} = -\dot{x} = \underbrace{\begin{bmatrix} -Q - P(x_2+e_2)x_2 - k_3(x_3+e_3)x_2 + R + k_1(x_1+e_1) - S \\ \hat{\omega}_{R2}x_1 + P(x_2+e_2)x_1 + k_3(x_3+e_3)x_1 - T \\ e_{\omega,2} + P(x_2+e_2) + k_3(x_3+e_3) \end{bmatrix}}_{= g^\delta(t,x,e,e_\omega) = -f^\delta(t,x,e,e_\omega)} \tag{49}$$

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and (compare with expression (18))

$$\bar{e} \preceq \underbrace{\begin{bmatrix} k_1 & k_2|\hat{v}_{R2}|M_2 & \max\{|\hat{v}_{R2}|, k_3 M_2\} \\ k_2|\hat{v}_{R2}|M_2 & k_2|\hat{v}_{R2}|M_1 & \max\{k_2|\hat{v}_{R2}|, k_3 M_1\} \\ \max\{|\hat{v}_{R2}|, k_3 M_2\} & \max\{k_2|\hat{v}_{R2}|, k_3 M_1\} & k_3 \end{bmatrix}}_{A = \text{initial point for the convex program (41)}} \bar{e} + \\ + \underbrace{\begin{bmatrix} k_2|\hat{v}_{R2}|x_2^2 + |k_1 x_1 + e_{\omega,1}\cos x_3 - \hat{\omega}_{R2} x_2 - k_3 x_2 x_3| \\ k_2|\hat{v}_{R2} x_1 x_2| + |\hat{\omega}_{R2} x_1 + k_3 x_1 x_3 - T| \\ k_2|\hat{v}_{R2} x_2| + |e_{\omega,2} + k_3 x_3| \end{bmatrix}}_{\tilde{y}(t, x, \hat{\omega}_p, e_\omega)} \tag{50}$$

where

$$\begin{aligned} P &= k_2 \hat{v}_{R2} \frac{\sin(x_3 + e_3)}{x_3 + e_3}, \\ Q &= \hat{\omega}_{R2} x_2, \\ R &= \hat{v}_{R2} \cos(x_3 + e_3), \\ S &= (\hat{v}_{R2} - e_{\omega,1}) \cos x_3, \\ T &= (\hat{v}_{R2} - e_{\omega,1}) \sin x_3, \end{aligned}$$

and $|x_1| \leq M_1$, $|x_2| \leq M_2$. Constants M_1 and M_2 are obtained from the sets $\mathcal{S}_i$'s (see the next paragraph for more details about computing $\mathcal{S}_i$'s). We choose $k_1 = 1.5$, $k_2 = 1.2$ and $k_3 = 1.1$. In addition, in order to make this example more realistic, ω_p takes values in $[-3, 3] \times [-3, 3]$. Consequently, reachability sets $\mathcal{S}_i$ are compact which, in turn, implies finite M_1 and M_2. Since scenarios including Case 3 are more realistic, this section includes numerical results for such a scenario. When emulating noisy environments, we use $e_\omega \in U([-0.3, 0.3] \times [-0.3, 0.3])$ and $h(t) \in U([-0.15, 0.15] \times [-0.15, 0.15] \times [-0.15, 0.15])$ where $U(\mathcal{B})$ denotes the uniform distribution over a set $\mathcal{B}$.

Before verifying that the hypotheses of Theorem 4 hold and presenting numerical results, let us provide details behind Steps 1-6. Since the bounds on ω_p are known, we confine measurements $\hat{\omega}_p$ to the same set, i.e., $r \in \mathcal{P} = [-3, 3] \times [-3, 3]$. In other words, if we obtain $\hat{\omega}_p(t_i) \notin \mathcal{P}$ (for instance, because of measurement noise), we use the closest value in $\mathcal{P}$ (with respect to the Euclidean distance) for $\hat{\omega}_p(t_i)$. Now, after receiving $\hat{\omega}_p(t_i)$ and $\hat{y}(t_i)$, we update control signal $u(t_i)$. Next, we need to determine when to sample again. Utilizing $\hat{y}(t_i)$, we obtain $\hat{x}(t_i) \in \hat{y}(t_i) \pm [-0.15, 0.15] \times [-0.15, 0.15] \times [-0.15, 0.15]$. Starting from these $\hat{x}(t_i)$, and due to the fact that $u(t_i)$ is the linear and angular velocity of $R1$ and remains constant until t_{i+1}, we readily compute reachable states $\hat{x}(t_i + \tau_i)$ for any $\tau_i \geq 0$ using $u(t_i)$ and the bounds on ω_p. Because of (47), reachable output errors $e(t_i + \tau_i)$ are immediately computed and sets $\mathcal{S}_i$ are obtained. Inspecting the form of A in (50), we infer that the first two components of x are important for (50) to hold on some $\mathcal{S}_i$. Due to the properties of $\preceq$, we choose the maximum values of $|x_1|$ and $|x_2|$ in $\mathcal{S}_i$ denoted M_1 and M_2, respectively.

Let us verify that the hypotheses of Theorem 4 hold. Due to (50), we infer that item (i) of Assumption 2 is fulfilled provided that M_1 and M_2 are finite. For this reason, we restrict the analysis to a set for the state x given by bounded $|x_1|$ and $|x_2|$. Next, combining the approach of [25] and the power iterations method [34], we estimate $\mathcal{L}_2$-gains Σ_n^δ over $\mathcal{P}$ and obtain $\overline{\gamma_n} = 96$. The upper bound $\overline{K_n}$ is obtained similarly. Hence, item (ii) of Assumption 2 holds. Item (iii) of Assumption 2 is inferred from (49) and (50) as follows. It can be shown that

$$\|x\|^2 \leq k\big(\|\tilde{y}\|^2 + \|(e, e_\omega)\|^2\big)$$

for any $k \geq 2$. Integrating both sides of the last inequality over $[t_0, t]$ for any $t \geq t_0$ and taking the square root, yields

$$\|x[t_0, t]\|_2 \leq \sqrt{k}\|\tilde{y}[t_0, t]\|_2 + \sqrt{k}\|(e, e_\omega)[t_0, t]\|_2.$$

In other words, the state x of the system Σ_n^r is $\mathcal{L}_2$ to $\mathcal{L}_2$ detectable from $(e, e_\omega, \tilde{y})$ with the upper bounds $K = 0$ and $\gamma = \sqrt{k}$. Finally, the condition (36) of Theorem 4 is easily verified for an arbitrary user-selected $\overline{K} \geq 0$, where $\overline{K}$ captures the desired impact of the initial conditions on the tracking performance according to (39), as the simulation progresses for a given $\delta : [t_0, \infty) \to \mathcal{P}$. Recall that the switching signal δ is in fact $\hat{\omega}_p$. Accordingly, only after the control law (4) is updated with the most recent information about ω_p, i.e., with $\hat{\omega}_p$ (and provided that this value is different from the current value of $\hat{\omega}_p$), that time instant becomes a switching instant and the corresponding $x(t_i^{\delta+})$ contributes to the left hand side of (36). In case (36) might get violated, simply decrease τ_i's in Step 6 or cease switching according to Step 3 (i.e., the left hand side of (36) remains unaltered) by using the current $\hat{\omega}_p$ until (36) is no longer compromised (if ever). Furthermore, even from the viewpoint that $\hat{\omega}_p$ is a noisy version of ω_p, it might not be advantageous (in terms of decreasing the tracking error) to always update the control law (4) with the incoming information about ω_p, i.e., with $\hat{\omega}_p$. This observation is found in the following paragraph as well.

In the simulations, we choose $\omega_p(t) = (1,1)t_{[0,2.26)} + (0.6, 0.15)t_{[2.26,9.25)} + (2,2)t_{[9.25,12]}$, where $t_{\mathcal{I}}$ is the indicator function on an interval $\mathcal{I}$, i.e., $t_{\mathcal{I}} = t$ when $t \in \mathcal{I}$ and zero otherwise. The corresponding $\mathcal{L}_2$-gains Σ_n^δ are as follows: $\gamma_n^{(0.6,0.15)} = 22$, $\gamma_n^{(1,1)} = 53$ and $\gamma_n^{(2,2)} = 56$. In order to illustrate Theorem 4 and the mechanism behind (34), we superpose a continuous signal $e_\omega(t) \in [-0.3, 0.3] \times [-0.3, 0.3]$, where $t \in [0, 12]$, onto the above $\omega_p(t)$, and update the control law with $\hat{\omega}_p$ being $(0.6, 0.15)$, $(1,1)$ or $(2,2)$. This way, we are able to use a fixed γ_n^δ between two switches so that the left hand side of (36) does not increase unnecessarily (and because the received values are corrupted by noise anyway). In addition, the impact of changes in $\hat{y}$ on τ_i's is easier to observe. The obtained numerical results are provided in Figure 5. As can be seen from Figure 5, intersampling intervals τ_i's tend to increase as $\|x\|$ approaches the origin because M_1 and M_2 decrease. In addition, the abrupt changes of τ_i at 2.26 s and 9.25 s, visible in Figure 5(c), are the consequence of the abrupt changes in $\hat{\omega}_p$. In other words, τ_i's adapt to the changes in $\hat{\omega}_p$. This adaptation of τ_i's follows from (34) where individual gains are considered instead of the unified gain [23]. The simulation results obtained using the unified gain $\sup_{r \in \mathcal{P}} \gamma_n^r = \overline{\gamma_n} = 96$, achieved when $r = (3,3)$, and corresponding A in (34) are shown in Figure 6. Apparently, the use of the unified gains decreases τ_i's, does not allow for adaptation of τ_i, and yet does not necessarily yield stability of the closed-loop system since (36) does not have to hold (a similar observation is found in [23]). Consequently, the number of transmissions in the scenario depicted in Figure 5 is 580, while in the scenario depicted in Figure 6 is 1377 (refer to Figure 3 as well). Finally, it should be mentioned that the oscillations of x in Figures 5(a) and 6(a) are an inherited property of the controller, and not a consequence of intermittent feedback.

6.1. *Comparison with Related Work*

In this subsection, we compare our methodology and the methodology presented in [18] and [19] in more detail. This work by Wang and Lemmon appears to be the most similar to our work. Nevertheless, [18] and [19] aim at a different goal under different assumptions than the work found herein. The design objective of state-triggering from [18] and [19] is to maintain the $\mathcal{L}_2$-gain from disturbance ω_p to x_p below a desired threshold. In addition, the approach from [18] and [19] is developed for linear time-invariant plants driven by full-information H_∞ controllers. In other words, the approach from [18] and [19] is not applicable to general nonlinear controllers and plants (5)-(6). In addition, we consider output feedback, while [18] and [19] do not. On the one hand, we take into account measurement noise, and on the other hand [18] and [19] take into consideration delays due to nontrivial executions times of control laws. Furthermore, we investigate $\mathcal{L}_p$-stability, where $p \in [1, \infty]$, while the related papers investigate $\mathcal{L}_2$-stability. The use of the $\mathcal{L}_\infty$ vector space allows us to apply our results to stable closed-loop systems in the presence of continuous feedback and zero disturbance (refer to [29, 35] and Theorem 3), while [18] and [19] require asymptotic stability when feedback is continuous and disturbance is zero.

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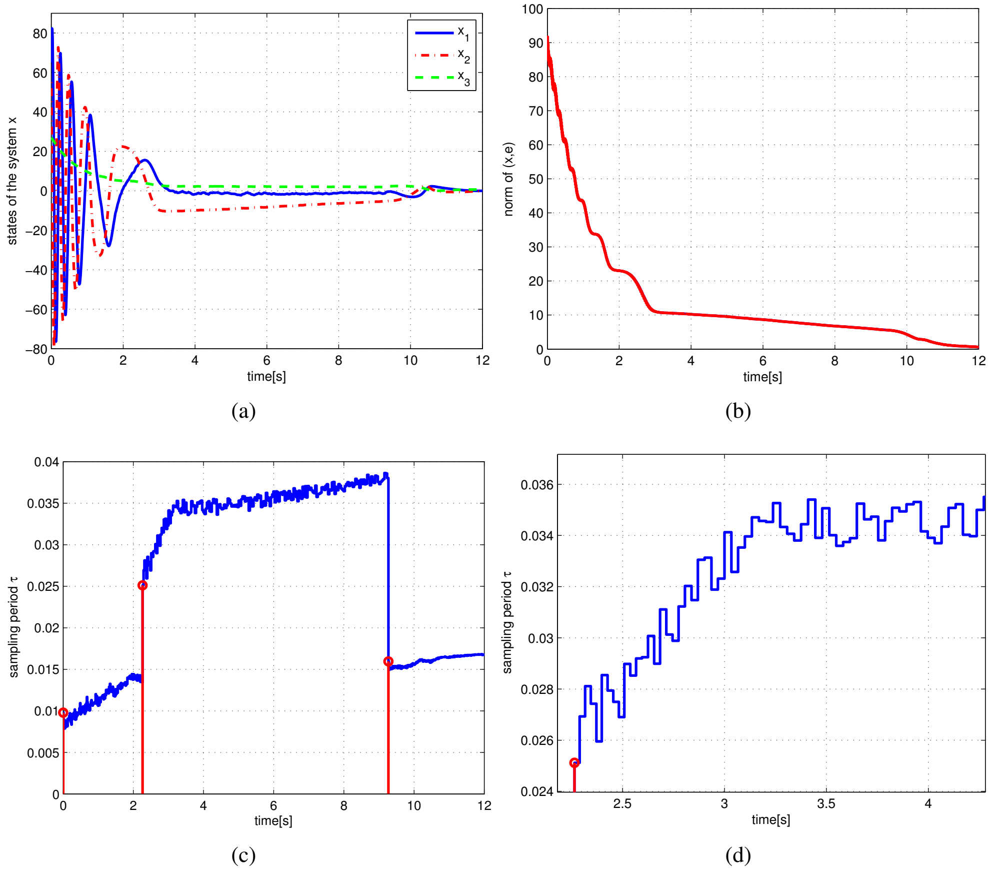

Figure 5. A realistic scenario illustrating input-output triggering: (a) States x of the tracking system; (b) Norm of (x, e); (c) Values of intersampling intervals τ_i's between two consecutive transmissions. Red stems indicate time instants when changes in δ happen; and, (d) A detail from Figure 5(c).

We now apply our framework to the numerical example from [19]. Let us consider a linearized inverted pendulum problem given by

$$\dot{x}_p = \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{-m_1 g}{m_2} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{g}{l} & 0 \end{bmatrix}}_{\hat{A}} x_p + \underbrace{\begin{bmatrix} 0 \\ \frac{1}{m_2} \\ 0 \\ \frac{-1}{m_2} \end{bmatrix}}_{\hat{B}} u + \underbrace{\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}}_{\hat{B}_d} \omega_p,$$

where m_1 is the mass of the pendulum bob, m_2 is the cart mass, l is the length of the pendulum arm, and g is gravitational acceleration. According to [19], we choose $m_1 = 1$, $m_2 = 10$, $l = 3$, and $g = 10$. The full-information H_∞ control law is given by $u = \hat{K} x_p$, where $\hat{K} = [2 \quad 12 \quad 378 \quad 210]$. The desired threshold on the $\mathcal{L}_2$-gain of the closed-loop system is chosen to be $\gamma^B = 4 \times 10^4$. The system state is a four dimensional vector consisting of the cart's position, the corresponding velocity, the pendulum bob's angle with respect to the vertical, and the corresponding velocity.

Next, let us consider the case without delays summarized in [19, Figure 1 & 2]. Since ω_p is not measured and the measurements are not sent to the controller, there is no switching. Hence, $\hat{\omega}_p \equiv 0$ is irrelevant and $e_\omega = -\omega_p$ in this example. From the setting in [19], we infer that $e_u \equiv 0$ and $x = x_p$. In addition, $y = x_p$ due to state feedback. Next, we introduce $e = \hat{x} - x = [e_1 \quad e_2 \quad e_3 \quad e_4]^\top$.

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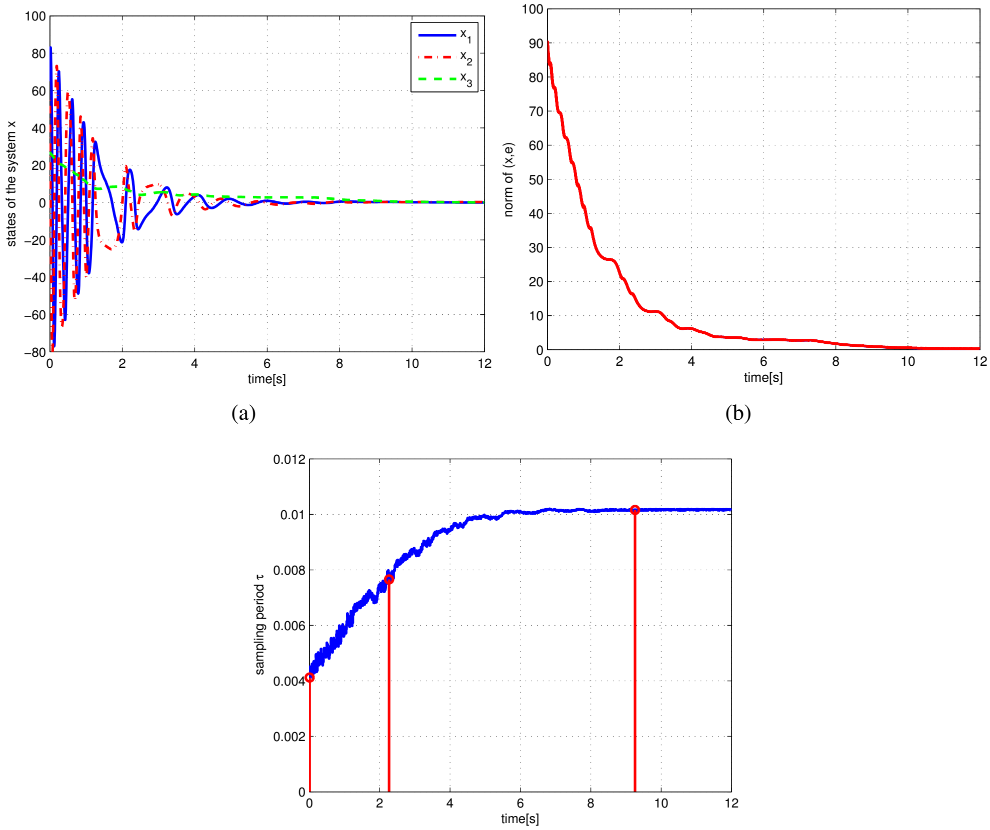

(a) (b)

(c)

Figure 6. A realistic scenario illustrating input-output triggering using the unified gains: (a) States x of the tracking system; (b) Norm of (x, e); and (c) Values of intersampling intervals τ_i's between two consecutive transmissions. Red stems indicate time instants when changes in δ happen.

Following the exposition at the beginning of Section 5, one obtains:

$$\dot{x} = (\hat{A} + \hat{B}\hat{K})x + [\hat{B}\hat{K} \quad -\hat{B}_d](e, e_\omega),$$
$$\dot{e} = -\hat{B}\hat{K}e - (\hat{A} + \hat{B}\hat{K})x + \hat{B}_d e_\omega.$$

Now, we define $\tilde{y} = \overline{-(\hat{A} + \hat{B}\hat{K})x + \hat{B}_d e_\omega}$. One way to proceed further is to find $\mathcal{L}_2$-gains from (e_ω, e) to $\tilde{y}$ and from (e_ω, e) to x utilizing [36, Theorem 5.4]. These gains are denoted γ and γ_d, respectively. It can be shown that the $\mathcal{L}_2$-gain from e_ω to (x, e), denoted γ_ω, equals

$$\gamma_\omega = \frac{\gamma_d + \kappa}{1 - \kappa}.$$

Notice that γ_ω is also an upper bound (although quite conservative) for the $\mathcal{L}_2$-gain from ω_p to state x. Hence, by choosing τ_i's such that $\gamma_\omega \le \gamma^B$, the goal of [19] is achieved. Because the problem of interest is linear and time-invariant, the upper bound (18) holds for all $(t, x(t), \hat{\omega}_p, e_\omega(t)) \in \mathbb{R} \times \mathbb{R}^{n_x} \times \mathbb{R}^{n_\omega} \times \mathbb{R}^{n_\omega}$ with the same A. Consequently, there is no need for the reachability analysis (which significantly simplifies the application of our methodology). Both the absence of switching and the fact that $\mathcal{S} = \mathbb{R} \times \mathbb{R}^{n_x} \times \mathbb{R}^{n_\omega} \times \mathbb{R}^{n_\omega}$ produce constant τ_i's. However, some other method for computation of $\mathcal{L}_p$-gains over a finite horizon (other than Theorem 2) may yield variable τ_i's for this example.

Using the MATLAB function norm(·,inf), we obtain $\gamma = 77$ and $\gamma_d = 287$. The value $\kappa = 0.9928$ corresponds to $\gamma_\omega = \gamma^B = 4 \times 10^4$ and $\tau_i = 10$ ms, $i \in \mathbb{N}_0$. This intersampling interval equals the minimal intersampling interval obtained in [19, Figure 1]. In addition, our intersampling interval is about an order of magnitude smaller than the maximal intersampling interval obtained in [19, Figure 1]. This slight conservativeness of our intersampling interval is due to Theorem 2 and the fact that γ_ω is an $\mathcal{L}_2$-gain from e_ω to (x, e) and not from e_ω to x as in [19]. This suggests a possible avenue for our future research.

7. CONCLUSION AND FUTURE WORK

In this paper we present a methodology for input-output triggered control of nonlinear systems. Based on the currently available measurements of the output and external input of the plant, a sampling policy yielding the closed-loop system stable in some sense is devised. Using the formalism of $\mathcal{L}_p$-gains and $\mathcal{L}_p$-gains over a finite horizon, the small gain theorem is employed to prove stability, asymptotic, and $\mathcal{L}_p$-stability (with bias) of the closed-loop system. Different types of stability are a consequence of different assumptions on the noise environment causing the mismatch between the actual external input and output of the plant, and the measurements available to the controller via feedback. The closed-loop systems are modeled as hybrid systems, and a novel result regarding $\mathcal{L}_p$-stability of such systems is presented. Finally, our input-output triggered sampling policy is exemplified on a trajectory tracking controller for velocity-controlled unicycles.

The future work is dedicated to applying scattering transformation between the controller and plant in order to eliminate detrimental effects of delays. Furthermore, actuators with saturation will be analyzed. In order to obtain larger intertransmission intervals, zero-order hold estimation strategies will be replaced with model-based estimation of control signals and plant outputs. Finally, we expect our results (with slight modifications) to hold for input-to-state stability of hybrid systems.

APPENDIX

7.1. *Properties of Matrix Functions*

Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. The work in [9] constructs a matrix A such that $\|f(A)\| = f(\|A\|)$ for any f by requiring that matrix A is symmetric, with nonnegative entries and positive semidefinite. While symmetry of the matrix and nonnegative entries are required throughout [9], positive semidefiniteness is required only in [9, Lemma 7.1]. Exploiting the fact that f in [9, Lemma 7.1] (i.e., in Theorem 2 herein) is the exponential function $f(\cdot) = \exp(\cdot)$ and A is a real symmetric matrix with nonnegative entries, the following lemmas show that the positive semidefiniteness requirement of [9, Theorem 5.1] can be relaxed.

Lemma 1
Suppose that $f(\cdot) = \exp(\cdot)$ and A is a real $n \times n$ symmetric matrix. The eigenvalue of A with the largest absolute value is real and nonnegative if and only if $\|f(A)\| = f(\|A\|)$.

Proof
It is well known that real symmetric matrices can be diagonalized, i.e., $A = UDU^T$ where D is a diagonal matrix and U is an orthogonal matrix. Since the spectral norm is unitarily invariant [37], it is straightforward to show that the following equalities hold:

$$\|f(A)\| = \|UDU^T\| = \|D\| = \|\text{diag}(f(\lambda_1(A)), f(\lambda_2(A)), \ldots, f(\lambda_n(A))\| = \max_k |f(\lambda_k(A))|. \tag{51}$$

On the other hand, using the definition of the induced matrix 2-norm $\|\cdot\|$, we have

$$f(\|A\|) = f(\max_k |\sigma_k(A)|). \tag{52}$$

In addition, real symmetric matrices have the following property

$$\sigma_k(A) = |\lambda_k(A)|. \tag{53}$$

Using (53), and monotonicity and positivity of the exponential function, we conclude that expressions (51) and (52) are equal if and only if the eigenvalue of A with the largest absolute value is real and nonnegative. □

For completeness, we now write the following well-known result for symmetric matrices (see, e.g., [38, Chapter 8]).

Lemma 2
If A is a symmetric matrix with nonnegative entries, then the eigenvalue of A with the largest absolute value is real and nonnegative.

7.2. *Proof of Theorem 1*

We prove this theorem for the case $p \in [1, \infty)$. The proof for the case $p = \infty$ is similar.

Let us start from some initial condition $\chi(t_0)$ and apply input ω to a hybrid system Σ^δ given by (12) to obtain the state trajectory $t \mapsto \chi(t)$ and associated output $t \mapsto y(t)$. Now, we can write for every $t \geq t_0$

$$\begin{aligned}\|y[t_0,t]\|_p^p &= \int_{t_0}^{t} \|y(s)\|^p \mathrm{d}s = \sum_{i=0}^{J-1} \int_{t_i^\delta}^{t_{i+1}^\delta} \|y(s)\|^p \mathrm{d}s + \int_{t_J^\delta}^{t} \|y(s)\|^p \mathrm{d}s \\ &= \sum_{i=0}^{J-1} \|y[t_i^\delta, t_{i+1}^\delta]\|_p^p + \|y[t_J^\delta, t]\|_p^p, \end{aligned} \tag{54}$$

where $J = \arg\max\{j : t_j^\delta \leq t\}$. From (14) we obtain

$$\begin{aligned}\|y[t_0,t]\|_p^p \leq \sum_{i=0}^{J-1} \Big(\widetilde{K}(\tau_i^\delta)) \|\chi(t_i^{\delta+})\| + \widetilde{\gamma}(\tau_i^\delta) \|\omega[t_i^\delta, t_{i+1}^\delta]\|_p \Big)^p + \\ + \Big(\widetilde{K}(\tau_J^\delta)) \|\chi(t_J^{\delta+})\| + \widetilde{\gamma}(\tau_J^\delta) \|\omega[t_J^\delta, t]\|_p \Big)^p. \end{aligned} \tag{55}$$

Using (15) and (16) yields

$$\|y[t_0,t]\|_p^p \leq \sum_{i=0}^{J-1} \Big(K_M \|\chi(t_i^{\delta+})\| + \gamma_M \|\omega[t_i^\delta, t_{i+1}^\delta]\|_p \Big)^p + \Big(K_M \|\chi(t_J^{\delta+})\| + \gamma_M \|\omega[t_J^\delta, t]\|_p \Big)^p. \tag{56}$$

In what follows we use the following version of the Minkowski inequality

$$\left(\sum_{i=1}^{M} (a_i + b_i)^p \right)^{1/p} \leq \left(\sum_{i=1}^{M} a_i^p \right)^{1/p} + \left(\sum_{i=1}^{M} b_i^p \right)^{1/p}, \tag{57}$$

where $a_i, b_i \geq 0$ and $M \in \mathbb{N} \cup \{\infty\}$. Taking the p^{th} root of (56) yields

$$\|y[t_0,t]\|_p \leq \left(\sum_{i=0}^{J-1} \Big(K_M \|\chi(t_i^{\delta+})\| + \gamma_M \|\omega[t_i^\delta, t_{i+1}^\delta]\|_p \Big)^p + \Big(K_M \|\chi(t_J^{\delta+})\| + \gamma_M \|\omega[t_J^\delta, t]\|_p \Big)^p \right)^{\frac{1}{p}}. \tag{58}$$

Applying (57) to the right hand side of (58) with $M = J$, $a_i = K_M \|\chi(t_i^{\delta+})\|$ and $b_i = \gamma_M \|\omega[t_i^\delta, t_{i+1}^\delta]\|_p$ for $i \leq J-1$, $b_J = \gamma_M \|\omega[t_J^\delta, t]\|_p$, leads to

$$\|y[t_0,t]\|_p \leq K_M \left(\sum_{i=0}^{J} \|\chi(t_i^{\delta+})\|^p \right)^{\frac{1}{p}} + \gamma_M \left(\sum_{i=0}^{J-1} \|\omega[t_i^\delta, t_{i+1}^\delta]\|_p^p + \|\omega[t_J^\delta, t]\|_p^p \right)^{\frac{1}{p}}. \tag{59}$$

Applying the inequality $(a+b)^{1/p} \leq a^{1/p} + b^{1/p}$, where $a, b \geq 0$, to the first term in (59) multiple times, and noting that, since $t_0^\delta = t_0$,

$$\sum_{i=0}^{J-1} \|\omega[t_i^\delta, t_{i+1}^\delta]\|_p^p + \|\omega[t_J^\delta, t]\|_p^p = \|\omega[t_0, t]\|_p^p$$

(as in (54)), we obtain

$$\|y[t_0, t]\|_p \leq K_M \Big(\sum_{i=0}^{J} \|\chi(t_i^{\delta+})\| \Big) + \gamma_M \|\omega[t_0, t]\|_p. \tag{60}$$

Applying (17) we obtain

$$\|y[t_0, t]\|_p \leq K_M \overline{K} \|\chi(t_0)\| + \gamma_M \|\omega[t_0, t]\|_p \tag{61}$$

for all $t \geq t_0$.

7.3. *Proof of Theorem 2*

Let $Df(t)$ denote the left-handed derivative of $f : \mathbb{R} \to \mathbb{R}^n$, i.e., $Df(t) = \lim_{h\to 0, h<0} \frac{f(t+h)-f(t)}{h}$. The following two lemmas and theorem are taken from [9] and slightly modified.

Lemma 3
Let $I = [t_0, t_1]$, $v \in \mathbb{R}^n$ and consider $Dv \preceq Av + d(t), \quad v(t_0) = v_0, \quad \forall t \in I$, where $A \in \mathcal{A}_{n_e}^+$, $\|A\| < \infty$, and $d(t) : I \to \mathbb{R}^n$ is continuous. Then, for all $t \in I$, $v(t)$ is bounded by

$$v(t) \preceq \exp(A(t-t_0))v_0 + \int_{t_0}^{t} \exp(A(t-s))d(s)\mathrm{d}s.$$

Lemma 4 (Young's Inequality)
Let $*$ denote convolution over the interval I, $f \in \mathcal{L}_p[I]$ and $f \in \mathcal{L}_q[I]$. The Young's inequality is $\|f * g\|_r \leq \|f\|_p \|g\|_q$ for $1/r = 1/p + 1/q - 1$ where $p, q, r > 0$.

Theorem 5 (Riesz-Thorin Interpolation Theorem)
Let $F : \mathcal{A}_n^+ \to \mathcal{A}_n^+$ be a linear operator and suppose that $p_0, p_1, q_0, q_1 \in [1, \infty]$ satisfy $p_0 < p_1$ and $q_0 < q_1$. For any $t \in [0, 1]$ define p_t, q_t by $1/p_t = (1-t)/p_0 + t/p_1$ and $1/q_t = (1 - t/q_0) + t/q_1$. Then, $\|F\|_{p_t \to q_t} \leq \|F\|_{p_0 \to q_0}^{1-t} \|F\|_{p_1 \to q_1}^{t}$. In particular, if $\|F\|_{p_0 \to q_0} \leq M_0$ and $\|F\|_{p_1 \to q_1} \leq M_1$, then $\|F\|_{p_t \to q_t} \leq M_0^{1-t} M_1^t$.

Now we are ready to prove Theorem 2. By the hypotheses of the theorem, we have $\dot{\overline{\chi}} = \widetilde{g}(t, \chi, v) \preceq A\overline{\chi} + \tilde{y}(t, \chi, v)$ for all $t \in [t_0, t_0 + \tau]$, and the i^{th} component of $\dot{\overline{\chi}}$ is given by:

$$\left|\tfrac{d}{dt}\chi_i(t)\right| = \left|\lim_{h\to 0, h<0} \tfrac{\chi_i(t+h) - \chi_i(t)}{h}\right| \geq \lim_{h\to 0, h<0} \frac{|\chi_i(t+h)| - |\chi_i(t)|}{h} = D\overline{\chi}_i(t).$$

Therefore, $D\overline{\chi} \preceq A\overline{\chi} + \tilde{y}(t)$. Using Lemma 3, we can write

$$\overline{\chi}(t) \preceq \exp(A(t-t_0))\overline{\chi}(t_0) + \int_{t_0}^{t} \exp(A(t-s))\tilde{y}(s)\mathrm{d}s. \tag{62}$$

Setting the input term $\tilde{y} \equiv 0$, we obtain $\overline{\chi}(t) \preceq \exp(A(t-t_0))\overline{\chi}(t_0)$. Taking the norm of both sides of this inequality and using Lemmas 1 and 2 we obtain:

$$\|\overline{\chi}(t)\| \leq \exp(\|A\|(t-t_0))\|\overline{\chi}(t_0)\|. \tag{63}$$

Raising to the $p^{\text{th}} \in [1, \infty)$ power and integrating over $[t_0, t]$ yields

$$\|\overline{\chi}[t_0, t]\|_p^p \leq \frac{\exp(\|A\|p(t-t_0)) - 1}{p\|A\|} \|\overline{\chi}(t_0)\|^p.$$

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Taking the p^{th} root yields

$$\|\overline{\chi}[t_0,t]\|_p \leq \left(\frac{\exp(\|A\|p(t-t_0))-1}{p\|A\|}\right)^{\frac{1}{p}} \|\overline{\chi}(t_0)\|, \quad p\in[1,\infty). \tag{64}$$

The $\mathcal{L}_\infty$ bound is easily obtained by taking $\lim_{p\to\infty}\|\overline{\chi}[t_0,t]\|_p$ obtaining

$$\|\overline{\chi}[t_0,t]\|_\infty \leq \exp(\|A\|(t-t_0))-1)\|\overline{\chi}(t_0)\|.$$

Let us now set $\overline{\chi}(t_0)=0$ and estimate the contribution from the input term. From (62) we have: $\overline{\chi}(t) \preceq \int_{t_0}^{t}\exp(A(t-s))\tilde{y}(s)\mathrm{d}s$. Using Lemmas 1 and 2 we obtain

$$\|\overline{\chi}(t)\| \leq \int_{t_0}^{t}\exp(\|A\|(t-s))\|\tilde{y}(s)\|\mathrm{d}s. \tag{65}$$

Let us denote $\phi(s)=\exp(\|A\|s)$. Integrating the previous inequality and using Lemma 4 with $p=q=r=1$ yields the $\mathcal{L}_1$-norm estimate:

$$\|\overline{\chi}[t_0,t]\|_1 \leq \|\phi[0,t-t_0]\|_1\|\tilde{y}[t_0,t]\|_1. \tag{66}$$

Taking the $\max$ over $[t_0,t]$ in (65) and using Lemma 4 with $q=r=\infty$ and $p=1$ yields the $\mathcal{L}_\infty$-norm estimate:

$$\|\overline{\chi}[t_0,t]\|_\infty \leq \|\phi[0,t-t_0]\|_1\|\tilde{y}[t_0,t]\|_\infty. \tag{67}$$

We can think of (62) as a linear operator G mapping $\tilde{y}$ to $\overline{\chi}$ with bound for the norms $\|G\|_1\leq\|G\|_1^*$ and $\|G\|_\infty\leq\|G\|_\infty^*$ where $\|G\|_1^*$ and $\|G\|_\infty^*$ are given by (66) and (67), respectively. Because $\|G\|_1^*=\|G\|_\infty^*$, Theorem 5 gives that $\|G\|_p\leq\|G\|_1^*=\|G\|_\infty^*$ for all $p\in[1,\infty]$. This yields

$$\|\overline{\chi}[t_0,t]\|_p \leq \|\phi[0,t-t_0]\|_1\|\tilde{y}[t_0,t]\|_p, \quad p\in[1,\infty].$$

Since $\|\phi[0,t-t_0]\|_1=\frac{\exp(\|A\|(t-t_0))-1}{\|A\|}$, we obtain

$$\|\overline{\chi}[t_0,t]\|_p \leq \frac{\exp(\|A\|(t-t_0))-1}{\|A\|}\|\tilde{y}[t_0,t]\|_p, \quad p\in[1,\infty]. \tag{68}$$

After summing up the contributions of (64) and (68), the statement of the theorem follows.

7.4. *Proof of Results in Section 5.3*

The proofs of the results in Section 5.3 use the property over an arbitrary finite interval with constant δ introduced in the next section. After that, this property is used sequentially, over a finite horizon of arbitrary length, to obtain an $\mathcal{L}_p$ bound on (x,e).

7.4.1. $\mathcal{L}_p$ property over an arbitrary finite interval with constant δ: Consider the nontrivial interval $I:=[\underline{t},\underline{t}+\tau)$, $\underline{t}\geq t_0$ on which $t\mapsto\delta(t)$ is constant and with $\tau>0$ to be defined. Let r be such that $\delta(t)=r$ for all $t\in I$. From item (ii) of Assumption 2, given initial condition $x(\underline{t})$, we have, for all $t\in I$,

$$\begin{aligned}\|\tilde{y}^r[\underline{t},t]\|_p &\leq K_n^r\|x(\underline{t})\|+\gamma_n^r\|(e,e_\omega)[\underline{t},t]\|_p\\ &\leq K_n^r\|x(\underline{t})\|+\gamma_n^r\|e[\underline{t},t]\|_p+\gamma_n^r\|e_\omega[\underline{t},t]\|_p.\end{aligned} \tag{69}$$

Item (i) of Assumption 2 allows us to invoke Theorem 2 for Σ_e^r on I. Then, we have, for all $t\in I$,

$$\|e[\underline{t},t]\|_p \leq \widetilde{K}_e^r(\tau)\|e(\underline{t})\|+\widetilde{\gamma}_e^r(\tau)\|\tilde{y}^r[\underline{t},t]\|_p \tag{70}$$

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for any τ chosen as

$$\tau \in (0, \tau^{r,*}], \tag{71}$$

where $\widetilde{K}_e^r(\tau), \widetilde{\gamma}_e^r(\tau)$ are given as in (30) and $\tau^{r,*}$ is given as in (34). Due to the construction of $\tau^{r,*}$ and the choice of τ in (71), the open-loop gain of the interconnection (see Figure 4) from e_ω to e is $\gamma_n^r \widetilde{\gamma}_e^r(\tau)$ is less than $\kappa \in (0,1)$. Then, combining (69) and (70), we have, for each $t \in I$,

$$\|e[\underline{t}, t]\|_p \leq \frac{\widetilde{K}_e^r(\tau)}{1 - \gamma_n^r \widetilde{\gamma}_e^r(\tau)} \|e(\underline{t})\| + \frac{\widetilde{\gamma}_e^r(\tau) K_n^r}{1 - \gamma_n^r \widetilde{\gamma}_e^r(\tau)} \|x(\underline{t})\| + \frac{\widetilde{\gamma}_e^r(\tau) \gamma_n^r}{1 - \gamma_n^r \widetilde{\gamma}_e^r(\tau)} \|e_\omega[\underline{t}, t]\|_p \tag{72}$$

Next, let us use $\tau_{\max}^*$ from Remark 2 instead of τ in order to obtain the following upper bound for $\widetilde{K}_e^r(\tau)$ over $(0, \tau^{r,*}]$ and for any $r \in \mathcal{P}$:

$$\widetilde{K}_e^r(\tau) \leq \sup_{r \in \mathcal{P}} \left(\frac{\exp\left(\|A^r\| p \tau_{\max}^*\right) - 1}{p \|A^r\|} \right)^{\frac{1}{p}} =: \widetilde{K}_e(\tau).$$

This supremum exists due to Assumption 2. Likewise, $\widetilde{\gamma}_e^r(\tau)$ can be upper bounded by a constant $\widetilde{\gamma}_e(\tau)$. Next, using the detectability property in item (iii) of x from $(\widetilde{y}^r, e_\omega, e)$ (with gains K and γ), and combining (72) and (69), gives

$$\|(x, e)[\underline{t}, t]\|_p \leq \widehat{K} \|(x(\underline{t}), e(\underline{t}))\| + \widehat{\gamma} \|e_\omega[\underline{t}, t]\|_p \tag{73}$$

for all $t \in I$, where the constants $\widehat{K}$ and $\widehat{\gamma}$ are given by

$$\widehat{K} = \sqrt{n_x + n_e} \max \left\{ K + \frac{\widetilde{\gamma}_e(\tau)\overline{K_n} + \gamma \overline{K_n} + \gamma \widetilde{\gamma}_e(\tau) \overline{K_n}}{1 - \kappa}, \frac{\widetilde{K}_e(\tau) + \gamma \overline{\gamma_n} \widetilde{K}_e(\tau) + \gamma \widetilde{K}_e(\tau)}{1 - \kappa} \right\},$$
$$\widehat{\gamma} = \gamma + \frac{\widetilde{\gamma}_e(\tau)\overline{\gamma_n} + \gamma \overline{\gamma_n} + \gamma \widetilde{\gamma}_e(\tau) \overline{\gamma_n}}{1 - \kappa},$$

where we have used the bounds on K_n^r and γ_n^r given in item (ii) of Assumption 2. Notice that the above expressions are independent of $r \in \mathcal{P}$.

7.4.2. Extending bounds to (arbitrarily long) finite horizon: Given initial time t_0 and initial condition x_0, we use analysis on an arbitrary interval in Section 7.4.1 to design the sampling instants $\mathcal{T}$ to update $(\hat{y}, \hat{u})$ and, in turn, define the instants in $\mathcal{T}^\delta$ at which the signal $t \mapsto \delta(t)$ is allowed to switch. In this way, using $\underline{t} = t_0$, $x(\underline{t}) = x_0$, $e(\underline{t}) = e_0$, pick τ_0 to satisfy (71) to define $I_0 := [t_0, t_0 + \tau_0)$, over which $t \mapsto \delta(t)$ is constant, i.e., $\delta(t) = r_0$ for all $t \in I_0$, for some $r_0 \in \mathcal{P}$. Then, as previously explained, we obtain, for all $t \in I_0$,

$$\|(x, e)[t_0, t]\|_p \leq K^{r_0} \|(x_0, e_0)\| + \gamma^{r_0} \|e_\omega[t_0, t]\|_p. \tag{74}$$

Now, at $t = t_0 + \tau_0$ a sampling event occurs, which updates (x, e) according to (25a), and, potentially, δ changes value. Then, using $\underline{t} = t_1$, $x(\underline{t}) = x(t_1^+)$, $e(\underline{t}) = \lim_{t \nearrow t_1} h(t) =: e(t_1)$, pick τ_1 to satisfy (71) to define $I_1 := [t_1, t_1 + \tau_1)$, over which $t \mapsto \delta(t)$ is constant, i.e., $\delta(t) = r_1$ for all $t \in I_1$, for some $r_1 \in \mathcal{P}$. Then, as before, we obtain, for all $t \in I_1$,

$$\|(x, e)[t_1, t]\|_p \leq K^{r_1} \|(x(t_1), e(t_1^+))\| + \gamma^{r_1} \|e_\omega[t_1, t]\|_p. \tag{75}$$

Proceeding in this way for subsequent sampling intervals, in particular, for the i-th sampling time, we use $\underline{t} = t_i$, $x(\underline{t}) = x(t_i)$, $e(\underline{t}) = \lim_{t \nearrow t_i} h(t) =: e(t_i^+)$, pick τ_i to satisfy (71) to define $I_i := [t_i, t_1 + \tau_i)$, over which $t \mapsto \delta(t)$ is constant, i.e., $\delta(t) = r_i$ for all $t \in I_i$, for some $r_i \in \mathcal{P}$. Now, we obtain, for all $t \in I_i$,

$$\|(x, e)[t_i, t]\|_p \leq K^{r_i} \|(x(t_i), e(t_i^+))\| + \gamma^{r_i} \|e_\omega[t_i, t]\|_p. \tag{76}$$

In fact, the above construction of t_i's can be performed for any $t \geq t_0$, combining (76) over each I_i interval with $i \in \{0, 1, 2, \ldots, N\}$, where N is such that $[0, t] \subset \cup_{i=0}^{N} I_i$ and $[0, t] \not\subset \cup_{i=0}^{N-1} I_i$.

Now we are ready to prove the results in Section 5.3.

7.4.3. Proof of Theorem 3: Let us now apply Theorem 1 with (x, e) playing the role of y and e_ω playing the role of ω. According to Theorem 1, we are interested only in instants $t_i \in \mathcal{T}$ that are followed by a change of the value of δ, i.e., in $t_i^\delta \in \mathcal{T}^\delta$. Recall that, due to (73), we have $K_M \leq \widehat{K}$ and $\gamma_M \leq \widehat{\gamma}$; hence, the suprema in (15) and (16) exist. Next, we need to verify that there exists $K_1 \geq 0$ such that $\sum_{t_i^\delta \in \mathcal{T}_0^\delta} \|(x(t_i^\delta), e(t_i^{\delta+}))\| \leq K_1 \|(x(t_0), e(t_0))\|$. Due to the perfect resets of e at $t_i^\delta \in \mathcal{T}^\delta$, given by (35a), and hypothesis (36), we infer that $K_1 = \overline{K}$ for Cases 1 and 2. In other words, (37) holds for Cases 1 and 2.

The case for $p = \infty$ follows similarly.

7.4.4. Proof of Theorem 4: Notice that $h(t)$ in Theorem 4 is more general than $h(t)$ in Remark 4. Consequently, the condition (ii) of Theorem 1 is no longer satisfied. In order to take into account the setting of Theorem 4, we rewrite (70) as follows

$$\|e[t_0, \bar{t}]\|_p \leq \widetilde{K}_e^r(\tau^{r_0,*})\|e(t_0)\| + \widetilde{\gamma}_e^r(\tau^{r_0,*})\|\tilde{y}^r[t_0, \bar{t}]\|_p,$$

for all $\bar{t} \in I_0 = [t_0, t_0 + \tau^{r_0,*})$, and

$$\begin{aligned}
\|e[t_i, \bar{t}]\|_p &\leq \widetilde{K}_e^r(\tau^{r_i,*})\|h(t_i)\| + \widetilde{\gamma}_e^r(\tau^{r_i,*})\|\tilde{y}^r[t_i, \bar{t}]\|_p \leq \widetilde{K}_e^r(\tau^{r_i,*})\overline{K}_h + \widetilde{\gamma}_e^r(\tau^{r_i,*})\|\tilde{y}^r[t_i, \bar{t}]\|_p \\
&= \left(\frac{\exp(\|A^{r_i}\| p\tau^{r_i,*}) - 1}{p\|A^{r_i}\|}\right)^{\frac{1}{p}} \overline{K}_h + \widetilde{\gamma}_e^r(\tau^{r_i,*})\|\tilde{y}^r[t_i, \bar{t}]\|_p \\
&= \left[\int_{t_i}^{t_i+\tau^{r_i,*}} \left\| \exp\left(\|A^{r_i}\|(s - t_i)\right)\overline{K}_h \right\|^p \mathrm{d}s\right]^{\frac{1}{p}} + \widetilde{\gamma}_e^r(\tau^{r_i,*})\|\tilde{y}^r[t_i, \bar{t}]\|_p \\
&\leq \left[\int_{t_i}^{t_i+\tau^{r_i,*}} \left\| \exp\left(\|A^{r_i}\|\tau^{r_i,*}\right)\overline{K}_h \right\|^p \mathrm{d}s\right]^{\frac{1}{p}} + \widetilde{\gamma}_e^r(\tau^{r_i,*})\|\tilde{y}^r[t_i, \bar{t}]\|_p \\
&= \|b^{r_i}[t_i, t_i + \tau^{r_i,*}]\|_p + \widetilde{\gamma}_e^r(\tau^{r_i,*})\|\tilde{y}^r[t_i, \bar{t}]\|_p,
\end{aligned}$$

for all $\bar{t} \in I_i = [t_i, t_i + \tau^{r_i,*})$ with $i \in \{0, 1, 2, \ldots, N\}$, where N is such that $[0, t] \subset \cup_{i=0}^N I_i$ and $[0, t] \not\subset \cup_{i=0}^{N-1} I_i$, and $b^{r_i}(t) \equiv b^{r_i} = \exp\left(\|A^{r_i}\|\tau^{r_i,*}\right)\overline{K}_h$. Notice that $\sup_{r\in\mathcal{P}} b^r \leq \overline{K}_h \exp(\overline{\eta}\tau^*_{\max}) =: b \equiv b(t)$, where $\overline{\eta}$ is defined in (ii) of Assumption 2 and $\tau^*_{\max}$ in Remark 2, is a suitable uniform choice over all I_i intervals. Next, let us use the following bounds over each interval:

$$\|e[t_i, \bar{t}]\|_p \leq \widetilde{K}_e^r(\tau^{r_i,*})\|e(t_i)\| + \widetilde{\gamma}_e^r(\tau^{r_i,*})\|\tilde{y}^r[t_i, \bar{t}]\|_p + \|b[t_i, \bar{t}]\|_p,$$

for all $\bar{t} \in I_i = [t_i, t_i + \tau^{r_i,*})$, where $\|e(t_i)\| = 0$ for $i \in \{1, 2, ..., N\}$. Notice that the above systems are $\mathcal{L}_p$-stable with the same bias $b(t) \equiv b$ over intervals I_i.

In the same manner as before, we now apply the small-gain theorem over each interval I_i and utilize item (iii) of Assumption 2. Afterwards, we apply Theorem 1 to the resulting $\mathcal{L}_p$-stable systems with bias over each interval I_i. [Notice that Theorem 1 readily extends to $\mathcal{L}_p$-stability with bias. By adding $\|b[t_i^\delta, t_{i+1}^\delta]\|_p$ and $\|b[t_J^\delta, t]\|_p$ to the respective terms on the right hand side of (55) and following the remainder of the proof in Section 7.2, one arrives at $\|y[t_0, t]\|_p \leq K_M\overline{K}\|\chi(t_0)\| + \gamma_M\|\omega[t_0, t]\|_p + \|b[t_0, t]\|_p$ for all $t \geq t_0$.] In other words, following the exposition below (70), one readily obtains:

$$\|(x, e)[t_0, t]\|_p \leq \overline{K}\widehat{K}\|(x, e)(t_0)\| + \widehat{\gamma}\|e_\omega[t_0, t]\|_p + \|\widehat{b}[t_0, t]\|, \qquad \forall t \geq t_0,$$

where

$$\widehat{b}(t) \equiv \widehat{b} = \frac{\gamma\overline{\gamma_n} + \gamma}{1 - \kappa}\overline{K}_h \exp(\overline{\eta}\tau^*_{\max}).$$

The case of $p = \infty$ is similar.

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