Title
BISE--A TWO-VERTEX KINEMATIC PROGRAM

Author
Hanerfeld, Harold.

Publication Date
1967-03-28

University of California

Ernest O. Lawrence Radiation Laboratory

BISE--A TWO-VERTEX KINEMATIC PROGRAM

Harold Hanerfeld

March 28, 1967

UCRL-17473
TID-4500 (50th Ed.)
UC-32 Math. and Comp.

UNIVERSITY OF CALIFORNIA

Lawrence Radiation Laboratory
Berkeley, California

AEC Contract No. W-7405-eng-48

BISE--A TWO-VERTEX KINEMATIC PROGRAM

Harold Hanerfeld

March 28, 1967

Printed in the United States of America
Available from
Clearinghouse for Federal Scientific and Technical Information
National Bureau of Standards, U.S. Department of Commerce
Springfield, Virginia 22151
Price: Printed Copy $3.00; Microfiche $0.65

BISE--A TWO-VERTEX KINEMATIC PROGRAM

Harold Hanerfeld

Lawrence Radiation Laboratory
University of California
Berkeley, California

March 28, 1967

ABSTRACT

BISE is a two-vertex kinematic least-square program written specifically for fitting events of the type

$$K^-p \to \Sigma^+ \left\{ \begin{array}{l} \pi^-\pi^0 \\ \pi^- \end{array} \right.$$

$$\Sigma^+ \longrightarrow \left\{ \begin{array}{l} p\,\pi^0 \\ \pi^+ n \end{array} \right.$$

This report is a complete description of the mathematical formulation on which the program is based.

BISE--A TWO-VERTEX KINEMATICS PROGRAM

Harold Hanerfeld

Lawrence Radiation Laboratory
University of California
Berkeley, California

March 28, 1967

BISE is a program (FORTRAN-MAP) for the kinematic reconstruction of certain types of bubble chamber events. The routine was written specifically with the following event types in mind:

$$K^-p \to \Sigma^+ \begin{cases} \pi^-\pi^0 \\ \pi^- \end{cases} \qquad \hookrightarrow \begin{cases} p\pi^0 \\ \pi^+ n \end{cases}$$

The main innovation in BISE is the simultaneous solution for all the variables--measured and unmeasured variables and Lagrangian multipliers. It has been the practice to solve the system of linear equations by partitioning the matrix obtained in the least-square technique. The partitioned system is then solved by a matrix inversion and sequence of matrix multiplication. Difficulties sometimes arise, as the matrix to be inverted may be singular or near singular. This known defect has in turn been treated by methods which themselves were not completely satisfactory. A second fault in earlier programs was the use of poor numerical methods, leading to inaccurate results. Methods such as matrix inversion by Cramer's Rule were used to solve a system of linear equations. By solving the complete system of equations and using a good numerical technique for the solution, both difficulties are avoided. Further, the algebraic simplification also simplifies the programming.

Another innovation in BISE is related to the question of when to accept a set of values as the solution to the least-square problem. In theory a solution exists when it satisfies exactly the system of equations (in general the equations are nonlinear).

In practice the solution is found by iteration with a linear system of equations which approximate the exact equations. A desirable solution is one which is both sufficiently accurate for our requirements and consistent with the accuracy of the data. In BISE we require that a solution satisfy the exact (nonlinear) equations to the degree of accuracy desired and be consistent with the errors in the measured variables.

When necessary, BISE directs the iteration by introducing bounds on the momentum of the particles and by step cutting. This may be necessary when the variables are not in the neighborhood of the solution.

The analysis of Σ decays is made difficult by a number of things--the presence of unmeasured uncharged particles, connected vertices, and most of all by the difficulty in measuring the short Σ connecting tracks. A number of practical decisions were made for dealing with this last condition. These include treating as unmeasured the momentum of the Σ. The initial guesses for the momentum of this track are the result of the zero-c solution (equal number of unknowns and constraint equations) to the constraint equations at the second vertex. In the zero-c fit, the expression for the momentum is a quadratic equation whose radicand frequently is negative. Physically this occurs because the momentum is in a region where only one solution to the quadratic exists, but because of errors in the measurements the radicand becomes negative. In this case setting the radicand to zero leads to a satisfactory initial guess for the momentum.

BISE has been used to reconstruct some 7000 events. Chi-square distributions, cross sections, and scattering-angle distributions compare well with known results.

Although applied to Σ decays the techniques used in BISE are not limited, and could be used as the basis for a more general program.

DEFINITIONS

x	The variables that are measured
x^m	The measured values of x
y	The variables that are not measured
ϵ	The Lagrangian multipliers
$F(x, y) = F$	The constraint equations
$x^k, y^k, \epsilon^k, F^k$	The values of x, y, ϵ, F at iteration k
U	The error matrix associated with x^m
F_x	$\frac{\partial F}{\partial X}(x, y)$
F^T	The transpose of F
$P_i, \alpha_i, \beta_i, E_i$	Momentum, azimuthal angle, dip angle, energy of particle whose track is number i.
λ_i, μ_i, ν_i	Direction cosines of track i
L	The length of the track which connects the production to the decay origin
M	The sum of χ^2 and the product of the constraints and their Lagrangian multipliers.
$\approx$	Symbol for approximately
$\equiv$	Symbol for a definition
$P^{\perp}$	Component of P perpendicular to the magnetic field
$P^{\parallel}$	Component of P parallel to the magnetic field

FORMULATION

We wish to find x, y which minimize $(x-x^m)^T U^{-1}(x-x^m)$ and which satisfy the constraints $F(x, y) = 0$. We use the technique of introducing Lagrange multipliers ϵ, and set

$$M = (x-x^m)^T U^{-1}(x-x^m) + 2 F(x, y)^T \epsilon.$$

The equations that must be satisfied are then

$$0 = \frac{\partial M}{\partial x} = U^{-1}(x-x^m) + F_x^T \epsilon , \tag{1}$$

$$0 = \frac{\partial M}{\partial y} = F_y^T \epsilon , \tag{2}$$

$$0 = \frac{\partial M}{\partial \epsilon} = F . \tag{3}$$

In general F is not linear in x and y, and Eqs. 1, 2, and 3 are not easily solved. To avoid nonlinearity F is approximated to first-order terms by expansion in a Taylor series about x^i, y^i:

$$F(x, y) = F(x^i, y^i) + (x-x^i) F_x(x^i, y^i) + (y-y^i) F_y(x^i, y^i),$$

$$M \approx (x-x^m)^T U^{-1}(x-x^m) + 2[F(x^i, y^i) + (x-x^i) F_x(x^i, y^i) + (y-y^i) F_y(x^i, y^i)]^T \epsilon.$$

We now obtain the linear system

$$0 = \frac{\partial M}{\partial x} = U^{-1}(x-x^m) + F_x^T(x^i, y^i) \epsilon, \tag{4}$$

$$0 = \frac{\partial M}{\partial y} = F_y^T (x^i, y^i) \epsilon, \tag{5}$$

$$0 = \frac{\partial M}{\partial \epsilon} = F(x^i, y^i) + (x-x^i) F_x(x^i, y^i) + (y-y^i) F_y(x^i, y^i). \tag{6}$$

Multiply Eq. 4 by U, and then, rearranging 4, 5, and 6, we have

$$x + UF_x^T(x^i, y^i)\,\epsilon = x^m,$$

$$F_y^T\,\epsilon = 0,$$

$$F_x x + F_y y = -F(x^i, y^i) + F_x(x^i, y^i)x^i + F_y(x^i, y^i)y^i,$$

or

$$\begin{pmatrix} I & 0 & UF_x^{i\,T} \\ 0 & 0 & F_y^{i\,T} \\ F_x^i & F_y^i & 0 \end{pmatrix} \begin{bmatrix} x \\ y \\ \epsilon \end{bmatrix} = \begin{bmatrix} x^m \\ 0 \\ -F + F_x^i x^i + F_y^i y^i \end{bmatrix} \qquad (7)$$

If x, y, ϵ satisfy the convergence criterion we have our solution. If not, x, y, ϵ become candidates for x^{i+1}, y^{i+1}, ϵ^{i+1}, and the iteration continues.

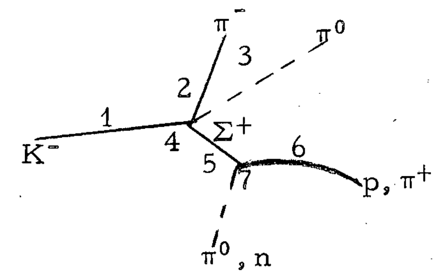
Solution of the System of Equations, 7

There are many packaged subroutines which solve systems of linear equations such as Eq. 7. The problems of maintaining accuracy and speed of solution have been studied by numerical analysts. Roughly speaking, Gaussian elimination and partial pivoting provide a means for attaining speed and accuracy.

EVENT TYPES CONSIDERED

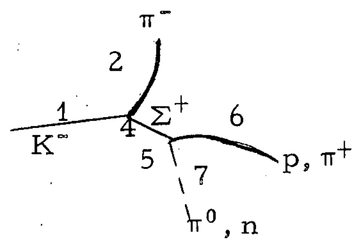
We consider events of the type

$$K^- p \longrightarrow \Sigma^+ \pi^- \pi^0$$
$$\Sigma^+ \longrightarrow \begin{cases} p & \pi^0 \\ \pi^+ & n \end{cases}$$



and $K^- p \longrightarrow \Sigma^+ \pi^-$

$$\Sigma^+ \longrightarrow \begin{cases} p & \pi^0 \\ \pi^+ & n \end{cases} .$$



For convenience we number the tracks as indicated in the diagrams. The connecting track is numbered 4 at the production origin and 5 at the decay origin. The following discussion relates to the case with the π^0 at the first origin, the other being a simple modification within the program.

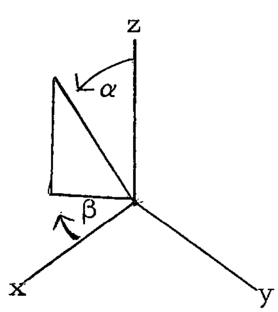
The coordinate system used is consistent with the variables that have been measured, that is, a right-handed system with the azimuthal angle β measured in the positive sense clockwise and the dip angle α measured in the positive sense from the z axis.

The following relationships exist:

$\lambda_i = \sin\alpha_i \cos\beta_i$, $\quad \xi_i = \cos\alpha_i \cos\beta_i$,

$\mu_i = -\sin\alpha_i \sin\beta_i$, $\quad \eta_i = -\cos\alpha_i \sin\beta_i$,

$\nu_i = \cos\alpha_i$, $\quad \delta_i = -\sin\alpha_i$,



$$E_i = \sqrt{P_i^2 + M_i^2} .$$

We set $\alpha_4 = \alpha_5$.

We set the error in the measured value of β_5 equal to that of β_4.

Measured Variables

The measured variables x are

$P_1, \alpha_1, \beta_1, P_2, \alpha_2, \beta_2, \alpha_4, \beta_4, \beta_5, P_6, \alpha_6, \beta_6.$

Unmeasured Variables

The unmeasured variables y are

$P_3, \alpha_3, \beta_3\ P_4, P_5, P_7, \alpha_7, \beta_7.$

Constraints

F_1 through F_4 and F_7 through F_{10} are the energy-momentum constraints at the production and decay origin respectively.

F_5 and F_6 are constraints upon the connecting track. F_5 constrains the connecting track momentum by its length L,

$$F_5: \qquad P_4 - \left(\frac{k_5 P_5^{a_5} + L}{k_4} \right)^{1/a_4} = 0.$$

Where L is a measured quantity and taken to be exact; k_4, k_5, a_4, a_5 are constants which arise in the approximation to the range-momentum tables (see Appendix II). A more sophisticated approach might treat L as a measured variable with an error.

F_6 constrains the azimuthal angle, β, of the connecting track over L:

$$F_6: \qquad \beta_4 - \beta_5 = \left[\left(\frac{K}{P^{\perp}} \right)_4 + 4 \left(\frac{\overline{K}}{P^{\perp}} \right) + \left(\frac{K}{P^{\perp}} \right)_5 \right] \frac{L}{6} .$$

Here $P^{\perp}$ is the component of P along the perpendicular to the magnetic field; $K = 3 \times 10^{-7}$ H, where H is the magnetic field in gauss.

$\left(\frac{\overline{K}}{P^{\perp}} \right)$ designates evaluation at the midpoint of the track. (See Appendix III.)

The dip angles, α_4 and α_5, are set equal throughout the program. This relation is not treated as a formal constraint.

THE ITERATION

From Eq. 7 we see that it is necessary to evaluate F, F_x, F_y at each stage of the iteration. These expressions are computed as follows.

F The equations of constraint

F_1: $$\sum_{i=2}^{4} P_i \lambda_i - P_1 \lambda_1 = 0$$

F_2: $$\sum_{i=2}^{4} P_i \mu_i - P_1 \mu_1 = 0$$

F_3: $$\sum_{i=2}^{4} P_i \nu_i - P_1 \nu_1 = 0$$

F_4: $$\sum_{i=2}^{4} E_i - E_1 - E_{proton} = 0$$

F_5: $$P_4 - \left(\frac{k_5 P_5^{a_5} + L}{k_4} \right)^{1/a_4} = 0$$

F_6: $$\beta_4 - \beta_5 - \left[\left(\frac{K}{P^{\perp}} \right)_4 + 4 \left(\frac{K}{P^{\perp}} \right) + \left(\frac{K}{P^{\perp}} \right)_5 \right] \frac{L}{6}$$

F_7: $$\sum_{i=6}^{7} P_i \lambda_i - P_5 \lambda_5 = 0$$

F_8: $$\sum_{i=6}^{7} P_i \mu_i - P_5 \mu_5 = 0$$

F_9: $$\sum_{i=6}^{7} P_i \nu_i - P_5 \nu_5 = 0$$

F_{10}: $$E_6 + E_7 - E_5 = 0$$

F_x^T is the matrix

$$
\begin{pmatrix}
-\lambda_1 & -\mu_1 & -\nu_1 & -P_1/E_1 & & & & & & \\
-P_1\xi_1 & -P_1\eta_1 & -P_1\delta_1 & 0 & & & & & & \\
-P_1\mu_1 & P_1\lambda_1 & 0 & 0 & & & & & & \\
\lambda_2 & \mu_2 & \nu_2 & P_2/E_2 & & & & \text{\Large 0} & & \\
P_2\xi_2 & P_2\eta_2 & P_2\delta_2 & 0 & & & & & & \\
P_2\mu_2 & -P_2\lambda_2 & 0 & 0 & & & & & & \\
P_4\xi_4 & P_4\eta_4 & P_4\delta_4 & 0 & & & & & & \\
P_4\mu_4 & P_4\lambda_4 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & & & & & -1 & -P_5\mu_5 & P_5\lambda_5 & 0 & 0 \\
 & & & & & 0 & \lambda_6 & \mu_6 & \nu_6 & P_6/E_6 \\
 & & \text{\Large 0} & & & 0 & P_6\xi_6 & P_6\eta_6 & P_6\delta_6 & 0 \\
 & & & & & 0 & P_6\mu_6 & -P_6\lambda_6 & 0 & 0
\end{pmatrix}
$$

F_y^T is the matrix

$$
\begin{pmatrix}
\lambda_3 & \mu_3 & \nu_3 & P_3/E_3 & & & & & & \\
P_3\xi_3 & P_3\eta_3 & P_3\delta_3 & 0 & & & & & & \\
P_3\mu_3 & -P_3\lambda_3 & 0 & 0 & & & & & & \\
\lambda_4 & \mu_4 & \nu_4 & P_4/E_4 & 1 & * & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & ** & * & -\lambda_5 & -\mu_5 & -\nu_5 & P_5/E_5 \\
 & & & & & & \lambda_7 & \mu_7 & \nu_7 & P_7/E_7 \\
 & & & & & & P_7\xi_7 & P_7\eta_7 & P_7\delta_7 & 0 \\
 & & & & & & P_7\mu_7 & -P_7\lambda_7 & 0 & 0
\end{pmatrix}
$$

Initial Values for y

Initial values for P_5, P_7, α_7, β_7 are obtained from the zero-c solution of the energy-momentum equations at the decay origin (F_7, F_8, F_9, F_{10}). The initial P_4 is obtained from the length of the Σ and P_5 by using the range-energy relation (constraint F_5). The initial values of P_3, α_3, β_3 are obtained from F_1, F_2, F_3 and the value of P_4 obtained above.

Let

$$M = \frac{\left(M_5^2 + M_6^2 - M_7^2\right)}{2} \quad \text{and} \quad \cos\theta_6 = \lambda_6\lambda_7 + \mu_5\mu_6 + \nu_5\nu_6,$$

then (see Appendix I for derivations)

$$P_5 = \frac{M\,P_6\cos\theta_6 + \left(E_6^2\,(M^2 - M_5^2[E_6^2 - P_6^2\cos^2\theta_6])\right)^{1/2}}{E_6^2 - P_6^2\cos^2\theta_6} .$$

*See Appendix III

**See Appendix II

From F_{10} we have

$$P_7 = \left((E_5 - E_6)^2 - M_7^2 \right)^{1/2} ,$$

from F_9,

$$\cos \alpha_7 = \frac{P_5 \nu_5 - P_6 \nu_6}{P_7} ,$$

$$\alpha_7 = \tan^{-1} \left[\left(\frac{P_7^{\,2}}{(P_5 \nu_5 - P_6 \nu_6)^2} - 1 \right)^{1/2} \right]$$ (if $\cos \alpha_7 < 0$ then $\alpha_7 = \alpha_7 + \pi/2$),

and from F_7, F_8, $\beta_7 = \tan^{-1} \left(\frac{P_5 \mu_5 - P_6 \mu_6}{P_5 \lambda_5 - P_6 \lambda_6} \right)$

(determine correct quadrant for β_7 from $P_7 \lambda_7 = P_7 \sin\alpha_7 \cos\beta_7 = P_5 \lambda_5 - P_6 \lambda_6$).

P_4 is determined from P_5 and 6 by the Range-Momentum Relation.

$$P_4 = P(R(P_5 + L))$$

From F_1, F_2, F_3,

$$P_3 \lambda_3 = P_1 \lambda_1 - P_2 \lambda_2 - P_4 \lambda_4,$$

$$P_3 \mu_3 = P_1 \mu_1 - P_2 \mu_2 - P_4 \mu_4,$$

$$P_3 \nu_3 = P_1 \nu_1 - P_2 \nu_2 - P_4 \nu_4,$$

we have $P_3 = \left((P_3 \lambda_3)^2 + (P_3 \mu_3)^2 + (P_3 \nu_3)^2 \right)^{1/2}.$

From $\cos \alpha_3 = P_3 \nu_3 / P_3$

we have $\alpha_3 = \tan^{-1} \left[\left(\frac{P_3^{\,2}}{(P_1 \nu_1 - P_2 \nu_2 - P_4 \nu_4)^2} - 1 \right)^{1/2} \right]$

(if $\cos \alpha_3 < 0$ then $\alpha_3 < 0$ then $\alpha_3 = \alpha_3 + \pi/2$),

and $\beta_3 = \tan^{-1} \left[\frac{P_1 \mu_1 - P_2 \mu_2 - P_4 \mu_4}{P_1 \lambda_1 - P_2 \lambda_2 - P_4 \lambda_4} \right]$

(determine correct quadrant for β_3 from $P_3 \lambda_3 = P_3 \sin\alpha_3 \cos\beta_3 = P_1 \lambda_1 - P_2 \lambda_2 - P_4 \lambda_4$).

Frequently, because of inaccuracies in the measured values the discriminant in the expression for P_5 becomes less than zero. In this case we set the discriminant to zero, solve for $\cos\theta_6$, and use it to determine P_5:

$$P_5 = \frac{M_5}{M} (M_5^2 E_6^2 - M^2)^{1/2} .$$

CALCULATION OF ERROR MATRICES

When the iteration converges and a solution x^s, y^s is obtained, we wish to know the error matrices associated with the solution.

We define

$U(x^s)$ as the 12×12 error matrix of the measured variables,

$U(y^s)$ as the 8×8 error matrix of the unmeasured variables,

$U(x^s, y^s)$ as the 12×8 error matrix of the measured variables correlated with the unmeasured variables.

The error matrices are then obtained from the formulas

$$U(x^s) = \left(\frac{\partial x^s}{\partial x^m}\right) U \left(\frac{\partial x^s}{\partial x^m}\right)^T ,$$

$$U(y^s) = \left(\frac{\partial y^s}{\partial x^m}\right) U \left(\frac{\partial y^s}{\partial x^m}\right)^T ,$$

$$U(x^s, y^s) = \left(\frac{\partial x^s}{\partial x^m}\right) U \left(\frac{\partial y^s}{\partial x^m}\right)^T .$$

The 12×12 matrix $\left(\frac{\partial x^s}{\partial x^m}\right)$ and the 8×12 matrix $\left(\frac{\partial y^s}{\partial x^m}\right)$ are obtained from Eqs. 7 by differentiation and solving the 12 systems of equations so obtained:

$$\begin{pmatrix} I & 0 & UF_x^{s\,T} \\ 0 & 0 & F_y^{s\,T} \\ F_x^s & F_y^s & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial x^s}{\partial x^m} \\ \frac{\partial y^s}{\partial x^m} \\ \frac{\partial \epsilon^s}{\partial x^m} \end{pmatrix} = \begin{pmatrix} I \\ 0 \\ 0 \end{pmatrix} ,$$

where I is the 12 X 12 identity matrix. In practice this last calculation is no more difficult then solving a single system of equations, as the solutions of all 12 systems of equations proceed simultaneously.

CONVERGENCE, DIVERGENCE

Before the first iteration the constraint equations are nearly all satisfied. This is because of the manner in which the initial guesses are obtained. Ordinarily the initial guesses are not in the neighborhood of the solution, and the result of the first iteration is to change those variables which will reduce the constraint equations, which are least satisfied. This often results in considerably increasing the sum of the constraints. We allow this to happen on the first iteration, but require all subsequent iterations to reduce the sum of the constraints when it is greater than 0.0005 BeV/c. When an iteration fails to reduce that sum we cut in half the increment by which the variables had been changed last. This last step may be repeated several times.

During the iteration, problems sometimes occur for angles near zero degrees where there is a jump discontinuity; care is taken to avoid introducing errors because of the discontinuity.

Because the momentum of the sigma is unknown we allow it for this experiment to range freely between 1.7 BeV/c and the momentum it must have to produce a track of the measured length, resulting in a sigma of zero momentum. In the first three iterations we allow any of the particles to assume a negative momentum. After three iterations we assume that a negative momentum implies that the event interpretation is wrong and this interpretation is terminated.

Ordinarily we allow ten iterations for convergence to be attained. However, when a solution lies in a steep valley it may take more iterations. To allow for this possibility, at the end of ten iterations we test the sum of the constraints. If it is sufficiently small, less than 0.005 BeV/c, we allow the program to continue for up to five more iterations.

We say the iteration has converged when

$$\sum_{i=1}^{10} |F_i| \leq 0.0005 \text{ BeV/c} \qquad (7)$$

and $$|x^k - x^{k+1}| \leq \epsilon' = 0.0001,$$

$$|y^k - y^{k+1}| \leq \epsilon' = 0.0001 \qquad (8)$$

for all x and y. Equation 7 assures us that each of the Eqs. 3 are satisfied to less than 1/2 MeV/c. As the measured values of x, (x^m) are less well known than the converged values of x in Eq. 8, continuing the iteration with the intent of further reducing χ^2 will not increase the accuracy of the result.

Although Eq. 6 satisfies Eq. 3 to second-order terms, Eqs. 4 and 5 satisfy Eqs. 1 and 2 only to first-order terms. Equations 8 are meant to keep these first-order terms small. From Eqs. 5 we can show that the constraint ϵ_4 dominates ϵ_1, ϵ_2, ϵ_3. Likewise ϵ_{10} dominates ϵ_7, ϵ_8, ϵ_9. To make certain that the first-order terms in Eqs. 1 and 2 are as small as desired one needs only to examine the result of multiplying ϵ_4, ϵ_5, ϵ_6, ϵ_{10} by the number ϵ' defined above.

Appendix I

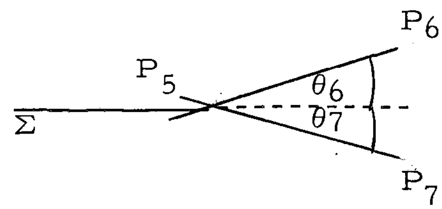
SOLUTION OF THE ZERO-CONSTRAINT PROBLEM

For the zero-c solution we confine our attention to a plane region. The energy momentum relations within the plane are

$$E_5 = E_6 + E_7, \qquad \text{(I.1)}$$

$$P_6 \cos\theta_6 + P_7 \cos\theta_7 = P_5, \qquad \text{(I.2)}$$

$$P_6 \sin\theta_6 + P_7 \sin\theta_7 = 0. \qquad \text{(I.3)}$$



From Eqs. I.2 and I.3 we have

$$(P_7 \cos\theta_7)^2 = (P_5 - P_6 \cos\theta_6)^2,$$

$$(P_7 \sin\theta_7)^2 = (-P_6 \sin\theta_6)^2,$$

$$P_7^2 = P_5^2 + P_6^2 - 2\, P_5 P_6 \cos\theta_6.$$

From Eq. I.1 and the above we can write

$$P_7^2 + M_7^2 = P_5^2 + M_5^2 - 2\, E_6(P_5^2 + M_5^2)^{1/2} + P_6^2 + M_6^2,$$

$$P_5^2 + P_6^2 - 2 P_5 P_6 \cos\theta_6 + M_7^2 = P_5^2 + M_5^2 - 2\, E_6 (P_5^2 + M_5^2)^{1/2} + P_6^2 + M_6^2,$$

$$E_6 (P_5^2 + M_5^2)^{1/2} = \left(\frac{M_5^2 + M_6^2 - M_7^2}{2}\right) + P_5 P_6 \cos\theta_6.$$

Let $M = \dfrac{M_5^2 + M_6^2 - M_7^2}{2}$, $\cos\theta_6 = \lambda_5\lambda_6 + \mu_5\mu_6 + \nu_5\nu_6.$

Then $E_6^2 (P_5^2 + M_5^2) = M^2 + 2 M P_5 P_6 \cos\theta_6 + P_5^2 P_6^2 \cos^2\theta_6,$

$$P_5^2 (E_6^2 - P_6^2 \cos^2\theta_6) - P_5 2 M P_6 \cos\theta_6 + E_6^2 M_5^2 - M^2 = 0,$$

$$P_5 = \frac{2 M P_6 \cos\theta_6 \pm \left[4M^2P_6^2 \cos\theta_6 - 4(E_6^2 - P_6^2 \cos^2\theta_6)(E_6^2 M_5^2 - M^2)\right]^{1/2}}{2(E_6^2 - P_6^2 \cos^2\theta_6)}$$

$$= \frac{M P_6 \cos\theta_6 \pm \{E_6^2[M^2 - M_5^2 (E_6^2 - P_6^2 \cos^2\theta_6)]\}^{1/2}}{E_6^2 - P_6^2 \cos^2\theta_6}\;. \qquad \text{(I.4)}$$

If in Eq. I.4 we have $\{E_6^2[M^2 - M_5^2(E_6^2 - P_6^2\cos^2\theta_6)]\} < 0$, we are in a region where there is only one possible real solution for the momentum. The negative radicand is caused by errors in the measurements. As an expedient we proceed as follows:

$$E_6^2[M^2 - M_5^2(E_6^2 - P_6^2\cos^2\theta_6)] = 0,$$

$$\cos^2\theta_6 = \frac{M_5^2 E_6^2 - M^2}{M_5^2 P_6^2} .$$

Using this value for $\cos^2\theta_6$ together with Eq. I.4 we obtain

$$P_5 = \frac{M_5}{M}(M_5^2 E_6^2 - M^2)^{1/2} . \tag{I.4'}$$

When $\qquad M^2 - M_5^2(E_6^2 - P_6^2\cos^2\theta_6) < 0$

we have $\qquad M^2 < M_5^2(E_6^2 - P_6^2\cos^2\theta_6) < M_5^2 E_6^2 ,$

so that P_5 in Eq. I.4' is real.

Appendix II

THE RANGE-MOMENTUM RELATION

There are tables of range vs momentum for Σ (and other particles) in hydrogen and helium. When plotted on ln - ln paper they may be approximated over tabular intervals by line segments. Having chosen line segments which cover the range of momentum of interest, we use linear interpolation to obtain range-momentum values.

Within a line segment we have ln R = a ln P + ln k, where a is the slope of the segment and ln k its R intercept when extended, so that $R = k\,P^a$.

We distinguish the production origin by the subscript 4 and the decay origin by the subscript 5:

$$P_4 = \left(\frac{R_4}{k_4}\right)^{1/a_4}, \text{ but } R_4 = k_5 P_5^{\,a_5} + L,$$

so that

$$P_4 = \left(\frac{k_5 P_5^{\,a_5} + L}{k_4}\right)^{1/a_4} .$$

F_5 may now be written

$$P_4 - \left(\frac{k_5 P_5^{\,a_5} + L}{k_4}\right)^{1/a_4} = 0,$$

from which

$$\frac{\partial F_5}{\partial P_4} = 1,$$

$$\frac{\partial F_5}{\partial P_5} = -\frac{1}{a_4}\left(\frac{k_5 P_5^{\,a_5} + L}{k_4}\right)^{\frac{1}{a_4}-1} \frac{k_5}{k_4}\, a_5 P_5^{\,a_5 - 1}$$

$$= -\frac{k_5 a_5}{k_4 a_4}\,\frac{P_5}{P_5}\,\frac{a_5\!\left(\frac{k_5 P_5^{\,a_5} + L}{k_4}\right)^{1/a_4}}{\left(\frac{k_5 P_5^{\,a_5} + L}{k_4}\right)} . \qquad \text{(II.1)}$$

We maintain the distinction between k_4 and k_5, a_4 and a_5, since they will be different when the line segment approximations to the range-momentum curve are overlapped by the length and momentum of the Σ. In practice Eq. II.1 is evaluated from

$$P_5^{a_5} = e^{a_5 \ln P_5}, \left(\frac{k_5 P_5^{a_5} + L}{k_4}\right)^{1/a_4} = e^{\frac{1}{a_4} \ln\left(\frac{k_5 P_5^{a_5} + L}{k_4}\right)} .$$

The coefficients a and k are obtained from line segment approximations to the ln - ln plot of the range-momentum table.

A sample calculation of a and k follows. From the range-momentum table we have

for P_{BeV} a = 0.1360260, k = 0.142689,

for R_{cm} a = 0.451137, k = 0.536981;

$$a = \frac{\ln(0.536981) - \ln(0.451137)}{\ln(0.142689) - \ln(0.1360260)} = \frac{\ln\left(\frac{0.536981}{0.451137}\right)}{\ln\left(\frac{0.142689}{0.136026}\right)} = 3.64254,$$

$$k = \frac{0.451137}{(0.136026)^{3.64254}} = \frac{0.451137}{e^{3.64254 \times \ln(0.136026)}} = 645.8492.$$

Over the region of interest to this program the table on the facing page give the values of a and k for Σ in hydrogen.

Table of coefficients derived from straight-line approximations to the ln-ln plot of the range momentum relation for Σ in hydrogen, $\ln R = a \ln P + \ln k$.

a	k	P_1	P_2	R_1	R_2
3.5248191	482.2912636	0.0576100	0.0604250	0.0206200	0.0243960
3.5364522	498.2963028	0.0604250	0.0633760	0.0243960	0.0288770
3.5497488	516.9136047	0.0633760	0.0697190	0.0288770	0.0405140
3.5630673	535.5781403	0.0697190	0.0731250	0.0405140	0.0480190
3.5708517	546.5946655	0.0731250	0.0766980	0.0480190	0.0569370
3.5784602	557.3787918	0.0766980	0.0804460	0.0569370	0.0675370
3.5863379	568.5551224	0.0804460	0.0843770	0.0675370	0.0801400
3.5928719	577.8147049	0.0843770	0.0885010	0.0801400	0.0951280
3.5997364	587.5127563	0.0885010	0.0928270	0.0951280	0.1129580
3.6060463	596.3912201	0.0928270	0.0973650	0.1129580	0.1341730
3.6115735	604.1189728	0.0973650	0.1021260	0.1341730	0.1594210
3.6170271	611.6828156	0.1021260	0.1071210	0.1594210	0.1894770
3.6223669	619.0227127	0.1071210	0.1123610	0.1894770	0.2252620
3.6274482	625.9371109	0.1123610	0.1178580	0.2252620	0.2678760
3.6321437	632.2533340	0.1178580	0.1236250	0.2678760	0.3186330
3.6354420	636.6278152	0.1236250	0.1296770	0.3186330	0.3790970
3.6397792	642.2931061	0.1296770	0.1360260	0.3790970	0.4511370
3.6425469	645.8492966	0.1360260	0.1426890	0.4511370	0.5369810
3.6458180	649.9759140	0.1426890	0.1496800	0.5369810	0.6392850
3.6484967	653.2910538	0.1496800	0.1570160	0.6392850	0.7612210
3.6499877	655.0969315	0.1570160	0.1647160	0.7612210	0.9065690
3.6525124	658.0866776	0.1647160	0.1812780	0.9065690	1.2864100
3.6537282	659.4544449	0.1812780	0.2093370	1.2864100	2.1764100
3.6526224	658.3151474	0.2093370	0.2196380	2.1764100	2.5938200
3.6508673	656.5661240	0.2196380	0.2304550	2.5938200	3.0914700
3.6488654	654.6398315	0.2304550	0.2418140	3.0914700	3.6847600
3.6454240	651.4495163	0.2418140	0.2537460	3.6847600	4.3920100
3.6420336	648.4275055	0.2537460	0.2662790	4.3920100	5.2350300
3.6375838	644.6207047	0.2662790	0.2794460	5.2350300	6.2397400
3.6321095	640.1373596	0.2794460	0.2932820	6.2397400	7.4369600
3.6256474	635.0832825	0.2932820	0.3078240	7.4369600	8.8633100
3.6183358	629.6357040	0.3078240	0.3231100	8.8633100	10.5622000
3.6100820	623.7917862	0.3231100	0.3391820	10.5622000	12.5853000
3.6004103	617.3026581	0.3391820	0.3560840	12.5853000	14.9935000
3.5894596	610.3617935	0.3560840	0.3738650	14.9935000	17.8592999
3.5775609	603.2581787	0.3738650	0.3925740	17.8592999	21.2683001
3.5637577	595.5223389	0.3925740	0.4122670	21.2683001	25.3213999
3.5485949	587.5746689	0.4122670	0.4330020	25.3213999	30.1379001
3.5319611	579.4507294	0.4330020	0.4548420	30.1379001	35.8586001
3.5134336	571.0544586	0.4548420	0.4778550	35.8586001	42.6487002
3.4930125	562.5076218	0.4778550	0.5021150	42.6487002	50.7026000
3.4707869	553.9601974	0.5021150	0.5277000	50.7026000	60.2480998
3.4462171	545.3278656	0.5277000	0.5546980	60.2480998	71.5518999
3.4197519	536.8884659	0.5546980	0.5832000	71.5518999	84.9254999
3.3908122	528.5753784	0.5832000	0.6133090	84.9254999	100.7320004
3.3594520	520.5332870	0.6133090	0.6451350	100.7320004	119.3929996
3.3255547	512.8568649	0.6451350	0.6787980	119.3929996	141.3969994
3.2892241	505.6886406	0.6787980	0.7144290	141.3969994	167.3099995
3.2500769	499.0753555	0.7144290	0.7521740	167.3099995	197.7840004
3.2082130	493.1605644	0.7521740	0.7921890	197.7840004	233.5660000
3.1634959	488.0499344	0.7921890	0.8346490	233.5660000	275.5130005
3.1161063	483.8874435	0.8346490	0.8797430	275.5130005	324.6020012
3.0658518	480.7817802	0.8797430	0.9276820	324.6020012	381.9430008
3.0128419	478.8724213	0.9276820	0.9786960	381.9430008	448.7900009
2.9571103	478.2980461	0.9786960	1.0330420	448.7900009	526.5589981
2.8988322	479.2050438	1.0330420	1.0910000	526.5589981	616.8349991
2.8380527	481.7484818	1.0910000	1.1528830	616.8349991	721.3899994
2.7749839	486.0904427	1.1528830	1.2190350	721.3899994	842.1910019
2.7098022	492.4064980	1.2190350	1.2898390	842.1910019	981.4160004
2.6427800	500.8781738	1.2898390	1.3657170	981.4160004	1141.4600067
2.5743159	511.6811981	1.3657170	1.4471380	1141.4600067	1324.9600067
2.5042764	525.0993729	1.4471380	1.5346220	1324.9600067	1534.7599945
2.4336700	541.2206879	1.5346220	1.6287430	1534.7599945	1774.0000000
2.3622147	560.4183960	1.6287430	1.7301410	1774.0000000	2046.0299988

Appendix III

DERIVATION OF THE CONSTRAINT F_6

We have $$\beta_4 - \beta_5 = \int_0^L \frac{1}{\rho}\, ds,$$

$$P^{\perp} = c\, B\rho \equiv K\rho,$$

where $P^{\perp}$ is the component of P along the perpendicular to the magnetic field B. When P is in BeV and B in gauss, the constant $c = 3\times 10^{-7}$.

Then $$\beta_4 - \beta_5 = \int_0^L \frac{K}{P^{\perp}}\, ds .$$

The integral is evaluated with a three-point Simpson's-rule approximation,

$$\int_0^{2h} f(x)\, dx \approx \frac{h}{3}\left[f(0) + 4\, f(h) + f(2h)\right].$$

Therefore

$$F_6: \quad \beta_4 - \beta_5 - \left[\left(\frac{K}{P^{\perp}}\right)_4 + 4\left(\overline{\frac{K}{P^{\perp}}}\right) + \left(\frac{K}{P^{\perp}}\right)_5\right]\frac{L}{6} = 0,$$

where $\left(\overline{\frac{K}{P^{\perp}}}\right)$ indicates evaluation at the middle of the connecting track. For a sigma decay the integral may be accurately approximated by using the trapezoid rule. This would simplify the programming and the derivations. However, as the Simpson's-rule approximation might be useful in some other event type we continue its use here.

$\overline{K}$ is computed at the midsection of a circular arc, L, joining the production and the decay origin. For the sake of symmetry we compute $\overline{P}$ as

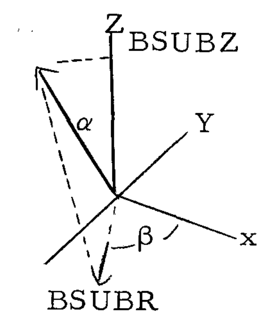
$$\overline{P} = 1/2\left[\left(\frac{k_4\, P_4^{\,a_4} - \frac{L}{2}}{k_-}\right)^{1/a_-} + \left(\frac{k_5\, P_5^{\,a_5} + \frac{L}{2}}{k_+}\right)^{1/a_+}\right].$$

(See Appendix II for the range-momentum relation.)

$\underline{P^{\perp}}$

We wish to find the component of P along the perpendicular, ⊥, to the magnetic field. We assume that the coordinate systems are parallel so that directions are the same in both systems.

We find the direction cosines of the field B in the Z, P plane:



$$\lambda_B = \sin\alpha \cos\beta,$$

$$\mu_B = -\sin\alpha \sin\beta,$$

$$\nu_B = \cos\alpha,$$

where $\tan\alpha = \dfrac{BSUBR}{BSUBZ}$ and $\beta = \beta_P$.

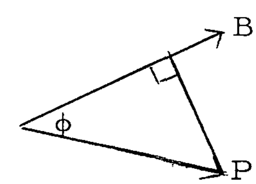
Then

$$\cos\phi = (P \cdot B) = \sin\alpha_P \cos\beta \sin\alpha \cos\beta + \sin\alpha_P \sin\beta \sin\alpha \sin\beta + \cos\alpha_P \cos\alpha = \sin\alpha_P \sin\alpha + \cos\alpha_P \cos\alpha = \cos(\alpha_P - \alpha)$$

so $\phi = \pm(\alpha_P - \alpha)$,

and $\sin\phi = \pm(\sin\alpha_P \cos\alpha - \cos\alpha_P \sin\alpha)$.

We choose $0 \leqslant \phi \leqslant \pi$ so that $\sin\phi \geqslant 0$.



Then we have

$$P^{\perp} = |P \sin\phi| = \left| \frac{P(\sin\alpha_P\, BSUBZ - \cos\alpha_P\, BSUBR)}{B} \right|,$$

or

$$\frac{B}{P^{\perp}} = \frac{B^2}{|P\, BSUBZ \sin\alpha_P - BSUBR \cos\alpha_P|}.$$

Computation of $\frac{\partial F_6}{\partial P_4}$, $\frac{\partial F_6}{\partial P_5}$

We have $$\frac{\partial F_6}{\partial P_4} = \frac{\partial}{\partial P_4}\left(\beta_4 - \beta_5 - \left[\left(\frac{K}{P^\perp}\right)_4 + 4\left(\frac{\overline{K}}{P^\perp}\right) + \left(\frac{K}{P^\perp}\right)_5\right]\frac{L}{6}\right)$$

$$= -\left(\frac{\partial}{\partial P_4}\left(\frac{K}{P^\perp}\right)_4 + 4\frac{\partial}{\partial P_4}\left(\frac{\overline{K}}{P^\perp}\right)\right)\frac{L}{6} .$$

Now, $$\frac{\partial}{\partial P_4}\left(\frac{K}{P^\perp}\right)_4 = -\frac{K_4}{\left(P_4^\perp\right)^2}\frac{\partial P_4^\perp}{\partial P_4} = -\frac{K_4}{\left(P_4^\perp\right)^2}\frac{\partial\left(P_4^2 - P_4^{\parallel 2}\right)^{1/2}}{\partial P_4}$$

$$= -\frac{K_4}{\left(P_4^\perp\right)^2}\frac{P_4}{P_4^\perp} = -\frac{K_4 P_4}{\left(P_4^\perp\right)^3}$$

and $$\frac{\partial}{\partial P_4}\left(\frac{\overline{K}}{P^\perp}\right) = -\frac{\overline{K}\ \overline{P}}{\left(\overline{P^\perp}\right)^3}\frac{\partial \overline{P}}{\partial P_4} .$$

Since

$$\overline{P} \equiv 1/2\left[\left(\frac{k_4 P_4^{a_4} - \frac{L}{2}}{k_-}\right)^{1/a_-} + \left(\frac{k_5 P_5^{a_5} + \frac{L}{2}}{k_+}\right)^{1/a_+}\right] \equiv 1/2\left[P_- + P_+\right],$$

it follows that

$$\frac{\partial \overline{P}}{\partial P_4} = \frac{1}{2a_-}\left(\frac{k_4 P_4^{a_4} - \frac{L}{2}}{k_-}\right)^{\frac{1}{a} - 1}\frac{k_4}{k_-}a_4 P_4^{a-1} = 1/2\frac{k_4 a_4}{k_- a_-}\frac{P_4^{a_4}}{\overline{P}_4}\frac{P_-}{P_-^{a_-}},$$

so that $$\frac{\partial \frac{\overline{K}}{P^{\perp}}}{\partial P_4} = - \frac{\overline{K} \quad \overline{P}}{\left(\overline{P}^{\perp}\right)^3} \times 1/2 \, \frac{k_4 a_4}{k_- a_-} \, \frac{P_4^{a_4}}{P_4} \, \frac{P_-}{P_-^{a_-}} \; .$$

Finally we have

$$\frac{\partial F_6}{\partial P_4} = \left[\frac{K_4 P_4}{\left(P_4^{\perp}\right)^3} + \frac{2 \overline{K} \quad \overline{P}}{\left(\overline{P}^{\perp}\right)^3} \, \frac{k_4 a_4}{k^- a^-} \, \frac{P_4^{a_4}}{P_4} \, \frac{P_-}{P_-^{a_-}} \right] \frac{L}{6} \; .$$

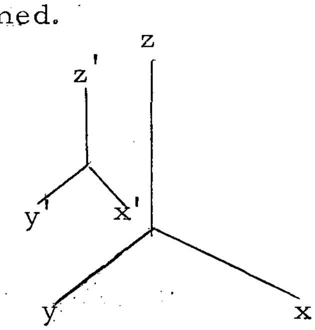
In like manner it follws that

$$\frac{\partial F_6}{\partial P_5} = \left[\frac{K_5 P_5}{\left(P_5^{\perp}\right)^3} + \frac{2 \overline{K} \overline{P}}{\left(\overline{P}^{\perp}\right)^3} \, \frac{k_5 a_5}{k_+ a_+} \, \frac{P_5^{a_5}}{P_5} \, \frac{P_+}{P_+^{a_+}} \right] \frac{L}{6} \; .$$

Magnetic Field Calculation for the 25-Inch Bubble Chamber

We compute the magnetic field by the polynomials given in the Trilling-Goldhaber Group internal report TG-74 written by J. L. Brown. The coordinate system used in this report is at the center of the magnet.

Let the coordinate system in which the measurements are given be primed and the magnetic field coordinate system be unprimed.



Then
$$\begin{aligned} x &= (x' - 50)/2.54 \text{ in.}, \\ y &= (y' - 50)/2.54 \text{ in.}, \\ z &= (z' - 37.3)/2.54 \text{ in.}, \\ r &= (x^2 + Y^2)^{1/2} . \end{aligned}$$

We rewrite the polynomials given on page 4 of TG-74 as follows: let $r^2/z^2 \equiv t$, let $z^2 \equiv \omega$;

$$f_z^0 \equiv FZ0 = 1,$$

$$\frac{1}{\omega} f_z^2 \equiv FZ2 = (-2+t),$$

$$\frac{1}{\omega^2} f_z^4 \equiv FZ4 = (8/3 + t(-8 + t)),$$

$$\frac{1}{\omega^3} f_z^6 \equiv FZ6 = (-3.2 + t(24 + t(-18 + t))),$$

$$\frac{1}{\omega^4} f_z^8 \equiv FZ8 = \left(\frac{128}{35} + t\left(-\frac{256}{5} + t(96 + t(-32 + t))\right)\right),$$

$$\frac{1}{\omega^5} f_z^{10} \equiv FZ10 = \left(\frac{256}{13} + t\left(\frac{640}{7} + t\left(-320 + t\left(\frac{800}{3} + t(-50 + t)\right)\right)\right)\right),$$

$$\frac{1}{\omega^6} f_z^{12} \equiv FZ12 = \left(\frac{1024}{231} + t\left(\frac{1024}{7} + t\left(\frac{5760}{7} + t(-1280 + t(600 + t(-72 + t)))\right)\right)\right),$$

then

$$B_z(r, z) \equiv BSUBZ = a_0 + \omega(a_2\, FZ2 + \omega(a_4\, FZ4 + \omega(a_6\, FZ6 + \omega(a_8\, FZ8$$

$$+ \omega(a_{10}\, FZ10 + \omega a_{12} FZ12))))),$$

$$\frac{1}{rz} f_r^2 = FR2 = 2,$$

$$\frac{1}{rz\omega} f_r^4 = FR4 = -\frac{16}{3} + 4t,$$

$$\frac{1}{rz\omega^2} f_r^6 = FR6 = 9.6 + t(-24 + 6t),$$

$$\frac{1}{rz\omega^3} f_r^8 = FR8 = -\frac{512}{35} + t(7.68 + t(-64 + 8t))$$

$$\frac{1}{rz\omega^4} f_r^{10} = FR10 = 20.317 + t(-182.86 + t(320 + t(-133.33 + 10\,t)))$$

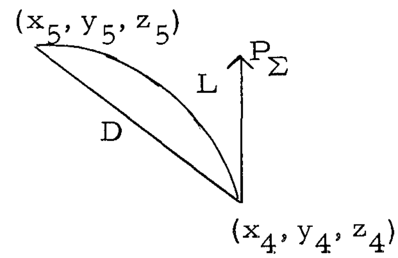
$$\frac{1}{rz\omega^5} f_r^{12} = FR12 = -26.597 + t(365.71 + t(-1097.1 + t(-240 + 12\,t)));$$

then $B_r(r, z) \equiv BSUBR$

$$= rz(a_2\, FR2 + \omega(a_4\, FR4 + \omega(a_6\, FR6 + \omega(a_8\, FR8 + \omega(a_{10}\, FR10 + \omega a_{12}\, FR12))))).$$

The Coordinates of the Midpoint of L

Let λ, μ, ν be the direction cosines of line D joining (x_4, y_4, z_4) and (x_5, y_5, z_5).



By definition,

$$\lambda = \frac{x_5 - x_4}{D}, \quad \mu = \frac{y_5 - y_4}{D}, \quad \nu = \frac{z_5 - z_4}{D} .$$

Where the length of

$$D = \left((x_5-x_4)^2 + (y_5-y_4)^2 + (z_5-z_4)^2\right)^{1/2} .$$

The equation of the line tangent to L at (x_4, y_4, z_4) is found from the equations

$$\frac{x-x_4}{\lambda_4} = \frac{y-y_4}{\mu_4} = \frac{z-z_4}{\nu_4} .$$

Then $x = \frac{\lambda_4}{\nu_4} z + x_4 - z_4 \frac{\lambda_4}{\nu_4}$, (III.1)

and $y = \frac{\mu_4}{\nu_4} z + y_4 - z_4 \frac{\mu_4}{\nu_4}$.

The equation of the line joining (x_4, y_4, z_4) and (x_5, y_5, z_5) is given by

$$x = \frac{\lambda}{\nu} z + x_4 - z_4 \frac{\lambda}{\nu} , \tag{III.2}$$

$$y = \frac{\mu}{\nu} z + y_4 - z_4 \frac{\mu}{\nu} .$$

The equation of the plane containing line III.1 and III.2 is given by

$$\left(\frac{\mu_4}{\nu_4} - \frac{\mu}{\nu}\right)\left(x - \frac{\lambda_4}{\nu_4} z - x_4 + z_4 \frac{\lambda_4}{\nu_4}\right) = \left(\frac{\lambda_4}{\nu_4} - \frac{\lambda}{\nu}\right)\left(y - \frac{\mu_4}{\nu_4} z - y_4 + z_4 \frac{\mu_4}{\nu_4}\right). \tag{III.3}$$

Collecting terms, we define A, B, C, E

$$\underbrace{\left(\frac{\mu_4}{\nu_4} - \frac{\mu}{\nu}\right)}_{A} x - \underbrace{\frac{\lambda_4}{\nu_4}\left(\frac{\mu_4}{\nu_4} - \frac{\mu}{\nu}\right)}_{C_1} z + \underbrace{\frac{\mu_4}{\nu_4}\left(\frac{\lambda_4}{\nu_4} - \frac{\lambda}{\nu}\right)}_{C_2} z - \underbrace{\left(\frac{\lambda_4}{\nu_4} - \frac{\lambda}{\nu}\right)}_{B} y$$

$$= \underbrace{\left(\frac{\mu_4}{\nu_4} - \frac{\mu}{\nu}\right)\left(x_4 - z_4 \frac{\lambda_4}{\nu_4}\right) + \left(\frac{\lambda_4}{\nu_4} - \frac{\lambda}{\nu}\right)\left(-y_4 + z_4 \frac{\mu_4}{\nu_4}\right)}_{E} .$$

Simplifying the expressions, we have

$$C_1 + C_2 = C = -\frac{\lambda_4}{\nu_4}\,\frac{\mu_4}{\nu_4} + \frac{\lambda_4}{\nu_4}\,\frac{\mu}{\nu} + \frac{\mu_4\,\lambda_4}{\nu_4^2} - \frac{\mu_4}{\nu_4}\,\frac{\lambda}{\nu} = \left(\frac{\lambda_4\,\mu - \mu_4\,\lambda}{\nu\,\nu_4}\right)$$

$$\text{and } E = \left(\frac{\mu_4}{\nu_4} - \frac{\mu}{\nu}\right) x_4 - \left(\frac{\lambda_4}{\nu_4} - \frac{\lambda}{\nu}\right) y_4 + z_4 \left[-\frac{\mu_4}{\nu_4}\,\frac{\lambda_4}{\nu_4} + \frac{\mu}{\nu}\,\frac{\lambda_4}{\nu_4} + \frac{\lambda_4\mu_4}{\nu_4\,\nu_4} - \frac{\lambda}{\nu}\,\frac{\mu_4}{\nu_4}\right]$$

$$= \left(\frac{\mu_4}{\nu_4} - \frac{\mu}{\nu}\right) x_4 - \left(\frac{\lambda_4}{\nu_4}\ \frac{\lambda}{\nu}\right) y_4 + \left(\frac{-\lambda\,\mu_4 + \mu\,\lambda_4}{\nu\,\nu_4}\right) z_4$$

$$= \left(\frac{\mu_4}{\nu_4} - \frac{\mu}{\nu}\right) x_4 - \left(\frac{\lambda_4}{\nu_4} - \frac{\lambda}{\nu}\right) y_4 + \left(\frac{\lambda_4\,\mu - \lambda\,\mu_4}{\nu\,\nu_4}\right) z_4 \ .$$

From the above we may write Eq. III.3 as

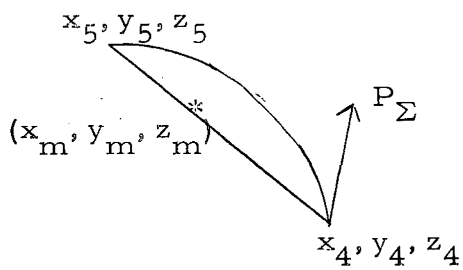
$$\left(\frac{\mu_4}{\nu_4} - \frac{\mu}{\nu}\right)\left(x - x_4\right) - \left(\frac{\lambda_4}{\nu_4} - \frac{\lambda}{\nu}\right)\left(y - y_4\right) + \left(\frac{\lambda_4\,\mu - \mu_4\,\lambda}{\nu\,\nu_4}\right)\left(z - z_4\right) = 0$$

or

$$\overbrace{(\nu\mu_4 - \mu\nu_4)}^{A}(x - x_4) + \overbrace{(\nu_4\lambda - \nu\lambda_4)}^{B}(y - y_4) + \overbrace{(\lambda_4\mu - \mu_4\lambda)}^{C}(z - z_4) = 0,$$

where A, B, C have been redefined.

The coordinates of the midpoint of the chord joining (x_4, y_4, z_4) and (x_5, y_5, z_5) are



$$x_m = x_4 + \frac{D}{2}\lambda = x_4 + \frac{x_5 - x_4}{2} = \frac{x_4 + x_5}{2},$$

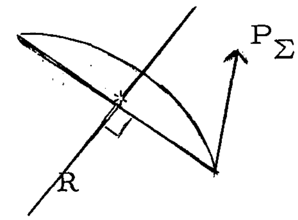
$$y_m = y_4 + \frac{D}{2}\mu = \frac{y_4 + y_5}{2}$$

$$z_m = z_4 + \frac{D}{2}\nu = \frac{z_4 + z_5}{2} \ .$$

We wish to find the equation of the line R in plane III.3 which is perpendicular to the line joining (x_4, y_4, z_4) and (x_5, y_5, z_5) at the point of (x_m, y_m, z_m).

Let the direction cosines of line R be ℓ, m, n,

then $\ell\lambda + m\mu + n\nu = 0$.



R passes through the point (x_m, y_m, z_m), so that

$$\frac{x - x_m}{\ell} = \frac{y - y_m}{m} = \frac{z - z_m}{n} .$$

R is the plane III.3, so

$$Ax + By + Cz = E, \text{ where } A = (\nu\mu_4 - \mu\nu_4), \text{ etc.}$$

From the above definition,

$$\ell = \frac{m(x - x_m)}{(y - y_m)} , \quad n = \frac{m(z - z_m)}{(y - y_m)} ,$$

and from the orthogonal relationship,

$$\lambda m \frac{(x - x_m)}{(y - y_m)} + m \mu + \nu m \frac{(z - z_m)}{(y - y_m)} = 0,$$

giving $$\lambda(x - z_m) + \mu(y - y_m) + \nu(z - z_m) = 0. \quad \text{III.4}$$

Also, x_m, y_m, z_m is in the plane III.3.

$$Ax_m + By_m + Cz_m = E,$$

so that $$A(x - x_m) + B(y - y_m) + C(z - z_m) = 0. \quad \text{III.5}$$

If we eliminate $z - z_m$ from Eqs. III.4 and III.5 we have

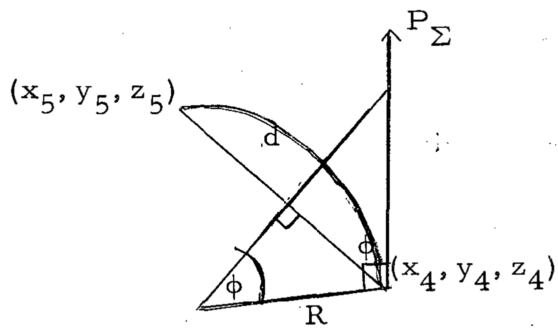
$$(\lambda C - \nu A)(x - x_m) + (\mu C - \nu B)(y - y_m) = 0$$

or $(x - x_m) = - \dfrac{(\mu c - \nu B)}{(\lambda C - \nu A)} (y - y_m)$.

Eliminating $(y - y_m)$ from Eqs. III.4 and III.5 we have

$$(\lambda B - \mu A)(x - x_m) + (\nu B - \mu C)(z - z_m) = 0$$

or $$(x - x_m) = - \frac{(\nu B - \mu C)}{(\lambda B - \mu A)} (z - z_m).$$



Now, $$\frac{x - x_m}{(\mu C - \nu B)} = \frac{y - y_m}{(\mu A - \lambda C)} = \frac{z - z_m}{(\lambda B - \mu A)} ;$$

then we have

$$\ell = \frac{\mu C - \nu B}{N} = \frac{\mu(\lambda_4 \mu - \mu_4 \lambda) + \nu(\lambda_4 \nu - \lambda \nu_4)}{N} = \frac{\lambda_4(\mu^2 + \nu^2) - \lambda(\mu_4 \mu + \nu_4 \nu)}{N}$$

$$= \frac{\lambda_4(1 - \lambda^2) - \lambda(\cos\phi - \lambda_4 \lambda)}{N} = \frac{\lambda_4 - \lambda \cos\phi}{N} ,$$

$$m = \frac{\nu A - \lambda C}{N} = \frac{\mu_4 - \mu \cos\phi}{N} ,$$

$$n = \frac{\lambda B - \mu A}{N} = \frac{\nu_4 - \nu \cos\phi}{N} .$$

Since

$$1 = \ell^2 + m^2 + n^2 = \frac{(\lambda_4 - \lambda \cos\phi)^2 + (\mu_4 - \mu \cos\phi)^2 + (\nu_4 - \nu \cos\phi)^2}{N^2} ,$$

$$N^2 = 1 - 2(\lambda_4 \lambda + \mu_4 \mu + \nu_4 \nu) \cos\phi + \cos^2\phi = 1 - \cos^2\phi = \sin^2\phi$$

and $$N = \sin\phi,$$

which gives

$$\ell = \frac{\lambda_4 - \lambda \cos\phi}{\sin\phi}, \quad m = \frac{\mu_4 - \mu \cos\phi}{\sin\phi} , \quad n = \frac{\nu_4 - \nu \cos\phi}{\sin\phi} .$$

Since

$$\cos\phi = (\lambda_4 \lambda + \mu_4 \mu + \nu_4 \nu) \qquad \text{for } 0 \leqslant \phi \leqslant \frac{\pi}{2} ,$$

and $$\sin\phi = (1 - \cos^2\phi)^{1/2},$$

The length of track between (x_4, y_4, z_4) and (x_5, y_5, z_5) is L. The following relationships exist for ϕ:

$$R\phi = \frac{L}{2}$$

and $$\sin\phi = \frac{\frac{D}{2}}{R} = \frac{\frac{D\phi}{2}}{R\phi} = \frac{D}{L} \phi ,$$

so that $\phi = \frac{L}{D} \sin \phi.$

We have $d = R - R \cos\phi = R(1 - \cos \phi) = \frac{L}{2\phi}(1 - \cos \phi).$

We wish to find a point $(\bar{x}, \bar{y}, \bar{z})$ on the line R at a distance d from (x_m, y_m, z_m),

$$\bar{x} = x_m + \ell d,$$

$$\bar{y} = y_m + md,$$

$$\bar{z} = z_m + nd.$$

The tangent to the circle at $\bar{x}, \bar{y}, \bar{z}$, is parallel to the chord $(x_4 y_4 z_4)$-(x_5, y_5, z_5), so it has the same direction cosines.

We wish to find the sin and cos of the dip angle at $\bar{x}, \bar{y}, \bar{z}$.

By definition, $\lambda = \sin \bar{\alpha} \cos \bar{\beta},$

$$\mu = - \sin \bar{\alpha} \sin \bar{\beta},$$

$$\nu = \cos \bar{\alpha},$$

so that $\cos \bar{\alpha} = \nu$

$$\sin \bar{\alpha} = + \sqrt{1 - \nu^2} \; .$$