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Mathematical Structures in Super Yang-Mills Amplitudes from Weak to Strong Coupling

By

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DISSERTATION

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Abstract

The last two decades have seen much progress in the field of scattering amplitudes. A big reason behind this revitalization of the program approximately 50 years after the initial formalization of quantum scattering has been our improved understanding of the rich mathematical structure behind amplitudes. In particular, mathematical properties of scattering amplitudes are being used to push calculations further and these new calculations are in turn used to look for new patterns. This virtuous cycle has been very fruitful especially in the context of $\mathcal{N} = 4$ Super Yang-Mills, which is a favorite playground for particle theorists. In this dissertation, we will be reviewing some of our novel contributions to this program. In particular, we explore: mathematical properties of diagrams that show up in a modern treatment of $\mathcal{N} = 4$ Super Yang-Mills amplitudes, consequences of a property called integrability on the perturbative side, and a novel diagrammatic technique for obtaining partial all-loop resummations. In order to set up the right ingredients before we go over what is new in each paper, this thesis is going to be structured as follows.

In chapter 1, we will be reviewing basics of scattering, introduce some notation and terminology used in the modern amplitudes program, and introduce Super Yang-Mills amplitudes. In chapter 2, we will go over loop-level methods, introduce objects called on-shell diagrams, and briefly go over how they connect to the positive Grassmannian. In chapter 3, we will explore our work on how poles at infinite momenta show up in on-shell diagrams, which is a topic unresolved thus far by the Positive Grassmannian framework. In chapter 4, we will go over yet another gap in the on-shell diagram/Grassmannian story, namely non-planar diagrams. We will associate new regions in the Grassmannian with these diagrams and explore their novel combinatorial properties. The two papers that these chapters are based on are part of a larger program initiated by my PhD advisor and his collaborators to understand on-shell diagrams and their mathematical properties. The novel results outlined in chapters 3 and 4 were thus taken from the accompanying papers, to which the author of this dissertation contributed by computing the forms associated with on-shell diagrams and geometries.

In chapter 5, we will change gears slightly and move away from the on-shell diagrams part of the story. We will go over special properties of planar Super Yang-Mills and how they connect to a property called integrability. This will lay the groundwork for the terminology and motivation necessary for chapters 6 and 7. In chapter 6, we will extend a framework that provides a novel

diagrammatic organization for the 4-point amplitude in planar $\mathcal{N} = 4$ Super Yang-Mills. Our work in chapter 7 further builds on this and examines how these diagrams can be used to compute a quantity called the cusp anomalous dimension, which controls the leading IR divergences of the amplitude. The two papers these chapters are based on reflect the author’s ongoing interest in finding implications of integrability on perturbative scattering amplitudes as well as looking for new ways to leverage positive geometry based methods in planar $\mathcal{N} = 4$ sYM. The novel results found in these chapters were also taken from the accompanying papers and contain most of the computations that were contributed by the author of this dissertation.

Finally, we include various appendices to provide useful details for the work we review. In appendix A, we include some of the proofs written by my co-authors to supplement the theorems in chapter 4. This appendix is taken from the paper this chapter is based on. In appendix B, we include a proof we wrote in the paper chapter 6 is based on to support one of the main claims we make. Finally, appendix C contains some lengthy computations done by the author to provide additional details relevant for chapter 7.

The results outlined in these chapters provide novel contributions to the amplitudes program in two different themes. In chapters 2 through 4, we extend the geometric understanding of planar $\mathcal{N} = 4$ sYM on-shell diagrams to cases where one loses the notion of planarity or the restricted pole structure imposed by the theory’s special properties. A big motivation behind studying planar $\mathcal{N} = 4$ sYM is to build tools for understanding more complicated theories, such as QCD and Gravity. Our work thus provides insight into what notions one would have to modify in order to initiate a geometric exploration of these less restricted theories. In the second theme, which is the subject of chapters 5 through 7, we are motivated by searching for implications of integrability on the amplitudes side. In particular, AdS/CFT duality and integrability programs have both computed quantities of planar $\mathcal{N} = 4$ sYM in the strong coupling. However, this regime is non-perturbative and thus the only way to access it through the toolkit of the standard perturbative amplitudes program is if we managed to resum everything. While there are subtle hints that planar $\mathcal{N} = 4$ sYM might just be restricted enough to allow for this, how such a program would look like is not yet known. Our work provides a novel contribution to this effort by extending on a diagrammatic toolkit that allows for partial all-loop resummations at 4-points.

Acknowledgments

I would like to start this dissertation by expressing gratitude to everyone to whom I owe debt for making this work possible.

First and foremost, I would like to thank my advisor Jaroslav Trnka. It is hard for me to sufficiently emphasize how much he has contributed to my intellectual growth as a scientist since I first joined the PhD program at UC Davis. Through his patient advising over the course of 6 years, I was able to learn and make novel contributions to topics I never thought I would ever grasp when I was learning about physics as an undergraduate. I will forever be grateful to Jaroslav for taking me under his mentorship and I hope I can repay some of this debt in the future by doing science as long as I can and discovering some cool facts about nature in the process.

Another special thanks goes to Shruti Paranjape, who was essentially a secondary advisor to me during her short stay at QMAP as a postdoc. Even with five projects simultaneously compiling in the back of her head, she always made time to answer my basic questions about quantum field theory in a more thoughtful, intuitive way than I have seen in any textbook. I will always be grateful to her for the mentorship and the collaboration.

I would also like to express gratitude to all the collaborators I have had over the years, whom I will list in alphabetical order like any good high energy theorist. In particular, I would like to thank Christoph Bartch, Taro Brown, Lance Dixon, Karol Kampf, Joris Koefer, Seth Koren, Artyom Lisitsyn, Melissa Sherman-Bennet, Yongqun Xu, Bailee Zacovic, and Shun-Qing Zhang (as well as Shruti and Jaroslav as mentioned above). A special thanks goes to Lance Dixon and Karol Kampf for writing my rec letters, which allowed me to miraculously get a postdoc position and buy another 3+ years in my academic journey. Another special thanks goes to Seth Koren, along with Nathaniel Craig, for introducing me to the wonderful world of high energy theory as a 20 year old. Despite the enormous amount of knowledge I have gained over the years, the seeds planted by Nate and Seth still shape how I think about particle physics. I would especially like to thank Taro Valentin Brown, who has been much more than a collaborator to me. I think it is fair to say I would not have survived graduate school without his friendship. I can't think of a better company to explore the ins and outs of grad school research with and I will forever be grateful for all the fun times we had.

I also owe tremendous intellectual debt to everyone in QMAP who collectively taught me a great amount of physics over the years, be it in courses I took, through small conversations in the hallway, or through the enormously helpful feedback I got during my qualifying exam. In particular, I would like to thank Andy Albrecht, Veronika Hubeny, Markus Luty, Mukund Rangamani, and Andrew Waldron.

On a more personal note, I would like to thank all the friends who made graduate school not only bearable but also a fun and enriching experience. I don't use the word "blessed" often but I can't think of another word to describe how lucky I was to coincide with the UCD Graduate Physics cohort of 2020. Among many others, I would like to give special thanks to Joseph Levine, Lena Korkeila, Rahul Hingorani, Sunny Rhoades, Griffin Heier, Rashad Kadado, Gabriel Lynch, Davis Zackaria, Adam Goga, Andrew Jackson, Can Görmez, and Cameron Chaffey. I feel very fortunate to have been surrounded by such supportive and fun people during my stay in Davis. I will do my best to keep in touch with you guys.

Of course it goes without saying that none of this would be possible without my family. I would like to thank my parents, Levent and Ruşen Öktem, for all their unwavering love and support through the years. I would not have even come close to attending graduate school if it was not for their lifelong encouragement of my interest in nature and science. Another special thanks goes to my sister Öykü Cansu Öktem for giving me a reason to be the best version of myself, even when life get hard.

Finally, I want to thank my girlfriend Megan. Thank you for being my best friend and kindred spirit. My time as a graduate student would've been quite a lonely one if it wasn't for the tremendous amount of love and support you gave me.

CHAPTER 1

Introduction

Many of the goals of theoretical physics can roughly be categorized under two camps. The first goal is a scientific one, we want to make predictive models that spit out parameters describing an observable for every input data we feed it. The second goal is more philosophical, we also want to use these models to enrich our qualitative understanding of the laws of nature. It is my personal belief that at least in particle physics, these goals are connected. In particular, mathematical structures that come up when we attempt to describe observables in particle physics elucidate novel aspects of the underlying theory. These insights can then be used to make progress in calculations and new calculations create new data in which we can search for new patterns. This type of virtuous cycle has driven much of the recent progress in the modern amplitudes program. Since the 1980s, the field has seen rapid progress both in efficient computation techniques for scattering amplitudes and also in their applications to other areas of physics as well as their connections to mathematics. Some important examples include the Parke-Taylor formula for n -point MHV gluon amplitudes [1], generalized unitarity techniques for multi-loop calculations [2,3,4], methods for computing classical observables relevant for gravitational wave physics [5,6,7], and last but not least, connections to positive geometries and the Amplituhedron [8,9,10,11,12,13,14,15,16,17,18,19,20], which will be relevant for a good chunk of this thesis. This list is far from complete, so we direct interested readers to [21] for a list of comprehensive reviews for each direction. In this thesis, we will be carving only a tiny bit of the amplitudes research space, nonetheless, it is still going to be a long and windy road. In particular, this dissertation is based on the following papers.

- Taro V. Brown, **Umut Oktem**, Jaroslav Trnka, *Poles at Infinity in On-Shell Diagrams*, Journal of High Energy Physics 02, 2023.
- Artyom Lisitsyn, **Umut Oktem**, Melissa Sherman-Bennett, Jaroslav Trnka, *Grassmannian Geometries for Non-Planar On-Shell Diagrams*, e-Print: 2512.25005 [hep-th].

- Taro V. Brown, **Umut Oktem**, Shruti Paranjape, Jaroslav Trnka, *Loops of loops expansion in the amplituhedron*, Journal of High Energy Physics 07, 2024.
- Lance Dixon, **Umut Oktem**, Shruti Paranjape, Jaroslav Trnka, Yongqun Xu, Shun-Qing Zhang, *Multi-loop Negative Geometries*, 2605.28926 [hep-th].

corresponding to chapters 3, 4, 6, and 7 respectively. In addition, the author has completed the following work as a part of the PhD, which will be omitted from this dissertation in order to preserve thematic continuity

- Taro V. Brown, Karol Kampf, **Umut Oktem**, Shruti Paranjape, Jaroslav Trnka, *Scalar Bern-Carrasco-Johansson bootstrap*, Physical Review D 108 (2023),
- Christoph Bartsch, Taro V. Brown, Karol Kampf, **Umut Oktem**, Shruti Paranjape, Jaroslav Trnka, *Hidden amplitude zeros from the double-copy map*, Physical Review D 111, 2025.
- Joris Koefer, **Umut Oktem**, Shruti Paranjape, Jaroslav Trnka, Bailee Zacovic, *Uniqueness of MHV Gravity Amplitudes*, Le Mat. 80, 2025.

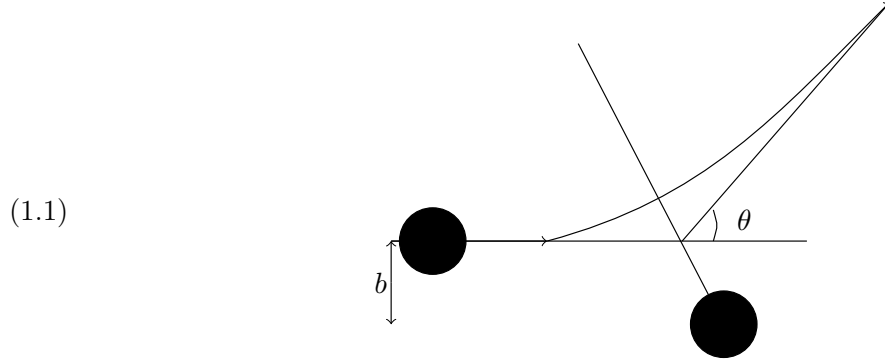
1.1 Classical Scattering

Why is scattering an interesting physical process to study? Well, the main sales pitch of particle physics is that if we understand constituents of our universe and their interactions, we will understand nature's laws at the deepest possible level. And scattering covers pretty much all fundamental particle interactions. In this thesis, we will be talking about the interactions of gluons within the framework of a theory called $\mathcal{N} = 4$ Super Yang-Mills.

It will be useful to start this dissertation by introducing some facts and tools regarding scattering in gauge theory that we will frequently use in this work. In this chapter, our main goal is to explain why singularities of scattering amplitudes are important to our work. But before we delve into the modern treatment of scattering, it might be easier to start with a simple example from classical physics to build intuition.

Say we have a point mass traveling towards a fixed source that scatters it away. This event is characterized by a scattering angle θ , and an impact parameter (denoted b), which is the closest distance between the trajectory if there was no scattering and the center of mass of the target. Such

a setup can be illustrated as



where the intersection of the line coming out of the right blob (representing source) with the incident particle's (left blob) trajectory is called the closest approach. While the impact parameter is easy to see from this diagram, in the atomic and subatomic limit this is not a measurable quantity, which motivates the definition of a cross section. If we just had a single incoming object, say a ball, scatter off some target, we wouldn't be able to get any useful information out of this process without knowing the impact parameter. So let's say we have a beam of incident particles that all have definite center of mass and momenta. For the time being, let's not worry about scattering angle but only focus on counting the number of collisions. And instead of one target, let's say we have a finite area with a finite number of targets, each targets having an area σ . If we have N targets in the target area A , the probability of an incident particle getting a hit is

$$(1.2) \quad P = \frac{N\sigma}{A}$$

and σ here is called cross-section. If the target number density is constant, then this probability is given by $P = n_t\sigma$. Even if we don't know the impact parameter b , we can observe the number of hits and this can be used with the number of incident particles and the target density to find σ , the target cross section.

Now let's take a look at what happens if we also care about scattering angle. Instead of counting the total number of hits, we want the number of scattered particles that fly off inside a small cone $d\Omega(\theta, \phi)$. For a narrow cone at fixed $r = R$ that varies by $d\theta$ and $d\phi$ in polar coordinates, the differential solid angle is $d\Omega = \sin(\theta)d\theta d\phi$. Associated with this differential solid angle, we can define a differential cross section $d\sigma$ for the target. To get an intuition for this, consider the

previous example except if we only counted particles that flew off into $d\Omega$ after hitting the target. Then we would replace σ with $d\sigma$.

Let's assume that the scattering is spherically symmetric and consider the setup shown in Figure 1.1. Say incident particles are allowed to pass through within the bounds of the circle with radius b . If we increase the radius by db , then the total area that the particles can pass through increases

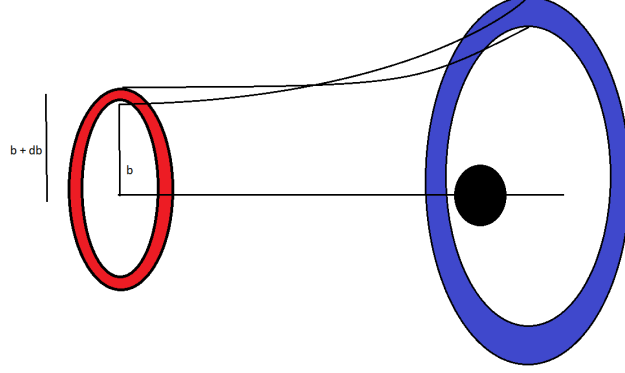


FIGURE 1.1

by $d\sigma = 2\pi b db$, illustrated by the red shaded region. If the particles passing through the original circle scatter into a solid angle cone Ω_0 , then the particles passing through $d\sigma$ will fly off into $d\Omega$, illustrated by the blue shaded region, where $d\Omega = 2\pi \sin \theta d\theta$. This leads to the following expression relating differential cross section to the solid angle the incident particles scatter into

$$(1.3) \quad \frac{d\sigma}{d\Omega} = \frac{b}{\sin(\theta)} \frac{db}{d\theta}$$

So the differential cross section is related to the impact parameter b . This expression gives the fraction of incident particles that go through differential area $d\sigma$ per unit time that scatter into the differential solid angle $d\Omega$. In other words, it computes the ratio of scattered particle flux to the incident flux.

In the illustration, the target is a physical object but this doesn't need to be the case. In fact, we can easily generalize the previous results if scattering was achieved by a potential instead. Let's apply this to a historically relevant scattering example, namely Rutherford scattering. In this experiment, a gold foil was bombarded with charged alpha particles and Rutherford, Geiger and

Marsden measured the scattering of the particles off of the gold nuclei [22, 23]. In the classical limit, scattering in this case is purely due to the coulomb potential $V(r) = \frac{k}{r}$. From the formula above, determining the scattering cross section boils down to finding the scattering angle changes with impact parameter. This is reminiscent of orbit problems, so we can use the same tools. Since the Coulomb force is conservative, total energy in this system is conserved (if we ignore the recoil of the sourcing charge) so we can write the total energy as

$$(1.4) \quad E = \frac{m\dot{r}^2}{2} + \frac{m\ell^2}{2r^2} + \frac{k}{r}$$

where ℓ in the second term is angular momentum per mass $\ell = L/m$. The orbit for an attractive potential $\frac{-k}{r}$ can be written as

$$(1.5) \quad r(\theta) = \frac{r_0}{1 + \epsilon \cos(\theta)}$$

where $r_0 = \frac{m\ell^2}{k}$ and e is some constant. Then by analogy, for a repulsive potential it should be

$$(1.6) \quad r(\theta) = \frac{r_0}{\epsilon \cos(\theta) - 1}$$

which tells us $\frac{dr}{d\theta} = \frac{r_0 \epsilon \sin \theta}{(\cos \theta - 1)^2}$. Plugging this back into the equation for total energy gives

$$(1.7) \quad E_i = \frac{mv^2}{2} = E_f = \frac{k^2(e^2 - 1)}{2m\ell^2}$$

where the first term is the particle's initial asymptotic energy before the interaction takes place. Just as with energy, the angular momentum is also conserved so we can write $\ell = L_i/m = \frac{mvb}{m} = vb$. Letting ϕ denote the angle between the closest approach and the asymptotic trajectory, $\theta = \pi - 2\phi$ [24] and $r \rightarrow \infty$ as $\theta = \phi$, which means $e \cos \phi = 1$. Then

$$(1.8) \quad E = \frac{mv^2}{2} = \frac{k^2(e^2 - 1)}{2m\ell^2} = \frac{k^2 \tan^2 \phi}{2mv^2 b^2} = \frac{k^2 \tan^2 ((\theta - \pi)/2)}{2mv^2 b^2}$$

which gives

$$(1.9) \quad b = \frac{k}{mv^2} \cot \theta/2$$

Putting everything together, we get the Rutherford scattering formula

$$(1.10) \quad \frac{d\sigma}{d\Omega} = \left(\frac{k}{4E \sin^2(\theta/2)} \right)^2$$

This formula is historically significant because it was used by Rutherford, Geiger and Marsden to measure how alpha particles scatter off gold atoms and they found that most of the atom's mass was concentrated in the positively charged nucleus. Thus, they were able to show that the hitherto assumed plum pudding model of the atom was wrong [22].

But why did we start with such an elementary example? At the first level, this example is a simple illustration of how scattering reveals the underlying physics of systems we are studying. Second, we can already see hints of a feature of gauge theories that will be a key player in our discussions later, namely IR singularities. If we plot the classical amplitude $f(\theta) = \frac{k}{4E \sin^2(\theta/2)}$ in θ space, we can immediately see that we see a divergence at the origin. Physically, $\theta = 0$ corresponds to large impact parameter, which means the photon being exchanged in the full QED theory that describes this interaction has a momentum that goes soft (ie $p \rightarrow 0$). This is an example of what is called an IR-divergence.

The divergence of the amplitude at $\theta = 0$ does not mean our model is unphysical in this limit. On the contrary, this is exactly what we should expect from our understanding of the underlying QED. Namely, the quantum theory of Coulomb interactions tells us that the Coulomb force Rutherford observed is mediated by photons, which are massless bosons. Because photons are massless, the Coulomb force they mediate is a long ranged force. In other words, we always have arbitrarily soft photon radiation effects that contribute to the interaction. The divergence in $\theta = 0$ then corresponds to the forward limit of a t -channel process in QED. This forward kinematics divergence is closely related to the IR-divergences we will see in massless gauge theories, although there are important distinctions as we will discuss later. The purpose of this example was to illustrate how singularities in scattering usually have a physical origin, but let's shelve this for now and introduce some of the standard machinery we need to discuss how scattering works in sYM before we look at IR divergences again.

1.2 Particles, Lagrangians, and Scattering

In the example we considered above, our model for particles was not too different than what most people imagine, namely “tiny balls” that make up matter around us. However, quantum field theory tells us that particles are not balls with definite position and momenta but rather excited states of a corresponding quantum field. Or as Wigner would put it, particles are irreducible unitary representations of the Poincare group that acts on the Hilbert space of the underlying theory. This is the definition I will use for the rest of this thesis.

To study how particles interact, we need to work with the fundamental objects they are associated with, namely fields. More specifically, particles are local excitations of underlying fields. In the standard treatment of quantum field theory, one works with Lagrangians constructed out of such fields, which tell us about the rules of particle interactions. To start simple, let’s consider a scalar theory ϕ with the Lagrangian

$$(1.11) \quad \mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 - V(\phi)$$

When $V(\phi) = 0$, what we have at hand is a free-theory. The equations of motion we get from this free Lagrangian, namely the Klein-Gordon equation $(\square + m^2)\phi = 0$, gives a mode expansion for the free field in terms of creation and annihilation operators. This can be written as

$$(1.12) \quad \phi(x, t) = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} (a_p e^{-ipx} + a_p^\dagger e^{ipx})$$

where the p that appears in the exponential is 4-momenta, so $px = -\vec{p} \cdot \vec{x} + \omega t$. Typically when we calculate scattering, one assumes that the theory is free at “infinite” past or future, thus the initial and final states of consideration are asymptotic states built out of the creation/annihilation operators of the free theory. For example, an initial 2-particle state with momenta p_1, p_2 can be defined by

$$(1.13) \quad |i\rangle = \lim_{t \rightarrow -\infty} a_{p_1}^\dagger(t) a_{p_2}^\dagger(t) |0\rangle$$

and the final state by

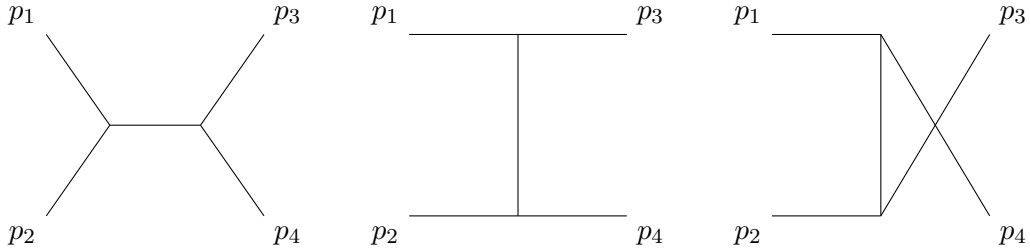
$$(1.14) \quad |f\rangle = \lim_{t \rightarrow \infty} a_{p_1'}^\dagger(t) a_{p_2'}^\dagger(t) |0\rangle$$

Then our scattering amplitude of interest, $\mathcal{A}$, is the object defined by $\langle f|iT|i\rangle = i(2\pi)^4\delta^4(P)\mathcal{A}$, where $P = p_1 + p_2 - p'_1 - p'_2$. The operator T here is defined by $T = S - 1$ and it describes the interacting part of the scattering matrix. To compute this object, we need to evaluate the path integral

$$(1.15) \quad \int \mathcal{D}\phi e^{i \int d^4x \left(\frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 - V(\phi) \right)}.$$

This integral can be performed for $V(\phi) = 0$ but not for a generic $V(\phi)$, so typically one works in the perturbative limit where couplings that enter $V(\phi)$ are assumed to be small. Then the exponential can be expanded in a perturbative series, which are organized by Feynman diagrams. The leading order terms in this expansion correspond to a saddle point of the path integral, which is given by the classical equations of motion that extremize the action $S = \int d^4x \mathcal{L}$. These are called tree-level contributions, for Feynman diagrammatic reasons. There are also higher order terms that carry powers of $\hbar$, which correspond to fluctuations around the classical equations of motion and are sometimes called quantum corrections as a result. These are represented by loops when one works with Feynman diagrams.

Let's look at a simple example, given by $V(\phi) = \frac{\lambda}{3!}\phi^3$. As the interaction potential suggests, this is usually called phi-cubed theory. For $2 \rightarrow 2$ scattering, the tree-level contributions are given by the following diagrams which are called s , t , and u diagrams respectively. They get their names



from the Mandelstam invariants that correspond to the internal propagators

$$(1.16) \quad s = (p_1 + p_2)^2, \quad t = (p_1 - p_3)^2, \quad u = (p_1 - p_4)^2$$

The corresponding tree-level amplitude one gets can be written in terms of these variables, as

$$(1.17) \quad \mathcal{A}^{tree} = \lambda^2 \left(\frac{1}{s - m^2} + \frac{1}{t - m^2} + \frac{1}{u - m^2} \right) + \mathcal{O}(\lambda^4).$$

Mandelstam invariants satisfy $t + s + u = m_1^2 + m_2^2 + m_3^2 + m_4^2$ (where m_i is the mass associated each external leg), thus the amplitude $\mathcal{A}^{tree}$ can be considered as a function of two of the Mandelstams, such as $\mathcal{A}^{tree}(s, u)$.

1.2.1 Spinor-Helicity Formalism

As we saw in the scalar example above, the kinematic part of the amplitude will be a function of 4-momenta p_i^μ . In the example above in particular, we could write the answer in terms of variables s, t and u , which are Lorentz invariant products of these p^μ . However, all the amplitudes in the rest of this work will be for particles that have spin, and thus before we move onto describing gluon amplitudes, we should first introduce notation of a more natural choice of variables. This is called spinor-helicity formalism, and it will be a more natural choice of variables for describing amplitudes of massless particles with non-zero helicity.

Say we have a particle with 4-momenta p^μ . Taking the dot product with Pauli matrices, we can write

$$(1.18) \quad p_\mu(\sigma^\mu)_{ab} = p_{ab} = \begin{pmatrix} -p^0 + p^3 & p^1 - ip^2 \\ p^1 + ip^2 & -p^0 - p^3 \end{pmatrix}$$

The determinant of this matrix $\det(p_{ab}) = -p^\mu p_\mu$ is invariant under applying $SL(2, \mathbb{C})$ transformations to p_{ab} . In particular, if we are talking about massless particles, then

$$(1.19) \quad \det \begin{pmatrix} -p^0 + p^3 & p^1 - ip^2 \\ p^1 + ip^2 & -p^0 - p^3 \end{pmatrix} = -p^2 = 0$$

Linear algebra tells us that if a 2×2 matrix has a vanishing determinant, then we can write it as an outer product of two 2-dimensional vectors, which we will refer to as spinors, namely

$$(1.20) \quad p_{ab} = \lambda_a \tilde{\lambda}_b$$

The Lorentz group acts on λ and $\tilde{\lambda}$ as separate 2×2 blocks, and furthermore there is a subgroup of the Lorentz group that sends $\lambda \rightarrow t\lambda$, $\tilde{\lambda} \rightarrow \frac{1}{t}\tilde{\lambda}$. This is called the little group. We will express the amplitudes in this thesis in helicity basis so that the helicity of each particle entering the process is specified, and in this basis the kinematic part of the amplitude can be written in terms of Lorentz

invariant products of the spinor-helicity variables introduced below, namely

$$(1.21) \quad \epsilon_{\alpha\beta}\lambda_i^\alpha\lambda_j^\beta = \langle ij \rangle, \quad \epsilon_{\dot{\alpha}\dot{\beta}}\tilde{\lambda}_i^{\dot{\alpha}}\tilde{\lambda}_j^{\dot{\beta}} = [ij]$$

Furthermore, these variables satisfy the following identities, which we will make frequent use of in the following chapters

$$(1.22) \quad 1) \quad \langle ij \rangle = -\langle ji \rangle, \quad [ij] = -[ji], \quad \langle ii \rangle = [ii] = 0,$$

$$(1.23) \quad 2) \quad \langle ij \rangle[ji] = (p_i + p_j)^2 = s_{ij},$$

$$(1.24) \quad 3) \quad \sum_{j=1}^n \langle ij \rangle[jk] = 0$$

$$(1.25) \quad 4) \quad \langle ij \rangle \langle kl \rangle - \langle ik \rangle \langle jl \rangle = \langle il \rangle \langle kj \rangle.$$

Identity 4) is called the Schouten identity and it is going to play an important role in our work.

Now that we got notation out of the way, we can move back to the physics. Gluon interactions are described by the Yang-Mills Lagrangian, given by

$$(1.26) \quad \mathcal{L}_{YM} = -\frac{1}{4} \text{Tr}\{F_{\mu\nu}F^{\mu\nu}\}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu]$ is the gluon field strength and $A_\mu = A_\mu^a T^a$ is the vector field describing gluons (where T^a are $SU(3)$ generators). The upper index in the gluon field A_μ^a corresponds to color and it is acted on by the adjoint representation of $SU(3)$ (thus $a = 1, 2, \dots, 8$). In the real world, gluons transform under the adjoint representation of $SU(3)$, however for our purposes we will study $SU(N)$ Yang-Mills. Thus, we will have N colors and $N^2 - 1$ generators. These generators are normalized by

$$(1.27) \quad \text{Tr}\{T^a T^b\} = \delta^{ab}$$

and

$$(1.28) \quad [T^a, T^b] = i\tilde{f}^{abc}T^c$$

A popular choice of gauge fixing is the Gervais-Neveu gauge in which the Lagrangian becomes

$$(1.29) \quad \mathcal{L} = \text{Tr} \left(-\frac{1}{2} \partial^\mu A^\nu \partial_\mu A_\nu - i\sqrt{2} g \partial^\mu A^\nu A_\nu A_\mu + \frac{1}{4} g^2 A^\mu A^\nu A_\mu A_\nu \right)$$

Since our Lagrangian allows 3-point interactions, let's go ahead and start by looking at the 3 particle vertex at tree-level. The 3-gluon vertex comes with a factor of $\tilde{f}^{abc}$, where the color ordering is set by the external legs. Since one needs to sum over all color indices to get the full amplitude, it is often convenient to write the amplitude as a sum over products of the generator traces and the kinematic part for each color ordering, called the partial amplitude. For n -particles this is written as

$$(1.30) \quad A_n^{tree} = g^{n-2} \sum_{\sigma} A_n[1\sigma(2 \dots n)] \text{Tr} \left\{ T^{a_1} T^{\sigma(a_2 \dots T^{a_n})} \right\}$$

where σ runs over all non-cyclic permutations. For the rest of this work, we will be computing the partial amplitudes, so we are only concerned with kinematics for now.

Then Feynman rules tells us that the 3-point vertex is

$$(1.31) \quad A_3[123] = -\sqrt{2} [\epsilon_1 \epsilon_2 (\epsilon_3 p_1) + \epsilon_2 \epsilon_3 (\epsilon_1 p_2) + (\epsilon_3 \epsilon_1) (\epsilon_2 p_3)]$$

in terms of external polarizations ϵ_i . Note that this isn't a physical amplitude, in the sense that momentum conservation $p_1 + p_2 + p_3 = 0$ and external legs being on-shell $p_1^2 = p_2^2 = p_3^2 = 0$ can't be solved if p_1, p_2 and p_3 are all taken to be real. Nonetheless, as we will see, the vertex factor is going to play a crucial part in our story so we can just analytically continue the momenta into the complex plane for now.

As advertised before, we can write this result in the helicity basis using spinor-helicity notation by considering all helicity configurations for gluons 1, 2, and 3. Doing so, we get

$$(1.32) \quad A_3^{MHV}[1^- 2^- 3^+] = \frac{\langle 12 \rangle^3}{\langle 23 \rangle \langle 31 \rangle}$$

and

$$(1.33) \quad A_3^{\overline{MHV}}[1^+ 2^+ 3^-] = \frac{[12]^3}{[23][31]}$$

whereas configurations where all external legs have the same helicity are 0 (this will be a general feature for n -point). This wasn't so bad, so let's take a look at our previous example from the scalar theory, namely $2 \rightarrow 2$ scattering. At tree-level, we only need to consider 4 different topologies, namely the s, t, u channels and the contact term, as shown in Figure 1.2 Using Feynman rules to

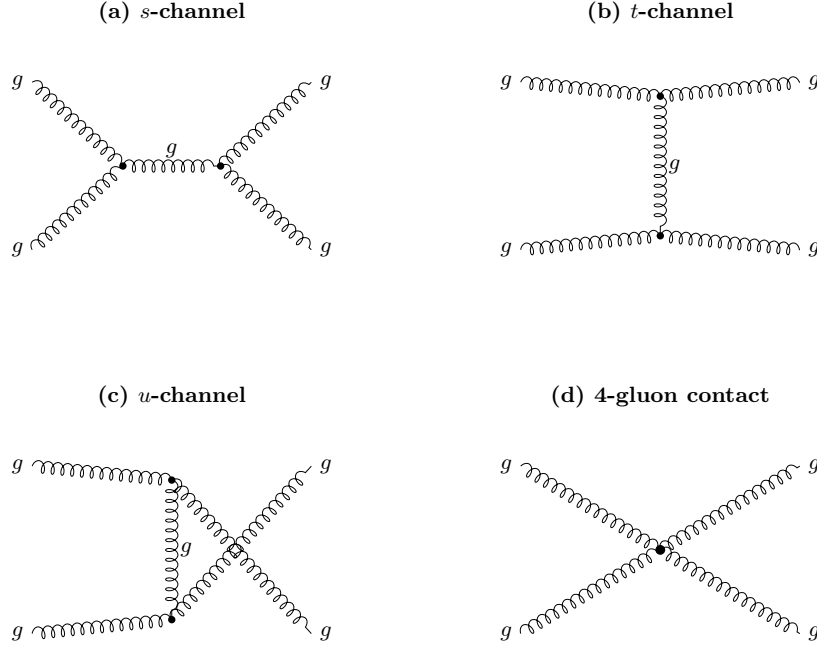


FIGURE 1.2. All diagrams contributing to $gg \rightarrow gg$ at tree level

compute each diagram and translating the result into helicity basis, one finds

$$(1.34) \quad A_4[1^- 2^- 3^+ 4^+] = \frac{\langle 12 \rangle^4}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 41 \rangle}$$

with $A_4[1^- 2^- 3^- 4^+] = A_4[1^+ 2^+ 3^+ 4^-] = A_4[1^- 2^- 3^- 4^-] = A_4[1^+ 2^+ 3^+ 4^+] = 0$. This is a maximally helicity violating (MHV for short) amplitude, meaning there are two negative helicities and $n - 2$ positive helicities. We will see the term MHV pop up again over and over through this work. When we have $k + 2$ negative helicities, the amplitude is said to be $N^k \text{MHV}$, where the letter N stands for “next-to”. Both the 3-point and 4-point tree level partial amplitude have a compact form when written in spinor-helicity variables. One might wonder, does this simplicity carry onto higher n ? As we will see now, the answer is a resounding yes.

1.3 Parke-Taylor Formula and On-Shell Recursion

Our previous example was only for $2 \rightarrow 2$ scattering, but it is easy to see that the complexity grows rather fast with n . If we had $gg \rightarrow ggg$, instead, we would have to consider 25 diagrams. Still doable (although annoying) by hand, but adding another final state g increases the number of diagrams needed to 220. And this is just for tree-level! This does not pose a philosophical problem yet. After all, nature does not have to be nice. However, this turns out to be one of the many cases in science where our initial approach to the problem hides simplifications. In spinor-helicity variables, the tree-level result for the 5- and 6-point MHV amplitudes take the forms

$$(1.35) \quad A_5[1^- 2^- 3^+ 4^+ 5^+] = \frac{\langle 12 \rangle^4}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 45 \rangle \langle 51 \rangle}$$

$$(1.36) \quad A_6[1^- 2^- 3^+ 4^+ 5^+ 6^+] = \frac{\langle 12 \rangle^4}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 45 \rangle \langle 56 \rangle \langle 61 \rangle}$$

respectively. Together with the 4-point result, we can already see a pattern emerging. This is not accidental. Turns out the n -point MHV gluon amplitude is indeed given by the Parke-Taylor formula

$$(1.37) \quad A_n[1^- 2^- 3^+ \dots n^+] = \frac{\langle 12 \rangle^4}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \dots \langle n1 \rangle}$$

When Parke and Taylor first stumbled upon this formula [1], they were interested in computing the $gg \rightarrow gggg$ amplitude at tree-level. While the full answer that came from Feynman diagrams took a whopping 9-pages, the final MHV answer simplified to the compact form shown above. But what is the physical mechanism responsible for this miracle? After all, drastic simplifications usually come from some symmetry. We will eventually see how the analytical structure of the amplitude as well as constraints from physical principles “force” the MHV amplitude to take this form, but to do that, we must first take a detour and revisit the story of IR divergences that we got a very brief taste of at the beginning.

1.3.1 Analyticity of Amplitudes

Based on the data we have so far, we have seen that the tree level amplitude is a rational function of kinematic variables, which in spinor-helicity notation take the form $[ij]$ and $\langle kl \rangle$. For

our amplitude to describe a physical process, it must respect locality, which in quantum field theory means any pole of the amplitude must correspond to the exchange of a particle that is physically produced. Such particles are said to be on-shell and they satisfy $p^2 = -m^2$. For massless particles, this condition becomes $p^2 = 0$. This is a consequence of locality being baked into the Lagrangian we use. This means for a tree-amplitude, poles can only come from a region in the kinematic space where $(\sum_I p_I)^2 = 0$, where I lists all the external legs that have momenta flowing through some internal propagator. But we can easily see that there are multiple ways to reach this region. To start off, let's consider two incoming gluons that create a propagator with momentum $(p_1 + p_2)$. The expression $(p_1 + p_2)^2$ can go to 0, which creates a singularity in the propagator factor $\frac{1}{(p_1 + p_2)^2}$, in two distinct ways. The first of this is if $p_1 \rightarrow 0$, which is called a soft singularity. In this limit, the tree amplitude factorizes into a piece that carries the singularity, called the soft function, and a $n - 1$ point non-singular part. For example if we have $A_n^{tree}[12 \cdots i - 1 \ i^\pm \ i + 1 \cdots n]$ with $p_i \rightarrow 0$, the amplitude factorizes as

$$(1.38) \quad A_n^{tree}[12 \cdots i - 1 \ i^\pm \ i + 1 \cdots n]_{p_i \rightarrow 0} \rightarrow S^\pm(i - 1 \ i^\pm \ i + 1) \times A_{n-1}[12 \cdots i - 1 \ i + 1 \cdots n]$$

where the soft functions are given as

$$(1.39) \quad S^-(i - 1 \ i \ i + 1) = -\frac{[i + 1 \ i - 1]}{[i + 1 \ i][i \ i - 1]}$$

$$(1.40) \quad S^+(i - 1 \ i \ i + 1) = -\frac{\langle i + 1 \ i - 1 \rangle}{\langle i + 1 \ i \rangle \langle i \ i - 1 \rangle}$$

Note that the entire singular piece is contained in $S^\pm(i - 1 \ i \ i + 1)$. The other type of pole comes from $p_1 \cdot p_2 = 0$ and it is called a collinear singularity. In this limit, the n -point amplitude factorizes into a piece that carries the singularity, called splitting function, times the $n - 1$ point amplitude. In particular, if propagator P has only p_{i+1} and p_i flowing in, in the limit where $p_i \cdot p_{i+1} \rightarrow 0$ one can parametrize $p_i = (1 - z)p_P$ and $p_{i+1} = zp_P$. If i and $i + 1$ are both positive helicity, then the amplitude "splits" as follows

$$(1.41) \quad A_n[1^- 2^- 3^+ \cdots i^+ (i + 1)^+ \cdots n]_{p_i \cdot p_{i+1} \rightarrow 0} \rightarrow \frac{1}{\sqrt{z(1 - z)} \langle i \ i + 1 \rangle} A_{n-1}[1^- 2^- \cdots P^+ \cdots n]$$

where i and $i + 1$ are substituted with P^+ and the splitting function is

$$(1.42) \quad Split_-(i^+ (i + 1)^+) = \frac{1}{\sqrt{z(1-z)}\langle i i + 1 \rangle}.$$

The splitting function is different if the collinear particles carry different helicity, but the part that is important to our discussion is that all splitting function configurations contain the singular part $\frac{1}{\langle i i + 1 \rangle}$.

Both of the poles that we have considered above are examples of IR-singularities. In fact, the Rutherford scattering divergence we encountered earlier is conceptually similar to the soft and collinear divergences we see here. In the Coulomb interaction case (Rutherford experiment setup), the IR divergence came from the forward region $t \rightarrow 0$, and it was due to massless photon exchange creating a long-range Coulomb force. Here in Yang-Mills, the soft/collinear divergences can physically be interpreted as the Lagrangian allowing for gluons that are arbitrarily soft or collinear with respect to one of the external legs to contribute to the process, even though a "detector" wouldn't be able to differentiate these modes. So in a sense, these IR divergences can be interpreted as a degeneracy in the states. Now, we have to face one issue we had thus far brushed under the table, which is the question of how we can define an S-matrix for gluons. If we can always get arbitrarily soft/collinear modes with each gluon, we can't really define an asymptotic gluon state in the naive sense. So how do we talk about physical scattering in a meaningful way?

The Rutherford scattering we saw earlier can be described by QED in terms of a t -channel massless photon exchange at tree-level. The $t \rightarrow 0$ divergence shows that QED is sensitive to exchange of arbitrarily soft modes, as a consequence of the force carrier (photon) being massless. The soft and virtual emission divergences we see in QED have their origin in the same IR-structure, although they arise in a different context (one of which is loop-level). This does not prevent us from computing observables, as we can get finite cross sections in QED by canceling the soft photon emission IR singularity against virtual photon loop one, which is called the Bloch-Nordsieck (BN) theorem [25]. However, the BN theorem fails for non-abelian theories, such as Yang-Mills. Luckily, a more general statement that applies to any massless theory handles this case and it goes by the name of the Kinoshita-Lee-Nauenberg theorem (KLN for short) [26, 27]. KLN says that in order to get IR-safe observables in gluon interactions we must sum over all degenerate states, both final and initial. Thus, one deals with IR-safe observables, such as jets, when dealing with gluon interactions

in the real world, such as those in the LHC. The non-abelian nature of gluon interactions gives rise to yet another departure from the QED analogy, namely the issue of confinement. While this is not relevant for the perturbative regime we are in, it is worth noting that the strong interaction forces asymptotic QCD states to only exist in color singlets at low energies. These color-singlets are called hadrons, and this is the reason why we observe hadrons but no quarks or gluons at low energies. The only time we talk about non-perturbative physics in this thesis will be in the context of $\mathcal{N} = 4$ sYM, which does not confine, unlike Yang-Mills. Therefore, we can shelve this fact for now.

The physical subtleties are out of the way, so we can go back to the task at hand, which is figuring out where the Parke-Taylor factor comes from. Here we should note that the IR-singularities we have considered thus far came from propagators with only two external legs attached. What happens if we have $(p_1 + p_2 + p_3)^2 = 0$? This is where things get interesting, because in this region the amplitude factorizes into two disconnected pieces. As we will see now, this allows us to write powerful recursion relations, and among other things, prove the Parke-Taylor formula.

1.3.2 BCFW Recursion

As discussed above, the tree amplitude A_n is a function of external momenta. Say we introduce $m \leq n$ complex valued 4-vectors r_i^μ and shift our external momenta by them such that $\hat{p}_i^\mu = p_i^\mu + z r_i^\mu$ for some complex parameter z . Furthermore, let's require that these vectors satisfy the following conditions

$$(1.43) \quad \sum_{i=1}^m r_i^\mu = 0$$

$$(1.44) \quad r_i \cdot r_j = 0$$

$$(1.45) \quad p_i \cdot r_i = 0$$

Since these shifting vectors satisfy momentum conservation among themselves, we still have momentum conservation among all the external momenta, such that

$$(1.46) \quad \sum_{i=1}^n \hat{p}_i^\mu = 0$$

where n is the number of external particles. And we still have $\hat{p}_i^2 = 0$ for all i . Now let's consider the shifted tree amplitude $\hat{A}_n(z)$, which is now a function of z as well as the external momenta. Since A_n is only describing a tree-level process, it has no branch cuts, and thus its analytic structure is completely captured by simple poles located on points in the z -plane such that some shifted propagator goes on-shell. Now let's consider the function $\frac{\hat{A}_n(z)}{z}$ in the z -plane. Since poles are at discrete locations in the z -plane (determined by external momenta), we can choose a contour around the origin such that it includes no poles other than

$$(1.47) \quad \text{Res}_{z=0} \frac{\hat{A}_n(z)}{z} = A_n$$

which is simply the unshifted amplitude. Then say we invert the contour such that it surrounds all other poles instead. If the shifted amplitude vanishes at $z \rightarrow \infty$, Cauchy's theorem tells us

$$(1.48) \quad \text{Res}_{z=0} \frac{\hat{A}_n(z)}{z} = - \sum_{z_I} \text{Res}_{z=z_I} \frac{\hat{A}_n(z)}{z}$$

which means

$$(1.49) \quad A_n = - \sum_{z_I} \text{Res}_{z=z_I} \frac{\hat{A}_n(z)}{z}$$

Where z_I goes over each pole of $\hat{A}(z)$ in the z -plane. From the discussion above, these correspond to regions in the kinematic space where some shifted propagator $\hat{P}_I^2 = (\sum_{i \in I} \hat{p}_i)^2$ goes on-shell. Since the amplitude factorizes into disjoint pieces on such poles, we now have a formula for the n -point amplitude in terms of a sum over the products of lower point ones. Putting everything together, this becomes

$$(1.50) \quad A_n = \sum_{z_I} \hat{A}_L(z_I) \frac{1}{P_I^2} \hat{A}_R(z_I)$$

where $\hat{A}_{L/R}(z_I)$ is the left and right pieces of the factorized amplitude evaluated on the support of $z = z_I = -\frac{P_I^2}{2P_I \cdot R_I}$, P_I is the propagator associated with the factorization channel I , and I indexes over all possible factorization channels. This is essentially the idea behind on-shell recursion. But we have to be cautious and note that this only works for "good shifts", meaning $\hat{A}(z) \rightarrow 0$ as $z \rightarrow \infty$. So one must be careful in how to choose the shift. To see how this works, let's look at one of the most famous shifts, which results in the Britto–Cachazo–Feng–Witten recursion relations (BCFW

for short) [28, 29]. To implement this BCFW shift, we will pick out two external legs, i and j and select our r_i^μ from these as well. In particular, let's have

$$(1.51) \quad |\hat{i}] = |i] + z|j], \quad |\hat{j}] = |j], \quad |\hat{i}\rangle = |i\rangle, \quad |\hat{j}\rangle = |j\rangle - z|i\rangle$$

Furthermore, to ensure that $\hat{A}_n(z) \rightarrow 0$ as $z \rightarrow \infty$, we must choose i and j such that the helicity configuration of $\{i, j\}$ is any of $\{-, -\}, \{-, +\}, \{+, +\}$. So let's go ahead and test this on our n -point MHV amplitude, namely $A_n^{tree}(1^- 2^- 3^+ \dots n^+)$, and choose $i = 1, j = 2$ such that

$$(1.52) \quad |\hat{1}] = |1] + z|2], \quad |\hat{2}] = |2], \quad |\hat{1}\rangle = |1\rangle, \quad |\hat{2}\rangle = |2\rangle - z|1\rangle$$

Then we have

$$(1.53) \quad A_n^{tree}(1^- 2^- 3^+ \dots n^+) = \sum_{k=4}^n \sum_{h_I=\pm} \hat{A}_{n-k+3}[\hat{1}^-, \hat{P}_I^{h_I}, k^+, \dots, n^+] \frac{1}{\hat{P}_I^2} \hat{A}_{k-1}[-\hat{P}_I^{-h_I}, \hat{2}^-, 3^+, \dots, (k-1)^+]$$

where $\hat{P}_I = \hat{p}_2 + p_3 + \dots + p_{k-1}$. If $\hat{1}$ and $\hat{2}$ are partitioned into the same sub-diagram, $\hat{P}_I = P_I$ and thus the residue vanishes. So the only valid factorization channels are ones with $\hat{1}$ and $\hat{2}$ on opposite sides. Furthermore, all single minus helicity amplitudes vanish for $n > 3$ and all plus helicity amplitudes vanish for any n . Then we have

$$A_n[1^- 2^- 3^+ \dots n^+] = \begin{array}{c} \hat{1}^- \\ \diagup \\ \textcircled{L} \\ \diagdown \\ n^+ \end{array} \xrightarrow{+\hat{P}_I} \begin{array}{c} \hat{2}^- \\ \diagup \\ \textcircled{R} \\ \diagdown \\ n-1^+ \end{array} \quad + \quad \begin{array}{c} \hat{1}^- \\ \diagup \\ \textcircled{L} \\ \diagdown \\ 4^+ \end{array} \xrightarrow{+\hat{P}_I} \begin{array}{c} \hat{2}^- \\ \diagup \\ \textcircled{R} \\ \diagdown \\ 3^+ \end{array}$$

If we evaluate $\hat{A}_3[\hat{1}^- \hat{P}_I^+ n^+]$ on the support of $\hat{P}_{1n}^2 = 0$, we find that the numerator vanishes and thus the expression reduces to

$$A_n[1^- 2^- 3^+ \dots n^+] = \begin{array}{c} \hat{1}^- \\ \diagup \\ \textcircled{L} \\ \diagdown \\ 4^+ \end{array} \xrightarrow{+\hat{P}_I} \begin{array}{c} \hat{2}^- \\ \diagup \\ \textcircled{R} \\ \diagdown \\ 3^+ \end{array}$$

Evaluating this diagram gives

$$(1.54) \quad A_n[1^- 2^- 3^+ \dots n^+] = \frac{\langle \hat{1} \hat{P}_{23} \rangle^4}{\langle \hat{1} \hat{P}_{23} \rangle \langle \hat{P}_{23} 4 \rangle \langle 45 \rangle \dots \langle n \hat{1} \rangle} \times \frac{1}{\langle 23 \rangle [23]} \times \frac{[3 \hat{P}_{23}]^3}{[\hat{P}_{23} \hat{2}] [\hat{2} \hat{3}]}$$

with

$$(1.55) \quad \begin{aligned} \langle \hat{1} \hat{P}_{23} \rangle [3 \hat{P}_{23}] &= -\langle \hat{1} \hat{P}_{23} \rangle [\hat{P}_{23} 3] = \langle \hat{1} | \hat{P}_{23} | 3 \rangle = \langle \hat{1} | (\hat{p}_2 + p_3) | 3 \rangle \\ &= \langle \hat{1} | \hat{p}_2 | 3 \rangle = -\langle \hat{1} \hat{2} \rangle [\hat{2} 3] = -\langle 12 \rangle [23] \end{aligned}$$

$$(1.56) \quad \langle \hat{P}_{23} 4 \rangle [\hat{P}_{23} \hat{2}] = \langle 4 | \hat{P}_{23} | \hat{2} \rangle = \langle 4 | 3 | 2 \rangle = -\langle 43 \rangle [32] = -\langle 34 \rangle [23]$$

Putting everything together, we get

$$(1.57) \quad A_n^{tree}(1^- 2^- 3^+ \dots n^+) = \frac{\langle 12 \rangle^4}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 45 \rangle \dots \langle n1 \rangle}$$

which gives the Parke-Taylor formula as advertised! The same method can be used also for more intricate helicity configurations, although in that case the final result isn't as compact. The power of on-shell recursion relations aren't just limited to Yang-Mills. In fact, even Gravity amplitudes can be computed by a smart choice of shifts. As fascinating as this discussion is in its own right, we refer the interested reader to [30] and move onto the last ingredient we need before we move from the tree-level story, which is Supersymmetric Yang-Mills theory.

1.4 $\mathcal{N} = 4$ Super Yang-Mills at Tree-Level

As mentioned in the beginning, $\mathcal{N} = 4$ sYM is one of the most well studied theories in high energy theory. While much can be said about the subject and its many motivations, in this section, our purpose is going to be just introducing what we need for later. For starters, let's meet our new fields. In the Yang-Mills example, we only had gluons but in $\mathcal{N} = 4$ sYM the requirement of Supersymmetry means we need to have a richer particle content. In particular, we have a single *CPT* self-conjugate supermultiplet that contains 16 massless states. These include a single +1 helicity gluon, 4 helicity $+\frac{1}{2}$ gluinos, 6 scalars, 4 helicity $-\frac{1}{2}$ gluinos, and a single helicity -1 gluon. To see where these come from, we can follow the treatment of [31] and consider general massless multiplets with helicity $|\lambda| \leq 1$. Then starting with a Clifford vacuum with helicity $\lambda = -1$, one can use the conjugate supercharge $Q^\dagger$ as a helicity raising operator and build

State	Helicity	number of states
$ \Omega_{-1}\rangle$	-1	1
$Q^\dagger \Omega_{-1}\rangle$	$-\frac{1}{2}$	4
$Q^\dagger Q^\dagger \Omega_{-1}\rangle$	0	6
$Q^\dagger Q^\dagger Q^\dagger \Omega_{-1}\rangle$	$\frac{1}{2}$	4
$Q^\dagger Q^\dagger Q^\dagger Q^\dagger \Omega_{-1}\rangle$	1	1

It is worth taking a digression here to consider the field content we would get if our theory had fewer supercharges. For example, let's consider $\mathcal{N} = 3$. Starting again with $|\Omega_{-1}\rangle$, we get

State	Helicity	Number of States
$ \Omega_{-1}\rangle$	-1	1
$Q^\dagger \Omega_{-1}\rangle$	$-\frac{1}{2}$	3
$Q^\dagger Q^\dagger \Omega_{-1}\rangle$	0	3
$Q^\dagger Q^\dagger Q^\dagger \Omega_{-1}\rangle$	$\frac{1}{2}$	1

But then we would also have to add the CPT conjugate, built out of $|\Omega_{-\frac{1}{2}}\rangle$

State	Helicity	Number of States
$ \Omega_{-\frac{1}{2}}\rangle$	$-\frac{1}{2}$	1
$Q^\dagger \Omega_{-\frac{1}{2}}\rangle$	0	3
$Q^\dagger Q^\dagger \Omega_{-\frac{1}{2}}\rangle$	$\frac{1}{2}$	3
$Q^\dagger Q^\dagger Q^\dagger \Omega_{-\frac{1}{2}}\rangle$	1	1

which gives the same field content as $\mathcal{N} = 4$. Thus, the $\mathcal{N} = 3$ and its CPT conjugate give the $\mathcal{N} = 4$ multiplet, they are the same theory (at the on-shell level at least)! We can shelve this discussion for now but it will be relevant for Chapter 3.

Before we talk about computing scattering amplitudes with the fields we have introduced, some comments are in order. In addition to super-symmetry, $\mathcal{N} = 4$ sYM has conformal symmetry. This means the theory has no intrinsic scale, and thus we can't define asymptotic states to scatter in the usual sense. Additionally, since our states are massless, we have to worry about IR-divergences. Similar to the Yang-Mills case, sYM gluons are always degenerate with extra soft or collinear modes. Thus, there are two ways in which people make sense of scattering in $\mathcal{N} = 4$ sYM. One can do dimensional regularization and go to $d = 4 - 2\epsilon$ dimensions to kill the IR-divergences. However, this breaks the conformal symmetry. Another option is giving the scalars vevs in a way that keeps supersymmetry. This is called going to the Coulomb Branch, which also breaks conformal

symmetry but preserves something called dual conformal symmetry of the integrand, which we discuss in Chapter 5. Now that this subtlety is out of the way, let's move onto the amplitudes.

Supersymmetry is a symmetry that relates bosonic degrees of freedom to fermionic ones. Thus, amplitudes with different external states in $\mathcal{N} = 4$ sYM are not independent. They are related to each other through a long list of linear relationships, which are called Supersymmetric Ward identities. Our usage of spinor helicity variables earlier was to hopefully convince the reader that it is often useful to work with variables that make symmetries/constraints manifest. This lesson still rings true here, and thus it is convenient to work with variables that make these SUSY ward identities manifest. Thus, amplitudeologists usually work in what is called the On-shell superspace formalism. This collects all 16 states into an on-shell chiral superfield Ω by introducing four Grassmann variables η^A , where A is an $SU(4)$ index for the R -symmetry that rotates the 4 sets of supersymmetry generators $Q^A, \tilde{Q}^A$ in the theory. The on-shell superfield is then written as

$$(1.58) \quad \Omega = g^+ + \eta_A \lambda^A - \frac{1}{2!} \eta_A \eta_B S^{AB} - \frac{1}{3!} \eta_A \eta_B \eta_C \lambda^{ABC} + \eta_1 \eta_2 \eta_3 \eta_4 g^-$$

where the fields that appear are

$$(1.59) \quad \underbrace{g^+}_{1 \text{ gluon}}, \quad \underbrace{\lambda^A}_{4 \text{ gluinos}}, \quad \underbrace{S^{AB}}_{6 \text{ scalars}}, \quad \underbrace{\lambda^{ABC}}_{4 \text{ gluinos}}, \quad \underbrace{g^-}_{1 \text{ gluon}}.$$

Then when we talk about amplitudes in $\mathcal{N} = 4$ sYM, we refer to the scattering of the superfields Ω_i , and the amplitudes read $A_n(\Omega_1, \Omega_2, \dots, \Omega_n)$ and is a function of the Grassmann variables η_i^A as well as the momenta p_i for each superfield Ω_i . Furthermore, we can take derivatives in the Grassmann variables η_i^A to project out the particles we want from the total superamplitude. For example if we wanted the scattering of two negative helicity gluons and $n-2$ positive helicity gluons, we could do

$$(1.60) \quad A_n(1^- 2^- \dots n^+) = \left(\prod_{A=1}^4 \frac{\partial}{\partial \eta_{1A}} \right) \left(\prod_{B=1}^4 \frac{\partial}{\partial \eta_{2B}} \right) \mathcal{A}_n(\Omega_1, \dots, \Omega_n) \Big|_{\eta_{kC}=0}$$

where the $\eta_{kC} = 0$ at the end means all the leftover Grassmann variables are set to 0 after the derivatives are taken. Same goes for other fields. This is because when expanded in Grassmann variables $A_n(\Omega_1, \dots, \Omega_n)$ is a sum over polynomials in η_i^A of degree $4(k+2)$, where k indexes each

helicity sector through $N^k MHV$. More specifically, we can write

$$(1.61) \quad \mathcal{A}_n = \mathcal{A}_n^{MHV} + \mathcal{A}_n^{NMHV} + \mathcal{A}_n^{N^2MHV} + \dots + \mathcal{A}_n^{\overline{MHV}}$$

The supercharges are

$$(1.62) \quad Q^A = \sum_{i=1}^n [i] \frac{\partial}{\partial \eta_{iA}}, \quad Q^{\dagger A} = \sum_{i=1}^n |i\rangle \eta_{iA}$$

and all the Ward identities can be condensed into

$$(1.63) \quad Q^A A_n = 0, \quad Q^{\dagger A} A_n = 0$$

Since the action of Q^A is multiplicative, amplitudes written in the on-shell superspace formalism carry a delta function with appropriate factors of η_i to make SUSY ward identities manifest. For example the n -point MHV amplitude in $\mathcal{N} = 4$ sYM, is written as

$$(1.64) \quad A_n^{MHV}(123 \dots n) = \frac{\delta^4(P) \delta^8(\tilde{\eta}_1|1\rangle + \tilde{\eta}_2|2\rangle + \dots + \tilde{\eta}_n|n\rangle)}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \dots \langle n-1n \rangle \langle n1 \rangle}$$

where $\delta^8(\tilde{\eta}_1|1\rangle + \tilde{\eta}_2|2\rangle + \dots + \tilde{\eta}_n|n\rangle)$ is often written as $\delta^8(\mathcal{Q})$ and $\delta^4(P), \delta^8(\mathcal{Q})$ together ensure that all supersymmetry constraints are satisfied. Note that in writing $\tilde{\eta}_i$ we have suppressed the $SU(4)$ index A . This just means the $SU(4)$ index is uncontracted and the dimension 8 in the delta function is precisely due to this ($2 \text{ spinor} \times 4 \text{ R-symmetry} = 8$). Finally, we must note that when we attempt on-shell recursion on supersymmetric amplitudes one needs to also shift the Grassmann parameters $\tilde{\eta}_i$. In particular, the “super” version of the BCFW shift from before now reads

$$(1.65) \quad |\hat{1}] = |1] + z|2], \quad |\hat{2}\rangle = |2\rangle - z|1\rangle, \quad \hat{\eta}_1 = \eta_1 + z\eta_2$$

and one needs to perform a Grassmann integral $\int d^4\eta_{P_I}$ for each factorization channel with propagator momenta P_I while performing the sub-amplitude products that appear in the recursion relations. Physically, this arises from the need to sum over all modes that can be exchanged on the internal line I . Whereas only on-shell gluons with helicity $\pm$ were available in pure Yang-Mills, now we have 16 states to worry about. As explained in [30], the Grassmann integral automatically takes care of this. And indeed, one can get formulas for n -point $N^k MHV$ amplitudes in $\mathcal{N} = 4$ sYM using BCFW recursion.

On-shell methods for tree-level amplitudes is a long and interesting subject. This chapter is meant to introduce the necessary ingredients for the formulas and notation that comes up over and over in the work reviewed here, and as a result we have not deeply explored any of the topics we have discussed thus far. The interested reader is encouraged to refer to some excellent reviews on the subject, listed in [30, 32]. Otherwise, we have met all the characters we are going to encounter from the tree-side of the story. So let's see what we need to know from the loop side of things.

CHAPTER 2

Loops, On-Shell Diagrams, and Geometry

The on-shell recursion relations we have discussed in the previous section impose such powerful constraints on tree amplitudes that we can in principle compute them for any n as long as the amplitude has no poles at $|z| \rightarrow \infty$. To set up the recursion relations, we have assumed that the amplitude has no branch cuts, which is true in the case of tree amplitudes. However, the story changes at loop-level, since we now have branch-cuts as well as other complications. Luckily, some of the tools we have already discussed can help us here too. In particular, we will be leveraging the unitarity of the S -matrix. Let's take a step back and remember the definition of unitarity. If an operator S , such as the scattering matrix, is unitary, it satisfies $SS^\dagger = 1$. We can decompose the scattering matrix S as $S = 1 + iT$, which means

$$(2.1) \quad SS^\dagger = (1 + iT)(1 - iT^\dagger) = 1 + TT^\dagger + i(T - T^\dagger) = 1$$

Canceling the identity on both sides, we get the following expression

$$(2.2) \quad i(T^\dagger - T) = TT^\dagger$$

We can then sandwich both sides between the states $\langle f|$ and $|i\rangle$ to translate the operator level equation, which tend to be harder to interpret, into functional one. Namely

$$(2.3) \quad i\langle f|T^\dagger - T|i\rangle = \sum_j \prod_k^j \int \frac{d^3\ell_k}{(2\pi)^3 2E_k} \langle f|T^\dagger|j\rangle \langle j|T|i\rangle$$

where we inserted a complete set of states that equal the identity in the middle. As mentioned above, loop amplitudes have branch cuts, meaning the amplitude now has non-zero discontinuity. Thus, the discontinuity of the amplitude A across a branch cut is given by its imaginary part, i.e. $Disc(M) = 2Im(M)$. Then the statement above relates the discontinuity of the L loop amplitude

to products of lower loop ones. Schematically, this is

$$(2.4) \quad \text{Disc}(T) = T^\dagger T$$

The operator T describes forward scattering, so we can write

$$(2.5) \quad \text{Disc}(M(i \rightarrow f)) = \sum_{j=2}^{\infty} \int d\Phi_j M(i \rightarrow j) M^\dagger(j \rightarrow f)$$

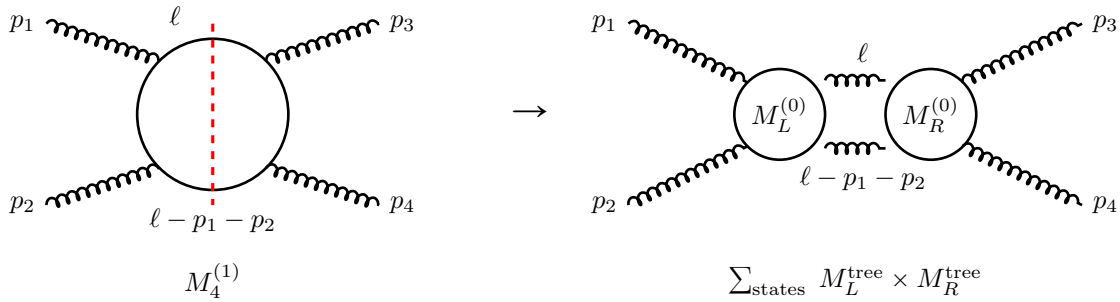
where $d\Phi_j$ is the phase space measure over j particle final states with total momentum equaling the incoming momenta P_i . Then expanding $M(i \rightarrow f)$ perturbatively, we have

$$(2.6) \quad \text{Disc}(M_{if}^{(0)}) = 0$$

as tree amplitudes have no branch cuts. For higher loop, we start to get non-trivial results, such as

$$(2.7) \quad \text{Disc}(M_{if}^{(1)}) = \int d\Phi_2 M_{i2}^{(0)} M_{2f}^{(0)\dagger}.$$

This equation tells us that the discontinuity of a one loop amplitude can be written as a product of tree amplitudes. In particular, the right hand-side corresponds to putting the internal loop propagators $\ell, \ell - p_1 - p_2$ on-shell such that $\ell^2 = (\ell - p_1 - p_2)^2 = 0$. Diagrammatically, this looks like the following.



This is called a 2-particle unitarity cut. At the integral level, performing this unitarity cut amounts to replacing the propagators with a delta function that enforces the on-shell condition, namely

$$(2.8) \quad \int d^D \ell \frac{1}{\ell^2 (\ell - p_1 - p_2)^2} \rightarrow \int d^D \ell \delta(\ell) \delta(\ell - p_1 - p_2).$$

The pattern continues at higher loop order, where we get

$$(2.9) \quad \text{Disc}(M_{if}^{(2)}) = \int d\Phi_2 (M_{i2}^{(1)} M_{2f}^{(0)\dagger} + M_{i2}^{(0)} M_{2f}^{(1)\dagger}) + \int d\Phi_3 M_{i3}^{(0)} M_{3f}^{(0)\dagger}$$

Now there are two different types of terms that contribute to the 2-loop amplitude's discontinuity. First one is the product of a 1-loop amplitude and a tree, which comes from a 2-particle cut. The second one is a product of two tree-level amplitudes that come from a 3-particle cut.

The punchline here is that the discontinuity of an L -loop amplitude can be written in terms of lower loop results. But how does this help us construct the amplitude itself? If we had to work non-perturbatively, we would use dispersion relations. Assuming the amplitude A is analytic in the upper half plane, Kramers-Konig relations tell us that we can set up the following dispersion relation

$$(2.10) \quad A(s) = \frac{1}{2\pi i} \int \frac{\text{Disc}(A(s'))}{s' - s} ds'$$

where the bounds go from where a branch cut begins to infinity. One caveat in addition to analyticity is that for this to work $A(s)$ has to be well behaved at $s \rightarrow \infty$. However, this equation is not very practical by itself if we want to apply it to the gauge theories we talked about in the previous chapter. This is due to numerous issues, such as needing subtractions to determine pieces that aren't discontinuous on the chosen variable and regulating the IR divergences we talked about, which come up stronger than ever at loop-level. Luckily, we can leverage the perturbative expansion to get something more practical. Based on the idea that unitarity cuts impose constraints on loop integrands, a powerful method was introduced in 1994 by Bern, Dixon, Dunbar, and Kosower [2, 3] to find loop amplitudes in gauge theory using their cuts and a suitable basis of integrals. The now broadened set of tools based on this idea are known in the amplitudes field as modern unitarity methods.

2.1 Unitarity Methods

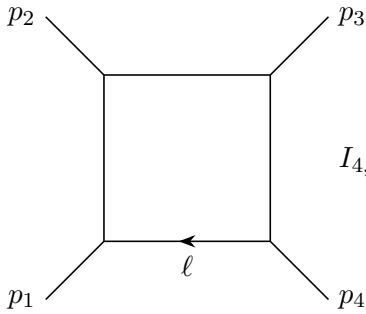
Let's think about the general structure of a loop diagram. For L loops, the kinematic part looks like

$$(2.11) \quad I = \int \prod_n^L \frac{d^D \ell_n}{(2\pi)^D} \prod_j \frac{1}{q_j(\ell, p)^2 + i\epsilon}$$

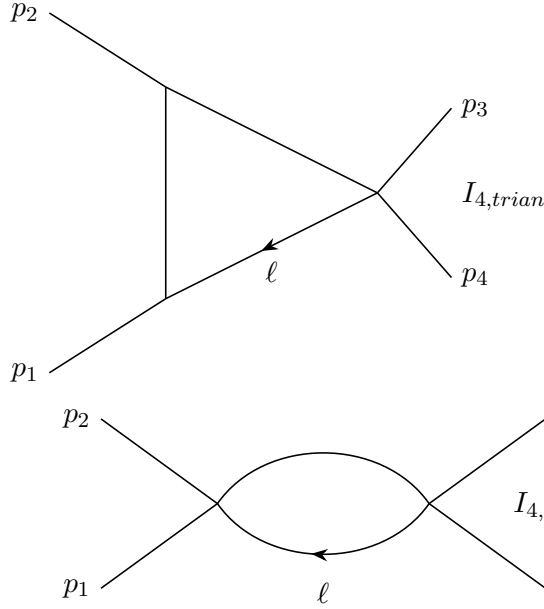
where n indexes over each loop, j goes over internal propagators, and $q_j(\ell, p)$ denotes the momentum flowing through each propagator. Normally we would include a m_j^2 after this but here we are interested in massless gauge theories, which we can afford to do without IR problems since we are keeping the number of spacetime dimensions unfixed for now. Now when we compute the full L -loop amplitude, we add all such topologically distinct diagrams together, weighed by the appropriate symmetry factors. In the planar limit of $\mathcal{N} = 4$ sYM, where $N \rightarrow \infty$, we are actually able to combine all the integrands into one. So let's set aside the problem of actually performing the integral for a minute and first focus on constructing the integrand. We add everything together and the result is a rational function that has all the propagators in the denominator. Then, the numerator of the combined object is what we need to determine in order to construct the integrand. As its name suggests, the unitarity method uses the unitarity constraints we discussed above to fix this numerator. Let's illustrate this with an example. Consider the 1-loop MHV amplitude $A_4^{(1)}(1^- 2^- 3^+ 4^+)$ in planar $\mathcal{N} = 4$ Super-Yang Mills. In terms of the numerator we wish to construct, it takes the form

$$(2.12) \quad A_4^{(1)}(1^- 2^- 3^+ 4^+) = -i A_4^{tree} \int \frac{d^{4-2\epsilon} \ell}{(2\pi)^{4-2\epsilon}} \frac{N}{\ell^2 (\ell - p_1)^2 (\ell - p_1 - p_2)^2 (\ell + p_4)^2}$$

To use the unitarity method, we also need a set of basis integrals. Of course for more complicated cases we need to be smart about how to choose a basis. For now, the object at hand is simple enough that we can consider everything. Namely, our basis consists of scalar bubbles, triangles, and boxes, shown below.



$$I_{4,box}(p_1, p_2, p_3, p_4) = \int \frac{d^D \ell}{(2\pi)^D} \frac{1}{\ell^2 (\ell + p_1)^2 (\ell + p_1 + p_2)^2 (\ell - p_4)^2}$$



$$I_{4,triangle}(p_1, p_2, p_3 + p_4) = \int \frac{d^D \ell}{(2\pi)^D} \frac{1}{\ell^2 (\ell + p_1)^2 (\ell + p_1 + p_2)^2}$$

$$I_{4,bubble}(p_1 + p_2, p_3 + p_4) = \int \frac{d^D \ell}{(2\pi)^D} \frac{1}{\ell^2 (\ell + p_1 + p_2)^2}$$

Then the procedure is as follows. Once we write $A_4^{(1)}(1^- 2^- 3^+ 4^+) = c_1 I_{box} + c_2 I_{bubble} + c_3 I_{triangle}$, all we have to do is compute the coefficients c_i . And to do this, we can systematically impose the unitarity cuts we have discussed above. Usually, we also have to include rational terms that are not detectable by these cuts but these are absent in the case of $\mathcal{N} = 4$ sYM. We start by systematically “cutting” internal propagators, which amount to putting them on-shell, such that $\frac{1}{\ell^2} \rightarrow \delta(\ell^2)$. We can start with the maximal cut, $\ell^2 = (\ell + p_1)^2 = (\ell + p_1 + p_2)^2 = (\ell - p_4)^2 = 0$. The triangle and the bubble diagrams both vanish and we get

$$(2.13) \quad c_{box} = \frac{1}{2} \sum_{\ell^*} A^{\text{tree}, (1)} A^{\text{tree}, (2)} A^{\text{tree}, (3)} A^{\text{tree}, (4)} \Big|_{\ell=\ell^*},$$

where ℓ^* denote the solutions to the maximal cut condition and $A^{(i)}$ represent the 3–point amplitude that corresponds to the i –th corner of the box. Evaluating the non-vanishing helicities, one finds

$$(2.14) \quad c_{box} = -i s t A^{\text{tree}}(1^- 2^- 3^+ 4^+)$$

Imposing the triple and double cut to isolate the triangle and bubble part respectively, one gets $c_2 = c_3 = 0$. Note that this is only true for $\mathcal{N} = 4$ sYM and that in pure Yang-Mills, we would get a non-zero coefficient for the triangle. Putting everything together, we get

$$(2.15) \quad A_4^{(1)}(1^- 2^- 3^+ 4^+) = -i s t A^{\text{tree}}(1^-, 2^-, 3^+, 4^+) I_4^{\text{box}}(s, t)$$

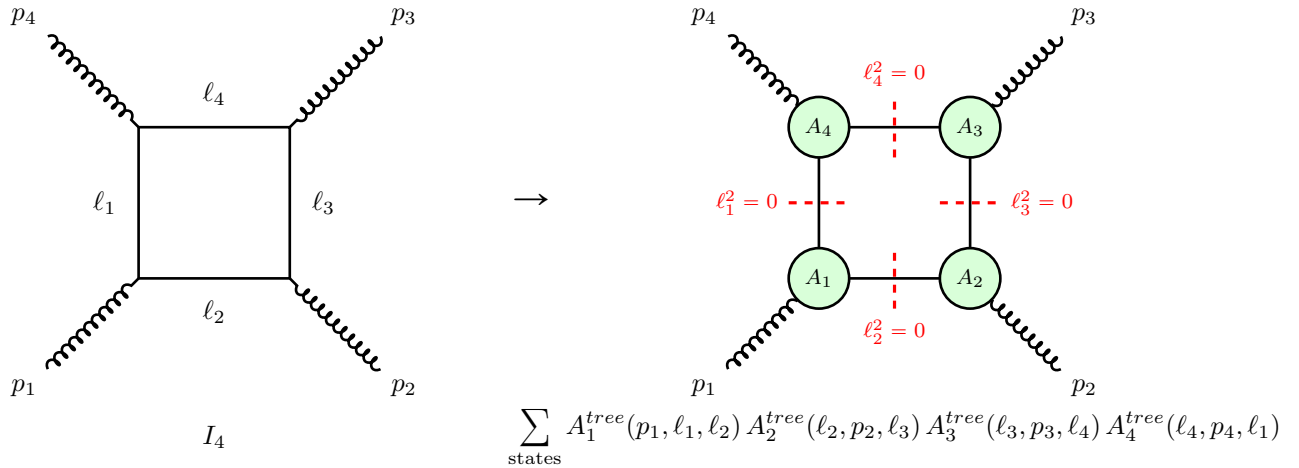
Evaluating this in $d = 4$ gives IR divergences, so doing $d = 4 - 2\epsilon$ instead we get

$$(2.16) \quad A_4^{(1)} = A^{\text{tree}} \left(\frac{\Gamma_{\text{cusp}}^{(1)}}{\epsilon^2} \left[\left(\frac{-s}{\mu^2} \right)^{-\epsilon} + \left(\frac{-t}{\mu^2} \right)^{-\epsilon} \right] - \left[\ln^2 \left(\frac{-s}{-t} \right) + \pi^2 \right] \right) + \mathcal{O}(\epsilon),$$

The quantity $\Gamma_{\text{cusp}}^{(1)}$ is the cusp anomalous dimension at one-loop and the more general object $\Gamma_{\text{cusp}} = \sum g^{2L} \Gamma_{\text{cusp}}^{(L)}$ is going to play a key role in our discussion starting at chapter 5 onwards. But for now, our purpose is just to briefly mention unitarity methods. While this example is quite simple and doesn't demonstrate the full power of unitarity methods, one of our key players for the next chapter, namely on-shell diagrams, already shows up here. In the next section, we explain how.

2.2 On-Shell Diagrams and Positive Geometries

The story so far has involved building computational tools for the efficient computation of scattering amplitudes, leveraging the assumed unitarity and locality of the underlying quantum field theory. However, as we will see, there is more to the story than merely making computations efficient. Scattering amplitudes have a rich mathematical structure, and some of this structure was only made more visible with these new methods. This is an example of what has been dubbed a virtuous cycle in the community [21], where modern methods for amplitude computations leads to the discovery of new underlying structure, and this underlying structure in turn can be used to discover new computational tools. The main body of this thesis is deeply connected to one such structure, called the Positive Grassmannian. As a gentle entry point, consider the quadruple cut we have used for the box in the previous section, which said to be a maximal cut (all propagators put on-shell).



This is an example of an On-Shell diagram. In its most broad definition (from a physics point of view at least), an on-shell diagram is a graph formed by gluing 3-point vertices together on a leg, conserving momentum at each merged leg. In particular, the fundamental building blocks are the following two objects

$$(2.17) \quad \begin{array}{c} 1 \\ | \\ \bullet \\ / \quad \backslash \\ 2 \quad 3 \end{array} \quad \tilde{\lambda}_1 \sim \tilde{\lambda}_2 \sim \tilde{\lambda}_3, \quad \begin{array}{c} 1 \\ | \\ \circ \\ / \quad \backslash \\ 2 \quad 3 \end{array} \quad \lambda_1 \sim \lambda_2 \sim \lambda_3$$

and thus every theory that has 3-point interactions admits these on-shell diagrams. Then, one can glue them as advertised

$$\begin{array}{c} 2 \quad 3 \\ \backslash \quad / \\ \circ \text{---} \bullet \\ / \quad \backslash \\ 1 \quad 4 \end{array} = \int d^4 \tilde{\eta}_1 \dots d^4 \tilde{\eta}_4 \int \frac{d^2 \lambda_{\ell_1} d^2 \tilde{\lambda}_{\ell_1}}{\text{GL}(1)} \dots \frac{d^2 \lambda_{\ell_2} d^2 \tilde{\lambda}_{\ell_2}}{\text{GL}(1)} A_3(p_1, p_2, -P) A_3(P, p_3, p_4).$$

where P is fixed to be on-shell and $P = p_1 + p_2$. Let's see how this generalizes when we glue a larger number of vertices.

2.2.1 Building on-shell diagrams

On-shell diagrams are given by the product of three-point amplitudes, multiplied by a Jacobian that stems from eliminating λ 's and η 's of internal on-shell legs in the momentum and super-momentum conservation delta functions. The final result depends on external kinematic data and possibly on free unfixed parameters.

Putting an internal edge on-shell gives one constraint equation, from which we can solve for the internal momenta in terms of the external kinematic data. Denoting the difference between the number of constraints and the internal degrees of freedom by n_δ , we get three distinct cases. $n_\delta = 0$ where we can express all internal momenta in terms of external data, $n_\delta > 0$ which places additional conditions on the external data, and $n_\delta < 0$ which leaves us with unsolved parameters that the amplitude will depend on. The simplest example of $n_\delta = 0$, is the four-point one-loop box

diagram,

$$(2.18) \quad \begin{array}{c} \begin{array}{ccc} & 2 & 3 \\ & \bullet & \circ \\ \ell_1 & \ell_2 & \\ \circ & & \circ \\ \ell_4 & \ell_3 & \\ & \bullet & \\ 1 & & 4 \end{array} \end{array} \quad \begin{array}{l} \ell_1 = \frac{\langle 23 \rangle}{\langle 13 \rangle} \lambda_1 \tilde{\lambda}_2, \quad \ell_3 = \frac{\langle 14 \rangle}{\langle 13 \rangle} \lambda_3 \tilde{\lambda}_4, \\ \ell_2 = \frac{\langle 12 \rangle}{\langle 13 \rangle} \lambda_3 \tilde{\lambda}_2, \quad \ell_4 = \frac{\langle 34 \rangle}{\langle 13 \rangle} \lambda_1 \tilde{\lambda}_4. \end{array}$$

We can see that each vertex imposes collinearity conditions on λ or $\tilde{\lambda}$. For example, the holomorphic part of momentum ℓ_2 is given by the λ_3 spinor (from the upper right white vertex) and the anti-holomorphic part is given by the $\tilde{\lambda}_2$ spinor (from the upper left black vertex). This itself solves for 3 of 4 on-shell conditions on internal on-shell propagators, $\ell_1^2 = \ell_2^2 = \ell_3^2 = 0$. The constraint from the final propagator $\ell_4^2 = 0$ fixes the overall constant in ℓ_2 to be $\langle 12 \rangle / \langle 13 \rangle$. In $\mathcal{N}=4$ SYM theory we do not need to specify helicities of internal and external legs. For $\mathcal{N}<4$ SYM theory we have to specify them (or equivalently helicity flows denoted by arrows) to fully define an on-shell diagram. Solutions for internal momenta are the same regardless of the number of supersymmetries or theory (it can also be gravity on-shell diagrams or other). However, the actual on-shell function associated with the on-shell diagram is sensitive to these details. In particular, the on-shell function in $\mathcal{N}=4$ SYM for (2.18) is

$$(2.19) \quad \Omega = \int d^4 \tilde{\eta}_1 \dots d^4 \tilde{\eta}_4 \int \frac{d^2 \lambda_{\ell_1} d^2 \tilde{\lambda}_{\ell_1}}{\text{GL}(1)} \dots \frac{d^2 \lambda_{\ell_4} d^2 \tilde{\lambda}_{\ell_4}}{\text{GL}(1)} \\ \times \left\{ A_3(1, \ell_1, \ell_4) A_3(2, \ell_1, \ell_2) A_3(3, \ell_2, \ell_3) A_3(4, \ell_3, \ell_4) \right\} = \frac{\delta^4(P) \delta^8(Q)}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 41 \rangle}.$$

All integrals are evaluated on the delta functions using

$$(2.20) \quad \int dx f(x) \delta(x - x_0) = f(x_0)$$

(see [13] for more details). All poles in the on-shell function correspond to sending one of the internal momenta of the diagram to zero as evident from (2.18). Taking the residue on this pole erases an edge and the result corresponds to a new on-shell diagram. For example, sending $\langle 12 \rangle = 0$

implies $\ell_2 = 0$ and we get

$$(2.21) \quad \begin{array}{c} 2 \\ \diagdown \\ \circ \\ \diagup \\ 1 \end{array} \text{---} \begin{array}{c} \bullet \\ \diagup \\ 3 \\ \diagdown \\ 4 \end{array} = \frac{\delta(\langle 12 \rangle) \delta^4(P) \delta^8(Q)}{\langle 23 \rangle \langle 34 \rangle \langle 41 \rangle}.$$

This diagram is an example of $n_\delta > 0$ where the extra constraint on the external kinematics $\langle 12 \rangle = 0$ is evident from the delta function $\delta(\langle 12 \rangle)$ in the on-shell form (6.22). The condition on external kinematics can be also seen directly in the diagram where legs 1, 2 meet in the same white vertex. At the same time, legs 3, 4 meet in the same black vertex which imposes $[34] = 0$ – which is indeed a consequence of $\langle 12 \rangle = 0$ for four-point kinematics. The simplest example of an on-shell diagram with $n_\delta < 0$, is the four-point double-box diagram,

$$(2.22) \quad \begin{array}{c} 2 \\ \diagdown \\ \bullet \\ \diagup \\ 1 \end{array} \begin{array}{c} \ell_1 \\ \diagdown \\ \circ \\ \diagup \\ \ell_5 \end{array} \begin{array}{c} \ell_2 \\ \diagdown \\ \circ \\ \diagup \\ \ell_3 \end{array} \begin{array}{c} \ell_3 \\ \diagdown \\ \bullet \\ \diagup \\ \ell_4 \end{array} \begin{array}{c} 3 \\ \diagdown \\ \bullet \\ \diagup \\ 4 \end{array} \begin{array}{c} \ell_4 \\ \diagdown \\ \circ \\ \diagup \\ \ell_6 \end{array} \begin{array}{c} z\tilde{\lambda}_3\lambda_4 \\ \diagdown \\ \circ \\ \diagup \\ \ell_6 \end{array}$$

$$\begin{aligned} \ell_1 &= \frac{\langle 23 \rangle + z\langle 24 \rangle}{\langle 13 \rangle + z\langle 14 \rangle} \lambda_1 \tilde{\lambda}_2, & \ell_4 &= \frac{\langle 14 \rangle}{\langle 13 \rangle + z\langle 14 \rangle} \hat{\lambda}_3 \hat{\lambda}_4, \\ \ell_2 &= \frac{\langle 12 \rangle}{\langle 13 \rangle + z\langle 14 \rangle} \hat{\lambda}_3 \tilde{\lambda}_2, & \ell_5 &= \frac{\langle 34 \rangle}{\langle 13 \rangle + z\langle 14 \rangle} \lambda_1 \hat{\lambda}_4, \\ \ell_3 &= \hat{\lambda}_3 \tilde{\lambda}_3, & \ell_6 &= \lambda_4 \hat{\lambda}_4. \end{aligned}$$

where $\hat{\lambda}_4 = \tilde{\lambda}_4 + z\tilde{\lambda}_3$ and $\hat{\lambda}_3 = \lambda_3 - z\lambda_4$. This diagram can be thought as a heptacut of the two-loop four-point amplitude in the $\mathcal{N}=4$ SYM theory, with z being the last parameter in ℓ_1, ℓ_2 not localized by on-shell conditions from the cut. Alternatively, we can also interpret the diagram as a BCFW shift of the one-loop four-point amplitude (on legs 3, 4) with z being the BCFW shift parameter. The on-shell function is given by

$$(2.23) \quad \Omega = \int \frac{dz \delta^4(P) \delta^8(Q)}{z\langle 12 \rangle (\langle 23 \rangle + z\langle 24 \rangle) \langle 34 \rangle \langle 41 \rangle}.$$

Each pole corresponds to erasing an edge in the diagram but not all edges are erasable (hence we have five poles rather than seven). The residue on the pole $z = 0$ corresponds the diagram (2.18) already encountered. Now that we have some intuition for these objects on the physics side, let's dive a little deeper into the mathematical structure.

2.2.2 The Positive Grassmannian

The connection between on-shell diagrams and positive geometries is a very deep one and like many other deep and beautiful subjects, we will only review the surface level story in this chapter. We will closely follow the treatment of [13] here. The main goal of this section will be to set-up the language and motivation we need to make sense of the works in chapters 3 and 4. Let's start with setting up some mathematical terminology.

Let's say someone gives us a vector space $\mathbf{V}$. The Grassmannian $G(k, \mathbf{V})$ is defined as the set of k -dimensional subspaces of the vector space $\mathbf{V}$. For our purposes, $\mathbf{V}$ is either $\mathbb{R}^n$ or $\mathbb{C}^n$ and we will refer the Grassmannian in either case as $G(k, n)$, where it is the space of k -planes passing through the origin. We can define any k -plane by gathering k vectors that live in $\mathbb{C}^n$ such that they span the plane. Finally, we can assemble these planes into a $(k \times n)$ matrix, which we will call C . But wait, any linear combination of these vectors will give us the same plane! So we can act with $GL(k)$ transformations on C and get the same plane, meaning we have some "gauge redundancy" that we need to fix. To do so, we can choose any k columns of C to be the $k \times k$ identity matrix. This gives us a coordinate chart for the Grassmannian $G(k, n)$, and we can cover the entire Grassmannian by repeating this procedure for every distinct set of k columns and taking the union. As an example, a point in $G(2, 4)$ can be represented by

$$(2.24) \quad C = \begin{pmatrix} 1 & 0 & a & b \\ 0 & 1 & c & d \end{pmatrix}$$

where $(a, b, c, d) \in \mathbb{C}^4$ are affine coordinates on this chart. The $k \times k$ minors of this matrix, which live in $\mathbb{P}^{\binom{n}{k}-1}$, are called Plücker coordinates. Since any k -vector can be written in terms of k linearly independent vectors, the minors of these matrices satisfy relations called Plücker relations

$$(2.25) \quad (i_1 \dots i_{k-1} j_1)(j_2 \dots j_{k+1}) + \dots + (-1)^{k-1} (i_1 \dots i_{k-1} j_{k+1})(j_1 \dots j_k) = 0$$

For example, the C matrix we have above gives the single Plücker relation

$$(2.26) \quad (12)(34) - (13)(24) + (14)(23) = 0.$$

These relations define an embedding $G(k, n) \rightarrow \mathbb{P}^{\binom{n}{k}-1}$ called the Plücker embedding. As we shall see in a moment, the Schouten relations we have talked about earlier are an example of these Plücker relations.

For any k -plane, we can find a $(n - k)$ -plane that is orthogonal, represented by $C^\perp$ and is defined by

$$(2.27) \quad C^\perp \cdot C = 0$$

This can be represented by a $(n - k) \times n$ matrix and its minors are determined by the minors of C . If we gauge fix C as

$$(2.28) \quad C = (\mathbb{1}_{k \times k} | c_{ab}).$$

Then, $C^\perp$ can be written as

$$(2.29) \quad C^\perp = (-c_{ba} | \mathbb{1}_{(n-k) \times (n-k)}).$$

So far we have not introduced any restrictions on the signs of the minors. As we will see, doing so is going to make the story much richer. The region in the Grassmannian where all Plücker coordinates are non-negative is called the totally non-negative Grassmannian, and it is defined as follows

$$(2.30) \quad G_+(k, n) := \{C \in G(k, n) : p_I(C) \geq 0 \quad \forall I \in \binom{[n]}{k}\}.$$

For a given C that lives in $G_+(k, n)$, we can define a set called a Matroid (which is an abstract generalization of linear independence) as

$$(2.31) \quad M := \{I \in \binom{[n]}{k}, \quad p_I(C) \neq 0\}$$

Matroids associated with non-negative Grassmannian are called Positroids. For a given positroid, we can look at elements in G_+ such that the Plückers indexed by $I \in M$ are positive and those associated with indices outside of M vanish, namely

$$(2.32) \quad S_M := \{C \in G_+(k, n) : \quad p_I(C) > 0 \text{ for } I \in M, \quad p_I(C) = 0 \text{ for } I \notin M\}$$

This is called a positroid cell and collection of such cells gives a matroid stratification of the non-negative part of the Grassmannian. We refer to the top-dimensional positroid cell as the top-cell. This object has, for all its ordered minors, $p_I(C) > 0$. So the non-negativity condition we had before is replaced with strict positivity. Thus, this object covers what is called the **positive Grassmannian**. Unfortunately, $G_+(k, n)$ is used to denote both the positive Grassmannian and the non-negative Grassmannian in physics literature. We will do the same here, but hopefully the distinction will be clear from context. Starting with the top-cell, codimension-1 boundaries are reached by setting single Plücker variables to 0. Then we can reach any cell that lives in the positive stratification of the top-cell by iteratively setting Plücker variables to 0.

By this point, it might feel like we got a little far from Physics. Before we get back to it, we need to introduce one more technology for which the preceding discussion has hopefully laid an adequate groundwork for, and this is the canonical form. Given a top-cell C that parametrizes a region of the $G_+(k, n)$, we can define a differential form called the Canonical Form on the positive Grassmannian as

$$(2.33) \quad \Omega(C) = \frac{d^{k \times n} C}{\text{vol}(GL(k))} \frac{1}{(1 \cdots k) \cdots (n \cdots k-1)}$$

If elements of C are parametrized by positive variables $\alpha_i > 0$ for $i = 1, \dots, k(n-k)$, then the canonical form becomes

$$(2.34) \quad \Omega(C) = \bigwedge_{i=1}^{n(n-k)} d \log \alpha_i$$

Okay, now we have introduced sufficient technology to see how all this relates to on-shell diagrams.

2.2.3 Grassmannian Formula for On-Shell Functions

Let's go back to our spinor-helicity variables. For an n -particle process, the set of spinors $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$ can be described by a $(2 \times n)$ matrix

$$(2.35) \quad \lambda = \begin{pmatrix} \lambda_1^1 & \lambda_2^1 & \cdots & \lambda_n^1 \\ \lambda_1^2 & \lambda_2^2 & \cdots & \lambda_n^2 \end{pmatrix}$$

Now if we take each row of this separately, we can view this as two n -dimensional vectors. Furthermore, they represent a 2-plane in n -dimensions, and Lorentz transformations that act on

the spinors leave this plane invariant. Same goes for the $\tilde{\lambda}$ spinors. We can already see hints of a connection to the Grassmannians we have introduced in the previous section. But wait, there is more! Recall that the momentum conservation equation is

$$(2.36) \quad \sum_i \lambda_i^\alpha \tilde{\lambda}_i^{\dot{\alpha}} = 0$$

Geometrically, this is just the statement that the 2-plane λ is orthogonal to the 2-plane $\tilde{\lambda}$. Splendid, now let's go back to our on-shell diagrams.

Recall that on-shell diagrams are made out of 3-point building blocks. These are the two (complex momenta) solutions to the equations $p_1 + p_2 + p_3 = 0$ and $p_1^2 = p_2^2 = p_3^2 = 0$, represented by black and white vertices.

$$(2.37) \quad \begin{array}{c} 1 \\ | \\ \bullet \\ / \quad \backslash \\ 2 \quad 3 \end{array} = \frac{\delta(P)\delta(Q)}{\langle 12 \rangle \langle 23 \rangle \langle 31 \rangle}, \quad \begin{array}{c} 1 \\ | \\ \circ \\ / \quad \backslash \\ 2 \quad 3 \end{array} = \frac{\delta(P)\delta(\tilde{Q})}{[12][23][31]}$$

Let's start with the white vertex, which will be easily generalizable to black vertices. A white vertex has $\lambda_1 \sim \lambda_2 \sim \lambda_3$. Now let's try to express this in terms of the orthogonal planes we introduced earlier. But wait a minute, 2-dimensional planes can't be orthogonal in 3 dimensions. So one of the $\lambda, \tilde{\lambda}$ planes has to be a 1-plane. Let's choose this to be the λ plane. Since momentum conservation requires

$$(2.38) \quad \lambda_1 \tilde{\lambda}_1 + \lambda_2 \tilde{\lambda}_2 + \lambda_3 \tilde{\lambda}_3 = 0$$

λ can be written as

$$(2.39) \quad \lambda = \begin{pmatrix} [23] & [31] & [12] \\ 0 & 0 & 0 \end{pmatrix}$$

Since $\lambda_1 \sim \lambda_2 \sim \lambda_3$, all other solutions live in the same plane spanned by these 3 vectors. Okay, so let's get back to our white-vertex, which has the on-shell function

$$(2.40) \quad \begin{array}{c} 1 \\ | \\ \circ \\ / \quad \backslash \\ 2 \quad 3 \end{array} = \frac{\delta^4(P)\delta^4(\tilde{Q})}{[12][23][31]}$$

We can re-write the momentum conservation in terms of an auxiliary 1-plane $C \in G_+(1, 3)$ such that C is orthogonal to $\tilde{\lambda}$ and contains λ . Since the Grassmann parameters $\tilde{\eta}^I$ in the supercharge conserving delta function have λ_i^1 as its coefficients, we can write $\delta([23]\tilde{\eta}_1 + [31]\tilde{\eta}_2 + [12]\tilde{\eta}_3) = \delta^{1 \times 4}(C \cdot \tilde{\eta})$. Then the on-shell function can be written as

$$(2.41) \quad \mathcal{A}_3^{(1)} = \int \frac{d^{1 \times 3} C}{\text{vol}(GL(1))} \frac{\delta^{1 \times 4}(C \cdot \tilde{\eta})}{(1)(2)(3)} \delta^{1 \times 2}(C \cdot \tilde{\lambda}) \delta^{2 \times 2}(\lambda \cdot C^\perp).$$

To recover the previous expression from this, we can fix the entries of C using $C \cdot \tilde{\lambda} = 0$ and solve $\delta(\lambda \cdot C^\perp) \delta(C \cdot \tilde{\eta})$ on the support of this solution to obtain the momentum conservation delta functions. We can do the same for the black vertex solution, which has $\tilde{\lambda}_1 \sim \tilde{\lambda}_2 \sim \tilde{\lambda}_3$ and is given by

$$(2.42) \quad \begin{array}{c} 1 \\ | \\ \bullet \\ / \quad \backslash \\ 2 \quad 3 \end{array} = \frac{\delta(P)\delta(Q)}{\langle 12 \rangle \langle 23 \rangle \langle 31 \rangle}$$

Now we have an auxiliary 2-plane $C \in G(2, 3)$ and write

$$(2.43) \quad \mathcal{A}_3^{(2)} = \int \frac{d^{2 \times 3} C}{\text{vol}(GL(2))} \frac{\delta^{2 \times 4}(C \cdot \tilde{\eta})}{(12)(23)(31)} \delta^{2 \times 2}(C \cdot \tilde{\lambda}) \delta^{2 \times 1}(\lambda \cdot C^\perp),$$

This formula generalizes to any on-shell function we can as an amalgamation of such black and white vertices. In particular, each on-shell diagram gives a parametrization of a cell in the Grassmannian. Furthermore, in planar $\mathcal{N} = 4$ sYM, the corresponding on-shell function $f^{(k)}$ can be written as a residue of the top-form, namely

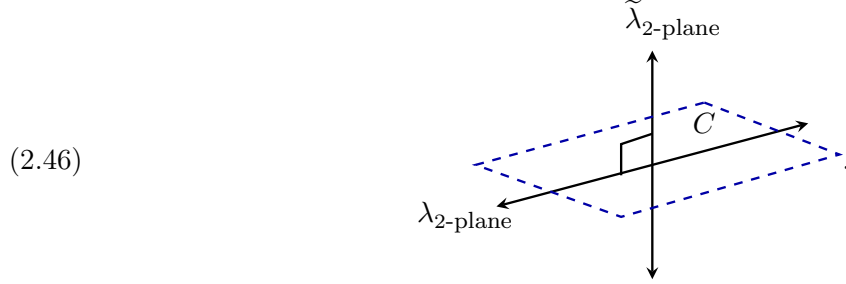
$$(2.44) \quad f^{(k)} = \oint \frac{d^{k \times n} C}{\text{vol}(GL(k))} \frac{\delta^{k \times 4}(C \cdot \tilde{\eta})}{(1 \cdots k) \cdots (n \cdots k-1)} \delta^{k \times 2}(C \cdot \tilde{\lambda}) \delta^{2 \times (n-k)}(\lambda \cdot C^\perp)$$

where the parametrization of C is given by the on-shell diagram in question. Let's now see in detail how this n -point generalization works.

Using a C matrix that lives in $G(k, n)$, we can re-write the n -point momentum conservation equation, namely $\delta^4(\lambda_1 \tilde{\lambda}_1 + \cdots + \lambda_n \tilde{\lambda}_n)$, as follows:

$$(2.45) \quad \delta(C \cdot Z) = \delta^{((n-k) \times 2)}(C^\perp \cdot \lambda) \delta^{(k \times 2)}(C \cdot \tilde{\lambda}) \delta^{(k \times \mathcal{N})}(C \cdot \tilde{\eta}),$$

where $C^\perp$ is an orthogonal complement to C satisfying $C^\perp \cdot C = 0$. We can think about this geometrically in the n -dimensional “particle space” [8]. In this space λ and $\tilde{\lambda}$ are 2-planes which are orthogonal to each other. We define a k -dimensional plane C which contains λ and is orthogonal to $\tilde{\lambda}$,



The C matrix is then used to linearize the momentum conservation but conditions (2.45) are in fact stronger, $C \cdot \tilde{\lambda} = C^\perp \cdot \lambda = 0$ impose $2n$ conditions on the parameters of C . This reproduces four conditions from the momentum conservation and leaves us with additional $2n-4$ additional conditions. Depending on the exact dimensionality of the C -matrix, we can be left with some free parameters in C (matrix is not completely fixed) or oppositely in order to satisfy (2.45) we need to impose additional conditions on external kinematics. Note that the last delta function in (2.45) involves fermionic variables $\tilde{\eta}$ and it does not impose any conditions on C – we should just think about it as a polynomial in $\tilde{\eta}$.

Each on-shell diagram provides a particular parametrization of the C -matrix using edge variables α_i , or face variables f_i . For planar diagrams, if the variables are real and take definite signs, all the main $(k \times k)$ minors are positive, and the C matrix represents a cell in the positive Grassmannian $G_+(k, n)$, as advertised. The connection between cells in positive Grassmannian and plabic graphs (name used for on-shell diagrams in the math literature) was found by Alexander Postnikov in [33], and later the connection to physics and scattering amplitudes was discovered in [13]. Here, we just shortly review the key ingredients.

For a given on-shell diagram, we choose an *orientation* by assigning arrows to all edges such that each black vertex has two incoming and one outgoing leg, while each white vertex has one incoming and two outgoing edges. Then we assign edge variables to all edges, fixing in each vertex one variable to one. It does not matter how we exactly choose the position of edge variables (fixing the $GL(1)$ redundancy) but in each vertex there should be at least one edge with an edge variable.

The diagram has k sources, external legs with incoming arrows, and $n-k$ sinks, external legs with outgoing arrows. Then the element of the $(k \times n)$ matrix is given by the following prescription:

$$(2.47) \quad C_{\alpha a} = \sum_{\Gamma_{\alpha \rightarrow a}} \prod_j \alpha_j.$$

For each path from a source α to a sink a we take a product of all edge variables along the path and sum over all paths. The element $C_{\alpha\alpha} = 1$ while $C_{\alpha\alpha'} = 0$ for the path between two different sources. For a simple, one-loop four-point box diagram we have

$$(2.48) \quad \begin{array}{c} \begin{array}{ccc} & 2 & \\ & \swarrow & \searrow \\ & \bullet & \circ \\ \alpha_1 \uparrow & \alpha_2 \leftarrow & \rightarrow \alpha_3 \\ \circ & \alpha_4 \rightarrow & \bullet \\ \swarrow & \searrow & \\ 1 & & 4 \end{array} \end{array} \Rightarrow C = \begin{pmatrix} 1 & \alpha_1 & 0 & \alpha_4 \\ 0 & \alpha_2 & 1 & \alpha_3 \end{pmatrix}, \quad C^\perp = \begin{pmatrix} -\alpha_1 & 1 & -\alpha_2 & 0 \\ -\alpha_4 & 0 & -\alpha_3 & 1 \end{pmatrix}.$$

Fixing the signs of edge variables to be

$$(2.49) \quad \alpha_1 > 0, \quad \alpha_2 > 0, \quad \alpha_3 > 0, \quad \alpha_4 < 0,$$

guarantees that all (2×2) minors are positive and the C -matrix represents a top cell of $G_+(2, 4)$. The on-shell function associated with the same on-shell diagram in $\mathcal{N} = 4$ SYM theory is given by

$$(2.50) \quad \Omega = \int \prod_i \frac{d\alpha_i}{\alpha_i} \delta(C \cdot Z),$$

where we are instructed to solve for the α_i variables from the delta functions which fix them as functions of external kinematics. For our example, the delta-function constraint on $C^\perp$ gives the following equations

$$(2.51) \quad \delta^4(C^\perp \cdot \lambda) = \delta^2(-\alpha_1 \lambda_1 + \lambda_2 - \alpha_3 \lambda_3) \delta^2(-\alpha_4 \lambda_1 - \alpha_2 \lambda_3 + \lambda_4),$$

which after contracting with λ_1 and λ_3 turns into 4 equations $\prod_i^4 \delta(E_i)$, that can then be rewritten as

$$(2.52) \quad J_\delta \times \langle 13 \rangle^2 \times \delta \left(\alpha_2 - \frac{\langle 12 \rangle}{\langle 13 \rangle} \right) \delta \left(\alpha_1 - \frac{\langle 23 \rangle}{\langle 13 \rangle} \right) \delta \left(\alpha_3 - \frac{\langle 14 \rangle}{\langle 13 \rangle} \right) \delta \left(\alpha_4 - \frac{\langle 43 \rangle}{\langle 13 \rangle} \right).$$

Here $J_\delta \equiv \left| \frac{\partial E_i}{\partial \alpha_j} \right|^{-1} = \frac{1}{\langle 13 \rangle^4}$ is the Jacobian that stems from solving the four δ -function equations, and the factor $\langle 13 \rangle^2$ stems from the contraction with λ_1 and λ_3 . Plugging the solutions for the edge-variables into the other δ -function, one finds:

$$(2.53) \quad \begin{aligned} \delta^4(C \cdot \tilde{\lambda}) &= \delta^2 \left(\tilde{\lambda}_1 + \frac{\langle 23 \rangle}{\langle 13 \rangle} \tilde{\lambda}_2 + \frac{\langle 43 \rangle}{\langle 13 \rangle} \tilde{\lambda}_4 \right) \delta^2 \left(\frac{\langle 12 \rangle}{\langle 13 \rangle} \tilde{\lambda}_2 + \tilde{\lambda}_3 + \frac{\langle 14 \rangle}{\langle 13 \rangle} \tilde{\lambda}_4 \right) \\ &= \langle 13 \rangle^4 \delta^2 \left(\langle 13 \rangle \tilde{\lambda}_1 + \langle 23 \rangle \tilde{\lambda}_2 + \langle 43 \rangle \tilde{\lambda}_4 \right) \delta^2 \left(\langle 12 \rangle \tilde{\lambda}_2 + \langle 13 \rangle \tilde{\lambda}_3 + \langle 14 \rangle \tilde{\lambda}_4 \right). \end{aligned}$$

The two equations can be obtained from a single momentum conservation equation by contracting with λ_3 and λ_1 respectively, i.e. we have,

$$(2.54) \quad \delta^4(C \cdot \tilde{\lambda}) = \langle 13 \rangle^2 \delta^4(\lambda_1 \tilde{\lambda}_1 + \lambda_2 \tilde{\lambda}_2 + \lambda_3 \tilde{\lambda}_3 + \lambda_4 \tilde{\lambda}_4) \equiv \langle 13 \rangle^2 \delta^4(P).$$

For the last delta-function we get the same thing except for replacing $\tilde{\lambda}_i \rightarrow \tilde{\eta}_i$,

$$(2.55) \quad \delta^8(C \cdot \tilde{\eta}) = \frac{1}{\langle 13 \rangle^4} \delta^8(\lambda_1 \tilde{\eta}_1 + \lambda_2 \tilde{\eta}_2 + \lambda_3 \tilde{\eta}_3 + \lambda_4 \tilde{\eta}_4) \equiv \frac{1}{\langle 13 \rangle^4} \delta^8(Q).$$

We are now in a position to calculate the on-shell function Ω ,

$$(2.56) \quad \begin{aligned} \Omega &= \int \frac{d\alpha_1 \dots d\alpha_4 \delta^8(Q) \delta^4(P)}{\alpha_1 \alpha_2 \alpha_3 \alpha_4 \langle 13 \rangle^4} \delta \left(\alpha_2 - \frac{\langle 12 \rangle}{\langle 13 \rangle} \right) \delta \left(\alpha_1 - \frac{\langle 23 \rangle}{\langle 13 \rangle} \right) \delta \left(\alpha_3 - \frac{\langle 14 \rangle}{\langle 13 \rangle} \right) \delta \left(\alpha_4 - \frac{\langle 43 \rangle}{\langle 13 \rangle} \right) \\ &= \frac{\delta^8(Q) \delta^4(P)}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 41 \rangle}, \end{aligned}$$

which reproduces the on-shell function (2.19) from the previous subsection. The only poles in the on-shell function Ω come from the edge variables being sent to $\alpha_i = 0$ or $\alpha_i = \infty$, which removes an edge and imposes conditions on external kinematics, in complete agreement with our earlier discussion. Note that flipping the orientation of an edge variable also flips $\alpha \leftrightarrow 1/\alpha$ and thus flips the pole at 0 and ∞ . Let us now consider another on-shell diagram,

$$C = \begin{pmatrix} 1 & \alpha_1 + \alpha_2 \alpha_3 & \alpha_3 & 0 \\ 0 & \alpha_2 \alpha_3 \alpha_4 & \alpha_5 + \alpha_3 \alpha_4 & 1 \end{pmatrix}.$$

(2.57)

The edge variables α_i 's are given by,

$$(2.58) \quad \alpha_1 = \frac{\langle 23 \rangle - \alpha_5 \langle 24 \rangle}{\langle 13 \rangle - \alpha_5 \langle 14 \rangle}, \quad \alpha_2 = \frac{\langle 12 \rangle}{\langle 13 \rangle - \alpha_5 \langle 14 \rangle}, \quad \alpha_3 = \frac{\langle 34 \rangle}{\langle 14 \rangle}, \quad \alpha_4 = \frac{\langle 13 \rangle - \alpha_5 \langle 14 \rangle}{\langle 34 \rangle}.$$

The Jacobian from solving the δ -functions is $J = \frac{1}{(\langle 13 \rangle - \alpha_5 \langle 14 \rangle) \langle 34 \rangle}$. From (2.50) we get

$$(2.59) \quad \Omega = \int \frac{d\alpha_5 \delta^4(P) \delta^8(Q)}{\alpha_5 \langle 12 \rangle (\langle 23 \rangle - \alpha_5 \langle 24 \rangle) \langle 34 \rangle \langle 14 \rangle},$$

which we recognize as (2.23) with $\alpha_5 \rightarrow -z$. In some cases the perfect orientation can involve an infinite loop. For example, if we choose the external arrows of the box diagram in the following way:

$$C = \begin{pmatrix} \alpha_2 \alpha_3 \alpha_4 \delta & 1 & \alpha_2 \delta & 0 \\ \alpha_4 \delta & 0 & \alpha_1 \alpha_2 \alpha_3 \delta & 1 \end{pmatrix}.$$

(2.60)

The δ that appears here corresponds to a geometric series coming from an infinite loop,

$$(2.61) \quad \delta = \sum_{j=0}^{\infty} (\alpha_1 \alpha_2 \alpha_3 \alpha_4)^j = \frac{1}{1 - \alpha_1 \alpha_2 \alpha_3 \alpha_4}.$$

Solving for edge variables we get

$$(2.62) \quad \alpha_1 = \frac{\langle 23 \rangle}{\langle 13 \rangle}, \quad \alpha_2 = \frac{\langle 13 \rangle}{\langle 12 \rangle}, \quad \alpha_3 = \frac{\langle 41 \rangle}{\langle 13 \rangle}, \quad \alpha_4 = \frac{\langle 13 \rangle}{\langle 34 \rangle} \quad \text{and also} \quad \delta = \frac{\langle 12 \rangle \langle 34 \rangle}{\langle 13 \rangle \langle 24 \rangle}$$

and plugging in (2.50) we get the same formula as in (2.56). Note that in all these cases we considered on-shell diagrams in $\mathcal{N} = 4$ SYM theory which do not depend on the assignment of helicities. Hence, the formulas for diagrams (2.48) and (2.60) gave the same on-shell functions. For $\mathcal{N} < 4$, this is not the case and the helicity assignments are important.

2.3 On-shell diagrams for $\mathcal{N} < 4$ SYM theory

If the theory has less than the maximal supersymmetry, we do need to assign both external and internal helicities to fully specify an on-shell diagram. The same on-shell diagrams with different helicity assignments correspond to the same kinematics (solutions for momenta, edge variables etc)

but they are given by different on-shell functions. The on-shell function for the $\mathcal{N}<4$ SYM on-shell diagram [13] is

$$(2.63) \quad \Omega = \int \frac{d\alpha_1}{\alpha_1} \frac{d\alpha_2}{\alpha_2} \dots \frac{d\alpha_m}{\alpha_m} \mathcal{J}^{\mathcal{N}-4} \delta(C \cdot Z),$$

where the Jacobian $\mathcal{J}$ is given by

$$(2.64) \quad \mathcal{J} = 1 + \sum_i f_i + \sum_{\substack{\text{disjoint} \\ \text{pairs } i,j}} f_i f_j + \sum_{\substack{\text{disjoint} \\ \text{pairs } i,j,k}} f_i f_j f_k + \dots,$$

with f_i denoting the face of a closed cycle, and the sums are over disjoint collections of these closed cycles. We mostly work with edge-variables, and each face f_i can be defined as a clockwise-oriented product of edge-variables. Let us compare the results for two different orientations of the one-loop box diagram. For the diagram with no internal loop (2.48) we have $\mathcal{J} = 1$ and

$$(2.65) \quad \Omega = \frac{\langle 13 \rangle^{4-\mathcal{N}} \delta^4(P) \delta^{2\mathcal{N}}(\mathcal{Q})}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 41 \rangle}.$$

Note that this result is identical up to a trivial helicity factor, $\delta^8(\mathcal{Q}) \rightarrow \langle 13 \rangle^{4-\mathcal{N}} \delta^{2\mathcal{N}}(\mathcal{Q})$, to the $\mathcal{N} = 4$ on-shell function (2.56). The second diagram (2.60) has an internal loop and the Jacobian reads

$$(2.66) \quad \mathcal{J} = 1 - \alpha_1 \alpha_2 \alpha_3 \alpha_4 = \frac{\langle 13 \rangle \langle 24 \rangle}{\langle 12 \rangle \langle 34 \rangle}.$$

The on-shell function is then given by

$$(2.67) \quad \Omega = \frac{\langle 24 \rangle^{4-\mathcal{N}} \delta^4(P) \delta^{2\mathcal{N}}(\mathcal{Q})}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 41 \rangle} \times \left(\frac{\langle 12 \rangle \langle 34 \rangle}{\langle 13 \rangle \langle 24 \rangle} \right)^{4-\mathcal{N}} = \frac{(\langle 12 \rangle \langle 34 \rangle)^{4-\mathcal{N}} \delta^4(P) \delta^{2\mathcal{N}}(\mathcal{Q})}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 41 \rangle \langle 13 \rangle^{4-\mathcal{N}}}.$$

Note that the on-shell function (2.67) has a new pole for $\mathcal{N}<4$ located at $\langle 13 \rangle = 0$. Looking back to (2.18) this sends $\ell_1 \rightarrow \infty$, hence we refer to it as a *pole at infinity* or a *UV pole*. This pole is substantially different from all other poles which correspond to erasing edges in the on-shell diagram – these poles correspond to sending the associated loop momenta to zero, we will refer to them as *IR poles*. The presence of the Jacobian in (2.63) dramatically changes the pole structure of the on-shell function Ω . We will often divide the form into the bare function Ω^{bare} and the appropriate power of the Jacobian,

$$(2.68) \quad \Omega = \Omega^{\text{bare}} \times \mathcal{J}^{\mathcal{N}-4}.$$

The IR poles come from Ω^{bare} while the UV poles always come from the Jacobian part. The role of UV poles is the main topic of the next chapter and we will elaborate on it much in 3. Before diving into this discussion, it is worth noting one important point. For $\mathcal{N}=4$ SYM theory each on-shell diagram directly corresponds to a cut of the loop integrand. For $\mathcal{N}<4$ SYM this is a collection of on-shell diagrams in general: with fixed helicities of external legs we have to sum over all possible orientations of internal legs. For the one-loop example, if legs 1, 3 are incoming there is one orientation and a quadruple cut of the one-loop four-point $\mathcal{N} < 4$ SYM integrand is given by a single diagram,

$$(2.69) \quad \begin{array}{c} \begin{array}{ccc} 2^+ & & 3^- \\ & \ell_2 & \\ \ell_1 & & \ell_3 \\ & \ell_4 & \\ 1^- & & 4^+ \end{array} \\ \text{[Diagram with dashed red lines and black/white vertices]} \end{array} = \begin{array}{c} \begin{array}{ccc} 2 & & 3 \\ & \alpha_2 & \\ \alpha_1 & & \alpha_3 \\ & \alpha_4 & \\ 1 & & 4 \end{array} \\ \text{[Diagram with solid black lines and arrows]} \end{array} .$$

For legs 2, 4 incoming there are two possible internal orientations,

$$(2.70) \quad \begin{array}{c} \begin{array}{ccc} 2^- & & 3^+ \\ & \ell_2 & \\ \ell_1 & & \ell_3 \\ & \ell_4 & \\ 1^+ & & 4^- \end{array} \\ \text{[Diagram with dashed red lines and black/white vertices]} \end{array} = \begin{array}{c} \begin{array}{ccc} 2 & & 3 \\ & \alpha_2 & \\ \alpha_1 & & \alpha_3 \\ & \alpha_4 & \\ 1 & & 4 \end{array} \\ \text{[Diagram with solid black lines and arrows, clockwise]} \end{array} + \begin{array}{c} \begin{array}{ccc} 2 & & 3 \\ & \alpha_2 & \\ \alpha_1 & & \alpha_3 \\ & \alpha_4 & \\ 1 & & 4 \end{array} \\ \text{[Diagram with solid black lines and arrows, anti-clockwise]} \end{array} .$$

This is exactly the case of on-shell diagrams with internal loops as we always have two different orientations (clockwise and anti-clockwise). While in a cut they only appear in the sum, we can still treat these on-shell diagrams as independent well-defined objects and study their individual properties.

This should give us all the technology we need to deal with the on-shell diagrams we are going to encounter in chapter 3.

2.4 Outlook

We have seen that planar $\mathcal{N} = 4$ sYM on-shell diagrams give a parametrization of a cell in the positive Grassmannian and we can calculate the corresponding on-shell function as a canonical form on this positroid cell. Furthermore, we can start off with the form associated with the top cell and go to the cell's boundaries, which correspond to taking residues of the form's poles. These poles

correspond to certain physical singularities of the theory, namely physical factorization channels and soft/collinear limits. When we go to a cell’s boundary, it corresponds to erasing an edge of the on-shell diagram and the resulting diagram is associated with the cell we reach by going to this co-dimension 1 boundary.

While this is a very nice story, there are some significant open questions that remain. As mentioned before, we can write on-shell diagrams for any theory that has 3–point interactions. However, these deep connections between the positive Grassmannian, on-shell diagrams, and scattering amplitudes have only been worked out completely for planar $\mathcal{N} = 4$ sYM so far. The story becomes much more complicated when we lose two of the assumptions that we have made: no UV poles, which is the subject of the next chapter, and planarity, which is the subject of the one after.

Unlike the IR where unitarity tells us that the amplitude must factorize, we don’t have an analogous statement about the UV. These only show up in the Grassmannian integral description through an extra Jacobian factor we add and thus don’t have an interpretation as any cell’s boundary. Thus, we don’t have a good understanding of these regions from neither the physics nor the math side. Yet these regions of the kinematic space have physical significance. For example, they are relevant if one wants to study on-shell diagrams in the context of Gravity [34, 35]. The work we review in chapter 3 is a modest step we took in exploring this problem, which is based on the following work.

- Taro V. Brown, **Umut Oktem**, Jaroslav Trnka, *Poles at Infinity in On-Shell Diagrams*, Journal of High Energy Physics 02, 2023.

As alluded to in the discussion regarding supersymmetry earlier, $\mathcal{N} = 4$ sYM can be written in terms of building blocks that manifest $\mathcal{N} = 3$, where the spectrum is divided into CPT conjugate sectors such that both halves combine to give a CPT invariant multiplet. In on-shell diagram language, this means each maximal cut gives an on-shell diagram that comes with a CPT conjugate and one must add these to get the full cut. So just like with the amplitudes, $\mathcal{N} = 3$ is just another way of talking about $\mathcal{N} = 4$ sYM. However, each on-shell diagram preserves $\mathcal{N} = 3$ supersymmetry individually. Thus, each contributing $\mathcal{N} = 3$ diagram has a UV pole, which cancels out when we sum over diagrams on a cut.

For example, in the case of the amplitude $A_4(1^+2^-3^+4^-)$ (legs 2, 4 incoming) in $\mathcal{N}=3$, each on-shell diagram has a single pole at infinity, located at $\langle 13 \rangle = 0$ while the cut does not have this

pole. Plugging (2.63) for $\mathcal{N} = 3$ we get,

$$\text{Cut } A_4(1^+ 2^- 3^+ 4^-) = \frac{\langle 12 \rangle \langle 23 \rangle \delta^4(P) \delta^6(Q)}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 41 \rangle \langle 13 \rangle} + \frac{\langle 23 \rangle \langle 14 \rangle \delta^4(P) \delta^6(Q)}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 41 \rangle \langle 13 \rangle} = \frac{\langle 24 \rangle \delta^4(P) \delta^6(Q)}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 41 \rangle}.$$

This is a general property: cuts of $\mathcal{N}=3$ amplitudes are identical to their $\mathcal{N}=4$ SYM counterparts (up to trivial helicity factors) while $\mathcal{N}=3$ on-shell diagrams are individually different and have poles at infinity. On a $\mathcal{N}=3$ cut we sum over contributing on-shell diagrams and any pole at infinity has to cancel. While these poles are spurious, they give an excellent playground for us to study UV poles. This will be the starting point of the work we review in the next chapter.

Another aspect that muddies the story is the loss of planarity, which is relevant both for gravity and $\mathcal{N} = 4$ sYM without the planar limit. On the math side, the minors we write our on-shell functions in terms of no longer come from the positive part of the Grassmannian $G(k, n)$. On the physics side, loop integrand reconstruction from cuts is harder in non-planar sYM, as we don't have global loop momentum variables anymore. In addition, leading singularities don't determine the integrand like they do in the planar case, as we no longer enjoy dual conformal symmetry to constrain the ansatz. This makes the non-planar extension of the story we outlined thus far a well motivated problem to study. This will be the subject of chapter 4, which is based on the following paper

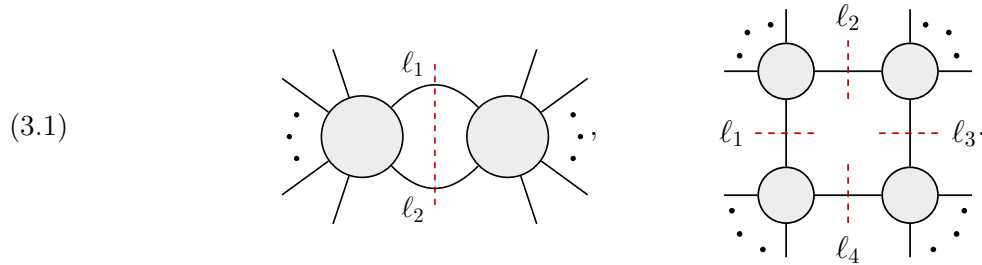
- Artyom Lisitsyn, **Umut Oktem**, Melissa Sherman-Bennett, Jaroslav Trnka, *Grassmannian Geometries for Non-Planar On-Shell Diagrams*, e-Print: 2512.25005 [hep-th].

CHAPTER 3

Poles at Infinity in On-Shell Diagrams

3.1 Introduction

Let's recap what we have learned so far. Modern on-shell methods for scattering amplitudes are based on a simple idea of building amplitudes from on-shell gauge invariant objects (see [21, 30, 36, 37, 38, 39] for recent reviews). At tree-level, this idea is materialized in the form of on-shell recursion relations where the n -point tree-level amplitude is built recursively from the lower point (on-shell) amplitudes. The primary example is the Britto-Cachazo-Feng-Witten (BCFW) recursion relations [28, 29] where we shift external momenta of two external legs. There are also generalizations to multi-line shifts [40, 41, 42], as well as theories with special soft limits [43, 44, 45, 46]. At loop level, unitarity methods allow us to write the amplitude as a linear combination of basis integrals with gauge-invariant on-shell prefactors [2, 3, 4, 47, 48, 49, 50, 51, 52, 53]. These prefactors are calculated using *cuts of the loop integrand*, where we set propagators to be on-shell and the loop integrand factorizes into simpler objects. The basic cut is the *unitarity cut* where two propagators from one loop are set to zero. This procedure can be iterated further and when the maximal number of propagators are cut, we talk about *maximal cuts*.



In these diagrams both internal and external lines are on-shell, and the result is given by the product of tree-level amplitudes in the corners. In theories with fundamental 3-point amplitudes we can proceed further and express tree-level amplitudes in the corners using BCFW recursion relations. In the end, we get *on-shell diagrams* [13, 34, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65] which are on-shell gauge invariant objects. Each diagram is defined as a product of on-shell 3-point

amplitudes (all lines, external and internal, are on-shell) integrated over momenta and Grassmann variables of internal legs,

$$(3.2) \quad \begin{array}{c} \text{Diagram: A 3-loop 6-point MHV amplitude diagram. It consists of a central square with two internal vertical lines. The top-left vertex is white, top-right is black, bottom-left is black, and bottom-right is white. External legs are: 1 (top), 2 (top-right), 3 (bottom-right), 4 (bottom), 5 (left), and 6 (right). Vertices are marked with black dots.} \end{array} = \prod_k \int d^{\mathcal{N}} \tilde{\eta}_k \int \frac{d^2 \lambda_k d^2 \tilde{\lambda}_k}{\text{GL}(1)} \left(\prod_j A_3^{(j)} \right)$$

where $\mathcal{N}$ is the number of supersymmetries. On-shell diagrams play multiple roles, such as giving us the kinematical solutions for cuts of loop integrands and reproducing the associated on-shell functions on these cuts, but a subset of them also play a role as building blocks in the recursion relations for tree-level amplitudes or loop integrands. For example, the same diagram represents a maximal cut of 3-loop 6-point MHV amplitude or a term in the BCFW construction of the 6-point MHV tree-level amplitude,

$$(3.3) \quad \begin{array}{c} \text{Diagram: A 3-loop 6-point MHV amplitude diagram. It consists of a central square with two internal vertical lines. The top-left vertex is white, top-right is black, bottom-left is black, and bottom-right is white. External legs are: 1 (top), 2 (top-right), 3 (bottom-right), 4 (bottom), 5 (left), and 6 (right). Vertices are marked with black dots.} \end{array}$$

In $\mathcal{N}=4$ SYM theory this diagram evaluates to the Parke-Taylor factor,

$$(3.4) \quad \Omega = \frac{\delta^4(\lambda \cdot \tilde{\lambda}) \delta^4(\lambda \cdot \tilde{\eta})}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 45 \rangle \langle 56 \rangle \langle 61 \rangle}.$$

Earlier, we saw that on-shell diagrams are fascinating objects and are tightly connected to the geometric formulations for on-shell scattering amplitudes using the positive Grassmannian and the Amplituhedron [14, 15, 16, 17, 18, 66, 67], which have been successfully used to obtain various explicit results including certain all-loop order quantities [56, 68, 69, 70, 71, 72, 73, 74, 75, 76, 77, 78, 79, 80, 81]. In particular, each planar diagram, known in the math literature as a plabic graph [33], is associated with a specific cell in the positive Grassmannian $G_+(k, n)$ represented by the matrix C . The matrix C is parametrized by the variables α_i which are associated with edges (or faces) of

the graph. In physics, the on-shell function in $\mathcal{N}=4$ SYM theory is reproduced by a canonical dlog form [13] in all α_i and delta functions which solve for α_i in terms of kinematical variables,

$$(3.5) \quad \Omega = \int \frac{d\alpha_1}{\alpha_1} \frac{d\alpha_2}{\alpha_2} \dots \frac{d\alpha_m}{\alpha_m} \delta(C \cdot \tilde{\lambda}) \delta(C^\perp \cdot \lambda) \delta(C \cdot \tilde{\eta}).$$

The kinematical poles in the on-shell function are related to the poles in the edge variables. One particular parametrization of the on-shell diagram (3.3) is

(3.6)

$$\begin{aligned} \alpha_1 &= \frac{\langle 23 \rangle}{\langle 13 \rangle}, & \alpha_2 &= \frac{\langle 12 \rangle}{\langle 13 \rangle}, \\ \alpha_3 &= \frac{\langle 46 \rangle}{\langle 36 \rangle}, & \alpha_4 &= \frac{\langle 34 \rangle}{\langle 36 \rangle}, \\ \alpha_5 &= \frac{\langle 36 \rangle}{\langle 13 \rangle}, & \alpha_6 &= \frac{\langle 16 \rangle}{\langle 36 \rangle}, \\ \alpha_7 &= \frac{\langle 56 \rangle}{\langle 46 \rangle}, & \alpha_8 &= \frac{\langle 45 \rangle}{\langle 46 \rangle}. \end{aligned}$$

and the formula (3.5) reproduces (3.4). The kinematical poles in Ω correspond to poles in edge variables α_j . In fact, taking a residue on the pole $\alpha_j=0$ erases a corresponding edge and produces a lower-loop on-shell diagram. Its on-shell function is equal to the residue of the original on-shell diagram on the kinematical pole. For example, in the diagram (3.6) the condition $\alpha_7=0$ sets the momentum flow in a corresponding edge to zero and forces $\langle 56 \rangle = 0$. The result is a lower-loop on-shell diagram where external legs 5,6 are attached to the same white vertex.

(3.7)

All poles in the on-shell function for $\mathcal{N}=4$ SYM are located deep in the IR region where the internal momenta are set to zero. For $\mathcal{N}<4$ there is an additional Jacobian factor in the Grassmannian form (3.5) which introduces new types of poles other than $\alpha_j=0$ [13]. These poles are in the UV where the loop momenta go to infinity. Nothing is known about these poles, i.e. what the precise locations in the loop momentum space are (other than being at infinity), what type of kinematical conditions

they impose on external momenta, and what the residues of on-shell diagrams on these poles are. Obviously, these poles also naturally arise when calculating the on-shell diagrams from the definition (3.2) but the Grassmannian picture nicely divides the well-understood IR poles (located at $\alpha_j=0$) from these new UV poles, which we also refer to as *poles at infinity*.

The nature of these UV poles is a very important conceptual question, and we address it in the context of on-shell diagrams which properties are inherited from the properties of scattering amplitudes. In fact, the behavior on the IR poles, such as erasing edges in on-shell diagrams, taking residues on $\alpha_j=0$ etc., is just a consequence of perturbative unitarity for scattering amplitudes. However, the unitarity of the S-matrix does not predict what exactly happens with amplitudes (tree-level amplitudes and loop integrands) on UV poles when the loop momenta go to infinity (either external momenta or loop momenta) in a certain direction. The behavior on UV poles might be also tied to symmetries or special properties, for example the complete absence of any poles at infinity in planar $\mathcal{N}=4$ SYM is a consequence of the dual conformal symmetry [82, 83]. The same property has been also conjectured to hold for the non-planar sector of the same theory [84, 85, 86, 87, 88, 89, 90, 91], pointing to a possibility of a hidden symmetry in the full theory. The surprising behavior at infinity has also been observed in gravitational theories. This includes the improved $1/z^2$ scaling under BCFW shift [92, 93, 94], enhanced cancelations on certain cuts [34, 35, 89, 95] or observed enhanced UV cancelations in higher loop amplitudes [35, 95, 96, 97, 98]. This is also likely related to surprisingly simple formulas for tree-level amplitudes [99, 100, 101, 102, 103, 104, 105, 106] or other special properties of gravity such as color-kinematics duality [107, 108, 109].

This naturally raises a question of a uniform framework to analyze amplitudes at infinite momenta, and search for implications of the *unitarity at infinity* for the perturbative S-matrix. In this work, we will make a modest step in this direction, and give a partial answer to this question in the context of planar on-shell diagrams in Yang-Mills theory. We will show where the poles at infinity are located in the kinematical space and what the kinematical constraints we have to impose to approach them are. The residue at infinity is then a diagrammatic operation on an on-shell diagram which we call a *non-planar twist*. The residue on this pole of a planar diagram is a lower-loop *non-planar diagram*. For $\mathcal{N}<3$ SYM we have multiple poles and the residue corresponds to a derivative operator acting on the lower-loop twisted on-shell diagram.

The chapter is organized as follows: In section 2, we study the poles at infinity in one-loop on-shell diagrams for $\mathcal{N}=3$ and show that the imposed kinematical condition produces a simple tree-level factorization. In section 3, we show that in planar diagrams all poles at infinity are localized in one-loop subgraphs and we discuss their embedding into a higher-loop on-shell diagrams. We also show a number of examples and discuss Global Residue Theorems which relate UV poles to IR poles. In section 4, we generalize our procedure to $\mathcal{N}<3$ SYM and discuss the multiple poles at infinity. In section 5, we show that in non-planar diagrams the situation is more complicated as one kinematical condition can send multiple loop momenta to infinity. We end with conclusion and outlook in section 6.

3.2 UV poles in one-loop graphs

The bare on-shell functions describe the dynamics of scattering amplitudes in the IR sector when the (cut) loop momenta are small. All poles in Ω^{bare} correspond to setting one of these momenta to zero and erasing an edge in the diagram, as a consequence of the perturbative unitarity. In the $\mathcal{N}=4$ SYM theory the on-shell function is given by Ω^{bare} , and hence we only have IR poles. This is true for both planar and non-planar sectors of the theory. In the planar sector, this is also a consequence of the dual conformal symmetry (which we discuss in 5), while the symmetry explanation of the absence of poles at infinity in the non-planar sector is yet-to-be found. Note that the absence of these poles is related, but not identical, to the statement of the UV finiteness of final (integrated) scattering amplitudes in $\mathcal{N}=4$ SYM theory. In fact, the absence of poles at infinity in on-shell diagrams is a stronger property than the UV finiteness of the S-matrix, see [84, 85, 89] for more details.

For $\mathcal{N}<4$ SYM theory, the bare function Ω^{bare} is decorated by a power of the Jacobian $\mathcal{J}$ as outlined in the previous section. The Jacobian changes the pole structure: it removes some of the existing IR poles, and adds new UV poles. Note that the UV poles are not fixed by the (standard) unitarity – unlike IR poles where tree-level factorization or unitarity cuts fully fix corresponding residues. As a result, we can not naively expect that the residue on a UV pole, where the loop momentum goes to infinity, is an operation on an on-shell diagram. In the following, we will show that despite these assumptions the real situation is quite contrary: the UV pole can be understood as a (rather simple) diagrammatic operation, not erasing an edge, but shrinking a corresponding

loop (which blows up on the pole) and rejoining internal legs via a “non-planar twist”. As a result, the residue on the UV pole is generally a lower-loop *non-planar* on-shell diagram. In our discussion, we first focus on the $\mathcal{N}=3$ case, where the on-shell function contains only one power of $\mathcal{J}$, and the pole at infinity in Ω is always a simple pole. For general $\mathcal{N}$ we get higher order poles at infinity and the residues correspond to derivatives of an on-shell function associated with the diagram.

3.2.1 First $\mathcal{N}=3$ examples

The simplest example is the one-loop box on-shell diagram with an internal loop discussed earlier. The on-shell function is given by,

$$(3.8) \quad \begin{array}{c} \begin{array}{c} 2 \\ \swarrow \\ \bullet \\ \nearrow \\ 3 \end{array} \quad \begin{array}{c} \alpha_2 \\ \leftarrow \\ \circ \\ \rightarrow \\ \alpha_3 \end{array} \\ \begin{array}{c} \alpha_1 \\ \leftarrow \\ \circ \\ \rightarrow \\ \alpha_4 \end{array} \quad \begin{array}{c} 1 \\ \swarrow \\ \bullet \\ \nearrow \\ 4 \end{array} \end{array} = \underbrace{\frac{\delta^4(P)\delta^6(\mathcal{Q})\langle 24 \rangle}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 14 \rangle}}_{\Omega^{\text{bare}}} \times \underbrace{\left(\frac{\langle 12 \rangle \langle 34 \rangle}{\langle 13 \rangle \langle 24 \rangle} \right)}_{\mathcal{J}^{-1}} = \frac{\delta^4(P)\delta^6(\mathcal{Q})}{\langle 23 \rangle \langle 14 \rangle \langle 13 \rangle}.$$

where the Grassmann variables are η_i^I , $I = 1, 2, 3$, and hence the supermomentum conservation is $\delta^6(\mathcal{Q})$. We can see that the Jacobian $\mathcal{J}$ introduces two new poles, $\langle 13 \rangle$ and $\langle 24 \rangle$ and removes poles $\langle 12 \rangle$ and $\langle 34 \rangle$. The absence of poles $\langle 12 \rangle$ and $\langle 34 \rangle$ is caused by the fact that corresponding edges are no longer removable. For example, removing edge α_2 sends $\langle 12 \rangle = 0$ which leads to an illegal helicity flow

$$(3.9) \quad \begin{array}{c} 2 \\ \swarrow \\ \circ \\ \nearrow \\ 1 \end{array} \quad \begin{array}{c} \leftarrow \\ \text{red} \\ \rightarrow \end{array} \quad \begin{array}{c} 3 \\ \swarrow \\ \bullet \\ \nearrow \\ 4 \end{array}$$

as we have a wrong number of incoming/outgoing legs in black/white vertices. This pole was perfectly legal pole in the $\mathcal{N}=4$ SYM case but now the Jacobian removes it. Same is true for the pole $\langle 23 \rangle = 0$ and the edge α_4 . Next, the pole $\langle 24 \rangle$ is canceled by the helicity factor in the numerator of Ω and only $\langle 13 \rangle = 0$ stays as a new valid pole. As noted before, this is a pole at infinity as it sends the on-shell loop momentum $\ell_1 \rightarrow \infty$. While this pole does not erase any edge, we can easily see what type of kinematical constraint it imposes:

$$(3.10) \quad \langle 13 \rangle = 0 \leftrightarrow \lambda_1 \sim \lambda_3, \text{ and by momentum conservation } [24] = 0 \leftrightarrow \tilde{\lambda}_2 \sim \tilde{\lambda}_4.$$

But this is nothing else than the factorization diagram,

$$(3.11) \quad \begin{array}{c} 3 \\ \swarrow \\ \circ \leftarrow \bullet \\ \searrow \\ 1 \end{array} \begin{array}{c} 2 \\ \swarrow \\ \bullet \leftarrow \circ \\ \searrow \\ 4 \end{array} = \frac{\delta^4(P)\delta^6(Q)\delta(\langle 13 \rangle)}{\langle 23 \rangle \langle 14 \rangle}.$$

The internal momentum flow is here fixed by the number of incoming/outgoing legs in white/black vertex. Note that our procedure was based solely on the analysis of the kinematical conditions between external legs, i.e. we imposed $\langle 13 \rangle = 0$ and searched for an on-shell diagram which materializes that. But very importantly, the on-shell function for this diagram (3.11) is equal to the residue of the original diagram (3.8) on the pole $\langle 13 \rangle = 0$. In this case it might look a bit trivial as we only replaced $1/\langle 13 \rangle \rightarrow \delta(\langle 13 \rangle)$ but the same procedure is valid for the general case.

Checking the equality of the residue of the original on-shell diagram evaluated on the pole with the new on-shell diagram (with extra delta function imposed) might sometimes be tricky. For example, if we consider the same diagram with the internal loop and the opposite orientation, we get

$$(3.12) \quad \begin{array}{c} 2 \\ \swarrow \\ \bullet \xrightarrow{\alpha_2} \circ \\ \searrow \\ 3 \end{array} \begin{array}{c} 3 \\ \swarrow \\ \circ \xrightarrow{\alpha_3} \bullet \\ \searrow \\ 4 \end{array} \begin{array}{c} 1 \\ \swarrow \\ \circ \xrightarrow{\alpha_1} \bullet \\ \searrow \\ 2 \end{array} \begin{array}{c} 4 \\ \swarrow \\ \bullet \xrightarrow{\alpha_4} \circ \\ \searrow \\ 1 \end{array} = \underbrace{\frac{\delta^4(P)\delta^6(Q)\langle 24 \rangle}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 14 \rangle}}_{\Omega^{\text{bare}}} \times \underbrace{\left(\frac{\langle 23 \rangle \langle 14 \rangle}{\langle 13 \rangle \langle 24 \rangle} \right)}_{\mathcal{J}^{-1}} = \frac{\delta^4(P)\delta^6(Q)}{\langle 12 \rangle \langle 34 \rangle \langle 13 \rangle}.$$

Taking the residue on the pole $\langle 13 \rangle = 0$ we arrive at the same diagram as before,

$$(3.13) \quad \begin{array}{c} 3 \\ \swarrow \\ \circ \leftarrow \bullet \\ \searrow \\ 1 \end{array} \begin{array}{c} 2 \\ \swarrow \\ \bullet \leftarrow \circ \\ \searrow \\ 4 \end{array} = \frac{\delta^4(P)\delta^6(Q)\delta(\langle 13 \rangle)}{\langle 12 \rangle \langle 34 \rangle}.$$

As you can see the on-shell functions look different in (3.11) and (3.13) but they are the same up to a sign, as $\langle 12 \rangle \langle 34 \rangle = \langle 23 \rangle \langle 14 \rangle$ for $\langle 13 \rangle = 0$. In fact, if we sum (3.11) and (3.13) we get zero which is consistent with $\mathcal{N} = 3$ cuts not having any poles at infinity.

Next, let us consider the four-point double box on-shell diagram we discussed in the previous section. There are multiple different orientations of the same diagram which lead to different structures of poles at infinity. We can see from the parametrization of the loop momenta (2.22)

that the left loop blows up for $z = -\langle 13 \rangle / \langle 14 \rangle$, while the right loop momentum goes to infinity for $z = \infty$. The accessibility of these poles depends on a particular orientation and the presence of closed loops. For the following orientation

$$(3.14) \quad \begin{array}{c} \begin{array}{ccc} 2 & & 3 \\ \swarrow & & \swarrow \\ \bullet & \leftarrow & \bullet \\ \downarrow & & \downarrow \\ \circ & \rightarrow & \circ \\ \swarrow & & \swarrow \\ 1 & & 4 \end{array} \end{array} = \int dz \underbrace{\frac{\langle 24 \rangle \delta^4(P) \delta^6(Q)}{z(\langle 23 \rangle + z\langle 24 \rangle) \langle 14 \rangle \langle 34 \rangle \langle 12 \rangle}}_{\Omega^{\text{bare}}} \times \underbrace{\frac{\langle 14 \rangle (\langle 23 \rangle + z\langle 24 \rangle)}{\langle 24 \rangle (\langle 13 \rangle + z\langle 14 \rangle)}}_{\mathcal{I}^{-1}} \\ = \int dz \frac{\delta^4(P) \delta^6(Q)}{z \langle 34 \rangle \langle 12 \rangle (\langle 13 \rangle + z\langle 14 \rangle)},$$

we have a closed cycle in the left loop and the on-shell function has a pole at $z = -\langle 13 \rangle / \langle 14 \rangle$. A different orientation has a closed cycle in the right loop and a pole at $z = \infty$,

$$(3.15) \quad \begin{array}{c} \begin{array}{ccc} 2 & & 3 \\ \swarrow & & \swarrow \\ \bullet & \leftarrow & \bullet \\ \downarrow & & \downarrow \\ \circ & \rightarrow & \circ \\ \swarrow & & \swarrow \\ 1 & & 4 \end{array} \end{array} = \int dz \frac{\langle 13 \rangle \delta^4(P) \delta^6(Q)}{z(\langle 23 \rangle + z\langle 24 \rangle) \langle 14 \rangle \langle 34 \rangle \langle 12 \rangle} \times \frac{z\langle 14 \rangle}{\langle 13 \rangle} \\ = \int dz \frac{\delta^4(P) \delta^6(Q)}{\langle 12 \rangle \langle 34 \rangle (\langle 23 \rangle + z\langle 24 \rangle)}$$

Finally, we can have an orientation with two internal cycles and both poles in Ω ,

$$(3.16) \quad \begin{array}{c} \begin{array}{ccc} 2 & & 3 \\ \swarrow & & \swarrow \\ \bullet & \leftarrow & \bullet \\ \downarrow & & \downarrow \\ \circ & \rightarrow & \circ \\ \swarrow & & \swarrow \\ 1 & & 4 \end{array} \end{array} = \int dz \frac{\langle 23 \rangle \delta^4(P) \delta^6(Q)}{z(\langle 23 \rangle + z\langle 24 \rangle) \langle 12 \rangle \langle 14 \rangle \langle 34 \rangle} \times \frac{z\langle 14 \rangle (\langle 23 \rangle + z\langle 24 \rangle)}{\langle 23 \rangle (\langle 13 \rangle + z\langle 14 \rangle)} \\ = \int dz \frac{\delta^4(P) \delta^6(Q)}{\langle 12 \rangle \langle 34 \rangle (\langle 13 \rangle + z\langle 14 \rangle)}$$

Note that in this case there is no IR pole present as erasing any edge would lead to a contradiction with the helicity assignment. As in the previous case, we want to interpret the residue on the pole at infinity as another (now one-loop) on-shell diagram. As both cycles are just one-loop on-shell diagrams, we can treat them locally, see section 3.3.1 for proof of this. This means that if we blow up the left loop we need to replace the box on-shell diagram with external legs $1, 2, \ell_3, \ell_6$ by the

tree-level factorization diagram with legs $2, \ell_6$ in the black vertex and $1, \ell_3$ in the white vertex,

$$(3.17) \quad \begin{array}{ccc} \begin{array}{c} \text{Diagram 1: A square loop with vertices } \ell_1, \ell_2, \ell_3, \ell_4 \text{ (top-left to top-right).} \\ \text{Legs: } 2 \text{ (in), } \ell_6 \text{ (out), } 1 \text{ (in), } \ell_3 \text{ (out).} \\ \text{A red dashed line connects } \ell_2 \text{ and } \ell_4. \end{array} & \longrightarrow & \begin{array}{c} \text{Diagram 2: A tree-level diagram with vertices } \ell_1, \ell_2 \text{ (top), } \ell_3, \ell_4 \text{ (bottom).} \\ \text{Legs: } 1 \text{ (in), } \ell_3 \text{ (out), } 2 \text{ (in), } \ell_6 \text{ (out).} \end{array} \end{array}$$

Making this replacement in (3.15) we get

$$(3.18) \quad \begin{array}{ccc} \begin{array}{c} \text{Diagram 1: A square loop with vertices } \ell_1, \ell_2, \ell_3, \ell_4 \text{ (top-left to top-right).} \\ \text{Legs: } 2 \text{ (in), } 3 \text{ (out), } 1 \text{ (in), } 4 \text{ (out).} \\ \text{A red dashed line connects } \ell_2 \text{ and } \ell_4. \end{array} & \xrightarrow{z = -\frac{\langle 13 \rangle}{\langle 14 \rangle}} & \begin{array}{c} \text{Diagram 2: A square loop with vertices } \ell_1, \ell_2, \ell_3, \ell_4 \text{ (top-left to top-right).} \\ \text{Legs: } 1 \text{ (in), } 3 \text{ (out), } 2 \text{ (in), } 4 \text{ (out).} \end{array} = \frac{\delta^4(P)\delta^6(Q)}{\langle 12 \rangle \langle 13 \rangle \langle 34 \rangle} \end{array}$$

The same expression for the on-shell function can be obtained by taking the residue of (3.14) on the $z = -\langle 13 \rangle / \langle 14 \rangle$ pole. If we blow up the same loop in the third diagram (3.16), we get a one-loop diagram with a closed cycle,

$$(3.19) \quad \begin{array}{ccc} \begin{array}{c} \text{Diagram 1: A square loop with vertices } \ell_1, \ell_2, \ell_3, \ell_4 \text{ (top-left to top-right).} \\ \text{Legs: } 1 \text{ (in), } 3 \text{ (out), } 2 \text{ (in), } 4 \text{ (out).} \\ \text{A red dashed line connects } \ell_2 \text{ and } \ell_4. \end{array} & = & \frac{\delta^4(P)\delta^6(Q)\langle 23 \rangle}{\langle 12 \rangle \langle 13 \rangle \langle 24 \rangle \langle 34 \rangle} \times \left(\frac{\langle 12 \rangle \langle 34 \rangle}{\langle 23 \rangle \langle 14 \rangle} \right) = \frac{\delta^4(P)\delta^6(Q)}{\langle 23 \rangle \langle 14 \rangle \langle 13 \rangle} \end{array}$$

which is precisely equal to the residue of (3.16) on the $z = -\langle 13 \rangle / \langle 14 \rangle$ pole. We can do the same analysis for the $z = \infty$ pole, again finding an agreement between the new on-shell diagram and the residue of the original on-shell function (3.16).

3.2.2 Two-loop pentabox diagram

Let us now look at a more complicated example: five-point MHV pentabox on-shell diagram. Solving for internal on-shell momenta gives

(3.20)

$$\begin{aligned} \ell_A &= \frac{\langle 23 \rangle}{\langle 13 \rangle} \lambda_1 \tilde{\lambda}_2, & \ell_E &= \frac{\langle 34 \rangle}{\langle 35 \rangle} \lambda_5 \tilde{\lambda}_4, \\ \ell_B &= \frac{\langle 12 \rangle}{\langle 13 \rangle} \lambda_3 \tilde{\lambda}_2, & \ell_F &= \frac{\lambda_5 (Q_{12} \cdot \lambda_3)}{\langle 35 \rangle}, \\ \ell_C &= \frac{\lambda_3 (Q_{23} \cdot \lambda_1)}{\langle 13 \rangle}, & \ell_G &= \frac{\lambda_1 (Q_{12} \cdot \lambda_3)}{\langle 13 \rangle}, \\ \ell_D &= \frac{\langle 45 \rangle}{\langle 35 \rangle} \lambda_3 \tilde{\lambda}_4, & \ell_H &= \frac{\langle 15 \rangle}{\langle 13 \rangle \langle 35 \rangle} \lambda_3 (Q_{12} \cdot \lambda_3). \end{aligned}$$

We see that the left (pentagon) loop has a pole at infinity for $\langle 13 \rangle = 0$, all momenta $\ell_A, \ell_B, \ell_C, \ell_G, \ell_H$ blow up, while the right (box) loop has a UV pole for $\langle 35 \rangle = 0$ when ℓ_D, ℓ_E, ℓ_F and ℓ_H go to infinity. The loop momentum ℓ_H contains both poles as it participates in both internal cycles. Note that the assignment of cycles is unique and it critically depends on the *planarity* of the diagram. In particular, we do not consider the external heptagon cycle – there is no pole at infinity associated with it, see Section 4.3 for elaboration,

(3.21)

Now we choose a particular orientation where both poles at infinity are present due to helicity assignments and solve for the edge variables,

$$(3.22) \quad \begin{array}{c} \begin{array}{c} \text{Diagram: A 2-loop 5pt amplitude diagram with external legs 1, 2, 3, 4, 5. Internal edges are labeled } \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6. \text{ A dashed red line connects } \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6. \end{array} \quad \begin{array}{l} \alpha_1 = \frac{\langle 13 \rangle}{\langle 23 \rangle}, \quad \alpha_2 = \frac{\langle 12 \rangle}{\langle 13 \rangle}, \quad \alpha_3 = \frac{\langle 45 \rangle}{\langle 35 \rangle}, \\ \alpha_4 = \frac{\langle 35 \rangle}{\langle 34 \rangle}, \quad \alpha_5 = \frac{\langle 13 \rangle}{\langle 35 \rangle}, \quad \alpha_6 = \frac{\langle 35 \rangle}{\langle 15 \rangle}. \end{array} \end{array}$$

This on-shell diagram is a part of the octacut of the 2-loop 5pt amplitude $A_5(1^+2^-3^+4^-5^+)$. In order to calculate the whole cut we would have to sum over all possible internal helicity assignments (for $\mathcal{N}=4$ SYM) as discussed earlier, but we can also study this one on-shell diagram individually. For this particular case (3.22) the bare on-shell function (which is identical for any assignment of internal helicities) is,

$$(3.23) \quad \Omega^{\text{bare}} = \int \frac{d\alpha_1 \dots d\alpha_6 \delta(C \cdot Z)}{\alpha_1 \alpha_2 \alpha_3 \alpha_4 \alpha_5 \alpha_6} = \frac{\langle 24 \rangle \delta^4(P) \delta^6(Q)}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 45 \rangle \langle 51 \rangle}$$

and the Jacobian $\mathcal{J}$ consists of two cycles,

$$(3.24) \quad \mathcal{J} = \frac{1}{1 - \alpha_1 \alpha_2 \alpha_6 - \alpha_3 \alpha_4 \alpha_5 \alpha_6} = \frac{\langle 13 \rangle \langle 24 \rangle \langle 35 \rangle}{\langle 23 \rangle \langle 15 \rangle \langle 34 \rangle}$$

The on-shell function Ω is then equal to

$$(3.25) \quad \Omega = \underbrace{\frac{\langle 24 \rangle \delta^4(P) \delta^6(Q)}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 45 \rangle \langle 51 \rangle}}_{\Omega^{\text{bare}}} \times \underbrace{\left(\frac{\langle 23 \rangle \langle 15 \rangle \langle 34 \rangle}{\langle 13 \rangle \langle 24 \rangle \langle 35 \rangle} \right)}_{\mathcal{J}^{-1}} = \frac{\delta^4(P) \delta^6(Q)}{\langle 12 \rangle \langle 45 \rangle \langle 13 \rangle \langle 35 \rangle}$$

We see that three of the IR poles $\langle 23 \rangle$, $\langle 34 \rangle$, $\langle 15 \rangle$ are removed as erasing α_1 , α_4 and α_6 edges would lead to inconsistent helicity flows. On the other hand, both UV poles $\langle 13 \rangle$, $\langle 35 \rangle$ are added. We first blow up the right loop and calculate the residue on the pole $\langle 35 \rangle = 0$. The right loop can be treated as a one-loop box diagram with external legs $C, 4, 5, G$. The residue is a following on-shell

diagram,

$$(3.26) \quad \begin{array}{ccc} \begin{array}{c} \text{C} \\ \swarrow \\ \text{white} \\ \uparrow \\ \text{black} \\ \swarrow \\ \text{G} \end{array} & \begin{array}{c} \text{4} \\ \swarrow \\ \text{black} \\ \uparrow \\ \text{white} \\ \swarrow \\ \text{5} \end{array} & \longrightarrow & \begin{array}{c} \text{C} \\ \swarrow \\ \text{white} \\ \uparrow \\ \text{black} \\ \swarrow \\ \text{G} \end{array} \\ & \begin{array}{c} \text{5} \\ \swarrow \\ \text{white} \\ \uparrow \\ \text{black} \\ \swarrow \\ \text{4} \end{array} & & \begin{array}{c} \text{5} \\ \swarrow \\ \text{white} \\ \uparrow \\ \text{black} \\ \swarrow \\ \text{4} \end{array} \end{array}$$

Replacing the box by this tree graph in (3.20) we get

$$(3.27) \quad \begin{array}{c} \begin{array}{c} \text{3} \\ \uparrow \\ \text{white} \\ \swarrow \\ \text{B} \\ \swarrow \\ \text{black} \\ \swarrow \\ \text{2} \end{array} \\ \begin{array}{c} \text{C} \\ \swarrow \\ \text{white} \\ \uparrow \\ \text{I} \\ \swarrow \\ \text{black} \\ \swarrow \\ \text{4} \end{array} \\ \begin{array}{c} \text{5} \\ \swarrow \\ \text{white} \\ \uparrow \\ \text{G} \\ \swarrow \\ \text{black} \\ \swarrow \\ \text{1} \end{array} \\ \begin{array}{c} \text{A} \\ \swarrow \\ \text{white} \\ \uparrow \\ \text{black} \\ \swarrow \\ \text{1} \end{array} \end{array}$$

Note that in this diagram the condition $\langle 35 \rangle$ is automatically imposed as evident from legs 3, 5 being connected by a chain of white vertices. We can now assign edge variables

$$(3.28) \quad \begin{array}{c} \begin{array}{c} \text{3} \\ \uparrow \\ \text{white} \\ \swarrow \\ \alpha_2 \\ \swarrow \\ \text{black} \\ \swarrow \\ \text{2} \end{array} \\ \begin{array}{c} \alpha_3 \\ \swarrow \\ \text{white} \\ \uparrow \\ \alpha_4 \\ \swarrow \\ \text{black} \\ \swarrow \\ \text{4} \end{array} \\ \begin{array}{c} \alpha_5 \\ \swarrow \\ \text{white} \\ \uparrow \\ \alpha_1 \\ \swarrow \\ \text{black} \\ \swarrow \\ \text{1} \end{array} \end{array} \quad \begin{array}{l} \alpha_1 = \frac{\langle 15 \rangle}{\langle 25 \rangle}, \quad \alpha_2 = \frac{\langle 12 \rangle}{\langle 13 \rangle}, \\ \alpha_3 = \frac{\langle 23 \rangle}{\langle 25 \rangle}, \quad \alpha_4 = \frac{\langle 15 \rangle}{\langle 14 \rangle}, \quad \alpha_5 = \frac{\langle 45 \rangle}{\langle 51 \rangle}. \end{array}$$

and calculate the on-shell function,

$$(3.29) \quad \Omega_{\text{UV}} = \int \frac{d\alpha_1 \dots d\alpha_5 \delta(C \cdot Z)}{\alpha_1 \alpha_2 \alpha_3 \alpha_4 \alpha_5} \times \frac{1}{1 - \alpha_1 \alpha_2 \alpha_3 \alpha_4 \alpha_5} = \frac{\delta^4(P) \delta^6(Q) \delta(\langle 35 \rangle)}{\langle 12 \rangle \langle 45 \rangle \langle 13 \rangle},$$

which is equal to a residue of (3.25) on $\langle 35 \rangle = 0$ pole. Let us now consider the same pentabox on-shell diagram but calculate the residue on the other pole at infinity $\langle 13 \rangle = 0$ which blows up the left pentagon. As we have already seen, the calculation of UV poles is a local operation and we need to figure out how to do it for the pentagon sub-diagram. As it turns out, the rule is very similar to what we saw for the box. The result is a tree-level on-shell diagram with legs 2, F in the

black vertex and legs 1, 3, D connected by a chain of white vertices,

$$(3.30) \quad \begin{array}{c} \text{Diagram 1: A pentagon with vertices 2 (black), 3 (white), 1 (white), F (black), and D (white). A red dashed line connects 2, 3, and 1.} \\ \text{Diagram 2: A pentagon with vertices 2 (black), 3 (white), 1 (white), F (black), and D (white). A red dashed line connects 2, 3, and 1.} \\ \text{Diagram 3: A tree-level diagram with a white vertex connected to 3, D, and 1. A black vertex is connected to 1, 2, and F.} \end{array}$$

where we first merge-expanded the pentagon diagram and then we used the same replacement rule as for the box. In the final tree-level on-shell diagram we merged all legs into one white vertex, which can be interpreted as a 4-point $k = 1$ tree-level amplitude, or alternatively can be further expanded as a chain of regular white three-point vertices,

$$(3.31) \quad \begin{array}{c} \text{Diagram 1: A tree-level diagram with a white vertex connected to 3, D, and 1. A black vertex is connected to 1, 2, and F.} \\ \text{Diagram 2: A chain of three white vertices connected by black vertices.} \\ \text{Diagram 3: A chain of three white vertices connected by black vertices.} \\ \text{Diagram 4: A chain of three white vertices connected by black vertices.} \end{array}$$

The identity between all these diagrams is a consequence of a usual merge-expand rule [13]. Using (3.31) the residue on the $\langle 13 \rangle = 0$ pole is

$$(3.32) \quad \begin{array}{c} \text{Diagram 1: A pentagon with vertices 2 (black), 3 (white), 4 (black), 5 (white), and 1 (white). A red dashed line connects 2, 3, and 4.} \\ \text{Diagram 2: A pentagon with vertices 2 (black), 3 (white), 4 (black), 5 (white), and 1 (white). A red dashed line connects 2, 3, and 4.} \end{array}$$

As expected this diagram automatically implements the condition $\langle 13 \rangle = 0$. Note that using other representations of the tree graph (3.31) we would get other one-loop pentagons which differ from (3.32) by the merge-expand move and do not change the on-shell function. It is easy to see that the

on-shell function for (3.32) is equal to

$$(3.33) \quad \Omega_{\text{UV}} = \frac{\delta^4(P)\delta^6(\mathcal{Q})\delta(\langle 13 \rangle)}{\langle 12 \rangle \langle 45 \rangle \langle 35 \rangle}$$

which is equal to the residue of (3.25) on $\langle 13 \rangle = 0$ pole. Naively, we could access the pole at infinity by solving for one of the edge variables from $\mathcal{J}^{-1} = 0$. However, the meaning of the new edge variables is unclear as they do not correspond to a parametrization of the lower-loop on-shell diagram. Hence we have to work directly in the kinematical space when accessing the poles at infinity.

3.2.3 MHV one-loop diagrams

As we have already seen, the calculation of the UV pole of an on-shell diagram reduced to the calculation of this pole in the one-loop sub-diagram. The result of this operation was glued into the rest of the graph. Our goal here is to calculate the UV pole for a general one-loop on-shell diagram. This is a direct generalization of what we have already seen for box and pentagon. Let us first look at the MHV hexagon,

$$(3.34) \quad \begin{aligned} \ell_A &= \frac{[13]}{[36]} \lambda_1 \tilde{\lambda}_6, & \ell_B &= \frac{[16]}{[36]} \lambda_1 \tilde{\lambda}_3, \\ \ell_C &= \frac{[26]}{[36]} \lambda_2 \tilde{\lambda}_3, & \ell_D &= \frac{[46]}{[36]} \lambda_4 \tilde{\lambda}_3, \\ \ell_E &= \frac{[34]}{[36]} \lambda_4 \tilde{\lambda}_6, & \ell_F &= \frac{[35]}{[36]} \lambda_5 \tilde{\lambda}_6. \end{aligned}$$

As we can see from the chains of white vertices, the external kinematics is subject to two constraints $\langle 12 \rangle = \langle 45 \rangle = 0$, i.e. $\lambda_2 = \alpha \lambda_1$, $\lambda_5 = \beta \lambda_4$. The pole at infinity is accessed by sending $[36] = 0$ as evident from explicit solutions for internal loop momenta,

$$(3.35) \quad 0 = s_{36} = s_{1245} = (p_1 + p_2) \cdot (p_4 + p_5) = \langle 14 \rangle ([14] + \alpha [24] + \beta [15] + \alpha \beta [25])$$

Hence $[36] = 0$ implies $\langle 14 \rangle = 0$ on the support of $\langle 12 \rangle = \langle 45 \rangle = 0$. In the end, all λ s of external legs connected to white vertices are collinear, $\lambda_1 \sim \lambda_2 \sim \lambda_4 \sim \lambda_5$, i.e. $\langle ij \rangle = 0$ for $i, j = 1, 2, 4, 5$.

This merges legs 1, 2, 4, 5 in the white vertex and 3, 6 into black vertex

(3.36)

where the five-point white vertex can be again expanded in any way we want. We could obtain the same conclusion if we first merge-expanded the original hexagon,

(3.37)

By accessing the pole at infinity in the box diagram, we arrive at one particular representation of the resulting tree graph,

(3.38)

This can be also verified by an explicit calculation. The on-shell function for the hexagon with closed internal loop (3.34) is

(3.39)

$$\Omega = \underbrace{\frac{\langle 36 \rangle \delta^4(P) \delta^6(Q) \delta(\langle 12 \rangle) \delta(\langle 45 \rangle)}{\langle 23 \rangle \langle 34 \rangle \langle 56 \rangle \langle 16 \rangle}}_{\Omega^{\text{bare}}} \times \underbrace{\frac{\langle 23 \rangle \langle 56 \rangle}{\langle 25 \rangle \langle 36 \rangle}}_{\mathcal{J}^{-1}} = \frac{\delta^4(P) \delta^6(Q) \delta(\langle 12 \rangle) \delta(\langle 45 \rangle)}{\langle 34 \rangle \langle 16 \rangle \langle 25 \rangle}$$

On the pole at infinity, $\langle 25 \rangle = 0$, and replacing $1/\langle 25 \rangle \rightarrow \delta(\langle 25 \rangle)$ we do get the correct on-shell function for the on-shell diagram (3.38). Note that this is all calculated on a very constrained kinematics so there are many equivalent ways how to write the formula (3.39). It is clear how the calculation of the UV pole generalizes for any one-loop MHV on-shell diagram,

(3.40)

where we conveniently reduced the n -gon into a “core” box diagram connected to white blobs of external legs. The λ -spinors of all external legs connected to the same white vertex are proportional, so the kinematics are highly constrained. As before, on the pole at infinity $[ij] = 0$ we replace the box diagram by a simple tree-level factorization graph, and all but legs i, j are now connected to the same white vertex,

(3.41)

We can expand the white vertex into three-point vertices in an arbitrary way without changing the kinematics or the on-shell function. In the original on-shell diagram many collinearity conditions are already imposed,

(3.42) $\lambda_{i+1} \sim \lambda_{i+2} \sim \dots \sim \lambda_{j-1} \quad \text{and} \quad \lambda_{j+1} \sim \lambda_{j+2} \sim \dots \sim \lambda_{i-1}$

The pole at infinity is accessed by $\langle pq \rangle = 0$ where p, q are from two different sets of labels of external legs attached to white vertices, $p \in \{i+1, \dots, j-1\}$ and $q \in \{j+1, \dots, i-1\}$, or alternatively we can access the pole by sending $[ij] = 0$, which are all equivalent conditions.

3.2.4 Arbitrary n -gons and helicities

Our goal now is to generalize the rule for the UV pole for an n -gon with an arbitrary configuration of white and black vertices, and arbitrary k degree. The first example is a very simple six-point NMHV hexagon

$$(3.43) \quad \begin{array}{c} \begin{array}{c} \text{Diagram 1: A hexagon with vertices labeled 1 to 6. Vertices 1, 3, 5 are white (circles) and 2, 4, 6 are black (filled circles). A dashed red line connects the white vertices 1, 3, 5 in a cycle.} \\ \text{Diagram 2: A hexagon with vertices labeled 1 to 6. Vertices 1, 3, 5 are black (filled circles) and 2, 4, 6 are white (circles). A dashed red line connects the white vertices 2, 4, 6 in a cycle.} \end{array} \end{array} =$$

where we used the merge-expand moves to redraw the diagram. We can see that the core diagram is again a box, hence the rule for the UV pole is the same as before and we have not learnt anything new. This makes it clear that the next qualitatively new case is the hexagon diagram where the white and black vertices are alternating, and the core diagram is a proper hexagon that can not be simplified,

$$(3.44) \quad \begin{array}{c} \begin{array}{c} \text{Diagram: A hexagon with alternating white and black vertices labeled A through F. A dashed red line connects the white vertices A, C, E in a cycle.} \end{array} \quad \begin{array}{l} \ell_A = \frac{\langle 23 \rangle}{\langle 13 \rangle} \lambda_1 \tilde{\lambda}_2 = \frac{[16]}{[26]} \lambda_1 \tilde{\lambda}_2, \quad \ell_B = \frac{\langle 12 \rangle}{\langle 13 \rangle} \lambda_3 \tilde{\lambda}_2 = \frac{[34]}{[24]} \lambda_3 \tilde{\lambda}_2, \\ \ell_C = \frac{\langle 45 \rangle}{\langle 35 \rangle} \lambda_3 \tilde{\lambda}_4 = \frac{[23]}{[24]} \lambda_3 \tilde{\lambda}_4, \quad \ell_D = \frac{\langle 34 \rangle}{\langle 35 \rangle} \lambda_5 \tilde{\lambda}_4 = \frac{[56]}{[46]} \lambda_4 \tilde{\lambda}_5, \\ \ell_E = \frac{\langle 16 \rangle}{\langle 15 \rangle} \lambda_5 \tilde{\lambda}_6 = \frac{[45]}{[46]} \lambda_5 \tilde{\lambda}_6, \quad \ell_F = \frac{\langle 56 \rangle}{\langle 15 \rangle} \lambda_1 \tilde{\lambda}_6 = \frac{[12]}{[26]} \lambda_1 \tilde{\lambda}_6. \end{array} \end{array}$$

The kinematics of this on-shell diagram is more involved. There is no chain of black/white vertices and no merge-expand operation can change it. There are two independent kinematical conditions imposed on external momenta but none of these conditions is a simple collinearity of λ or $\tilde{\lambda}$ spinors. Solving for internal momenta we learn that there are three cyclically related kinematical conditions

imposed,

$$(3.45) \quad \langle 1|2+3|4] = \langle 3|4+5|6] = \langle 5|6+1|2] = 0$$

One of these conditions is redundant, i.e. any two of them imply the third one. Looking at the denominator of on-shell loop momenta we can see that the pole at infinity can be approached by setting $\langle 13 \rangle = 0$. On the support of (3.45) this also sends

$$(3.46) \quad \langle 13 \rangle = \langle 15 \rangle = \langle 35 \rangle = 0 = [24] = [26] = [46]$$

In other words, $\lambda_1 \sim \lambda_3 \sim \lambda_5$ and $\tilde{\lambda}_2 \sim \tilde{\lambda}_4 \sim \tilde{\lambda}_6$ on the pole at infinity. The corresponding on-shell diagram is then

$$(3.47) \quad \begin{array}{c} \begin{array}{ccc} & 5 & 2 \\ & \uparrow & \downarrow \\ 3 & \leftarrow \bigcirc & \leftarrow \bullet & \leftarrow 4 \\ & \downarrow & \uparrow \\ & 1 & 6 \end{array} \end{array}$$

and the on-shell function is

$$(3.48) \quad \Omega_{\text{UV}} = \frac{[23]\delta(\langle 13 \rangle)\delta(\langle 35 \rangle)\delta([24])\delta(P)\delta(Q)\delta([56]\tilde{\eta}_4 + [45]\tilde{\eta}_6)}{[34]\langle 12 \rangle \langle 25 \rangle [25][56]^2}$$

It is very interesting that the kinematics in (3.44) is already quite constrained (3.45), but it is only after imposing one more condition and approaching the UV pole that it completely factorizes into $\lambda, \tilde{\lambda}$ collinearities (3.46).

We now evaluate the on-shell function for this diagram. The topology allows us to fix the GL(1)s in a symmetric way and choose the edge variables as

$$(3.49) \quad \begin{array}{c} \begin{array}{ccccc} & & 3 & & \\ & & \uparrow & & \\ 2 & \rightarrow & \bullet & \xrightarrow{\alpha_2} & \circ & \xrightarrow{\alpha_3} & 4 \\ & \nwarrow & & \nearrow & \nwarrow & \nearrow & \\ & \alpha_1 & & \alpha_4 & & \\ 1 & \rightarrow & \circ & \xrightarrow{\alpha_5} & \bullet & \rightarrow & 5 \\ & \nwarrow & & \nearrow & \nwarrow & \nearrow & \\ & \alpha_6 & & & \alpha_5 & & \\ & & 6 & & \uparrow & & \end{array} \end{array} \quad \begin{array}{l} \alpha_1 = \frac{\langle 23 \rangle}{\langle 13 \rangle}, \quad \alpha_2 = \frac{\langle 13 \rangle}{\langle 12 \rangle}, \\ \alpha_3 = \frac{\langle 45 \rangle}{\langle 35 \rangle}, \quad \alpha_4 = \frac{\langle 35 \rangle}{\langle 34 \rangle}, \\ \alpha_5 = \frac{\langle 16 \rangle}{\langle 15 \rangle}, \quad \alpha_6 = \frac{\langle 15 \rangle}{\langle 56 \rangle}. \end{array}$$

The Jacobian of the diagram is given by

$$(3.50) \quad \mathcal{J} = 1 - \alpha_1 \alpha_2 \alpha_3 \alpha_4 \alpha_5 \alpha_6 = \frac{\langle 12 \rangle \langle 34 \rangle \langle 56 \rangle - \langle 23 \rangle \langle 45 \rangle \langle 16 \rangle}{\langle 12 \rangle \langle 34 \rangle \langle 56 \rangle} = \frac{\langle 13 \rangle \langle 4 | 3+5 | 1 \rangle}{\langle 12 \rangle \langle 34 \rangle [12]}$$

where in the last equality we used the special kinematics (3.46). The on-shell functions can be then written in many equivalent ways thanks to (3.46), one representation is

$$(3.51) \quad \Omega = \underbrace{\frac{\langle 4 | 3+5 | 1 \rangle \langle 34 \rangle \langle 15 \rangle \delta(\Xi)}{[56]^2 [12] \langle 12 \rangle \langle 16 \rangle \langle 23 \rangle \langle 45 \rangle \langle 35 \rangle}}_{\Omega^{\text{bare}}} \times \underbrace{\frac{\langle 12 \rangle \langle 34 \rangle [12]}{\langle 13 \rangle \langle 4 | 3+5 | 1 \rangle}}_{\mathcal{J}^{-1}} = \frac{\langle 15 \rangle \langle 34 \rangle^2 \delta(\Xi)}{[56]^2 \langle 13 \rangle \langle 16 \rangle \langle 23 \rangle \langle 45 \rangle \langle 35 \rangle}$$

where we denoted all delta functions as

$$(3.52) \quad \delta(\Xi) \equiv \delta^4(P) \delta^6(\mathcal{Q}) \delta(\langle 1 | 2+3 | 4 \rangle) \delta(\langle 3 | 4+5 | 6 \rangle) \delta^3([56] \tilde{\eta}_4 + [46] \tilde{\eta}_5 + [45] \tilde{\eta}_6)$$

Looking at the denominator of the inverse Jacobian $\mathcal{J}^{-1}$ we can see two terms: the first is $\langle 4 | 3+5 | 1 \rangle$ – this is just a helicity factor which is canceled by the numerator of the bare form (and would continue to be canceled also for $\mathcal{N} < 3$), and then the pole at infinity $\langle 13 \rangle = 0$. Note that due to the special kinematics, the pole structure of Ω is not very transparent. The singularities, $\langle 23 \rangle = 0$, $\langle 45 \rangle = 0$, $\langle 16 \rangle = 0$, are all physical poles and correspond to erasing edges α_1 , α_3 , α_5 . Pole $[56] = 0$ is not present because it also implies $\langle 34 \rangle = 0$ and there is a cancelation between the numerator and the denominator (it also corresponds to non-erasable edge α_4 which is forbidden). We can make manifest an absence of this pole but at the expense of introducing some other spurious pole. This is a general feature of such expressions, because of mass dimension and little group weights any representation has some spurious terms that can be removed by trading them for another spurious object.

Let us look in more detail at the pole at infinity located at $\langle 13 \rangle = \langle 15 \rangle = \langle 35 \rangle = 0$. This pole is not present in Ω^{bare} as the factor $\langle 15 \rangle / \langle 35 \rangle$ is finite and no singularity is generated. This can be made manifest when choosing a particular parametrization. The Jacobian $\mathcal{J}^{-1}$ does introduce this pole and we can show after some work that the residue on the UV pole is equal to

$$(3.53) \quad \Omega_{\text{UV}} = \frac{[23] \delta(\langle 13 \rangle) \delta(\langle 35 \rangle) \delta([24]) \delta^4(P) \delta^6(\mathcal{Q}) \delta([56] \tilde{\eta}_4 + [45] \tilde{\eta}_6)}{[34] \langle 12 \rangle \langle 25 \rangle [25] [56]^2}$$

where the delta functions indicate the simple collinearity conditions. The expression (3.53) is equal to (3.48) which proves our conjecture. Note that in (3.53) the kinematics is incredibly constrained

(three extra conditions) and there are many equivalent ways how to write it. For arbitrary $\mathcal{N}$ the formula (3.51) is generalized to

$$(3.54) \quad \Omega = \frac{\langle 4|3+5|1 \rangle^{4-\mathcal{N}} \langle 34 \times 15 \rangle \delta(\Xi)}{[56]^2 [12] \langle 12 \times 16 \rangle \langle 23 \times 45 \rangle \langle 35 \rangle} \times \left(\frac{\langle 12 \times 34 \rangle [12]}{\langle 13 \times 4|3+5|1 \rangle} \right)^{4-\mathcal{N}}$$

and while the helicity factor $\langle 4|3+5|1 \rangle$ is canceled, we do generate a higher pole at infinity for $\langle 13 \rangle = 0$. We will discuss it in more details in the next section.

The next case is an arbitrary NMHV n -gon, with three black and $n-3$ white vertices. We can use the merge-expand moves to redraw this diagram as a core hexagon with three extra white blobs,

$$(3.55) \quad \begin{array}{c} \text{Diagram 1: A circular diagram with } n \text{ vertices. Three vertices are black (labeled } i, j, k \text{) and } n-3 \text{ are white. External lines are labeled } i-1, i+1, j-1, j+1, k-1, k+1, n, 1. \end{array} = \begin{array}{c} \text{Diagram 2: A core hexagon with three black vertices (labeled } i, j, k \text{) and three white vertices. External lines are labeled } i-1, i+1, j-1, j+1, k-1, k+1. \end{array}$$

All the λ spinors in each group are proportional,

$$(3.56) \quad \lambda_{i+1} \sim \lambda_{i+2} \sim \dots \sim \lambda_{j-1} \text{ etc.}$$

and in addition we have two independent mixed conditions, for example,

$$(3.57) \quad \langle i-1|P_i|j \rangle = \langle j-1|P_j|k \rangle = 0$$

where $P_i = p_i + p_{i+1} + \dots + p_{j-1}$ and $P_j = p_j + p_{j+1} + \dots + p_{k-1}$. The pole at infinity is accessed by a single condition when $\langle pq \rangle = 0$ where p, q are from two different white sectors. This collapses the whole kinematics and all external λ s attached to white vertices are proportional, and all external

$\tilde{\lambda}$ s attached to black vertices are proportional. As a result, we get an on-shell diagram

$$(3.58) \quad \begin{array}{c} \begin{array}{c} j-1 \\ \nearrow \\ \bullet \\ \nearrow \\ i+1 \end{array} \quad \begin{array}{c} j+1 \\ \nearrow \\ \bullet \\ \nearrow \\ i \end{array} \\ \begin{array}{c} \bullet \\ \bullet \\ \bullet \\ \bullet \\ \bullet \end{array} \quad \begin{array}{c} \bullet \\ \bullet \\ \bullet \\ \bullet \\ \bullet \end{array} \\ \begin{array}{c} i-1 \\ \nwarrow \\ \bullet \\ \nwarrow \\ k-1 \end{array} \quad \begin{array}{c} k \\ \nwarrow \\ \bullet \\ \nwarrow \\ k+1 \end{array} \end{array}$$

Note that all such tree-level on-shell diagrams are *non-planar* in a sense of the canonical ordering $1, 2, \dots, n$ we started with. The same procedure generalizes to higher k degrees. The “core” N^2 MHV one-loop on-shell diagram is an octagon,

$$(3.59) \quad \begin{array}{c} \begin{array}{c} 2 \\ \nearrow \\ \bullet \\ \nearrow \\ 1 \end{array} \quad \begin{array}{c} 3 \\ \nearrow \\ \bullet \\ \nearrow \\ 4 \end{array} \\ \begin{array}{c} \bullet \\ \bullet \\ \bullet \\ \bullet \\ \bullet \end{array} \quad \begin{array}{c} \bullet \\ \bullet \\ \bullet \\ \bullet \\ \bullet \end{array} \\ \begin{array}{c} 8 \\ \nwarrow \\ \bullet \\ \nwarrow \\ 7 \end{array} \quad \begin{array}{c} 6 \\ \nwarrow \\ \bullet \\ \nwarrow \\ 5 \end{array} \end{array}$$

Here we have four independent kinematical conditions imposed on the external kinematics, for example

$$(3.60) \quad \langle 1|2+3|4] = \langle 3|1+2|8] = \langle 7|8+1|2] = \langle 5|3+4|2] = 0$$

There are no collinearity conditions between λ s and $\tilde{\lambda}$ s. The four independent constraints (3.60) are quadratic in nature and there is no simplification. The solution for the internal on-shell loop momenta is

$$(3.61) \quad \begin{array}{c} \begin{array}{c} 2 \\ \nearrow \\ \bullet \\ \nearrow \\ 1 \end{array} \quad \begin{array}{c} 3 \\ \nearrow \\ \bullet \\ \nearrow \\ 4 \end{array} \\ \begin{array}{c} \bullet \\ \bullet \\ \bullet \\ \bullet \\ \bullet \end{array} \quad \begin{array}{c} \bullet \\ \bullet \\ \bullet \\ \bullet \\ \bullet \end{array} \\ \begin{array}{c} 8 \\ \nwarrow \\ \bullet \\ \nwarrow \\ 7 \end{array} \quad \begin{array}{c} 6 \\ \nwarrow \\ \bullet \\ \nwarrow \\ 5 \end{array} \end{array} \quad \begin{array}{l} \ell_A = \frac{\langle 23 \rangle}{\langle 13 \rangle} \lambda_1 \tilde{\lambda}_2, \quad \ell_B = \frac{\langle 12 \rangle}{\langle 13 \rangle} \lambda_3 \tilde{\lambda}_2, \\ \ell_C = \frac{\langle 45 \rangle}{\langle 35 \rangle} \lambda_3 \tilde{\lambda}_4, \quad \ell_D = \frac{\langle 34 \rangle}{\langle 35 \rangle} \lambda_5 \tilde{\lambda}_4, \\ \ell_E = \frac{\langle 67 \rangle}{\langle 57 \rangle} \lambda_5 \tilde{\lambda}_6, \quad \ell_F = \frac{\langle 56 \rangle}{\langle 57 \rangle} \lambda_7 \tilde{\lambda}_6, \\ \ell_G = \frac{\langle 18 \rangle}{\langle 17 \rangle} \lambda_7 \tilde{\lambda}_8, \quad \ell_H = \frac{\langle 78 \rangle}{\langle 17 \rangle} \lambda_1 \tilde{\lambda}_8. \end{array}$$

We see that the pole at infinity is accessed by imposing a simple condition $\langle 13 \rangle = 0$ which again collapses the whole kinematics to two sectors,

$$(3.62) \quad \lambda_1 \sim \lambda_3 \sim \lambda_5 \sim \lambda_7 \quad \text{and} \quad \tilde{\lambda}_2 \sim \tilde{\lambda}_4 \sim \tilde{\lambda}_6 \sim \tilde{\lambda}_8$$

and the corresponding on-shell diagram is

$$(3.63) \quad \begin{array}{c} \begin{array}{ccc} & 7 & 2 \\ & \uparrow & \downarrow \\ 5 & \nearrow & \nwarrow 4 \\ & \downarrow & \uparrow \\ 3 & \nearrow & \nwarrow 6 \\ & \downarrow & \uparrow \\ & 1 & 8 \end{array} \end{array}$$

We can see that the diagram again does not respect the canonical ordering of the original octagon (3.59). The on-shell function for the octagon is

$$(3.64) \quad \Omega = \underbrace{\frac{[18][14][23]\langle 56 \rangle^2 \langle 34 \rangle \langle 78 \rangle \langle 4|6+7|8][17]\delta(\Xi)}{[56]^2[78]^3[12]^3[34]\langle 23 \rangle^3 \langle 45 \rangle \langle 67 \rangle \langle 81 \rangle}}_{\Omega^{\text{bare}}} \times \underbrace{\frac{\langle 12 \rangle \langle 34 \rangle [87][12]}{\langle 13 \rangle \langle 4|6+7|8][17]}}_{\mathcal{J}^{-1}}$$

where the delta functions are

$$(3.65) \quad \begin{aligned} \delta(\Xi) &= \delta^4(P) \delta^6(\mathcal{Q}) \delta(\tilde{\eta}_4[56] + \tilde{\eta}_5[46] + \tilde{\eta}_6[45]) \delta(\tilde{\eta}_6[78] + \tilde{\eta}_7[68] + \tilde{\eta}_8[67]) \\ &\times \delta(\langle 3|4+5|6]) \delta(\langle 7|5+6|4]) \delta(\langle 5|6+7|8]) \delta(\langle 1|7+8|6]) \end{aligned}$$

The kinematics is incredibly constrained and there are many equivalent ways how to write Ω . Also many of the poles are equivalent, as it is evident from the figure, e.g. setting $\langle 12 \rangle = 0$ (erasing edge B) also sends $[34] = 0$ etc. The pole at infinity is again not present in the bare form – two factors of $\langle 35 \rangle \langle 57 \rangle$ are canceled by $\langle 17 \rangle^2$ in the numerator, and the form is regular for (3.62). The UV pole is produced by the inverse Jacobian and the residue on the pole (3.62) gives

$$(3.66) \quad \Omega_{\text{UV}} = \frac{\langle 34 \rangle \delta(\Xi_a)}{\langle 12 \rangle \langle 23 \rangle \langle 47 \rangle [14]^4 [16][18]}$$

where the delta functions are now denoted

$$(3.67) \quad \delta(\Xi_a) = \delta^4(P) \delta^6(\mathcal{Q}) \delta(\tilde{\eta}_4[18] + \tilde{\eta}_8[14]) \delta(\tilde{\eta}_4[16] + \tilde{\eta}_6[14]) \delta(\langle 13 \rangle) \delta(\langle 35 \rangle) \delta(\langle 57 \rangle) \delta([46]) \delta([28])$$

which reproduces the on-shell function for the tree on-shell diagram (3.63). Now we are ready to generalize to arbitrary one-loop diagram. It is enough to solve the problem only for a “core diagram” which for $N^{k-2}\text{MHV}$ is a $2k$ -gon with k black and k white alternating vertices.

(3.68)

In this diagram $2k - 4$ conditions on external kinematics are imposed. We can characterize them by two cyclic classes

(3.69)
$$\langle j | p_{j+1} + p_{j+2} | j+3 \rangle = \langle j | p_{j-1} + p_{j-2} | j-3 \rangle = 0$$

for all odd $j = 1, 3, \dots, 2k-1$. These are $2k$ conditions, 4 of them are redundant. The pole at infinity is approached by imposing $\langle pq \rangle = 0$ where p, q are attached to two different white vertices (alternatively $[cd] = 0$ where c, d are attached to black vertices). This completely collapses the kinematics into two sectors where all λ , resp. $\tilde{\lambda}$ are proportional,

(3.70)
$$\lambda_1 \sim \lambda_3 \sim \lambda_5 \sim \dots \sim \lambda_{2k-1}, \quad \tilde{\lambda}_2 \sim \tilde{\lambda}_4 \sim \tilde{\lambda}_6 \sim \dots \sim \tilde{\lambda}_{2k}$$

and the resulting on-shell diagram is

(3.71)

where all legs originally connected to $\overline{\text{MHV}}$ white vertices in (3.68) are now in the white vertex and all legs originally connected to MHV black vertices are now in the black vertex. For any other n -gon with chains of white and black vertices we can always use merge-expand moves to identify the core

polygon which contains the pole at infinity, and apply the same rule, i.e.

$$(3.72) \quad \begin{array}{c} \text{Diagram 1: A circular one-loop on-shell diagram with alternating black and white vertices. External legs are labeled with momenta like } i-1, i, i+1, j, j+1, k-1, k, k+1, l, l+1, m, m+1, n, n-1. \end{array} = \begin{array}{c} \text{Diagram 2: A tree-level graph where all external legs initially attached to white vertices are now attached together in the same white vertex, and the same applies to black vertices.} \end{array}$$

For an arbitrary one-loop on-shell diagram, such as the one above, the residue at infinity corresponds to a tree-level graph where all external legs initially attached to white vertices are now attached together in the same white vertex, and the same applies to black vertices,

$$(3.73) \quad \begin{array}{c} \text{Diagram 3: A tree-level graph with two vertices, one white and one black, connected by a single line. The white vertex has external legs labeled } i, k+1, l, k-1, j+1, n-1, m+1, 1. \text{ The black vertex has external legs labeled } k, i-1, 2, i+1, j, m, l+1, n. \end{array}$$

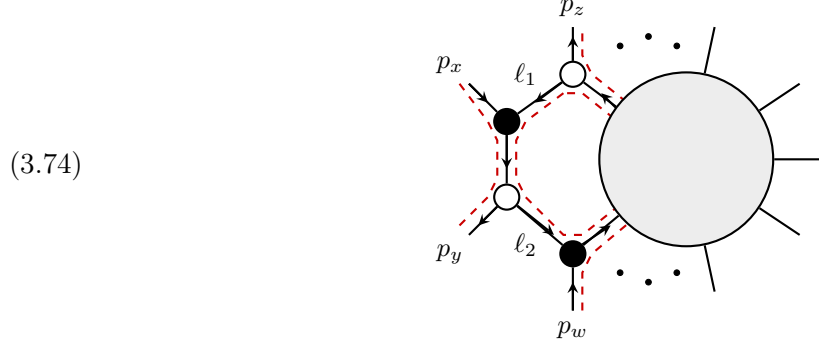
and the external momenta are then divided into two sets: in the first set (attached to white vertex) all λ 's are proportional, in the other set (attached to black vertex) all $\tilde{\lambda}$'s are proportional. Note that the cyclic ordering of the original graph is not preserved: the legs connected to black or white vertices are not ordered in any particular way, but the overall canonical ordering is broken. Hence we refer to this operation as a *non-planar twist*. As a result, the residue on the UV pole of a planar on-shell diagram is a lower-loop non-planar on-shell diagram.

3.3 All planar diagrams

In the simple examples we discussed earlier, the UV pole was always localized in a one-loop subgraph. We will show that this is a general property of planar on-shell diagrams, and using the rules of the previous section we can then calculate poles at infinity for an arbitrary higher loop planar graph.

3.3.1 Localization of UV poles in planar diagrams

Let us consider a core n -gon one-loop subdiagram somewhere in the middle of a larger on-shell diagram. As discussed we can always put any one-loop subdiagram in this form by merge-expand moves. Because the whole on-shell diagram is planar this n -gon is connected to any other one-loop subdiagram through a pair of adjacent legs. If the legs are not adjacent then the diagram is not planar and our proof does not apply.



where the central n -gon has a pole at infinity for $\langle zy \rangle = 0$ which is equivalent to $[xw] = 0$. This n -gon is glued into the rest of the diagram, so all momenta p_x, p_y, p_z, p_w are in general internal momenta. We want to prove that the UV pole does not propagate beyond this one-loop n -gon. In other words, external momenta p_x, p_y, p_z, p_w (from the point of view of the n -gon, otherwise these are internal momenta of the whole diagram) do not blow up for $\langle zy \rangle = [xw] = 0$. We label

$$(3.75) \quad p_x = \lambda_x \tilde{\lambda}_x, \quad p_y = \lambda_y \tilde{\lambda}_y, \quad p_z = \lambda_z \tilde{\lambda}_z, \quad p_w = \lambda_w \tilde{\lambda}_w$$

We focus on p_x, p_y but an analogous argument applies to other loops and we can also prove that p_z, p_w and other momenta attached to our n -gon. Using the kinematical properties of three point vertices we can write internal momenta ℓ_1, ℓ_2 in the following way,

$$(3.76) \quad \ell_1 = \frac{\alpha}{\langle zy \rangle} \lambda_z \tilde{\lambda}_x, \quad \ell_2 = \frac{\beta}{\langle zy \rangle} \lambda_y \tilde{\lambda}_w$$

where α, β are coefficients. We see that both ℓ_1, ℓ_2 blow up for $\langle zy \rangle = 0$ as expected. Note that α, β and spinors $\lambda_y, \lambda_z, \tilde{\lambda}_x, \tilde{\lambda}_w$ do not have a pole in $\langle zy \rangle = 0$. We need to prove that $\lambda_x, \tilde{\lambda}_y$ also do not have this pole, hence p_x, p_y are regular for $\langle zy \rangle = 0$. Using the momentum conservation

$$(3.77) \quad \ell_1 + p_x = \ell_2 + p_y$$

and the expressions (3.75), (3.76) we contract (3.77) with $\tilde{\lambda}_x$, resp. $\tilde{\lambda}_w$ and get

$$(3.78) \quad [xy] = -\frac{\beta[xw]}{\langle zy \rangle}, \quad \lambda_y[yw] = [xw]\lambda_x + \frac{\alpha[xw]}{\langle zy \rangle}\lambda_z$$

The ratio $[xw]/\langle zy \rangle$ is regular for $\langle zy \rangle = 0$ and has no pole. This means that both $[xy]$ and $[yw]$ are free of the UV pole, and hence also $\tilde{\lambda}_y$ and p_y are regular for $\langle zy \rangle = 0$. Very similarly, we can contract (3.77) with λ_y , resp. λ_z and conclude that $\langle xz \rangle$ and $\langle xy \rangle$ have no pole at infinity, hence also λ_x and p_x are free of this pole. This concludes the proof. We can use the same logic and prove that the UV pole $\langle zy \rangle = 0$ does not propagate to any other loop attached to our n -gon, hence this pole is only localized in this one-loop subgraph.

3.3.2 Three-loop example

We will demonstrate the procedure of calculating poles at infinity on several higher loop examples. First, we take a three-loop six-point NMHV leading singularity on-shell diagram and solve for the edge variables,

$$(3.79) \quad \begin{aligned} \alpha_1 &= -\frac{\langle 2|1+5|6 \rangle}{\langle 2|5+6|1 \rangle}, \quad \alpha_2 = -\frac{\langle 2|5+6|1 \rangle}{s_{156}}, \\ \alpha_3 &= \frac{\langle 2|5+6|1 \rangle}{\langle 3|5+6|1 \rangle}, \quad \alpha_5 = -\frac{\langle 2|1+5|6 \rangle}{\langle 2|1+6|5 \rangle}, \\ \alpha_4 &= \frac{\langle 2|1+5|6 \rangle \langle 3|5+6|1 \rangle}{\langle 34 \rangle [56] \langle 2|5+6|1 \rangle}, \\ \alpha_6 &= \frac{\langle 3|5+6|1 \rangle}{\langle 4|5+6|1 \rangle}, \quad \alpha_7 = \frac{[56]}{[16]}, \quad \alpha_8 = -\frac{\langle 34 \rangle}{\langle 23 \rangle}. \end{aligned}$$

In $\mathcal{N}=4$ SYM the on-shell function associated with this diagram is a famous R -invariant [83, 110] which appears in the context of tree-level recursion relations [110], leading singularities of loop integrands and Grassmannians [9, 10, 111, 112], and coefficients of polylogs in the hexagon bootstrap [113, 114, 115, 116],

$$(3.80) \quad \Omega = \int \frac{d\alpha_1 \dots d\alpha_8 \delta(C \cdot Z)}{\alpha_1 \alpha_2 \dots \alpha_8} = \frac{\delta^8(\mathcal{Q}) \delta^4(P) \delta([56] \tilde{\eta}_1 + [61] \tilde{\eta}_5 + [15] \tilde{\eta}_6)}{s_{156} \langle 23 \rangle \langle 34 \rangle \langle 2|3+4|5 \rangle \langle 4|5+6|1 \rangle [16] [56]} \equiv R[4, 5, 6, 1, 2]$$

Note that this expression does not depend on any helicity assignments because of the maximal supersymmetry. In $\mathcal{N}=3$ with the helicity assignment (3.79), we get

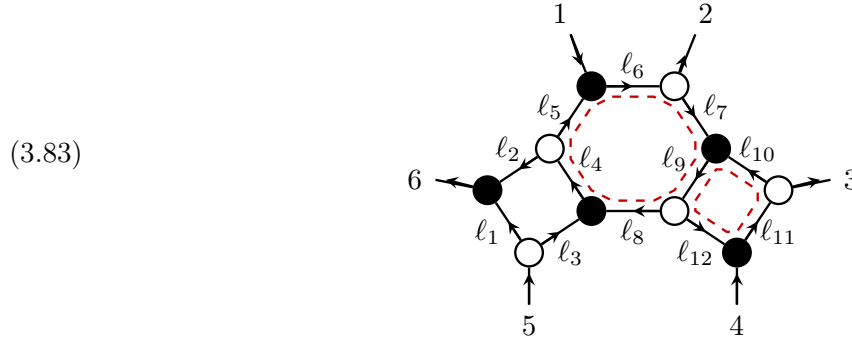
$$(3.81) \quad \Omega = \int \frac{d\alpha_1 d\alpha_2 \dots d\alpha_8}{\underbrace{\alpha_1 \alpha_2 \alpha_3 \alpha_4 \alpha_5 \alpha_6 \alpha_7 \alpha_8}_{\Omega^{\text{bare}}}} \times \frac{1}{\underbrace{1 - \alpha_2 \alpha_4 \alpha_7 \alpha_8 - \alpha_3 \alpha_6 \alpha_8}_{\mathcal{J}^{-1}}}.$$

which evaluates to

$$(3.82) \quad \Omega = \frac{\langle 4|1+5|6] \delta^6(\mathcal{Q}) \delta^4(P) \delta([56]\tilde{\eta}_1 + [61]\tilde{\eta}_5 + [15]\tilde{\eta}_6)}{\underbrace{s_{156} \langle 23 \rangle \langle 34 \rangle \langle 2|3+4|5] \langle 4|5+6|1] [16] [56]}_{\Omega^{\text{bare}}}} \times \frac{s_{156} \langle 23 \rangle \langle 4|5+6|1] [16]}{\underbrace{\langle 2|5+6|1] \langle 3|5+6|1] \langle 4|1+5|6]}_{\mathcal{J}^{-1}}}$$

$$= \frac{\delta^6(\mathcal{Q}) \delta^4(P) \delta([56]\tilde{\eta}_1 + [61]\tilde{\eta}_5 + [15]\tilde{\eta}_6)}{\langle 34 \rangle \langle 2|3+4|1] \langle 3|5+6|1] \langle 2|1+6|5] [56]}$$

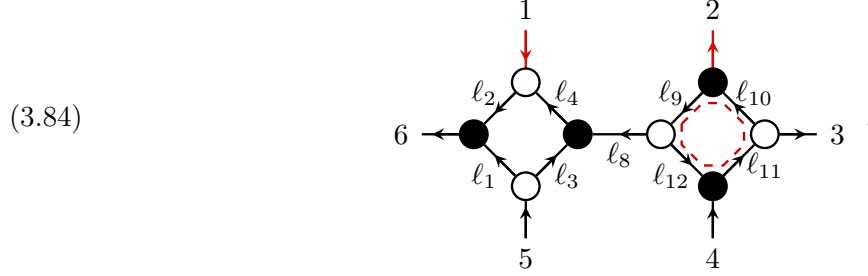
The inverse Jacobian adds two new poles $\langle 2|5+6|1]$ and $\langle 3|5+6|1]$ (the other term $\langle 4|1+5|6]$ is just a helicity factor). These are UV poles and originate from closed cycles in (3.79). In order to see which UV pole corresponds to a hexagon and which one corresponds to a box, we solve for the internal momenta (where we denoted $Q_{156} = p_1 + p_5 + p_6$ in all formulas),



$$\begin{aligned} \ell_1 &= \frac{\langle 2|Q_{156}|5]}{\langle 2|Q_{156}|6]} \tilde{\lambda}_6 \lambda_5, & \ell_2 &= \frac{\tilde{\lambda}_6 (\lambda_2 \cdot Q_{34}) \cdot Q_{56}}{\langle 2|Q_{156}|6]}, & \ell_3 &= \frac{[56] \lambda_5 (\lambda_2 \cdot Q_{34})}{\langle 2|Q_{156}|6]} \\ \ell_4 &= \frac{[16] (\lambda_2 \cdot Q_{34}) \cdot Q_{56} (\lambda_2 \cdot Q_{34})}{\langle 2|Q_{156}|6] \langle 2|Q_{156}|1]}, & \ell_5 &= \frac{\tilde{\lambda}_1 (\lambda_2 \cdot Q_{34}) \cdot Q_{56}}{\langle 2|Q_{156}|1]}, & \ell_6 &= \frac{s_{156}}{\langle 2|Q_{156}|1]} \tilde{\lambda}_1 \lambda_2 \\ \ell_7 &= \frac{\lambda_2 (\tilde{\lambda}_1 \cdot Q_{56}) \cdot Q_{34}}{\langle 2|Q_{156}|1]}, & \ell_8 &= \frac{(\lambda_2 \cdot Q_{34}) (\tilde{\lambda}_1 \cdot Q_{56})}{\langle 2|Q_{156}|1]}, & \ell_9 &= \frac{\langle 23 \rangle (\tilde{\lambda}_1 \cdot Q_{56}) \cdot Q_{34} (\tilde{\lambda}_1 \cdot Q_{56})}{\langle 3|Q_{156}|1] \langle 2|Q_{156}|1]} \\ \ell_{10} &= \frac{\lambda_3 (\tilde{\lambda}_1 \cdot Q_{56}) \cdot Q_{34}}{\langle 3|Q_{156}|1]}, & \ell_{11} &= \frac{\langle 4|Q_{156}|1]}{\langle 3|Q_{156}|1]} \tilde{\lambda}_4 \lambda_3, & \ell_{12} &= \frac{\langle 34 \rangle \tilde{\lambda}_4 (\tilde{\lambda}_1 \cdot Q_{56})}{\langle 3|Q_{156}|1]} \end{aligned}$$

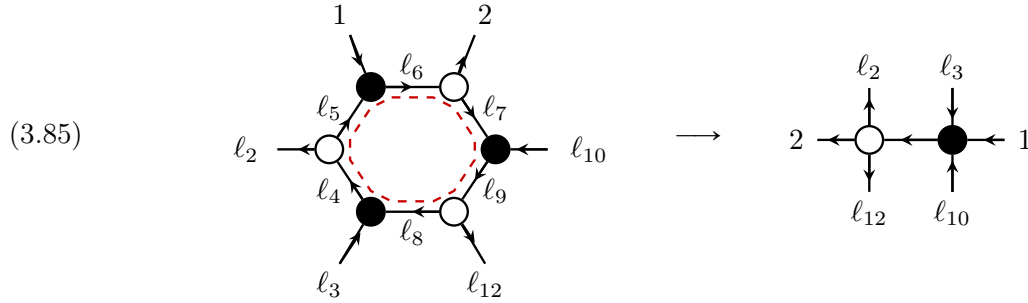
Looking at the denominator of solved on-shell loop momenta we see that $\langle 2|Q_{156}|6] = 0$ is the pole at infinity in the left box diagram, the pole at infinity in the hexagon is $\langle 2|Q_{156}|1] = 0$, and the pole

at infinity in the right box is $\langle 3|Q_{156}|1\rangle = 0$. The last two poles are accessible for this orientation as evident from the diagram (presence of both closed cycles) and the poles in the inverse Jacobian $\mathcal{J}^{-1}$, while the pole $\langle 2|1+5|6\rangle = 0$ can not be reached in this orientation. The inverse Jacobian also removes some of the irremovable edges. For example we can not remove α_2 edge (by sending $\alpha_2 \rightarrow \infty$) forcing $s_{156} = 0$ as it would lead to an illegal diagram,

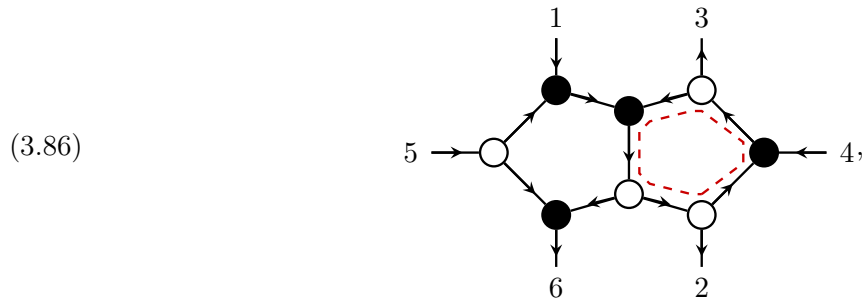


where we violated the rules about incoming/outgoing legs in black/white vertex.

Going back to the diagram (3.83) we concentrate on the pole at infinity in the hexagon subdiagram. There are two kinematical constraints imposed on the external legs of this hexagon $p_1, p_2, \ell_2, \ell_3, \ell_{10}, \ell_{12}$, while in the context of the whole diagram the last four momenta are internal and the actual external momenta $p_1, \dots, p_6$ are unconstrained. Using the rules developed in the last section, the residue on the UV pole turns the hexagon into a tree subdiagram,



Gluing back into the rest of the diagram we get,



where we used one particular way to expand the four-point black and white vertices. We can now treat it as a new on-shell diagram and assign edge variables, construct the C matrix and solve for them using delta functions

$$\begin{aligned}
 \alpha_1 &= \frac{[56]}{[16]}, \quad \alpha_2 = \frac{[15]}{[16]}, \quad \alpha_3 = \frac{\langle 23 \rangle}{\langle 42 \rangle}, \\
 \alpha_4 &= \frac{\langle 43 \rangle}{\langle 23 \rangle}, \quad \alpha_5 = \frac{\langle 2|1+5|6]}{\langle 23 \rangle[16]}, \\
 \alpha_6 &= \frac{\langle 3|5+6|1]}{\langle 23 \rangle[16]}, \quad \alpha_7 = \frac{\langle 23 \rangle[16]}{\langle 3|1+5|6]}.
 \end{aligned}$$

Note that in this diagram there is one extra kinematical condition imposed: $\langle 2|5+6|1] = 0$ which was exactly the hexagon UV pole in (3.83). The edge variables in (3.87) are not related in any obvious way with the edge variables in (3.86). Calculating the on-shell function for (3.87) we get

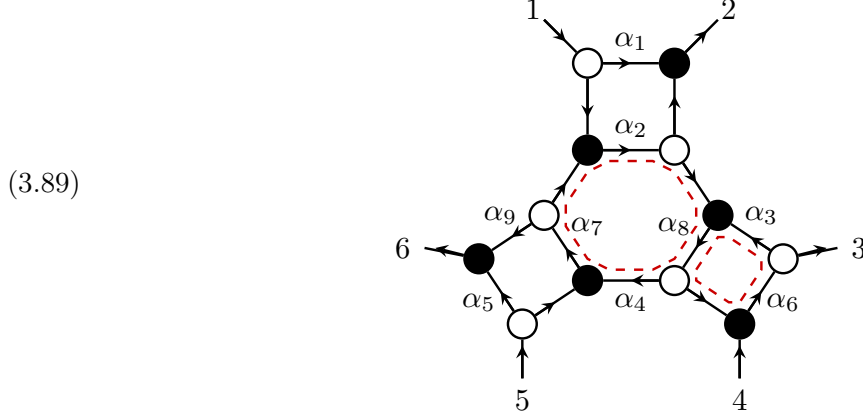
$$\begin{aligned}
 \Omega &= \int \frac{d\alpha_1 \dots d\alpha_7 \delta(C \cdot Z)}{\alpha_1 \alpha_2 \alpha_3 \alpha_4 \alpha_5 \alpha_6 \alpha_7} \times \frac{1}{1 - \alpha_3 \alpha_4 \alpha_5 \alpha_7} \\
 (3.88) \quad &= \frac{\langle 4|1+5|6] \delta^4(P) \delta^6(Q) \delta([56]\tilde{\eta}_1 + [61]\tilde{\eta}_5 + [15]\tilde{\eta}_6) \delta(\langle 2|5+6|1])}{s_{156} \langle 24 \rangle \langle 34 \rangle \langle 3|1+5|6] [56] [15]} \times \frac{\langle 24 \rangle \langle 3|1+5|6]}{\langle 23 \rangle \langle 4|1+5|6]} \\
 &= \frac{\delta^4(P) \delta^6(Q) \delta([56]\tilde{\eta}_1 + [61]\tilde{\eta}_5 + [15]\tilde{\eta}_6) \delta(\langle 2|5+6|1])}{s_{156} \langle 23 \rangle \langle 34 \rangle [56] [15]}
 \end{aligned}$$

which is the same as the residue of (3.82) on $\langle 2|5+6|1] = 0$ showing explicitly our procedure in action. Note that since we are on special kinematics, $s_{156} \langle 2q \rangle [1p] = \langle 2|1+5+6|p] \langle q|1+5+6|1]$ which is needed to show the equality.

3.3.3 Non-planarity from the UV pole

The pole at infinity does not preserve the planarity of the original on-shell diagram. We could already see it in our previous example where the original on-shell diagram (3.79) had a canonical ordering of external legs, while the on-shell diagrams we obtained as a result of the UV pole calculation had external legs ordered differently $123456 \rightarrow 134265$. In fact, this was just a special case of a more general phenomenon that the residue on a UV pole of a planar on-shell diagram is a

non-planar on-shell diagram. Let us consider the following example,



which is an on-shell diagram for a top cell of $G_+(3,6)$. This is a unique planar 9-dimensional (has 9 independent edge variables α_j) on-shell diagram and all other cells in $G_+(3,6)$ are associated with descendants of this diagram where one or more edges are removed. See [13] for more details on the stratification. We can also obtain it by adding a BCFW bridge on external legs 1, 2 of the previous on-shell diagram (3.79) – which was an eight-dimensional leading singularity diagram (with no free parameters and generic kinematics). Hence in our diagram there is one unfixed parameter α_1 . The solution for the edge variables is

(3.90)

$$\begin{aligned} \alpha_2 &= -\frac{\langle \hat{2}|5+6|\hat{1}\rangle}{s_{\hat{1}56}}, & \alpha_3 &= \frac{\langle \hat{2}|5+6|\hat{1}\rangle}{\langle 3|5+6|\hat{1}\rangle}, & \alpha_4 &= -\frac{\langle 34\rangle[56]\langle \hat{2}|5+6|\hat{1}\rangle}{\langle \hat{2}|3+4|6\rangle\langle 3|5+6|\hat{1}\rangle} \\ \alpha_5 &= -\frac{\langle \hat{2}|3+4|5\rangle}{\langle \hat{2}|3+4|6\rangle}, & \alpha_6 &= \frac{\langle 4|5+6|\hat{1}\rangle}{\langle 3|5+6|\hat{1}\rangle}, & \alpha_7 &= \frac{[\hat{1}6]}{[56]}, & \alpha_8 &= -\frac{\langle \hat{2}3\rangle}{\langle 34\rangle}, & \alpha_9 &= \frac{\langle \hat{2}|5+6|\hat{1}\rangle}{\langle \hat{2}|3+4|6\rangle}. \end{aligned}$$

where we denoted $\hat{\lambda}_1 = \tilde{\lambda}_1 + \alpha_1 \tilde{\lambda}_2$ and $\hat{\lambda}_2 = \lambda_2 - \alpha_1 \lambda_1$. The on-shell function is

(3.91)

$$\Omega = \int \frac{d\alpha_1 \dots d\alpha_9 \delta(C \cdot \mathcal{Z})}{\alpha_1 \alpha_2 \alpha_3 \alpha_4 \alpha_5 \alpha_6 \alpha_7 \alpha_8 \alpha_9} \times \frac{1}{1 - \alpha_2 \alpha_4 \alpha_7 \alpha_8 - \alpha_3 \alpha_6 \alpha_8}$$

Plugging in the edge variables we get,

(3.92)

$$\Omega = \int \frac{d\alpha_1 \langle 4|\hat{1}+5|6\rangle \delta(\Xi)}{\underbrace{\alpha_1 s_{\hat{1}56} \langle \hat{2}3\rangle \langle 34\rangle \langle \hat{2}|3+4|5\rangle \langle 4|5+6|\hat{1}\rangle [\hat{1}6] [56]}_{\Omega^{\text{bare}}}} \underbrace{\frac{s_{\hat{1}56} \langle \hat{2}3\rangle \langle 4|5+6|\hat{1}\rangle [\hat{1}6]}{\langle \hat{2}|5+6|\hat{1}\rangle \langle 3|5+6|\hat{1}\rangle (\langle 4|\hat{1}+5|6\rangle)}}_{\mathcal{J}^{-1}}$$

(3.93)

$$= \int \frac{d\alpha_1 \delta(\Xi)}{\alpha_1 \langle 34\rangle \langle \hat{2}|5+6|\hat{1}\rangle \langle 3|5+6|\hat{1}\rangle \langle \hat{2}|3+4|5\rangle [56]},$$

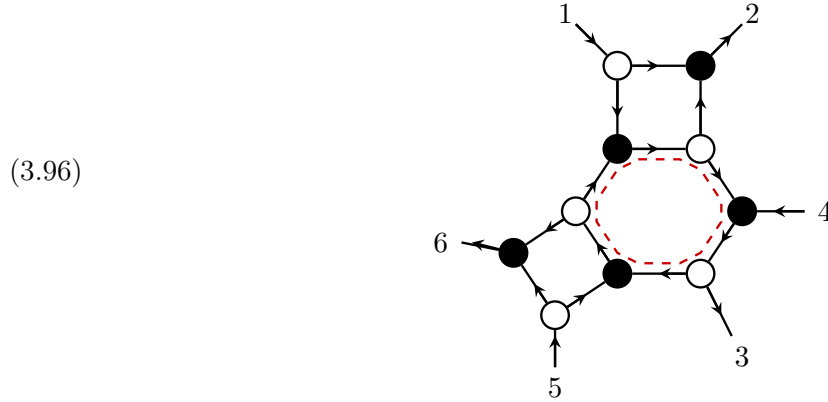
where the delta function

$$(3.94) \quad \delta(\Xi) \equiv \delta^6(\mathcal{Q}) \delta^4(P) \delta([56]\tilde{\eta}_1 + \alpha_1[56]\tilde{\eta}_2 + [6\hat{1}]\tilde{\eta}_5 + [\hat{1}5]\tilde{\eta}_6).$$

We can recognize the usual BCFW shift for $\tilde{\eta}_1$ in the super delta function. Note that the $\alpha_1 = 0$ pole erases a corresponding edge and we reproduce the diagram already encountered (3.79). In $\mathcal{N}=4$ SYM theory there are six erasable edges which do not impose constraints on external kinematics (poles in α_1 in the Ω^{bare}). They correspond to six co-dimension-1 (i.e. eight-dimensional) cells of $G_+(3,6)$ and their on-shell functions are the six R -invariants. For our orientation (3.89) in $\mathcal{N}<3$ SYM we have three IR poles and two UV poles from the inverse Jacobian. The UV pole in the box is accessed by $\langle 3|5+6|\hat{1}\rangle = 0$, the residue is

$$(3.95) \quad \text{Res}_{\alpha_1 = -\frac{\langle 3|5+6|1\rangle}{\langle 3|5+6|2\rangle}} [\Omega] = \frac{\delta^6(\mathcal{Q})\delta^4(P)\delta([24]\tilde{\eta}_1 + [14]\tilde{\eta}_2 + [12]\tilde{\eta}_4)}{\langle 56\rangle[12]\langle 3|5+6|1\rangle\langle 3|1+2|4\rangle\langle 6|1+2|4\rangle}$$

Based on our diagrammatic rule this UV pole corresponds to an on-shell diagram,



Calculating the on-shell function we get

$$(3.97) \quad \Omega_{\text{UV}} = \underbrace{\frac{\langle 5|1+4|2\rangle \delta^6(\mathcal{Q})\delta^4(P)\delta([24]\tilde{\eta}_1 + [14]\tilde{\eta}_2 + [12]\tilde{\eta}_4)}{s_{124}\langle 35\rangle\langle 56\rangle\langle 3|2+4|1\rangle\langle 6|1+2|4\rangle[12][24]}}_{\Omega^{\text{bare}}} \times \underbrace{\frac{s_{124}\langle 35\rangle[24]}{\langle 3|1+2|4\rangle\langle 5|1+4|2\rangle}}_{\mathcal{J}^{-1}} \\ = \frac{\delta^6(\mathcal{Q})\delta^4(P)\delta([24]\tilde{\eta}_1 + [14]\tilde{\eta}_2 + [12]\tilde{\eta}_4)}{\langle 56\rangle[12]\langle 3|5+6|1\rangle\langle 3|1+2|4\rangle\langle 6|1+2|4\rangle}$$

which agrees with (3.95). The UV pole in the hexagon subdiagram corresponds to $\langle \hat{2}|5+6|\hat{1}\rangle = 0$ which is a quadratic condition in α_1 ,

$$(3.98) \quad \langle 2|5+6|1\rangle - \alpha_1\langle 1|5+6|1\rangle + \alpha_1\langle 2|5+6|2\rangle + \alpha_1^2\langle 1|5+6|2\rangle = 0$$

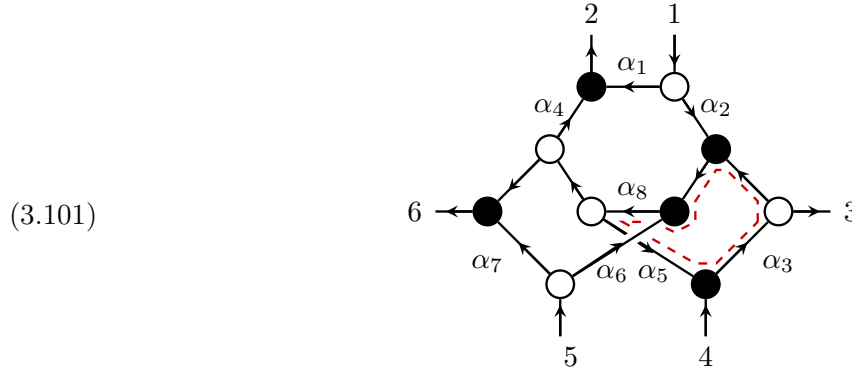
and leads to two solutions,

$$(3.99) \quad \alpha_1^\pm = \frac{(\langle 2|5+6|2\rangle - \langle 1|5+6|1\rangle) \pm \sqrt{(\langle 1|5+6|1\rangle + \langle 2|5+6|2\rangle)^2 - 4s_{12}s_{56}}}{2\langle 1|5+6|2\rangle}.$$

Plugging back into the form for the top cell (3.93) using $\langle \hat{2}|5+6|\hat{1}\rangle = \langle 1|5+6|2\rangle(\alpha_1 - \alpha_1^+)(\alpha_1 - \alpha_1^-)$, we get an on-shell function which contains square-roots of kinematical variables.

$$(3.100) \quad \Omega = \int \frac{d\alpha_1 \delta^6(\mathcal{Q}) \delta^4(P) \delta([56]\tilde{\eta}_1 + \alpha_1[56]\tilde{\eta}_2 + [\hat{6}\hat{1}]\tilde{\eta}_5 + [\hat{1}5]\tilde{\eta}_6)}{\alpha_1 \langle 34 \rangle \langle 1|5+6|2\rangle (\alpha_1 - \alpha_1^+) (\alpha_1 - \alpha_1^-) (\langle 3|5+6|1\rangle + \alpha_1 \langle 3|5+6|2\rangle) (\langle 2|3+4|5\rangle - \alpha_1 \langle 1|3+4|5\rangle) [56]}.$$

The quadratic (or higher order) equation for a parameter we localize and the presence of roots is not something unusual. For example, it also happens for the four-mass-box leading singularity, and in general we get very complicated structures. The UV pole in (3.79) can be again calculated diagrammatically using our operation: replacing the hexagon with the tree diagram (3.47). But now, something new happens: as a result, we get a *non-planar on-shell diagram*



The expression for the on-shell diagram using edge variables is the same as in the planar case and we can evaluate

$$(3.102) \quad \Omega_{UV} = \pm \frac{\delta^6(\mathcal{Q}) \delta^4(P) \delta([56]\tilde{\eta}_1 + \alpha_1^\pm [56]\tilde{\eta}_2 + [\hat{6}\hat{1}]\tilde{\eta}_5 + [\hat{1}5]\tilde{\eta}_6)}{\alpha_1^\pm \langle 34 \rangle \langle 1|5+6|2\rangle (\alpha_1^- - \alpha_1^+) s_{\hat{1}56} \langle \hat{2}3 \rangle [\hat{1}5] [56]}$$

where $\hat{1}$ and $\hat{2}$ are defined by the same convention as (3.90) and $\alpha_1^\pm$ is:

$$(3.103) \quad \alpha_1^\pm = \frac{(\langle 2|5+6|2\rangle - \langle 1|5+6|1\rangle) \pm \sqrt{(\langle 1|5+6|1\rangle + \langle 2|5+6|2\rangle)^2 - 4s_{12}s_{56}}}{2\langle 1|5+6|2\rangle}.$$

which agrees with the residue on either the plus or minus pole of (3.100). This six-point NMHV non-planar diagram corresponds to one of the eight dimensional cells in $G(3,6)$ (not the positive

part). All these cells for $G(3, 6)$ were classified in [60], and the subset of them relevant for BCFW recursion relations (for higher k and n) in [65].

3.3.4 UV poles and modified GRTs

One of the most important features of on-shell functions Ω for leading singularity on-shell diagrams is that they satisfy Global Residue Theorems (GRTs). The leading singularity on-shell diagram has a dimensionality $2n-4$ and the on-shell function has neither free parameters, nor are there any constraints imposed on external kinematics (other than momentum conservation). For the on-shell diagram of dimensionality $2n-3$, the on-shell function (after solving for edge variables) has one free parameter z , i.e. $\Omega(z)$. We can write Cauchy's integral theorem for this function,

$$(3.104) \quad \oint dz \Omega(z) = 0.$$

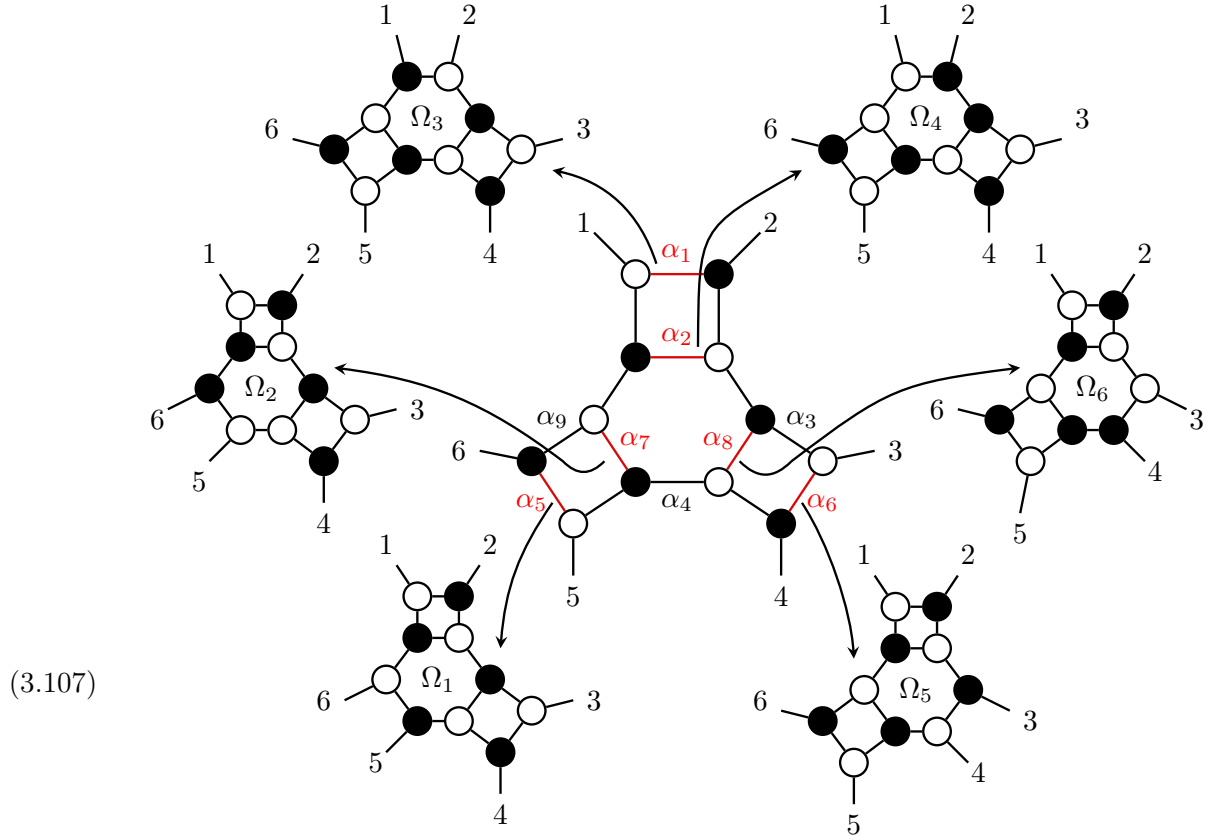
If the on-shell diagram has no pole at infinity, then all poles in z come from erasable edges and we can write

$$(3.105) \quad \text{GRT} = \sum_j \Omega_j = 0 \quad \text{where} \quad \Omega_j = \text{Res}_{z=z_j} \Omega(z)$$

Let us look at the most famous example which is our diagram (3.89). In $\mathcal{N}=4$ SYM theory the on-shell form is given by $\Omega = \Omega^{\text{bare}}$,

$$(3.106) \quad \Omega = \int \frac{d\alpha_1 \dots d\alpha_9 \delta(C \cdot Z)}{\alpha_1 \alpha_2 \dots \alpha_9} = \int \frac{d\alpha_1 \delta(\Xi)}{\alpha_1 s_{156} \langle \hat{2}3 \rangle \times 34 \times \langle \hat{2} | 3+4 | 5 \rangle \langle 4 | 5+6 | \hat{1} \rangle [\hat{1}6] [56]}.$$

There are six poles in this on-shell function corresponding to erasing edges $\alpha_1, \alpha_2, \alpha_5, \alpha_6, \alpha_7, \alpha_8$. Going to each of these six boundaries can be illustrated as follows,



where each arrow starts at the edge that is being erased, and all erasable edges are shown in red.

This leads to the GRT in the form

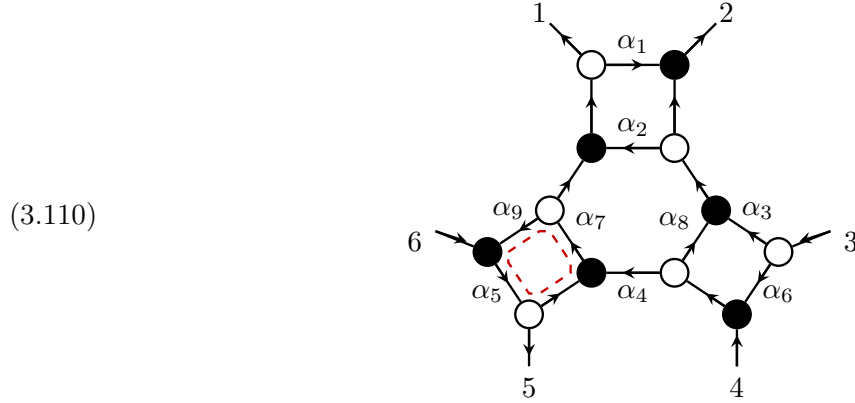
$$(3.108) \quad \sum_{i=1}^6 \Omega_i = 0, \quad \text{with} \quad \Omega_1 = \frac{\delta^4(P) \delta^8(Q) \delta([34] \tilde{\eta}_5 + [45] \tilde{\eta}_3 + [53] \tilde{\eta}_4)}{s_{345} [34] [45] \langle 16 \rangle \times 12 \times 2 | 3+4 | 5 \rangle \langle 6 | 1+2 | 3 \rangle}$$

where $\Omega_1 = R[23456]$ is a famous R -invariant, and all other Ω_j are related by cyclic shifts. The BCFW recursion relations for two different shifts represent the six-point NMHV tree-level as

$$(3.109) \quad \mathcal{A}_{6,3}^{\text{tree}} = \Omega_1 + \Omega_3 + \Omega_5 = -\Omega_2 - \Omega_4 - \Omega_6$$

where the equality of two representations is guaranteed by the GRT (3.108). All these relations were crucial in the early discovery of momentum twistor polytopes [117, 118], on-shell diagrams and positive Grassmannians [8, 9, 10, 13] and later the Amplituhedron construction [14, 16].

For $\mathcal{N} < 4$ SYM the Global Residue Theorems change dramatically. The Jacobian removes some of the poles in $\Omega(z)$ but also adds poles at infinity. The on-shell functions associated with IR poles also change as they are corrected by the Jacobian. We work in the context of $\mathcal{N} = 3$ and consider the same on-shell diagram for the top cell $G_+(3,6)$. At first, we choose the orientation with one pole at infinity,



Solving for edge variables and plugging in the formula we get for the on-shell function,

(3.111)

$$\begin{aligned} \Omega &= \int \frac{d\alpha_1 \langle 34 \rangle [\hat{1}5] \delta(\Xi)}{\underbrace{\alpha_1 s_{\hat{1}56} \langle \hat{2}3 \rangle \times 34 \times \langle \hat{2} | 3+4 | 5 \rangle \langle 4 | 5+6 | \hat{1} \rangle [\hat{1}6] [56]}_{\Omega^{\text{bare}}}} \underbrace{\frac{\langle \hat{2} | 3+4 | 5 \rangle [\hat{1}6]}{\langle \hat{2} | 3+4 | 6 \rangle [\hat{1}5]}}_{\mathcal{J}^{-1}} \\ &= \int \frac{d\alpha_1 \delta(\Xi)}{\alpha_1 s_{\hat{1}56} \langle \hat{2}3 \rangle \times 4 | 5+6 | \hat{1} \rangle \langle \hat{2} | 3+4 | 6 \rangle [56]}, \end{aligned}$$

where, again, $\delta(\Xi) \equiv \delta^6(\mathcal{Q})\delta^4(P)\delta([56]\tilde{\eta}_1 + \alpha_1[56]\tilde{\eta}_2 + [6\hat{1}]\tilde{\eta}_5 + [\hat{1}5]\tilde{\eta}_6)$. Taking the residue on the UV pole $\alpha_1 = \frac{\langle 2|3+4|6\rangle}{\langle 1|3+4|6\rangle}$ one obtains

(3.112)

$$\Omega_{\text{UV}} = \frac{\delta^6(\mathcal{Q})\delta^4(P)\delta([46]\tilde{\eta}_3 + [36]\tilde{\eta}_4 + [34]\tilde{\eta}_6)}{\langle 15 \rangle \times 12 \times \langle 2 | 3+4 | 6 \rangle \langle 5 | 2+4 | 3 \rangle [34] [46]},$$

which matches the form obtained by collapsing the box into a tree as per the prescription. We can use this to define a modified global residue theorem (GRT):

(3.113)

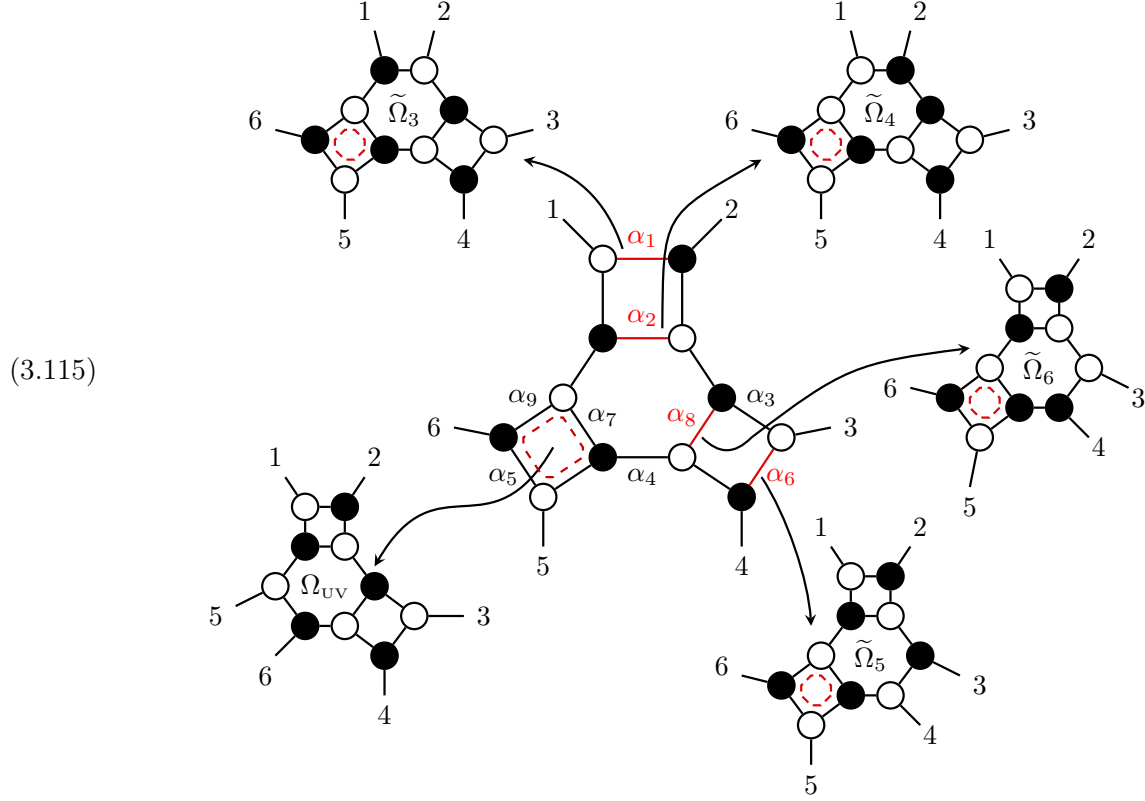
$$\text{GRT} = \tilde{\Omega}_3 + \tilde{\Omega}_4 + \tilde{\Omega}_5 + \tilde{\Omega}_6 + \Omega_{\text{UV}} = 0$$

with the IR poles given by

$$\tilde{\Omega}_3 = \frac{\delta^6(\mathcal{Q})\delta^4(P)\delta([56]\tilde{\eta}_1 + [16]\tilde{\eta}_5 + [15]\tilde{\eta}_6)}{s_{156}\langle 23 \rangle \times 4 | 5+6 | 1 \rangle \langle 2 | 3+4 | 6 \rangle [56]}, \quad \tilde{\Omega}_4 = \frac{\delta^6(\mathcal{Q})\delta^4(P)\delta([23]\tilde{\eta}_2 + [24]\tilde{\eta}_3 + [23]\tilde{\eta}_4)}{s_{156}\langle 1 | 5+6 | 4 \rangle [23] [34] \langle 15 \rangle \times 56 \rangle},$$

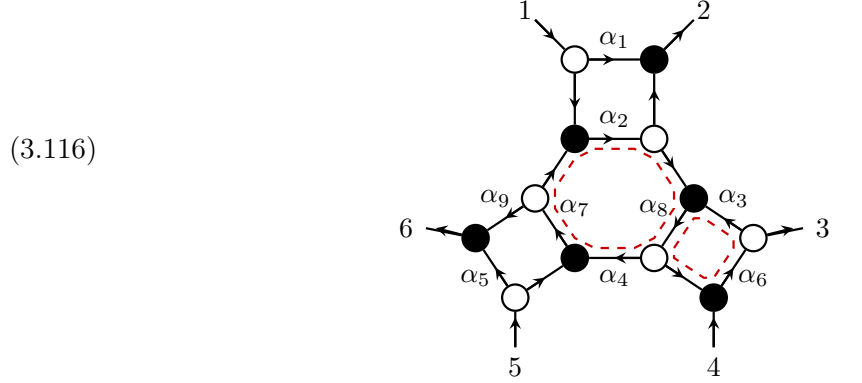
$$(3.114) \quad \tilde{\Omega}_5 = \frac{\delta^6(\mathcal{Q})\delta^4(P)\delta([23]\tilde{\eta}_1+[13]\tilde{\eta}_2+[12]\tilde{\eta}_3)}{s_{123}\langle 56\rangle\langle 5|1+2|3\rangle\langle 4|5+6|2\rangle[13][12]}, \quad \tilde{\Omega}_6 = \frac{\delta^6(\mathcal{Q})\delta^4(P)\delta([56]\tilde{\eta}_4+[46]\tilde{\eta}_5+[45]\tilde{\eta}_6)}{s_{456}\langle 23\rangle\langle 12\rangle\langle 1|5+6|4\rangle[56][46]}.$$

We can illustrate the GRT as before,



where each arrow starts at the edge that is being erased, and all erasable edges are shown in red, while the accessible UV pole is shown using the dashed internal cycle. Notice, how α_5 and α_7 now no longer are removable compared to the normal GRT in (3.107). Also note that the on-shell functions $\tilde{\Omega}$ contain Jacobians from the internal cycle in the box. This is a new type of GRT that relates the on-shell function for the UV pole to IR poles. Note that in this case all diagrams are planar, but the on-shell diagram for the UV pole does not respect the original canonical ordering with $5 \leftrightarrow 6$. Now we consider a more complicated case of the same diagram with the orientation discussed in

the previous subsection,



where the pole at infinity in the hexagon subdiagram is also accessible. In that case the Jacobian removes the IR poles for edges α_2 , α_6 , α_7 , α_8 leaving only two IR poles $\alpha_5 = \langle \hat{2}|3+4|5] = 0$ and $\alpha_1 = 0$. On the other hand we add 3 UV poles – a pair of poles from the hexagon loop for $\langle \hat{2}|5+6|\hat{1}] = 0$ and one UV pole from the right box subdiagram $\langle 3|5+6|\hat{1}] = 0$ as before. The global residue theorem in this case can be written as

$$(3.117) \quad \text{GRT} = \tilde{\Omega}_1 + \tilde{\Omega}_3 + \Omega_{\text{UV}_1}^+ + \Omega_{\text{UV}_1}^- + \Omega_{\text{UV}_2} = 0$$

where

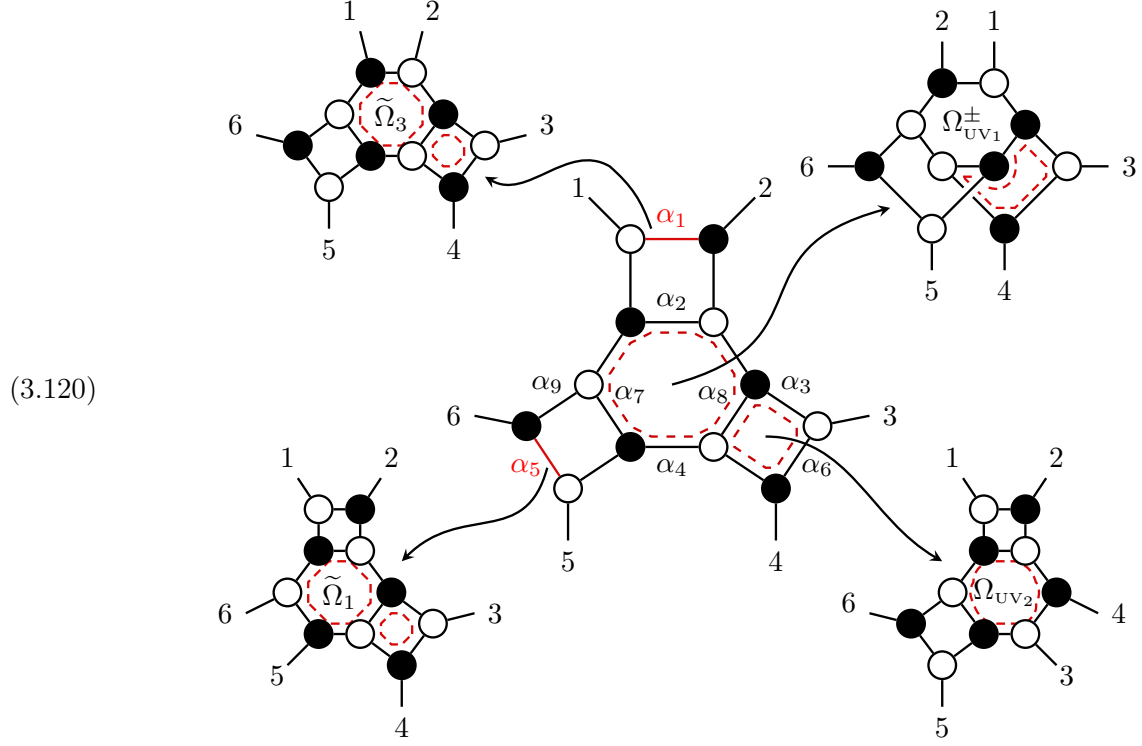
$$(3.118) \quad \begin{aligned} \tilde{\Omega}_1 &= \frac{\delta^4(P)\delta^6(\mathcal{Q})\delta([45]\tilde{\eta}_3 + [35]\tilde{\eta}_4 + [34]\tilde{\eta}_5)}{[34]\langle 12\rangle\langle 6|3+5|4]\langle 2|3+4|5]\langle 6|3+4|5]}, \quad \tilde{\Omega}_3 = \frac{\delta^6(\mathcal{Q})\delta^4(P)\delta([56]\tilde{\eta}_1 + [61]\tilde{\eta}_5 + [15]\tilde{\eta}_6)}{\langle 34\rangle\langle 2|5+6|1]\langle 3|5+6|1]\langle 2|1+6|5][56]} \\ \Omega_{\text{UV}_1}^\pm &= \pm \frac{\delta^6(\mathcal{Q})\delta^4(P)\delta([56]\tilde{\eta}_1 + \alpha_1^\pm[56]\tilde{\eta}_2 + [6\hat{1}]\tilde{\eta}_5 + [\hat{1}5]\tilde{\eta}_6)}{\langle 1|5+6|2](\alpha_1^- - \alpha_1^+)s_{\hat{1}56}\langle \hat{2}3\rangle\langle 34\rangle[56][\hat{1}5]\alpha_1^\pm} \\ \Omega_{\text{UV}_2} &= \frac{\delta^6(\mathcal{Q})\delta^4(P)\delta([24]\tilde{\eta}_1 + [14]\tilde{\eta}_2 + [12]\tilde{\eta}_4)}{\langle 56\rangle[12]\langle 3|5+6|1]\langle 3|1+2|4]\langle 6|1+2|4]}, \end{aligned}$$

with

$$(3.119) \quad \alpha_1^\pm = \frac{(\langle 2|5+6|2] + \langle 1|5+6|1]) \pm \sqrt{(\langle 1|5+6|1] + \langle 2|5+6|2])^2 - 4s_{12}s_{56}}}{2\langle 1|5+6|2]}.$$

This is a new type of residue theorem for on-shell functions for two IR poles (modified R -invariants) and two UV poles – one is a planar R -invariant for a different ordering and the other is a pair of non-planar on-shell functions. Note that while each on-shell function $\Omega_{\text{UV}_1}^\pm$ contains square-roots, they do cancel in the sum as evident from the GRT (3.117). Once again we can illustrate the GRT

in this case,



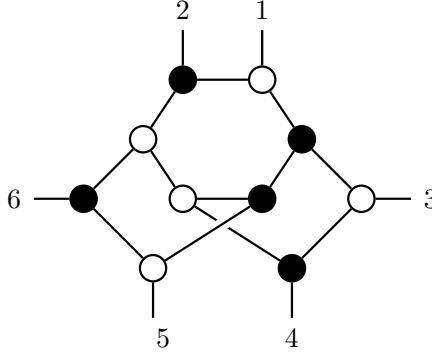
from which it is clear that the only two removable edges left are α_5 and α_1 .

In $\mathcal{N}=4$ SYM theory we do not have any poles at infinity whatsoever, so we can never generate non-planar diagrams from planar diagrams. However, even in $\mathcal{N} = 4$ SYM theory we can just directly consider non-planar on-shell diagrams, but with no orientations and internal cycles. The on-shell functions in $\mathcal{N} = 4$ SYM for the same diagrams are then

(3.121)

$$4 = \frac{\delta^4(P)\delta^8(Q)\delta([24]\tilde{\eta}_1 + [14]\tilde{\eta}_2 + [12]\tilde{\eta}_4)}{s_{124}\langle 35\rangle\langle 56\rangle\langle 3|5+6|1]\langle 6|1+2|4][12][24]}$$

(3.122)



$$= \pm \frac{\delta^4(P) \delta^8(Q) \delta([56] \tilde{\eta}_1 + \alpha_1^\pm [56] \tilde{\eta}_2 + [\hat{6}1] \tilde{\eta}_5 + [\hat{1}5] \tilde{\eta}_6)}{(\alpha_1^- - \alpha_1^+) s_{\hat{1}56} \langle \hat{2}4 \rangle \langle 34 \rangle \langle 1|5+6|2 \rangle \langle 3|\hat{1}+5|6 \rangle [56] [\hat{1}5] \alpha_1^\pm}$$

where the first on-shell function is just a regular R-invariant with $5 \leftrightarrow 6$ from canonical ordering, as also evident from the on-shell diagram. The on-shell function for the second, genuine non-planar, on-shell diagram is of a new type and is not related to an R-invariant as also evident from the presence of square-roots. It would be interesting to identify this form with the items in the list of appendix B of [60].

Having thoroughly gone through the procedure of taking UV poles of general n -gons and how to incorporate this into larger on-shell diagrams in $\mathcal{N}=3$ we are ready to tackle the higher order polls at infinity that arise in $\mathcal{N}<3$ in the next section.

3.4 Higher order poles at infinity

So far we have studied the minimal version of the problem of UV poles where individual on-shell diagrams preserve $\mathcal{N} = 3$ SUSY and all poles at infinity are simple poles. We want to generalize it to gauge theories with lower supersymmetries, including pure Yang-Mills theory. As explained in previous sections, there are two parts of the story: (i) locations of poles at infinity and the kinematics of on-shell diagrams which are produced on these poles; and (ii) the actual residues at infinity, i.e. on-shell functions associated with on-shell diagrams. The first part of the story is general and not specific to any theory, hence our discussion is valid for all relevant QFTs (with fundamental 3-point vertices) including gravity. The second part, the on-shell function, is specific to a particular theory and here, indeed, we solved only the case of $\mathcal{N} = 3$ and need to find a generalization to other theories.

3.4.1 UV pole as a residue

Let us consider $\mathcal{N} = 2$ SYM theory, the case of general $\mathcal{N}$ is just a trivial generalization. The main challenge is to figure out how to deal with higher poles at infinity which come from the inverse Jacobian $\mathcal{J}^{-1}$. For $\mathcal{N} = 2$ these are double poles. First, we redo the box on-shell diagram (3.8) and get

$$(3.123) \quad \begin{array}{c} \begin{array}{ccc} & 2 & \\ & \swarrow & \searrow \\ \bullet & & \circ \\ \downarrow & & \uparrow \\ \circ & & \bullet \\ \swarrow & & \searrow \\ 1 & & 4 \end{array} \end{array} = \frac{\delta^4(P)\delta^4(Q)\langle 24 \rangle^2}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 14 \rangle} \times \left(\frac{\langle 23 \rangle \langle 14 \rangle}{\langle 13 \rangle \langle 24 \rangle} \right)^2 = \frac{\langle 23 \rangle \langle 14 \rangle \delta^4(P)\delta^4(Q)}{\langle 12 \rangle \langle 34 \rangle \langle 13 \rangle^2}$$

where the supermomentum delta function has the same form, but now in η_i^I , $I = 1, 2$. Our goal is to calculate the residue on the UV pole $\langle 13 \rangle = 0$. Just based on the little group weights and the mass dimension, the resulting expression must be

$$(3.124) \quad \text{Res}_{\langle 13 \rangle=0} \frac{\langle 23 \rangle \langle 14 \rangle \delta^4(P)\delta^4(Q)}{\langle 12 \rangle \langle 34 \rangle \langle 13 \rangle^2} \sim \frac{\langle 24 \rangle \delta^4(P)\delta^4(Q)\delta(\langle 13 \rangle)}{\langle 12 \rangle \langle 34 \rangle}$$

Note that on the support of $\langle 13 \rangle = 0$ we have a relation $\langle 12 \rangle \langle 34 \rangle = \langle 23 \rangle \langle 14 \rangle$. We want to derive this result by taking the residue of (3.123).

The first issue is to find a good parametrization that allows us to approach the pole in a nice way. One approach is to write $\langle 13 \rangle = \lambda_1^{(1)}\lambda_3^{(2)} - \lambda_1^{(2)}\lambda_3^{(1)}$, fix three of the components and then solve for the last one, e.g. $\lambda_3^{(2)}$, making sure that $\langle 13 \rangle = 0$. However, we need to have in mind that through momentum conservation this parameter also appears in other momenta, which makes this approach less clean. Therefore, it is simplest to shift the external kinematics, e.g. using a BCFW shift $\lambda_3 \rightarrow \lambda_3 + z\lambda_4$, $\tilde{\lambda}_4 \rightarrow \tilde{\lambda}_4 - z\tilde{\lambda}_3$, we get $\Omega(z)$ but with an extra $\delta(z)$ ensuring we preserve the same kinematics. Then we take a residue on $\langle 13 \rangle \rightarrow \langle 13 \rangle + z\langle 14 \rangle = 0$ using the formula (3.126),

$$(3.125) \quad \Omega_{\text{UV}} = \text{Res}_{z=-\frac{\langle 13 \rangle}{\langle 14 \rangle}} \frac{\langle 14 \rangle (\langle 23 \rangle + z\langle 24 \rangle) \delta^4(P)\delta^4(Q)\delta(z)}{\langle 12 \rangle \langle 34 \rangle (\langle 13 \rangle + z\langle 14 \rangle)^2} = \frac{\langle 24 \rangle \delta^4(P)\delta^4(Q)\delta(\langle 13 \rangle)}{\langle 12 \rangle \langle 34 \rangle}$$

where we used $\delta\left(-\frac{\langle 13 \rangle}{\langle 14 \rangle}\right) = -\frac{\delta(\langle 13 \rangle)}{\langle 14 \rangle}$ and for the calculation of the residue on the higher pole we used the definition

$$(3.126) \quad \text{Res}_{x=x_0} f(x) = \frac{1}{(m-1)!} \lim_{x \rightarrow x_0} \frac{d^{m-1}}{dx^{m-1}} ((x-x_0)^m f(x))$$

Same result is obtained using the shift $\lambda_3 \rightarrow \lambda_3 + z\lambda_2$, $\tilde{\lambda}_2 \rightarrow \tilde{\lambda}_2 - z\tilde{\lambda}_3$,

$$(3.127) \quad \Omega_{\text{UV}} = \text{Res}_{z=-\frac{\langle 13 \rangle}{\langle 12 \rangle}} \frac{\langle 14 \rangle \langle 23 \rangle \delta^4(P) \delta^4(Q) \delta(z)}{\langle 12 \rangle (\langle 34 \rangle + z \langle 24 \rangle) (\langle 13 \rangle + z \langle 12 \rangle)^2} = \frac{\langle 24 \rangle \delta^4(P) \delta^4(Q) \delta(\langle 13 \rangle)}{\langle 23 \rangle \langle 14 \rangle}$$

which is the same as (3.125) using $\langle 12 \rangle \langle 34 \rangle = \langle 14 \rangle \langle 23 \rangle$ on the support of $\delta(\langle 13 \rangle)$. We can show that the result is independent of the shift. For general $\mathcal{N} < 4$ we can repeat the same exercise and get,

$$(3.128) \quad \Omega_{\text{UV}} = \frac{\langle 24 \rangle^{3-\mathcal{N}} \delta^4(P) \delta^{2\mathcal{N}}(Q) \delta(\langle 13 \rangle)}{\langle 12 \rangle \langle 34 \rangle}$$

Note that m-shell function for the on-shell diagram (3.13) with additional helicity factor for $\mathcal{N} < 3$ supersymmetry. Therefore, our diagrammatic rule works perfectly here for any $\mathcal{N}$ for this case. Let us look at the double box on-shell diagram in $\mathcal{N} = 2$ SYM,

$$(3.129) \quad = \int dz \frac{\langle 24 \rangle^2 \delta^4(P) \delta^4(Q)}{z(\langle 23 \rangle + z \langle 24 \rangle) \langle 14 \rangle \langle 34 \rangle \langle 12 \rangle} \times \left(\frac{\langle 14 \rangle (\langle 23 \rangle + z \langle 24 \rangle)}{\langle 24 \rangle (\langle 13 \rangle + z \langle 14 \rangle)} \right)^2$$

$$= \int dz \frac{\langle 14 \rangle (\langle 23 \rangle + z \langle 24 \rangle) \delta^4(P) \delta^4(Q)}{z \langle 12 \rangle \langle 34 \rangle (\langle 13 \rangle + z \langle 14 \rangle)^2}$$

This on-shell diagram has only two poles: IR pole for $z = 0$ (deleting the far right edge) and the double UV pole which blows up the left loop. The residue on the IR pole is

$$(3.130) \quad \Omega_{\text{IR}} = \text{Res}_{z=0} \Omega(z) = \frac{\delta^4(P) \delta^4(Q) \langle 14 \rangle \langle 23 \rangle}{\langle 12 \rangle \langle 34 \rangle \langle 13 \rangle^2}$$

which is a box with an internal loop (3.123) as expected. The residue on the UV pole is

$$(3.131) \quad \Omega_{\text{UV}} = \text{Res}_{z=-\frac{\langle 13 \rangle}{\langle 14 \rangle}} \Omega(z) = -\frac{\delta^4(P) \delta^4(Q) \langle 14 \rangle \langle 23 \rangle}{\langle 12 \rangle \langle 34 \rangle \langle 13 \rangle^2}$$

and the sum $\Omega_{\text{IR}} + \Omega_{\text{UV}} = 0$ as dictated by the GRT for Ω . Note that the on-shell diagram associated with the UV pole (based on our rule) should be just a box without an internal loop and

an ordering (1342). However, the on-shell function obtained in (3.131) is not an on-shell function for this diagram,

(3.132)

But rather, is its given by

(3.133)

$$\Omega_{UV} = \underbrace{\frac{\langle 24 \rangle^2 \delta^4(P) \delta^4(Q)}{\langle 13 \rangle \langle 34 \rangle \langle 42 \rangle \langle 21 \rangle}}_{\Omega_D} \times \underbrace{\frac{\langle 14 \rangle \langle 23 \rangle}{\langle 13 \rangle \langle 24 \rangle}}_{\mathcal{K}}$$

where Ω_D is the on-shell function associated with the diagram and $\mathcal{K}$ is some prefactor. This formula generalizes to arbitrary $\mathcal{N}$,

(3.134)

$$\Omega_{UV} = \frac{\delta^4(P) \delta^{2\mathcal{N}}(Q) \langle 24 \rangle^{4-\mathcal{N}}}{\langle 13 \rangle \langle 34 \rangle \langle 42 \rangle \langle 21 \rangle} \times \left(\frac{\langle 14 \rangle \langle 23 \rangle}{\langle 13 \rangle \langle 24 \rangle} \right)^{3-\mathcal{N}} = \frac{(\langle 14 \rangle \langle 23 \rangle)^{3-\mathcal{N}} \delta^4(P) \delta^{2\mathcal{N}}(Q)}{\langle 13 \rangle^{4-\mathcal{N}} \langle 34 \rangle \langle 42 \rangle \langle 21 \rangle}$$

We can see that the situation is not as simple as in the $\mathcal{N}=3$ SYM case, the on-shell function is now corrected by some prefactor. This prefactor does not add any new poles, it only increases the degree of some of the poles. We will see shortly that it is generated by a derivative operator acting on this on-shell diagram.

3.4.2 Infinity operator

To start with, let us consider a generalization of the two-loop double box on-shell diagram,

(3.135)

where the one-loop box on-shell diagram with an internal loop is glued to an arbitrary other on-shell diagram through two on-shell internal legs ℓ_3, ℓ_4 . The on-shell function for the diagram for $\mathcal{N} = 2$

SYM can be then written as

$$(3.136) \quad \Omega = \int d^2\tilde{\eta}_{\ell_3} d^2\tilde{\eta}_{\ell_4} \int \frac{d^2\lambda_{\ell_3} d^2\tilde{\lambda}_{\ell_3}}{\text{GL}(1)} \int \frac{d^2\lambda_{\ell_4} d^2\tilde{\lambda}_{\ell_4}}{\text{GL}(1)} \left\{ \Omega_1 \times \Omega_2 \right\}$$

The on-shell functions for left and right parts of the diagram are:

$$(3.137) \quad \Omega_1 = \frac{\langle 2\ell_4 \rangle^2 \delta^4(P_1) \delta^4(Q_1)}{\langle 12 \rangle \langle 2\ell_3 \rangle \langle \ell_3 \ell_4 \rangle \langle \ell_4 1 \rangle} \left(\frac{\langle 1\ell_4 \rangle \langle 2\ell_3 \rangle}{\langle 2\ell_4 \rangle \langle 1\ell_3 \rangle} \right)^2 = \frac{\langle 2\ell_3 \rangle \langle 1\ell_4 \rangle \delta^4(P_1) \delta^4(Q_1)}{\langle 12 \rangle \langle \ell_3 \ell_4 \rangle \langle 1\ell_3 \rangle^2}$$

$$(3.138) \quad \Omega_2 = \delta^4(P_2) \delta^4(Q_2) \omega_2$$

where P_1, P_2 are momentum conservation and Q_1, Q_2 are supermomentum conservation for each part of the diagram. ω_2 for the right part of the diagram is arbitrary, and while it can contain additional delta functions or extra parameters, it does not depend on legs 1, 2 (only through ℓ_3, ℓ_4). Now we want to calculate the residue on the pole at infinity $\langle 1\ell_3 \rangle = 0$ of the box diagram. In order to do that we have to take λ_{ℓ_3} to be collinear with λ_1 . However, due to momentum conservation any changes in λ_{ℓ_3} also propagate into other momenta, and it is relevant when we calculate the derivative of the function in (3.126). We can remove this dependence by integrating over ℓ_4 , then ℓ_3 becomes independent,

$$(3.139) \quad \Omega = \int d^2\tilde{\eta}_{\ell_3} \int \frac{d^2\lambda_{\ell_3} d^2\tilde{\lambda}_{\ell_3}}{\text{GL}(1)} \left\{ \frac{\langle 12 \rangle}{\langle 1\ell_3 \rangle^2} \times \underbrace{\int d^2\tilde{\eta}_{\ell_4} \int \frac{d^2\lambda_{\ell_4} d^2\tilde{\lambda}_{\ell_4}}{\text{GL}(1)} \frac{\langle 2\ell_3 \rangle \langle 1\ell_4 \rangle \delta^4(P_1) \delta^4(Q_1)}{\langle 12 \rangle^2 \langle \ell_3 \ell_4 \rangle} \times \Omega_2}_{\omega} \right\}$$

where we defined an on-shell function ω which automatically implements the momentum conservation between ℓ_3, ℓ_4 and external legs, and the momentum ℓ_3 is now independent and not subject to any constraints. In this form we can take a residue on the pole $\langle 1\ell_3 \rangle = 0$. In order to approach the pole we parametrize

$$(3.140) \quad \lambda_{\ell_3} = \alpha \lambda_1 + \beta \lambda_2$$

and rewrite

$$(3.141) \quad \Omega = \int d^2\tilde{\eta}_{\ell_3} \int \frac{d\alpha d\beta d^2\tilde{\lambda}_{\ell_3}}{\text{GL}(1)} \left\{ \frac{1}{\beta^2} \times \omega \right\}$$

The UV pole is then given by the residue on $\beta = 0$,

$$(3.142) \quad \Omega_{\text{UV}} = \int d^2 \tilde{\eta}_{\ell_3} \int \frac{d\alpha d^2 \tilde{\lambda}_{\ell_3}}{\text{GL}(1)} \left\{ \frac{d\omega}{d\beta} \Big|_{\beta=0} \right\}$$

We can also undo the parametrization (3.140) and rewrite the result in a more invariant way,

$$(3.143) \quad \Omega_{\text{UV}} = \int d^2 \tilde{\eta}_{\ell_3} \int \frac{d^2 \lambda_{\ell_3} d^2 \tilde{\lambda}_{\ell_3}}{\text{GL}(1)} \left\{ \delta(\langle 1\ell_3 \rangle) \left\langle \lambda_2 \frac{d\omega}{d\lambda_{\ell_3}} \right\rangle \right\}$$

In the angle bracket we take the derivative of ω with respect to λ_{ℓ_3} and then contract with λ_2 . This is the on-shell function associated with the pole at infinity in (3.135). Now we need to compare (3.143) with the on-shell diagram for the collapsed box diagram,

$$(3.144) \quad \text{Diagram: A large shaded circle labeled } \Omega_2 \text{ with several external lines. To its left, a vertical line segment connects a white circle (top) and a black circle (bottom). The white circle has an incoming line labeled 1 and an outgoing line labeled } \ell_3 \text{ to the } \Omega_2 \text{ circle. The black circle has an incoming line labeled 2 and an outgoing line labeled } \ell_4 \text{ to the } \Omega_2 \text{ circle.}$$

The on-shell function for this diagram is

$$(3.145) \quad \Omega_{\text{UV}}^{\text{bare}} = \int d^2 \tilde{\eta}_{\ell_3} \int \frac{d^2 \lambda_{\ell_3} d^2 \tilde{\lambda}_{\ell_3}}{\text{GL}(1)} \left\{ \delta(\langle 1\ell_3 \rangle) \underbrace{\int d^2 \tilde{\eta}_{\ell_4} \int \frac{d^2 \lambda_{\ell_4} d^2 \tilde{\lambda}_{\ell_4}}{\text{GL}(1)} \frac{\langle 2\ell_4 \rangle \delta^4(P_1) \delta^4(Q_1)}{\langle 12 \rangle \langle \ell_3 \ell_4 \rangle} \times \Omega_2}_{\omega^{\text{bare}}} \right\}$$

where we again eliminated ℓ_4 . Comparing (3.143) and (3.145) we find the relation

$$(3.146) \quad \left\langle \lambda_2 \frac{d\omega}{d\lambda_{\ell_3}} \right\rangle = \frac{\langle 2\ell_3 \rangle [2\ell_3]}{\langle 12 \rangle [1\ell_3]} \left\langle \lambda_2 \frac{d\omega^{\text{bare}}}{d\lambda_{\ell_3}} \right\rangle$$

Plugging back we get

$$(3.147) \quad \Omega_{\text{UV}} = \int d^2 \tilde{\eta}_{\ell_3} \int \frac{d^2 \lambda_{\ell_3} d^2 \tilde{\lambda}_{\ell_3}}{\text{GL}(1)} \left\{ \delta(\langle 1\ell_3 \rangle) \frac{\langle 2\ell_3 \rangle [2\ell_3]}{\langle 12 \rangle [1\ell_3]} \left\langle \lambda_2 \frac{d\omega^{\text{bare}}}{d\lambda_{\ell_3}} \right\rangle \right\}$$

We can then interpret the result in the following way: the pole at infinity of the original on-shell diagram (3.135) for $\mathcal{N} = 2$ SYM is equal to the “infinity operator”

$$(3.148) \quad \mathcal{O} = \frac{\langle 2\ell_3 \rangle [2\ell_3]}{\langle 12 \rangle [1\ell_3]} \left\langle \lambda_2 \frac{d}{d\lambda_{\ell_3}} \right\rangle$$

acting on the bare on-shell function

$$(3.149) \quad \Omega_{\text{UV}} = \frac{\langle 2\ell_3 \rangle [2\ell_3]}{\langle 12 \rangle [1\ell_3]} \left\langle \lambda_2 \frac{d}{d\lambda_{\ell_3}} \right\rangle \otimes \text{Diagram}$$

or, more schematically $\Omega_{\text{UV}} = \mathcal{O} \otimes \Omega_{\text{UV}}^{\text{bare}}$ in the sense of the formulas (3.145) and (3.146). Note that for $\mathcal{N} = 3$ the operator $\mathcal{O} = \mathbb{I}$ and the residue of Ω on the pole at infinity is equal to $\Omega_{\text{UV}} = \Omega_{\text{UV}}^{\text{bare}}$. This relation is very easy to generalize for arbitrary $\mathcal{N} < 4$. In that case the operator $\mathcal{O}$ is given by

$$(3.150) \quad \mathcal{O}^{\mathcal{N}} = \frac{1}{(3-\mathcal{N})!} \left(\frac{\langle 2\ell_3 \rangle [2\ell_3]}{\langle 12 \rangle [1\ell_3]} \left\langle \lambda_2 \frac{d}{d\lambda_{\ell_3}} \right\rangle \right)^{3-\mathcal{N}}$$

Note that the derivative commutes with the prefactor. As an example of our procedure, we look at the double box on-shell diagram studied in the previous section. We interpret the diagram in the context of (3.135) as a one-loop box with legs 1, 2, ℓ_3 , ℓ_4 glued to the tree-level diagram with legs 3, 4, ℓ_3 , ℓ_4 ,

$$(3.151) \quad \text{Diagram 1} \times \text{Diagram 2}$$

The on-shell function Ω_2 is now

$$(3.152) \quad \Omega_2 = \frac{\langle \ell_3 4 \rangle^2 \delta^4(P_2) \delta^4(Q_2) \delta(\langle 4\ell_4 \rangle)}{\langle \ell_3 3 \rangle \langle 34 \rangle \langle \ell_3 \ell_4 \rangle} = \frac{\langle \ell_3 4 \rangle [3\ell_4] \delta^4(P_2) \delta^4(Q_2) \delta([3\ell_3])}{\langle \ell_3 3 \rangle \langle 34 \rangle \langle \ell_3 \ell_4 \rangle}$$

where we rewrote the delta function on the support of the momentum conservation,

$$(3.153) \quad \delta(\langle 4\ell_4 \rangle) = [\ell_4 3] \delta(\langle 4|\ell_4|3 \rangle) = -[\ell_4 3] \delta(\langle 4|\ell_3|3 \rangle) = \frac{[3\ell_4]}{\langle \ell_3 4 \rangle} \delta([3\ell_3])$$

where the function ω , further integrated over $\tilde{\eta}_{\ell_3}$ and $\tilde{\lambda}_{\ell_3}$ for simplicity (this does not interfere with λ_{ℓ_3} derivative) is equal to

$$\begin{aligned}
& \int d^2 \tilde{\eta}_{\ell_3} \int \frac{d^2 \tilde{\lambda}_{\ell_3}}{\text{GL}(1)} \omega = \\
& = \int d^2 \tilde{\eta}_{\ell_3} d^2 \tilde{\eta}_{\ell_4} \int \frac{d^2 \tilde{\lambda}_{\ell_3}}{\text{GL}(1)} \int \frac{d^2 \lambda_{\ell_4} d^2 \tilde{\lambda}_{\ell_4}}{\text{GL}(1)} \frac{\langle 1\ell_4 \rangle [\ell_4 3] \langle 2\ell_3 \rangle \langle 4\ell_3 \rangle \delta^4(P_1) \delta^4(Q_1) \delta^4(P_2) \delta^4(Q_2) \delta([3\ell_3])}{\langle 12 \rangle^2 \langle 34 \rangle \langle 3\ell_3 \rangle \langle \ell_3 \ell_4 \rangle^2} \\
(3.154) \quad & \int \frac{d^2 \tilde{\lambda}_{\ell_3}}{\text{GL}(1)} \frac{(\langle 1\ell_3 \rangle [\ell_3 3] + \langle 14 \rangle [34]) \langle 2\ell_3 \rangle \langle 4\ell_3 \rangle \delta^4(P) \delta^4(Q) \delta((p_3 + p_4 - \ell_3)^2) \delta([3\ell_3])}{\langle 12 \rangle^2 \langle 34 \rangle \langle 3\ell_3 \rangle}
\end{aligned}$$

where the integration over $\tilde{\eta}_{\ell_3}$, $\tilde{\eta}_{\ell_4}$ reproduced an overall supermomentum conservation $\delta^4(Q)$ and an extra factor $\langle \ell_3 \ell_4 \rangle^2$ while the integral over ℓ_4 lead to an overall momentum conservation $\delta^4(P)$ and an extra delta function $\delta((p_3 + p_4 - \ell_3)^2)$. We can also integrate over $\tilde{\lambda}_{\ell_3}$ which fixes $\tilde{\lambda}_{\ell_3} = \tilde{\lambda}_3$ and factorizes the delta function

$$(3.155) \quad \delta((p_3 + p_4 - \ell_3)^2) = \frac{1}{[34]} \delta(\langle 34 \rangle - \langle 4\ell_3 \rangle)$$

Plugging back we get

$$(3.156) \quad \int d^2 \tilde{\eta}_{\ell_3} \int \frac{d^2 \tilde{\lambda}_{\ell_3}}{\text{GL}(1)} \omega = \frac{\langle 14 \rangle \langle 2\ell_3 \rangle \langle 4\ell_3 \rangle \delta^4(P) \delta^4(Q) \delta(\langle 34 \rangle - \langle 4\ell_3 \rangle)}{\langle 12 \rangle^2 \langle 34 \rangle \langle 3\ell_3 \rangle}$$

Now we calculate the derivative with respect to λ_{ℓ_3} and contract with λ_2 ,

$$\begin{aligned}
(3.157) \quad & \int d^2 \tilde{\eta}_{\ell_3} \int \frac{d^2 \tilde{\lambda}_{\ell_3}}{\text{GL}(1)} \left\langle \lambda_2 \frac{d\omega}{d\lambda_{\ell_3}} \right\rangle = \frac{\langle 14 \rangle \langle 2\ell_3 \rangle (\langle 2\ell_3 \rangle \langle 34 \rangle^2 + \langle 23 \rangle \langle 4\ell_3 \rangle^2) \delta^4(P) \delta^4(Q) \delta((\langle 34 \rangle - \langle 4\ell_3 \rangle)^2)}{\langle 12 \rangle^2 \langle 34 \rangle \langle 3\ell_3 \rangle^2}
\end{aligned}$$

where all delta functions are here considered in the context of contour integrals and residues, i.e. under the integral $\delta(z) \sim \frac{1}{z}$. This is important since the derivative also acts on the delta function.

Plugging back into (3.143) and integrating over λ_{ℓ_3} we get

$$\begin{aligned}
(3.158) \quad & \Omega_{\text{UV}} = \delta^4(P) \delta^4(Q) \int d^2 \lambda_{\ell_3} \frac{\langle 14 \rangle \langle 2\ell_3 \rangle (\langle 2\ell_3 \rangle \langle 34 \rangle^2 + \langle 23 \rangle \langle 4\ell_3 \rangle^2) \delta^4(P) \delta^4(Q) \delta((\langle 34 \rangle - \langle 4\ell_3 \rangle)^2) \delta(\langle 1\ell_3 \rangle)}{\langle 12 \rangle^2 \langle 34 \rangle \langle 3\ell_3 \rangle^2}
\end{aligned}$$

In order to make the integration easier, we can expand ℓ_3 using (3.140),

$$\begin{aligned}
(3.159) \quad & \Omega_{\text{UV}} = \frac{\langle 14 \rangle}{\langle 12 \rangle \langle 34 \rangle} \delta^4(P) \delta^4(Q) \oint d\alpha d\beta \frac{\alpha^2 \langle 12 \rangle \langle 34 \rangle^2 + \alpha \langle 23 \rangle (\alpha \langle 14 \rangle + \beta \langle 24 \rangle)^2}{\beta (\langle 34 \rangle + \alpha \langle 14 \rangle + \beta \langle 24 \rangle)^2 (\alpha \langle 13 \rangle + \beta \langle 23 \rangle)^2} \\
& = - \frac{\langle 14 \rangle \langle 23 \rangle \delta^4(P) \delta^4(Q)}{\langle 12 \rangle \langle 34 \rangle \langle 13 \rangle^2}
\end{aligned}$$

where the integral was evaluated on the contour given by the delta functions, $\beta = 0$ and $\alpha = -\langle 34 \rangle / \langle 14 \rangle$ (this was a double pole). Now we compare (3.159) with the infinity operator (3.148) acting on the on-shell diagram

(3.160)

which is obtained by collapsing the left box with the UV pole in (3.151). The on-shell function ω^{bare} for Ω_2 for the diagram (3.160), again integrated over $\tilde{\eta}_{\ell_3}$ and $\tilde{\lambda}_{\ell_3}$ is

(3.161)

$$\begin{aligned} \int d^2 \tilde{\eta}_{\ell_3} \int \frac{d^2 \tilde{\lambda}_{\ell_3}}{\text{GL}(1)} \omega^{\text{bare}} &= \\ &= \int d^2 \tilde{\eta}_{\ell_3} d^2 \tilde{\eta}_{\ell_4} \int \frac{d^2 \tilde{\lambda}_{\ell_3}}{\text{GL}(1)} \int \frac{d^2 \lambda_{\ell_4} d^2 \tilde{\lambda}_{\ell_4}}{\text{GL}(1)} \frac{\langle 2\ell_4 \rangle \langle 4\ell_3 \rangle [3\ell_4] \delta^4(P_1) \delta^4(Q_1) \delta^4(P_2) \delta^4(Q_2) \delta([3\ell_3])}{\langle 12 \rangle \langle 34 \rangle \langle \ell_3 \ell_4 \rangle^2 \langle 3\ell_3 \rangle} \\ &= \frac{\langle 24 \rangle \langle 4\ell_3 \rangle \delta^4(P) \delta^4(Q) \delta(\langle 34 \rangle - \langle 4\ell_3 \rangle)}{\langle 12 \rangle \langle 34 \rangle \langle 3\ell_3 \rangle} \end{aligned}$$

(3.162)

If we now integrated over λ_{ℓ_3} we would just recover the ordinary on-shell function for the box on-shell diagram (3.160),

(3.163)

$$\Omega^{\text{bare}} = \int d^2 \lambda_{\ell_3} \frac{\langle 24 \rangle \langle 4\ell_3 \rangle \delta^4(P) \delta^4(Q) \delta(\langle 34 \rangle - \langle 4\ell_3 \rangle) \delta(\langle 1\ell_3 \rangle)}{\langle 12 \rangle \langle 34 \rangle \langle 3\ell_3 \rangle} = \frac{\langle 24 \rangle \delta^4(P) \delta^4(Q)}{\langle 12 \rangle \langle 13 \rangle \langle 34 \rangle}$$

where the contour integral over the delta functions fixed $\lambda_{\ell_3} = \frac{\langle 34 \rangle}{\langle 14 \rangle} \lambda_1$. But instead of that we have to calculate the derivative,

(3.164)

$$\begin{aligned} \int d^2 \tilde{\eta}_{\ell_3} \int \frac{d^2 \tilde{\lambda}_{\ell_3}}{\text{GL}(1)} \frac{\langle 2\ell_3 \rangle [2\ell_3]}{\langle 12 \rangle [1\ell_3]} \left\langle \lambda_2 \frac{d\omega^{\text{bare}}}{d\lambda_{\ell_3}} \right\rangle \\ = \frac{[23] \langle 24 \rangle \langle 2\ell_3 \rangle (\langle 34 \rangle^2 \langle 2\ell_3 \rangle + \langle 23 \rangle \langle 4\ell_3 \rangle^2) \delta^4(P) \delta^4(Q) \delta((\langle 34 \rangle - \langle 4\ell_3 \rangle)^2)}{[13] \langle 12 \rangle^2 \langle 34 \rangle \langle 3\ell_3 \rangle^2} \end{aligned}$$

and then integrate over λ_{ℓ_3} ,

$$\Omega_{\text{UV}} = \int d^2 \lambda_{\ell_3} \frac{[23] \langle 24 \rangle \langle 2\ell_3 \rangle (\langle 34 \rangle^2 \langle 2\ell_3 \rangle + \langle 23 \rangle \langle 4\ell_3 \rangle^2) \delta^4(P) \delta^4(Q) \delta((\langle 34 \rangle - \langle 4\ell_3 \rangle)^2) \delta(\langle 1\ell_3 \rangle)}{[13] \langle 12 \rangle^2 \langle 34 \rangle \langle 3\ell_3 \rangle^2}$$

$$(3.165) \quad = - \frac{\langle 14 \rangle \langle 23 \rangle \delta^4(P) \delta^4(Q)}{\langle 12 \rangle \langle 34 \rangle \langle 13 \rangle^2}$$

with a perfect agreement with (3.159). In pure Yang-Mills theory ($\mathcal{N}=0$) we act with the operator $\mathcal{O}$ three times, and get

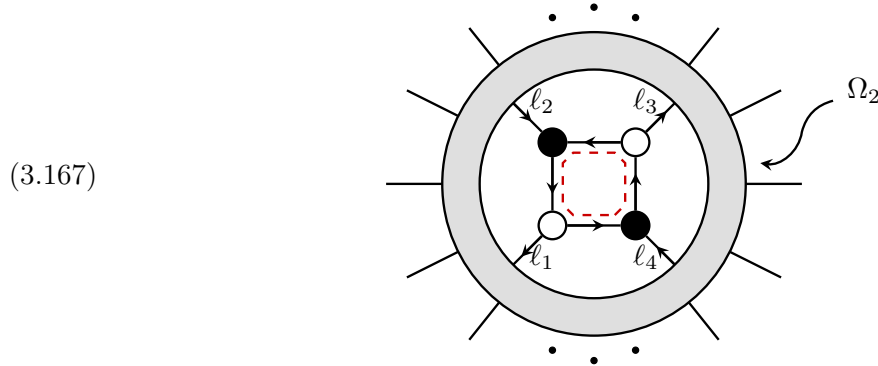
$$(3.166) \quad \begin{aligned} \Omega_{\text{UV}} &= \frac{1}{6} \int d^2 \tilde{\eta}_{\ell_3} \int \frac{d^2 \lambda_{\ell_3} d^2 \tilde{\lambda}_{\ell_3}}{\text{GL}(1)} \left(\frac{\langle 2 \ell_3 \rangle [2 \ell_3]}{\langle 12 \rangle [1 \ell_3]} \right)^3 \left\langle \lambda_2 \frac{d}{d \lambda_{\ell_3}} \left\langle \lambda_2 \frac{d}{d \lambda_{\ell_3}} \left\langle \lambda_2 \frac{d \omega^{\text{bare}}}{d \lambda_{\ell_3}} \right\rangle \right\rangle \right\rangle \\ &= - \frac{\langle 14 \rangle^3 \langle 23 \rangle^3 \delta^4(P)}{\langle 12 \rangle \langle 34 \rangle \langle 13 \rangle^4} \end{aligned}$$

where ω^{bare} is the same as before up to an extra $\langle 24 \rangle^2$ helicity factor and supermomentum delta function. Finally, let us summarize the procedure of the calculation of the UV pole:

- Calculate the bare on-shell function ω^{bare} of a lower-loop on-shell diagram obtained by diagrammatic rules: a collapse of the loop and a non-planar twist. This includes an integration over the internal leg ℓ_4 which eliminates the dependencies from the momentum conservation.
- Take the appropriate number of derivatives with respect to λ_{ℓ_3} and plug into (3.166). This is the value of the original on-shell diagram on the UV pole.

3.4.3 General on-shell diagram

We are now ready to solve a more general problem where the box diagram with the UV pole is inside an arbitrary on-shell diagram,



The on-shell function for the diagram is

$$(3.168) \quad \Omega = \int d^2 \tilde{\eta}_{\ell_1} \dots d^2 \tilde{\eta}_{\ell_4} \int \frac{d^2 \lambda_{\ell_1} d^2 \tilde{\lambda}_{\ell_1}}{\text{GL}(1)} \dots \int \frac{d^2 \lambda_{\ell_4} d^2 \tilde{\lambda}_{\ell_4}}{\text{GL}(1)} \{ \Omega_1 \times \Omega_2 \}$$

where on-shell functions for the internal box and the rest of the diagram are

$$(3.169) \quad \Omega_1 = \frac{\langle \ell_2 \ell_4 \rangle^2 \delta^4(P_1) \delta^4(Q_1)}{\langle \ell_1 \ell_2 \rangle \langle \ell_2 \ell_3 \rangle \langle \ell_3 \ell_4 \rangle \langle \ell_4 \ell_1 \rangle} \left(\frac{\langle \ell_1 \ell_4 \rangle \langle \ell_2 \ell_3 \rangle}{\langle \ell_2 \ell_4 \rangle \langle \ell_1 \ell_3 \rangle} \right)^2, \quad \Omega_2 = \delta^4(P_2) \delta^4(Q_2) \omega_2$$

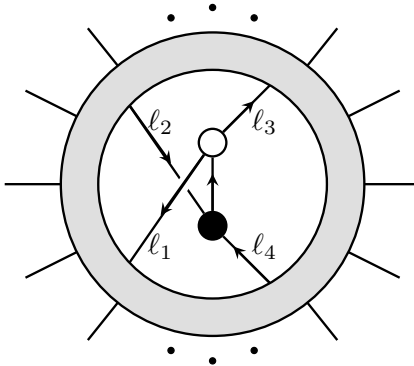
We rewrite this expression as

$$(3.170) \quad \Omega = \int d^2 \tilde{\eta}_{\ell_1} \dots d^2 \tilde{\eta}_{\ell_3} \int \frac{d^2 \lambda_{\ell_1} d^2 \tilde{\lambda}_{\ell_1}}{\text{GL}(1)} \dots \int \frac{d^2 \lambda_{\ell_3} d^2 \tilde{\lambda}_{\ell_3}}{\text{GL}(1)} \left\{ \frac{\langle \ell_1 \ell_2 \rangle}{\langle \ell_1 \ell_3 \rangle^2} \times \omega \right\}$$

where the function ω is defined as

$$(3.171) \quad \omega = \int d^2 \tilde{\eta}_{\ell_4} \int \frac{d^2 \lambda_{\ell_4} d^2 \tilde{\lambda}_{\ell_4}}{\text{GL}(1)} \left\{ \frac{\langle \ell_2 \ell_3 \rangle \langle \ell_1 \ell_4 \rangle \delta^4(P_1) \delta^4(Q_1)}{\langle \ell_1 \ell_2 \rangle^2 \langle \ell_3 \ell_4 \rangle} \times \Omega_2 \right\}$$

This is a straight generalization of the case discussed in the last subsection. We included the integration over internal particle ℓ_4 in the definition of ω in order to eliminate the momentum conservation on four internal legs, such that we can then take a derivative of the expression without any ambiguities. But this is just one choice, we can obviously choose another internal leg to be eliminated in this way. The residue on the pole at infinity $\langle \ell_1 \ell_3 \rangle = 0$ is then given by

$$(3.172) \quad \Omega_{\text{UV}} = \mathcal{O} \otimes \Omega_{\text{UV}}^{\text{bare}} \quad \text{where} \quad \Omega_{\text{UV}}^{\text{bare}} =$$


where the operator $\mathcal{O}$ is given by

$$(3.173) \quad \mathcal{O} = \left(\frac{\langle \ell_2 \ell_3 \rangle [\ell_2 \ell_3]}{\langle \ell_1 \ell_2 \rangle [\ell_1 \ell_3]} \right) \left\langle \lambda_{\ell_2} \frac{d}{d\lambda_{\ell_3}} \right\rangle$$

and the action on the (bare) on-shell function in (3.149) is given by the analogous formula to (3.156),

$$(3.174) \quad \Omega_{\text{UV}} = \int d^2 \tilde{\eta}_{\ell_1} \dots d^2 \tilde{\eta}_{\ell_3} \int \frac{d^2 \lambda_{\ell_1} d^2 \tilde{\lambda}_{\ell_1}}{\text{GL}(1)} \dots \int \frac{d^2 \lambda_{\ell_3} d^2 \tilde{\lambda}_{\ell_3}}{\text{GL}(1)} \left\{ \delta(\langle \ell_1 \ell_3 \rangle) \frac{\langle \ell_2 \ell_3 \rangle [\ell_2 \ell_3]}{\langle \ell_1 \ell_2 \rangle [\ell_1 \ell_3]} \left\langle \lambda_{\ell_2} \frac{d\omega^{\text{bare}}}{d\lambda_{\ell_3}} \right\rangle \right\}$$

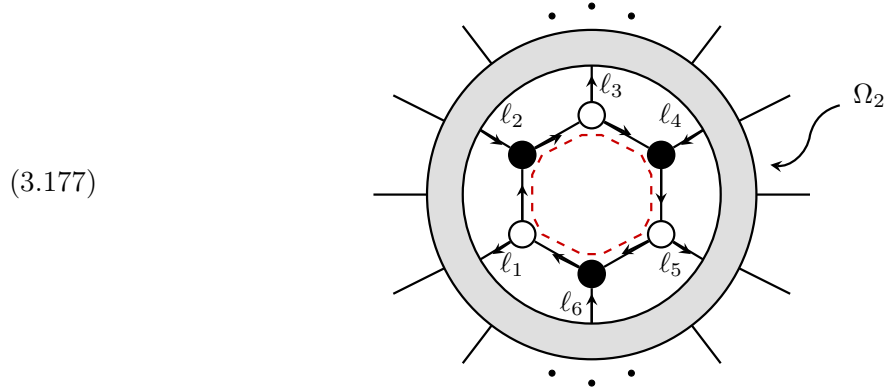
where the bare on-shell function ω^{bare} for (3.172) is

$$(3.175) \quad \omega^{\text{bare}} = \int d^2 \tilde{\eta}_{\ell_4} \int \frac{d^2 \lambda_{\ell_4} d^2 \tilde{\lambda}_{\ell_4}}{\text{GL}(1)} \left\{ \frac{\langle \ell_2 \ell_4 \rangle \delta^4(P_1) \delta^4(Q_1)}{\langle \ell_1 \ell_2 \rangle \langle \ell_3 \ell_4 \rangle} \times \Omega_2 \right\}$$

which is just a straightforward generalization of our discussion before. Note that the only limitation is that we required that the leg ℓ_4 is an internal leg. In case it is an external leg, and we do not integrate over it (hence we do not resolve the momentum conservation), we just have to choose another leg to integrate over first. The generalization to an arbitrary number of supersymmetries is again straightforward, we just act with $3-\mathcal{N}$ copies of $\mathcal{O}$, i.e.

$$(3.176) \quad \Omega_{\text{UV}} = \mathcal{O}^{3-\mathcal{N}} \otimes \Omega_{\text{UV}}^{\text{bare}}$$

We are now ready to generalize this procedure to higher n -gons. Let us now consider the NMHV hexagon subdiagram,



The on-shell function for $\mathcal{N} = 2$ SYM is

$$(3.178) \quad \Omega = \int d^2 \tilde{\eta}_{\ell_1} \dots d^2 \tilde{\eta}_{\ell_6} \int \frac{d^2 \lambda_{\ell_1} d^2 \tilde{\lambda}_{\ell_1}}{\text{GL}(1)} \dots \int \frac{d^2 \lambda_{\ell_6} d^2 \tilde{\lambda}_{\ell_6}}{\text{GL}(1)} \left\{ \Omega_1 \times \Omega_2 \right\},$$

where the on-shell function for the hexagon can be written as

$$(3.179) \quad \begin{aligned} \Omega_1 &= \frac{\langle \ell_4 | \ell_3 + \ell_5 | \ell_1 \rangle^2 \langle \ell_3 \ell_4 \rangle \langle \ell_1 \ell_5 \rangle \delta(\Xi)}{[\ell_5 \ell_6]^2 [\ell_1 \ell_2] \langle \ell_1 \ell_2 \rangle \langle \ell_1 \ell_6 \rangle \langle \ell_2 \ell_3 \rangle \langle \ell_4 \ell_5 \rangle \langle \ell_3 \ell_5 \rangle} \times \left(\frac{\langle \ell_1 \ell_2 \rangle \langle \ell_3 \ell_4 \rangle [\ell_1 \ell_2]}{\langle \ell_1 \ell_3 \rangle \langle \ell_4 | \ell_3 + \ell_5 | \ell_1 \rangle} \right)^2 \\ &= \frac{\langle \ell_3 \ell_4 \rangle^3 \langle \ell_1 \ell_5 \rangle \langle \ell_1 \ell_2 \rangle [\ell_1 \ell_2] \delta(\Xi)}{[\ell_5 \ell_6]^2 \langle \ell_1 \ell_6 \rangle \langle \ell_2 \ell_3 \rangle \langle \ell_4 \ell_5 \rangle \langle \ell_3 \ell_5 \rangle \langle \ell_1 \ell_3 \rangle^2} \end{aligned}$$

where the delta functions are

$$(3.180) \quad \delta(\Xi_1) \equiv \delta^4(P_1) \delta^4(Q_1) \delta(\langle \ell_1 | \ell_2 + \ell_3 | \ell_4 \rangle) \delta(\langle \ell_3 | \ell_4 + \ell_5 | \ell_6 \rangle) \delta([\ell_5 \ell_6] \tilde{\eta}_{\ell_4} + [\ell_6 \ell_4] \tilde{\eta}_{\ell_5} + [\ell_4 \ell_5] \tilde{\eta}_{\ell_6})$$

The residue on the pole at infinity $\langle \ell_1 \ell_3 \rangle = 0$ is then given by

$$(3.181) \quad \Omega_{\text{UV}} = \int d^2 \tilde{\eta}_{\ell_1} \dots d^2 \tilde{\eta}_{\ell_5} \int \frac{d^2 \lambda_{\ell_1} d^2 \tilde{\lambda}_{\ell_1}}{\text{GL}(1)} \dots \int \frac{d^2 \lambda_{\ell_5} d^2 \tilde{\lambda}_{\ell_5}}{\text{GL}(1)} \left\{ \delta(\langle \ell_1 \ell_3 \rangle) \left(\frac{\langle \ell_2 \ell_3 \rangle [\ell_2 \ell_3]}{\langle \ell_1 \ell_2 \rangle [\ell_1 \ell_3]} \right) \left\langle \lambda_{\ell_2} \frac{d\omega^{\text{bare}}}{d\lambda_{\ell_3}} \right\rangle \right\}$$

where ω^{bare} is now the on-shell function associated with the graph

$$(3.182) \quad \text{Diagram: A circular diagram with a thick gray ring. Inside the ring, there are two vertices: a white circle at the top and a black circle at the bottom. Six lines connect these two vertices, forming a hexagon. The lines are labeled with momenta: ℓ_1 (bottom-left), ℓ_2 (top-left), ℓ_3 (top), ℓ_4 (top-right), ℓ_5 (bottom-right), and ℓ_6 (bottom). The diagram is surrounded by several external lines and dots, indicating it is part of a larger graph.$$

before we integrate over momenta $\ell_1, \dots, \ell_5$ (in fact, we can integrate over all momenta except λ_{ℓ_3} in which we take the derivative). This straightforwardly generalizes to any one-loop n -gon, and also to any number of supersymmetries – we again act $(3-\mathcal{N})$ times with the operator $\mathcal{O}$.

It is worth noting that in our procedure of approaching and calculating the higher pole at infinity we made a certain choice: we parametrized $\lambda_{\ell_3} = \alpha \lambda_{\ell_1} + \beta \lambda_{\ell_2}$ and calculated the pole for $\beta = 0$ which then transformed into a certain form of the $\mathcal{O}$. Note that the final result does not depend on this choice (the calculation of the residue is independent on the parametrization). One assumption we made in our calculation was that the leg ℓ_3 was internal (as we integrated over it and removed any dependencies between ℓ_j through momentum conservation). If this was an external leg we would have to approach the pole via another leg. For the general on-shell diagram with $L > 1$ at least one of the legs of the n -gon (which UV pole we approach) is internal and we can use it in our procedure. The only exception is the one-loop n -gon itself as an on-shell diagram, in this case we can just attach a BCFW bridge as we did for the box diagram in the previous section. The result for the UV pole does not depend on the particular procedure used to approach that pole.

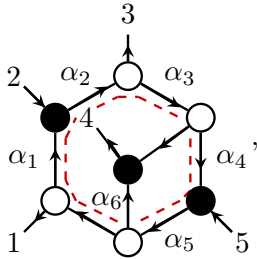
3.5 Non-planar diagrams

The non-planar on-shell diagrams have been extensively studied in the past. Based on the principles of generalized unitarity they represent the cuts of non-planar loop integrands. In the dual

description, the on-shell function is reproduced by the simple dlog form in $\mathcal{N} = 4$ SYM theory while in the lower SUSY Yang-Mills theories we have to include the Jacobians [13], and the formula is the same as for the planar diagrams. Unlike in the planar case not much is known about the connection of non-planar on-shell diagrams to the Grassmannian geometry, permutations, stratifications, identity moves etc. Some interesting observations were made for MHV on-shell diagrams in the context of $\mathcal{N} = 4$ SYM theory [55] where the on-shell function for each diagram was written as a special linear combination of Parke-Taylor factors. A more systematic study was performed in [57] and the classification of all on-shell diagrams for 6 point NMHV amplitudes was performed in [60]. The first attempts to associate NMHV on-shell diagrams with Grassmannian geometries were provided in [65] in the context of non-adjacent BCFW recursion relations.

In our discussion of UV poles in on-shell diagrams, we restricted only to planar diagrams and also our proof of the localization of UV poles in one loop subdiagrams required the planarity of the whole diagram. We will show that indeed the same argument does not hold for non-planar on-shell diagrams and the poles at infinity propagate beyond just a one-loop subgraph. We explore one particular five-point non-planar diagram,

(3.183)



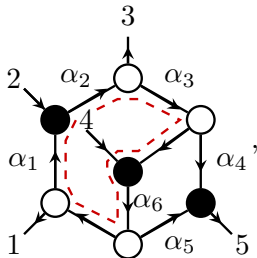
$$\begin{aligned} \alpha_1 &= \frac{\langle 32 \rangle}{\langle 13 \rangle}, & \alpha_2 &= \frac{\langle 13 \rangle}{\langle 12 \rangle}, \\ \alpha_3 &= \frac{\langle 14 \rangle}{\langle 13 \rangle}, & \alpha_4 &= \frac{\langle 51 \rangle}{\langle 14 \rangle}, \\ \alpha_5 &= \frac{\langle 13 \rangle}{\langle 53 \rangle}, & \alpha_6 &= \frac{\langle 43 \rangle}{\langle 13 \rangle}. \end{aligned}$$

The Jacobian from the cycle is

(3.184)
$$\mathcal{J} = 1 - \alpha_1 \alpha_2 \alpha_3 \alpha_4 \alpha_5 = \frac{\langle 13 \rangle \langle 25 \rangle}{\langle 12 \rangle \langle 35 \rangle},$$

where we can identify the UV pole as $\langle 13 \rangle \rightarrow 0$, while $\langle 25 \rangle$ is the helicity factor. We could instead have picked an orientation with a different internal cycle as follows,

(3.185)



$$\begin{aligned} \alpha_1 &= \frac{\langle 32 \rangle}{\langle 13 \rangle}, & \alpha_2 &= \frac{\langle 13 \rangle}{\langle 12 \rangle}, \\ \alpha_3 &= \frac{\langle 14 \rangle}{\langle 13 \rangle}, & \alpha_4 &= \frac{\langle 51 \rangle}{\langle 14 \rangle}, \\ \alpha_5 &= \frac{\langle 35 \rangle}{\langle 13 \rangle}, & \alpha_6 &= \frac{\langle 13 \rangle}{\langle 34 \rangle}. \end{aligned}$$

Here the Jacobian from the cycle is

$$(3.186) \quad \mathcal{J} = 1 - \alpha_1 \alpha_2 \alpha_3 \alpha_6 = \frac{\langle 13 \rangle \langle 24 \rangle}{\langle 12 \rangle \langle 34 \rangle}.$$

We find once again that the pole at infinity is at $\langle 13 \rangle \rightarrow 0$, even though we are looking at different loop. We can uncover the same problem by solving for internal momenta,

$$(3.187) \quad \begin{array}{c} \begin{array}{c} \text{Diagram: A graph with 5 external vertices (1, 2, 3, 4, 5) and 8 internal vertices. Vertices 1, 2, 4, 5 are black; 3 and the central vertex are white. Edges are labeled } \ell_1 \text{ through } \ell_8. \end{array} \\ \begin{array}{l} \ell_1 = \frac{\langle 23 \rangle}{\langle 13 \rangle} \lambda_1 \tilde{\lambda}_2, \quad \ell_2 = \frac{\langle 12 \rangle}{\langle 13 \rangle} \lambda_3 \tilde{\lambda}_2, \quad \ell_3 = \frac{(Q_{23} \cdot \lambda_1)}{\langle 13 \rangle} \lambda_3, \\ \ell_4 = \frac{\langle 15 \rangle}{\langle 13 \rangle} \lambda_3 \tilde{\lambda}_5, \quad \ell_5 = \frac{\langle 35 \rangle}{\langle 13 \rangle} \lambda_1 \tilde{\lambda}_5, \quad \ell_6 = \frac{(Q_{12} \cdot \lambda_3)}{\langle 13 \rangle} \lambda_1, \\ \ell_7 = \frac{\langle 34 \rangle}{\langle 13 \rangle} \lambda_1 \tilde{\lambda}_4, \quad \ell_8 = \frac{\langle 14 \rangle}{\langle 13 \rangle} \lambda_3 \tilde{\lambda}_4. \end{array} \end{array}$$

and we see that the UV pole $\langle 13 \rangle$ is present in multiple loops rather than only one-loop subgraph. There is a purely graph theoretic reason for these subtleties: we can not invariantly talk about “fundamental” and “overlapping” internal loops. In the planar case, the first type corresponds to poles at infinity, while the second type does not. Take e.g. the pentabox (3.21) with the orientation in the box flipped. Schematically we can show the pole and cycle structure as,

$$(3.188) \quad \begin{array}{c} \text{Overlapping cycle} \rightarrow \text{no UV pole} \\ \text{No cycle} \rightarrow \text{no UV pole} \\ \ell_{\text{pentagon}} \sim \frac{1}{\langle 13 \rangle} \\ \text{Inner cycle} \rightarrow \text{UV pole} \\ \ell_{\text{box}} \sim \frac{1}{\langle 35 \rangle} \end{array}$$

Even with the overlapping cycle, the Jacobian still only introduces the UV pole of the box $\langle 35 \rangle$. For the non-planar graphs there is no difference between these two types of cycles, and we can always

redraw a diagram in a way which turns one type into the other,

(3.189)

This makes it clear that we need to generalize our rule to capture the UV poles in non-planar diagrams where multiple loops blow up at the same time. This also concerns the gravity on-shell diagrams [34, 61, 62, 63] which are intrinsically non-planar.

3.6 Conclusion and Outlook

In this chapter, we studied the poles at infinity in on-shell diagrams for $\mathcal{N}=4$ SYM theory. We showed that for planar diagrams the poles at infinity are located in one-loop subgraphs and the kinematical condition on the pole reduces the graph to a simpler on-shell diagram with a *non-planar twist*, hence the diagram is generally non-planar. The on-shell function on the UV pole is then given by a certain derivative operator acting on the on-shell function for a new non-planar on-shell diagram. This comes naturally from the definition of the residue on a higher pole as a derivative. We also showed that for non-planar on-shell diagrams the situation is more complicated as the poles at infinity are not localized in one-loop subgraphs and a different procedure is needed.

Our work is a direct attempt to systematically study the poles at infinity in scattering amplitudes. While the residues of tree-level amplitudes and loop integrands in the deep IR are governed by principles of unitarity and lead to factorizations and unitarity cuts, the poles in the UV are naively not constrained by similar principles. We do not face this problem directly in the context of scattering amplitudes, but we chose simpler on-shell gauge-invariant objects – on-shell diagrams, and solve the problem here. Note that tree-level amplitudes (and in special cases also loop integrands) can be expressed as sums of on-shell diagrams, so it is suggestive that our results would be relevant to amplitudes problem as well. The two main lessons from our analysis are: (i) the kinematics of UV poles correspond to non-planar objects, in this case non-planar on-shell diagrams, (ii) on-shell

functions are given by derivatives of the bare on-shell diagrams. It would be interesting if this insight can eventually be used to help address the problem directly in the context of tree-level amplitudes and loop integrands.

The further goal is to formulate the general prescription for the calculation of poles at infinity for arbitrary tree-level amplitudes, and loop integrands – similar to rules of factorizations and cuts. This would have major implications on our ability to calculate amplitudes using recursion relations, and possibly search for positive geometries that could capture amplitudes in theories with non-trivial UV physics.

CHAPTER 4

Grassmannian Geometries for Non-planar Diagrams

4.1 Introduction

We saw in the previous chapter that even when we start with planar diagrams, going to the poles at infinity leaves us with a non-planar diagram. This brings us to an important question that we have not addressed yet, namely how do the mathematical structures we have discussed so far extend beyond the planar limit. More specifically, what geometry can we associate to amplitudes in the non-planar theory? We have some indications from physics [86, 89, 91, 119] that non-planar amplitudes do enjoy very special analytic properties, though this is not obvious from any known physical principles. However, on-shell diagrams are well-defined both in planar and non-planar sectors of the theory, and can be calculated as products of three-point amplitudes. Furthermore, let's remember that the connection between on-shell diagrams and the Grassmannian $G(k, n)$ allows for the dual formulation and the evaluation of the canonical dlog form using parameters of the cell in the Grassmannian. This connection holds beyond the planar limit and it will be a cornerstone of our exploration in this chapter. This was noted in the original paper on scattering amplitudes and the positive Grassmannian [13], but in the context of non-planar diagrams was further studied in [55, 57, 60, 64, 65] (see also [34, 61, 63, 120, 121] for work on on-shell diagrams in Einstein gravity, though the forms there are non-logarithmic and depend explicitly in kinematics in addition to Grassmannian variables).

In particular, in [55] it was shown that all MHV (maximal-helicity-violating) on-shell diagrams can be evaluated in a remarkably simple way. On one hand, the expression for each diagram can be expanded in a basis of Parke-Taylor factors, that are relevant for planar diagrams with different orderings. Alternatively, the same expression can be obtained using an intriguing determinant formula. In the Grassmannian language, the MHV case corresponds to $G(2, n)$ which exhibits major simplifications. Higher degree cases were studied in [60] and already for $G(3, 6)$ there are qualitatively new features.

In this chapter, we focus on the following question: what geometries can be associated to MHV non-planar on-shell diagrams? That is, are there natural geometries whose canonical forms match those from non-planar on-shell diagrams? For planar diagrams, there is an elegant answer to this question: the positive Grassmannian $G_+(k, n)$. Each planar diagram is associated with a certain cell in $G_+(k, n)$, a region defined by setting some minors to zero and asking that others are nonnegative. If we calculate a canonical form of this region, we reproduce the expression for the diagram calculated from physics.

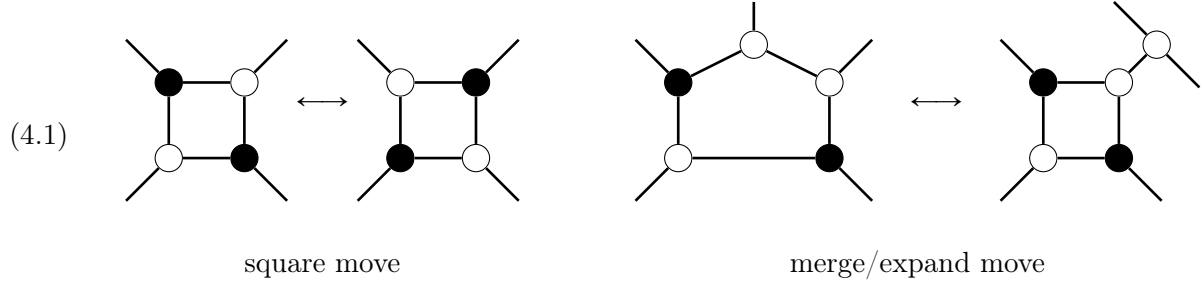
In the non-planar sector, we do have an expression for each diagram, written in terms of minors of the (no longer positive part of the) Grassmannian $G(k, n)$. But we do not know a region whose canonical form is this expression. In our work, we find such a region in the $G(2, n)$ case, which is relevant for n -pt MHV amplitudes. This is interesting from a mathematical perspective: we will associate geometries with non-planar graphs, and obtain certain generalizations of the positive Grassmannian. From a physical perspective, this is a major step in the exploration of non-planar amplitudes using new mathematical tools. A detailed understanding of underlying combinatorial structures should lead to insights on non-planar amplitudes, similar to those we obtained in planar amplitudes from the positive Grassmannian and the amplituhedron formulations.

The chapter is organized as follows: In Section 4.2, we review the identity moves of on-shell diagrams and set the notation we will use for the rest of this work. In Section 4.3, we discuss the generalization to the non-planar sector and review the triplet formula for the MHV on-shell diagrams. In Section 4.4, we leverage this formula to prove that the previously known sphere move introduced in [64] are the only identity move that does not change the canonical form. Section 4.5 contains the main result of this chapter, and we study pseudo-positive geometries associated with MHV on-shell diagrams. In particular, we show that for the class of internally planar diagrams, the geometry is always connected. We end with Conclusions and Outlook. We leave the proofs of various statements to three appendices.

4.2 Planar On-shell Diagrams, Identity Moves, and Canonical Functions

Many different on-shell diagrams will give a parametrization of the same cell. There are two identity moves that change the diagram but do not change the cell (and also do not change the

associated differential form).



If boundary measurement is injective, the plabic graph is called *reduced*. A graph is reduced if and only if doing identity moves to it never produces a bubble, shown below.



When a graph corresponds to the top cell, the matrix $C(\alpha_i)$ has all Plücker coordinates positive when $\alpha_i > 0$. Lower-dimensional cells can be obtained from the top cell by setting certain edge variables to zero, which removes the corresponding internal edges in the graph and also makes some Plücker coordinates zero.

Note that the parameters of the positive Grassmannian n, k are encoded respectively by the number of external legs of the plabic graph, and

$$(4.3) \quad k = 2n_B + n_W - n_I$$

where n_B, n_W are the numbers of black and white vertices and n_I is the number of internal edges of the graph. Our case of interest in this chapter is MHV amplitudes, where $k = 2$. The top cell of $G_+(2, n)$ has dimension $2n-4$, hence the associated on-shell diagram is a leading singularity. The geometric constraint of the delta functions identifies the C -plane with the λ -plane as this is an unique 2-plane which contains a fixed 2-plane λ . So we have

$$(4.4) \quad C = \lambda$$

and the (2×2) minors of the C matrix (ij) are identified with the angle brackets as

$$(4.5) \quad (ij) = \langle ij \rangle$$

The canonical form on the top cell of $G_+(2, n)$ is given by

$$(4.6) \quad \omega_G = \frac{dC}{(12)(23)(34) \dots (n-1 \ n)(n1)}$$

where no residue is taken because C is a top cell. The associated superfunction for the on-shell diagram is

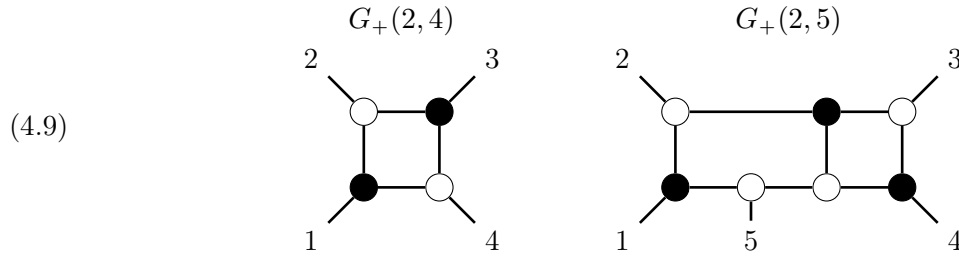
$$(4.7) \quad \mathcal{F}_G = \frac{\delta(P) \delta(\mathcal{Q})}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \dots \langle n1 \rangle}.$$

We can talk about three different objects: the canonical form ω_G , the canonical function f_G (which is just ω_G without a measure) and the superfunction $\mathcal{F}_G$. For $k = 2$, the differences are unimportant, and hence it is enough to consider only the canonical function

$$(4.8) \quad f_G = \frac{1}{(12)(23)(34) \dots (n1)}.$$

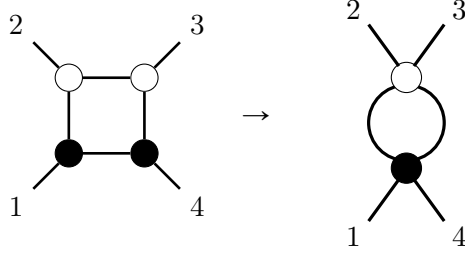
In the remainder of the chapter, we will consider canonical functions almost exclusively. We draw the reader's attention to some particularly pertinent points.

- For each n there is only **one** cell of $G_+(2, n)$ with canonical function as in (4.8), which is the top cell. This cell has many different on-shell diagrams, related by the identity moves. Examples of these on-shell diagrams for $n = 4, 5$ are



All diagrams for the top cell give rise to the same canonical function. That is, the canonical function is also invariant under identity moves.

- Not every on-shell diagram with $m = 2n-4$ edge variables is a leading singularity diagram—only the reduced ones are. Consider the following non-reduced diagram



where we used the merge-expand move to expose the bubble. Because the dimension is still $m = 2n-4 = 4$, the presence of the bubble leads to an edge variable α_4 which is not solved for from delta functions, leading to an ‘excess’ delta function which imposes the below constraints on external kinematics

$$\begin{aligned}
 \mathcal{F}_G &= \int \frac{d\alpha_1}{\alpha_1} \frac{d\alpha_2}{\alpha_2} \frac{d\alpha_3}{\alpha_3} \frac{d\alpha_4}{\alpha_4} \delta^{2 \times 2}(C \cdot \tilde{\lambda}) \delta^{2 \times 2}(C^\perp \cdot \lambda) \delta^{2 \times 4}(C \cdot \tilde{\eta}) \\
 (4.10) \quad &= \int \frac{d\alpha_4}{\alpha_4} \frac{\delta(P) \delta(Q)}{\langle 23 \rangle \langle 34 \rangle \langle 41 \rangle} \delta(\langle 12 \rangle).
 \end{aligned}$$

This is in contrast with the reduced diagram where the edge variable α_4 would be solved for from the delta function.

4.2.1 Boundary structure of $G_+(2, n)$

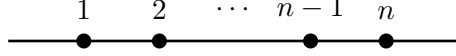
It will be useful later to recall the following interpretation of the cells of $G_+(2, n)$ as point configurations on the projective line $\mathbb{P}^1$. Suppose C is in the top cell, so has all positive Plücker coordinates. If we rescale each column of C by a coefficient $c_i > 0$ to make the first entry 1, we obtain

$$(4.11) \quad C' = \begin{pmatrix} 1 & 1 & 1 & \dots & 1 & 1 \\ x_1 & x_2 & x_3 & \dots & x_{n-1} & x_n \end{pmatrix}$$

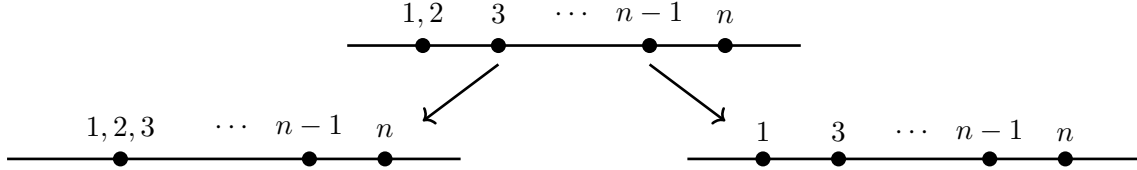
The minor (ij) of the matrix for $i < j$ is

$$(4.12) \quad (ij) = c_i c_j (x_j - x_i) > 0 \quad \text{and hence} \quad x_j > x_i$$

Therefore, we can interpret C' as a configuration of n ordered points on $\mathbb{P}^1$, as below.



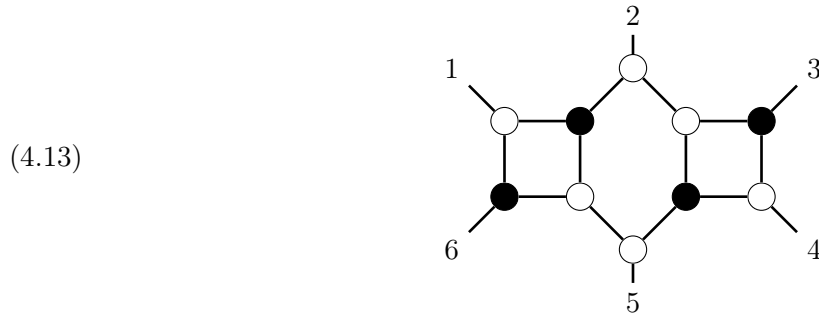
The only codimension 1 boundaries of the space correspond to merging of two adjacent points, $x_i = x_{i+1}$ (note that this includes $x_n = x_1$). There are two types of codimension-2 boundaries of the space, e.g. if $(12) = 0$, we can set $(23) = 0$ by either merging all three points $x_1 = x_2 = x_3$ or removing the point 2 completely by setting $c_2 = 0$.



This simple picture provides a complete stratification of the space, all the way down to the lowest zero-dimensional boundaries which correspond to two points on the line (all other points are removed). At the level of on-shell diagrams, going to the boundary of the positive Grassmannian and taking the residue of f_G corresponds to erasing an edge.

4.3 Non-planar On-shell Diagrams

On-shell diagrams are well-defined objects even when the legs of the graph (cuts of loop amplitude) do not form a planar diagram. Here, a planar diagram is one that can be drawn in the disc with no crossing edges *and* with the external legs on the boundary of the disk, as in the example below.



Note that this notion of planarity differs from the standard definition in graph theory, where there are no external legs.

In the physics setup, each on-shell diagram is dressed with a *color factor* which can be written as a product of structure constants of the $SU(N)$ color group. We can also write same color functions in terms of traces of generators of $SU(N)$. In the planar $N \rightarrow \infty$ limit, the single traces dominate, while multiple traces are subleading. Only planar on-shell diagrams contain single trace

contributions, while non-planar diagrams start the expansion at a subleading order (multiple trace). This limit is also called the *planar limit* as the planar diagrams dominate for large N . The leading singularity at finite N is given by the on-shell diagram where each vertex is dressed by the $SU(N)$ structure constant f^{abc} . The result is a product of the group part (given by the product of structure constants) and the kinematical function $\mathcal{F}_G$. For large N we can expand the group part in powers of $1/N$ where the dominating piece is a single trace of $SU(N)$ generators T^a , while the subleading pieces are multiple traces (products of traces). For the planar diagrams we then get

$$(4.14) \quad \begin{array}{c} \begin{array}{c} \text{Diagram: A planar on-shell diagram with 5 external legs labeled 1, 2, 3, 4, 5. The diagram consists of a central square with two additional vertices connected to the top and bottom vertices of the square. The vertices are colored: top (white), bottom (white), left (black), right (white), and center (black). The edges are labeled with structure constants: $f^{a_7 a_3 a_8}$ (top), $f^{a_8 a_9 a_{10}}$ (top-right), $f^{a_{10} a_4 a_{11}}$ (right), $f^{a_{11} a_5 a_{12}}$ (bottom-right), $f^{a_{12} a_9 a_{13}}$ (bottom), $f^{a_{13} a_1 a_6}$ (bottom-left), $f^{a_6 a_2 a_7}$ (left), and $f^{a_7 a_3 a_8}$ (top-left).} \end{array} \end{array} = [\text{Tr}(T^{a_1} T^{a_2} \dots T^{a_n}) + \dots] \times \mathcal{F}_G$$

where we sum over internal labels $a_6, \dots, a_{13}$. The non-planar diagrams are subleading at large N , they only contain multiple traces. In other words, they are suppressed by $\mathcal{O}(1/N)$ or higher when compared to their planar counterparts

$$(4.15) \quad \begin{array}{c} \begin{array}{c} \text{Diagram: A non-planar on-shell diagram with 5 external legs labeled 1, 2, 3, 4, 5. The diagram consists of a central square with two additional vertices connected to the top and bottom vertices of the square. The vertices are colored: top (white), bottom (white), left (black), right (white), and center (black). The edges are labeled with structure constants: $f^{a_8 a_3 a_9}$ (top), $f^{a_9 a_{10} a_{11}}$ (top-right), $f^{a_{11} a_5 a_{12}}$ (right), $f^{a_{12} a_{13} a_6}$ (bottom), $f^{a_6 a_1 a_7}$ (bottom-left), $f^{a_7 a_2 a_8}$ (left), and $f^{a_{10} a_4 a_{13}}$ (center).} \end{array} \end{array} = \mathcal{O}\left(\frac{1}{N}\right) \times \mathcal{F}_G$$

However, for finite N we have to consider both planar and non-planar on-shell diagrams, as they all represent valid cuts of the amplitude. Planar diagrams come with an ordering of external legs on the disc, but non-planar diagrams do not have any natural ordering.

Non-planar on-shell diagrams provide a way into the problem of $\mathcal{N} = 4$ SYM amplitudes, which is challenging for a number of reasons: difficulty in defining a unique non-planar integrand, absence

of dual conformal symmetry, no amplituhedron construction, no powerful symbol methods and no known integrability. On the other, the principles of generalized unitarity are universal, and non-planar on-shell diagrams are non-planar amplitudes evaluated on the cuts. Hence, they provide a perfect window to study non-planar $\mathcal{N} = 4$ SYM amplitudes in a setup which allow one to more easily search for new symmetries and connections to mathematics.

4.3.1 First look: Grassmannian formula

In physics, non-planar on-shell diagrams are defined as a product of three-point amplitudes in the same way as their planar counterparts. The result is a superfunction with the same $\delta(P)\delta(Q)$ conserving delta functions

$$(4.16) \quad \begin{array}{c} \text{Diagram: A non-planar on-shell diagram with 5 external legs labeled 1, 2, 3, 4, 5. Vertices are represented by black and white circles. The diagram consists of two internal vertices (white circles) and three external vertices (black circles). The connections are: 1 (white) to 2 (black), 2 (black) to 3 (white), 3 (white) to 4 (black), 4 (black) to 5 (white), 5 (white) to 1 (black), and 1 (white) to 4 (black).} \end{array} = \frac{\langle 13 \rangle \delta(P) \delta(Q)}{\langle 12 \rangle \langle 23 \rangle \langle 14 \rangle \langle 15 \rangle \langle 34 \rangle \langle 35 \rangle}$$

but no cyclic symmetry due to the loss of ordering. The general amalgamation prescription allows us to evaluate any non-planar on-shell diagram as the kinematical function (and also the color factor to get the complete dressed formula). For MHV leading singularity diagrams, the superconformal invariance dictates (as in the planar sector) that the result is a function of angle brackets $\langle \dots \rangle$ only,

$$(4.17) \quad \mathcal{F}_G = \frac{\mathcal{N}}{\prod_{ij} \langle ij \rangle} \delta(P) \delta(Q)$$

where the only poles are physical $\langle ij \rangle$ (not true for NMHV where spurious poles can appear in leading singularities, even in the planar sector). The numerator $\mathcal{N}$ is a polynomial in angle brackets $\langle \dots \rangle$ which guarantees correct little group and mass dimension of f_G . Note that there is only one planar MHV leading singularity for each n , for which the numerator is just $\mathcal{N} = 1$.

The connection between non-planar on-shell diagrams and the Grassmannian retains some features from the planar case. In a non-planar graph, we can assign edge variables α_j and construct the C -matrix using a boundary measurement. The matrix $C(\alpha)$ no longer has any special positivity properties for $\alpha_j > 0$ and there is no connection to cells in $G_+(k, n)$ but the C -matrix does label

a subset of the Grassmannian $G(k, n)$. The amalgamation procedure (and its independence on the planarity of the graph) ensures that the canonical function f_G can be obtained by the same formula as in the planar case. For the MHV case, $G(2, n)$, the delta functions again identify the C -matrix with the λ -matrix, $C = \lambda$, and each on-shell diagram is then given by the rational function f_G on the Grassmannian $G(2, n)$. For the 5pt example above, analogously to the planar case (4.8),

$$(4.18) \quad f_G = \frac{(13)}{(12)(23)(14)(15)(34)(35)}.$$

Note that the role of edge variables and minors (ij) are very similar to the planar case: $\alpha_k = 0$ sends some minor $(ij) = 0$ and erases an edge in an on-shell diagram. This strongly suggests there should be a positive geometry, like the positive Grassmannian $G_+(k, n)$ for planar on-shell diagrams, which is associated for each non-planar diagram.

4.3.2 MHV diagrams and the triplet formula

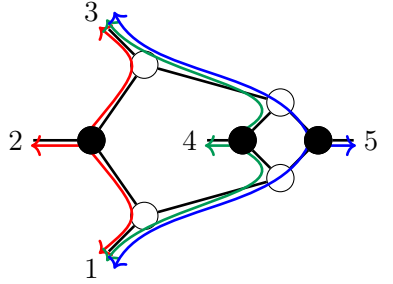
The prescription to calculate f_G for a leading singularity using edge variables and boundary measurements works for any on-shell diagram, planar or non-planar, for any k , as it follows from the purely local amalgamation procedure. MHV on-shell diagrams are more special as explored in detail in [55]. In particular, it was shown that there is an important shortcut to calculate f_G using a *triplet formula*. For an MHV leading singularity on-shell diagram with n external legs, n_W white vertices, n_B black vertices, and n_I internal edges,

$$(4.19) \quad k = 2n_B + n_W - n_I = 2.$$

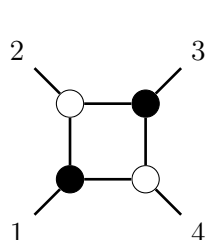
Combining this with two expressions $n_W + n_B = 3n - 8$, $n_I = n_B + n_W + n - 4$ which are valid for any leading singularity graph, we get

$$(4.20) \quad n_B = n - 2.$$

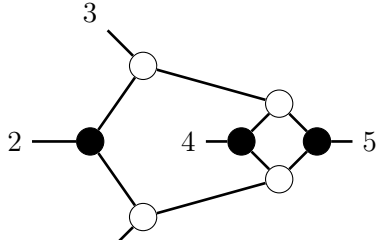
It can be shown that each of these black vertices are attached to exactly three external legs via white vertices.

(4.21)  $T = \{(\textcolor{red}{123}), (\textcolor{green}{134}), (\textcolor{blue}{135})\}$

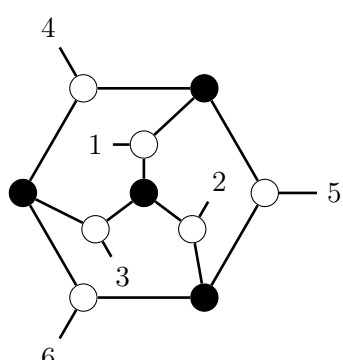
Thus, labeling the black vertices with $1, \dots, n-2$, an MHV leading singularity diagram is encoded by $n-2$ triplets τ_i consisting of the three external legs black vertex i is attached to. We can see how this works for the following examples for $n = 4, 5, 6$



$T = \{(124), (234)\}$



$T = \{(123), (134), (135)\}$



$T = \{(123), (145), (256), (346)\}$

The triplets allow us to construct a particular parametrization of the $(n-2) \times n$ matrix $C^\perp(\vec{\alpha}^*)$ where α^* are evaluated in terms of λ spinors. Each triplet corresponds to a row in this matrix, and three non-zero entries are given by the labels in the triplet. For the triplet (abc) we put the bracket $\langle bc \rangle$ in the a^{th} column, $\langle ac \rangle$ in the b^{th} column and $\langle ab \rangle$ in the c^{th} column. We do this operation for

all triplets, giving us the parametrization of the matrix $C^\perp$. For the simplest 4pt example

$$C^\perp = \begin{pmatrix} \langle 24 \rangle & \langle 14 \rangle & 0 & \langle 12 \rangle \\ 0 & \langle 34 \rangle & \langle 24 \rangle & \langle 23 \rangle \end{pmatrix}$$

$$(4.22) \quad T = \{(124), (234)\}$$

Next, we construct a matrix M_{ab} by deleting two columns a, b . Choosing $a, b = 1, 3$ gives

$$(4.23) \quad M_{13} = \begin{pmatrix} \langle 14 \rangle & \langle 12 \rangle \\ \langle 34 \rangle & \langle 23 \rangle \end{pmatrix}.$$

Now we calculate the determinant of this matrix divided by $\langle ab \rangle$,

$$(4.24) \quad \frac{\det(M_{ab})}{\langle ab \rangle}.$$

This quantity does not depend on the choice of a, b . We also note that $\det(M_{ab})$ can be written to have a factor of $\langle ab \rangle$, so dividing by $\langle ab \rangle$ yields a polynomial. The formula for the on-shell diagram superfunction is then given by

$$(4.25) \quad \mathcal{F}_G = \frac{(\det(M_{ab})/\langle ab \rangle)^2}{\prod_{\tau \in T} \langle \tau_1 \tau_2 \rangle \langle \tau_2 \tau_3 \rangle \langle \tau_3 \tau_1 \rangle} \delta(P) \delta(Q)$$

where the denominator is given by the product of three poles for each triplet $(\tau_1 \tau_2 \tau_3)$. In the 4pt example, $\det(M_{13}) = \langle 13 \rangle \langle 24 \rangle$. The triplet (124) contributes the poles $\frac{1}{\langle 12 \rangle \langle 14 \rangle \langle 24 \rangle}$ and (234) contributes $\frac{1}{\langle 23 \rangle \langle 24 \rangle \langle 34 \rangle}$. Then bringing everything together, we get the superfunction

$$(4.26) \quad \mathcal{F}_G = \frac{\langle 13 \rangle^2 \langle 24 \rangle^2}{\langle 13 \rangle^2} \frac{\delta(P) \delta(Q)}{\langle 12 \rangle \langle 14 \rangle \langle 24 \rangle \langle 23 \rangle \langle 24 \rangle \langle 34 \rangle} = \frac{\delta(P) \delta(Q)}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 14 \rangle}$$

as expected. We can use the same prescription to compute the form for any planar or non-planar MHV on-shell diagrams. As discussed earlier, minors of the C -matrix are equal to $(ij) = \langle ij \rangle$ after imposing the delta functions. Hence, we can also here just use (ij) instead of $\langle ij \rangle$ and work with the rational function f_G , rather than the superfunction $\mathcal{F}_G$ for each graph. And also the entries of $C^\perp$ and M_{ab} will be minors (ij) of the C matrix.

Note that the numerator plays a crucial role in making sure that an associated form has only logarithmic singularities. We can demonstrate it on the 5pt example (4.16) with triplets (123), (134), (135). If we choose to delete columns 1, 3 the M_{13} matrix is given by (remaining columns are 2, 4, 5),

$$(4.27) \quad M_{13} = \begin{pmatrix} \langle 13 \rangle & 0 & 0 \\ 0 & \langle 13 \rangle & 0 \\ 0 & 0 & \langle 13 \rangle \end{pmatrix}$$

and we get

$$(4.28) \quad f_G = \frac{(13)^4}{(13)^3(12)(14)(15)(23)(34)(35)} = \frac{(13)}{(12)(23)(14)(15)(34)(35)}$$

in agreement with (4.18). Apart from the trivial cancelation of the pole $(13)^3$, the numerator $N = (13)$ also guarantees that all lower-dimensional residues of ω_G stay logarithmic. For example, if we send $(12) = 0$ by setting $1 = \alpha 2$, ie. columns 1 and 2 in the C matrix are proportional, we get for the associated canonical form (with the measure) $\omega_G = f_G dC$,

$$(4.29) \quad \text{Res}_{1=\alpha 2} \omega_G = \frac{(23)\alpha^2 d\alpha \wedge d^4 C'}{\alpha^3 (23)(24)(25)(34)(35)} = \frac{d\alpha \wedge d^4 C'}{\alpha (24)(25)(34)(35)},$$

where $d^4 C'$ is the measure on the $G(2, 4)$ of the four independent columns. The numerator was crucial in canceling the double pole α^2 in the denominator: one factor came from the measure, but the second came from $(13) \rightarrow \alpha(23)$. This is much more involved for complicated higher pole examples, but the numerator always does the ‘magic’ to preserve logarithmic singularities. From now on, we will denote the canonical function f_T just to indicate that we derive it from the triplet formula rather than the on-shell diagram, but they are obviously equal, $f_T = f_G$.

Note that there are also on-shell diagrams with the same number of edges and vertices as the MHV leading singularity diagrams, which are not described by triplets. The canonical functions for these diagrams vanish because of the constraints imposed on external kinematics. In the planar sector, these diagrams are not reduced due to the presence of internal bubbles (4.2), but we do not have the same notion of reducedness for non-planar diagrams yet. In any case, these diagrams are not described by triplets, have vanishing canonical functions, and are not part of our discussion.

4.3.3 Parke-Taylor expansion

The canonical function f_T can be also expanded in terms of elementary building blocks called Parke-Taylor factors,

$$(4.30) \quad f_T = \sum_{\sigma} \text{PT}(\sigma) \quad \text{where} \quad \text{PT}(\sigma) = \frac{1}{(\sigma_1\sigma_2)(\sigma_2\sigma_3)(\sigma_3\sigma_4)\dots(\sigma_n\sigma_1)}$$

where the Parke-Taylor factor is written for a given permutation $\sigma = \sigma_1\sigma_2\dots\sigma_n$ and the sum is over a subset of all permutations given in (4.32).

Note that the Parke-Taylor factor $\text{PT}(\sigma)$ is the canonical function of the positive Grassmannian $G_+(2, n)$ where the columns are ordered using σ . In fact, the collection of all such spaces for all orderings tiles the full Grassmannian $G(2, n)$. This is also evident from the picture of points on a projective line $\mathbb{P}^1$. The positive Grassmannian $G_+(2, n)$ with a canonical ordering $123\dots n$ corresponds to a set of ordered n points, as discussed in Section 4.2. For any point in $G_+(2, n)$ the associated points are ordered in some way and hence the point also belongs to one of the permuted $G_+(2, n)$ subspaces. As a result, the full Grassmannian $G(2, n)$ is tiled by all permuted $G_+(2, n)$ s, which also do not overlap.

This is no longer true for $k > 2$ but $k = 2$ is special because of this feature. Note that the expansion of f_T in terms of Parke-Taylor factors makes manifest that all singularities of the form are logarithmic – this is not manifest in the determinant form as the numerator conspires with the denominator to remove all non-logarithmic poles.

The prescription how to obtain a Parke-Taylor decomposition of a canonical function f_T for a general set of triplets T was given in [55]. The first step is to fix the *orientation*, which is a permutation of labels for each triplet, modulo cyclic ordering. For a triplet (123) there are two orientations: (123), (132). The orientation of the triplet changes the sign of f_T in (4.30) as in the denominator

$$(4.31) \quad (123) \rightarrow (12)(23)(13), \quad (132) \rightarrow (13)(32)(12) = -(12)(23)(13)$$

while the numerator (a square) does not depend on the orientation. For a fixed orientation, f_T is given by

$$(4.32) \quad f_T = \sum_{\sigma \in S_n^{(T)}} \text{PT}(\sigma).$$

where the sum is given over all orderings σ which are compatible with oriented triplets T

$$(4.33) \quad S_n^{(T)} = \{\sigma \in S_n : \sigma_1 = 1, \forall (ijk) \in T \text{ } i, j, k \text{ appear in that order up to rotation}\}.$$

As an example, let us take the 5pt non-planar diagram with triplets (123), (134), (135). For this orientation we get

$$(4.34) \quad f_T = \text{PT}(12345) + \text{PT}(12354)$$

while for the orientation (123), (134), (153) it is

$$(4.35) \quad \tilde{f}_T = \text{PT}(12534) + \text{PT}(15234).$$

Parke-Taylor factors are not independent but satisfy Kleiss-Kuijf relations and the $U(1)$ decoupling identity,

$$(4.36) \quad \text{PT}(12534) + \text{PT}(15234) + \text{PT}(12345) + \text{PT}(12354) = 0$$

and hence $\tilde{f}_T = -f_T$. This is a general feature of triplets: different orientations give the same expressions up to a sign.

4.3.4 Towards a positive geometry

The basic question we want to address in this work is if there exists a positive geometry associated with a non-planar on-shell diagram. Namely,

Is there a positive geometry for which f_T is the canonical function?

For planar on-shell diagrams (plabic graphs) the answer is positive. In that case the canonical function is a single Parke-Taylor factor $\text{PT}(123 \dots n)$ which is the canonical function of the positive Grassmannian $G_+(2, n)$. For non-planar on-shell diagrams, we use the Parke-Taylor expansion (4.32) and interpret it geometrically as the union of the corresponding spaces. The space corresponding

to $\text{PT}(\sigma)$ is a top cell of the positive Grassmannian $G_+(2, n)$ with a particular ordering σ . Note that there are many options here, both because of different orientations of triplets and different Parke-Taylor expansions, but also because of additional choices to be discussed in Section 5.

Here as an example, we consider just a planar 4pt graph (4.22), the choice of the orientation of triplets (123), (134) and using (4.32) gives us $\text{PT}(1234)$ as the canonical function. We can characterize the positive geometry, the top cell of $G_+(2, 4)$ by the positivity conditions on minors,

$$(4.37) \quad (12) > 0, (13) > 0, (14) > 0, (23) > 0, (24) > 0, (34) > 0$$

If we now change the orientation to (123), (143) we get

$$(4.38) \quad f_T = \text{PT}(1423) + \text{PT}(1243)$$

which is equal to $\text{PT}(1234)$ up to a sign using the $U(1)$ decoupling identity. If we now associate geometries for each Parke-Taylor factor,

$$(4.39) \quad \text{PT}(1423) : (12) > 0, (13) > 0, (14) > 0, (23) > 0, (24) < 0, (34) < 0$$

$$(4.40) \quad \text{PT}(1243) : (12) > 0, (13) > 0, (14) > 0, (23) > 0, (24) > 0, (34) < 0$$

the union of these spaces is the region in $G(2, 4)$ given by the following inequalities:

$$(4.41) \quad (12) > 0, (13) > 0, (14) > 0, (23) > 0, (34) < 0.$$

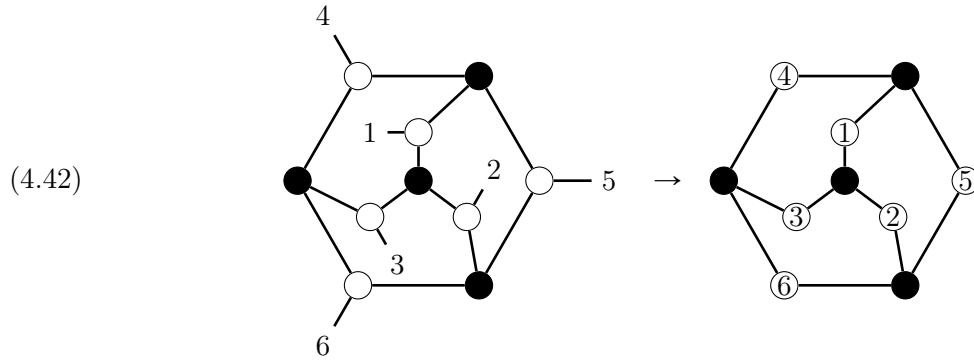
Note that the sign of the minor (24) is unconstrained. This space has a very different topology than the positive Grassmannian $G_+(2, 4)$. As a result, we see that even for a simple planar diagram we can associate multiple spaces with its canonical function f_T . In this case, the space (4.37) is strongly preferable to than (4.41), but for a general non-planar diagram the situation is less clear. We illustrate this in Section 4.5.

4.4 Determinant Form and Identity Moves for Non-Planar Diagrams

In this section, we investigate the triplet formula (4.25) for non-planar diagrams further from a more mathematical perspective. While all on-shell MHV canonical functions from diagrams must come from triplets, the converse is not quite true. Some triplets give a form which is identically zero. We would like to exclude these triplets from our consideration. With this in mind, we first give

a combinatorial characterization for when f_T is not the zero polynomial, using the determinantal formula. Second, we use the determinantal formula to give a factorization algorithm for f_T following from work of Castravet–Tevelev [122]. Third, and most significantly, we turn to identity moves, meaning moves on triplets which preserve the form. For the planar case, the only identity move is the square move from Section 2. For arbitrary triplets, there are additional identity moves, discovered by [64, 122], called *sphere moves*. We show that in fact the only identity moves are sphere moves. In the process, we introduce a new object (“doublets”) which encode the same information as the triplets but in a sphere-move-invariant way.

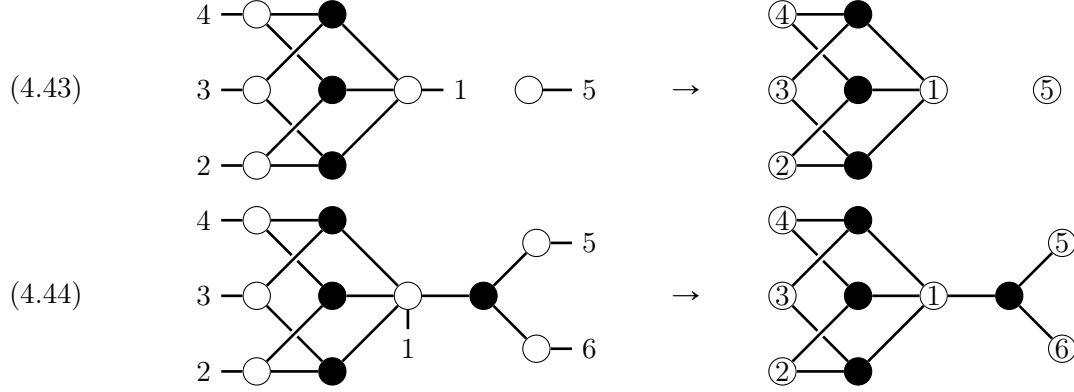
From this point on, except for a few examples, we use a simplified notation for on-shell diagrams shown below, where we omit the external legs and instead number the white vertices adjacent to external legs.



4.4.1 Non-vanishing function and factorization

First, we identify which triplets give a non-vanishing function f_T . One can show that if any Plücker coordinate of C is identically zero, then in fact there is a triple $(ijk) \in T$ such that all three Plücker coordinates $(ij), (ik), (jk)$ are identically zero. In this case, the matrix $C^\perp(\vec{\alpha}^*)$ is never full-rank, as it has a row of zeros, and $\det(M_{ab}) = 0 = f_T$. On the other hand, if all Plücker coordinates of C are not identically zero, then $C^\perp(\vec{\alpha}^*)$ is generically full-rank, and represents the subspace perpendicular to C . So there is a polynomial g in the Plücker coordinates such that for all a, b , $\det(M_{ab}) = \pm(ab)g$. We have that g is generically nonzero, as M_{ab} is generically full-rank. So in this case, f_T is not identically zero.

We can also characterize entirely using the triples T when f_T is not the zero polynomial. Roughly, f_T is nonvanishing if every subset of T contains “enough points”. The formal statement is below, but first we consider a pair of examples:



The top diagram is disconnected, with the external edge 5 attached to a white vertex that has no other edges. The bottom diagram amends this by introducing an extra triplet and external edge. The canonical function f_T vanishes for both as a consequence of there being the same group of three triplets that contain only four indices.

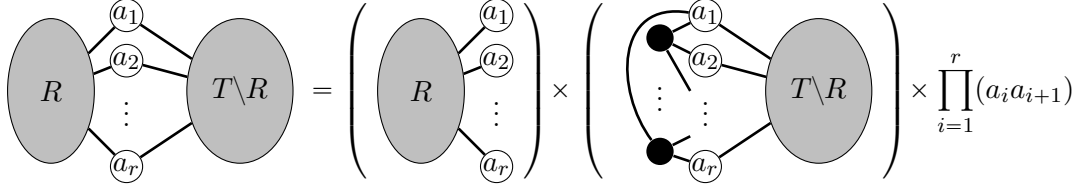
LEMMA 4.4.0.1 (Non-vanishing condition). *The polynomial f_T is not the zero polynomial if and only if any subset of d triplets contains at least $d + 2$ indices. Formally, for any $S \subset \{1, \dots, n - 2\}$,*

$$(4.45) \quad \left| \bigcup_{i \in S} \tau_i \right| \geq |S| + 2.$$

PROOF. Consider a subset $S \subset \{1, \dots, n - 2\}$ of rows in the matrix $C^\perp(\vec{\alpha}^*)$. In order to have a non-vanishing f_T , the number of columns with non-zero entries in at least one of these rows must be at least $|S| + 2$. Otherwise, one could choose $\{a, b\}$ so that in M_{ab} , rows S have rank at most $|S| - 1$ and thus the determinant of M_{ab} is zero. Rows correspond to triplets, and non-zero entries correspond to indices in the triplets, so this shows f_T is nonzero only if (4.45) holds.

On the other hand, if (4.45) holds, then we can apply the factorization algorithm below. The algorithm terminates with the collection of irreducible factors of the numerator of f_T , which are either individual Plückers or the irreducible “hypertree divisors” of Castravet–Tevelev [122]. In particular, all the factors are nonzero, so f_T is also nonzero. $\square$

We now turn to factorization. We call a subset R of T a *valid subset* if R consists of $d < n - 2$ triplets and uses exactly $d + 2$ indices. If R is valid, then R again corresponds to an on-shell diagram on a smaller number of points. As illustrated below, if T has a valid subset R , then f_T factors.



Stated more formally, the factorization is as follows.

LEMMA 4.4.0.2 (Factorization of the form). *Let T be a set of $n - 2$ triplets on $[n]$. Suppose R is a valid subset of T . Then*

$$(4.46) \quad f_T = f_R \times f_{(T \setminus R) \cup P} \times \prod_{i=1}^r (a_i a_{i+1}),$$

where $A = \{a_1, \dots, a_r\}$ consists of the indices appearing both in R and T (we set $a_{r+1} = a_1$) and $P = \{(a_1, a_i, a_{i+1}) | i \in \{2, \dots, r - 1\}\}$ is a set of triplets defining a triangulation of a polygon with vertices A .

PROOF. The main idea of the proof lies within the determinantal formula. By choosing the removed columns to be within R , one finds a block-triangular matrix, factorizing the determinant. In order to get back a set of triplets corresponding to a diagram, one of these factors can be rewritten to include the auxiliary set of triplets P , whose contribution is corrected by the extra linear factors. Details of the proof can be found in Appendix A.1. $\square$

We make a few remarks on the factorization formula. If all the indices of R are contained in $T \setminus R$, and R is planar¹, then the factorization is trivial. That is, one may choose $P = R$, so the first and last factors are inverses and the middle factor is again f_T .

There are two distinct cases when the factorization is nontrivial. Of course we may assume that the form is nonzero. The first case of nontrivial factorization is when R consists of a single triplet (ijk) and one index, say k , does not appear in any other triplet of T . We have that i, j will both appear in T (otherwise the form would be zero), so $A = \{i, j\}$ and P is empty. We obtain the

¹We include the trivial case where R is a single triplet.

factorization

$$\begin{aligned}
 f_T &= f_R \times f_{T \setminus R} \times (ij)^2 \\
 (4.47) \quad &= f_{T \setminus R} \times \frac{(ij)}{(ik)(kj)}.
 \end{aligned}$$

So the triplet (ijk) contributes a monomial in Plücker coordinates to the form. The second case of nontrivial factorization is when R is non-planar. In this case, the numerator of f_R is a high-degree, possibly quite complicated, polynomial in Plückers.

Applying (4.46) repeatedly in the two nontrivial cases, one eventually arrives at a product whose factors are Plücker coordinates, their inverses, and forms for T where T has no lone indices and no valid subsets which are non-planar. We call the latter sets of triplets *irreducible*, and they are in direct correspondence with the *irreducible hypertrees* studied by [122]. For such irreducible T , when f_T is written in lowest terms, the numerator is the square of an irreducible polynomial. This irreducible polynomial vanishes on a *hypertree divisor* of $\mathcal{M}_{0,n}$. We note that the factorization of f_T is not necessarily unique, since there may be some cancellation between distinct factors.

4.4.2 Sphere moves

As is already clear from the planar case, there may be many different triplets which give rise to the same nonzero form. It is well-known that all planar diagrams with the same form are related by square moves:

$$\begin{aligned}
 (4.48) \quad & \begin{array}{ccc} \text{Diagram 1} & \leftrightarrow & \text{Diagram 2} \\ \text{Diagram 1: A square with vertices } (1,2) \text{ on the left, } (3,4) \text{ on the right, and } (1,3) \text{ on the top.} & & \text{Diagram 2: A square with vertices } (1,2) \text{ on the left, } (3,4) \text{ on the right, and } (1,4) \text{ on the top.} \end{array} \\
 & f_{(124),(234)} = f_{(123),(134)}
 \end{aligned}$$

In terms of triplets, T has a square if $(ijk), (ikl) \in T$ for some i, j, k, l . The square move replaces these two triplets with $(jil), (jkl)$. For non-planar diagrams, some diagrams with the same form are *not* related by square moves. Castravet–Tevelev show in [122] that the more general *sphere moves* also preserve the numerator of f_T ; using very different techniques, [64] showed that the whole form is preserved under sphere moves. We will show that in fact the sphere moves are the only moves on triplets preserving the form.

To describe sphere moves most concisely, we switch from thinking of a triplet (ijk) as a “tripod” (one black vertex attached to the three white vertices i, j, k) to thinking of (ijk) as a triangle with vertices labeled i, j, k . A sphere move comes from triangulating the sphere in any way, labeling the vertices with some subset of $[n]$, and then coloring half of the triangles black and the other half white so that every vertex is in both a black and a white triangle. The sphere move replaces the triplets B in T corresponding to the black triangles with the triplets W corresponding to the white triangles.

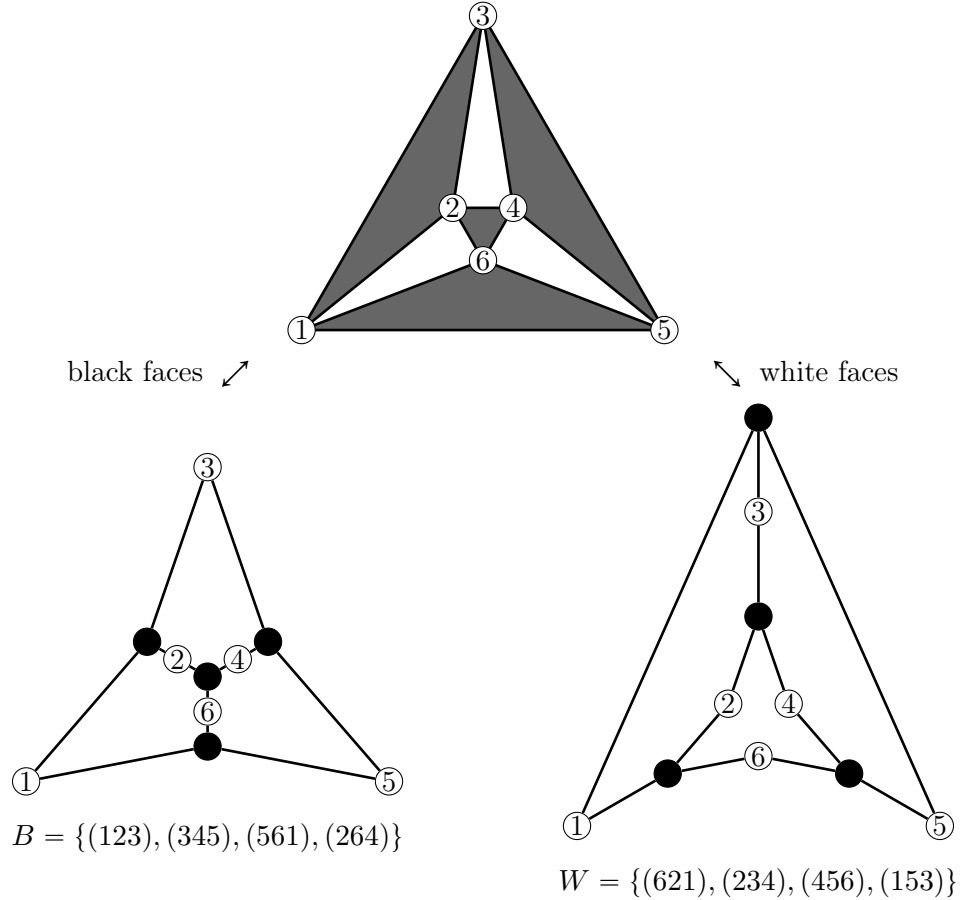


FIGURE 4.1. The 6-point sphere move. For convenience, in the bottom row we omit the external legs and label the white vertex they were attached to instead.

For example, consider the triangulation of the sphere into 8 triangles shown above in Figure 4.1 (since we draw the sphere on the plane, one triangle is the unbounded region). The black triangles correspond to the triplets $B = \{(123), (345), (561), (264)\}$, and the white to $W = \{(621), (234), (456), (153)\}$. The corresponding sphere move can be applied to any T containing the triplets B

and replaces B with W and does not change the associated form. Notice that the square move is a particularly special kind of sphere move. In this case, the triangulation consists of 4 triangles, and looks like a tetrahedron with two faces colored black and two colored white.

Note also that every sphere move changes a valid subset of T into a different valid subset. This is easy to see by a simple counting argument using the Euler characteristic. With $|B| = |W| = d-2$, the triangulation will have $2d-4$ faces and $3d-6$ edges, so requiring that the Euler characteristic is $\chi = 2$ reveals that the number of vertices is d . So both B and W are valid. One could define a similar move for any triangulation of any surface, with half the triangles colored white and half black. However, if this surface is not a sphere, such a move would involve triplets which fail the nonvanishing condition. Indeed, for a surface which is not a sphere, we have $\chi = 2 - 2g \leq 0$. We also have $\chi = 2|B| - 3|B| + V$ so the number of vertices is at most $|B|$. Thus any set of triples T containing B has vanishing form.

4.4.3 Sphere moves are the only moves

To show that triplets have the same form if and only if they are related by sphere moves, it is useful to introduce another combinatorial object, inspired by the derivation of the sphere moves in [64]. For T a collection of triplets, the corresponding set of *doublets* is

$$(4.49) \quad D(T) = \{ij \mid i \text{ and } j \text{ appear together in an odd number of triplets}\}.$$

The idea is that the doublet sets are themselves invariant under sphere moves, so they are a better labeling set for the form than the set of triplets. With these definitions, we state the following theorem.

THEOREM 4.4.1. *Let T and T' be two sets of triplets with nonvanishing forms (i.e. T, T' satisfy (4.45)). The following are equivalent:*

- (1) *There is a sequence of sphere moves relating T and T' .*
- (2) *The corresponding forms are equal: $f_T = f_{T'}$.*
- (3) *The corresponding doublets are the same: $D(T) = D(T')$.*

We give a sketch of the proof here; see the appendix for additional details. (1) $\implies$ (2) follows from [64, 122].

For (2) $\implies$ (3), we claim that $D(T)$ is also equal to

$$\{ij|(ij) \text{ appears with an odd exponent in } f_T\}.$$

For example, taking the set of triplets $T = \{(123), (134), (135)\}$, one has

$$(4.50) \quad f_T = \frac{(13)}{(12)(32)(14)(34)(15)(35)}, \quad D(T) = \{12, 23, 14, 34, 15, 35, 13\}.$$

We prove this statement in general by examining residues and applying factorization in Lemmas [A.2.0.1](#), [A.2.0.2](#), and [A.2.0.3](#).

To prove (3) $\implies$ (1), we give an algorithm in Lemma [A.2.0.4](#) to build sets $R \subset T, R' \subset T'$ so that $R \cup R'$ triangulate the sphere. As an example, we illustrate the algorithm for the 6-point non-planar diagram. We have two triplets $B = \{(123), (345), (561), (264)\}$ and $W = \{(621), (234), (456), (153)\}$ that give the same doublets, namely

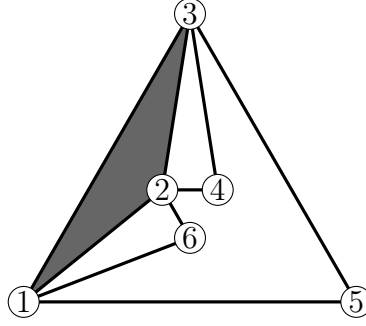
$$D(B) = D(W) = \{12, 23, 13, 34, 24, 35, 45, 56, 15, 26, 24, 46\}$$

Now we construct a sphere move from B to W by running the algorithm in Lemma [A.2.0.4](#). We start by picking a random triplet (123) from set B . So now $R = \{(123)\}, R' = \{\}$, and $D(R) = \{12, 23, 13\}$. Treating R as a collection of black triangles, so far we have



Moving onto step 2), $W \setminus R' = \{(621), (234), (456), (153)\}$. All triplets except (456) have a pair of indices $\{i, j\}$ in the doublet set $D(R) \setminus D(R') = \{12, 23, 13\}$. These triplets are added to R' , so $R' = \{(621), (234), (153)\}$. Interpreting R' as a collection of white triangles, we have attached three white triangles to the lone black triangle, (the triangle corresponding to (135) is the unbounded region).

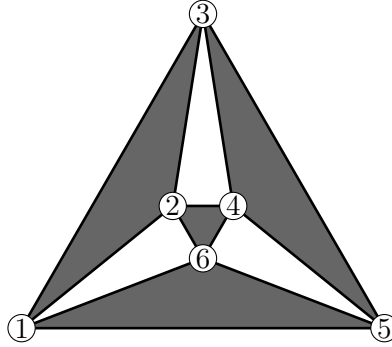
(4.52)



There aren't any triplets in $W \setminus R' = \{(456)\}$ containing a pair of indices in $D(R') \setminus D(R)$ so we skip step 3). All the triplets in $B \setminus R = \{(345), (561), (264)\}$ contain a pair of indices in $D(R \cup R') = \{26, 34, 24, 15, 53\}$ so we now have $R = \{(123), (345), (561), (264)\}$.

Cycling around, we skip step 1) since $W \setminus R = \{\}$. Following step 2), we add (456) to R' , which gives us a complete triangulation as show below

(4.53)



The algorithm terminates here and we can see from the graph that R and R' form a complete bicolored triangulation of a sphere.

The upshot of Theorem 4.4.1 is that two general non-vanishing MHV on-shell diagrams have the same canonical functions f_G if and only if they are related by sphere moves and merge/expand moves. This is a generalization of the analogous statement for the planar case, where two diagrams have the same canonical functions if and only if they are related by square moves and merge/expand moves.

4.5 Oriented Regions and Geometries for Non-planar Diagrams

Our goal in this section is to associate regions of the Grassmannian with MHV on-shell diagrams G (equivalently triplets T), just as $G_+(2, n)$ is associated with planar on-shell diagrams. These regions will have canonical form $\omega_G = \omega_T$, in the sense of [123].

4.5.1 Parke-Taylor Factors as Canonical Functions of Oriented Regions

To prepare for a geometric interpretation of the decomposition in (4.30), we identify geometries whose canonical functions are the Parke-Taylor factors. Recall that $G_+(2, n)$ has canonical function $\text{PT}(12 \dots n)$. We also recall from Section 2 that the interior of $G_+(2, n)$ has a nice interpretation in terms of n ordered points on $\mathbb{P}^1$. If we reorder these points via a permutation σ , or equivalently reorder the columns of C using σ , we obtain another space isomorphic to $G_+(2, n)$, which is described by all Plücker coordinates having some fixed sign and whose canonical function is $\text{PT}(\sigma)$. We also have the freedom of exchanging any column of C for its negative; this gives a space with the same canonical function. We call these spaces *oriented regions*.

We can specify an oriented region by specifying a way to permute and negate the columns of $C \in G_+(2, n)$. We use the notation

$$(4.54) \quad R(\varepsilon_{i_1} i_1, \varepsilon_{i_2} i_2, \dots, \varepsilon_{i_n} i_n) = \{C \in G(2, n) : \varepsilon_{i_a} \varepsilon_{i_b} (i_a i_b) \geq 0 \ \forall \ a < b\},$$

where $i_1 \dots i_n$ is a permutation of $1, \dots, n$ and $\varepsilon_i \in \{\pm\}$ is a choice of sign.

This notation is redundant, allowing for cyclic shifts, overall sign-flips, and reversals up to a sign,

$$(4.55) \quad R(\varepsilon_{i_1} i_1, \varepsilon_{i_2} i_2, \dots, \varepsilon_{i_n} i_n) = R(\varepsilon_{i_2} i_2, \dots, \varepsilon_{i_n} i_n, -\varepsilon_{i_1} i_1),$$

$$(4.56) \quad R(\varepsilon_{i_1} i_1, \varepsilon_{i_2} i_2, \dots, \varepsilon_{i_n} i_n) = R(-\varepsilon_{i_1} i_1, -\varepsilon_{i_2} i_2, \dots, -\varepsilon_{i_n} i_n),$$

$$(4.57) \quad R(\varepsilon_{i_1} i_1, \varepsilon_{i_2} i_2, \dots, \varepsilon_{i_n} i_n) = R(\varepsilon_{i_1} i_1, -\varepsilon_{i_n} i_n, \dots, -\varepsilon_{i_2} i_2).$$

We fix the redundancy by making the following convention choice: we fix $i_1 = 1$, and $\varepsilon_1 = \varepsilon_2 = +$. This leaves us with $2^{n-2} \cdot (n-1)!$ distinct oriented regions that tile the real Grassmannian. As a shorthand for oriented regions denoted with this convention, we will write $R(1, \varepsilon\sigma)$, where ε is understood to denote the signs $\varepsilon_3, \dots, \varepsilon_n$ and σ denotes the permutation of $2, \dots, n$.

Noting that $R(1, 2, \dots, n) = G_+(2, n)$ has canonical function $\text{PT}(1, 2, \dots, n)$, it is easy to see that $R(1, \varepsilon\sigma)$ has canonical function $\text{PT}(1, \sigma)$. Consequently, each Parke-Taylor factor is the canonical function of 2^{n-2} distinct oriented regions. It is important to note that in both cases, the canonical function is unique only up to an overall sign; however, this subtlety will not be relevant to our further discussion as the decompositions from (4.30) contain only Parke-Taylor factors with positive sign.

As each oriented region is simply a relabeling and reflection of the positive Grassmannian, its boundary structure is as discussed in Section 4.2. We call two oriented regions *adjacent* if they share a codimension 1 boundary. This occurs when the signed permutations labeling the regions differ by a simple transposition:

$$(4.58) \quad R(1, \dots, \varepsilon_i i, \varepsilon_j j, \dots) \Big|_{(ij)=0} = R(1, \dots, \varepsilon_j j, \varepsilon_i i, \dots) \Big|_{(ij)=0},$$

$$(4.59) \quad R(1, \dots, \varepsilon_i i) \Big|_{(1i)=0} = R(1, -\varepsilon_i i, \dots) \Big|_{(1i)=0},$$

where $|_{f=0}$ denotes the intersection with $\{f = 0\}$. The second case is the same as the first, but looks different because of our conventions on how to notate oriented regions. This geometric statement about adjacent regions is in direct correspondence with a statement about residues of adjacent Parke-Taylor factors,

$$(4.60) \quad \text{Res}_{(ij)=0} \text{PT}(1 \dots ij \dots) = - \text{Res}_{(ij)=0} \text{PT}(1 \dots ji \dots).$$

4.5.2 Unions of Oriented Regions

We now use oriented regions to obtain spaces with canonical function f_T . Since for each f_T there are a multitude of Parke-Taylor decompositions (one for each orientation of the triplets, and then more if one may apply sphere moves), and each Parke-Taylor factor is the canonical function of 2^{n-2} different oriented regions, there is a very large set of geometries with canonical function f_T , most of which have undesirable properties such as unnecessary spurious boundaries. We will restrict our attention as follows.

- We consider only those geometries that come directly from the formula (4.32) for f_T in terms of triplets

$$(4.61) \quad f_T = \sum_{\sigma \in S_n^{(T)}} \text{PT}(\sigma) \rightsquigarrow R_T = \bigcup_{\sigma \in S_n^{(T)}} R(\varepsilon_\sigma \sigma).$$

There are many other formulas for f_T . For example, one may add a sum of PT factors which is equal to 0. We do not consider geometries from such formulas.

- We consider geometries only up to re-signing all of the appearances of a given index.
- We choose signs for oriented regions so that the number of adjacencies is maximized.

To convince the reader that these restrictions are reasonable, we will quickly illustrate some bad features that occur already at 4 points when we do *not* place these restrictions. Then, with the restrictions in mind, we analyze all possible geometries for the simplest non-planar on-shell diagram, which serves as a starting point for understanding more general unions.

4-point square The unique, up to relabeling, 4-point on-shell diagram has canonical function $\text{PT}(1234)$, which can be thought of as coming from triplets in the sense of (4.32)

$$T = \{(123), (134)\} \rightarrow S_4^{(123), (134)} = \{1234\}.$$

Consequently, one positive geometry with this form is simply $R(1234) = G_+(2, 4)$ as one expects from this being a planar diagram. Since the canonical function is not affected by reflections, the positive geometries $R(1, 2, -3, 4)$ or $R(1, 2, -3, -4)$ also have the desired form, and are isomorphic. This motivates our constraint to consider geometries only up to re-signing all appearances of a single index.

Applying a Kleiss-Kuijf relation, we can find a more complicated geometry. As one has $\text{PT}(1234) = -(\text{PT}(1243) + \text{PT}(1423))$, the positive geometry $R(1243) \cup R(1423)$ also has canonical function $\text{PT}(1234)$. This geometry has an undesirable property, called a *null boundary*: there is a codimension-1 boundary along $(13) = 0$ but (13) is not a pole of the original form. Nonetheless, we still consider this geometry as it does come from an orientation of the triplets in the sense of (4.32) as

$$T' = \{(123), (143)\} \rightarrow S_4^{(123), (143)} = \{1243, 1423\}.$$

If we flip some signs, say to obtain $R(1, 2, 4, 3) \cup R(1, -4, 2, 3)$, we do not change the form. As this is not an overall reflection, the topology of the resulting geometry has changed. In particular, we now have a second null boundary when $(24) = 0$ as the two regions are no longer glued on the corresponding facet. This motivates our constraint to consider only those geometries where signs are chosen to make as many regions adjacent as possible.

Furthermore, one could in principle include extra oriented regions for which the canonical function adds to zero. For example, $\text{PT}(1432) + \text{PT}(1342) + \text{PT}(1234) = 0$, so we can take any of the aforementioned geometries and then take the union with $R(1432) \cup R(1342) \cup R(1324)$, which introduces many null boundaries. This motivates our constraint to consider only those geometries that directly come from triplets.

5-point non-planar The unique 5-point nonplanar on-shell diagram (up to relabeling) is

$$(4.62) \quad \begin{array}{c} \text{Diagram 1: A diamond shape with vertices 2 (left), 3 (top), 1 (bottom), and 4 (center). Vertex 2 is black, 3 and 1 are white, and 4 is black. Edges connect 2-3, 2-1, 3-4, 1-4, and 4-5.} \\ \text{Diagram 2: A diamond shape with vertices 2 (left), 3 (top), 1 (bottom), and 4 (center). Vertices 2, 3, and 1 are black, and 4 is white. Edges connect 2-3, 2-1, 3-4, 1-4, and 4-5.} \end{array} =$$

with triplets $T = \{(123), (134), (135)\}$ or, applying a square move, $T' = \{(124), (234), (135)\}$. The canonical function is

$$(4.63) \quad f_T = \frac{(13)}{(12)(23)(14)(34)(15)(35)},$$

and (up to a sign) has many PT expressions induced by different orientations of T or T' ,

$$(4.64) \quad \begin{aligned} f_T &= \text{PT}(12345) + \text{PT}(12354) \\ &= \text{PT}(13245) + \text{PT}(13254) + \text{PT}(13425) + \text{PT}(13452) + \text{PT}(13524) + \text{PT}(13542) \\ &= \text{PT}(13425) + \text{PT}(13452) + \text{PT}(13542) + \text{PT}(14235). \end{aligned}$$

The first line comes from both $S_5^{(123),(134),(135)}$ and $S_5^{(124),(234),(135)}$, the second comes from $S_5^{(132),(134),(135)}$, and the third from $S_5^{(142),(234),(135)}$. This list exhausts the possible decompositions up to relabeling. The first decomposition can come from either choice of triplets for a particular choice of orientations; this is caused by $(123), (134)$ and $(124), (234)$ making the same requirement that the indices (1234) to appear in that cyclic order. Corresponding to these decompositions, we

have several geometries that have the correct canonical function:

$$(4.65) \quad R_{(123),(134),(135)} = R_{(124),(234),(135)} = R(12345) \cup R(12354),$$

$$(4.66) \quad R_{(132),(134),(135)} = R(13245) \cup R(13254) \cup R(13425) \cup R(13452) \cup R(13524) \cup R(13542),$$

$$(4.67) \quad R_{(142),(234),(135)} = R(13425) \cup R(13452) \cup R(13542) \cup R(14235).$$

The first two geometries (4.65, 4.66) are strongly connected, i.e. the oriented regions are connected by their facets; we will find in Section 4.5.3 that this is because we chose the orientation of triplets coming from reading around each black vertex in (4.16) clockwise. The last geometry (4.67) is not strongly connected, as $R(14235)$ does not share any facets with the remaining regions.

The latter two geometries (4.66, 4.67) both have a null codimension-1 boundary at $(13) = 0$, with the oriented regions having it as a facet despite it not being a pole of the canonical function. The first geometry (4.65) appears not to have such a problem, but it turns out that a closer look reveals null boundaries at higher codimension. Taking the residue as $2 = \alpha 1$ (or $2 = \alpha 3$), one finds a residue (in the sense of (4.29))

$$(4.68) \quad \text{Res}_{2=\alpha 1} f_T = \frac{1}{\alpha(14)(34)(15)(35)},$$

which has no pole at $(13) = 0$. However, finding the boundary of $R_{(123),(134),(135)}$ at $2 = \alpha 1$, one can write it as the closure of an affinely parametrized set,

$$(4.69) \quad R_{(123),(134),(135)} \Big|_{2=\alpha 1} = \overline{\left\{ \begin{pmatrix} 1 & \alpha & x & 0 & y \\ 0 & 0 & z & 1 & w \end{pmatrix} \in G(2, 5) : \alpha, x, z, w \geq 0, xw - yz \geq 0 \right\}},$$

which does have a boundary at $(13) = 0$. Note that the variable y intentionally does not have an associated inequality in (4.69), as the sign of the minor (45) is not fixed in $R_{(123),(134),(135)}$. Studying this boundary region one does find a further boundary at $(13) = 0$, reached by $z \rightarrow 0$ in the parametrization above. More specifically, we get

$$(4.70) \quad R_{(123),(134),(135)} \Big|_{2=\alpha 1, 3=\beta 1} = \overline{\left\{ \begin{pmatrix} 1 & \alpha & \beta & 0 & y \\ 0 & 0 & 0 & 1 & w \end{pmatrix} \in G(2, 5) : \alpha, \beta, w \geq 0 \right\}}$$

Note that the parameter y is completely free once we set $(123) = 0$. Then the geometry becomes $(\alpha, \beta, w) \in \mathbb{R}_{\geq 0}^3 \times y \in \mathbb{R}$. When we take the projective closure, the geometry associated with the

y factor is $\mathbb{P}^1(\mathbb{R})$. This is a circle topologically and hence has no boundary points, meaning the canonical function is 0. Thus we can see from the geometry that the boundary lying on $(13) = 0$ is null and the canonical function has no pole there.

Even for the simplest non-planar on-shell diagram, it is *impossible* to find a Grassmannian region R_T with correct canonical form f_T which is a positive geometry.

We now turn to the topology of $R(12345) \cup R(12354)$. The regions $R(12345)$ and $R(12354)$ glue on the facet where $(45) = 0$. They also glue on the (codimension-6) vertex where the first three columns vanish:

$$(4.71) \quad R(12345) \Big|_{\text{cols } 1,2,3 \text{ vanish}} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \stackrel{GL(2)}{=} \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix} = R(12354) \Big|_{\text{cols } 1,2,3 \text{ vanish}}.$$

As each oriented region is topologically a ball, gluing them together on a facet and a vertex results in something like a pinched torus. A caricature of this geometry is illustrated in Figure 4.2. One must keep in mind that $R(12345)$ and $R(12354)$ are 6-dimensional, the intersection with $(45) = 0$ is 5-dimensional, the intersection with $(13) = 0$ is 4-dimensional, and the point where the first three columns vanish is 0-dimensional.

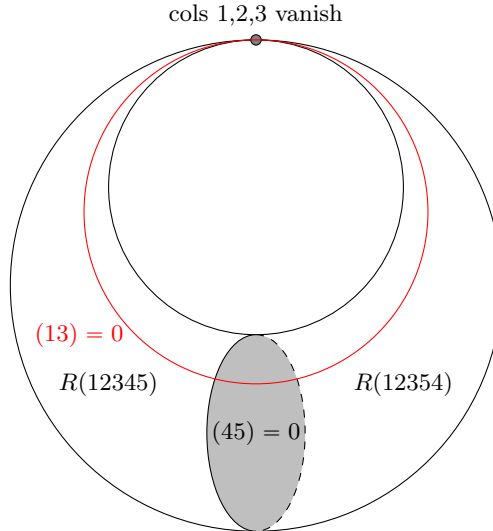


FIGURE 4.2. A caricature of $R(12345) \cup R(12354)$.

Another feature to consider is how PT-like residues are represented in the geometry. Taking the residue at $\alpha = 0$ of (4.68), one finds PT(1435). The label 2 has vanished as we have sent the entire column to zero, so we might expect the corresponding geometry to have reduced to the oriented region $R(1435)$ embedded into $G(2, 5)$ by inserting a zero-column. Unfortunately, this is not the case for any of the geometries², as taking the boundary when columns 1 and 2 are parallel followed by the vanishing of column 2 still leaves us with a union of oriented regions with the canonical function decomposed as $\text{PT}(1345) + \text{PT}(1354)$. Consequently, the boundaries of the geometries for non-planar diagrams do not generally keep the nice recursive structure that appeared for planar diagram geometries.

4.5.3 Hierarchy of Diagrams and Corresponding Results

In this section, we discuss the behavior of the geometries for arbitrary collections of triplets T , and so for arbitrary on-shell diagrams. As mentioned above, there are *many* geometries for each collection of triplets, and some are substantially better behaved than others. We will focus on two kinds of well-behavedness, which require some definitions.

For a collection T of triplets, the region R_T has facets along hypersurfaces $(ij) = 0$. If f_T has a pole along a facet, we call the facet *physical*; otherwise, we call the facet *null*. Note that the residue of f_T along a null facet is zero. More generally, a boundary of R_T is a *null boundary* if, taking iterated residues of f_T , one obtains the zero function on the boundary. Positive geometries have no null boundaries; if a geometry with null boundaries otherwise satisfies the definition of positive geometries, it is a *pseudo-positive geometry*. We will look for geometries which are as close to positive geometries as possible, meaning that null boundaries occur in codimension at least 2. We call a pseudo-positive geometry a *good-facet geometry* if all facets are physical. Positive geometries are good-facet geometries.

We would like the decomposition of R_T into oriented regions to be as analogous to a triangulation of a polytope as possible. We call R_T *strongly connected* if one can walk in R_T from any region $R(1, \epsilon\sigma) \subset R_T$ to any other $R(1, \epsilon'\sigma')$ passing only through facets.

²Technically, one can find a geometry by relabeling the indices that does give a suitable boundary that is a single oriented region. For example, $R(14325) \cup R(14352)$ has boundary $R(1435)$ as desired. However, this will always introduce an analogous problem for another index, as with the vanishing of column 4 for this relabeling.

- To summarize, for an arbitrary collection T of triplets, our main two questions are:
- (1) Can we find a good-facet geometry R_T ? That is, a geometry where f_T has a pole along *every* facet?
 - (2) Can we find a strongly connected geometry R_T ?

Note that if the diagram is planar, the geometry $G_+(2, n)$ is both a good-facet geometry and strongly connected. So we are looking for geometries which are in some sense comparably nice to the planar geometry³. We also note that the two properties are quite different. The first is about how well the geometry reflects the canonical function f_T , while the second is about how “tightly” the geometry is packed together.

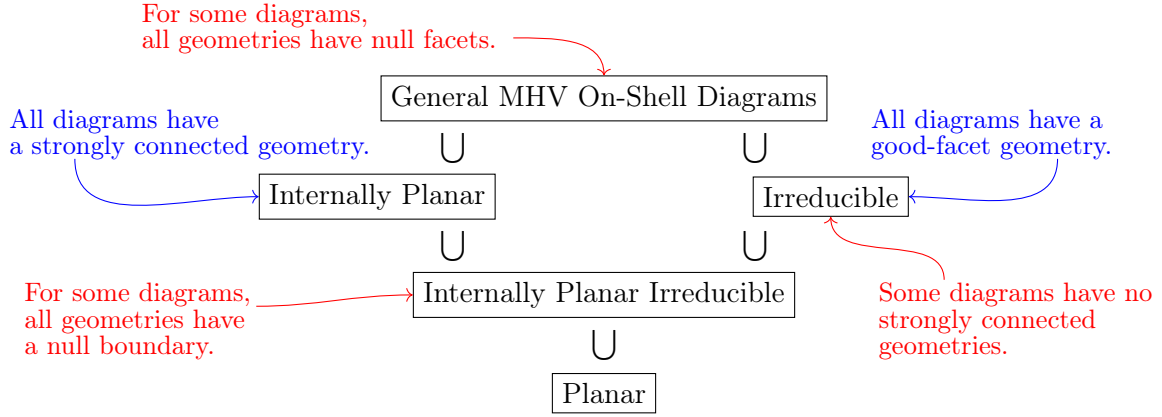
We find two particularly nice classes of diagrams which have good-facet or strongly connected geometries. Recall from Section 4.1 that a diagram is irreducible if f_T does not admit a non-trivial factorization⁴ from Lemma 4.4.0.2. We explain below that if T is irreducible, then for any orientation of T , there is a good-facet geometry R_T .

The second nice class of diagrams are the *internally planar* diagrams. A diagram is internally planar if it can be drawn in a planar way once the external legs have been deleted. One example is the 6-point diagram in (4.42). We emphasize that, while all planar diagrams are also internally planar, most internally planar diagrams are *not* planar on-shell diagrams. This is because planar on-shell diagrams by definition must admit a planar drawing with external legs ending on the boundary of the disk. We prove in Appendix C that all internally planar diagrams have a strongly-connected geometry R_T . (In fact, the orientation of the triplets which gives this geometry is dictated by the planar drawing of the diagram without legs; one reads around each black vertex clockwise.)

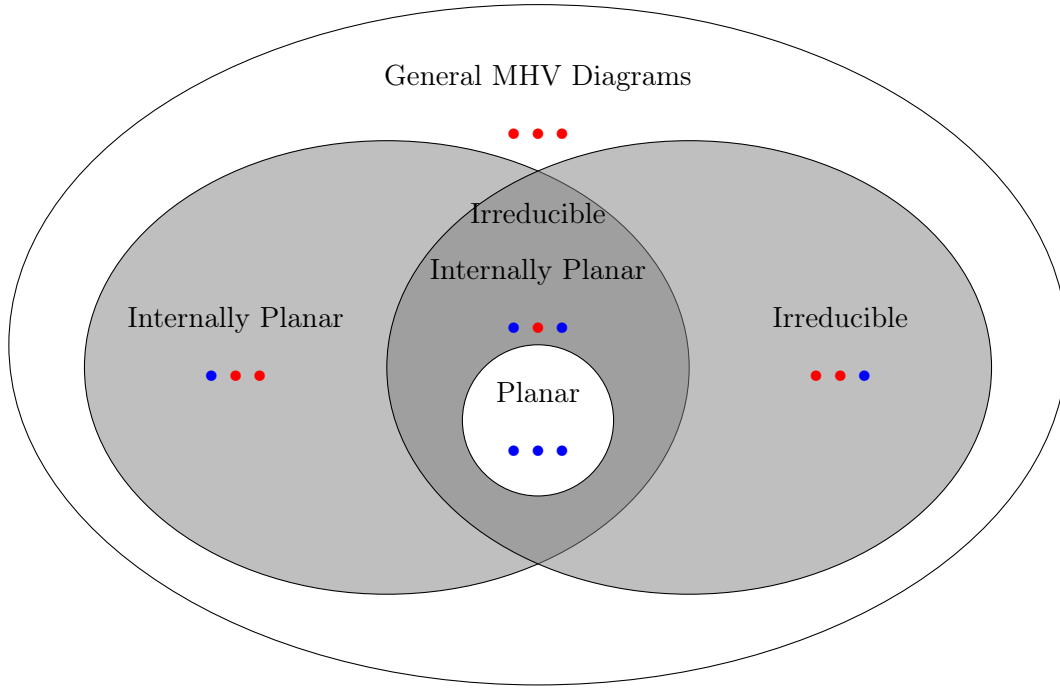
It is a fact of life, however, that for some on-shell diagrams, no geometries are strongly connected (see Section 4.5.4.2), or no geometries are good-facet (see Section 4.5.4.4). Even for internally planar irreducible diagrams, which have strongly connected good-facet geometries, all geometries may have null boundaries of higher codimension and so the diagram admits no positive geometry (see Section 4.5.4.1). Our results are summarized in the diagrams below. We note that planar diagrams are internally planar but are technically *not* irreducible; however, they admit strongly connected good-facet geometries, so we treat them as “morally” irreducible.

³Other nice properties of $G_+(2, n)$, such as being topologically a ball, are too restrictive to ask for.

⁴A more precise characterization is given by irreducible hypertrees [122].



We can also visualize the different sets of on-shell diagrams by the Venn diagram:



In the image we use dots $\bullet\bullet\bullet$ to represent the three properties of our interest in this order:

- All diagrams in the set admit strongly connected geometries.
- All diagrams in the set have good-boundary geometries.
- All diagrams in the set have good-facet geometries.

Blue • signifies the property is always present, and red • signifies that there exists some counterexample.

We can see that the only diagrams which have all three properties are planar diagrams, while the generic (non-planar) MHV diagrams fail to have any of these three. The ‘best’ set of non-planar on-shell diagrams are irreducible internally planar diagrams which only fail to satisfy the second property.

Before moving to examples, we discuss why irreducible diagrams have good-facet geometries for any orientation of the triplets T . Dividing the triplets into maximal planar subdiagrams⁵, e.g.

$$(4.72) \quad T = \{(123)\} \cup \{(345)\} \cup \{(567)\} \cup \{(178)\} \cup \{(284), (246)\},$$

one finds (see Lemma A.2.0.2) that the denominator of the form f_T is precisely the product of the denominators from each planar subdiagram:

$$(4.73) \quad f_T = \frac{((17)(28)(34)(56) - (18)(26)(34)(57) + (18)(24)(35)(67))^2}{(12)(23)(31) \times (34)(45)(53) \times (56)(67)(75) \times (17)(78)(81) \times (28)(86)(64)(42)}.$$

If (ab) is not a pole of f_T , then either a, b do not appear together in any triplet, or they are non-adjacent indices in a maximal planar subdiagram. In the former case, for each oriented region $R(\cdots ab \cdots)$ there is a corresponding $R(\cdots ba \cdots)$ glued on the facet $(ab) = 0$. In the latter case, the natural orientation of the triplets in the planar subdiagram keeps the indices non-adjacent in the oriented regions $R(\cdots a \cdots b \cdots)$. So, choosing suitable orientations within each maximal planar subdiagram, one can ensure that $(ab) = 0$ is not a facet of f_T , and thus one can always find a good-facet geometry.

We note that we do not characterize the diagrams admitting strongly connected or good-facet geometries, though we show that this certainly includes internally planar and irreducible diagrams, respectively. We also do not investigate which diagrams admit positive geometries. However, the previous analysis of the simplest non-planar diagram, the 5-point example in (4.62), shows that all geometries R_T have null boundaries and so are not positive geometries. It seems likely that more generally, diagrams with a 5-point non-planar subdiagram admit only pseudo-positive R_T . This motivates the following question:

⁵This can always be done uniquely. The only way for it to not be unique would be if three triplets share the same doublet $(abc), (abd), (abe)$ with any pair being part of some maximal planar diagram.

Are planar on-shell diagrams the only diagrams which admit a positive geometry R_T ?

If such a geometry exists for a non-planar diagram, it should be connected as a disconnected geometry does not admit an unique canonical form. For two disconnected positive geometries, the canonical form for its union would be

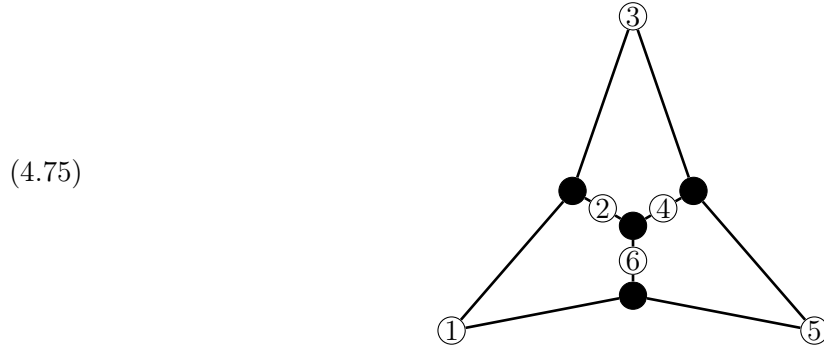
$$(4.74) \quad \Omega = \Omega_1 \pm \Omega_2$$

and there is no way how to fix a relative sign. In fact, the situation is more complicated as for some connected geometries we can have a similar problem. Namely, if two positive geometries share only a vertex, and not a higher dimensional boundary, then the ambiguity of the canonical form is present too. In any case, from our explorations (and using the 5pt argument above) it seems likely that the only positive geometries are positive Grassmannians $G_+(2, n)$, associated with planar diagrams.

4.5.4 Examples

In the remainder of the section, we take a closer look at strongly connected and god-facet examples, and provide examples of geometries lacking these properties.

4.5.4.1 6-point internally planar irreducible The smallest irreducible diagram appears at 6 points, defined by the set of triplets $T = \{(123), (345), (561), (264)\}$. It is also the unique 6-point irreducible diagram, and is internally planar as shown in this embedding:



With the orientation of triplets induced by this embedding, one finds a decomposition labeled by

$$(4.76) \quad S_6^{(T)} = \{142536, 142563, 145236, 145263, 125364, 125634, 152364, 152634\}$$

and the canonical function is

$$\begin{aligned}
(4.77) \quad f_T &= \frac{((15)(26)(34) - (16)(24)(35))^2}{(12)(13)(15)(16)(23)(24)(26)(34)(35)(45)(46)(56)} \\
&= \text{PT}(142536) + \text{PT}(142563) + \text{PT}(145236) + \text{PT}(145263) \\
&\quad + \text{PT}(412536) + \text{PT}(412563) + \text{PT}(415236) + \text{PT}(415263).
\end{aligned}$$

This canonical function is the canonical function of the union of oriented regions

$$\begin{aligned}
(4.78) \quad R_T &= R(142536) \cup R(142563) \cup R(145236) \cup R(145263) \\
&\quad \cup R(412536) \cup R(412563) \cup R(415236) \cup R(415263) \\
&= \left\{ \begin{array}{llll} (12) \geq 0, & (13) \geq 0, & (15) \geq 0, & (16) \geq 0, \\ & (23) \geq 0, & (24) \leq 0, & (26) \geq 0, \\ & & (34) \leq 0, & (35) \leq 0, \\ & & & (45) \geq 0, & (46) \geq 0, \\ & & & & (56) \geq 0. \end{array} \right\}
\end{aligned}$$

This geometry is strongly connected, as one can see by the gluing on the facets $(14), (25), (36) = 0$. This property is specific to the orientation chosen; flipping one of the triplets can lead to a geometry that is not strongly connected:

$$\begin{aligned}
(4.79) \quad R_{(123),(345),(561),(246)} &= [R(123456)] \cup [R(124536) \cup R(124563)] \\
&\quad \cup [R(125346) \cup R(152346)] \cup [R(145623) \cup R(415623)],
\end{aligned}$$

where the strongly connected components have been grouped with braces.

This is a good-facet geometry, as one can see from the defining inequalities matching the poles of the canonical function. This property is independent of the orientation chosen; each triplet is its own maximal planar subdiagram, so any orientation avoids null facets by making the non-pole facets $(14), (25), (36) = 0$ be only internal boundaries of the union.

However, this geometry has null boundaries. Taking the residues on $6 = \alpha 5$ and $\alpha = 0$, the canonical function is

$$(4.80) \quad \text{res}_{\alpha=0} \text{res}_{6=\alpha 5} f_T = \frac{(23)}{(12)(13)(24)(34)(25)(35)},$$

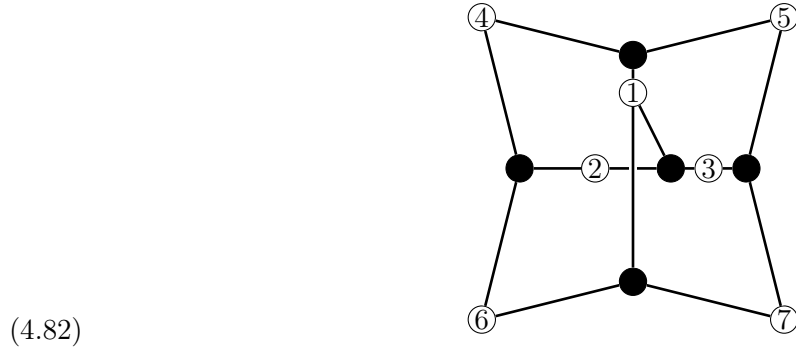
and the corresponding geometry living in $G(2, 5) \cong \{C \in G(2, 6) : \text{column 6 vanished}\}$ is

$$(4.81) \quad R_T \Big|_{6=\alpha 5} \Big|_{\alpha=0} = R(14253) \cup R(41253).$$

This is just a relabeling of the canonical function (4.63) and first geometry (4.65) from the 5-point non-planar diagram. As analyzed earlier, the 5-point non-planar diagram has a null boundary for any geometry that is chosen. Since the 6-point irreducible diagram contains the 5-point non-planar diagram on a boundary, one deduces that *any* geometry chosen for the 6-point irreducible diagram must also have a null boundary at some codimension.

4.5.4.2 7-point irreducible The unique irreducible diagram at 7 points is defined by the set of triplets

$$T = \{(123), (145), (167), (246), (357)\}$$



and has canonical function

$$(4.83) \quad f_T = \frac{((14)(17)(26)(35) - (15)(16)(24)(37))^2}{((12)(13)(14)(15)(16)(17)(23)(24)(26)(35)(37)(45)(46)(57)(67))}.$$

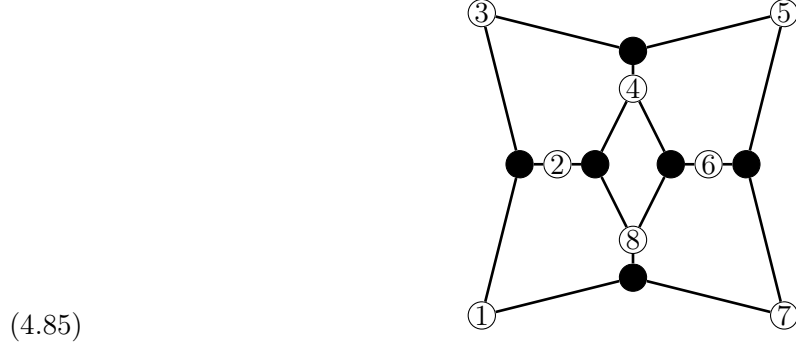
The diagram is not internally planar, so there is no preferred orientation of the triplets. Using the orientation above, $|S_7^{(T)}| = 24$ and we can write it as

$$(4.84) \quad \begin{aligned} S_7^{(T)} = & \{1234567, 1234657, 1243567, 1243657, 1246357\} \\ & \cup \{1624735, 1627435, 1627345, 1672435, 1672345\} \\ & \cup \{1456273, 1456723, 1465273, 1465723, 1462573\} \\ & \cup \{1245673, 1246573\} \cup \{1462735, 1467235\} \cup \{1623457, 1624357\} \\ & \cup \{1462357\} \cup \{1246735\} \cup \{1624573\}, \end{aligned}$$

where the permutations have been grouped by the maximal strongly connected components. We do not show the many other possible decompositions / geometries here, but it is easy to check (with computer assistance) that all possible orientations also give a non-strongly connected geometry.

This is however a good-facet geometry. Like with the 6-point irreducible diagram, this property is independent of the orientation chosen.

4.5.4.3 *8-point internally planar irreducible* One 8-point irreducible diagram is



from which one may read the oriented triplets $T = \{(123), (345), (567), (178), (284), (486)\}$. Then, $|S_8^{(T)}| = 47$, and can be described as follows:

$$(4.86) \quad S_8^{(T)} = \{\sigma \in S_8 / \mathbb{Z}_8 \mid \sigma \text{ related to } 14725836 \text{ by adjacent transpositions} \\ \text{such that no index is swapped twice}\}.$$

For example, $17428536 = \widehat{14725836}$ is in the set, but 14257836 is not. The adjacent transpositions also include swapping 1 and 6, as the permutations defined only up to cyclic shifts. The corresponding canonical function is

$$(4.87) \quad f_T = \frac{((17)(28)(34)(56) + (18)((24)(35)(67) - (26)(34)(57)))^2}{(12)(13)(17)(18)(23)(24)(28)(34)(35)(45)(46)(56)(57)(67)(68)(78)}$$

The resulting geometry is strongly connected, with $R(14725836)$ being ‘at the center’ of the clump of positive regions, so none of its facets are boundaries of R_T .

One may write

$$(4.88) \quad R_T = \left\{ \begin{array}{lllll} (12) \geq 0, & (13) \geq 0, & (15) \geq 0, & (17) \geq 0, & (18) \geq 0, \\ & (23) \geq 0, & (24) \leq 0, & (26) \geq 0, & (28) \geq 0, \\ & & (34) \leq 0, & (35) \leq 0, & (37) \leq 0, \\ & & & (45) \geq 0, & (46) \geq 0, & (48) \geq 0, \\ & & & & (56) \geq 0, & (57) \leq 0, \\ & & & & & (67) \leq 0, & (68) \leq 0, \\ & & & & & & (78) \geq 0. \end{array} \right\}.$$

These are the inequalities defining $R(14725836)$ with those involving adjacent indices removed, as they are allowed to swap. This is also a good-facet-geometry, which is expected as the diagram is irreducible. Unlike the 6 and 7-point cases in 4.5.4.1 and 4.5.4.2, this does depend on the orientation: only those orientations where $(284), (486)$ are oriented as a square produce a geometry with no null facets.

4.5.4.4 *8-point reducible with null facets* Consider the diagram defined by triplets

$$T = \{(123), (124), (125), (346), (357), (458)\},$$



This diagram has many lone indices, so it is clearly reducible, and the canonical function is easy to compute

$$(4.90) \quad f_T = \frac{(34)(35)(45)(12)}{(13)(23)(14)(24)(15)(25)(36)(46)(37)(57)(38)(58)}.$$

Accounting for possible square moves, and for every possible orientation of triplets, one finds for every geometry that at least one of the following is a null facet: $(34), (35), (45) = 0$.

Furthermore, since the diagram is not internally planar, we do not have an orientation induced by a corresponding embedding. Checking all possible geometries, one finds that none are strongly connected.

4.5.5 Geometries without a Diagram

What makes the forms, or corresponding geometries, coming from on-shell diagrams special? From the analysis done earlier in the section it is evident that properties corresponding to planar diagrams are not preserved for a general on-shell diagram; the geometries one generally finds are not strongly connected and have null facets. To begin answering this question, one may study those forms/geometries that do not come from on-shell diagrams to see what may be missing.

Enumerating the canonical functions for pseudo-positive geometries consisting of unions of oriented regions in $G(2, 5)$, one finds (up to permutation of indices) three that do not come from on-shell diagrams:

$$(4.91) \quad R_A = R(15352) \cup R(15423) \rightsquigarrow f_A = \frac{(12)(34)(35) + (13)(23)(45)}{(12)(13)(15)(23)(24)(34)(35)(45)}$$

$$(4.92) \quad R_B = R(15234) \cup R(15423) \cup R(15342) \rightsquigarrow f_B = \frac{(13)(23)(25)(45) - (12)(24)(35)^2}{(12)(13)(15)(23)(24)(25)(34)(35)(45)}$$

$$(4.93) \quad R_C = R(13452) \cup R(15324) \rightsquigarrow f_C = \frac{(14)(15)(23)(24)(35) - (12)(13)(25)(34)(45)}{(12)(13)(14)(15)(23)(24)(25)(34)(35)(45)}$$

It is easy to check that none of these regions are strongly connected, and they all have null facets. Interestingly, though R_A, R_B, R_C do not come from 5-point on-shell diagrams (equivalently, sets of three triplets), they still arise from choosing a larger collection of triplets and taking a union as in (4.61):

$$(4.94) \quad T_A = \{(153), (154), (142), (423)\},$$

$$(4.95) \quad T_B = \{(152), (153), (234), (154)\},$$

$$(4.96) \quad T_C = \{(132), (134), (152), (345)\}.$$

This is interesting, because obviously not all unions of regions come from oriented triplets. For example, consider the following geometry

$$(4.97) \quad R(12345) \cup R(12354) \cup R(12453).$$

Here the first two regions have labels 4, 5 swapped, but the region found by swapping those indices in the third region, $R(12543)$, is not included. This makes the geometry inconsistent with any triplets. However, the canonical function for this region is equal (up to relabeling) to f_B . The task of enumerating all distinct classes of forms is not feasible at 6pt, so we leave it as an open question: Is there a canonical function of a union of oriented regions that is not represented by triplets? Note that in the absence of the on-shell diagram connection, it is ambiguous if the canonical function contains $+PT(\dots)$ or $-PT(\dots)$. In fact, for the region (4.97) we need to consider

$$(4.98) \quad f = PT(12345) + PT(12354) - PT(12453)$$

with these particular signs to guarantee that all highest dimensional residues are ± 1 . If we flip the sign,

$$(4.99) \quad f' = PT(12345) + PT(12354) + PT(12453)$$

some of the residues are ± 2 and we can no longer talk about pseudo-positive geometries. In a different context, a similar feature was observed in [81] where the authors defined the notion of *graded positive geometries*. It seems conceivable that using a ‘wrong’ orientation of the region $R(12453)$ in the union (4.97) yields such geometry. Note that on-shell diagrams by definition never lead to residues other than ± 1 and we never need to discuss the ‘graded version’ of pseudo-positive geometries in this context.

The purpose of studying the canonical functions associated with Grassmannian regions – unions of oriented regions – is motivated by another, even more important, question which in some sense motivated the study of non-planar on-shell diagrams in the first place:

Question: Are the regions in the Grassmannian associated with non-planar on-shell diagrams special?

In our previous discussion, we concluded that the geometries for generic non-planar on-shell diagram (internally non-planar and non-irreducible) do not exhibit any nice properties: some of their facets and boundaries are null and the spaces are disconnected. This is exactly what we expect from a generic union of R -regions. So at the level of geometry, we do not see anything

special. Nevertheless, the canonical functions f_T are very special thanks to the formula (4.25). In particular, the numerator of the canonical function is a perfect square of a polynomial $\mathcal{P}[\dots]$ in Plücker coordinates multiplied by linear factors,

$$(4.100) \quad f = \frac{\prod(\dots) \times \mathcal{P}[\dots]^2}{\prod(\dots)}$$

We can see that the 5pt canonical functions f_A, f_B, f_C not associated with on-shell diagrams do *not* have this form. The same is true for the class of functions we were able to check at 6pt. Hence it is natural to conjecture that only the regions associated with oriented triplets produce the canonical function of the form (4.100). The big open question is what are the geometric implications of this very special property.

4.6 Conclusions and Outlook

In this chapter, we explored the MHV non-planar on-shell diagrams and the correspondence to Grassmannian geometries in $G(2, n)$. In the planar case there is only one top cell of $G_+(2, n)$ and one planar on-shell diagram, but there are many non-planar diagrams of the same (maximal) dimensionality for general n . Our analysis relied mostly on the triplet description, which determines the canonical function either using the determinant formula or the Parke-Taylor expansion. Our first result is an algorithm for identifying diagrams with vanishing forms. In the planar sector, the vanishing of the form indicates the presence of internal bubbles and the fact that the diagram is not reduced. The same type of diagrams do not have a triplet representation. But there is a new class of diagrams which are described by triplets but still have vanishing form, which we are able to identify. Our second result concerns the identity moves. In the planar case, there are only square moves (and trivial merge-expand move) while in the non-planar case there are also sphere moves proposed in [64]. We show using a novel doublet representation that indeed square moves and sphere moves are the only identity moves.

Our main result concerns the (pseudo-)positive Grassmannian geometries that we associate with non-planar on-shell diagrams with non-vanishing canonical functions. The Parke-Taylor decompositions provide candidate spaces, which are unions of permuted positive parts $G_+(2, n)$ s. However, these spaces are generally disconnected and the canonical function does not encode properly the boundary structure, i.e. there are boundaries for which the form vanishes – we call them null facets. We showed

that for internally planar diagrams there is a choice of Parke-Taylor decomposition such that the space is connected, and for irreducible diagrams there are no null facets. The privileged class of internally planar irreducible diagrams enjoys both of these special properties, and they provide an interesting non-planar extension of the planar on-shell diagrams associated with the positive Grassmannian.

Our work reviewed here is just an invitation into the very complex problem of non-planar on-shell diagrams and the Grassmannian geometries. There are two main future directions on the mathematical side: 1) Further study of MHV diagrams and $G(2, n)$ geometries, their boundary structures, topological and combinatorial properties. 2) Generalization to N^{k-2} MHV on-shell diagrams starting with six-point NMHV case, ie. $G(3, 6)$. The complete classification of all non-planar on-shell forms was given in [60], and the list contains exotic terms with polynomial and even algebraic poles. The first step is the classification of all terms with only monomial poles, which is closely related to problems considered in [124, 125]. On the physics side, the canonical forms of non-planar on-shell diagrams are leading singularities of loop integrands, and hence they play a role of coefficients that multiply polylogarithms (and generalizations) in the expansion of non-planar $\mathcal{N} = 4$ SYM amplitudes. If the dual conformal symmetry that we discuss in the next chapter extends in some form to the non-planar sector, such kinematic structures should be most accessible to study precisely at the level of leading singularities.

CHAPTER 5

Searching for Integrability in Amplitudes

In the past two chapters, we have outlined some of the work we did to understand how the special mathematical structures found in planar $\mathcal{N} = 4$ sYM can be extended when one loses the notion of planarity and restrictions on the types of poles that can appear. However, we left out some aspects of what makes planar $\mathcal{N} = 4$ sYM so special in the previous discussion. Thus, we make a detour back in this chapter. After all, our motivation for studying Super Yang-Mills comes from its potential to teach us about the structure of gauge theories, so we must leave no stone unturned. Previously, we have touched upon the fact that $\mathcal{N} = 4$ sYM enjoys a high number of symmetries. In addition to supersymmetry, we have seen that it is a conformally invariant theory. What makes the restricted pole structure and the connection with the positive Grassmannian possible is another symmetry that we haven't discussed in detail yet, called dual superconformal symmetry. In particular, superconformal and dual superconformal symmetry combine to form an infinite dimensional algebra called the Yangian. While this Yangian plays a key role in the geometric story we have told thus far, it also connects sYM to another special property called integrability. Roughly, integrability can be stated as the property that certain quantities in a theory can be determined to all order thanks to an infinite number of independent, mutually commuting conserved charges. As we will see, integrability of planar $\mathcal{N} = 4$ sYM can be used to compute a quantity called the cusp anomalous dimension non-perturbatively. This quantity controls the leading IR divergences of the amplitude. Even more surprisingly, the planar $\mathcal{N} = 4$ sYM amplitude is dual to null polygonal Wilson loops, where the same cusp anomalous dimension controls the divergences at the null cusps. The work we are about to review in the next two chapters will be motivated in large part by trying to draw new bridges between the structures found in perturbative scattering amplitudes, the duality with Wilson loops, and the integrability of planar sYM, where we believe the cusp anomalous dimension acts as a natural entry point. Before we delve deeper into the motivation from the physics side, it will be useful to introduce some of the formalism and mathematical tools

we are going to use in our work. In particular, we will be using the Amplituhedron formalism in the next two chapters, which connects naturally to the earlier discussion on the positive Grassmannian.

5.1 The Amplituhedron

Introduced in 2013 by Jaroslav Trnka and Nima Arkani-Hamed [14], the Amplituhedron plays a key role in the long and elaborate story of what positive geometries have to do with scattering amplitudes. As with many other topics we have discussed here so far, we shall only introduce what we need for the work we are going to review to make sense for the reader, and we refer those interested in further explorations to [13, 66].

In earlier chapters we saw that each on-shell diagram, formed by gluing 3-point vertices, parametrize positroid cells in $G_+(k, n)$, and the canonical form on this positroid cell gives the corresponding on-shell function. At tree-level, the amplituhedron goes a step further and associates a canonical form with the full tree-amplitude. So without further ado, we list the definition of the Amplituhedron below

$$(5.1) \quad \mathcal{A}_{m,n,k} := \{Y \in G(k, k+m) \text{ s.t. } Y = C \cdot Z, C \in G_+(k, n)\}$$

where Z is a $(k+m) \times n$ matrix with positive ordered minors. Rows of this matrix represent momentum twistors Z_i for each external particle $i = 1, 2, 3, \dots, n$. Then, the Amplituhedron is the image of a map from the positive Grassmannian $G_+(k, n)$ we know to the ambient space $G(k, k+m)$ through a matrix encoding external momentum data. In physics, we are interested in $m = 4$. And the canonical form defined on $\mathcal{A}_{4,n,k}$ give us tree-level $N^k MHV$ amplitudes of $\mathcal{N} = 4$ sYM. It is typically said that the amplitude is calculated as a volume in this space. What one does in practice is compute this “volume” through triangulating the amplituhedron. More specifically, one triangulates $\mathcal{A}_{n,k}$ into cells Δ_j such that each cell is an image of a cell in the positive Grassmannian $G_+(k, n)$ (through the map $Y = C \cdot Z$) and Ω_{Δ_j} is the canonical form of this positroid cell. Then the canonical form on the amplituhedron is

$$(5.2) \quad \Omega_{\mathcal{A}_{n,k}} = \sum_j \Omega_{\Delta_j}$$

Just like the canonical forms we discussed earlier, the only singularities of this form are logarithmic and located on the boundaries of $\mathcal{A}_{n,k}$. The BCFW recursion procedure we have discussed earlier

give one such triangulation, where each Ω_{Δ_j} correspond to a BCFW term. While these individual terms, and the corresponding forms, don't necessarily have boundaries that correspond to physical factorization channels, these singularities cancel out in the total sum.

This formalism also generalizes to loop-level, where the canonical form describes the integrand. To see this construction in detail, it will be useful to introduce momentum twistor variables first, which we heavily use in chapters 6 and 7.

5.1.1 Momentum twistors

Scattering amplitudes in planar $\mathcal{N} = 4$ SYM theory and the underlying Amplituhedron geometry [14] are naturally described using momentum twistor variables [117]. Let's first introduce the dual momentum variables x_i , which are associated with a particular cyclic ordering of the n external momenta p_i in $\mathbb{R}^{3,1}$, according to

$$(5.3) \quad p_i = x_i - x_{i-1} ,$$

where $i = 1, 2, \dots, n$ labels the external particles.

Momentum conservation is automatically satisfied if we identify $x_{n+1} \equiv x_1$ and hence the dual points x_i form a closed polygon with light-like edges. The complexified momenta for n particles can also be encoded in n complex vectors $Z_i \in \mathbb{C}^4$, called *momentum twistors*. The dual conformal symmetry [126], which we review later in this chapter, acts as $SL(4)$ and the invariant multi-linear form is given by the twistor bracket

$$(5.4) \quad \langle ijkl \rangle := \det(Z_i Z_j Z_k Z_l) = \epsilon_{IJKL} Z_i^I Z_j^J Z_k^K Z_l^L ,$$

where ϵ_{IJKL} is the four-dimensional Levi-Civita tensor. More concretely, we can think about these *four-brackets* as 4×4 determinants of four momentum twistors. Any n generic momentum twistors correspond to cyclically ordered lightlike momenta that satisfy momentum conservation. Hence, the Z_i are unconstrained and all four-brackets are generically non-zero. The four-brackets do satisfy certain identities as derived from the properties of Levi-Civita tensors, for example, the Schouten identity

$$(5.5) \quad \langle abcd \rangle \langle abef \rangle + \langle abc f \rangle \langle abde \rangle = \langle abce \rangle \langle abdf \rangle .$$

Due to dual conformal invariance, we can treat the Z_j 's as points in the projective space $\mathbb{P}^3$. In order to describe the loop integrands, we introduce (dual) loop momenta $y_i \in \mathbb{R}^{3,1}$, $i = 1, 2, \dots, L$, which in momentum-twistor space are associated with lines AB_ℓ in $\mathbb{P}^3$. The choice of points A_i , B_i on this line is arbitrary, and the collection of all lines in $\mathbb{P}^3$ is the *Grassmannian* $G(2, 4)$ we encountered before.

While we can't interpret the loop amplitude, which has branch cuts as well as IR divergences, in terms of a nice geometry, the loop integrand itself only has simple poles that correspond to factorization channels. As mentioned before, the planar limit of $\mathcal{N} = 4$ sYM makes it possible to talk about the L -loop integrand as a single object that gets integrated over, instead over a sum. And thus, the Amplituhedron can be generalized for the loop-level to represent integrands for all n, k and L . To do so, one enlarges the C matrix into a $(k + 2L) \times n$ one such that we have the space of k -planes in n -dimensions and on top also L number of 2-planes D^ℓ that live in $C^\perp$. This can be represented as

$$\mathcal{C} = \begin{pmatrix} D^{(1)} \\ D^{(2)} \\ \vdots \\ D^{(L)} \\ C \end{pmatrix}$$

The L -loop Amplituhedron is then the image of the map $\mathcal{C} \rightarrow \mathcal{Y} = \mathcal{C} \cdot Z$ such that

$$\mathcal{Y} = \begin{pmatrix} \mathcal{L}^{(1)} \\ \mathcal{L}^{(2)} \\ \vdots \\ \mathcal{L}^{(L)} \\ Y \end{pmatrix}$$

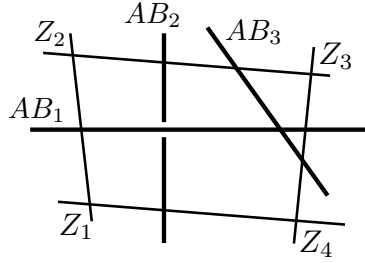
and $\mathcal{L}_\alpha^{(\ell)} = (D^{(\ell)})^i_\alpha Z_{i\alpha}$. So essentially we have added the kinematic data for L loops into our description, encoded by $\mathcal{L}^{(\ell)}$ for each loop momenta ℓ . The loop generalization also comes with additional positivity conditions, given by

$$(5.6) \quad \langle (AB)_i \ j \ j + 1 \rangle > 0, \quad \langle (AB)_i (AB)_j \rangle > 0$$

where $(AB)_i$ encode loop-momenta in momentum twistor space. We note that while the loop Amplituhedron provides a geometric description for the problem of finding a loop integrand, the actual computation of it is still an unsolved problem for general L . Especially for $n > 4$, the computational complexity grows quite fast with L . But for our purposes at the moment, we are only interested in $n = 4$, so let's see how this construction looks.

5.1.2 Definition of the four-point Amplituhedron

The four-point L -loop Amplituhedron [16] is defined as a configuration of L lines in the momentum twistor space [117],



The external data are represented by four momentum twistors Z_1, Z_2, Z_3, Z_4 and must satisfy

$$(5.7) \quad \langle 1234 \rangle \equiv \epsilon_{IJKL} Z_1^I Z_2^J Z_3^K Z_4^L > 0.$$

Each loop is represented by one line AB_i where A_i, B_i are two arbitrary points on that line. The configuration space of the line AB_i is restricted to satisfy the following inequalities

$$(5.8) \quad \langle AB_i 12 \rangle, \langle AB_i 23 \rangle, \langle AB_i 34 \rangle, \langle AB_i 14 \rangle > 0, \quad \langle AB_i 13 \rangle, \langle AB_i 24 \rangle < 0.$$

We can represent the loop lines using the external momentum twistors, and will often use the standard parametrization

$$(5.9) \quad Z_{A_i} = Z_1 + x_i Z_2 - y_i Z_4, \quad Z_{B_i} = Z_3 + z_i Z_2 + w_i Z_4.$$

In that case, the inequalities reduce to $x_i, y_i, z_i, w_i > 0$. Furthermore, for each pair of lines AB_i, AB_j we have a mutual positivity condition,

$$(5.10) \quad \langle AB_i AB_j \rangle > 0 \quad \text{for each pair } i, j.$$

In the parametrization (5.9) this corresponds to

$$(5.11) \quad -(x_i - x_j)(w_i - w_j) - (y_i - y_j)(z_i - z_j) > 0.$$

These mutual inequalities make the Amplituhedron space very complicated at higher L . We define a canonical form Ω_L with logarithmic singularities on the boundaries of this space, which directly reproduces the L -loop integrand form,

$$(5.12) \quad \Omega_L = \mathcal{I}_L d\mu_L \quad \text{where} \quad d\mu_L = \prod_{i=1}^L \langle AB_i d^2 A_i \rangle \langle AB_i d^2 B_i \rangle$$

where $d\mu_L$ is the integration measure on the configuration of L lines in the momentum twistor space (we often drop this factor for convenience). At the low loop order, we often use a more traditional notation for the loop lines, $AB \equiv AB_1$, $CD \equiv AB_2$, $EF \equiv AB_3$ to make the formulas more transparent.

5.1.3 Calculating the differential form

The four-point integrand is equal to the canonical differential form with logarithmic singularities on the boundaries of the space. At one-loop the space is extremely simple, given just by a set of inequalities $x, y, z, w > 0$ and the dlog form is just

$$(5.13) \quad \Omega_1 = \frac{dx}{x} \frac{dy}{y} \frac{dz}{z} \frac{dw}{w} = \frac{\langle 1234 \rangle^2 d\mu}{\langle AB12 \rangle \langle AB23 \rangle \langle AB34 \rangle \langle AB14 \rangle},$$

where $d\mu = \langle AB d^2 A \rangle \langle AB d^2 B \rangle$ is the integration measure of a line in $\mathbb{P}^3$, analogue of $d^4\ell$ in momentum space. At higher loops, the situation is more complicated and we can pursue multiple strategies to obtain the form. The most straightforward is to triangulate the space into elementary “simplices”, write the dlog forms for them and sum.

At two loops, we need to divide the space into eight elementary regions when solving the mutual inequality condition [15]. The space is defined in the standard parametrization

$$(5.14) \quad x_1, y_1, w_1, z_1, x_2, y_2, w_2, z_2 > 0, \quad -(x_1 - x_2)(w_1 - w_2) - (y_1 - y_2)(z_1 - z_2) > 0.$$

We divide the space into eight simplices by giving definite signs to differences (x_1-x_2) , (y_1-y_2) and (z_1-z_2) . One of them is given by the following inequalities

$$(5.15) \quad x_1 > x_2 > 0, \quad y_1 > y_2 > 0, \quad z_1 > z_2 > 0, \quad w_1 > 0, \quad w_2 > w_1 + \frac{(y_1-y_2)(z_1-z_2)}{(x_1-x_2)},$$

and the canonical dlog form for this simplex is

$$(5.16) \quad \Omega_2^{(1)} = \frac{dx_1}{(x_1-x_2)} \frac{dx_2}{x_2} \frac{dy_1}{(y_1-y_2)} \frac{dy_2}{y_2} \frac{dz_1}{(z_1-z_2)} \frac{dz_2}{z_2} \frac{dw_1}{w_1} \frac{dw_2}{w_2 - w_1 - \frac{(y_1-y_2)(z_1-z_2)}{(x_1-x_2)}}.$$

The poles (x_1-x_2) , (y_1-y_2) and (z_1-z_2) are just artifacts of the triangulation, and when summing over all pieces they go away. Summing all the contributions,

$$(5.17) \quad \Omega_2 = \sum_{i=1}^8 \Omega_2^{(i)} = \frac{dx_1 dx_2 dy_1 dy_2 dz_1 dz_2 dw_1 dw_2 (x_1 w_2 + x_2 w_1 + y_1 z_2 + z_1 y_2)}{x_1 x_2 y_1 y_2 w_1 w_2 z_1 z_2 [(x_1-x_2)(w_1-w_2) + (y_1-y_2)(z_1-z_2)]}.$$

When writing back in terms of momentum twistors we get

$$(5.18) \quad \Omega_2 = \frac{d\mu_2 (\langle AB12 \rangle \langle CD34 \rangle + \langle AB23 \rangle \langle CD14 \rangle + \langle AB34 \rangle \langle CD12 \rangle + \langle AB14 \rangle \langle CD23 \rangle) \langle 1234 \rangle^3}{\langle AB12 \rangle \langle AB23 \rangle \langle AB34 \rangle \langle AB14 \rangle \langle ABCD \rangle \langle CD12 \rangle \langle CD23 \rangle \langle CD34 \rangle \langle CD14 \rangle},$$

where we use the usual notation for two loop lines AB and CD , rather than the generic AB_i . When expanding the numerator and canceling poles in the denominator we get an expansion of Ω_2 in terms of four double-box integrands (s -double box, t -double box + symmetrization in AB, CD).

It is easy to see that this construction gets complicated rather quickly for higher loops. This will be dealt with in due time, with a strategy we outline in chapter 6. However, it will be good to not digress too much here and remind ourselves of the physical motivation. In particular, we are interested in finding a bridge between the mathematical structures we have seen in perturbative amplitudes, and their duality with Wilson loops, which we haven't discussed yet. One of the most crucial such bridges for our story starts with a 2005 paper by Zvi Bern, Lance Dixon, and Vladimir Smirnov [47]. This is now known as the BDS ansatz.

5.2 The BDS Ansatz

Postulated first in 1997, the AdS/CFT correspondence [127] lead many to believe that sYM might be a solvable theory. In particular, this duality tells us that results at strong 't Hooft coupling take a simple form. This hints at hidden simplifications on the perturbative level that would allow for all-loop resummations, as this is another way to get strong coupling results without dualities.

Another such hint was found in [47] on the perturbative side. In particular, the authors found that the 4 and 5 point planar $\mathcal{N} = 4$ sYM amplitude has an iterative form that allows for an all-loop representation¹. More specifically, the amplitude (for $n = 4, 5$) takes the form

$$(5.19) \quad A_n = \exp \left\{ \left[\sum_{l=1}^{\infty} a^l(f^{(l)}(\epsilon) A_n^{(1)}(l\epsilon) + h_n^{(l)}(\epsilon)) \right] \right\}$$

where $A_n^{(1)}(l\epsilon)$ is the $\frac{1}{\epsilon^2}$ divergent 1-loop amplitude and f, h are functions of ϵ and transcendental constants. This is now known as the BDS ansatz. While it was known that IR-divergences exponentiate in Gauge Theories, the BDS ansatz makes a powerful statement about the finite part of the amplitude. The leading IR-divergence in sYM is controlled by a quantity called the cusp anomalous dimension in the following way

$$(5.20) \quad \log \left(\frac{A_n}{A_n^{\text{tree}}} \right) = -\frac{1}{2} \sum_{\ell=1}^{\infty} \left(\frac{g_{YM}^2 N}{16\pi^2} \right)^{\ell} \left[\frac{\Gamma_{\text{cusp}}^{(\ell)}}{(\ell\epsilon)^2} + \frac{2G_0^{(\ell)}}{\ell\epsilon} \right] \sum_{i=1}^n \left(\frac{\mu_{IR}^2}{-s_{i,i+1}} \right)^{\ell\epsilon} + F_n^{MHV}(p_1, p_2, \dots, p_n, g)$$

where $F_n^{MHV}(p_1, p_2, \dots, p_n, g)$ is the finite part of the amplitude. Writing $g^2 = \frac{g_{YM}^2 N}{16\pi^2}$ for the 't Hooft coupling, we define the cusp anomalous dimension as

$$(5.21) \quad \Gamma_{\text{cusp}}(g) = \sum_{L=1}^{\infty} g^{2L} \Gamma_{\text{cusp}}^{(L)}.$$

The BDS statement is even stronger. In particular, the BDS ansatz says the finite part is also controlled by Γ_{cusp} through

$$(5.22) \quad F_4(p_1, p_2, p_3, p_4, g) = \frac{\Gamma_{\text{cusp}}(g)}{4} \left(\ln^2 \left(\frac{s}{t} \right) + \text{const} \right)$$

$$(5.23) \quad F_5(p_1, p_2, p_3, p_4, p_5, g) = \frac{\Gamma_{\text{cusp}}(g)}{4} \left(\sum_{i=1}^5 \ln \frac{s_{i,i+1}}{s_{i+1,i+2}} \ln \frac{s_{i+2,i+3}}{s_{i+3,i+4}} + \text{const} \right).$$

While this ansatz no longer holds at 6-point, the surprisingly simple form we see at 4 and 5 points hints at additional structure in the perturbative regime. To understand why this happens, we need to first study a duality between Wilson loops and amplitudes in $\mathcal{N} = 4$ sYM.

¹The initial ansatz was for any n but it was found in later work to fail starting at $n = 6$

5.3 Amplitudes and Wilson-Loops

In [128], Alday and Maldacena computed the gluon amplitude in strong coupling using the AdS/CFT correspondence, where the leading order result is a minimal surface calculation. Furthermore, they found that the result matched the expectation value of the null polygonal Wilson loop. This lead them to conjecture a duality between Wilson loops and scattering amplitudes in planar $\mathcal{N} = 4$ sYM that was eventually generalized to the weak coupling (perturbative) limit. In particular, the duality can be states as follows

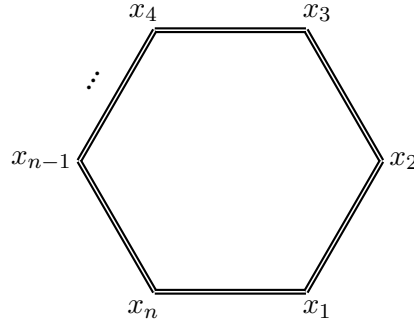
$$(5.24) \quad \ln W(C_n) = D_{n,UV} + F_n^{WL}(a, x_{ij}^2) + \mathcal{O}(\epsilon)$$

$$(5.25) \quad \ln\left(\frac{A_n}{A_{tree}}\right) = D_{n,IR} + F_n^{WL} + const + \mathcal{O}\left(\frac{1}{N}\right) + \mathcal{O}(\epsilon)$$

where the null polygonal Wilson loop is defined as

$$(5.26) \quad W(C_n) = \frac{1}{N} \langle 0 | Tr(\mathcal{P} \exp \left\{ \left(ig \int_{C_N} dx_\mu A^\mu \right) | 0 \rangle \right\}.$$

An interesting fact about this duality that we haven't emphasized thus far is that it relates the IR divergences of amplitudes to UV divergences of Wilson loops. To see this in more detail, let's look at how these UV divergences arise. The Wilson-loop $W(C_n)$ is defined on an n-side polygon with each vertex assigned a coordinate in dual momenta, related to null momenta p_i by $x_{i+1} - x_i = p_i$.



The contour is guaranteed to be closed by momentum conservation $\sum_i^n p_i = 0$. Since each momenta is null, we have $x_{i,i+1}^2 = 0$. These vertices where the null momenta meet are called cusps. Each cusp carries a double logarithmic divergence whose coefficient is given by Γ_{cusp} , and this is where the cusp anomalous dimension gets its name from. More specifically, divergences enter $\log W(C_n)$

as

$$(5.27) \quad \langle \log W(C_n) \rangle = \sum_{i=1}^n D(x_{i-1,i+1}^2) + F_n(x_{ij})$$

where $D(x_{i-1,i+1}^2)$ encodes the UV divergences and the finite part is a function of dimensionless ratios constructed out of x_{ij}^2 . This object is gonna play a big part in Chapter 7. After dimensionally regularizing by a parameter ϵ , one gets

$$(5.28) \quad D(x^2) = -\frac{1}{4} \sum_{\ell=1} g^{2\ell} (-x^2 \mu^2)^{\ell\epsilon} \left(\frac{\Gamma_{cusp}}{(\ell\epsilon)^2} + \mathcal{O}\left(\frac{1}{\epsilon}\right) \right)$$

The divergence takes the same form for all n , thus we can take appropriate ratios of Wilson loops to get finite quantities out of them. We can do the same thing for the planar $\mathcal{N} = 4$ sYM amplitude, as its IR divergences are also universal and controlled by Γ_{cusp} . Since such ratios are regulator independent, they can be used to state the duality with the scattering amplitude in a more clean way. For example, let's look at $n = 4$, which is the case central to our discussion in Chapters 6 and 7. On the Wilson-loop side, we have

$$(5.29) \quad \log \left(\frac{\langle W(x_{13}^2, x_{24}^2) \rangle}{\sqrt{\langle W(x_{13}^2, x_{13}^2) \rangle \langle W(x_{24}^2, x_{24}^2) \rangle}} \right) = \frac{1}{2} g^2 \log^2 \left(\frac{x_{13}^2}{x_{24}^2} \right) + \mathcal{O}\left(\left(\frac{g^2 N}{8\pi^2}\right)^2\right)$$

which matches the part of the 4-point amplitude after we subtract IR-divergent and constant terms, namely

$$(5.30) \quad R_4 = \frac{1}{2} g^2 \log^2 \left(\frac{s}{t} \right) + \mathcal{O}\left(\left(\frac{g^2 N}{8\pi^2}\right)^2\right).$$

The Wilson-loop duality also makes manifest another symmetry that was hidden in the initial amplitudes formulation, which is called dual superconformal symmetry. Let's first unpack the conformal side of this story.

5.3.1 Dual Conformal Symmetry

The null polygonal Wilson loop is most naturally expressed in dual momentum coordinates x_i , on which conformal transformations act as follows.

$$(5.31) \quad x_i^\mu \rightarrow \frac{x_i^\mu}{x_i^2}, \quad x_{ij}^2 \rightarrow \frac{x_{ij}^2}{x_i^2 x_j^2}.$$

These transformations leave null separated $x_{i,j}$ invariant, thus the full Wilson loop should be invariant under these dual conformal transformations. However, we need to regularize the UV divergences when computing expectation values, which breaks the conformal invariance. In fact, something similar happens for the amplitudes too. Let's consider the 1-loop amplitude.

$$(5.32) \quad A_4^{1-loop} = A_4^{tree} \times \int \frac{d^4 k}{(2\pi)^4} \frac{(p_1 + p_2)^2 (p_2 + p_3)^2}{k^2 (k + p_1)^2 (k + p_1 + p_2)^2 (k - p_4)^2}$$

We can now switch to dual coordinates x_i , where $p_i^\mu = x_i^\mu - x_{i-1}^\mu$. In this notation, writing $x_{jk} = (x_j^\mu - x_k^\mu)$ and $k^\mu = x_5^\mu - x_1^\mu$, the integral becomes

$$(5.33) \quad \int \frac{d^4 x_5}{(2\pi)^4} \frac{x_{13}^2 x_{24}^2}{x_{15}^2 x_{25}^2 x_{35}^2 x_{45}^2}$$

The integrand is invariant under conformal transformations in dual space. But wait, if the momenta are on-shell, then $x_{i,i+1}^2 = 0$, so we can't set $d = 4$ if we want to actually compute the integral. If we regulate by going to $d = 4 - 2\epsilon$ then the integral is no longer invariant under dual conformal transformations, as the Jacobian from transforming the measure no longer cancels with the 4 propagators. So once again, dual conformal invariance is broken, this time by an IR-regulator.

To recap, the null polygonal Wilson loop contain UV poles that come from cusps of the contour, dual to IR divergences in the sYM loop amplitude. Just like with the amplitude, regulating these divergences breaks the dual conformal invariance. This leads to an anomalous contribution to the conformal boost ward identity (this is indeed where "anomalous" in the cusp anomalous dimension comes from)

$$(5.34) \quad \sum_{i=1}^n (2x_i^\mu (x_i \partial_{x_i}) - x_i^2 \partial_{x_i}^\mu) F_n^{WL} = \frac{\Gamma_{cusp}(g)}{2} \sum_{i=1}^n x_{i,i+1}^\mu \ln \frac{x_{i,i+2}^2}{x_{i-1,i+1}^2}$$

The homogenous solution to this differential equation only exists for $n \geq 6$, and the particular solution gives the full F_n^{WL} when $n = 4, 5$. Thus, this constraint is enough to fix F_n^{WL} for $n = 4$ and $n = 5$ as follows

$$(5.35) \quad F_4^{(WL)} = \frac{\Gamma_{cusp}(g)}{4} \ln^2 \frac{x_{13}^2}{x_{24}^2} + const$$

$$(5.36) \quad F_5^{(WL)} = \frac{\Gamma_{cusp}(g)}{4} \sum_{i=1}^5 \ln \frac{x_{i,i+2}^2}{x_{i+1,i+3}^2} \ln \frac{x_{i+2,i+4}^2}{x_{i+3,i}^2} + const$$

which matches the BDS ansatz from before. Furthermore, as we will discuss soon, $\Gamma_{cusp}(g)$ can be computed from integrability. Starting at $n = 6$, one can build cross ratios $\frac{x_{ij}^2 x_{kl}^2}{x_{ik}^2 x_{jl}^2}$ that enter into the homogenous solution, and thus the finite part is no longer fixed by the Ward identities. If the conjectured Wilson loop-Amplitudes duality holds, this explains why the BDS ansatz is true for $n = 4, 5$ but fails starting at $n = 6$. More concretely, starting at $n = 6$, one has the BDS ansatz plus an arbitrary function of all the conformal invariants one can build. For the 6-point case, there are 3 of them

$$(5.37) \quad u_1 = \frac{x_{13}^2 x_{46}^2}{x_{14}^2 x_{36}^2}, \quad u_2 = \frac{x_{24}^2 x_{15}^2}{x_{25}^2 x_{14}^2}, \quad u_3 = \frac{x_{35}^2 x_{26}^2}{x_{36}^2 x_{25}^2}$$

so we get

$$(5.38) \quad F_6^{WL} = F_6^{BDS} + f(u_1, u_2, u_3, g)$$

While the question of how to constrain this remainder is an interesting open question on its own, our focus for this thesis is going to be when $n = 4$.

5.3.2 The Yangian

In $\mathcal{N} = 4$ sYM, R -symmetry, supersymmetry, and conformal symmetry combine to form a superconformal algebra that lives in $PSU(2, 2|4)$. The generators j_a annihilate the superamplitude such that

$$(5.39) \quad j_a A_n^{tree} = 0.$$

Let's see if we can do a similar analysis for the dual conformal symmetry discussed in the previous section, namely work out how dual superconformal symmetry acts on λ , $\tilde{\lambda}$, and η . The derivation, done in [83] is lengthy so here we will just list the results from the reviews in [37, 129]. First we note that just like dual variables can be used to write momenta as $p_i = x_{i+1} - x_i$, one can do the same for supercharges and introduce dual supervariables θ_i such that

$$(5.40) \quad \theta_i^{\alpha A} - \theta_{i+1}^{\alpha A} = \lambda_i^\alpha \eta_i^A$$

Then, one can define a dual superconformal symmetry that acts on variables $\{\lambda, \tilde{\lambda}, \eta, x, \theta\}$. The explicit form of the dual superconformal generators $K_{\alpha\dot{\alpha}}$ and $S_{\alpha A}$ won't be relevant to our discussion

here, but they can be found in the original paper [83]. What is important for our purposes here is that they act on tree amplitudes through

$$(5.41) \quad K^{\alpha\dot{\alpha}} A_n = - \sum_i x_i^{\alpha\dot{\alpha}} A_n^{tree}$$

$$(5.42) \quad S^{\alpha A} A_n = - \sum_i \theta_i^{\alpha A} A_n$$

In order to derive the analogue of $j_a A_n = 0$ for dual superconformal symmetry, one can define the new operators

$$(5.43) \quad K^{\alpha\dot{\alpha}'} = K^{\alpha\dot{\alpha}} + \sum_i x_i^{\alpha\dot{\alpha}}$$

$$(5.44) \quad S^{\alpha A'} = S^{\alpha A} + \sum_i \theta_i^{\alpha A}$$

$$(5.45) \quad D' = D + n$$

where $DA_n = nA_n$. The new operators satisfy the dual superconformal algebra and now for any generator J'_a of the dual $PSU(2, 2|4)$ we can write

$$(5.46) \quad J'_a A_n^{tree} = 0$$

Furthermore, if we only act on on-shell superspace variables, the action of all generators of superconformal symmetry coincide with those of dual superconformal symmetry, except for K' and S' . Then all generators of the original superconformal algebra, j_a , can be combined with K' and S' to form the Yangian algebra of $PSU(2, 2|4)$, denoted by $Y(PSU(2, 2|4))$. This Yangian is defined by zero-level commutators

$$(5.47) \quad [J_a^{(0)}, J_b^{(0)}] = [j_a, j_b] = f_{ab}^c j_c$$

and level-one commutators

$$(5.48) \quad [J_a^{(0)}, J_b^{(1)}] = [j_a, j_b^{(1)}] = f_{ab}^c j_c^{(1)}$$

as well as higher order commutators given by Serre relations [129]. These relations generate an infinite tower of conserved charges, hence the Yangian is an infinite dimensional algebra. For any $y \in Y(psu(2, 2|4))$, one has

$$(5.49) \quad yA_n^{tree} = 0$$

which represents all the symmetries of the tree-level amplitude in sYM. The Grassmannian formulation of tree amplitudes we have discussed in the very beginning has deep connections to the appearance of this Yangian. In particular, the Grassmannian integral

$$(5.50) \quad \mathcal{L}_{n,k}(\mathcal{W}_i) = \int \frac{d^{n \times k} C_{ai}}{GL(k) \prod_{j=1}^n M_j} \prod_{a=1}^k \delta^{4|4} \left(\sum_{l=1}^n C_{al} \mathcal{W}_l^A \right)$$

gives a generic form of Yangian invariants. Furthermore, Yangian symmetry is said to be a hallmark of integrability [130], which is the reason why we have discussed it here. We are now in a good position to revisit integrability and talk about the motivation behind our work.

5.4 Integrability, Γ_{cusp} , and Open Questions

The calculation of Γ_{cusp} from integrability is quite technical and introduces more machinery than we need to motivate our work. Thus, we will only attempt a surface level review of the story here. So let's start from where we left off with the Yangian. While we saw the Yangian in the context of planar $\mathcal{N} = 4$ sYM amplitudes at tree-level, this is not the only theory where it shows up. For example, the Heisenberg spin chain model also has a Yangian symmetry. This model describes spins arranged in a periodic 1-dimensional lattice, with the Hamiltonian

$$(5.51) \quad H = 4 \sum_{i=1}^L \left(\frac{1}{4} - S_i \cdot S_{i+1} \right), \quad S_i = \frac{\vec{\sigma}_i}{2}$$

where $\vec{\sigma}_i$ are the three Pauli-matrices. In 1931, Hans Bethe found a solution to the eigenvalue problem $H|\psi\rangle = E|\psi\rangle$ for this Hamiltonian, which now goes by the name of Bethe ansatz [131]. In particular, for a spin chain of length L with M down spins, Bethe found

$$(5.52) \quad E = 2 \sum_{k=1}^M \frac{1}{u_k^2 + \frac{1}{4}}$$

with complex variables u_k , called Bethe roots, being determined by the following equation.

$$(5.53) \quad \left(\frac{u_k + \frac{i}{2}}{u_k - \frac{i}{2}} \right)^L = \prod_{j=1, j \neq k}^M \frac{u_k - u_j + i}{u_k - u_j - i}$$

As explained in the review article [132], one can also derive these results by using an operator called transfer matrix, denoted by $\mathbf{T}(u)$. This method allows for the generalization of the result above to other representations of spin and also makes the Yangian connection we alluded earlier manifest. In particular, $\mathbf{T}(u)$ generates an infinite tower of commuting charges whose algebra closes into the Yangian $Y(su(2))$. The level 0 generators are none other than total spin $\sum_i S_i$ and the level 1 generators are built bilocally (pairs of products of level zero generators on different sites). Furthermore, the Hamiltonian we saw above is the second conserved charge generated by $\mathbf{T}(u)$. Thus, the conserved quantities produced by the Yangian symmetry justifies the Heisenberg spin chain model's integrability. In this context, integrability refers to the fact that for a spin chain of length L we only have to solve M Bethe roots instead of diagonalize the full $2^L \times 2^L$ matrix, as a direct consequence of having L mutually commuting conserved charges we can simultaneously diagonalize. But wait, there is more. As it turns out, single trace operators in $\mathcal{N} = 4$ sYM can be arranged in a spin chain just like this. Furthermore, planar $\mathcal{N} = 4$ sYM carries the same integrable spin-chain structure [133, 134]!

Following the example of [132], let's consider the simplest $su(2)$ scalar sector, which turns out to be exactly the Heisenberg spin chain at one-loop. We can identify the complex scalars X and Z with the spins such that

$$(5.54) \quad Z = |\uparrow\rangle, \quad X = |\downarrow\rangle$$

Since trace is cyclic, one must impose periodic boundary conditions, which in the spin-chain side looks like $S_{L+1} = S_1$. Then, for example, the single trace operator $Tr(ZZXZZ)$ would correspond to the periodic spin chain state $|\uparrow\uparrow\downarrow\uparrow\uparrow\rangle$. To one-loop, the dilatation generator $D \in su(2) \in psu(2, 2|4)$ takes the form

$$(5.55) \quad D = L + g^2 H + \mathcal{O}(g^4)$$

where H is the spin-chain Hamiltonian from before. Then, the anomalous dimension spectra problem $D\mathcal{O} = \Delta\mathcal{O}$ maps to the spin-chain eigenvalue problem $H|\psi\rangle = E|\psi\rangle$! In particular, one-loop anomalous dimensions of the single trace operators of the scalar $su(2)$ sector are given by the spin chain energy eigenvalues, up to a multiplicative factor with 't Hooft coupling. While the exact mapping to the Heisenberg Hamiltonian doesn't extend beyond one-loop, the integrable structure does!

In the previous section, we saw that a quantity called Γ_{cusp} controlled the leading IR-divergences of the planar $\mathcal{N} = 4$ sYM amplitude, as well as the cusp divergences of the dual Wilson-loop. Another context in which this quantity comes up is by controlling the large spin behavior of twist-two operators $Tr(D_+^{M-k} Z D_+^k Z)$, which live in the $sl(2)$ sector of $\mathcal{N} = 4$ sYM. This can be mapped onto an $sl(2)$ spin chain by treating each lightcone covariant derivative as an excitation on the state $Tr(Z^L)$, where L is the spin-chain length [135]. Written in the notation from [136], the asymptotic anomalous dimension spectrum problem is then again solved by the Bethe roots u_i that obey

$$(5.56) \quad \left(\frac{x_k^+}{x_k^-} \right)^L = \prod_{j=1}^M \frac{u_k - u_j - i}{u_k - u_j + i} \left(\frac{1 - g^2/x_k^+ x_j^-}{1 - g^2/x_k^- x_j^+} \right)^2 e^{2i\theta(u_k, u_j)}$$

where $g^2 = \frac{g_{YM}^2 N}{16\pi^2}$ and $\theta(u_k, u_j)$ is a dressing phase added by Beisert, Eden, and Staudacher [137] after they noticed a discrepancy between [138] and the earlier work [135]. Furthermore, in [137], they found an integral equation for the Γ_{cusp} (which they call scaling function), given by

$$(5.57) \quad \hat{\sigma}(t) = \frac{t}{e^t - 1} \left[\hat{K}(2gt, 0) - 4g^2 \int_0^\infty dt' \hat{K}(2gt, 2gt') \hat{\sigma}(t') \right]$$

and

$$(5.58) \quad \Gamma(g) = 8g^2 \hat{\sigma}(0)$$

where the Kernel is written in terms of Bessel functions. While there is no closed form solution for this equation, one can use it to compute $\Gamma(g)$ asymptotically to any order in g for both weak and strong coupling, a truly impressive discovery! In particular, one can use this equation to find

$$(5.59) \quad \Gamma_{cusp}(g)|_{g \ll 1} = 4g^2 - \frac{4\pi^2}{3}g^4 + \frac{44\pi^4}{45}g^6 - 8 \left(\frac{73\pi^6}{630} + 4\zeta_3^2 \right) g^8 + \mathcal{O}(g^{10})$$

$$(5.60) \quad \Gamma_{cusp}(g)|_{g \gg 1} = 2g - \frac{3 \log 2}{2\pi} + \mathcal{O}\left(\frac{1}{g}\right).$$

This is a good place to stop and review the pieces of the story we have so far. We have seen that there are two narratives at play here, one regarding the duality between Wilson-loops and scattering amplitudes, and another regarding the integrability of planar $\mathcal{N} = 4$ sYM thanks to the infinite tower of conserved charges generated by its Yangian symmetry. And the cusp anomalous dimension points us at a very clean connection between these two narratives. Both the universal IR divergences of the amplitude and the cusp divergences of the Wilson-loop are controlled by this quantity, allowing us to make a statement about the duality without worrying about regulator dependent changes. Furthermore, this quantity is known exactly to all orders from the integrability of single trace operators in $\mathcal{N} = 4$ sYM. However, we still don't have a complete understanding of how all these pieces fit together. It is almost like the integrable structure of sYM and AdS/CFT is trying to tell us that we should be able to solve sYM perturbatively. In particular, the theory might just have enough structure to let us resum everything in the perturbation theory side of things. Despite all the advanced technology we have reviewed thus far, this is still quite an ambitious task. So we need to choose a good entry point to see how such a program would even be possible. Let's consider the 4-point amplitude. Its logarithm is of great interest to us for two reasons. For starters, its leading IR divergence is quite tame at $\frac{1}{\epsilon^2}$ for any loop order and it is controlled by Γ_{cusp} , for which we have an understanding of through the BES equation. Secondly, it is dual to the null polygonal Wilson-loop with a lagrangian insertion. Inspired by these two facts, [77] found a novel expansion for the integrand of the logarithm of the amplitude in terms of diagrams called "negative-geometries", which we explain in full detail in the next chapter. Furthermore, certain subsets of these diagrams obey differential equations that make partial resummations possible! This allows one to both resum contributions and find the Wilson loop's finite part at strong coupling, and also get contributions to Γ_{cusp} in the non-perturbative limit from the amplitudes side. Thus, this technology might be a promising new approach to the program of bridging amplitudes, wilson-loops, and integrability all together. With this goal in mind, our work in chapters 6 and 7 will review the progress we made in using this technology in the context of $\log M_4 \sim W_4$ and Γ_{cusp} . In Chapter 6, we go over a method we found for computing contributions to the integrand of $\log M_4$ for more complicated diagram topologies than those considered in [77]. This chapter is based on the following paper

- Taro V. Brown, **Umut Oktem**, Shruti Paranjape, Jaroslav Trnka, *Loops of loops expansion in the amplituhedron*, Journal of High Energy Physics 07, 2024.

In Chapter 7, we use these tools to find new contributions to the Wilson-loop ratios and Γ_{cusp} at strong coupling. This is based on the upcoming paper

- Lance Dixon, **Umut Oktem**, Shruti Paranjape, Jaroslav Trnka, Yongqun Xu, Shun-Qing Zhang *Multi-loop negative geometries*, 2605.28926 [hep-th].

CHAPTER 6

Loops of Loops Expansion of the Amplituhedron

6.1 Introduction

Let's review what we learned in the previous chapter. Planar $\mathcal{N} = 4$ super Yang-Mills (SYM) theory is a great playground for testing new theoretical ideas, including new methods for and approaches to the calculation of scattering amplitudes [139]. This involves many different studies of the amplitudes - Wilson loop duality and dual conformal symmetry [126, 128, 140, 141], symbols [142, 143, 144], hexagon bootstrap and antipodal duality [113, 116, 145, 146, 147, 148], flux tube S-matrix approach [149, 150, 151], all-loop recursion relations and momentum twistor integrands [111, 112], the geometric reformulation of the problem using the positive Grassmannian and on-shell diagrams [8, 9, 10, 11, 12, 13, 34, 60, 65], and the Amplituhedron [14, 15, 16], see also [20, 56, 68, 69, 70, 71, 74, 75, 76, 77, 78, 80, 81, 86, 115, 152, 153, 154, 155, 156, 157, 158, 159, 160, 161] and [162, 163, 164, 165, 166, 167, 168] for exciting recent work on geometries in other theories.

In the Amplituhedron picture, the all-loop integrand is obtained as a canonical differential form on a positive geometry. This translates the physics problem of particle scattering into the mathematical problem of triangulation of a geometric space (or some other method of calculating the canonical form). The Amplituhedron has been studied from a purely mathematical perspective as a natural generalization of the positive Grassmannian with many exciting properties [72, 169, 170, 171, 172, 173, 174, 175, 176, 177, 178, 179, 180, 181, 182, 183, 184, 185]. It has been used to obtain a number of interesting results, but the main computational ambition is still left unresolved: calculating all-loop integrands. While this is something very difficult (or impossible) to do using more traditional methods, the Amplituhedron provides a clear mathematical definition of the problem, which still remains unresolved. There is also an interesting redefinition of the Amplituhedron in momentum space, ie. Momentum Amplituhedron, which provides a great window into generalizations of this geometric picture to theories beyond planar limit [17, 73, 186, 187, 188, 189, 190, 191].

Recently, the Amplituhedron pictures has been also obtained for the ABJM theory [19, 192, 193, 194, 195, 196].

The simplest case is a four-point scattering amplitude which has a particular simple integrated form – the leading IR divergence is given by the cusp anomalous dimension [137] and the full kinematical part is given by the Bern-Dixon-Smirnov (BDS) ansatz [47]. Yet the integrand at higher loops is incredibly complex and involves a lot of physical information hidden in its cuts. Explicit calculations have allowed the determination of the answer up to 7-loops using unitarity methods [138, 197, 198] and up to 10-loops using a certain hidden symmetry of the correlator in the dual Wilson loop picture [53, 199, 200, 201]. The Amplituhedron definition of the four-point problem is strikingly simple and can be written on two lines: at L -loops the integrand is a collection of L lines in momentum twistor space which satisfy certain quadratic conditions. While the problem can be easily stated in geometric language, it is still extremely difficult and a general approach has not been found.

In [77], Arkani-Hamed, Henn and one of the authors proposed a curious expansion using *negative geometries* of the L -loop integrand of the logarithm of the amplitude (which is essentially the same as the L -loop amplitude up to products of lower-loop terms). These geometries are analogous to the positive Amplituhedron geometry up to a sign change which has major implications in the structure of the answer. In particular, when integrated over real Minkowski space they only yield a very mild $1/\epsilon^2$ divergence (in dimensional regularization). This can be seen in the singularity structure of the canonical form (i.e. cut structure of the integrand) and the behavior in the collinear regions. In this framework, the L -loop integrand is then a sum of canonical forms for negative geometries. There is an emergent organizational principle among these geometries related to their graph-theoretical representation and the counting of internal loops (or cycles). In [77], the authors considered only geometries with no internal cycles, and solved the problem exactly for this case. Moreover, they showed that when integrating over all-but-one loops, the result is IR finite and equal to a Wilson loop with a Lagrangian insertion [202, 203, 204, 205, 206, 207, 208]. Using differential equation methods they managed to resum these contributions and get an exact result (in this geometric approximation).

In this work, we continue this effort and consider geometries with one internal cycle and solve the problem completely for all canonical forms to any loop order L . We show that there is a powerful

method for determining the form, without doing any triangulation, which instead relies on imposing constraints from vanishing cuts and cancellation of double poles. We outline how this method can be used for the calculation of geometries with more internal cycles and provide an explicit example. The main open problem is the analogue of the differential operator in [77] which could be used for these ‘one-loop’ geometries, and to see whether similar resummation procedure can be used. The chapter is organized as follows: in section 6.2, we provide some background on how the IR finite objects we are interested in relate to Amplituhedron, in section 6.3 we review the basic ideas behind negative geometries and the solution for all tree graphs of [77]. In section 6.4, we enumerate consistency conditions on canonical forms and in section 6.5 we use these conditions to solve all one-cycle geometries. In section 6.6, we provide a useful decomposition for a geometry with more cycles and in section 6.7 we comment on the integrated IR finite expressions. We conclude with some open problems in section 6.8.

6.2 Background

Remember that the all-loop integrand in planar $\mathcal{N} = 4$ SYM theory can be obtained from a canonical differential form on the *Amplituhedron* space. As discussed before, one way of computing this canonical differential form is through finding a triangulation of the Amplituhedron and add the differential form associated with each piece. However, the triangulation strategy becomes difficult already at 3-loops, so a new and more sophisticated approach is needed. An alternative strategy is to directly target the numerator $\mathcal{N}_L$ of the form Ω_L . The denominator is known a priori, so we can write the form as

$$(6.1) \quad \Omega_L = \frac{d\mu_L \cdot \mathcal{N}_L}{\prod_i D_i \cdot \prod_{i,j} \langle AB_i AB_j \rangle},$$

where we denoted the sequence of poles for each loop as

$$(6.2) \quad D_i \equiv \langle AB_i 12 \rangle \langle AB_i 23 \rangle \langle AB_i 34 \rangle \langle AB_i 14 \rangle.$$

Now we can impose various consistency conditions and try to fix $\mathcal{N}_L$ uniquely. Our strategy is to use *cuts*. There are three kinds of conditions that we need $\mathcal{N}_L$ to satisfy in the end:

- (1) Vanishing on unphysical cuts.
- (2) Canceling multiple poles generated by the denominator.
- (3) Reducing to known functions on certain cuts.

In the first case, the denominator generates a singularity on a cut where the amplitude should not have any – we often call it an *unphysical singularity* – hence the numerator must vanish. In the second case, the singularity itself might be physical, but the denominator contains a higher-order pole. As the form should only have logarithmic singularities, the numerator must again vanish (sometimes it must even contain a higher degree zero). Both of these conditions are constraints on the *zeroes* of the numerator. The final condition is that the form Ω_L should reduce to some known function, i.e. the numerator does not vanish on physical cuts, but rather is known and predictable.

6.2.1 From differential form to amplitude

Once we have the integrand form Ω_L we might want to integrate it to obtain the four-point L -loop amplitude,

$$(6.3) \quad M_L = \int \Omega_L .$$

The contour of integration corresponds to real momenta in the momentum space, which translates into a certain complex contour in the momentum twistor space where the lines AB_i live. The L -loop amplitude M_L suffers from IR divergences and needs to be regulated. In fact, in dimensional regularization, where we shift the dimension to $4 - 2\epsilon$, the IR divergence is

$$(6.4) \quad M_L \sim \frac{1}{\epsilon^{2L}} .$$

It is very well-known that the IR divergence in gauge theories exponentiates, and the invariant new divergence at each order in perturbation theory is at $1/\epsilon^2$ order, the rest is iterative from lower loops. In the planar $\mathcal{N} = 4$ SYM there is an even stronger statement: the BDS ansatz tells us that the full kinematical dependence of the amplitude (at 4-point and 5-point) is given by an iteration of the one-loop amplitude, and the only new information at every loop order reduces to certain constants [47],

$$(6.5) \quad M = 1 + \sum_{L=1}^{\infty} g^L M_L(\epsilon) = \exp \left[\sum_{l=1}^{\infty} g^l \left(f^{(l)}(\epsilon) M^{(1)}(l\epsilon) + C^{(l)}(\epsilon) + E^{(l)}(\epsilon) \right) \right] ,$$

where g is the t'Hooft coupling. The unknown functions f, C and E only depend on ϵ and some transcendental constants. We can also expand $f^{(l)} = f_0^{(l)} + \epsilon f_1^{(l)} + \epsilon^2 f_2^{(l)}$, where $f_0^{(l)}$ then corresponds to the cusp anomalous dimension γ_{cusp} . Indeed it is more convenient to work with the logarithm of

the amplitude (which is very natural because of the exponentiation in (6.5)),

$$(6.6) \quad \ln M = \sum_{l=1}^{\infty} g^l f^{(l)}(\epsilon) M^{(1)}(l\epsilon) + C^{(l)}(\epsilon) + E^{(l)}(\epsilon) = \frac{\Gamma_{\text{cusp}}}{\epsilon^2} + \mathcal{O}\left(\frac{1}{\epsilon}\right).$$

The leading term has a very mild $1/\epsilon^2$ divergence to any loop order and the kinematic dependence is only in the subleading terms. While the cusp anomalous dimension Γ_{cusp} for the planar $\mathcal{N} = 4$ SYM is predicted by integrability [137] to all loop orders, as we discussed, a derivation from all-loop amplitudes is still missing.

It is striking that some hints of the exponentiation of IR divergencies can be seen from the structure of the amplitude integrand, i.e. the L -loop canonical differential form Ω_L on the Amplituhedron. To see that, we explicitly write the perturbative expansion of the logarithm of the amplitude,

$$(6.7) \quad \ln M = \ln(1 + g^2 M_1 + g^4 M_2 + g^6 M_3 + \dots) = g^2 \ln M_1 + g^4 \ln M_2 + g^6 \ln M_3 + \dots,$$

where $\ln M_L$ denotes the L -loop logarithm corresponding to sums of products of amplitudes. In particular, the first few loop orders are

$$(6.8) \quad \ln M_1 = M_1, \quad \ln M_2 = M_2 - \frac{1}{2} M_1^2, \quad \ln M_3 = M_3 - M_2 M_1 + \frac{1}{3} M_1^3, \quad \dots$$

Each term in $\ln M_L$ is maximally $1/\epsilon^{2L}$ IR divergent while in the sum all terms up to $1/\epsilon^2$ cancel. Note that $\ln M_L$ is a combination of products of planar amplitudes, but it itself is not planar, i.e. cannot be expanded as a sum of planar Feynman diagrams. This can be seen already at two-loops where the square of the one-loop amplitude does not correspond to a planar two-loop diagram. However, we can still use global dual variables and momentum twistors and define

$$(6.9) \quad \ln M_L = \int \tilde{\Omega}_L$$

where $\tilde{\Omega}_L$ is the integrand for the L -loop logarithm. It can be obtained from summing over integrand forms for amplitudes (and their products), for example,

$$(6.10) \quad \tilde{\Omega}_2 = \Omega_2(AB, CD) - \Omega_1(AB)\Omega_1(CD),$$

$$(6.11) \quad \begin{aligned} \tilde{\Omega}_3 = & \Omega_3(AB, CD, EF) - \Omega_2(AB, CD)\Omega_1(EF) - \Omega_2(AB, EF)\Omega_1(CD) \\ & - \Omega_2(CD, EF)\Omega_1(AB) + 2\Omega_1(AB)\Omega_1(CD)\Omega_1(EF), \end{aligned}$$

where the difference in numerical factors here and in (6.8) is given by symmetrization in the integrand which is completely symmetric in all loop lines AB_i . Here we labeled AB , CD and EF as three loops, but in general we use the AB_i labeling. The L -loop logarithm integrand form $\tilde{\Omega}_L$ has very special properties in the collinear regions which control IR divergencies [15]. The idea is that the integrand for the logarithm $\tilde{\Omega}_L$ is also a canonical form on a certain positive geometry [209]. The answer was given in [77] and indeed, $\tilde{\Omega}_L$ is a sum of canonical forms on connected *negative geometries*, we will discuss it in section 6.3.

6.2.2 IR finite object

The logarithm of the amplitude $\ln M_L$ is almost an IR finite object, and the IR divergencies only appear if we integrate over all loop momenta – this is also evident from the cuts of the integrand $\tilde{\Omega}_L$ which do not allow the loops AB_i to access the collinear region unless all loops are involved in the cut (none of the loops can be left uncut). Hence, we decide to do the maximal “safe” integration of $\tilde{\Omega}_L$ and integrate over $AB_2, \dots, AB_L$ while freezing (not integrating over) the loop momentum $AB \equiv AB_1$ in the momentum twistor language [77]. The dual conformal symmetry dictates that the result can only depend on a single cross-ratio for $AB \equiv AB_1$,

$$(6.12) \quad z = \frac{\langle AB12 \rangle \langle AB23 \rangle}{\langle AB23 \rangle \langle AB14 \rangle}.$$

We can then define an IR function $\mathcal{F}_L(z)$,

$$(6.13) \quad \mathcal{F}_L(z) = \int \tilde{\Omega}_L \quad \text{where } AB \text{ is not integrated over.}$$

Further, we can perform a resummation and introduce a formal non-perturbative function $\mathcal{F}(g, z)$ by dressing $\mathcal{F}_L(z)$ with g^{2L} and summing over all L ,

$$(6.14) \quad \mathcal{F}(g, z) = \sum_{L=1}^{\infty} g^{2L} \mathcal{F}_L(z).$$

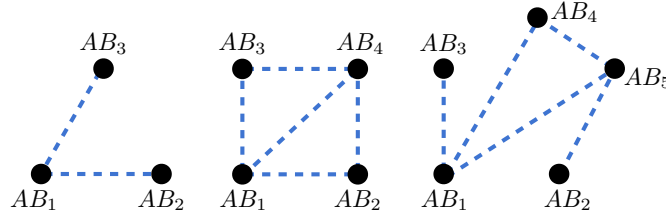
This object has been studied before [202, 203, 204, 205, 206, 207, 208] and it is equal to the ratio of a null-polygonal Wilson loop with a Lagrangian insertion divided by (an empty) null-polygonal Wilson loop,

$$\mathcal{F}(g, z) = \frac{\text{Diagram 1}}{\text{Diagram 2}}$$

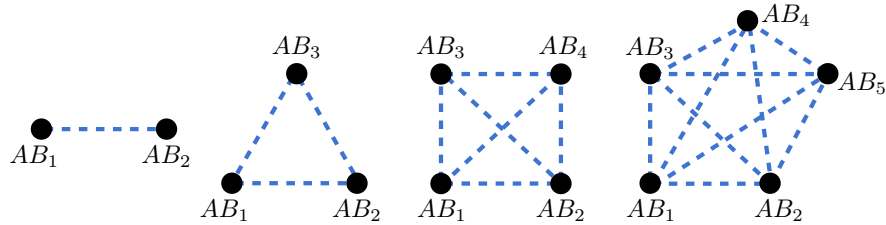
The object $\mathcal{F}_L(z)$ has a very natural expansion in terms of *marked* positive geometries as shown in [77] and we discuss it more in chapter 7.

6.3 General positive and negative geometries

We start our discussion with the generalization of the Amplituhedron geometry to a larger class of positive geometries. They correspond to configurations of L lines AB_i , where each line satisfies the one-loop Amplituhedron conditions (5.8) and in addition, we have a collection of mutual positivity conditions $\langle AB_i AB_j \rangle > 0$ but not necessarily for all pairs of AB_i and AB_j . We can graphically represent such geometries as graphs with L nodes – one for each loop line AB_i , where some of the nodes are connected by a blue dashed link, representing a mutual positivity condition $\langle AB_i AB_j \rangle > 0$. Here are some examples of these positive geometries:



Sometimes it is convenient to explicitly label AB_i associated with a given node. The most complicated, complete graph, corresponds to the L -loop Amplituhedron where the mutual positivity conditions are imposed between any pair of lines. Here are examples of Amplituhedron geometries up to $L = 5$:

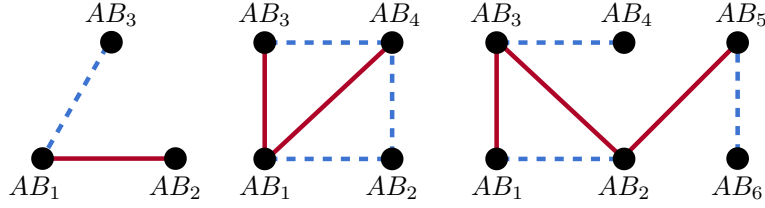


We could use the graphic notation for specifying a positive geometry, but for us it is more convenient to associate a logarithmic form Ω_Γ on the positive geometry with the graph (rather than just the

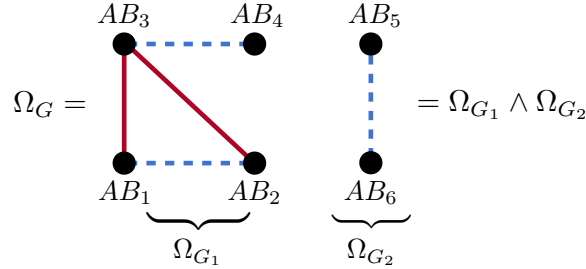
geometry). We now generalize our discussion slightly and also consider graphs where some of the links correspond to mutual positive conditions $\langle AB_i AB_j \rangle > 0$ and others to mutual negative conditions $\langle AB_i AB_j \rangle < 0$.



For nodes that are not connected by any link, the sign of the corresponding term $\langle AB_i AB_j \rangle$ is unspecified. Here are a few examples of these more general geometries:



Note that if the graph is not connected, the differential form is given by a (wedge) product of differential forms for the subgraphs which are connected. For example,



For simplicity, we skip the wedge symbol $\wedge$ in the following discussion. Note that the graphical notation we introduced is for the differential form Ω_G of the corresponding geometry. Nonetheless the graph also gives us the geometry itself. To summarize:

- Each dot is one loop line AB_i which satisfies one-loop Amplituhedron conditions (5.8).
- The dashed blue link between dots AB_i and AB_j imposes a mutual positive condition $\langle AB_i AB_j \rangle > 0$, while the red solid line represents a condition $\langle AB_i AB_j \rangle < 0$.

In some cases, we do not label the AB_i vertices at all – in this case we implicitly *symmetrize* over all AB_i .

6.3.1 From positive to negative

The positive and negative links are not unrelated, they do satisfy a simple sum rule,

$$(6.15) \quad \begin{array}{ccc} \begin{array}{c} \bullet \text{---} \bullet \\ AB_1 \quad AB_2 \\ \langle AB_1 AB_2 \rangle > 0 \end{array} & + & \begin{array}{c} \bullet \text{---} \bullet \\ AB_1 \quad AB_2 \\ \langle AB_1 AB_2 \rangle < 0 \end{array} = \begin{array}{c} \bullet \quad \bullet \\ AB_1 \quad AB_2 \\ \text{no sign restriction} \end{array} \end{array}$$

This relation makes sense as a union of two geometries, and as the sum of their canonical forms Ω_{G_1} and Ω_{G_2} . The resulting geometry is simpler, and we expect the canonical form to have simpler structure as well. This relation makes sense for $L = 2$ but can be also used inside an arbitrary complicated graph. For example,

$$(6.16) \quad \begin{array}{ccc} \begin{array}{c} EF \quad GH \\ \bullet \text{---} \bullet \\ \bullet \text{---} \bullet \\ AB \quad CD \end{array} & + & \begin{array}{c} EF \quad GH \\ \bullet \text{---} \bullet \\ \bullet \text{---} \bullet \\ AB \quad CD \end{array} = \begin{array}{c} EF \quad GH \\ \bullet \text{---} \bullet \\ \bullet \text{---} \bullet \\ AB \quad CD \end{array} \end{array}$$

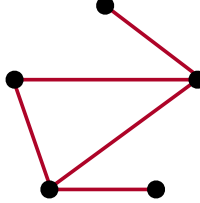
We can use the relation between the links (6.15), and express all positive links in the graph for the L -loop Amplituhedron in terms of negative links and no links. As a result, we express the L -loop Amplituhedron form Ω_L as a sum of canonical forms for *negative geometries* which have only negative links,

$$(6.17) \quad \Omega_L = \begin{array}{c} \bullet \text{---} \bullet \text{---} \bullet \\ \bullet \text{---} \bullet \text{---} \bullet \\ \bullet \text{---} \bullet \text{---} \bullet \end{array} = \sum_{\text{all } G} (-1)^{E(G)} \begin{array}{c} \bullet \text{---} \bullet \text{---} \bullet \\ \bullet \text{---} \bullet \end{array}$$

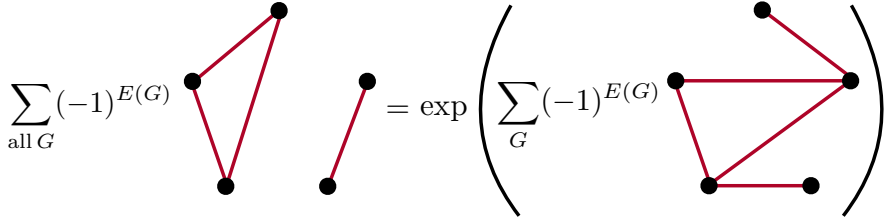
The sum is over all graphs G , and $E(G)$ is the number of edges in the graph G . This particular relation will be important later on in our discussion. Note that the relation (6.17) is not between geometries (the geometries on both sides of the equation are very different, with different positive or negative links) but between their canonical differential forms. The right hand side of (6.17) contains all graphs, connected and disconnected, with any number of links, and also with all possible assignments of loop labels AB_i .

6.3.2 Logarithm of the amplitude from negative geometries

Let us now consider a restriction of the sum on the right-hand-side of (6.17) and sum only over the differential forms of *connected* graphs with negative links, and define the form $\tilde{\Omega}_L$,

$$(6.18) \quad \tilde{\Omega}_L = \sum_G (-1)^{E(G)} \text{graph}$$


where $E(G)$ is the number of edges in the graph. It was shown in [77] that this object is in fact the integrand for the *logarithm of the amplitude* which we also discussed in section 6.2.1. This is a simple consequence of the relation between the collection of all graphs and all connected graphs.

$$(6.19) \quad \sum_{\text{all } G} (-1)^{E(G)} \text{graph} = \exp \left(\sum_G (-1)^{E(G)} \text{graph} \right)$$


More heuristically, any given graph in (6.17) can be obtained as the product (at the level of differential forms) of connected graphs, hence knowing the forms for all connected graphs clearly has all information about the forms for all graphs.

6.3.3 Loops of loops expansion

There is a natural hierarchy in the expansion for $\tilde{\Omega}_L$ which we presented at the end of the last section: the number of *internal loops* or *cycles*. We will mostly use the word “cycle” wherever we can, in order not to overuse the word *loop*, which can be confused with loop lines AB_i (that are represented by nodes in the graphs),

$$(6.20) \quad \tilde{\Omega}_L = \tilde{\Omega}_L^{\text{tree}} + \tilde{\Omega}_L^{1\text{-cycle}} + \tilde{\Omega}_L^{2\text{-cycle}} + \tilde{\Omega}_L^{3\text{-cycle}} + \dots = \sum_{\ell=0}^{\max} \tilde{\Omega}_L^{\ell\text{-cycle}}$$

where $\tilde{\Omega}_L^{\ell\text{-cycle}}$ is the sum of differential forms for all graphs with ℓ internal cycles. Tree graphs refer to graphs with no internal cycles. In general for the L -loop case with L nodes, the number of links in the graph is $L + \ell - 1$. As the maximal number of links is $L(L-1)/2$, the maximal number of

internal cycles is $(L-1)(L-2)/2$. We call the expansion (6.20) the *loops of loops expansion* and it is crucial for our discussion. The idea is that for fixed L , the expansion in terms of cycles (internal loops) is meaningful. The higher the number of cycles, the more complicated the geometry is and the more complicated the corresponding differential form Ω_Γ is. This is very natural as the number of cycles grows with the number of links in the graph, and clearly the more the links, the more complicated a differential form we expect.

Let us provide some explicit examples. For $L = 2$ we only get one tree graph (two nodes and one link), while for $L = 3$ we get both a tree graph and a one-cycle graph,

$$(6.21) \quad \tilde{\Omega}_3 = \underbrace{\text{[Tree graph with 3 nodes and 2 edges]}}_{\tilde{\Omega}_3^{\text{tree}}} + \underbrace{\text{[Triangle graph with 3 nodes and 3 edges]}}_{\tilde{\Omega}_3^{1\text{-cycle}}}$$

For $L = 4$ we get graphs up to 3 cycles.

$$(6.22) \quad \tilde{\Omega}_4 = \underbrace{\text{[Square tree graph with 4 nodes and 3 edges]}}_{\tilde{\Omega}_4^{\text{tree}}} + \underbrace{\text{[Square with one diagonal tree graph with 4 nodes and 4 edges]}}_{\tilde{\Omega}_4^{1\text{-cycle}}} + \underbrace{\text{[Square with two diagonals 2-cycle graph with 4 nodes and 5 edges]}}_{\tilde{\Omega}_4^{2\text{-cycle}}} + \underbrace{\text{[Square with both diagonals 3-cycle graph with 4 nodes and 6 edges]}}_{\tilde{\Omega}_4^{3\text{-cycle}}}$$

where in both (6.21) and (6.22) we need to label AB_i in all possible ways and sum. The task is to find differential forms for these geometries. The problem of tree forms was solved in [77], in this work we solve for all canonical forms of geometries with one cycle and also outline how to continue to geometries with more cycles.

6.3.4 All tree canonical forms

The solution for canonical forms for all tree graphs is particularly simple. For $L = 2$ we have only one graph topology (two nodes and one link),

$$\begin{array}{c} \bullet \text{---} \bullet \\ AB \quad CD \end{array} = \frac{d\mu N_2^{(+)}}{D_1 D_2 \langle ABCD \rangle} \quad \begin{array}{c} \bullet \text{---} \bullet \\ AB \quad CD \end{array} = \frac{d\mu N_2^{(-)}}{D_1 D_2 \langle ABCD \rangle}$$

where we denoted (up to an overall factor $\langle 1234 \rangle^3$ which we drop),

$$(6.23) \quad D_1 = \langle AB12 \rangle \langle AB23 \rangle \langle AB34 \rangle \langle AB14 \rangle, \quad D_2 = \langle CD12 \rangle \langle CD23 \rangle \langle CD34 \rangle \langle CD14 \rangle,$$

$$(6.24) \quad N_2^{(+)} = \langle AB12 \rangle \langle CD34 \rangle + \langle AB23 \rangle \langle CD14 \rangle + \langle AB34 \rangle \langle CD12 \rangle + \langle AB14 \rangle \langle CD23 \rangle,$$

$$(6.25) \quad N_2^{(-)} = \langle AB13 \rangle \langle CD24 \rangle + \langle AB24 \rangle \langle CD13 \rangle.$$

The union of both spaces are just two one-loop Amplituhedra, for each AB and CD . At the level of the numerators this gives

$$(6.26) \quad N_2^{(+)} + N_2^{(-)} = \langle ABCD \rangle \langle 1234 \rangle.$$

These two numerators will be used as the building blocks for the general L -loop case. We define two objects $N_{ij}^{(+)}$ and $N_{ij}^{(-)}$ which are two-loop numerators for positive, (resp. negative) links between AB_i and AB_j ,

$$\begin{aligned} N_{ij}^{(+)} &\equiv \langle AB_i 12 \rangle \langle AB_j 34 \rangle + \langle AB_i 23 \rangle \langle AB_j 14 \rangle + \langle AB_i 34 \rangle \langle AB_j 12 \rangle + \langle AB_i 14 \rangle \langle AB_j 23 \rangle, \\ (6.27) \quad N_{ij}^{(-)} &\equiv \langle AB_i 13 \rangle \langle AB_j 24 \rangle + \langle AB_i 24 \rangle \langle AB_j 13 \rangle. \end{aligned}$$

Then for an arbitrary tree graph with arbitrary assignments of positive and negative links the differential form can be written as

$$\Omega_G^{\text{tree}} = \begin{array}{c} \bullet \text{---} \bullet \\ \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet \\ \bullet \text{---} \bullet \end{array} = \frac{d\mu \mathcal{N}_G}{\prod_i D_i \cdot \prod_{\text{links}} \langle AB_i AB_j \rangle}$$

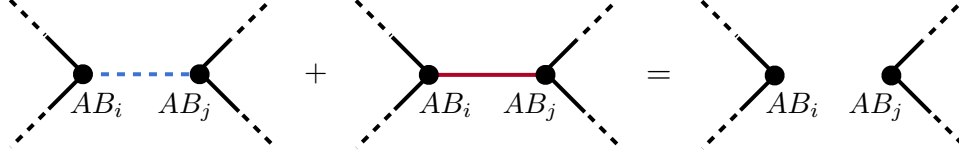
where the denominator has the expected structure of the product of one-loop numerators

$$(6.28) \quad D_i = \langle AB_i 12 \rangle \langle AB_i 23 \rangle \langle AB_i 34 \rangle \langle AB_i 14 \rangle,$$

and the mutual propagators $\langle AB_i AB_j \rangle$ for all links. The numerator carries information about the sign of the mutual conditions, and takes a very simple factorized form:

$$(6.29) \quad \boxed{\mathcal{N}_G = \langle 1234 \rangle^{L+1} \times \prod_{\text{pos links}} N_{ij}^{(+)} \times \prod_{\text{neg links}} N_{ij}^{(-)}}$$

It is worth noting that because of the factorized form of the numerator, any two tree graphs which differ only by a sign of a single link, satisfy the same summation rule as the $L = 2$ case (6.15),



This is just a trivial consequence of the relation between numerators,

$$(6.30) \quad N_{ij}^{(+)} + N_{ij}^{(-)} = \langle 1234 \rangle \langle AB_i AB_j \rangle.$$

Now for the special case of tree graphs with only negative links the differential form is

$$\Omega_G^{\text{tree}} = \text{[Tree Graph with 7 nodes and 6 red links]} = \frac{d\mu \prod_{\text{links}} N_{ij}^{(-)}}{\prod_i D_i \cdot \prod_{\text{links}} \langle AB_i AB_j \rangle}$$

The diagram shows a tree graph with 7 nodes and 6 red links. The graph consists of a central node connected to three other nodes, one of which is further connected to two more nodes, and another node connected to two more nodes. The entire graph is drawn with red lines. To the right of the graph is an equals sign followed by a fraction: the numerator is $d\mu \prod_{\text{links}} N_{ij}^{(-)}$ and the denominator is $\prod_i D_i \cdot \prod_{\text{links}} \langle AB_i AB_j \rangle$.

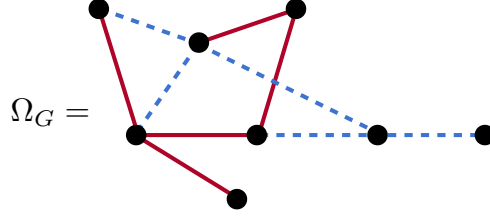
as also presented in [77]. Using this formula for an arbitrary tree graph, we can completely determine the first term in the expansion

$$(6.31) \quad \tilde{\Omega}_L^{\text{tree}} = \sum_{\text{trees } G} \tilde{\Omega}_G.$$

Furthermore, in section 6.7 we will show that we can take the next step and integrate (6.31) using a differential equation method.

6.4 Consistency conditions

Let us now consider a general connected graph,



with some labeling of vertices AB_i . There are important consistency conditions on the associated canonical form that come from cuts. Let us remind ourselves that the geometry satisfies

$$(6.32) \quad \langle AB_i 12 \rangle, \langle AB_i 23 \rangle, \langle AB_i 34 \rangle, \langle AB_i 14 \rangle > 0, \quad \langle AB_i 13 \rangle, \langle AB_i 24 \rangle < 0,$$

for any AB_i , and $\langle AB_i AB_j \rangle \leq 0$ for any link. In this section, we perform cuts on Ω_G , localize one or more loop lines and read off the inequalities of the reduced geometry.

6.4.1 Singlet conditions

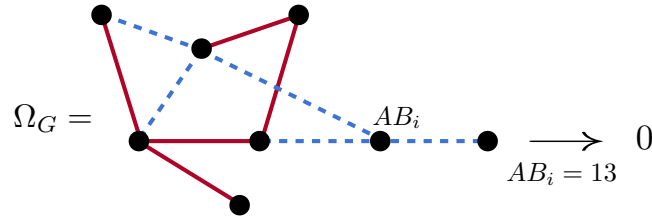
Let us first discuss cuts where only one of the loops gets localized on a cut – we call them *singlet* conditions. We start with localizing AB_i on the leading singularity

$$(6.33) \quad \langle AB_i 12 \rangle = \langle AB_i 23 \rangle = \langle AB_i 34 \rangle = \langle AB_i 14 \rangle = 0,$$

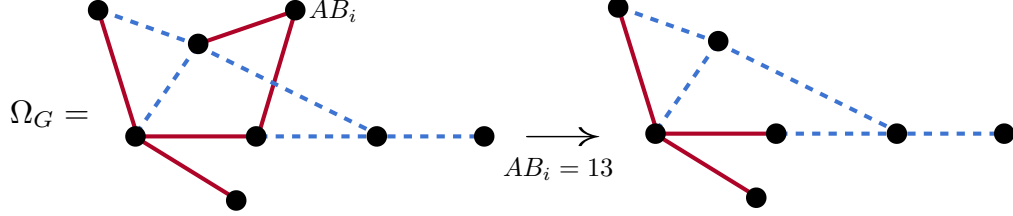
which fixes $AB_i = 13$ (or $AB_i = 24$, the discussion is analogous). If the vertex AB_i is connected to any other vertex AB_j through a positive link, we get

$$(6.34) \quad \langle AB_i AB_j \rangle = \langle AB_j 13 \rangle > 0 \quad \text{manifestly negative}.$$

This is in contradiction with the one-loop Amplituhedron conditions for AB_j which require $\langle AB_j 13 \rangle < 0$. Hence if AB_i is connected to any other vertex through at least one positive link, the canonical form Ω_G and the numerator $\mathcal{N}_G$ must vanish.



If the AB_i is connected to other vertices only through negative links, the line AB_i is eliminated from the graph without any additional condition added,



We refer to this type of condition as **P1** because it is illegal for positive links and involves only one of the loops. Another singlet condition is a composite triple cut

$$(6.35) \quad \langle AB_i 12 \rangle = \langle AB_i 23 \rangle = 0 \text{ including the Jacobian,}$$

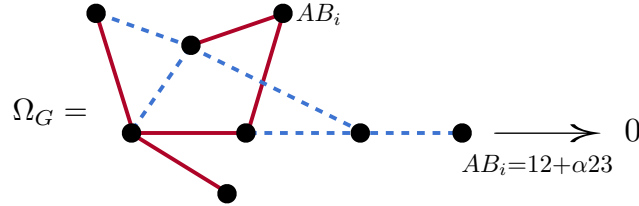
which localizes $AB_i = 12 + \alpha 23$, in other words the line AB_i passes through vertex 2 and lies in a plane (123). In the momentum space this corresponds to a collinear region when $\ell \sim p_2$. The one-loop Amplituhedron condition fixes the sign of α ,

$$(6.36) \quad \langle AB_i 14 \rangle = \alpha \langle 1234 \rangle > 0 \rightarrow \alpha > 0.$$

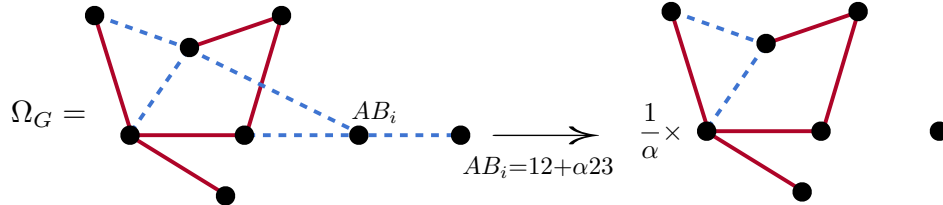
Then for a link $\langle AB_i AB_j \rangle$ we get

$$(6.37) \quad \langle AB_i AB_j \rangle = \langle AB_j 12 \rangle + \alpha \langle AB_j 23 \rangle > 0 \quad \text{manifestly positive.}$$

If AB_i is connected to at least one negative link,



If all links are positive the residue is



The $1/\alpha$ factor comes from the Jacobian $1/\langle AB_i 14 \rangle$ which is uncut. We refer to this type of condition as **N1** because it is illegal for negative links and involves only one loop. If the node AB_i is connected through both positive and negative links, then both **P1** and **N1** are illegal and the canonical form should vanish on these cuts.

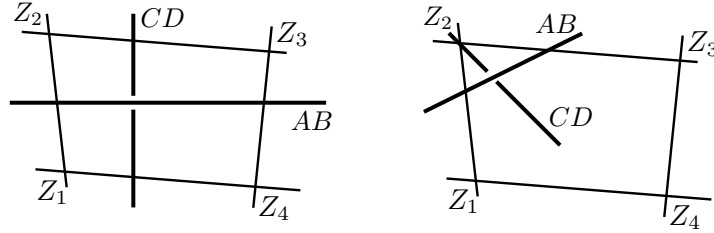
6.4.2 Doublet conditions

There are also *doublet* conditions which involve cutting two lines, say (AB) and (CD) and finding a contradiction between the fixed sign of $\langle ABCD \rangle$ and the solution of the cut. In other words, if we have a positive link $\langle ABCD \rangle > 0$ but the cut forces $\langle ABCD \rangle < 0$ this cut is illegal, and vice versa for a negative link $\langle ABCD \rangle < 0$ – any cut which forces $\langle ABCD \rangle > 0$ is illegal. Examples of the first type of cuts, which force $\langle ABCD \rangle < 0$, are:

(P2a) The cut when AB cuts lines 12 and 34 while CD cuts lines 23 and 14.

(P2b) The cut when AB lies in the plane (123) and CD passes through point 2. This can be achieved by cutting $\langle AB12 \rangle = \langle AB23 \rangle = \langle CD12 \rangle = \langle CD23 \rangle = 0$, this is the same as the cut used to detect double poles, but now choosing other cut solution.

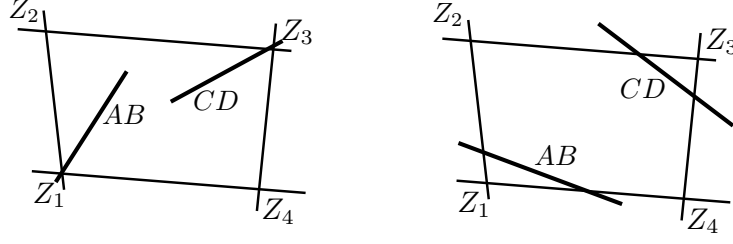
which can also be easily seen in any parametrization and where we have used the same naming convention as before. We can see from figures



that both these cuts violate “two-loop planarity” and they are not present in any planar diagrams (the loop lines ‘cross each other’). This makes sense because the $L = 2$ space with $\langle ABCD \rangle > 0$ corresponds to a planar two-loop amplitude. There are also the type of cuts which are incompatible with a negative link, ie. force $\langle ABCD \rangle > 0$,

(N2a) The cut when AB passes through 1 and CD passes through 3. To arrive at this configuration we have to cut $\langle AB14 \rangle = \langle AB12 \rangle = \langle CD23 \rangle = \langle CD34 \rangle = 0$ and choose an appropriate solution.

(N2b) Cutting the same propagators but choosing a different solution when AB is in the plane (412) and CD is in the plane (234) we again arrive at the illegal configuration.



Here we have less intuition of what this means from a physics perspective, but they are nevertheless illegal for the $\langle ABCD \rangle < 0$ space, which for $L = 2$ is $\tilde{\Omega}_2$, ie. the two-loop logarithm of the amplitude.

In principle, there are also doublet conditions where we cut certain loops AB and CD but it is the inequalities with another loop $EF - \langle ABEF \rangle, \langle CDEF \rangle$ which are violated. These cuts do exist in higher-cycle geometries, but they are harder to identify and we will not need them for our discussion here.

6.4.3 Tree numerator and double pole

It is easy to see that for an arbitrary graph, the factorized numerator (6.29) satisfies all singlet and doublet conditions,

$$(6.38) \quad \mathcal{N}_G^{\text{tree}} = \langle 1234 \rangle^{L+1} \times \prod_{\text{pos links}} N_{ij}^{(+)} \times \prod_{\text{neg links}} N_{ij}^{(-)}.$$

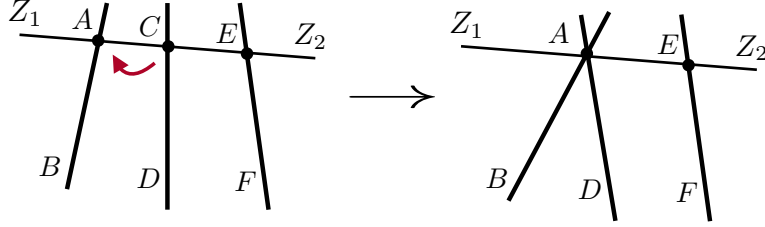
We refer to it as the *tree numerator*, as it gives the correct expression for all tree graphs. The numerator $\mathcal{N}_G^{\text{tree}}$ is not correct for a general graph (which contains closed cycles) for one simple reason – it fails to satisfy an important consistency condition: logarithmic singularities of the canonical form. To see this explicitly, let us consider a simplest positive geometry with one closed cycle,

$$\mathcal{N}_G^{\text{tree}} = N_{12}^{(+)} N_{23}^{(+)} N_{13}^{(+)}$$

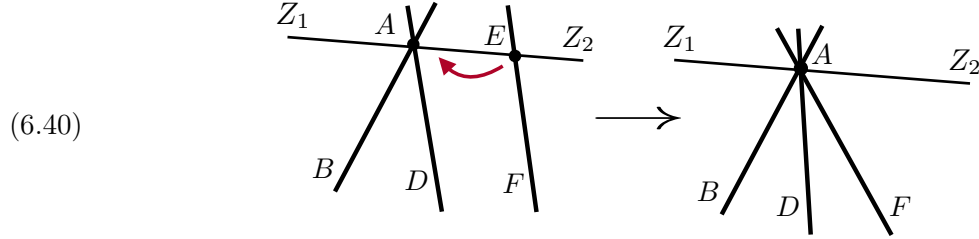
where though we chose all links to be positive, the signs are in fact irrelevant. We perform a cut when all three lines AB , CD and EF cut the line (12) and in addition AB intersects CD ,

$$(6.39) \quad \langle AB12 \rangle = \langle CD12 \rangle = \langle EF12 \rangle = \langle ABCD \rangle = 0.$$

To see the momentum twistor geometry of the solution to this cut we first let all three lines intersect line (12), and then let the lines AB and CD intersect each other on (12). This merges their intersection points and we can label them as $A = C$,



This is an allowed cut for any sign assignments, ie. the signs of $\langle ABEF \rangle$ and $\langle CDEF \rangle$ are not fixed on this cut. Now we further cut $\langle ABEF \rangle = 0$ by merging the intersection points A and E on line (12),



As we can see in this configuration, there is one accidental relation

$$(6.41) \quad \langle CDEF \rangle = 0,$$

where lines CD and EF also intersect. This means that in the canonical form we generate a double pole on the cut (6.40)

$$(6.42) \quad \Omega_G \xrightarrow{\text{cut}} \frac{d\alpha \mathcal{N}_G}{\alpha^2 \dots},$$

where α merges points A and E in some parametrization. Hence the numerator $\mathcal{N}_G$ has to vanish on the configuration (6.40) in order to cancel the double pole and preserve logarithmic singularities. The tree numerator $\mathcal{N}_G^{\text{tree}}$ fails to vanish on this cut and hence it is not the correct expression. The same applies for all other sign assignments in the triangle.

6.5 All one-cycle geometries

The main result of this chapter is the canonical form for all positive and negative geometries with one internal loop (cycle). As we saw in the previous section, this not just given by the naive formula (6.29). We will refer to these numerators as the “tree part” as this is exactly what the correct result would be if the graph was a tree. The formula (6.29) is actually a very good start that only needs to be corrected by adding an extra “remainder”,

$$(6.43) \quad \mathcal{N}_G^{1-\text{cycle}} = \mathcal{N}_G^{\text{tree}} + R_G^{1-\text{cycle}},$$

where the remainder $R_G^{1-\text{cycle}}$ is yet-to-be fixed.

6.5.1 Triangle geometry

Let us go back to the positive geometry (6.38) and fix the form from the conditions we mentioned in section 6.4. We write the dlog form as

$$(6.44) \quad \Omega_3^{(+++)} = \begin{array}{c} \text{Diagram: A triangle with vertices AB, CD, and EF. All three edges (AB, CD, EF) are represented by dashed blue lines.} \end{array} = \frac{N_{12}^{(+)} N_{23}^{(+)} N_{13}^{(+)} + R_{123}}{\langle AB12 \rangle \langle AB23 \rangle \langle AB34 \rangle \langle AB14 \rangle \langle ABCD \rangle \langle CD12 \rangle \langle CD23 \rangle \langle CD34 \rangle \langle CD14 \rangle \langle CDEF \rangle \langle EF12 \rangle \langle EF23 \rangle \langle EF34 \rangle \langle EF14 \rangle \langle ABEF \rangle}$$

We write the numerator for this sign assignment on the links as

$$(6.45) \quad \mathcal{N}_{\text{triangle}}^{(+++)} = N_{12}^{(+)} N_{23}^{(+)} N_{13}^{(+)} + R_{123},$$

where the remainder R_{123} is yet to be fixed. Naively, for other geometries where some of the positive links are flipped to negative, the remainder is different. For example,

$$(6.46) \quad \Omega_3^{(++-)} = \begin{array}{c} \text{Diagram: A triangle with vertices AB, CD, and EF. The edge AB is a solid red line, while the edges CD and EF are dashed blue lines.} \end{array} = \frac{N_{12}^{(+)} N_{23}^{(+)} N_{13}^{(-)} + \tilde{R}_{123}}{\langle AB12 \rangle \langle AB23 \rangle \langle AB34 \rangle \langle AB14 \rangle \langle ABCD \rangle \langle CD12 \rangle \langle CD23 \rangle \langle CD34 \rangle \langle CD14 \rangle \langle CDEF \rangle \langle EF12 \rangle \langle EF23 \rangle \langle EF34 \rangle \langle EF14 \rangle \langle ABEF \rangle}$$

The union of these two spaces has no $\langle ABEF \rangle$ link, hence it is a tree graph whose canonical form has a simple factorized numerator,

$$\Omega_3^{(++\pm)} = \begin{array}{c} \text{EF} \\ \bullet \\ \text{---} \\ \bullet \text{ AB} \quad \bullet \text{ CD} \end{array} = \frac{N_{12}^{(+)} N_{23}^{(+)}}{\langle AB12 \rangle \langle AB23 \rangle \langle AB34 \rangle \langle AB14 \rangle \langle ABCD \rangle \langle CD12 \rangle \langle CD23 \rangle \langle CD34 \rangle \langle CD14 \rangle \langle CDEF \rangle \langle EF12 \rangle \langle EF23 \rangle \langle EF34 \rangle \langle EF14 \rangle}$$

We can also get this form by summing (6.44) and (6.46), which puts a constraint on the one-cycle remainder terms R_{123} and $\tilde{R}_{123}$,

$$(6.47) \quad \mathcal{N}_G = N_{12}^{(+)} N_{23}^{(+)} + \frac{R_{123} + \tilde{R}_{123}}{\langle ABEF \rangle \langle 1234 \rangle} = N_{12}^{(+)} N_{23}^{(+)},$$

which forces $\tilde{R}_{123} = -R_{123}$. The same also applies to the other two assignments of signs to internal links (modulo relabeling of loops),

$$(6.48) \quad \begin{aligned} \Omega_3^{(+-)} &= \begin{array}{c} \text{EF} \\ \bullet \\ \text{---} \\ \bullet \text{ AB} \quad \bullet \text{ CD} \end{array} = \frac{N_{12}^{(+)} N_{23}^{(-)} N_{13}^{(-)} + R_{123}}{\langle AB12 \rangle \langle AB23 \rangle \langle AB34 \rangle \langle AB14 \rangle \langle ABCD \rangle \langle CD12 \rangle \langle CD23 \rangle \langle CD34 \rangle \langle CD14 \rangle \langle CDEF \rangle \langle EF12 \rangle \langle EF23 \rangle \langle EF34 \rangle \langle EF14 \rangle \langle ABEF \rangle} \\ \Omega_3^{(---)} &= \begin{array}{c} \text{EF} \\ \bullet \\ \text{---} \\ \bullet \text{ AB} \quad \bullet \text{ CD} \end{array} = \frac{N_{12}^{(-)} N_{23}^{(-)} N_{13}^{(-)} - R_{123}}{\langle AB12 \rangle \langle AB23 \rangle \langle AB34 \rangle \langle AB14 \rangle \langle ABCD \rangle \langle CD12 \rangle \langle CD23 \rangle \langle CD34 \rangle \langle CD14 \rangle \langle CDEF \rangle \langle EF12 \rangle \langle EF23 \rangle \langle EF34 \rangle \langle EF14 \rangle \langle ABEF \rangle} \end{aligned}$$

This remainder R_{123} is thus universal for all triangle geometries (up to a relative sign). We know that all these geometries suffer from the presence of double poles, and the purpose of R_{123} is to cancel them. The explicit form for R_{123} was already presented in [77],

$$(6.49) \quad R_{123} = 4M_1 - M_2^a - M_2^b - M_2^c,$$

where we defined

$$\begin{aligned} M_1 &= \langle AB12 \rangle \langle AB34 \rangle \langle CD12 \rangle \langle CD34 \rangle \langle EF12 \rangle \langle EF34 \rangle \\ &\quad + \langle AB23 \rangle \langle AB14 \rangle \langle CD23 \rangle \langle CD14 \rangle \langle EF23 \rangle \langle EF14 \rangle, \\ M_2^a &= \langle AB12 \rangle \langle AB34 \rangle \times (\langle CD12 \rangle \langle EF34 \rangle + \langle CD34 \rangle \langle EF12 \rangle) \\ &\quad \times (\langle CD13 \rangle \langle EF24 \rangle + \langle CD24 \rangle \langle EF13 \rangle) \\ &\quad + \langle AB23 \rangle \langle AB14 \rangle \times (\langle CD23 \rangle \langle EF14 \rangle + \langle CD14 \rangle \langle EF23 \rangle) \end{aligned}$$

$$\times (\langle CD13 \rangle \langle EF24 \rangle + \langle CD24 \rangle \langle EF13 \rangle),$$

where M_2^b and M_2^c differ from M_2^a by permutation of lines $(AB), (CD)$ and (EF) (M_2^a is already symmetric in $(CD) \leftrightarrow (EF)$). This can be obtained by a brute-force method: write down a complete ansatz for R_{123} and impose the double pole cancellation constraint on the numerator $\mathcal{N}_G$. Interestingly, this fixes the form R_{123} uniquely.

6.5.2 Detailed construction of the remainder

Let us now construct the remainder R_{123} systematically and provide more details. We know that the tree numerators built from $N_2^{(+)}$ and $N_2^{(-)}$ automatically satisfy all singlet and doublet conditions for any graph. Therefore, the remainder R_{123} must also satisfy all these conditions. But we also know that R_{123} is universal for all sign assignments of links, hence it must actually vanish for *all* singlet and doublet conditions: **P1**, **N1**, **P2a**, **P2b**, **N2a** and **N2b** for *any pair* of lines $(AB), (CD)$ and (EF) . To build this remainder R_{123} we use a particular set of building blocks

$$(6.50) \quad N_{ij}^{(a)} \equiv \langle AB_i 12 \rangle \langle AB_j 34 \rangle + \langle AB_i 34 \rangle \langle AB_j 12 \rangle,$$

$$(6.51) \quad N_{ij}^{(b)} \equiv \langle AB_i 23 \rangle \langle AB_j 14 \rangle + \langle AB_i 14 \rangle \langle AB_j 23 \rangle,$$

$$(6.52) \quad N_{ij}^{(c)} \equiv \langle AB_i 13 \rangle \langle AB_j 24 \rangle + \langle AB_i 24 \rangle \langle AB_j 13 \rangle.$$

It will also be useful to use a short-hand notation for the “diagonal elements”

$$(6.53) \quad N_i^{(a)} \equiv \frac{1}{2} N_{ii}^{(a)} = \langle AB_i 12 \rangle \langle AB_i 34 \rangle,$$

$$(6.54) \quad N_i^{(b)} \equiv \frac{1}{2} N_{ii}^{(b)} = \langle AB_i 23 \rangle \langle AB_i 14 \rangle.$$

Note that the two-loop numerators are just

$$(6.55) \quad N_{ij}^{(+)} = N_{ij}^{(a)} + N_{ij}^{(b)}, \quad N_{ij}^{(-)} = N_{ij}^{(c)}.$$

We can then write a general ansatz in terms of these objects $N_{ij}^{(a,b,c)}$ as

$$(6.56) \quad R_{123} = \sum c_k N_{xy}^{(a,b,c)} N_{zw}^{(a,b,c)} N_{pq}^{(a,b,c)},$$

where 1, 2 and 3 each appear twice in $\{x, y, z, w, p, q\}$ with some choices of a, b, c superscripts, and c_k are so far undetermined numerical factors. We *demand* that all terms in our basis satisfy all

singlet and doublet conditions *individually* – this is our assumption. Analyzing individual building blocks we learn that

- (1) $N_{ij}^{(a)}$ and $N_{ij}^{(b)}$ are designed to satisfy conditions **P1** for both i and j , and **P2a** and **P2b** for a link (i, j) .
- (2) $N_{ij}^{(c)}$ is designed to satisfy conditions **N1** for both i and j , and **N2a** and **N2b** for a link (i, j) .
- (3) Products $N_i^{(a)}$ or $N_i^{(b)}$ satisfies **P1** and **N1** for i , and also **P2b**, **N2a** and **N2b** for any link $(i, *)$.
- (4) Products $N_i^{(a)} N_{j*}^{(a)}$ and $N_i^{(b)} N_{j*}^{(b)}$ satisfy further condition **P2a** for a link (i, j) .
- (5) Products $N_i^{(a)} N_j^{(a)}$ and $N_i^{(b)} N_j^{(b)}$ satisfy **all** conditions for both i, j and the link (i, j) .

All conditions must be satisfied for all three pairs 12, 13 and 23 (because our graph has all three links), and the complete basis of such monomials is (modulo relabeling of 1,2,3)

$$(6.57) \quad \left\{ N_1^{(a)} N_2^{(a)} N_3^{(a)}, N_1^{(b)} N_2^{(b)} N_3^{(b)}, N_1^{(a)} N_{23}^{(a)} N_{23}^{(c)}, N_1^{(b)} N_{23}^{(b)} N_{23}^{(c)} \right\}.$$

Nothing else satisfies all constraints. Our ansatz for the remainder R_{123} is then

$$(6.58) \quad \begin{aligned} R_{123} = & c_1 \cdot N_1^{(a)} N_2^{(a)} N_3^{(a)} + c_2 \cdot N_1^{(b)} N_2^{(b)} N_3^{(b)} \\ & + c_3 \cdot \left\{ N_1^{(a)} N_{23}^{(a)} N_{23}^{(c)} + N_2^{(a)} N_{13}^{(a)} N_{13}^{(c)} + N_3^{(a)} N_{12}^{(a)} N_{12}^{(c)} \right\} \\ & + c_4 \cdot \left\{ N_1^{(b)} N_{23}^{(b)} N_{23}^{(c)} + N_2^{(b)} N_{13}^{(b)} N_{13}^{(c)} + N_3^{(b)} N_{12}^{(b)} N_{12}^{(c)} \right\}, \end{aligned}$$

and the full numerator is

$$(6.59) \quad \mathcal{N}_{\text{triangle}}^{(+++)} = N_{12}^{(+)} N_{23}^{(+)} N_{13}^{(+)} + R_{123}.$$

Now we impose the double pole cancellation condition on $\mathcal{N}_{\text{triangle}}^{(+++)}$. This gives a unique solution which fixes the coefficients in the ansatz (6.58) to be

$$(6.60) \quad c_1 = c_2 = 4, \quad c_3 = c_4 = -1.$$

Note that the cyclic symmetry (in Z_1, Z_2, Z_3, Z_4) forces us to have $c_1 = c_2$ and $c_3 = c_4$, but this is automatic in the solution. We can then write the result in a compact form,

$$(6.61) \quad R_{123} = \left\{ 4N_1^{(a)}N_2^{(a)}N_3^{(a)} - \left(N_1^{(a)}N_{23}^{(a)}N_{23}^{(c)} + N_2^{(a)}N_{13}^{(a)}N_{13}^{(c)} + N_3^{(a)}N_{12}^{(a)}N_{12}^{(c)} \right) \right\} + (a \rightarrow b),$$

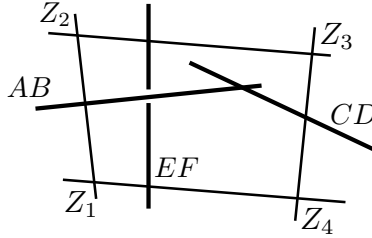
where $a \rightarrow b$ sends the “ a ” superscript to “ b ”, ie. $N_i^{(a)} \rightarrow N_i^{(b)}$ and $N_{ij}^{(a)} \rightarrow N_{ij}^{(b)}$, leaving the terms with superscript “ c ” untouched. This is exactly the formula (6.49) now written in this new notation. We can easily check that the same R_{123} expression trivially takes care of the double poles in the $(++-)$, $(+-)$ and $(---)$ spaces.

As a consistency check we can verify that $\Omega_{\text{triangle}}^{(+++)}$ reproduces exactly the integrand for the 3-loop amplitude Ω_3 , but now written in a very different way than the usual sum of double boxes and tennis court integrands. Similarly, $\Omega_{\text{triangle}}^{(---)}$ together with tree graphs reproduces $\tilde{\Omega}_3$, ie. the three-loop logarithm of the amplitude.

Note that there are actually more constraints involving the cut of all three loops than we used – we only used the double pole cancellation condition. We do not need the other constraints, as the numerator is already fully fixed. For example, for the $(+++)$ space we can consider a cut

$$(6.62) \quad \langle AB12 \rangle = \langle ABCD \rangle = \langle CD34 \rangle = \langle EF23 \rangle = \langle EF14 \rangle = 0$$

which is another example of a “non-planar cut” that is not supported by any planar diagram.

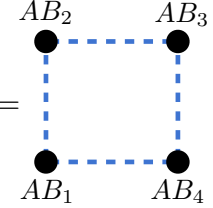


Indeed on this cut $\langle ABEF \rangle < 0$ and $\langle CDEF \rangle < 0$ and the numerator $\mathcal{N}_{\text{triang}}^{(+++)}$ must vanish. This requires non-trivial cancellations in the numerator which do indeed happen. These conditions are harder to classify, so we are lucky that singlet and doublet conditions along with double pole cancellation are enough to fix the remainder R_{123} completely.

The next set of geometries we consider are polygons for any L . We already solved the problem for $L = 3$ and before writing a general solution, we look at $L = 4$ and $L = 5$ examples.

6.5.3 Square and pentagon geometries

Let us now consider the next case of a simple geometry with one internal cycle, it is the $L = 4$ (four loops) ‘square’ geometry with only four links which form a closed cycle. For all links positive, we have

$$\Omega_{\text{square}}^{(++++)} = \frac{\mathcal{N}_{\text{square}}^{(++++)}}{D_1 D_2 D_3 D_4 \langle AB_1 AB_2 \rangle \langle AB_2 AB_3 \rangle \langle AB_3 AB_4 \rangle \langle AB_4 AB_1 \rangle}$$


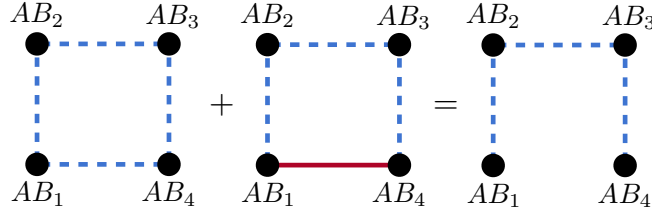
We start with writing the numerator as the sum of the tree part and the remainder, and for all links positive we have

$$(6.63) \quad \mathcal{N}_{\text{square}}^{(++++)} = N_{12}^{(+)} N_{23}^{(+)} N_{34}^{(+)} N_{14}^{(+)} + R_{1234}.$$

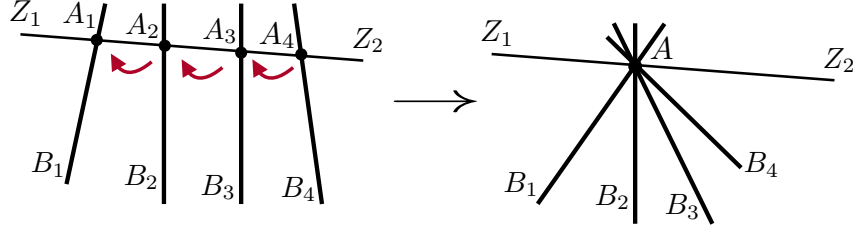
The remainder is again universal for any sign assignments on links. For example, if we flip the sign of the $\langle AB_1 AB_4 \rangle$ link we have the same denominator, but

$$(6.64) \quad \mathcal{N}_{\text{square}}^{(+++ -)} = N_{12}^{(+)} N_{23}^{(+)} N_{34}^{(+)} N_{14}^{(-)} - R_{1234},$$

such that the union of two geometries is a tree graph,



and indeed the sum of (6.63) and (6.64) reproduce the correct tree form. The remainder must satisfy all singlet and doublet conditions for the links in the graph (not all links – since links $\langle AB_1 AB_3 \rangle$ and $\langle AB_2 AB_4 \rangle$ are missing, their signs are unrestricted) and the double pole cancellation. The double pole is again generated in the denominator of the form Ω_G if all lines AB_i cut line (12) (or any other external line) and intersect each other on that line – we denote the intersection point A ,



Merging the intersection points on line (12) of lines $AB_1 \rightarrow AB_2 \rightarrow AB_3 \rightarrow AB_4$ also accidentally cuts the propagator $\langle AB_1 AB_4 \rangle$ which produces a double pole unless the numerator cancels it. As in the triangle case, we use the same building blocks (6.50)-(6.54) and same rules for constructing the basis of terms which satisfy the singlet conditions **P1** and **N1** and doublet conditions **P2a**, **P2b**, **N2a** and **N2b**. The basis now has two types of terms:

$$(6.65) \quad N_1^{(a)} N_2^{(a)} N_3^{(a)} N_4^{(a)} \quad \text{and} \quad N_1^{(a)} N_2^{(a)} N_{34}^{(a)} N_{34}^{(c)}$$

Both terms cancel for all singlet conditions trivially, and also for all doublet conditions – this is true not only for four links in the graph, but actually for all six possible links including $\langle AB_1 AB_3 \rangle$ and $\langle AB_2 AB_4 \rangle$ which are not in the graph! The cyclicity of the diagram then gives us the ansatz

$$(6.66) \quad R_{1234} = c_1 N_1^{(a)} N_2^{(a)} N_3^{(a)} N_4^{(a)} + c_2 \left\{ N_1^{(a)} N_3^{(a)} N_{24}^{(a)} N_{24}^{(c)} + N_2^{(a)} N_4^{(a)} N_{13}^{(a)} N_{13}^{(c)} \right\} \\ + c_3 \left\{ N_1^{(a)} N_2^{(a)} N_{34}^{(a)} N_{34}^{(c)} + N_2^{(a)} N_3^{(a)} N_{14}^{(a)} N_{14}^{(c)} + N_3^{(a)} N_4^{(a)} N_{12}^{(a)} N_{12}^{(c)} + N_1^{(a)} N_4^{(a)} N_{23}^{(a)} N_{23}^{(c)} \right\} \\ + (a \rightarrow b).$$

We also implicitly cycle in external legs, which translates into switching a labels to b labels. Note that the second and third terms in (6.66) are structurally the same but the labels are distributed differently and we have to consider them separately. Imposing the condition **D** on the numerator (6.63) reveals no solution. This means that our ansatz is not big enough. Indeed, there are more building blocks we can define which are analogous to $N_{ij}^{(a,b,c)}$ but now involve four lines AB_i rather than just two:

$$(6.67) \quad N_{1234}^{(a)} = \langle AB_1 12 \rangle \langle AB_2 34 \rangle \langle AB_3 12 \rangle \langle AB_4 34 \rangle + \langle AB_1 34 \rangle \langle AB_2 12 \rangle \langle AB_3 34 \rangle \langle AB_4 12 \rangle,$$

$$(6.68) \quad N_{1234}^{(b)} = \langle AB_1 23 \rangle \langle AB_2 14 \rangle \langle AB_3 23 \rangle \langle AB_4 14 \rangle + \langle AB_1 14 \rangle \langle AB_2 23 \rangle \langle AB_3 14 \rangle \langle AB_4 23 \rangle,$$

$$(6.69) \quad N_{1234}^{(c)} = \langle AB_1 13 \rangle \langle AB_2 24 \rangle \langle AB_3 13 \rangle \langle AB_4 24 \rangle + \langle AB_1 24 \rangle \langle AB_2 13 \rangle \langle AB_3 24 \rangle \langle AB_4 13 \rangle.$$

We see an intriguing pattern of labels here which will later generalize to higher L . Because of the number of lines involved in the building blocks, we refer to $N_i^{(a,b)}$ as *one-line invariants*, $N_{ij}^{(a,b,c)}$ as *two-line invariants* and $N_{ijkl}^{(a,b,c)}$ as *four-line invariants*. There are no three-line invariants and we indeed did not need any extra building blocks to solve the triangle problem.

Note that the four-line invariants are cyclically invariant (in internal lines AB_1, AB_2, AB_3 and AB_4 – in this ordering), and do not enjoy the full permutational symmetry in AB_j . As for satisfying conditions **P1**, **N1**, **P2a**, **P2b**, **N2a** and **N2b**, they perform a similar job to $N_{ij}^{(a,b,c)}$: they do satisfy singlet conditions for all lines AB_i , but the doublet conditions are only satisfied for pairs (AB_1, AB_2) , (AB_2, AB_3) , (AB_3, AB_4) and (AB_4, AB_1) . This is exactly what is needed for the topology of the graph. Hence the new building blocks to add to our ansatz are $N_{1234}^{(a)} N_{1234}^{(c)}$ and $N_{1234}^{(b)} N_{1234}^{(c)}$. Our corrected ansatz is then

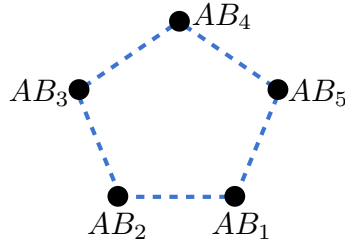
$$\begin{aligned}
(6.70) \quad R_{1234} = & c_1 N_1^{(a)} N_2^{(a)} N_3^{(a)} N_4^{(a)} + c_2 \left\{ N_1^{(a)} N_3^{(a)} N_{24}^{(a)} N_{24}^{(c)} + N_2^{(a)} N_4^{(a)} N_{13}^{(a)} N_{13}^{(c)} \right\} \\
& + c_3 \left\{ N_1^{(a)} N_2^{(a)} N_{34}^{(a)} N_{34}^{(c)} + N_2^{(a)} N_3^{(a)} N_{14}^{(a)} N_{14}^{(c)} + N_3^{(a)} N_4^{(a)} N_{12}^{(a)} N_{12}^{(c)} + N_1^{(a)} N_4^{(a)} N_{23}^{(a)} N_{23}^{(c)} \right\} \\
& + c_4 N_{1234}^{(a)} N_{1234}^{(c)} + (a \rightarrow b).
\end{aligned}$$

Imposing the double pole cancellation condition on the full numerator $\mathcal{N}_{\text{square}}^{(++++)}$, we find a unique solution,

$$(6.71) \quad c_1 = 12, \quad c_2 = c_3 = c_4 = -1.$$

We can now also check many other cuts which involve more than two lines where the numerator should vanish and find always a perfect match. Note that going from one sign assignment to another changes the sign of the remainder, each flip from positive to negative changes the sign of R_{1234} .

Let us do one more explicit example before writing down the solution for a general polygon. Let us consider $L = 5$ which is a pentagon,



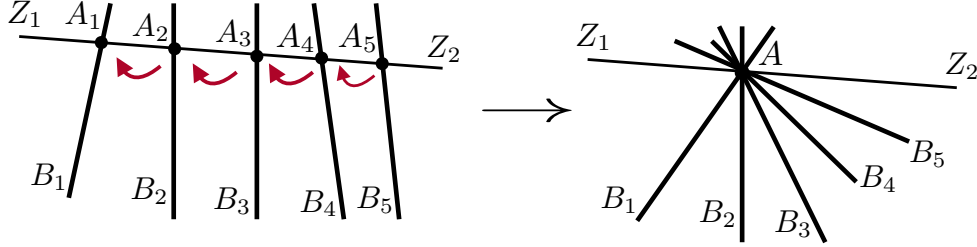
The ansatz for the numerator of a geometry with all links positive is

$$(6.72) \quad \mathcal{N}_{\text{pentagon}}^{(+++++)} = N_{12}^{(+)} N_{23}^{(+)} N_{34}^{(+)} N_{45}^{(+)} N_{15}^{(+)} + R_{12345} ,$$

where the ansatz for the remainder is now

$$(6.73) \quad R_{12345} = c_1 N_1^{(a)} N_2^{(a)} N_3^{(a)} N_4^{(a)} N_5^{(a)} + c_2 \left\{ N_1^{(a)} N_2^{(a)} N_3^{(a)} N_{45}^{(a)} N_{45}^{(c)} + \text{cycl} \right\} \\ + c_3 \left\{ N_1^{(a)} N_2^{(a)} N_4^{(a)} N_{35}^{(a)} N_{35}^{(c)} + \text{cycl} \right\} + c_4 \left\{ N_1^{(a)} N_{2345}^{(a)} N_{2345}^{(c)} + \text{cycl} \right\} + (a \rightarrow b) ,$$

where the ‘cycl’ refers to cycling in the loop lines $AB_i \rightarrow AB_{i+1}$. Note that the ansatz does not involve any five-line invariants. The dangerous configuration that produces the double pole in the canonical form is similar to the previous cases,

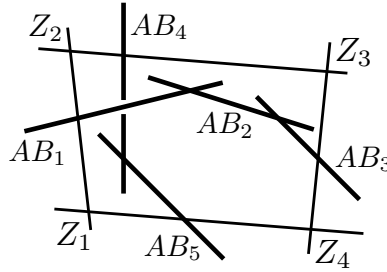


and we again find a unique solution,

$$(6.74) \quad c_1 = 28, \quad c_2 = c_3 = c_4 = -1 .$$

We can now check many additional cuts which are illegal in a given positive geometry and verify that our numerator vanishes where it should. For example, one ‘planarity condition’ is the following cut:

$$\langle AB_1 12 \rangle = \langle AB_1 AB_2 \rangle = \langle AB_2 AB_3 \rangle = \langle AB_3 34 \rangle = \langle AB_4 23 \rangle = \langle AB_4 AB_5 \rangle = \langle AB_5 14 \rangle = 0$$

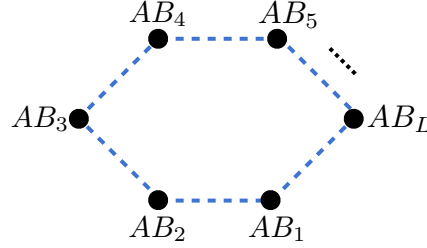


Very non-trivially, the numerator $\mathcal{N}_{\text{pentagon}}^{(+++++)}$ vanishes on this cut. Flipping a sign of a link again changes $N_{ij}^{(+)} \rightarrow N_{ij}^{(-)}$ in the tree part of the numerator and flips the sign of the remainder R_{12345} . For the totally negative geometry (with all links negative) we have

$$(6.75) \quad \mathcal{N}_{\text{pentagon}}^{(-----)} = N_{12}^{(-)} N_{23}^{(-)} N_{34}^{(-)} N_{45}^{(-)} N_{15}^{(-)} - R_{12345}.$$

6.5.4 General one-cycle graph

Let us now consider a general polygon geometry at L -loops (with L vertices). For all links positive



the numerator of the form is

$$(6.76) \quad \mathcal{N}_G = \prod_{i=1}^L N_{i,i+1}^{(+)} + R_L^{1\text{-loop}},$$

where the remainder $R^{1\text{-loop}}$ takes this form

$$(6.77) \quad R_L^{1\text{-loop}} = \left\{ (2^L - 4) N_1^{(a)} N_2^{(a)} \cdots N_L^{(a)} - \sum_{p \in \text{even}} \sum_i \{ N_{i_1 i_2 \dots i_p}^{(a)} N_{i_1 i_2 \dots i_p}^{(c)} N_{i_p+1}^{(a)} \cdots N_{i_L}^{(a)} \} \right\} + (a \rightarrow b)$$

We use the superscript ‘1-loop’ here and later denoting that the object is associated with the cycle (loop) in the diagram. The sum in (6.77) is over orderings i are all selections of p labels where the p labels are ordered within the argument of the factors $N_{i_1 i_2 \dots i_p}^{(a,b,c)}$. These are defined as

$$(6.78) \quad N_{12 \dots p}^{(a)} = \prod_{j \in \text{odd}} \langle AB_j 12 \rangle \prod_{k \in \text{even}} \langle AB_k 34 \rangle + \prod_{j \in \text{odd}} \langle AB_j 34 \rangle \prod_{k \in \text{even}} \langle AB_k 12 \rangle$$

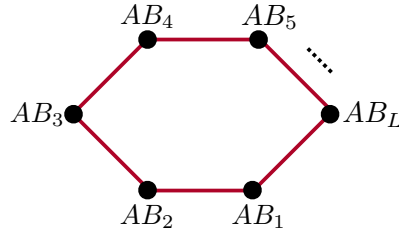
$$(6.79) \quad N_{12 \dots p}^{(b)} = \prod_{j \in \text{odd}} \langle AB_j 23 \rangle \prod_{k \in \text{even}} \langle AB_k 14 \rangle + \prod_{j \in \text{odd}} \langle AB_j 14 \rangle \prod_{k \in \text{even}} \langle AB_k 23 \rangle$$

$$(6.80) \quad N_{12 \dots p}^{(c)} = \prod_{j \in \text{odd}} \langle AB_j 13 \rangle \prod_{k \in \text{even}} \langle AB_k 24 \rangle + \prod_{j \in \text{odd}} \langle AB_j 24 \rangle \prod_{k \in \text{even}} \langle AB_k 13 \rangle$$

For example, for the $L = 6$ hexagon we have

$$(6.81) \quad R_{123456}^{1-\text{loop}} = \left\{ 60 N_1^{(a)} N_2^{(a)} N_3^{(a)} N_4^{(a)} N_5^{(a)} N_6^{(a)} - \{ N_1^{(a)} N_2^{(a)} N_3^{(a)} N_4^{(a)} N_{56}^{(a)} N_{56}^{(c)} + \text{perm} \} \right. \\ \left. - \{ N_1^{(a)} N_2^{(a)} N_{3456}^{(a)} N_{3456}^{(c)} + \text{perm} \} - N_{123456}^{(a)} N_{123456}^{(c)} \right\} + (a \rightarrow b)$$

Note in the second and third terms the sums are over all permutations of lines AB_i (we only take distinct terms, no duplicates), only the last fourth term does not respect permutational symmetry and it is only cyclically invariant. For the totally negative polygon we have

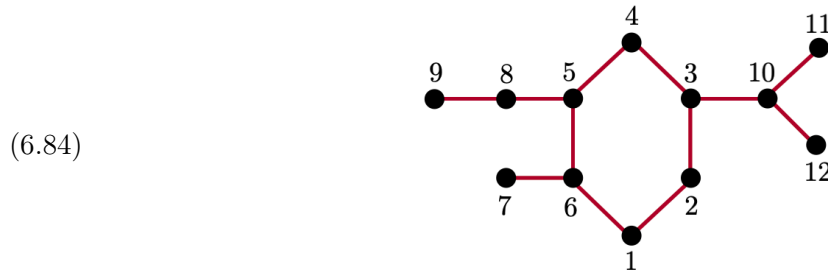


$$(6.82) \quad \mathcal{N}_G = \prod_{i=1}^L N_{i,i+1}^{(-)} + (-1)^L R_L^{1-\text{loop}}$$

Finally, let us consider a completely generic one-loop graph, ie. connected graph with one cycle (here for negative links only). There is always a one-loop core and the tree-level branches attached to it. The denominator is given by all the one-loop factors D_i and all internal propagators $\langle AB_i AB_j \rangle$ corresponding to the links as for any graph. The numerator can be organized as follows:

$$(6.83) \quad \mathcal{N}_G = \prod_{\text{links}} N_{ij}^{(-)} + (-1)^{L'} \prod_{\text{branches}} N_{ij}^{(-)} \times R_{L'}^{1-\text{loop}}$$

where the remainder $R_{L'}^{1-\text{loop}}$ is now for the L' -loop core with AB_i in that particular ordering. We can simply start from the numerator for the core and multiply it by $N_{ij}^{(-)}$ for any extra link attached to it. For an explicit $L = 12$ example we have (using just label k for AB_k for simplicity),



The numerator is now

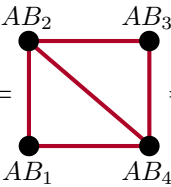
$$(6.85) \quad \mathcal{N}_G = N_{67}^{(-)} N_{58}^{(-)} N_{89}^{(-)} N_{3,10}^{(-)} N_{10,11}^{(-)} N_{10,12}^{(-)} \times \left\{ N_{12}^{(-)} N_{23}^{(-)} N_{34}^{(-)} N_{45}^{(-)} N_{56}^{(-)} N_{61}^{(-)} + R_{123456}^{1\text{-loop}} \right\}$$

where the loop function $R_{123456}^{1\text{-loop}}$ was given in (6.81). The rules for the graph with both positive and negative links is again the same up to a sign of $R^{1\text{-loop}}$. This completes a construction of a canonical form for an arbitrary graph with one internal cycle with any assignment of positive and negative links.

6.6 Canonical form decomposition

As evident from (6.22) even for $L = 4$ to get the complete result for $\tilde{\Omega}$ we need to calculate the canonical forms for higher-loop geometries (graphs with more than one cycles). While the general principle still should work: write down the ansatz and impose vanishing on illegal cuts as constraints, it is much harder to find a similar organizational principle to the one that allowed us to solve the one-loop geometries. Our approach of loops of loops expansion is very effective in solving infinite class of graphs (and provide an approximation of the full result), but if we want an exact result at a fixed loop order L , we can not avoid writing down the form for a complete graph (with all $\langle AB_i AB_j \rangle$ links) which is then as hard as calculating the integrand of the original amplitude Ω_L (which is just one graph with all positive links)

Our approach to attack the problem of higher-loop graphs (with more than one internal cycle) is a certain *decomposition* which reduces the problem of finding the canonical form to a purely higher-loop term (in the same sense as the $R_G^{1\text{-loop}}$ was the one-loop piece while the product of $N_{ij}^{(\pm)}$ was a tree piece. This approach separates ‘easy’ or ‘known’ terms from a truly new kind of contribution we have to calculate separately. Let us see the logic of this decomposition in the example from (6.22), and start solving the canonical form for the geometry with two cycles – this is the only contributing term to $\tilde{\Omega}_4^{2\text{-loop}}$.

$$(6.86) \quad \Omega_G = \begin{array}{c} \begin{array}{cc} AB_2 & AB_3 \\ \bullet & \bullet \\ | & | \\ \bullet & \bullet \\ AB_1 & AB_4 \end{array} \end{array} = \frac{\mathcal{N}_G}{D_1 D_2 D_3 D_4 \langle AB_1 AB_2 \rangle \langle AB_2 AB_3 \rangle \langle AB_3 AB_4 \rangle \langle AB_1 AB_4 \rangle \langle AB_2 AB_4 \rangle}$$


The numerator can be decomposed in this way,

$$(6.87) \quad \mathcal{N}_G = N_{12}^{(-)} N_{23}^{(-)} N_{34}^{(-)} N_{14}^{(-)} N_{24}^{(-)} - R_{124}^{1\text{-loop}} N_{23}^{(-)} N_{34}^{(-)} - R_{234}^{1\text{-loop}} N_{12}^{(-)} N_{23}^{(-)} + R_{1234}^{1\text{-loop}} N_{24}^{(-)} - R_{1234,13}^{2\text{-loop}}$$

We can introduce a graphic decomposition

$$(6.88) \quad \mathcal{N}_G = \begin{array}{c} AB_2 \quad AB_3 \\ \bullet \quad \bullet \\ \text{---} \quad \text{---} \\ \bullet \quad \bullet \\ AB_1 \quad AB_4 \end{array} + \begin{array}{c} AB_2 \quad AB_3 \\ \bullet \quad \bullet \\ \text{---} \quad \text{---} \\ \bullet \quad \bullet \\ AB_1 \quad AB_4 \end{array} + \begin{array}{c} AB_2 \quad AB_3 \\ \bullet \quad \bullet \\ \text{---} \quad \text{---} \\ \bullet \quad \bullet \\ AB_1 \quad AB_4 \end{array} + \begin{array}{c} AB_2 \quad AB_3 \\ \bullet \quad \bullet \\ \text{---} \quad \text{---} \\ \bullet \quad \bullet \\ AB_1 \quad AB_4 \end{array} + \begin{array}{c} AB_2 \quad AB_3 \\ \bullet \quad \bullet \\ \text{---} \quad \text{---} \\ \bullet \quad \bullet \\ AB_1 \quad AB_4 \end{array}$$

where the thick lines compose irreducible closed cycles (one-cycle, two-cycle) and the dashed lines are additional tree links. The minus signs in (6.87) follow the same logic as in the one-cycle geometry case (terms with an even number of negative links has relative plus sign and odd number of negative links leads to a minus sign). The last term in (6.88) is an irreducible two-loop (two-cycle) term $R_{1234,13}^{2\text{-loop}}$ which is universal for all sign assignments. This purely two-loop factor must satisfy an important constraint

$$(6.89) \quad \boxed{R_{1234,13}^{2\text{-loop}} = 0 \quad \text{on any cut when } \langle AB_i AB_j \rangle \text{ has a definite sign}}$$

for all the links in the two-loop graph. This is the same universality argument we used in the discussion of the one-loop function $R^{1\text{-loop}}$.

This decomposition of $\mathcal{N}_G$ in terms of tree, one-loop and two-loop parts makes various properties of the canonical form manifest. If we consider a totally negative geometry (6.86) then if we localize one of the lines AB_i on the leading singularity $AB_i = 13$ or $AB_i = 24$, the geometry reduces to something simpler which is also built in the form. For example, for $AB_3 = 13$ the geometry reduces to

$$(6.90) \quad \begin{array}{c} AB_2 \quad AB_3 \\ \bullet \quad \bullet \\ \text{---} \quad \text{---} \\ \bullet \quad \bullet \\ AB_1 \quad AB_4 \end{array} \xrightarrow{AB_3=13} \begin{array}{c} AB_2 \\ \bullet \\ \text{---} \\ \bullet \quad \bullet \\ AB_1 \quad AB_4 \end{array}$$

In the canonical form all terms with label 3 going away: the two-loop numerators together with the mutual pole reduce to constant

$$(6.91) \quad \frac{N_{3k}^{(-)}}{\langle AB_3 AB_k \rangle} = \frac{\langle AB_3 13 \rangle \langle AB_k 24 \rangle + \langle AB_3 24 \rangle \langle AB_k 13 \rangle}{\langle AB_3 AB_k \rangle} \xrightarrow{AB_3=13} \langle 1234 \rangle$$

while any one-loop and two-loop functions involving label 3 vanishes,

$$(6.92) \quad R_{134}^{1\text{-loop}} = R_{1234}^{1\text{-loop}} = R_{1234,13}^{2\text{-loop}} = 0$$

As a result the canonical form reduces to

$$(6.93) \quad \Omega_G \xrightarrow{AB_3=13} \Omega_{G_{\text{red}}} = \frac{N_{12}^{(-)} N_{24}^{(-)} N_{14}^{(-)} - R_{124}^{1\text{-loop}}}{D_1 D_2 D_4 \langle AB_1 AB_2 \rangle \langle AB_1 AB_4 \rangle \langle AB_2 AB_4 \rangle}$$

which is exactly the correct form of the reduced graph (6.48). Note that if instead of fixing $AB_3 = 13$ we fix now another loop line $AB_4 = 13$, the result is different but again correct. The graph reduces to

$$(6.94) \quad \begin{array}{ccc} \begin{array}{cc} AB_2 & AB_3 \\ \bullet & \bullet \\ | & | \\ \bullet & \bullet \\ AB_1 & AB_4 \end{array} & \xrightarrow{AB_4=13} & \begin{array}{cc} AB_2 & AB_3 \\ \bullet & \bullet \\ | & \\ \bullet & \\ AB_1 & \end{array} \end{array}$$

We get the same relation (6.91) now for AB_4 , but now all one-loop and two-loop functions are zero

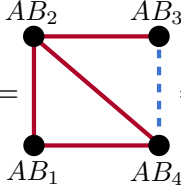
$$(6.95) \quad R_{134}^{1\text{-loop}} = R_{124}^{1\text{-loop}} = R_{1234}^{1\text{-loop}} = R_{1234,13}^{2\text{-loop}} = 0$$

and the canonical form reduces just to the tree form,

$$(6.96) \quad \Omega_G \xrightarrow{AB_4=13} \Omega_{G'_{\text{red}}} = \frac{N_{12}^{(-)} N_{23}^{(-)} N_{13}^{(-)}}{D_1 D_2 D_3 \langle AB_1 AB_2 \rangle \langle AB_1 AB_3 \rangle \langle AB_2 AB_3 \rangle}$$

This is precisely correct because (6.96) is a tree graph. The analogous procedure can be also applied to collinear limit cuts, which survive for positive links and such decomposition correctly reduces to the canonical form of a simpler graph. Another check we can perform is the sum rule, where we take a union of two geometries which differ by a sign of one link, and sum their canonical forms.

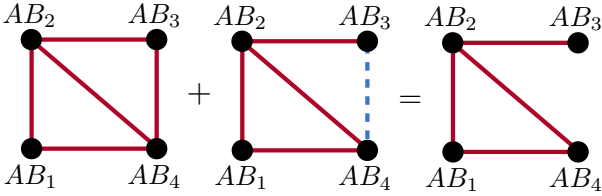
For example, the analogous graph where we flip the sign of $\langle AB_3 AB_4 \rangle$ link to positive, is

$$(6.97) \quad \Omega_G = \frac{\mathcal{N}'_G}{D_1 D_2 D_3 D_4 \langle AB_1 AB_2 \rangle \langle AB_2 AB_3 \rangle \langle AB_3 AB_4 \rangle \langle AB_1 AB_4 \rangle \langle AB_2 AB_4 \rangle}$$


has an analogous decomposition

$$(6.98) \quad \begin{aligned} \mathcal{N}'_G = & N_{12}^{(-)} N_{23}^{(-)} N_{34}^{(+)} N_{14}^{(-)} N_{24}^{(-)} \\ & - R_{124}^{1\text{-loop}} N_{23}^{(-)} N_{34}^{(+)} + R_{134}^{1\text{-loop}} N_{12}^{(-)} N_{23}^{(-)} - R_{1234}^{1\text{-loop}} N_{24}^{(-)} + R_{1234,13}^{2\text{-loop}} \end{aligned}$$

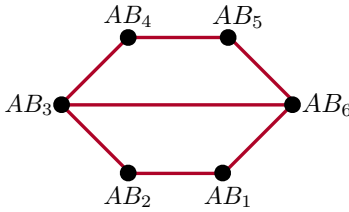
We can again see a geometric sum rule in action here: as the two geometries (6.86) and (6.97) only differ by a single sign of one link, we can take a union which gives us a one-loop geometry,

$$(6.99) \quad \text{[Diagram 1]} + \text{[Diagram 2]} = \text{[Diagram 3]}$$


At the level of numerators, we can sum two terms (6.87) and (6.98) and generate an inverse propagator $\langle AB_3 AB_4 \rangle$ and the numerator $\mathcal{N}''_G$ of the new one-loop graph,

$$(6.100) \quad \mathcal{N}_G + \mathcal{N}'_G = \underbrace{\left(N_{34}^{(+)} + N_{34}^{(-)} \right)}_{\langle AB_3 AB_4 \rangle \langle 1234 \rangle} \times \underbrace{\left(N_{12}^{(-)} N_{23}^{(-)} N_{14}^{(-)} N_{24}^{(-)} - R_{124}^{1\text{-loop}} N_{23}^{(-)} \right)}_{\mathcal{N}''_G}$$

We see that the whole problem reduces to the calculation of $R_{1234,13}^{2\text{-loop}}$. The same decomposition extends to any two-loop diagram. For example, one contribution to $\tilde{\Omega}_6^{2\text{-loop}}$ is a following negative geometry,

$$(6.101) \quad \text{[Diagram]}$$


for which the numerator can be decomposed as

$$\begin{aligned}
\mathcal{N}_G = & N_{12}^{(-)} N_{23}^{(-)} N_{34}^{(+)} N_{45}^{(-)} N_{56}^{(-)} N_{16}^{(-)} N_{36}^{(-)} + R_{1234}^{1\text{-loop}} N_{45}^{(-)} N_{56}^{(-)} N_{16}^{(-)} + R_{3456}^{1\text{-loop}} N_{16}^{(-)} N_{12}^{(-)} N_{23}^{(-)} \\
(6.102) \quad & + R_{123456}^{1\text{-loop}} N_{36}^{(-)} + R_{123456,36}^{2\text{-loop}}
\end{aligned}$$

where all relative signs are + because the cycles have even number of vertices each. The same decomposition also applies for canonical forms of graphs with more than two cycles.

Now assuming we know how to find the two-loop remainder $R^{2\text{-loop}}$, we can perform this decomposition for the three-loop graph, for example the only one in $\tilde{\Omega}_4^{3\text{-loop}}$ and the last piece needed to fully calculate $\tilde{\Omega}_4$.

$$(6.103) \quad \tilde{\Omega}_4^{3\text{-cycle}} = \text{Diagram of a square with vertices } AB_1, AB_2, AB_3, AB_4 \text{ and all four internal diagonals.}$$

As we already know, once we determine this canonical form, we can also easily calculate the canonical form Ω_4 for the $L = 4$ amplitude which graph has only positive links. The decomposition of the canonical form into tree, one-loop, two-loop and three-loop terms is:

$$\begin{aligned}
\Omega_G = & N_{12}^{(-)} N_{23}^{(-)} N_{34}^{(-)} N_{14}^{(-)} N_{24}^{(-)} N_{13}^{(-)} - \sum_{\pi_1} R_{123}^{1\text{-loop}} N_{14}^{(-)} N_{24}^{(-)} N_{34}^{(-)} + \sum_{\pi_2} R_{1234}^{1\text{-loop}} N_{13}^{(-)} N_{24}^{(-)} \\
(6.104) \quad & - \sum_{\pi_3} R_{1234,13}^{2\text{-loop}} N_{24}^{(-)} + R_{1234}^{3\text{-loop}}
\end{aligned}$$

$$\begin{aligned}
\mathcal{N}_G = & \text{Diagram of a square with vertices } AB_1, AB_2, AB_3, AB_4 \text{ and all four internal diagonals.} + \sum_{\pi_1} \text{Diagram of a square with vertices } AB_1, AB_2, AB_3, AB_4 \text{ and all four internal diagonals.} \\
& + \sum_{\pi_2} \text{Diagram of a square with vertices } AB_1, AB_2, AB_3, AB_4 \text{ and all four internal diagonals.} \\
(6.105) \quad & + \sum_{\pi_3} \text{Diagram of a square with vertices } AB_1, AB_2, AB_3, AB_4 \text{ and all four internal diagonals.} + \text{Diagram of a square with vertices } AB_1, AB_2, AB_3, AB_4 \text{ and all four internal diagonals.}
\end{aligned}$$

The permutation π_1 is over all embedded one-loop triangles, there are 4 of them (123), (124), (134), (234). The permutation π_2 is over all embedded one-loop squares (with a particular ordering), there are 3 of them (1234), (1243), (1324). The permutation π_3 is over all embedded two-loop subtopologies (where we ignore one of the links), this is 6 terms (ignoring one of the links each). The final purely three-loop piece is again universal for any link sign assignment and satisfies

$$(6.106) \quad \boxed{R_{1234}^{3-\text{loop}} = 0 \quad \text{on any cut when } \langle AB_i AB_j \rangle \text{ has a definite sign}}$$

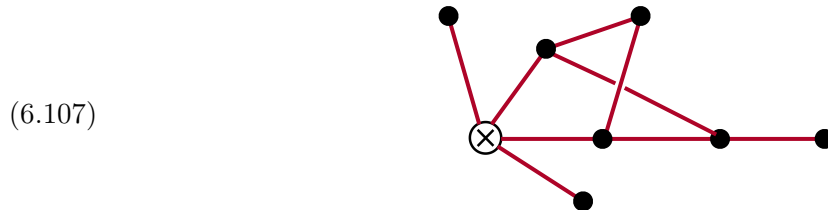
Now this is true for all links as the graph as well as $R_{1234}^{3-\text{loop}}$ is permutationally invariant in all four loop lines AB_i . The same logic can be used for higher loops at any L .

Our procedure seems reminiscent of ‘Feynman rules’ and ‘Feynman diagrams’ but now in the loops of loops space – where the role of external particles are played by loop lines AB_i , the functions $N_{ij}^{(\pm)}$ are free propagators and $R^{\ell-\text{loop}}$ are some kind of interactions vertices. The connection is very vague, and obviously the rules how to construct our graphs are quite different, but some analogy can be found here. In either case, the problem of finding the canonical form Ω_G for a given graph is reduced to finding these ℓ -loop irreducible functions (remainders) $R^{\ell-\text{loop}}$.

6.7 From canonical forms to polylogarithms

So far we have studied the canonical forms for positive and negative geometries, and showed that certain sets of them reproduce the integrands for the L -loop amplitude Ω_L and the logarithm of the L -loop amplitude $\tilde{\Omega}_L$. When integrating Ω_L or $\tilde{\Omega}_L$ or canonical forms for individual geometries, we encounter IR divergencies which need to be regulated.

As was pointed in [77], for negative geometries i.e. graphs with only negative links, we can define an *IR finite* object by freezing one of the loops and integrating over all others. The frozen loop is marked by a cross, as shown below:



This is all based on the integrand-level statement about the presence of IR divergencies: in order to generate an IR divergence during integration, the integrand needs to have a singularity when one or

more loop momenta access the collinear region. In the momentum twistor space, this collinear region corresponds to the line AB being in a collinear configuration in the plane $(i-1 \ i \ i+1)$ and passing through vertex Z_i . We have discussed this configuration multiple times in this chapter, because in this position for AB , any mutual bracket $\langle ABCD \rangle > 0$. Hence it is an illegal configuration if the vertex AB in the graph is connected to any other vertex CD – which it definitely is in any connected graph.

The only way around this is to place all loop lines AB_i , where none of the loops are left to violate the mutual negativity conditions. Hence freezing one of the loops ensures that the result after integration is IR finite. This was discussed in greater detail in [77].

6.7.1 Differential equation method

The special form of the tree numerator allows us to find a powerful differential equation. Suppose the frozen loop AB is connected to the rest of the graph through a link to another loop CD . Then we can write for the whole integral

$$(6.108) \quad I(z) = \int_{CD} \frac{\langle AB13 \rangle \langle CD24 \rangle \langle 1234 \rangle}{\langle CD12 \rangle \langle CD23 \rangle \langle CD34 \rangle \langle CD41 \rangle \langle ABCD \rangle} H(z_{CD}),$$

where $H(z_{CD})$ is the rest of the diagram integrated with the loop CD frozen, with $z_{CD} = \frac{\langle CD12 \rangle \langle CD34 \rangle}{\langle CD14 \rangle \langle CD23 \rangle}$. We can now act with the Laplacian $(z\partial_z)^2$ on the integral $I(z)$ and get

$$(6.109) \quad (z\partial_z)^2 I(z) = -H(z).$$

Effectively, the action of the operator reduces the graph to a simpler one with the link $(ABCD)$ removed. In the dual variables, this is

$$(6.110) \quad \square_{x_0} \otimes \text{---} \bigcirc = \otimes \bigcirc$$

Not only does this work for any tree graph where the frozen loop AB is connected to the rest of the graph through one link, but it can also be used for the resummation of all ladder graphs,

$$(6.111) \quad \mathcal{F}_{\text{ladder}}(g, z) = \otimes - (g^2) \otimes \text{---} \bullet + (g^2)^2 \otimes \text{---} \bullet \text{---} \bullet \\ - (g^2)^3 \otimes \text{---} \bullet \text{---} \bullet \text{---} \bullet + (g^2)^4 \otimes \text{---} \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet + \dots$$

Imposing the boundary condition that the result must vanish for $z = -1$ – which is just a simple consequence of the vanishing of the integrand for $\langle AB13 \rangle = \langle AB24 \rangle = 0$, we get [77]

$$(6.112) \quad \mathcal{F}_{\text{ladder}} = \frac{\cos(\sqrt{2}g \log z)}{\cosh(\sqrt{2}g\pi)}.$$

However, this is not a consistent approximation in our setup since we need to sum over all tree graphs to get the ‘leading order’ contribution in the loops of loops expansion.

$$(6.113) \quad \begin{aligned} \mathcal{F}_{\text{tree}}(g, z) = & \bigcirc - (g^2) \bigcirc \text{---} \bullet + (g^2)^2 \left\{ \bigcirc \text{---} \bullet \text{---} \bullet + \frac{1}{2!} \bigcirc \begin{array}{c} \bullet \\ \diagup \diagdown \\ \bullet \end{array} \right\} \\ & - (g^2)^3 \left\{ \bigcirc \text{---} \bullet \text{---} \bullet \text{---} \bullet + \bigcirc \begin{array}{c} \bullet \text{---} \bullet \\ \diagup \diagdown \\ \bullet \end{array} + \frac{1}{2!} \bigcirc \text{---} \bullet \begin{array}{c} \bullet \\ \diagup \diagdown \\ \bullet \end{array} + \frac{1}{3!} \bigcirc \begin{array}{c} \bullet \text{---} \bullet \text{---} \bullet \\ \diagup \diagdown \diagup \diagdown \\ \bullet \end{array} \right\} + \dots \end{aligned}$$

For a general tree graph the marked point can be somewhere inside and connected to other vertices through multiple links. To resolve that we realize that the sum of all tree graphs satisfies a simple exponentiation,

$$(6.114) \quad \mathcal{F}_{\text{tree}}(g, z) = e^{\mathcal{H}_{\text{tree}}(g)},$$

where $\mathcal{F}_{\text{tree}}(g, z)$ and $\mathcal{H}_{\text{tree}}(g)$ differ in the way that marked point AB can appear,

$$(6.115) \quad \mathcal{F}_{\text{tree}}(g, z) = \bigcirc \quad \mathcal{H}_{\text{tree}}(g, z) = \bigcirc \text{---} \bullet$$

and the generating functional $\mathcal{H}_{\text{tree}}(g, z)$ now satisfies the same Laplace equation because the marked point AB is now connected through a single link to the rest of the graph,

$$(6.116) \quad \frac{1}{2}(z\partial_z)^2 \mathcal{H}_{\text{tree}} + g^2 e^{\mathcal{H}_{\text{tree}}} = 0.$$

This was solved in [77] by

$$(6.117) \quad \mathcal{F}_{\text{tree}}(g, z) = \frac{A^2 z^A}{g^2 (z^A + 1)^2}$$

with $\frac{A}{2g \cos \pi A/2} = 1$. The differential equation results can be used to obtain $\mathcal{F}_{\text{tree}}^n(g, z)$ for any order n by asymptotically expanding the solution in powers of g . The refer the reader to [77] for more details. It is important to note that the Laplacian will collapse the propagator connecting

the frozen loop to rest of the diagram even if the rest is not a tree. However, this is not true if the unintegrated loop is attached to multiple propagators, such as the frozen corner of the 3-loop triangle, since the boxing trick described earlier no longer works. One needs to search for new types of differential operators, perhaps analogous to [210] where the operators acted on the external momentum twistors, or using methods in [196]. It is also clear that new ideas are needed to resum diagrams containing cycles. We leave this investigation to chapter 7.

6.8 Conclusion and Outlook

In this work we studied the Amplituhedron picture for four-point scattering amplitudes at all loops. We used the negative geometry expansion of [77] to determine the canonical form for the all-loop integrand in the (subleading) one-cycle approximation in the context of the loops of loops expansion. We determine the canonical forms by demanding that the numerator vanishes on illegal cuts, producing a unique solution. In the process of determining the form we came up with an interesting organizational principle for the canonical form by further expanding the numerator in terms of certain irreducible building blocks. We believe that the same procedure can be used for the canonical forms of negative geometries with more internal cycles, and perhaps more generally for other positive geometries too.

There are two main open questions motivated by our work: (1) calculation of higher-cycle canonical forms, (2) determination of the contribution of one-cycle geometries to $\mathcal{F}(g, z)$, behavior at strong coupling and the gamma cusp. The main tool to tackle (2) will probably be the differential equation method used in [77] to solve the tree-graph problem, which would require a different differential operator. If this mission is accomplished, it would open many directions of research and we could start to see a new emergent geometric picture for strong coupling amplitudes.

The related question is how well the loops of loops expansion approximates the full result $\mathcal{F}(g, z)$. The main reason for dividing the collection of all negative geometries for $\tilde{\Omega}_L$ into the sectors with fixed number of cycles is obvious: more cycles = more complicated forms, and we wanted to focus on sectors for which we can actually find the solution. But if we really believe this is a good expansion, we might try to introduce a deformation parameter ξ and weigh different ℓ -cycle sectors with powers ξ^ℓ ,

$$(6.118) \quad \mathcal{F}(g, z, \xi) = \mathcal{F}^{(0)}(g, z) + \xi \mathcal{F}^{(1)}(g, z) + \xi^2 \mathcal{F}^{(2)}(g, z) + \xi^3 \mathcal{F}^{(3)}(g, z) \dots$$

where $\mathcal{F}^{(k)}(g, z)$ is the k -cycle contribution. The physical case is $\xi = 1$ where all geometries are weighted equally, while our expansion is around $\xi = 0$. Hence a priori it is not clear why this should be a good expansion, but the results of [77] for the $k = 0$ case are optimistic. It would be interesting to see if the purely geometric parameter ξ has some physics analogue as some deformation of the Wilson loop picture.

Now that we know how to build integrands using negative geometries, albeit for a limited class of diagrams, let's see if we can use them to compute Wilson loops and contributions to Γ_{cusp} . This will be the subject of the next chapter.

CHAPTER 7

Multi-Loop Negative Geometries

7.1 Introduction

The main motivation of the technology we have developed in the previous chapter can be roughly stated as follows: the Wilson loop ratio $\mathcal{F}(g, z)$ is the *simplest* non-trivial object that we eventually aim to calculate to all loop orders, and we also want to explore the role of integrability and exact solutions in amplitudes. The entry point to this ambitious goal could be provided by Γ_{cusp} . It is worth emphasizing that while integrability provides Γ_{cusp} from the BES equation, it is not clear how it can be used to calculate more complicated objects. One successful program is the flux tube S-matrix bootstrap [149, 150, 151], which aims to calculate amplitudes and Wilson loops at finite coupling by doing an expansion around special kinematical regions, the near-collinear limits. While this program provides excellent numerical control over the amplitude, it does not incorporate many mathematical structures discovered in perturbation theory and believed to be applicable to the full non-perturbative object.

The study of this particular Wilson loop with a Lagrangian insertion has been a very active and fruitful area of research. At weak coupling, explicit calculations using the Wilson loop technology have been done up to three loops [205, 206, 211], while at strong coupling the leading order contribution was calculated via the AdS/CFT correspondence as a minimal surface problem [128]. We will stick to the perturbation theory and use the method of *negative geometries* discovered in ref. [77], and later used in refs. [19, 192, 196, 212, 213, 214, 215, 216, 217, 218, 219] to obtain interesting results both in the planar maximally supersymmetric Yang-Mills and ABJM theories. The main idea is to use the refinement of the Amplituhedron picture for the integrand of the logarithm of the amplitude. As a result, we get an expansion in terms of canonical dlog forms on Amplituhedron-like geometries. This provides an alternative expansion to Feynman diagrams with very different term-wise properties. Most importantly, these objects are IR finite term by term and have a pure maximal transcendental weight when integrated over all loop momenta (except

the frozen one). Note that in the standard approaches it is often highly non-trivial to expand the IR finite object in terms of finite building blocks; $\mathcal{F}(g, z)$ is “barely finite” and any standard diagrammatic expansion will contain divergent terms that only cancel in the sum. IR finiteness is essential in our approach, because the whole geometric problem is formulated in $D = 4$ and there is no space for any $D = 4 - 2\epsilon$ terms.

The negative geometry expansion can be recast using graphs with vertices (loop momenta) and edges (quadratic geometric constraints). The set of all such graphs is naturally organized in terms of the number of internal cycles (in this constrained space). The more cycles there are, the more complicated functions one gets. While this organization has no physical analogue, it provides a very powerful mathematical tool. In particular, certain subsets of graphs can be calculated to all loop orders [77, 212], and in this work we will push this frontier even further. As with any amplitude computation, the problem of calculating $\mathcal{F}(g, z)$ boils down to two parts: calculating the integrand for all negative geometries that contribute at a fixed loop order, and integrating them. The mathematical definition of negative geometries provides us with a set of inequalities that we need to solve. In certain cases this problem is solvable for arbitrary loop order; in other cases we have to work diligently to determine the integrand form. How to do it in general is an open problem and a big challenge, and in this work we push the technology to the next level. The second part, performing the integration, is an even less explored subject. In cases where the contribution to $\mathcal{F}_{\text{geom}}(z)$ from a given negative geometry satisfies a differential equation, we can solve for the contribution directly, without integrating over the frozen loop momentum. In other cases, we have to rely on the integration technology developed in the context of Feynman integrals. The differential equation method is very natural in this context, although finding an equation for each individual geometry is an open problem.

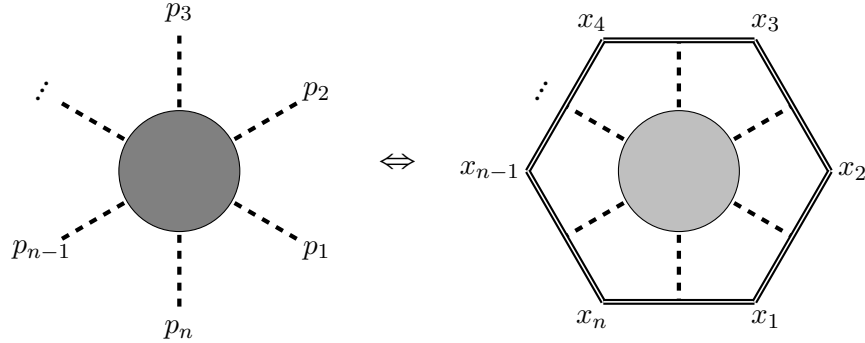
In the work reviewed here, we focus on two different aspects of the negative geometry expansion of the Wilson loop $\mathcal{F}(g, z)$. We will construct the integrands for all negative geometries contributing at three loops. We will then evaluate these integrals explicitly, both at the level of the symbol (in the chapter) and the actual functions, and confirm that the sum agrees with the result in the literature [205, 206, 211]. This agreement not only provides strong evidence that the negative geometry expansion conjecture is correct, but it will also give us very important insights into the hierarchy of transcendental functions that contribute to geometries with no cycles, one cycle, etc.

The second topic of this chapter is the resummation of certain classes of negative geometries. In ref. [77], all “tree” geometries, i.e. with no cycles, were resummed to all orders. Here we will discuss the resummation procedure in more generality, and provide a number of examples involving geometries with more cycles. Both of these threads, explicit fixed loop order calculations and all-loop resummations, are important components of the problem and move us a bit closer to the final goal of determining $\mathcal{F}(g, z)$ exactly.

The chapter is organized as follows: in section 7.2, we will give an overview of the Wilson loop with a Lagrangian insertion. In section 7.3, we discuss the integrated negative geometries for the three-loop WLI and the correspondence between geometries and the depths of polylogarithms. In section 7.4, we resum certain classes of one-cycle geometries. In section 7.5, we discuss the contributions of individual negative geometries to Γ_{cusp} . We conclude in section 7.6. More technical parts are moved to appendices that cover details of the triangle-ladder computation (C.2).

7.2 Wilson loop and IR finite object

One of the most surprising properties of planar $\mathcal{N} = 4$ SYM amplitudes is the duality to expectation values of null polygonal Wilson loops [126, 128, 140]. Namely, the n -point scattering amplitude $A_n(p_1, \dots, p_n)$ is dual to $W_n(x_1, \dots, x_n)$, the n -sided light-like polygonal Wilson loop, with cusps located at points x_i in the dual momentum space.



The conformal symmetry of the Wilson loop then translates to the dual conformal symmetry of the amplitude. Also, the UV divergence of the Wilson loop (from virtual gluons emitted near each corner of the polygon) corresponds to the IR divergence of the amplitude (from virtual gluons exchanged at long distances between color-adjacent external particles). The same duality also holds

for the logarithm of the amplitude, which is related to a ratio of two Wilson loops,

$$(7.1) \quad F_n(x_1, \dots, x_n; z_0) := \pi^2 \frac{\langle W_n(x_1, \dots, x_n) \mathcal{L}(z_0) \rangle}{\langle W_n(x_1, \dots, x_n) \rangle},$$

where $\mathcal{L}(z_0)$ denotes the Lagrangian inserted at a point z_0 . This observable has been considered extensively in previous works [202, 203, 204, 205, 206, 207, 208, 211, 215, 220, 221, 222] and it enjoys a number of special properties. The leading term at strong coupling was calculated as a minimal surface problem using the AdS/CFT correspondence [202]. More specifically, it corresponds to classical solutions in supergravity. However, the computation of the sub-leading term in the strong-coupling expansion is very difficult as it would require quantizing the Nambu-Goto action in the curved AdS background.

Most importantly, eq. (7.1) is IR finite at each order in perturbation theory, and because of the duality between Wilson loops and scattering amplitudes [126, 128, 140], the integrand of $\mathcal{F}_n$ at L loops is equal to the integrand of the logarithm of the n -point MHV amplitude at $(L+1)$ loops, after treating z_0 as an integration variable of the “frozen” loop momentum. In the four-point case, $\mathcal{F}_n$ reduces to the object $\mathcal{F}(g, z)$ we constructed earlier. Furthermore, the function $\mathcal{F}(g, z)$ is related to the object on the left-hand side above, namely $F(x_1, \dots, x_n; z_0)$, through

$$(7.2) \quad F(g, z) = -g^2 \mathcal{F}(g, z).$$

As a result, $\mathcal{F}(g, z)$ is an ideal object for testing new ideas and exploring features of integrated expressions that are related to planar $\mathcal{N} = 4$ SYM amplitudes. It is a function of a single cross ratio, it has a direct definition in terms of Wilson loops, and it is only one integration away from being the MHV amplitude’s logarithm. When we talk about strong coupling results however, we will consider $F(g, z)$ instead of $\mathcal{F}(g, z)$ as it is more directly related to Wilson loops.

7.3 Integrated negative geometries

Starting from the $L+1$ loop integrand for the amplitude logarithm, $\tilde{\mathcal{I}}_4^{(L+1)}$, we can now integrate over L of the loop momenta AB_i , $i = 2, 3, \dots, L+1$ and get $\mathcal{F}^{(L)}(z)$, which also can be interpreted as the expectation value of the Wilson loop with a Lagrangian insertion, as discussed in the previous section. This function is IR finite and has a uniform (and maximal) transcendentality $2L$. The same integration operation can be performed for each negative geometry G separately. We get an IR finite

function $\mathcal{F}_G(z)$ of the same uniform maximal transcendentality. If we sum over all functions $\mathcal{F}_G(z)$ for all graphs in the negative geometry expansion of the integrand at $L + 1$ loops, we get $\mathcal{F}^{(L)}(z)$ as expected.

To distinguish between the canonical form for a graph Ω_G and the transcendental function $\mathcal{F}_G(z)$, we use a modified graphical notation for $\mathcal{F}_G(z)$ where the frozen loop $AB_1 \equiv AB$ is denoted by the cross circle (marked point), while all other vertices correspond to loops over which we integrate. For most of the graphs, there are multiple ways to freeze the loop, and hence one negative geometry generates several integrated negative geometries. At $L = 1$, we get only one term,

$$(7.3) \quad \bullet \text{---} \bullet \rightarrow \bigotimes \text{---} \bullet$$

At $L = 2$, we have two terms: the tree graph produces two different integrated geometries and the loop graph produces one integrated geometry,

$$(7.4) \quad \begin{array}{c} \bullet \text{---} \bullet \text{---} \bullet \rightarrow \bigotimes \text{---} \bullet \text{---} \bullet + \frac{1}{2!} \bigotimes \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \end{array} \\ \bullet \text{---} \bullet \text{---} \bullet \rightarrow \frac{1}{2!} \bullet \text{---} \bullet \text{---} \bigotimes \end{array}$$

There are also symmetry factors involved which are given by the number of repetitions we get when marking the point, divided by $L!$ from the symmetrization of integrated loops. Such an analysis is easy to do for any negative geometry. The integration over all unfrozen loop momenta generally gives a rational function (leading singularity) multiplied by a transcendental function,

$$(7.5) \quad \begin{array}{c} \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet \\ \bullet \text{---} \bullet \text{---} \bullet \end{array} \rightarrow \bigotimes = \mathcal{R} \times \mathcal{T}(z).$$

At four points, the little group weights allow only one leading singularity, which is precisely the four-point one-loop amplitude,

$$(7.6) \quad \mathcal{R} = \frac{\langle 1234 \rangle^2}{\langle AB12 \rangle \langle AB23 \rangle \langle AB34 \rangle \langle AB14 \rangle}.$$

Higher point leading singularities were explored and for the full $\mathcal{F}(z)$ they were fully classified in ref. [217]. Because all negative geometries produce the same (7.6), we just factor it out, and do not consider it in our discussion here. Hence, the 0-loop integrated geometry (just the marked point itself) is normalized to 1,

$$(7.7) \quad \otimes = 1,$$

while the one-loop negative geometry gives

$$(7.8) \quad \otimes \text{---} \bullet = \int_{CD} \frac{-(\langle AB13 \rangle \langle CD24 \rangle + \langle AB24 \rangle \langle CD13 \rangle) \langle 1234 \rangle}{\langle CD12 \rangle \langle CD23 \rangle \langle CD34 \rangle \langle CD14 \rangle \langle ABCD \rangle} = \log^2 z + \pi^2.$$

where z is defined in eq. (6.12), and the actual integral was an IR finite pentagon considered already in refs. [111, 112].

Generically, we denote the contribution of an individual graph G_C with C cycles by $\mathcal{G}_{G_C}(z)$, not including the overall factor of $(-1)^{E(G)}$. Including that factor gives a contribution to $\mathcal{F}(z)$. Then, contribution to the L -loop $\mathcal{F}$ is given by

$$(7.9) \quad \mathcal{F}^{(L)}(z) = \sum_{C=0}^{C_{max}} \sum_{G_C} (-1)^{E(G)} \mathcal{G}_{G_C}(z)$$

7.3.1 Differential equation method

In the previous chapter, we briefly mentioned how certain subclasses of negative geometries satisfy differential equations that allow for the resummation of $\mathcal{F}(g, z)$. In this section, we revisit this idea in more detail. The special form of the tree numerator allows us to find a powerful differential equation [77]. Suppose the frozen loop AB is connected to the rest of the graph through a link to another loop CD . Then we can write for the whole integral

$$(7.10) \quad \mathcal{G}(z) = - \int_{CD} \frac{(\langle AB13 \rangle \langle CD24 \rangle + \langle AB24 \rangle \langle CD13 \rangle) \langle 1234 \rangle}{\langle CD12 \rangle \langle CD23 \rangle \langle CD34 \rangle \langle CD41 \rangle \langle ABCD \rangle} H(z_{CD}),$$

where $H(z_{CD})$ is the rest of the diagram integrated with the loop CD frozen, with $z_{CD} = \frac{\langle CD12 \rangle \langle CD34 \rangle}{\langle CD14 \rangle \langle CD23 \rangle}$. Let's consider the first piece of this integral, namely

$$(7.11) \quad I(z) = - \int_{CD} \frac{(\langle AB13 \rangle \langle CD24 \rangle) \langle 1234 \rangle}{\langle CD12 \rangle \langle CD23 \rangle \langle CD34 \rangle \langle CD41 \rangle \langle ABCD \rangle} H(z_{CD}),$$

We can now act with the Laplacian

$$(7.12) \quad \square = \frac{1}{2} (z \partial_z)^2,$$

such that the action on the integral $I(z)$ is

$$(7.13) \quad (z \partial_z)^2 I(z) = H(z).$$

Note that this equation also holds separately for the second piece with $\langle AB24 \rangle \langle CD13 \rangle$ in the numerator, hence we normalize the operator $\square$ with a factor $\frac{1}{2}$ in front and get

$$(7.14) \quad \square \mathcal{G}(z) = H(z).$$

Effectively, the action of the operator produces a delta function which reduces the graph to a simpler one with the link $(ABCD)$ removed. In the dual variables, this is

$$(7.15) \quad \square_{x_0} \otimes \text{[Diagram: a circle with a red line segment attached to its left side]} = \otimes \text{[Diagram: a circle]}.$$

We can demonstrate it on eq. (7.8). In this case the right hand side of the equation is just given by the $L = 0$ diagram, which evaluates to 1. Since the $L = 1$ diagram enters $\mathcal{F}$ with an extra minus sign, the equation at hand is

$$(7.16) \quad \square \mathcal{F}(z) = -1,$$

which has a general solution

$$(7.17) \quad \mathcal{F}(z) = -(\log^2 z + A).$$

The boundary condition for every diagram that has a branch, such as the LHS diagram in (7.15), is

$$(7.18) \quad \mathcal{F}(z = -1) = 0.$$

This follows from the factorized form of the numerator, and the fact that $\langle AB13 \rangle = \langle AB24 \rangle = 0$ sets $z = -1$ (see eq. (6.12)), for which the integrand vanishes identically. Using (7.18) the solution (7.17) then reads

$$(7.19) \quad \mathcal{F}(z) = -(\log^2 z + \pi^2),$$

in agreement with a direct integration of eq. (7.8) – note that the sign difference comes from the extra minus sign in the numerator of (7.8) compared to the full 2-point integrand. In fact, all tree graphs can be solved using the same differential equation method. Consider a general diagram where the marked point is located somewhere in the middle of the diagram,

$$(7.20) \quad \begin{array}{c} \bullet \quad \bullet \\ \diagup \quad \diagdown \\ \bullet \quad \otimes \quad \bullet \\ \diagdown \quad \diagup \\ \bullet \quad \bullet \end{array} = \begin{array}{c} \bullet \quad \bullet \\ \diagup \quad \diagdown \\ \bullet \quad \otimes \end{array} \times \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \otimes \quad \bullet \\ \diagdown \quad \diagup \\ \bullet \quad \bullet \end{array}$$

where the integrand factorizes as

$$(7.21) \quad \mathcal{I}(AB_1, \dots, AB_L) = \mathcal{R}^{-1} \times \mathcal{I}_1(AB_1, \dots, AB_{L'}, AB_0) \times \mathcal{I}_2(AB_0, AB_{L'+1}, \dots, AB_L),$$

where $\mathcal{R}^{-1}$ accounts for double counting of the denominator factor of the frozen loop AB_0 . After integration we get

$$(7.22) \quad \mathcal{F}(z) = \mathcal{F}_1(z) \times \mathcal{F}_2(z),$$

where we stripped off the overall $\mathcal{R}$ factor as usual. Because both factorized parts of the diagrams now have the marked point in the correct position to use eq. (7.15), both $\mathcal{F}_1(z)$ and $\mathcal{F}_2(z)$ now satisfy the differential equation (7.13). Of course, if there are more legs connected to the frozen loop in (7.20), the integrand initially factorizes into more components, each satisfying the differential equation. As a result, the function $\mathcal{F}(z)$ for any tree graph factorizes into a product of functions $\mathcal{F}_k(z)$, each satisfying the differential equation (7.13).

7.3.2 Two-loop results and the depth of the polylogarithm

The full two-loop Wilson-loop result was computed in ref. [205] and the negative geometry expansion was evaluated term-by-term in ref. [77]. This expansion involves three terms, and the

explicit evaluation reveals the major difference between two tree graphs and the one-cycle graph. The two tree graphs can be evaluated using the differential equation method. For the ‘ladder’ geometry, the equation reads

$$(7.23) \quad \square \left[\text{⊗} \text{---} \bullet \text{---} \bullet \right] = \text{⊗} \text{---} \bullet$$

which translates into

$$(7.24) \quad \square \mathcal{F}(z) = (\log^2 z + \pi^2),$$

with the same boundary condition (7.18). As a result, we get

$$(7.25) \quad \text{⊗} \text{---} \bullet \text{---} \bullet = \frac{1}{6} [\pi^2 + \log^2 z] \times [5\pi^2 + \log^2 z].$$

The second diagram is even simpler, as it factorizes as a product of two (7.8) diagrams,

$$(7.26) \quad \text{⊗} \begin{array}{c} \bullet \\ \diagup \diagdown \\ \bullet \end{array} = [\pi^2 + \log^2 z]^2.$$

The one-cycle diagram does not satisfy the same differential equation, and we have to perform a direct integration. As a result, we get

$$(7.27) \quad \begin{array}{c} \bullet \\ \diagup \diagdown \\ \bullet \end{array} \text{⊗} = 8H_{0,0,0,0} + 8H_{-1,0,0,0} - 16H_{-1,-1,0,0} + 8H_{-2,0,0} - 8\zeta_3 (2H_{-1} - H_0) \\ + 4\pi^2 (H_{-1,0} - 2H_{-1,-1} + H_{-2}) + \frac{13\pi^4}{45},$$

where we use the notation for the harmonic polylogarithms (HPLs) [223] (with the argument z omitted for brevity). If an HPL does not have a trailing 0 index, then it admits a convergent power series expansion for real z with $-1 < z < 1$,

$$(7.28) \quad H_{-a_d, \dots, -a_1}(z) \equiv H_{\underbrace{0, \dots, 0}_{a_d-1}, -1, \dots, \underbrace{0, \dots, 0}_{a_1-1}, -1}(z) = (-1)^d \sum_{j_d > \dots > j_1 \geq 1} \frac{(-z)^{j_d}}{j_d^{a_d}} \frac{1}{j_{d-1}^{a_{d-1}}} \cdots \frac{1}{j_1^{a_1}},$$

where the number of nonzero indices is the *depth* d . Trailing 0's can be removed using the shuffle algebra and the fact that

$$(7.29) \quad \underbrace{H_{0,\dots,0}}_n(z) = \frac{1}{n!} \log^n z.$$

In the case of (7.27), all HPLs reduce to classical polylogarithms [77],

$$(7.30) \quad \begin{aligned} \text{Diagram} &= \frac{1}{3} \log^4 z - 2 \log^2 z \left[-\frac{2}{3} \text{Li}_2 \left(\frac{1}{1+z} \right) - \frac{2}{3} \text{Li}_2 \left(\frac{z}{1+z} \right) + \frac{\pi^2}{9} \right] \\ &- 2 \log z \left[4 \text{Li}_3 \left(\frac{z}{1+z} \right) - 4 \text{Li}_3 \left(\frac{1}{1+z} \right) \right] + \frac{4}{3} \left[\text{Li}_2 \left(\frac{1}{1+z} \right) + \text{Li}_2 \left(\frac{z}{1+z} \right) - \frac{\pi^2}{6} \right]^2 \\ &+ \frac{16}{3} \pi^2 \left[\text{Li}_2 \left(\frac{1}{1+z} \right) + \text{Li}_2 \left(\frac{z}{1+z} \right) - \frac{\pi^2}{6} \right] + 16 \text{Li}_4 \left(\frac{1}{1+z} \right) + 16 \text{Li}_4 \left(\frac{z}{1+z} \right) + \frac{\pi^4}{9}. \end{aligned}$$

The reduction to classical polylogarithms does not generalize to higher loops. Note that all three diagrams produce transcendentality four functions, but the two tree graphs are much simpler than the triangle one-cycle diagram.

The main difference between the tree result (which are just powers of $\log z$) and the one-cycle formula is the *depth* of the polylogarithms. This is best evident from the *symbol* [142]. The symbol can be read off from the uncompressed list of HPL indices belonging to $\{0, -1\}$, by reversing the order and letting $0 \rightarrow z$ and $-1 \rightarrow 1+z$. We also write $\text{SB}(a, b, c, d)$ for $a \otimes b \otimes c \otimes d$. The symbols of the contributions to $\mathcal{F}(z)$ at two loops are,

$$(7.31) \quad \text{Symbol} \left[\text{Diagram} \right] = 4 \text{SB}(z, z, z, z),$$

$$(7.32) \quad \text{Symbol} \left[\text{Diagram} \right] = 24 \text{SB}(z, z, z, z),$$

$$(7.33) \quad \begin{aligned} \text{Symbol} \left[\text{Diagram} \right] &= 8 \left(\text{SB}(z, z, z, z) + \text{SB}(z, z, z, 1+z) \right. \\ &\quad \left. + \text{SB}(z, z, 1+z, z) - 2 \text{SB}(z, z, 1+z, 1+z) \right), \end{aligned}$$

where the last term of (7.33), $\text{SB}(z, z, 1+z, 1+z)$, corresponds to $H_{-1,-1,0,0}$ in eq. (7.27); the latter has (higher) depth two. We will see that this depth property generalizes to more cycles and higher depths.

7.3.3 New three-loop results

One of the main results of this chapter is the computation of the full three-loop Wilson loop $\mathcal{F}(z)$ using negative geometries. The result was obtained using a traditional method of Feynman diagrams (for Wilson loops) in ref. [206]. This expansion contained UV divergent integrals that needed to be regulated, and the finite result was only obtained after performing the whole sum. In our approach, each term is individually finite. The integrand for all contributing negative geometries was provided in the previous chapter as the four-loop integrand for the amplitude's logarithm. As before, we need to freeze one of the loops in each diagram and integrate over the others. At the level of diagrams, we have six $L = 4$ negative geometry integrands which produce 11 three-loop contributions to $\mathcal{F}(g, z)$.

$$\begin{aligned}
 & \text{Diagram 1} \rightarrow \text{Diagram 1.1} + \text{Diagram 1.2} \\
 & \text{Diagram 2} \rightarrow \frac{1}{2!} \text{Diagram 2.1} + \frac{1}{3!} \text{Diagram 2.2} \\
 & \text{Diagram 3} \rightarrow \text{Diagram 3.1} + \frac{1}{2!} \text{Diagram 3.2} + \frac{1}{2!} \text{Diagram 3.3} \\
 & \text{Diagram 4} \rightarrow \frac{1}{2!} \text{Diagram 4.1} \\
 & \text{Diagram 5} \rightarrow \frac{1}{2!} \text{Diagram 5.1} + \frac{1}{2!} \text{Diagram 5.2} \\
 & \text{Diagram 6} \rightarrow \frac{1}{3!} \text{Diagram 6.1}
 \end{aligned}
 \tag{7.34}$$

where we have included the correct symmetrization factors.

All zero-cycle (tree) diagrams are easy to evaluate using the differential equation method, we obtain for the $\mathcal{G}(z)$ contributions

$$\begin{aligned}
(7.35) \quad & \begin{array}{c} \text{Diagram 1: } \bigotimes \text{---} \bullet \text{---} \bullet \text{---} \bullet \\ \text{Diagram 2: } \bigotimes \text{---} \bullet \text{---} \bullet \\ \text{Diagram 3: } \bigotimes \text{---} \bullet \text{---} \bullet \\ \text{Diagram 4: } \bigotimes \text{---} \bullet \text{---} \bullet \end{array} \\
& \begin{aligned} &= 8H_{0,0,0,0,0,0} + 4\pi^2 H_{0,0,0,0} + \frac{5}{3}\pi^4 H_{0,0} + \frac{61}{90}\pi^6, \\ &= 120H_{0,0,0,0,0,0} + 28\pi^2 H_{0,0,0,0} + \frac{11}{3}\pi^4 H_{0,0} + \frac{5\pi^6}{6}, \\ &= 48H_{0,0,0,0,0,0} + 8\pi^2 H_{0,0,0,0} + 2\pi^4 H_{0,0} + \frac{11\pi^6}{15}, \\ &= 720H_{0,0,0,0,0,0} + 72\pi^2 H_{0,0,0,0} + 6\pi^4 H_{0,0} + \pi^6. \end{aligned} \end{aligned}$$

As a consistency check, the sum over all tree graphs, weighted by combinatoric factors and the overall minus sign from $(-1)^3$, should reproduce the result from the generating function at order g^6 [77], *i.e.*

$$\begin{aligned}
(7.36) \quad & \mathcal{F}_{\text{tree}}(z) \Big|_{g^6} = -\frac{1}{45} [\log^2 z + \pi^2] \times [17 \log^4 z + 73\pi^2 \log^2 z + 92\pi^4] \\
& = -272H_{0,0,0,0,0,0} - 48\pi^2 H_{0,0,0,0} - \frac{22}{3}\pi^4 H_{0,0} - \frac{92\pi^6}{45},
\end{aligned}$$

which is indeed the case.

One-cycle diagrams The one-cycle graphs are more involved to evaluate. One of them is just a trivial product of the 2-loop triangle (7.27) and the one-loop result (7.8) using (7.20),

$$\begin{aligned}
& \begin{array}{c} \text{Diagram: } \bullet \text{---} \bigotimes \text{---} \bullet \\ \text{Diagram: } \bullet \text{---} \bigotimes \text{---} \bullet \end{array} \\
& = 16 \left(-2\zeta_3 H_{-2,0} - 2\zeta_3 H_{-1,0,0} + 3\zeta_3 H_{0,0,0} - \pi^2 H_{-3,-1} + \frac{3}{2}\pi^2 H_{-3,0} \right. \\
& \quad - \pi^2 H_{-2,-2} - \pi^2 H_{-1,-3} - \frac{1}{2}\pi^4 H_{-1,-1} + \frac{1}{4}\pi^4 H_{-1,0} + \frac{13}{360}\pi^4 H_{0,0} + 3H_{-4,0,0} - \pi^2 H_{-2,-1,0} \\
& \quad + 2\pi^2 H_{-2,0,0} - \pi^2 H_{-1,-2,0} - 2H_{-3,-1,0,0} + 7H_{-3,0,0,0} - 2H_{-2,-2,0,0} - 2H_{-1,-3,0,0} \\
& \quad - 2\pi^2 H_{-1,-1,0,0} + 2\pi^2 H_{-1,0,0,0} + \frac{1}{2}\pi^2 H_{0,0,0,0} - 6H_{-2,-1,0,0,0} + 10H_{-2,0,0,0,0} - 6H_{-1,-2,0,0,0} \\
& \quad \left. - 12H_{-1,-1,0,0,0,0} + 10H_{-1,0,0,0,0,0} + 15H_{0,0,0,0,0,0} - 2H_{-3}\zeta_3 - \pi^2 H_{-1}\zeta_3 + \frac{1}{2}\pi^2 H_0\zeta_3 \right)
\end{aligned}$$

$$(7.37) \quad + \frac{3}{2}\pi^2 H_{-4} + \frac{1}{4}\pi^4 H_{-2} + \frac{13\pi^6}{720} \Big).$$

At the level of the symbol [142] this is

$$(7.38) \quad \text{Symbol} \left[\begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \quad \bigotimes \quad \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array} \right] =$$

$$16 \left(15 \text{SB}(z, z, z, z, z, z) + 10 \text{SB}(z, z, z, z, z, 1+z) + 10 \text{SB}(z, z, z, z, 1+z, z) \right.$$

$$- 12 \text{SB}(z, z, z, z, 1+z, 1+z) + 7 \text{SB}(z, z, z, 1+z, z, z) - 6 \text{SB}(z, z, z, 1+z, z, 1+z)$$

$$- 6 \text{SB}(z, z, z, 1+z, 1+z, z) + 3 \text{SB}(z, z, 1+z, z, z, z) - 2 \text{SB}(z, z, 1+z, z, z, 1+z)$$

$$\left. - 2 \text{SB}(z, z, 1+z, z, 1+z, z) - 2 \text{SB}(z, z, 1+z, 1+z, z, z) \right),$$

Another one does satisfy the boxing equation and can also be evaluated easily by canonical differential equations [206, 224]

$$(7.39) \quad \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \quad \bigotimes \\ \diagdown \quad \diagup \\ \bullet \end{array} = 2 \left(8H_{0,0,0,0,0,0} + 8H_{0,0,-1,0,0,0} - 16H_{0,0,-1,-1,0,0} + 8H_{0,0,-2,0,0} \right.$$

$$- 8\zeta_3(2H_{0,0,-1} - H_{0,0,0}) + 4\pi^2(H_{0,0,-1,0} - 2H_{0,0,-1,-1} + H_{0,0,-2})$$

$$\left. + \frac{13\pi^4}{90} \log(z)^2 + (4\pi^2\zeta_3 + 8\zeta_5) \log(z) + \left(\frac{7\pi^6}{45} - 8\zeta_3^2 \right) \right),$$

which reproduces the answer in [212] ¹ Its symbol is given by

$$\text{Symbol} \left[\begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \quad \bigotimes \\ \diagdown \quad \diagup \\ \bullet \end{array} \right]$$

$$= 2 \left(8 \text{SB}(z, z, z, z, z, z) + 8 \text{SB}(z, z, z, 1+z, z, z) + 8 \text{SB}(z, z, 1+z, z, z, z) \right)$$

¹Here we write the HPL in the uncompressed form to make the boxing equation to (7.27) and the comparison with [212] manifest.

$$(7.40) \quad -16 \text{SB}(z, z, 1+z, 1+z, z, z).$$

All the other one-cycle diagrams, as well as both two-cycle and the single three-cycle diagrams, do not satisfy a boxing equation. While these integrals are still very special IR finite integrals, currently there is no technology tailored for their computation. Here we rely on the technology of [206] and the explicit evaluation using canonical differential equations (not of a boxing type). This generic method involves the expansion of the integrand in terms of 257 uniform transcendental (UT) basis calculated in [206]. The integral reduction is performed using the **NeatIBP** package [225]. As a result, we get the following function for the last one-cycle box graph

$$\begin{aligned}
& \text{Diagram} = 16 \left(3H_{-1,-1,0,0,0,0} + 2H_{-1,0,0,0,0,0} + 3H_{0,0,0,0,0,0} - 4H_{-2,-1,0,0,0} - \frac{1}{2}H_{-2,0,0,0,0} \right. \\
& - 2H_{-1,-2,0,0,0} + 3H_{-3,0,0,0} + 2H_{-2,-2,0,0} - 5H_{-1,-3,0,0} + \frac{1}{2}\pi^2 H_{-1,-1,0,0} + \frac{7}{6}\pi^2 H_{-1,0,0,0} \\
& + \frac{1}{6}\pi^2 H_{0,0,0,0} + 4\zeta_3 H_{-1,0,0} + 2\zeta_3 H_{0,0,0} + \frac{3}{2}H_{-4,0,0} - \frac{2}{3}\pi^2 H_{-2,-1,0} - \frac{5}{12}\pi^2 H_{-2,0,0} \\
& - \frac{1}{3}\pi^2 H_{-1,-2,0} - 8\zeta_3 H_{-2,0} + \frac{1}{2}\pi^2 H_{-3,0} + \pi^2 H_{-2,-2} - \frac{5}{2}\pi^2 H_{-1,-3} + \frac{1}{8}\pi^4 H_{-1,-1} \\
& + \frac{11}{45}\pi^4 H_{-1,0} + \frac{5}{72}\pi^4 H_{0,0} - \frac{1}{3}\pi^2 H_{-1}\zeta_3 + \frac{3}{2}\pi^2 H_0\zeta_3 - 19H_{-1}\zeta_5 + \frac{31H_0\zeta_5}{2} \\
& \left. + \frac{3}{4}\pi^2 H_{-4} - \frac{43}{144}\pi^4 H_{-2} + \frac{505\pi^6}{9072} + 11\zeta_3^2 \right).
\end{aligned}
\tag{7.41}$$

The expression in eq. (7.41), has been numerically checked using AMFlow [226] and pySecDec [227]. The symbol is much more compact and given by

$$\begin{aligned}
& \text{Symbol} \left[\text{Diagram} \right] = \\
& 16 \left(3 \text{SB}(z, z, z, z, z, z) + 2 \text{SB}(z, z, z, z, z, 1+z) - \frac{1}{2} \text{SB}(z, z, z, z, 1+z, z) \right. \\
& + 3 \text{SB}(z, z, z, z, 1+z, 1+z) + 3 \text{SB}(z, z, z, 1+z, z, z) - 2 \text{SB}(z, z, z, 1+z, z, 1+z) \\
& \left. - 4 \text{SB}(z, z, z, 1+z, 1+z, z) + \frac{3}{2} \text{SB}(z, z, 1+z, z, z, z) - 5 \text{SB}(z, z, 1+z, z, z, 1+z) \right)
\end{aligned}$$

$$(7.42) \quad + 2 \text{SB}(z, z, 1+z, z, 1+z, z) \Big) .$$

The expression for the remaining one cycle diagram is more complicated and we only present the symbol expression here. One new important feature is that there is a *spurious symbol letter* $b(4)$

$$(7.43) \quad b(4) = \frac{1 + i\sqrt{z}}{1 - i\sqrt{z}}, \quad z = s/t .$$

The letter $b(4)$ shows up in individual negative geometry contributions (starting from the 4th entry), but it drops out of the sum for $\mathcal{F}^{(3)}$ [206]. For the remaining one cycle graph, the symbol result is

$$(7.44) \quad \begin{aligned} & \text{Symbol} \left[\begin{array}{c} \text{Diagram: A triangle with vertices at top (circle with cross), bottom-left (black dot), and bottom-right (black dot). Red lines connect top to bottom-left, top to bottom-right, and bottom-left to bottom-right.} \end{array} \right] = \\ & 2 \left(8 \text{SB}(z, z, z, z, b(4), b(4)) + 2 \text{SB}(z, z, z, b(4), z, b(4)) + 2 \text{SB}(z, z, z, b(4), b(4), z) \right. \\ & + 28 \text{SB}(z, z, z, z, z, z) + 8 \text{SB}(z, z, z, z, z, 1+z) + 16 \text{SB}(z, z, z, z, 1+z, z) \\ & - 8 \text{SB}(z, z, z, z, 1+z, 1+z) + 18 \text{SB}(z, z, z, 1+z, z, z) - 2 \text{SB}(z, z, z, 1+z, z, 1+z) \\ & - 2 \text{SB}(z, z, z, 1+z, 1+z, z) + 30 \text{SB}(z, z, 1+z, z, z, z) - 6 \text{SB}(z, z, 1+z, z, z, 1+z) \\ & \left. - 22 \text{SB}(z, z, 1+z, z, 1+z, z) - 32 \text{SB}(z, z, 1+z, 1+z, z, z) \right) . \end{aligned}$$

In order to obtain a representation in terms of HPLs [223] for the full function, we need to rationalize the $b(4)$ letter by defining a new variable w , such that

$$(7.45) \quad z = -s/t = -w^2 .$$

Two-cycle diagrams The result for the symbol of the first 2-cycle graph is

$$\text{Symbol} \left[\begin{array}{c} \text{Diagram: A square with vertices at top-left (black dot), top-right (black dot), bottom-left (circle with cross), and bottom-right (black dot). Red lines connect top-left to top-right, top-right to bottom-right, bottom-right to bottom-left, bottom-left to top-left, and a diagonal from top-left to bottom-right.} \end{array} \right] =$$

$$\begin{aligned}
& 8 \left(4 \text{SB}(z, z, z, z, b(4), b(4)) + \text{SB}(z, z, z, b(4), z, b(4)) + \text{SB}(z, z, z, b(4), b(4), z) \right. \\
& + 4 \text{SB}(z, z, z, z, z, z) + \text{SB}(z, z, z, z, z, 1+z) + \text{SB}(z, z, z, z, 1+z, z) \\
& + 6 \text{SB}(z, z, z, z, 1+z, 1+z) + 9 \text{SB}(z, z, z, 1+z, z, z) - \text{SB}(z, z, z, 1+z, z, 1+z) \\
& - 5 \text{SB}(z, z, z, 1+z, 1+z, z) - 4 \text{SB}(z, z, z, 1+z, 1+z, 1+z) + 7 \text{SB}(z, z, 1+z, z, z, z) \\
& - 3 \text{SB}(z, z, 1+z, z, z, 1+z) + \text{SB}(z, z, 1+z, z, 1+z, z) - 4 \text{SB}(z, z, 1+z, z, 1+z, 1+z) \\
& - 8 \text{SB}(z, z, 1+z, 1+z, z, z) - 4 \text{SB}(z, z, 1+z, 1+z, z, 1+z) - 4 \text{SB}(z, z, 1+z, 1+z, 1+z, z) \\
& \left. + 8 \text{SB}(z, z, 1+z, 1+z, 1+z, 1+z) \right) .
\end{aligned}
\tag{7.46}$$

The other 2-cycle contribution gives the following symbol result,

$$\begin{aligned}
& \text{Symbol} \left[\begin{array}{c} \text{Diagram: A square with vertices at (0,0), (1,0), (1,1), (0,1). The top-left vertex is a white circle with a cross. The other three vertices are black circles. Red lines connect (0,0) to (1,0), (1,0) to (1,1), (1,1) to (0,1), and (0,1) to (0,0). A red diagonal line also connects (0,0) to (1,1).} \end{array} \right] = \\
& 8 \left(4 \text{SB}(z, z, z, z, b(4), b(4)) + \text{SB}(z, z, z, b(4), z, b(4)) + 3 \text{SB}(z, z, z, b(4), b(4), z) \right. \\
& + 12 \text{SB}(z, z, z, z, z, z) + 16 \text{SB}(z, z, z, z, z, 1+z) + 27 \text{SB}(z, z, z, z, 1+z, z) \\
& - 26 \text{SB}(z, z, z, z, 1+z, 1+z) + 21 \text{SB}(z, z, z, 1+z, z, z) - 3 \text{SB}(z, z, z, 1+z, z, 1+z) \\
& - 19 \text{SB}(z, z, z, 1+z, 1+z, z) - 4 \text{SB}(z, z, z, 1+z, 1+z, 1+z) + 12 \text{SB}(z, z, 1+z, z, z, z) \\
& + \text{SB}(z, z, 1+z, z, z, 1+z) - 5 \text{SB}(z, z, 1+z, z, 1+z, z) - 4 \text{SB}(z, z, 1+z, z, 1+z, 1+z) \\
& - 16 \text{SB}(z, z, 1+z, 1+z, z, z) - 4 \text{SB}(z, z, 1+z, 1+z, z, 1+z) \\
& \left. - 4 \text{SB}(z, z, 1+z, 1+z, 1+z, z) + 8 \text{SB}(z, z, 1+z, 1+z, 1+z, 1+z) \right) , .
\end{aligned}
\tag{7.47}$$

Three-cycle diagram Finally, the 3-cycle graph evaluates at the level of the symbol to

$$\text{Symbol} \left[\begin{array}{c} \text{Diagram: A square with vertices at (0,0), (1,0), (1,1), (0,1). The top-left vertex is a black circle. The other three vertices are white circles with a cross. Red lines connect (0,0) to (1,0), (1,0) to (1,1), (1,1) to (0,1), and (0,1) to (0,0). Red diagonal lines also connect (0,0) to (1,1) and (0,1) to (1,0).} \end{array} \right] =$$

$$\begin{aligned}
& 24 \left(4 \text{SB}(z, z, z, z, b(4), b(4)) + \text{SB}(z, z, z, b(4), z, b(4)) + 3 \text{SB}(z, z, z, b(4), b(4), z) \right. \\
& + 2 \text{SB}(z, z, z, z, z, z) + \text{SB}(z, z, z, z, z, 1+z) + 11 \text{SB}(z, z, z, z, 1+z, z) \\
& - 10 \text{SB}(z, z, z, z, 1+z, 1+z) + 9 \text{SB}(z, z, z, 1+z, z, z) + 5 \text{SB}(z, z, z, 1+z, z, 1+z) \\
& - 7 \text{SB}(z, z, z, 1+z, 1+z, z) - 4 \text{SB}(z, z, z, 1+z, 1+z, 1+z) + 5 \text{SB}(z, z, 1+z, z, z, z) \\
& + 7 \text{SB}(z, z, 1+z, z, z, 1+z) - \text{SB}(z, z, 1+z, z, 1+z, z) - 4 \text{SB}(z, z, 1+z, z, 1+z, 1+z) \\
& - 12 \text{SB}(z, z, 1+z, 1+z, z, z) - 4 \text{SB}(z, z, 1+z, 1+z, z, 1+z) - 4 \text{SB}(z, z, 1+z, 1+z, 1+z, z) \\
& \left. + 8 \text{SB}(z, z, 1+z, 1+z, 1+z, 1+z) \right). \tag{7.48}
\end{aligned}$$

Comparison with Wilson loop If we sum all contributions at four loops, the terms with the spurious symbol letter $b(4)$ cancel and the result is given by

$$\begin{aligned}
& -8 \left(15 \text{SB}(z, z, z, z, z, z) - 6 \text{SB}(z, z, z, z, z, 1+z) - 5 \text{SB}(z, z, z, z, 1+z, z) \right. \\
& + 6 \text{SB}(z, z, z, z, 1+z, 1+z) - 5 \text{SB}(z, z, z, 1+z, z, z) + 4 \text{SB}(z, z, z, 1+z, z, 1+z) \\
& + 2 \text{SB}(z, z, z, 1+z, 1+z, z) - 2 \text{SB}(z, z, z, 1+z, 1+z, 1+z) - 6 \text{SB}(z, z, 1+z, z, z, z) \\
& + 4 \text{SB}(z, z, 1+z, z, z, 1+z) + 4 \text{SB}(z, z, 1+z, z, 1+z, z) - 2 \text{SB}(z, z, 1+z, z, 1+z, 1+z) \\
& + 6 \text{SB}(z, z, 1+z, 1+z, z, z) - 2 \text{SB}(z, z, 1+z, 1+z, z, 1+z) - 2 \text{SB}(z, z, 1+z, 1+z, 1+z, z) \\
& \left. + 4 \text{SB}(z, z, 1+z, 1+z, 1+z, 1+z) \right), \tag{7.49}
\end{aligned}$$

which agrees with the symbol of the Wilson loop calculated in the literature [206] (with an overall minus sign). An important open question is the cancellation mechanism for the spurious symbol letter $b(4)$. Namely, how exactly is this cancellation encoded in the cuts of the integrands?

7.4 Resummation and strong coupling

The negative geometries provide a new set of building blocks for the amplitude, and for the Wilson loop $\mathcal{F}(g, z)$. The important feature of the expansion is its term-wise IR finiteness, which is not manifest in the standard Feynman diagram approach for the Wilson loop. While the integrand is presented in this very special way, the actual process of integration is still cumbersome, as we do

not currently have a specific method which would exploit the simplicity of the integrand. In fact, we rely on the canonical differential equation method [206, 224] which is very powerful, but also very general, and not tailored to our problem.

However, there are certain classes of negative geometries which are solvable using a differential (boxing) equation method described in section 7.3.1. It was shown that this method can be used for any negative geometry with an appropriate topology: one where the marked point sticks out like a lollipop from the graph. The differential equation

$$(7.50) \quad \square \otimes \text{---} \bigcirc = \otimes \bigcirc$$

is supplemented by the vanishing boundary condition at $z = -1$. In section 7.3.1, we showed that this method can be used to solve for any tree negative geometry.

7.4.1 Tree resummation

As discussed in the previous chapter, this method works not only for any tree graph where the frozen loop AB is connected to the rest of the graph through one link, but it can also be used for resumming all ladder graphs.

$$(7.51) \quad \begin{aligned} \mathcal{F}_{\text{ladder}}(g, z) = & \otimes - (g^2) \otimes \text{---} \bullet + (g^2)^2 \otimes \text{---} \bullet \text{---} \bullet \\ & - (g^2)^3 \otimes \text{---} \bullet \text{---} \bullet \text{---} \bullet + (g^2)^4 \otimes \text{---} \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet + \dots \end{aligned}$$

which satisfies

$$(7.52) \quad \square \mathcal{F}_{\text{ladder}}(g, z) = \frac{1}{2}(z\partial_z)^2 \mathcal{F}_{\text{ladder}}(g, z) = -g^2 \mathcal{F}_{\text{ladder}}(g, z).$$

Imposing the boundary condition that the g -dependent part of the result must vanish for $z = -1$, which is just a simple consequence of the vanishing of the integrand for $\langle AB13 \rangle = \langle AB24 \rangle = 0$, we get [77]

$$(7.53) \quad \mathcal{F}_{\text{ladder}}(g, z) = \frac{\cos(\sqrt{2}g \log z)}{\cosh(\sqrt{2}g\pi)},$$

with $\mathcal{F}_{\text{ladder}}(g, -1) = 1$ as required.

For a general tree graph, the marked point can be located inside the graph, and it can be connected to other vertices through multiple links. To resolve that situation, we realize that the

7.4.2 Triangle plus ladder

While it would be desirable to evaluate all one cycle diagrams and resum them to all loop order, this problem is too difficult with our current tools. We can instead turn to a much more modest task: resum a certain class of all one cycle graphs which have the same triangle core solved already in (7.27). In particular, we are interested in the infinite set of diagrams of the form

$$(7.59) \quad \mathcal{F}_{\text{tri-lad}} = g^4 \underbrace{\text{triangle} \otimes \text{circle}}_{\mathcal{F}_3} - g^6 \underbrace{\text{triangle} \text{---} \text{circle} \otimes \text{circle}}_{\mathcal{F}_{3,1}} + g^8 \underbrace{\text{triangle} \text{---} \text{circle} \text{---} \text{circle} \otimes \text{circle}}_{\mathcal{F}_{3,2}} + \dots$$

We already saw two of these contributions in the previous section, the first one in eq. (7.27) at $L = 2$, and the second one in eq. (7.39) as one of the diagrams in the $L = 3$ result. To obtain (7.39), we evaluated the diagram using the differential equation,

$$(7.60) \quad \square \left(\text{triangle} \text{---} \text{circle} \otimes \text{circle} \right) = \text{triangle} \otimes \text{circle}$$

hence we solve

$$(7.61) \quad \square \mathcal{F}_{3,1}(z) = -\mathcal{F}_3(z),$$

where $\mathcal{F}_3(z)$ denotes the triangle $L = 2$ formula (7.27), and $\mathcal{F}_{3,1}(z)$ the triangle diagram with one extra segment. The boundary condition is again the vanishing of the result at $z = -1$ which fixes $\mathcal{F}_{3,1}(z)$ to (7.39). The same approach continues to work for the next case of the triangle with two segments $\mathcal{F}_{3,2}(z)$,

$$(7.62) \quad \square \left(\text{triangle} \text{---} \text{circle} \text{---} \text{circle} \otimes \text{circle} \right) = \text{triangle} \text{---} \text{circle} \otimes \text{circle}$$

with the solution [212]

$$\mathcal{F}_{3,2}(z) = 4 \left[8H_{0,0,0,0,0,0,0,0}(z) + 8H_{0,0,0,0,-1,0,0,0}(z) - 16H_{0,0,0,0,-1,-1,0,0}(z) + 8H_{0,0,0,0,-2,0,0}(z) \right]$$

$$(7.63) \quad -8\zeta_3(2H_{0,0,0,0,-1}(z) - H_{0,0,0,0,0}(z)) + 4\pi^2(H_{0,0,0,0,-1,0}(z) - 2H_{0,0,0,0,-1,-1}(z) \\ + H_{0,0,0,0,-2}(z)) + \frac{13\pi^4}{1080} \log(z)^4 + \frac{C_{3,1}}{6} \log(z)^3 + \frac{D_{3,1}}{2} \log(z)^2 + C_{3,2} \log(z) + D_{3,2} \Big],$$

where

$$(7.64) \quad C_{3,1} = 4\pi^2\zeta_3 + 8\zeta_5,$$

$$(7.65) \quad D_{3,1} = -8\zeta_3^2 + \frac{7\pi^6}{45}.$$

Evaluating $\mathcal{F}_{3,2}(z \rightarrow -1)$ and setting it to 0, we can solve for $C_{3,2}$ and $D_{3,2}$ and get

$$(7.66) \quad C_{3,2} = \frac{17\pi^4\zeta_3}{15} + 4\pi^2\zeta_5 + 8\zeta_7,$$

$$(7.67) \quad D_{3,2} = -16\zeta_{5,3} - 4\pi^2\zeta_3^2 - 96\zeta_5\zeta_3 + \frac{4891\pi^8}{75600}.$$

The same method extends to the general case of a triangle plus a ladder of arbitrary length n , $\mathcal{F}_{3,n}(z)$, which satisfies the equation,

$$(7.68) \quad \square \mathcal{F}_{3,n}(z) = -\mathcal{F}_{3,n-1}(z).$$

This equation is easy to solve as the $\square$ acts easily on the HPLs on the right hand side of the equation. For the harmonic polylogarithms, “unboxing” amounts to adding two 0’s on the vector argument, namely $(z\partial_z)^2 H_{\{0,0,\vec{n}\}}(z) = H_{\vec{n}}(z)$ (remember that $\square = \frac{1}{2}(z\partial_z)^2$). For the constant and $(\log z)^n$ terms, we can use the fact that $(z\partial_z)^2(\log z)^n = n(n-1)(\log z)^{n-2}$. Finally, there are terms of the form $C \log z + D$ that vanish under $(z\partial_z)^2$, which we can fix by the boundary condition that any diagram with branches vanishes as $z \rightarrow -1$. The result then reads,

$$(7.69) \quad \mathcal{F}_{3,n}(z) = (-2)^n \left[8H_{\vec{0}_{2n},0,0,0,0}(z) + 8H_{\vec{0}_{2n},-1,0,0,0}(z) - 16H_{\vec{0}_{2n},-1,-1,0,0}(z) + 8H_{\vec{0}_{2n},0,-1,0,0}(z) \right. \\ - 16\zeta_3 H_{\vec{0}_{2n},-1}(z) + 4\pi^2 \left(H_{\vec{0}_{2n},-1,0}(z) - 2H_{\vec{0}_{2n},-1,-1}(z) + H_{\vec{0}_{2n},-2}(z) \right) \\ + \frac{13\pi^4}{45} \frac{1}{(2n)!} \log^{2n}(z) + 8\zeta_3 \frac{1}{(2n+1)!} \log^{2n+1}(z) \\ \left. + \sum_{k=1}^n (-1)^k \left(\frac{C_k}{[2(n-k)+1]!} \log^{2(n-k)+1}(z) + \frac{D_k}{[2(n-k)]!} \log^{2(n-k)}(z) \right) \right].$$

where the constants C_k and D_k satisfy recursive formulas as detailed in Appendix C.2.3. Let us now consider the following object

$$(7.70) \quad \mathcal{F}_{\text{tri-lad}}(g, z) = g^4 \left(\text{triangle} \otimes \right) - g^6 \left(\text{triangle} - \text{triangle} \otimes \right) + g^8 \left(\text{triangle} - \text{triangle} - \text{triangle} \otimes \right) + \dots$$

where the index ‘tri-lad’ refers to the resummed triangle-ladder series. The resummation can be done in two different ways: either the direct resummation, or using the same differential operator applied on the whole series as was done in the earlier case of the “tree level” ladder resummation. Denoting the resummed series by $\mathcal{F}_{\text{tri-lad}}(g, z)$, we have the differential equation

$$(7.71) \quad \square \mathcal{F}_{\text{tri-lad}}(g, z) + g^2 \mathcal{F}_{\text{tri-lad}}(g, z) = g^4 (\square \mathcal{F}_3(z)),$$

where the right hand side is the differential operator acting on the triangle function $\mathcal{F}_3(z)$ – note that $\mathcal{F}_3(z)$ does not satisfy the Laplace equation, so $(\square \mathcal{F}_3(z))$ here plays a role of the source term. Explicitly, this source term is

$$(7.72) \quad \square \mathcal{F}_3(z) = \frac{4z \left(-2H_{-1,0,0} + H_{0,0,0} - 2\zeta_3 + \pi^2 \right) - 4\pi^2 z H_{-1} + 2\pi^2 z H_0 + 4(z(z+4)+1)H_{0,0}}{(z+1)^2}.$$

The differential equation (7.71) is too complicated for us to solve exactly using the standard toolkit. One approach we can take is to solve the equation in Mellin space, by replacing $f(z) \rightarrow f(\nu) = \int_0^1 dz z^\nu f(z)$. In this space, $\square$ acts as scalar multiplication by some parameter $\frac{1}{2}\nu^2$, and the HPLs turn into harmonic sums. We can then take the inverse transform and express the solution in terms of infinite sums. We can express most of these sums in an analytical form, and take the limit as $g \gg 1$ to get a first order approximation of those we can’t. This procedure is outlined in further detail in Appendix (C.2.1).

For strong g and $z > 0$, we switch to the $F(g, z)$ notation. Recall that $F(g, z)$ differs from $\mathcal{F}(g, z)$ just by the overall factor of $-g^2$ in eq. (7.2), but the F normalization is used in other literature. The final result then goes like

$$(7.73) \quad F_{\text{tri-lad}}(g, z)|_{g \gg 1} \sim -g^4 \frac{(2(z(z+4)+1)\log^2(z) + 2\pi^2 z \log(z) - 4\pi^2 z \log(z+1))}{(z+1)^2} - \frac{2g^4 z (12\text{Li}_3(-z) - 12\text{Li}_2(-z) \log(z) + \log^3(z) - 6\log(z+1) \log^2(z) - 12\zeta_3 + 6\pi^2)}{3(z+1)^2} + \mathcal{O}(g^3).$$

This expression, at the leading order in g , when put into HPL form, turns out to be the source term (7.72), suppressed by a factor of $\frac{1}{g^2}$. We defer the discussion of why this is the case to Appendix (C.2.1). It is worth noting here that the leading behavior at $g \gg 1$ is $F_{\text{tri-lad}} \sim g^4$, which grows much faster than the full Wilson loop behavior $F_{WLI} \sim g$. On the other hand, the series is convergent and the final behavior is weaker than the behavior of the leading term at weak coupling, which is the triangle diagram $\mathcal{F}_3$, which contributes to F as $\sim g^6$. We call such a series *partially summable*.

7.4.3 Box plus ladder

The same procedure can be also used to determine the negative geometries that have the form of a box diagram attached to a ladder:

(7.74)

Just like with the triangle, we are interested in a series of the form

(7.75)
$$\mathcal{F}_{rs,4}(g, z) = g^6 \underbrace{\text{Box}}_{\mathcal{F}_4} - g^8 \underbrace{\text{Box-Ladder}}_{\mathcal{F}_{4,1}} + g^{10} \underbrace{\text{Box-Ladder-Ladder}}_{\mathcal{F}_{4,2}} + \dots$$

We can start with the simplest diagram of the box with one segment, $\mathcal{F}_{4,1}(z)$ which satisfies a differential equation

(7.76)
$$\square \left(\text{Box-Ladder} \right) = \text{Box}$$

or $\square \mathcal{F}_{4,1}(z) = -\mathcal{F}_4(z)$ where $\mathcal{F}_4(z)$ is the box $L = 3$ diagram evaluated in (7.41). The solution for $\mathcal{F}_{4,1}(z)$ is then, given the same vanishing boundary condition,

$$\begin{aligned} \mathcal{F}_{4,1}(z) = - \bigg(& -\frac{32}{3} \pi^2 \zeta_3 H_{-3} - 256 \zeta_3 H_{-4,0} + 128 \zeta_3 H_{-3,0,0} + 48 \pi^2 \zeta_3 H_{0,0,0} + 64 \zeta_3 H_{0,0,0,0} \\ & - 608 \zeta_5 H_{-3} + 496 \zeta_5 H_{0,0,0} + 24 \pi^2 H_{-6} - \frac{86}{9} \pi^4 H_{-4} + 16 \pi^2 H_{-5,0} + 32 \pi^2 H_{-4,-2} - 80 \pi^2 H_{-3,-3} \\ & + 4 \pi^4 H_{-3,-1} + \frac{352}{45} \pi^4 H_{-3,0} + 48 H_{-6,0,0} - \frac{64}{3} \pi^2 H_{-4,-1,0} - \frac{40}{3} \pi^2 H_{-4,0,0} - \frac{32}{3} \pi^2 H_{-3,-2,0} \end{aligned}$$

$$\begin{aligned}
& +96H_{-5,0,0,0}+64H_{-4,-2,0,0}-160H_{-3,-3,0,0}+16\pi^2H_{-3,-1,0,0}+\frac{112}{3}\pi^2H_{-3,0,0,0}+\frac{20}{9}\pi^4H_{0,0,0,0} \\
& -128H_{-4,-1,0,0,0}-16H_{-4,0,0,0,0}-64H_{-3,-2,0,0,0}+96H_{-3,-1,0,0,0,0}+64H_{-3,0,0,0,0,0} \\
& +\frac{16}{3}\pi^2H_{0,0,0,0,0,0}+96H_{0,0,0,0,0,0,0}+176\zeta_3^2\log^2(z)+\frac{208}{15}\pi^4\zeta_3\log(z)+\frac{856}{3}\pi^2\zeta_5\log(z) \\
(7.77) \quad & +1216\zeta_7\log(z)+\frac{505}{567}\pi^6\log^2(z)-608\zeta_{5,3}+160\pi^2\zeta_3^2-1248\zeta_3\zeta_5+\frac{157517\pi^8}{170100}
\end{aligned}$$

The general diagram $\mathcal{F}_{4,n}(z)$ again satisfies the same differential equation $\square\mathcal{F}_{4,n}(z)=-\mathcal{F}_{4,n-1}(z)$ with the solution:

$$\begin{aligned}
\mathcal{F}_{4,n}(z) = & (-2)^{n+4} \left[3H_{\vec{0}_{2n+6}}(z) + 2H_{\vec{0}_{2n,-1},\vec{0}_5}(z) + 3H_{\vec{0}_{2n,-1,-1},\vec{0}_4}(z) - \frac{1}{2}H_{\vec{0}_{2n+1,-1},\vec{0}_4}(z) \right. \\
& -4H_{\vec{0}_{2n+1,-1,-1},\vec{0}_3}(z) + 3H_{\vec{0}_{2n+2,-1},\vec{0}_3}(z) + \frac{3}{2}H_{\vec{0}_{2n+3,-1,0,0}}(z) - 2H_{\vec{0}_{2n,-1,-2},\vec{0}_3}(z) \\
& + 2H_{\vec{0}_{2n+1,-1,-2,0,0}}(z) + \frac{1}{6}\pi^2H_{\vec{0}_{2n+4}}(z) + \frac{7}{6}\pi^2H_{\vec{0}_{2n,-1},\vec{0}_3}(z) + \frac{1}{2}\pi^2H_{\vec{0}_{2n,-1,-1,0,0}}(z) \\
& - 5H_{\vec{0}_{2n,-1,-3,0,0}}(z) - \frac{5}{12}\pi^2H_{\vec{0}_{2n+1,-1,0,0}}(z) - \frac{2}{3}\pi^2H_{\vec{0}_{2n+1,-1,-1,0}}(z) \\
& + \frac{1}{2}\pi^2H_{\vec{0}_{2n+2,-1,0}}(z) + \frac{3}{4}\pi^2H_{\vec{0}_{2n+3,-1}}(z) + 2\zeta_3H_{\vec{0}_{2n+3}}(z) + 4\zeta_3H_{\vec{0}_{2n,-1,0,0}}(z) \\
& - 8\zeta_3H_{\vec{0}_{2n+1,-1,0}}(z) - \frac{1}{3}\pi^2H_{\vec{0}_{2n,-1,-2,0}}(z) + \pi^2H_{\vec{0}_{2n+1,-1,-2}}(z) + \frac{5}{72}\pi^4H_{\vec{0}_{2n+2}}(z) \\
& + \frac{11}{45}\pi^4H_{\vec{0}_{2n,-1,0}}(z) + \frac{1}{8}\pi^4H_{\vec{0}_{2n,-1,-1}}(z) - \frac{5}{2}\pi^2H_{\vec{0}_{2n,-1,-3}}(z) - \frac{43}{144}\pi^4H_{\vec{0}_{2n+1,-1}}(z) \\
& + \frac{31}{2}\zeta_5H_{\vec{0}_{2n+1}}(z) - 19\zeta_5H_{\vec{0}_{2n,-1}}(z) + \frac{3}{2}\pi^2\zeta_3H_{\vec{0}_{2n+1}}(z) - \frac{1}{3}\pi^2\zeta_3H_{\vec{0}_{2n,-1}}(z) \\
& \left. + 11\zeta_3^2H_{\vec{0}_{2n}}(z) + \frac{505\pi^6H_{\vec{0}_{2n}}(z)}{9072} + \sum_{k=1}^n \hat{C}_k \frac{\log(z)^{2(n-k)+1}}{(2(n-k)+1)!} + \sum_{k=1}^n \hat{D}_k \frac{\log(z)^{2(n-k)}}{(2(n-k))!} \right].
\end{aligned}
\tag{7.78}$$

The coefficients $\hat{C}_n$ and $\hat{D}_n$ are determined recursively using the boundary condition $I_{4,n}(z=-1)=0$ in (7.18) and are given in Appendix C.2.3. In principle, we can use the same procedure to ‘extend’ any $L=3$ diagram by attaching a ladder to the marked point and evaluate the all-loop series, but a more detailed understanding of the infinite series is necessary to write some more explicit formulas for the resummed $\mathcal{F}$.

7.5 Leading IR divergence

The Wilson loop with Lagrangian insertion $\mathcal{F}(g,z)$ is an IR finite object. When we integrate over the insertion point z (the frozen loop momentum), we recover the logarithm of the amplitude

which is $1/\epsilon^2$ divergent. The coefficient of this divergence is the cusp anomalous dimension Γ_{cusp} ,

$$(7.79) \quad \log A_4 = \int_{AB} \mathcal{F}(g, z) \propto \frac{\Gamma_{\text{cusp}}(g)}{\epsilon^2} + \mathcal{O}\left(\frac{1}{\epsilon}\right).$$

The cusp anomalous dimension is a fascinating object; see refs. [137, 138, 206, 228, 229, 230, 231] for detailed studies at both weak and strong coupling. In a sense, it is a reduction of the full $\mathcal{F}(g, z)$ to a function of only the coupling, with no kinematical dependence. Fortunately, we have an access to all-loop order expression for this object from integrability [137]. On the other hand, it is not currently known how to obtain $\mathcal{F}(g, z)$ from integrability.

Current amplitude-based methods are not powerful enough to reproduce Γ_{cusp} to all loop orders from first principles. That goal would require constructing an all-loop order integrand and performing integration (of the logarithm). It remains a big open question to derive the formula for Γ_{cusp} from an amplitudes perspective. We would need some deeper insight into exactly how Γ_{cusp} is encoded in the amplitude – and an ability to extract Γ_{cusp} without constructing the all-loop order integrand and integrating it. The negative-geometry expansion, and the construction of the amplitude in momentum-twistor space in general, require a strict $D = 4$ framework. Hence we are not allowed to use dimensional regularization, which in principle requires the knowledge of $D \neq 4$ information about the integrand.

Fortunately, the Wilson loop function $\mathcal{F}(g, z)$ is IR finite and hence it can be evaluated strictly in $D = 4$, for example using our geometry expansion. If we were to integrate over the insertion point, regularization would again be needed. Of course, one is not required to use dimensional-regularization. One can potentially choose another framework such as a deformed amplituhedron, which is linked to Coulomb branch amplitudes [20, 232, 233]. Fortunately, there is another way to extract Γ_{cusp} from $\mathcal{F}(g, z)$, without the need for *any* regularization. The prescription was first found in refs [205, 206]; we use another version of the prescription [77], which introduces an operator $\mathcal{I}$ that implements an integration over z .³

The first step is to determine the action of the $\mathcal{I}$ operator on the $\mathcal{F}$ function as a contour integral in the complex z -plane,

$$(7.80) \quad \mathcal{I}(\mathcal{F}) = -\frac{g^2}{2\pi i} \int_C \frac{dz}{z} \mathcal{F}.$$

³The operator $\mathcal{I}$ should not be confused with integrands $\mathcal{I}$ elsewhere in this chapter.

Assuming that $\mathcal{F}$ only has branch cuts on the negative z -axis and is analytic elsewhere, the contour can be deformed to a unit circle around the origin,

$$(7.81) \quad \mathcal{I}(\mathcal{F}) = -\frac{g^2}{2\pi} \int_{-\pi}^{\pi} d\phi \mathcal{F}(e^{i\phi}).$$

This function then enters a simple first order differential equation for $\Gamma_{\text{cusp}}(g)$,

$$(7.82) \quad g \frac{\partial}{\partial g} \Gamma_{\text{cusp}}(g) = -8 \mathcal{I}(\mathcal{F}(g, z)).$$

This equation does not require any regularization: It can be solved for $\Gamma_{\text{cusp}}(g)$ without using dimensional regularization, staying entirely in $D = 4$. In perturbation theory, we expand

$$(7.83) \quad \Gamma_{\text{cusp}}(g) = \sum_{L=1}^{\infty} g^{2L} \Gamma_{\text{cusp}}^{(L)},$$

so a contribution to the L -loop cusp anomalous dimension is given by dividing by $2L$,

$$(7.84) \quad \Gamma_{\text{cusp}}^{(L)} = -\frac{4}{L} \mathcal{I}(\mathcal{F}^{(L-1)}(z)) = \frac{2}{\pi L} \int_{-\pi}^{\pi} d\phi \mathcal{F}^{(L-1)}(e^{i\phi}).$$

We quickly review below how $\mathcal{I}$ from (7.81) acts on the functions we see in negative geometries.

All the individual $\mathcal{F}(z)$ we consider here contain HPLs, natural logs, and transcendental constants. For constants, we simply have $\mathcal{I}[c] = -g^2 c$. The contour that we use for $\mathcal{I}$ avoids the negative z -axis that logs and their powers have branch cuts on, so we can integrate them along the circle as

$$(7.85) \quad \mathcal{I}(\log(z)^n) = -\frac{g^2}{2\pi} \int_{-\pi}^{\pi} (i\phi)^n d\phi = \frac{-g^2 \pi^n \cos(\frac{\pi n}{2})}{n+1}$$

Finally, we have HPLs that show up in our expressions, starting with single-cycle diagrams. The HPLs we find either have z in their argument, or the spurious letter $b(4) = \frac{1+i\sqrt{z}}{1-i\sqrt{z}}$. Let's first discuss how to deal with the former, for which we use the notation $H_{-m_n, \dots, -m_1}(z) = H_{-m_n, \dots, -m_1}$ (the argument is omitted for brevity). Only the HPLs that have 0 as their rightmost index are singular at $z \rightarrow 0$, so we first use the shuffle identities [223] to trade them for logs. For example, we can re-write $H_{-m,0}(z)$ as

$$(7.86) \quad H_{-m,0}(z) = \log(z) H_{-m} - m H_{0,-m}(z),$$

and $H_{-m,0,0}(z)$ as

$$(7.87) \quad H_{-m,0,0}(z) = \frac{1}{2} \log^2(z) H_{-m}(z) - m \log(z) H_{0,-m}(z) + \frac{m(m+1)}{2} H_{0,0,-m}(z),$$

and so forth. After getting rid of all the trailing 0's, we have powers of logs multiplied by HPLs that are regular at $z = 0$. Thus, all the remaining HPLs have a convergent Taylor series expansion for $z \in (-1, 1)$. So we can apply eq. (5.4) of ref. [206] to these terms, obtaining

$$(7.88) \quad \mathcal{I}(\log^a(z) H_{-m_n, \dots, -m_1}) = g^2 (-1)^{-a} a! \sum_{k=0}^a \frac{\pi^{k-1}}{k!} \sin\left(\frac{\pi k}{2}\right) H_{-(a+1-k+m_n), \dots, -m_1}(-1).$$

Starting at four loops, we also get diagrams with the spurious letter $b(4)$, such as (7.46) and (7.48). In such cases, we can find a combination of diagrams that is free of the spurious letter. We first apply the above procedure to find how $\mathcal{I}$ acts on the combination. Then we subtract a piece we know, using the independence of $\mathcal{I}$ on which node is frozen. For example, consider (7.46). The following sum can be expressed in a form that only contains HPLs with z in the argument

$$(7.89) \quad - \text{Diagram 1} + \frac{1}{2!} \text{Diagram 2}.$$

While the diagram on the left also contains a spurious letter, $\mathcal{I}$ should act on it the same way it does on (7.39), which we know how to do using (7.88). Then, we get

$$(7.90) \quad \mathcal{I} \left(\frac{1}{2!} \text{Diagram 2} \right) = \mathcal{I} \left(- \text{Diagram 1} + \frac{1}{2!} \text{Diagram 2} \right) + \mathcal{I} \left(\text{Diagram 3} \right)$$

Similarly, (7.48) can be combined with the 2-cycle diagram we get from above to form a spurious letter free combination that we can easily integrate as follows

$$(7.91) \quad \mathcal{I} \left(\frac{1}{2!} \text{Diagram 2} - \frac{1}{3!} \text{Diagram 4} \right) = 40\zeta_3^2 + \frac{4\pi^6}{135}.$$

Once we determine the 2-cycle piece, we can feed it into the equation above to extract the 3-cycle one.

The purpose of the remainder of this section is to compare how various negative geometries contribute to $\Gamma_{\text{cusp}}(g)$, and to see whether any of them dominate over the others.

7.5.1 Three-loop Γ_{cusp}

The negative geometry expansion up to three loops reads,

$$\begin{aligned}
\mathcal{F}(g, z) = & \bigotimes - g^2 \bigotimes \text{---} \bullet \\
& + g^4 \left\{ \bigotimes \text{---} \bullet \text{---} \bullet + \frac{1}{2!} \bigotimes \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \end{array} - \frac{1}{2!} \begin{array}{c} \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array} \bigotimes \right\} \\
& - g^6 \left\{ \bigotimes \text{---} \bullet \text{---} \bullet \text{---} \bullet + \bigotimes \begin{array}{c} \bullet \text{---} \bullet \\ \diagup \quad \diagdown \\ \bullet \end{array} + \frac{1}{2!} \bigotimes \begin{array}{c} \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array} \text{---} \bullet \right. \\
& + \frac{1}{3!} \bigotimes \begin{array}{c} \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array} \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \end{array} - \frac{1}{2!} \begin{array}{c} \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array} \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \end{array} \bigotimes - \frac{1}{2!} \begin{array}{c} \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array} \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \end{array} \bigotimes \\
& \left. - \frac{1}{2!} \begin{array}{c} \bullet \text{---} \bullet \\ \diagup \quad \diagdown \\ \bullet \end{array} \begin{array}{c} \bullet \text{---} \bullet \\ \diagup \quad \diagdown \\ \bullet \end{array} \bigotimes + \frac{1}{2!} \begin{array}{c} \bullet \text{---} \bullet \\ \diagup \quad \diagdown \\ \bullet \end{array} \begin{array}{c} \bullet \text{---} \bullet \\ \diagup \quad \diagdown \\ \bullet \end{array} \bigotimes + \frac{1}{2!} \begin{array}{c} \bullet \text{---} \bullet \\ \diagup \quad \diagdown \\ \bullet \end{array} \begin{array}{c} \bullet \text{---} \bullet \\ \diagup \quad \diagdown \\ \bullet \end{array} \bigotimes - \frac{1}{3!} \begin{array}{c} \bullet \text{---} \bullet \\ \diagup \quad \diagdown \\ \bullet \end{array} \begin{array}{c} \bullet \text{---} \bullet \\ \diagup \quad \diagdown \\ \bullet \end{array} \begin{array}{c} \bullet \text{---} \bullet \\ \diagup \quad \diagdown \\ \bullet \end{array} \bigotimes \right\}.
\end{aligned}
\tag{7.92}$$

At $L = 1$, a single diagram represents the whole result. At $L = 2$, we have three different diagrams, two zero-cycle and one one-cycle, which give the contributions summarized in table 7.1.

We can see from table 7.1 that the tree contributions dominate and the one-cycle diagram gives only a small correction to Γ_{cusp} . At $L = 3$, we have 11 diagrams in table 7.2. While numerically the tree diagrams still dominate overall, the one-cycle graphs give a smaller contribution than the two-cycle graphs. It makes sense to divide the result into the contributions to π^6 and ζ_3^2 as shown in table 7.3, and look at the stratification of the result in both sectors separately. Here we see that in the even zeta, ie. π^6 sector, the tree graphs dominate, one-cycle graphs are sub-leading, two-cycle graphs are sub-sub-leading and the three-cycle graph gives a smallest contribution. In the ζ_3^2 sector, the tree graphs do not contribute at all, and the one-cycle graphs dominate. The two-cycle graphs are sub-leading and the three-cycle graph again gives a smallest contribution. We see that the sign of the π^6 contribution is always opposite to the sign of the ζ_3^2 contribution.

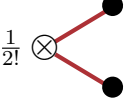
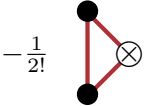
Graph	$\Gamma_{\text{cusp}}^{(3)}$	numerical $\Gamma_{\text{cusp}}^{(3)}$	Fraction $\frac{\Gamma_{\text{diagram}}^{(3)}}{\Gamma_{\text{actual}}^{(3)}}$
	$\frac{32\pi^4}{45}$	69.27	0.545
$\frac{1}{2!}$ 	$\frac{16\pi^4}{45}$	34.63	0.273
All tree graphs	$\frac{16\pi^4}{15}$	103.90	1.09
$-\frac{1}{2!}$  Only 1-cycle graph	$-\frac{4\pi^4}{45}$	-8.66	-0.09
Full result	$\frac{44\pi^4}{45}$	95.24	1

TABLE 7.1. Contributions of individual negative geometries to the three-loop coefficient of the cusp anomalous dimension, $\Gamma_{\text{cusp}}^{(3)}$. We see that the one-cycle geometry contribution to Γ_{cusp} is numerically suppressed in comparison to the tree graphs contribution.

As a result, we can conclude here that at least up to $L = 3$ we see the dominance of lower-cycle graphs over the higher-cycle graphs in the contribution to Γ_{cusp} if separated into the even and odd zeta sectors.

For $L = 4$ and higher, we do not have a systematic way how to evaluate $\mathcal{F}(z)$ and extract the cusp anomalous dimension. We can only do it for diagrams which can be solved using the differential equation discussed in section 7.3.

Graph	$\Gamma_{\text{cusp}}^{(4)}$	Numerical $\Gamma_{\text{cusp}}^{(4)}$	Fraction $\frac{\Gamma_{\text{graph}}^{(4)}}{\Gamma_{\text{actual}}^{(4)}}$
$-\otimes$	$-\frac{136}{315}\pi^6$	-415.075	0.443
$-\otimes$	$-\frac{136}{315}\pi^6$	-415.075	0.443
$-\frac{1}{2!}\otimes$	$-\frac{8}{35}\pi^6$	-219.746	0.234
$-\frac{1}{3!}\otimes$	$-\frac{8}{105}\pi^6$	-73.249	0.078
All tree graphs	$-\frac{368}{315}\pi^6$	-1123.15	1.1981
$\frac{1}{2!}$	$\frac{176\pi^6}{2835} - 24\zeta_3^2$	25.005	-0.0267
	$\frac{352\pi^6}{2835} - 48\zeta_3^2$	50.01	-0.0533
$\frac{1}{2!}$	$\frac{176\pi^6}{2835} - 24\zeta_3^2$	25.005	-0.0267
$\frac{1}{2!}$	$-\frac{164}{2835}\pi^6$	-55.61	0.0593
All 1-cycle graphs	$\frac{540\pi^6}{2835} - 96\zeta_3^2$	44.41	-0.047
$-\frac{1}{2!}$	$\frac{4\pi^6}{189} + 24\zeta_3^2$	55.025	-0.0587
$-\frac{1}{2!}$	$\frac{4\pi^6}{189} + 24\zeta_3^2$	55.025	-0.0587
All 2-cycle graphs	$\frac{8\pi^6}{189} + 48\zeta_3^2$	110.05	-0.117
$\frac{1}{3!}$ Only 3-cycle graph	$\frac{8\pi^6}{945} + 16\zeta_3^2$	31.26	-0.0333
Full result	$-8\left(\frac{73\pi^6}{630} + 4\zeta_3^2\right)$	-937.43	1

TABLE 7.2. Contributions of individual negative geometries to $\Gamma_{\text{cusp}}^{(4)}$. The tree graphs give the dominant contribution. The contribution of one-cycle, two-cycle and three-cycle graphs are numerically similar.

Graph	π^6 coefficient	Fraction $\frac{\text{graph}}{\text{actual}}$	ζ_3^2 coefficient	Fraction $\frac{\text{graph}}{\text{actual}}$
$-\otimes$	$-\frac{136}{315}$	0.466	0	0
$-\otimes$	$-\frac{136}{315}$	0.466	0	0
$-\frac{1}{2}\otimes$	$-\frac{8}{35}$	0.247	0	0
$-\frac{1}{3!}\otimes$	$-\frac{8}{105}$	0.082	0	0
All tree graphs	$-\frac{368}{315}$	1.26	0	0
$\frac{1}{2!}$	$\frac{176}{2835}$	-0.067	-24	0.75
	$\frac{352}{2835}$	-0.134	-48	1.5
$\frac{1}{2!}$	$\frac{176}{2835}$	-0.067	-24	0.75
$\frac{1}{2!}$	$-\frac{164}{2835}$	0.062	0	0
All 1-cycle graphs	$\frac{540}{2835}$	-0.205	-96	3
$-\frac{1}{2!}$	$\frac{4}{189}$	-0.023	24	-0.75
$-\frac{1}{2!}$	$\frac{4}{189}$	-0.023	24	-0.75
All 2-cycle graphs	$\frac{8}{189}$	-0.046	48	-1.5
$\frac{1}{3!}$	$\frac{8}{945}$	-0.009	16	-0.5
Only 3-cycle graph				
Full result	$-\frac{292}{315}$	1	-32	1

TABLE 7.3. When dividing the contributions to Γ_{cusp} into ζ_3^2 and π^6 parts, the hierarchy becomes more clear. In both sectors separately, the higher cycle graphs are suppressed.

One of these solvable diagrams is the following

$$(7.93) \quad \mathcal{F}_{3,2} = \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array} \text{---} \bullet \text{---} \bullet \text{---} \otimes$$

We will discuss this diagram in more details as a part of the fully solvable series in the next section. But here we just note that the contribution to Γ_{cusp} is

$$(7.94) \quad \Gamma_{\text{cusp}} = \frac{4g^{10}}{5} \left(-\frac{448}{5}\zeta_{5,3} - \frac{1}{3}32\pi^2\zeta_3^2 - 640\zeta_5\zeta_3 + \frac{9944\pi^8}{70875} \right)$$

where a **multiple-zeta value** (MZV) $\zeta_{5,3}$ appears for the first time. For this object, we use the convention that a multiple-zeta value of depth- k is equal to

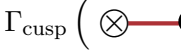
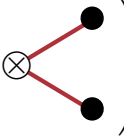
$$(7.95) \quad \zeta_{m_1, m_2, \dots, m_k} = \sum_{n_1 > n_2 > \dots > n_k > 0} \frac{1}{n_1^{m_1} \dots n_k^{m_k}}.$$

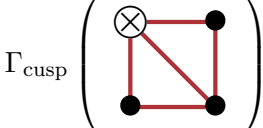
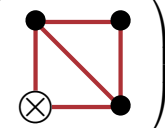
For n ladders attached to the triangle, the Γ_{cusp} contribution can be seen in (C.70) and for $n \geq 2$ we get irreducible multiple-zeta values. From the point of view of the full Γ_{cusp} , the term with $\zeta_{5,3}$ is spurious and cancels in the sum over all negative geometries at $L = 4$. We strongly suspect that *any representation* of the amplitude would generate MZVs in the intermediate steps, and only the final result is free of them. Understanding the mechanism of the cancellation of MZVs might be an important clue to how the integrability arises in the context of amplitudes, and how is the presence of MZVs in violation with integrability.

7.5.2 Gamma cusp of infinite classes

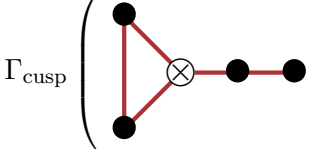
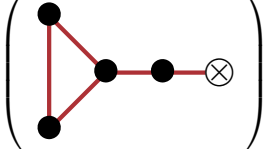
The task of calculating the Γ_{cusp} contribution for the infinite classes of negative geometries is simpler than for the full $F(g, z)$ function. The main reason is that for the purpose of Γ_{cusp} , it does not matter where the marked point is located in the graph. This is obvious from the fact that Γ_{cusp} can be also recovered from the integration over all of the loops of the integrand (with no marked points) as the leading IR divergence, though this is not the method we use here. Hence for this purpose, labeling a marked point in the graph is just an intermediate step (in the end all loops are integrated over), and the result for Γ_{cusp} of the given graph does not depend on where this marked point was located.

This also explains why different diagrams contribute to Γ_{cusp} in the same way in $L = 2$ and $L = 3$ tables (modulo symmetry factors). In particular,

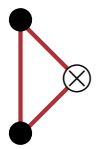
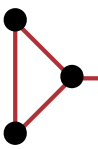
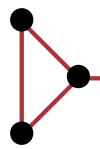
$$(7.96) \quad \Gamma_{\text{cusp}} \left(\text{diag}_1 \right) = \Gamma_{\text{cusp}} \left(\text{diag}_2 \right) = \frac{32\pi^4 g^6}{45},$$



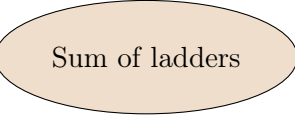
$$(7.97) \quad \Gamma_{\text{cusp}} \left(\text{diag}_3 \right) = \Gamma_{\text{cusp}} \left(\text{diag}_4 \right) = \left(24\zeta_3^2 + \frac{4\pi^6}{189} \right) g^8.$$



Similarly, we can apply this method on the triangle plus ladder diagrams in the form

$$(7.98) \quad \Gamma_{\text{cusp}} \left(\text{diag}_5 \right) = \Gamma_{\text{cusp}} \left(\text{diag}_6 \right).$$



We can also apply the same relation for the full sum of all graphs,

$$(7.99) \quad \Gamma_{\text{cusp}} \left(g^4 \text{diag}_7 - g^6 \text{diag}_8 + g^8 \text{diag}_9 + \dots \right) =$$




$$\Gamma_{\text{cusp}} \left(g^4 \text{diag}_7 \times \text{blob} \right)$$


where the “Ladder” blob corresponds to the series in (7.51). The $F(g, z)$ function for both the triangle and the resummed ladder diagrams are known (7.27), (7.53), hence the differential equation for Γ_{cusp} contribution is

$$(7.100) \quad g \frac{\partial}{\partial g} \Gamma_{\text{cusp}} = -8I[g^4 \mathcal{F}_3(z) \mathcal{F}_{\text{ladder}}(g, z)],$$

where the I operation acts on the product of the triangle, given by $\mathcal{F}_3(z)$ and the resummed ladder function $\mathcal{F}_{\text{ladder}}(g, z)$. There is an addition factor of g^4 which originates from g^6 factor of the triangle, but divided by $1/g^2$ to avoid double counting of the marked point (which is already counted in the resummed ladder). We leave the details of this computation to Appendix C.1. In the end, we find that $\Gamma_{\text{cusp}}(g^4 \mathcal{F}_3(z) \mathcal{F}_{\text{ladder}}(g, z)) \propto g^5 + \mathcal{O}(g^4)$ at $g \gg 1$ (see (C.5) for the full result). While this is suppressed by a factor of $\frac{1}{g}$ compared to the leading term in the weak coupling expansion, namely $F_3(g, z) \propto g^6$, it still shows polynomial growth in g and thus isn't resumable. As mentioned before, this is an example of a partially summable class. Attaching the ladder class to the single-cycle box, we again find suppression only by $\frac{1}{g}$ compared to the leading term. Namely, we find $\Gamma_{\text{cusp}}(g^6 \mathcal{F}_4(z) \mathcal{F}_{\text{ladder}}(g, z)) \propto g^7 + \mathcal{O}(g^6)$ at $g \gg 1$ (see (C.8) for the full result). This suggests that one might need to find a larger class of diagrams that includes these two classes as a subset in order to get suppression in g at strong coupling.

7.5.3 Geometric Series

At the moment, we have examples of *summable* negative geometries: these are ladders or all tree graphs, where the $g \gg 1$ is $\Gamma_{\text{cusp}} \sim g$ as in the full Wilson loop, and then partially summable geometries that we discussed above. There are also classes of *super-summable* geometries which are suppressed at large g : these are geometric series. In fact, any negative geometry participates in such a series. For example, we can consider a following sequence,

$$(7.101) \quad \bigotimes - g^2 \bigotimes \text{---} \bullet + \frac{g^4}{2!} \bigotimes \begin{array}{c} \bullet \\ \diagup \diagdown \\ \bullet \end{array} - \frac{g^6}{3!} \bullet \text{---} \bigotimes \begin{array}{c} \bullet \\ \diagup \diagdown \\ \bullet \end{array} + \cdots + \frac{g^{2n}}{n!} \begin{array}{c} \bullet \cdots \bullet \\ \diagup \quad \diagdown \\ \bigotimes \\ \diagdown \quad \diagup \\ \bullet \end{array}$$

with the $\mathcal{F}(g, z)$ function

$$(7.102) \quad \mathcal{F}_{\text{sun}}(g, z) = \sum_{n=0}^{\infty} \frac{(-g^2)^n}{n!} (\pi^2 + \log(z)^2)^n.$$

This sums into an exponential, namely

$$(7.103) \quad \mathcal{F}_{\text{sun}}(g, z) = e^{-g^2(\log^2(z) + \pi^2)}.$$

It is easy to see that at $g \gg 1$, $\mathcal{F}$ is exponentially suppressed. We can easily take the $\mathcal{I}$ integral of $\mathcal{F}_{\text{sun}}(g, z)$ and compute

$$(7.104) \quad \Gamma_{\text{cusp}}(\mathcal{F}_{\text{sun}}(g, z))_{g \gg 1} \sim -\frac{4 \log(\pi g)}{\pi^2}.$$

We can also attach 1-cycle diagrams together in a similar fashion and get another series that sums into an exponential. The simplest example is the following

$$(7.105) \quad \otimes - \frac{g^4}{2!} \text{ (triangle)} + \frac{g^8}{8} \text{ (bow-tie)} - \frac{g^{12}}{48} \text{ (pentagon)} + \frac{g^{16}}{384} \text{ (heptagon)} \dots$$

which contains the triangle function $\mathcal{F}_3(z)$ and its powers. We can think about the first term as the zero power and then it sums to

$$(7.106) \quad \mathcal{F}(g, z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n n!} (g^4 \mathcal{F}_3(z))^n = e^{-\frac{1}{2}(g^4 \mathcal{F}_3(z))}.$$

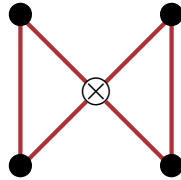
The extraction of Γ_{cusp} is trickier. We note that for all points in the integration domain $\phi \in -\pi, \pi$ relevant for the Γ_{cusp} contribution, $e^{-\frac{g^4}{2} \mathcal{F}_3(e^{i\phi})}$ is exponentially suppressed. $\mathcal{F}_3(e^{i\phi})$ has a global minimum at $\phi = 0$ in this domain, so we can do a saddle point approximation and compute (for large g)

$$(7.107) \quad \mathcal{I}[e^{-\frac{1}{2}(g^4 \mathcal{F}_3(z))}]_{g \gg 1} \sim \frac{e^{-\frac{1}{2}g^4 \mathcal{F}_3(1)}}{\sqrt{2\pi \mathcal{F}_3''(1)}} + \mathcal{O}\left(\frac{e^{-g^4}}{g^4}\right).$$

This gives us the Γ_{cusp} contribution

$$(7.108) \quad \Gamma_{\text{cusp}} \left(\sum_{n=0}^{\infty} \frac{(-1)^n}{2^n n!} (g^4 \mathcal{F}_3(z))^n \right)_{g \gg 1} \sim -\frac{0.87 e^{-1.2g^4}}{g^4} + \mathcal{O}\left(\frac{e^{-g^4}}{g^8}\right),$$

which is heavily suppressed in large g . We can also compute the individual terms in this expansion using the HPL shuffle identities. Let's see how this works for the bow-tie diagram.



Clearly this simply evaluates to $\mathcal{F}_3(z)^2$. In order to calculate $\mathcal{I}[\mathcal{F}_3(z)^2]$ we must use the shuffle identities for harmonic polylogarithms:

$$(7.109) \quad H_{\vec{p}}(z)H_{\vec{q}}(z) = \sum_{\vec{r}=\vec{p} \sqcup \vec{q}} H_{\vec{r}}(z),$$

where $\sqcup$ denotes the shuffle product of the two ordered lists $\vec{p}$ and $\vec{q}$. We can then re-write $\mathcal{F}_3(z)^2$ to be linear in HPLs and calculate $\Gamma_{\text{cusp}}(\mathcal{F}_3(z)^2)$

$$(7.110) \quad \begin{aligned} & \frac{4g^{10}}{5} \left[-512\zeta_3(\zeta_{3,2}-2\zeta_{2,1,2}-3\zeta_{2,2,1}) + \frac{64}{3}\pi^2(4(\zeta_{2,4} + \zeta_{3,3} + 3\zeta_{2,3,1} + 4\zeta_{3,1,2} + 6\zeta_{3,2,1} + 9\zeta_{4,1,1}) \right. \\ & + \zeta_3^2) - 256\zeta_{5,3} - 1024\zeta_{6,2} - 512\zeta_{2,2,4} - 1536\zeta_{2,3,3} - 3584\zeta_{2,4,2} - 7168\zeta_{2,5,1} - 1024\zeta_{3,1,4} - 2560\zeta_{3,2,3} \\ & - 5632\zeta_{3,3,2} - 11264\zeta_{3,4,1} - 3072\zeta_{4,1,3} - 6144\zeta_{4,2,2} - 12288\zeta_{4,3,1} - 5632\zeta_{5,1,2} - 10752\zeta_{5,2,1} \\ & - 7680\zeta_{6,1,1} - 1024\zeta_{2,1,2,3} - 3072\zeta_{2,1,3,2} - 6144\zeta_{2,1,4,1} - 1536\zeta_{2,2,1,3} - 6656\zeta_{2,2,3,1} - 4608\zeta_{2,3,1,2} \\ & - 3072\zeta_{3,1,1,3} - 7168\zeta_{3,1,2,2} - 9216\zeta_{3,2,1,2} - 15360\zeta_{3,2,2,1} - 18432\zeta_{3,3,1,1} - 13824\zeta_{4,1,1,2} \\ & \left. - 9216\zeta_{4,1,1,1} - 7680\zeta_{2,3,2,1} - 23040\zeta_{4,1,2,1} - 27648\zeta_{4,2,1,1} - 36864\zeta_{5,1,1,1} + 5120\zeta_3\zeta_5 - \frac{10196\pi^8}{14175} \right]. \end{aligned}$$

Note that while the Γ_{cusp} contribution of the single cycle diagrams we have considered above had at most depth-2 multiple zeta values, here we have depth-4. We can see from the shuffle identities that each cycle we add increases the maximum depth of MZVs that appear in the Γ_{cusp} contribution by 2. In general, a diagram with L 1-cycles glued together will contribute to Γ_{cusp} with at most n^L depth multiple zeta values, where n is the maximum depth of the HPLs that appear in the $\mathcal{F}(z)$ function of the 1-cycle diagram at hand. As these multiple zetas do not appear in the full Γ_{cusp} , there must be some mechanism that cancels them out. It is possible to re-write these higher depth zetas using relations between multiple-zeta values. For example, the expression above can be written as

$$(7.111) \quad \Gamma_{\text{cusp}}(\mathcal{F}_3(z)^2) \sim \frac{4g^{10}}{5} \left(\frac{406888}{45}\zeta_8 + 384\zeta_2\zeta_3^2 - 7680\zeta_3\zeta_5 - \frac{13056}{5}\zeta_{5,3} \right),$$

in terms of lower depth zetas. However, we still have the depth-2 $\zeta_{5,3}$ that we can't reduce further, which doesn't appear in the full cusp.

7.6 Conclusion

In this chapter, we continued our study of the 4-point planar $\mathcal{N} = 4$ SYM amplitude in the Amplituhedron framework. The logarithm of the amplitude is dual to the Wilson loop with a Lagrangian insertion (WLI), an IR finite object, a perfect laboratory to explore analytic structures of amplitudes. The negative geometry expansion is a new basis of objects, analogous to the Amplituhedron, which makes the IR properties manifest.

In this chapter, we pushed the exploration of these objects further by computing all negative geometries that contribute to the three-loop WLI. We showed that the number of internal cycles of associated graphs is closely related to the depth of polylogarithms. We also discussed how these negative geometries contribute to the leading IR divergence given by the cusp anomalous dimension, and found some evidence of the low-cycle dominance. Furthermore, we resummed certain classes of one-cycle negative geometries to all loop orders, and provided strong coupling expansions of these formulas.

There are many natural questions which directions to explore further: the four-loop computation of $\mathcal{F}_{WLI}^{4\text{-loop}}$ would require new tailored methods for IR finite objects and negative geometries, as for the more general integration technology the four-loops is not within the reach. One option is to find new differential equations, beyond the Laplace equation, for more general negative geometries. Another direction is to do more resummations. Our work only scratched the surface of this important question, and only considered special classes of one-cycle geometries. The real breakthrough would be to sum *all* one-cycle geometries and study the properties of such object, mainly the strong coupling expansion. We know that the summation of all zero-cycle (tree) negative geometries has a good strong coupling behavior, and it would be very interesting to see if this extends to the set of all geometries with fixed number of cycles. This would also necessarily require a new integration method, likely new set of differential equations.

The most important open problem is related to the cusp anomalous dimension $\Gamma_{\text{cusp}}(g)$. This object is predicted from integrability via the BES equation [137], but how to derive it from amplitudes is a big challenge, which would very likely uncover an imprint of $\mathcal{N} = 4$ SYM integrability in amplitudes. While this seems like an extremely difficult problem, even in the context of negative geometries, we have one particular technical aspect to study and understand – the presence of only single zeta values in $\Gamma_{\text{cusp}}(g)$. As we have seen in section 7.5, the general negative geometries (even

with only one cycle) do contribute with multiple zeta values to Γ_{cusp} . These MZVs must cancel while summing over multiple (or all) negative geometries. Exploring the precise mechanism how these cancellations happen would be an important first step towards the better understanding of integrability constraints on negative geometries and amplitudes in general.

CHAPTER 8

Summary and Outlook

Some of the most active research programs in modern high energy theoretical physics have branched off from forays into $\mathcal{N} = 4$ sYM. Even if it turns out the world we live in isn't supersymmetric at any scale, such explorations will still have contributed to our understanding of gauge theories. The work we reviewed in this dissertation has been centered around one such corner of the supersymmetry research space, namely its amplitudes and their relation to mathematical structures.

In the planar limit, $\mathcal{N} = 4$ sYM tree amplitudes and loop integrands can be described by positive geometries. This story is complete at the tree-level and an active area of research for the loop-level. In chapters 3 and 4, we reviewed how one would have to modify the story in the context of on-shell diagrams if one loses one of the two properties: lack of UV poles, and planarity. Furthermore, we found that these two aspects are connected, since planar on-shell diagrams often turn into non-planar ones at UV poles. On the non-planar side, we were able to find regions in the Grassmannian that we could associate with on-shell diagrams. These are formed by unions of positive Grassmannia and they classify under pseudo-positive geometries in the math literature. While enriching our understanding of the relationship between amplitudes and positive geometries, these two works have also spawned new exciting questions, as is often the case in physics research.

Going back to the planar limit, the past two decades have been full of hints that $\mathcal{N} = 4$ sYM might be a solvable theory. In particular, AdS/CFT tells us we can work out quantities of sYM in strong coupling, meaning the infinite number of terms we see in weak coupling should resum into something we can make sense of. The conjectured integrability of sYM, as evidenced by its Yangian symmetry, corroborates this account further.

In particular, sYM amplitudes have a dual description in terms of Wilson-loops and both objects have a universal structure of divergences controlled by a quantity called the cusp anomalous dimension, Γ_{cusp} for short. The Yangian symmetry of sYM generates an infinite tower of conserved charges, which makes it possible to write an integral equation that determines this object to all orders in coupling asymptotically. This is an example of integrability. On the amplitudes side

of things, the same Yangian appears as manifest when the tree-level amplitude is written as an integral over the Grassmannian. However, we don't currently have a unified picture that connects the integrability side of the story with the mathematical structures that we have discussed in the earlier chapters. Such a connection would possibly allow us to determine objects with kinematic dependence in the theory to all order. Thus, this makes searching for consequences of integrability in amplitudes a well motivated program. In chapters 6 and 7, we extended an application of the positive geometry technology to the computation of Wilson loop ratios and Γ_{cusp} , the one object we know of that talks to both amplitudes and integrability in a clean way. This work pushes the program further and helps us develop tools we then use to collect "strong coupling data" from the amplitudes side.

In this dissertation, we have started with a brief review of the connection between amplitudes and positive geometries, and added our own contributions to this growing program. This journey took us from a diverse set of topics, such as exploring what happens to on-shell diagrams on UV poles, associating new mathematical structures with non-planar diagrams, using positive geometries to construct IR-safe objects, and performing partial resummations in the planar limit to search for integrability. Our results help outline new exciting questions to tackle in the future, and hopefully the work behind this dissertation has uncovered new facts about $\mathcal{N} = 4$ sYM for the high energy theoretical physics community to learn and put to use.

APPENDIX A

Appendix for Chapter 4

In this appendix, we recite some of the proofs written by the author's co-authors in [234] to support the claims and theorems found in chapter 4.

A.1 The factorization formula

LEMMA 4.4.0.2 (Factorization of the form). *Let T be a set of $n - 2$ triplets on $[n]$. Suppose R is a valid subset of T . Then*

$$(A.1) \quad f_T = f_R \times f_{(T \setminus R) \cup P} \times \prod_{i=1}^r (a_i a_{i+1}),$$

where $A = \{a_1, \dots, a_r\}$ consists of the indices appearing both in R and T (we set $a_{r+1} = a_1$) and $P = \{(a_1, a_i, a_{i+1}) | i \in \{2, \dots, r-1\}\}$ is a set of triplets defining a triangulation of a polygon with vertices A .

PROOF. Let $S = T \setminus R$ to simplify the notation. Since R is a valid subset, it consists of $m - 2$ triplets on m indices. Then, S must contain the remaining $n - m$ triplets of T , as well as at least $n - m + r$ vertices. One must have $r \geq 2$, as otherwise T would fail the non-vanishing condition.

Then, the matrix M_T can be written as

$$(A.2) \quad M_T = \begin{pmatrix} M_{R,1} & M_{R,2} & 0 \\ 0 & M_{S,1} & M_{S,2} \end{pmatrix},$$

where $M_{R,1}$ is $(m-2) \times (m-r)$, $M_{R,2}$ is $(m-2) \times r$, $M_{S,1}$ is $(n-m) \times r$, and $M_{S,2}$ is $(n-m) \times (n-m)$.

Then, computing the form f_T one finds

$$(A.3) \quad \begin{aligned} f_T &= \frac{[\det(M_{T,ab})/(ab)]^2}{\prod_{\tau \in T} (\tau_1 \tau_2)(\tau_2 \tau_3)(\tau_3 \tau_1)} \\ &= \frac{[\det(M_{R,ab})/(ab)]^2}{\prod_{\tau \in R} (\tau_1 \tau_2)(\tau_2 \tau_3)(\tau_3 \tau_1)} \times \frac{[\det(M_{S,2})]^2}{\prod_{\tau \in S} (\tau_1 \tau_2)(\tau_2 \tau_3)(\tau_3 \tau_1)} \\ &= f_R \times \frac{[\det(M_{S,2})]^2}{\prod_{\tau \in S} (\tau_1 \tau_2)(\tau_2 \tau_3)(\tau_3 \tau_1)}, \end{aligned}$$

where we begin to see the factorization of the form into a piece dependent on the valid subset of triplets and some term dependent on the rest. This latter term may be rewritten further by introducing a set of extra triplets $P = \{(a_1, a_i, a_{i+1}) | i \in \{2, \dots, r-1\}\}$ that form an arbitrary polygon over the indices a_i shared by R and S . Then,

$$\begin{aligned}
\det(M_{P \cup S, a_1 a_2}) / (a_1 a_2) &= \det \begin{pmatrix} M_{P, a_1 a_2} & 0 \\ M_{S, 1, a_1 a_2} & M_{S, 2} \end{pmatrix} / (a_1 a_2) \\
&= \det(M_{S, 2}) \times [\det(M_{P, a_1 a_2}) / (a_1 a_2)] \\
&= \det(M_{S, 2}) \times \prod_{i=2}^{r-1} (a_1 a_i).
\end{aligned}
\tag{A.4}$$

This allows us to rewrite

$$\begin{aligned}
f_T &= f_R \times \frac{[\det(M_{S, 2})]^2}{\prod_{\tau \in S} (\tau_1 \tau_2) (\tau_2 \tau_3) (\tau_3 \tau_1)} \\
&= f_R \times \frac{[\det(M_{P \cup S, a_1 a_2}) / (a_1 a_2)]^2}{\prod_{\tau \in S} (\tau_1 \tau_2) (\tau_2 \tau_3) (\tau_3 \tau_1) \times \prod_{i=2}^{r+1} (a_1 a_i)^2} \\
&= f_R \times \frac{[\det(M_{P \cup S, a_1 a_2}) / (a_1 a_2)]^2}{\prod_{\tau \in P \cup S} (\tau_1 \tau_2) (\tau_2 \tau_3) (\tau_3 \tau_1)} \times \prod_{i=1}^r (a_i a_{i+1}) \\
&= f_R \times f_{P \cup S} \times \prod_{i=1}^r (a_i a_{i+1}),
\end{aligned}
\tag{A.5}$$

so the form corresponding to T factorizes into two other forms, with an extra factor from the polygon on which R and S were glued. $\square$

A.2 Sphere moves are the only moves

LEMMA A.2.0.1 (Pole for lone pair). *Suppose that in the triplets T , the indices i, j appear together only in one triplet (ijk) . Then, defining $T' = (T \setminus \{(ijk)\}) \Big|_{j \mapsto i}$, one has*

$$[(ij) \times f_T] \Big|_{j \mapsto i} = -f_{T'}.
\tag{A.6}$$

PROOF. Computing the LHS prior to mapping the indices,

$$\begin{aligned}
(ij) \times f_T &= \frac{[\det(M_{ij})/(ij)]^2}{(jk)(ki) \prod_{(a,b,c) \in T \setminus \{(ijk)\}} (ab)(bc)(ca)} \\
&= \frac{\left[\det \begin{pmatrix} (ij) & 0 \\ \vdots & M_{ijk} \end{pmatrix} / (ij) \right]^2}{(jk)(ki) \prod_{(a,b,c) \in T \setminus \{(ijk)\}} (ab)(bc)(ca)} \\
&= \frac{\det(M_{ijk})^2}{(jk)(ki) \prod_{(a,b,c) \in T \setminus \{(ijk)\}} (ab)(bc)(ca)}
\end{aligned}
\tag{A.7}$$

where M_{ijk} is the matrix M with columns i, j, k and the row corresponding to triplet (ijk) removed.

Then, mapping the index $j \mapsto i$, one finds

$$\begin{aligned}
[(ij) \times f_T] \Big|_{j \mapsto i} &= - \frac{[\det(M_{ijk})/(ik)]^2}{\prod_{(a,b,c) \in T \setminus \{(ijk)\}} (ab)(bc)(ca)} \Big|_{j \mapsto i} \\
&= - \frac{[\det(M'_{ik})/(ik)^2]}{\prod_{(a,b,c) \in T'} (ab)(bc)(ca)} \\
&= -f_{T'},
\end{aligned}
\tag{A.8}$$

as desired. $\square$

LEMMA A.2.0.2 (Poles for irreducible hypertrees). *For T a set of triplets corresponding to an irreducible hypertree, the denominator of f_T matches the doublets $D(T)$.*

PROOF. For $\{i, j\} \in D(T)$, we can repeatedly apply square moves $(ija), (ijb) \mapsto (iab), (jab)$, rewriting T such that i, j appear in only one triplet (ijk) . This allows us to apply Lemma A.2.0.1 to conclude that (ij) must be a pole of f_T if $T' = (T \setminus \{(ijk)\}) \Big|_{j \mapsto i}$ satisfies the non-vanishing condition (4.45).

Suppose the triplets T' fail the non-vanishing condition. Then, there exists some subset of triplets $R' \subset T'$ such that

$$\left| \bigcup_{(abc) \in R'} \{a, b, c\} \right| < |R'| + 2.$$

Denote by $R \subset T$ the subset of triplets that maps to R' after the substitution $j \mapsto i$. Note that $(ijk) \notin R$.

If the index j does not appear in R , then $R = R'$, and consequently T also fails the non-vanishing condition, leading to a contradiction with T coming from an irreducible hypertree.

Using that j is the only index in R but not in R' , one has

$$\left| \bigcup_{(abc) \in R} \{a, b, c\} \right| = \left| \bigcup_{(abc) \in R'} \{a, b, c\} \right| + 1 \leq |R'| + 2 = |R| + 2.$$

As T satisfies the nonvanishing condition, the left-hand side and right-hand side must be equal. So R is a valid subset of T .

If R does not correspond to a planar diagram, then T has a valid non-planar subset, contradicting that it came from an irreducible hypertree.

If R does correspond to a planar diagram, we consider two cases both of which lead to contradictions. If the index k is contained in R , then $R \cup \{(ijk)\}$ causes T to fail the non-vanishing condition by adding a triplet but not an index. If the index k is not contained in R , we either have that $R \cup \{(ijk)\} = T$ leading to a factorization of $f_T = f_R \times \frac{(ij)}{(ik)(kj)}$, or $R \cup \{(ijk)\} \subset T$ is a valid non-planar subset.

Thus, T' does not fail the non-vanishing condition, making (ij) a pole of f_T . This shows that if ij is a doublet of T , then (ij) is a pole of f_T .

To show the opposite, note that one can use a square move to replace any triplets with a repeated doublet $(ija), (ijb)$ with two triplets that do not have this doublet $(iab), (jab)$. For any ij that is not a doublet of T , it appears in an even number of triplets, to which the square move can be applied to remove any appearances of the doublet. Using the set of triplets where the doublet ij does not appear, one can see from the determinant formula that (ij) cannot appear in the denominator. $\square$

LEMMA A.2.0.3 (Doublets from the form). *For a set of triplets T , the following definitions of the doublets are equivalent:*

- $D(T) = \{\{i, j\} | i \text{ and } j \text{ appear together in an odd number of triplets}\}.$
- $D(T) = \{\{i, j\} | (ij) \text{ appears with an odd power in } f_T\}.$

PROOF. For diagrams corresponding to irreducible hypertrees, the numerator of the form has no monomials, so this statement is a corollary of Lemma A.2.0.2.

Otherwise, suppose the statement is true for all sets of triplets satisfying $|T| \leq n$. We show the statement is then true for arbitrary triplets T where $|T| = n$. Supposing that T not an irreducible hypertree, we apply the factorization formula (4.46) for some proper subset R that does not share

all its indices with $T \setminus R$,

$$(A.9) \quad f_T = f_R \times f_{(T \setminus R) \cup P} \times \prod_{i=1}^r (a_i a_{i+1}).$$

Since $|R| < |T| = n$ and $|(T \setminus R) \cup P| < |T| = n$, we can apply our inductive hypothesis. The parity of odd powers of monomials on the RHS is exactly the parity of the doublets appearances in $D(R)$ and $D(T \setminus R)$, where the parity coming from $D(P)$ is canceled by $\prod_{i=1}^r (a_i a_{i+1})$. Using a single triplet as the base case, the statement is true for all n . $\square$

LEMMA A.2.0.4 (Same doublets imply sphere moves). *Suppose two sets of triplets T and T' with nonvanishing forms have the same doublets, $D(T) = D(T')$. Then, these sets of triplets are related by a sequence of sphere moves.*

PROOF. Without loss of generality, suppose T and T' have no triplets in common, $T \cap T' = \emptyset$. If they did share triplets, the shared triplets $T \cap T'$ can simply be ignored as we still have $D(T \setminus (T \cap T')) = D(T' \setminus (T \cap T'))$, and can construct sphere moves for the remaining triplets. Now, since T, T' are disjoint, the equality $D(T) = D(T')$ implies that $D(T \cup T') = \emptyset$.

We will now construct sets $R \subset T$ and $R' \subset T'$ related by a sphere move. We will do this by starting with $R, R' = \emptyset$, and incrementally adding triplets to each via the algorithm below. As this algorithm unfolds, it correspondingly constructs a cell complex with black faces labeled by R , white faces labeled by R' , and vertices labeled by indices in $R \cup R'$.

At any step, each edge must appear in at most two faces. Each time a triplet is added in the algorithm below, a face is added to the complex. This face is to be glued to any edge that does not yet have two faces attached; otherwise new edges are created. Note that this means that complex may not be simplicial: the same pair of vertices may have multiple edges between them if they appear together in more than two faces. This procedure ensures that the boundary of the complex consists of edges identified with $D(R \cup R')$.

To start, we choose a single triplet $t \in T$ and add it to R . Now, we proceed with the following cycle, repeating until we complete a full cycle without adding any new triplets to R or R' .

- (1) Suppose there exists a triplet in $T \setminus R$ containing a pair of indices $ij \in D(R) \setminus D(R')$; add this triplet to R . Repeat this step until no such triplet exists. (Correspondingly in the complex, attach as many black faces to existing black faces as possible.)

- (2) Suppose there exists a triplet in $T' \setminus R'$ containing a pair of indices $ij \in D(R) \setminus D(R')$; add this triplet to R' . Repeat this step until no such triplet exists. (Correspondingly in the complex, attach as many white faces to existing black faces as possible.)
- (3) Suppose there exists a triplet in $T' \setminus R'$ containing a pair of indices $ij \in D(R') \setminus D(R)$; add this triplet to R' . Repeat this step until no such triplet exists. (Correspondingly in the complex, attach as many white faces to existing white faces as possible.)
- (4) Suppose there exists a triplet in $T \setminus R$ containing a pair of indices $ij \in D(R') \setminus D(R)$; add this triplet to R . Repeat this step until no such triplet exists. (Correspondingly in the complex, attach as many black faces to existing white faces as possible.)

Since T and T' are finite, this procedure must terminate. We now seek to show that the resulting complex has no boundary, i.e. $D(R \cup R') = \emptyset$.

Suppose $\exists ij \in D(R \cup R')$. Then, either $ij \in D(R) \setminus D(R')$ or $ij \in D(R') \setminus D(R)$. Since the algorithm terminating implies no triplet satisfying the necessary conditions existed in each step, one finds that $T \setminus R$ and $T' \setminus R'$ contain no triplets with indices ij . Consequently, $ij \notin D(T \setminus R)$ and $ij \notin D(T' \setminus R')$. But then,

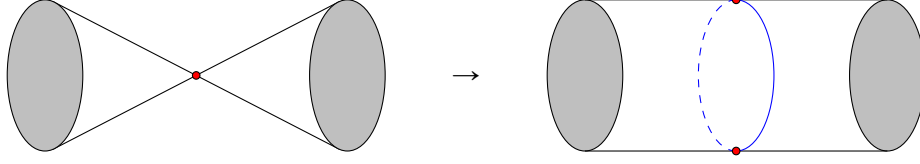
$$(A.10) \quad ij \in D([R \cup R'] \cup [T \setminus R] \cup [T' \setminus R']) = D(T \cup T'),$$

which is a contradiction with $D(T \cup T') = \emptyset$.

So, the complex with black faces R and white faces R' has no boundary, and each edge is in exactly two faces. The number of faces is $F = |R| + |R'|$ and the number of edges is $E = \frac{3}{2}F$. The number of vertices V is equal to the number of distinct indices in $R \cup R'$. The non-vanishing condition for either T or T' requires that $|R| \leq V - 2$ and $|R'| \leq V - 2$. The Euler characteristic of the complex is $\chi = V - E + F = V - (|R| + |R'|)/2 \geq V - (V - 2) = 2$, implying that it is a sphere.

There is a possible obstruction to identifying the complex as the triangulation of a sphere: one might have had a pinch-point, i.e. a vertex with multiple cycles of faces around it. However, a similar counting argument to the main text reveals that the existence of a pinch-point would contradict the non-vanishing condition. We remove the pinch-points by replacing each vertex by a pair of vertices

with an extra edge for each cycle, e.g.



Suppose this procedure introduces k new vertices, l new handles, and $k + l$ new edges (at each vertex, there is one more new edge than there are new handles). Then, the number of faces is still $\tilde{F} = |R| + |R'|$, the number of edges is now $\tilde{E} = \frac{3}{2}F + k + l$, and the number of vertices is $\tilde{V} = V + k$. Using the non-vanishing condition, one finds $\tilde{\chi} = \tilde{V} - \tilde{E} + \tilde{F} \geq 2 - l$. But with the l extra handles one also has genus at least l , so $\tilde{\chi} \leq 2 - 2l$. So, we reach a contradiction unless $l = 0$, the case when no such pinch-points existed.

Consequently, R and R' together form a bicolored triangulation of the sphere, and are related by a sphere move. Thus, for any T and T' that have the same doublets, there is a sphere move that allows some subset of the different triplets to be related. Repeating until no triplets differ we find a sequence of sphere moves that relate T and T' .

□

We note that in the previous proof, we gloss over some topological details on when a pure 2-dimensional cell complex is a manifold.

THEOREM 4.4.1. *Let T and T' be two sets of triplets with nonvanishing forms (i.e. T, T' satisfy (4.45)). The following are equivalent:*

- (1) *There is a sequence of sphere moves relating T and T' .*
- (2) *The corresponding forms are equal: $f_T = f_{T'}$.*
- (3) *The corresponding doublets are the same: $D(T) = D(T')$.*

PROOF. (1) $\implies$ (2) is a consequence of [?].

(2) $\implies$ (3) is a corollary of Lemma A.2.0.3.

(3) $\implies$ (1) is proven by Lemma A.2.0.4.

□

A.3 Geometries induced by planar embeddings are strongly connected

Before tackling the full question to do with geometries that come from orientations of triplets, we prove a more general result for geometries that come from a partial order on $\{1, \dots, n\}$. For a

partial order $P = \{i < j, \dots\}$, the set of linear extensions $S_n^{(P)}$ is

$$(A.11) \quad S_n^{(P)} = \{\sigma \in S_n : \sigma_i < \sigma_j \ \forall \ i < j \in P\}.$$

These define a corresponding geometry via a union of oriented regions,

$$(A.12) \quad R_P(\varepsilon) = \bigcup_{\sigma \in S_n^{(P)}} R(\varepsilon\sigma),$$

where $\varepsilon \in \{+, -\}^n$ encodes the sign of each label.

LEMMA A.3.0.1 (Linear extensions of a poset are connected). *Any element of $S_n^{(P)}$ can be transformed into any other using only adjacent transpositions without leaving the set.*

PROOF. By relabeling the indices if necessary, suppose without loss of generality that the identity permutation $12 \cdots n$ is in $S_n^{(P)}$.

The pair of indices $i < j$ is an inversion of a permutation σ if $\sigma_i > \sigma_j$. We denote the number of inversions by $\text{inv}(\sigma)$. The only permutation with $\text{inv}(\sigma) = 0$ is $12 \cdots n$.

We show that for any $\sigma \in S_n^{(P)}$ with $\text{inv}(\sigma) > 0$, there exists a permutation $\sigma' \in S_n^{(P)}$ related by an adjacent transposition such that $\text{inv}(\sigma') = \text{inv}(\sigma) - 1$.

Because $\sigma \neq 12 \cdots n$, there is some i such that $\sigma_i > \sigma_{i+1}$. Notice that $i < i+1 \notin P$ as these elements are oppositely ordered in $12 \cdots n \in S_n^{(P)}$. Consequently, the permutation $\sigma' = \sigma(i \ i+1)$ is also in $S_n^{(P)}$. This permutation has the same inversions as σ , except without the pair $i, i+1$, so $\text{inv}(\sigma') = \text{inv}(\sigma) - 1$ as desired.

By induction on the inversion number, with the identity as the base case, every permutation in $S_n^{(P)}$ can be brought to the identity via adjacent transpositions without leaving $S_n^{(P)}$. $\square$

COROLLARY A.3.1. *The geometry R_P corresponding to the poset P is strongly connected.*

We observe that the set of permutations defined by an orientation of triplets $S^{(T)}$ can be decomposed into a disjoint union of permutations defined by posets $S^{(P)}$ by choosing each possible cyclic shift of the triplets, except those involving 1 which is fixed to be the lowest index. For example, with oriented triplets $T = \{(123), (345), (156), (264)\}$, we select a cyclic shift for each

triplet not involving 1,

$$\begin{aligned}
P_1 &= \{1 < 2 < 3, 3 < 4 < 5, 1 < 5 < 6, 2 < 6 < 4, 1 < j \forall j\}, \\
P_2 &= \{1 < 2 < 3, 4 < 5 < 3, 1 < 5 < 6, 2 < 6 < 4, 1 < j \forall j\}, \\
P_3 &= \{1 < 2 < 3, 5 < 3 < 4, 1 < 5 < 6, 2 < 6 < 4, 1 < j \forall j\}, \\
P_4 &= \{1 < 2 < 3, 3 < 4 < 5, 1 < 5 < 6, 4 < 2 < 6, 1 < j \forall j\}, \\
P_5 &= \{1 < 2 < 3, 4 < 5 < 3, 1 < 5 < 6, 4 < 2 < 6, 1 < j \forall j\}, \\
P_6 &= \{1 < 2 < 3, 5 < 3 < 4, 1 < 5 < 6, 4 < 2 < 6, 1 < j \forall j\}, \\
P_7 &= \{1 < 2 < 3, 3 < 4 < 5, 1 < 5 < 6, 4 < 2 < 6, 1 < j \forall j\}, \\
P_8 &= \{1 < 2 < 3, 4 < 5 < 3, 1 < 5 < 6, 4 < 2 < 6, 1 < j \forall j\}, \\
P_9 &= \{1 < 2 < 3, 5 < 3 < 4, 1 < 5 < 6, 4 < 2 < 6, 1 < j \forall j\}.
\end{aligned}
\tag{A.13}$$

Most of these end up not being valid posets, due to there being a cycle of inequalities, e.g. $4 < 5 < 6 < 4 \in P_1$. In this example, the only ones that are posets are P_5 and P_9 , with linear extensions

$$\begin{aligned}
S_6^{(P_5)} &= \{142536, 145236, 142563, 145263\}, \\
S_6^{(P_9)} &= \{125364, 152364, 125634, 152634\}.
\end{aligned}
\tag{A.14}$$

Since we enumerated all possible cyclic shifts of the triplets, one finds that $S_6^{(P_5)} \cup S_6^{(P_9)} = S_6^{(T)}$, such that

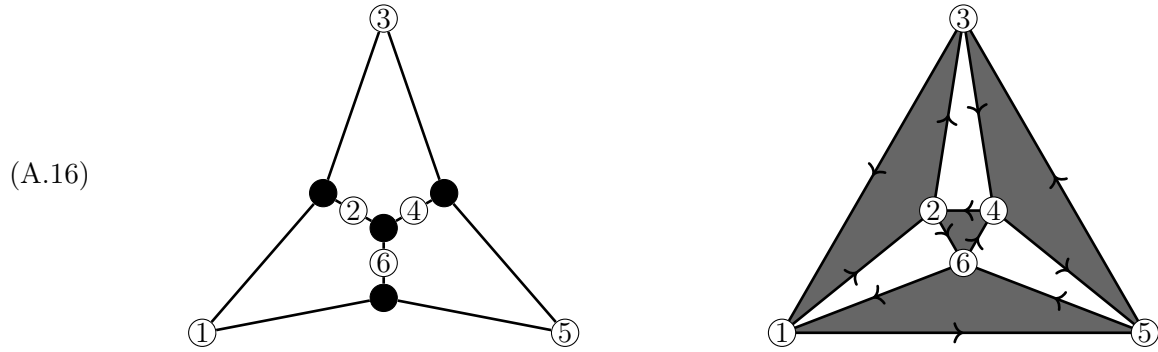
$$R_T = R_{P_5}(\varepsilon_5) \cup R_{P_9}(\varepsilon_9),
\tag{A.15}$$

where ε_i parametrize the freedom to reflect the regions corresponding to each poset. Corollary A.3.1 is sufficient to find the strong connectedness within each piece R_{P_i} , so one might ask whether we can choose ε_i such that the entire R_T is strongly connected. In this example, choosing $\varepsilon_5 = (++++++)$ and $\varepsilon_9 = (+++--++)$, the region defined by (A.15) is strongly connected, as each region in $R_{P_5}(\varepsilon_5)$ is adjacent to a region $R_{P_9}(\varepsilon_9)$ on the boundary (14) = 0. With the idea of regions corresponding to posets in mind, we now return to proving the following theorem:

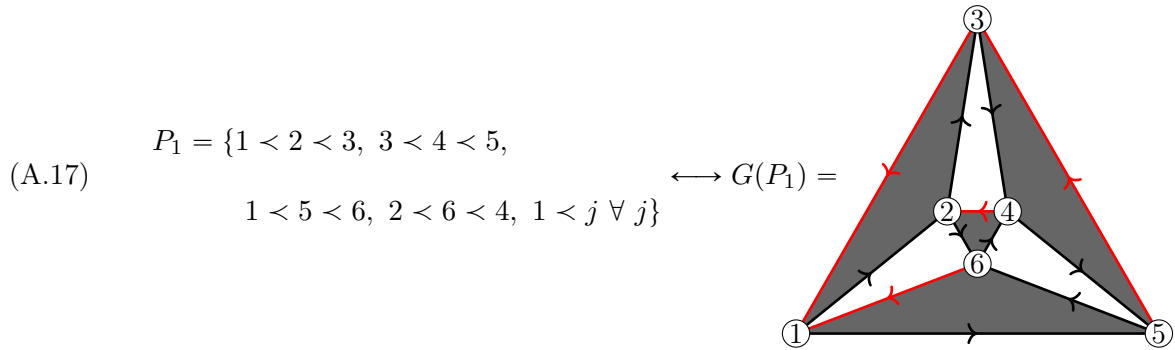
THEOREM A.3.1 (Internally planar diagrams admit strongly connected geometries). *The orientation of triplets T induced by a planar embedding of the corresponding on-shell diagram defines a strongly connected geometry R_T for an appropriate choice of signs.*

PROOF. To prove this theorem, we demonstrate a bijection between the posets with non-empty sets of linear extensions, and perfect matchings of a particular graph. To make the procedure clear, we follow an example with the 6-point irreducible internally planar diagram with triplets $T = \{(123), (345), (156), (264)\}$.

Starting with a planar embedding of the internal edges of the diagram, one may replace each black vertex with a triangular face,



We fix an orientation of the triplets by reading counterclockwise around each black triangle. Correspondingly, reading the edges counterclockwise gives us most of the inequalities used by the posets necessary to describe these triplets. To get a specific choice of cyclic shifts of the triplets P_i , we must throw away one inequality for each triplet, which we denote by decorating the corresponding edge in a graph $G(P_i)$.



$$\begin{aligned}
(A.18) \quad P_5 = \{ & 1 < 2 < 3, \ 4 < 5 < 3, \\
& 1 < 5 < 6, \ 4 < 2 < 6, \ 1 < j \ \forall \ j \} & \longleftrightarrow G(P_5) =
\end{aligned}$$

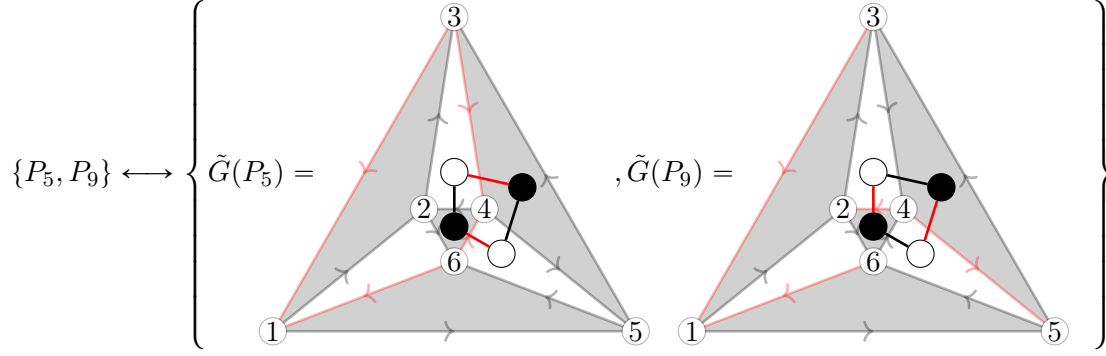
Those sets of inequalities that form poset cannot contain a cycle of inequalities. In the diagram, this corresponds to there being a directed cycle of non-decorated edges.

$$\begin{aligned}
(A.19) \quad 4 < 5 < 6 < 4 \in P_1 & \longleftrightarrow (4 \rightarrow 5 \rightarrow 6 \rightarrow 4 \text{ non-decorated cycle in (A.17)}) \\
& \implies S_n^{(P_1)} = \emptyset
\end{aligned}$$

Suppose we have two triangles (ijk) and (ilj) . To prevent the cycles $i < j < i$ and $i < l < j < k < i$, the decorated edges for these triangles must include one from each group. Generalizing to larger groups of black triangles glued into a black polygon, we conclude that exactly one of the edges of the polygon is decorated, with the remaining decorations spent on preventing 2-cycles in the polygon's triangulation. (No such polygons appear in the tracked example.)

In order to prevent the existence of such cycles around the white faces, each white face must have at least one decorated edge. Since there are an equal number of white and black faces, the only way this is possible is if each white face actually has exactly one decorated edge. On the bipartite dual graph, this corresponds to a perfect matching. Note all edges which are incoming to vertex 1 are decorated, since we have the convention that 1 is the first index labeling an oriented region. So we only need to consider the dual graph $\tilde{G}(P_i)$ of the faces that do not have 1 as a vertex.

$$\{P_i : S_n^{(P_i)} \neq \emptyset\} \longleftrightarrow \{\text{perfect matchings of the graph dual to the faces that don't include 1}\}$$



From this point forward, we consider only those posets that have non-empty sets of linear extensions, and the corresponding suitable decorations that are dual to a perfect matching.

Consider an index i , such that in the graph $G(P_i)$ (with edges that are inside of black faces removed) the corresponding vertex is not on the same face as 1. Suppose we find a valid decoration where all edges ingoing to i are decorated. Then, in the poset P corresponding to this decoration, one has $j < i \forall j$, from which we see that $S_n^{(P)}$ includes some element $1 \cdots i$ where i is the last index. The region $R(1, \cdots, i) \in R_P$ shares a codimension-1 boundary with the region $R(1, -i, \cdots)$. This latter region is included in $R_{P'}$, where P' is the same poset as P except instead of i being a maximal element, it is the smallest element larger than 1. This corresponds to the decoration where all outgoing edges to i are decorated instead. So, to show the strong connectedness of R_T , we must show that the posets used to construct subsets R_P are adjacent to each other by the move of ‘swapping all-ingoing decorations for all-outgoing decorations’.

Luckily, this move has a neat interpretation in terms of the perfect matchings. Swapping the all-ingoing and all-outgoing decorations in $G(P_i)$ correspond to rotating a face in $\tilde{G}(P_i)$ when that face has alternating edges. It has been shown that for bipartite planar diagrams, all perfect matchings are related to each other by such moves [235]. $\square$

APPENDIX B

Appendix for Chapter 6

In this appendix, we go over some proofs we wrote with our collaborators in [236] to support the main claim we make in chapter 6 regarding the constructibility of one-cycle integrands.

B.1 Proof of One-Cycle Result

In our construction of the one-cycle integrand, we start with an ansatz that is the sum of a product of $N^{+(-)}$ on the corresponding positive (negative) links F and a remainder R . The latter is built out of sums of products of the following objects:

$$(B.1) \quad N_i^a, N^b i, N_{ij}^a, N_{ij}^b, N_{ij}^c, N_{ijkl}^a, N_{ijkl}^b, N^c ijkl, N_{ijklmn}^a, \dots$$

We then require that this ansatz for R vanishes on all illegal 2-loop cuts, which disallows certain monomials. Finally, we place the constraint that the whole numerator (fixed links F + remainder R) must vanish on the L -loop double pole cut. This last condition uniquely fixes all free coefficients and we claim the result (6.83) to be the full one-cycle integrand.

In principle, if a general ansatz is uniquely fixed by requiring it vanishes on all illegal cuts, then the resulting integrand is correct. That being said, constructing a remainder out of the factors (B.1) is not the most general ansatz. Indeed it is not even the most general ansatz that satisfies all 2-loop conditions. As a result, in order to be sure that our result (6.83) is correct, we need to make a comprehensive list of illegal cuts and then prove that the our result indeed vanishes on all of these cuts.

Let us begin with a classification of all illegal cuts. There are two types:

- (1) Since the same (up to sign) remainder R works for any set of positive and negative links, the remainder must vanish on any cut that forces a link to have definite sign.
- (2) Locality requires that the integrand has no double poles. Any cut that exposes such a double pole (by setting two factors in the denominator equal) is also illegal.

The first type of cut is easy to classify. We simply express the links in some parametrization e.g. (5.9). In these variables, $x_i, y_i, w_i, z_i > 0$ in order to be within the 1-loop amplituhedron. As a result we can rewrite a general link suggestively:

$$(B.2) \quad \langle AB_i AB_j \rangle = (w_2 x_1 + w_1 x_2 + y_2 z_1 + y_1 z_2) - (w_1 x_1 + w_2 x_2 + y_1 z_1 + y_2 z_2) .$$

It is now clear that the list of cuts that force $\langle AB_i AB_j \rangle$ to be negative (positive) set the first (second) set of terms to zero. We refer to these as

- (1) **P1(i)**: This is a condition on x_i, y_i, w_i and z_i alone that leads to $\langle AB_i AB_j \rangle < 0$, e.g. $x_i = y_i = z_i = w_i = 0$ i.e. the cut where $\langle AB_i k k+1 \rangle = 0$.
- (2) **P2(i,j)**: This is a condition on both variables that sets to $\langle AB_i AB_j \rangle < 0$, e.g. $y_i = w_i = y_j = w_j = 0$ i.e. the cut where $\langle AB_i 12 \rangle = \langle AB_i 23 \rangle = \langle AB_j 12 \rangle = \langle AB_j 23 \rangle = 0$.
- (3) **N1(i)**: This is a condition on x_i, y_i, w_i and z_i alone that leads to $\langle AB_i AB_j \rangle > 0$, e.g. $x_i = y_i = 0$ while $w_i, z_i \rightarrow \infty$ i.e. the cut where $\langle AB_i 12 \rangle = \langle AB_i 14 \rangle = 0$.
- (4) **N2(i,j)**: This is a condition on both sets of variables that sets to $\langle AB_i AB_j \rangle > 0$, e.g. $x_i, y_i, z_j, w_j \rightarrow \infty$ i.e. the cut where $\langle AB_i 23 \rangle = \langle AB_i 34 \rangle = \langle AB_j 12 \rangle = \langle AB_j 14 \rangle = 0$.

Importantly, the building blocks (B.1) vanish on many of the cuts:

- (1) $N_i^{a(b)}$ vanishes on **P1(i)**, **N1(i)** and **N2(i,j)**.
- (2) $N_i^{a(b)} N_j^{a(b)}$ vanishes on **P1(i)**, **N1(i)**, **P1(j)**, **N1(j)**, **P2(i,j)** and **N2(i,j)**.
- (3) $N_{ij\dots}^{a(b)}$ vanishes on **P1(i)**, **P1(j)** and **P2(i,j)**.
- (4) $N_{ij\dots}^c$ vanishes on **N1(i)**, **N1(j)** and **N2(i,j)**.

Clearly the final result for the remainder (6.77), only contains monomials in N that vanish on all illegal cuts up to 2-loop order:

$$R_L^{1-\text{loop}} = \left\{ (2^L - 4) N_1^{(a)} N_2^{(a)} \dots N_L^{(a)} - \sum_{p \in \text{even}} \sum_i \{ N_{i_1 i_2 \dots i_p}^{(a)} N_{i_1 i_2 \dots i_p}^{(c)} N_{i_{p+1}}^{(a)} \dots N_{i_L}^{(a)} \} \right\} + (a \rightarrow b)$$

The second type of cut is a bit more involved. At 3-loop, we can see what types of cuts are illegal by starting with the zeros of a two links, without loss of generality we choose $\langle ABCD \rangle$ and $\langle CDEF \rangle$. We then evaluate $\langle ABEF \rangle$ on these zeros to find cuts on which setting $\langle ABCD \rangle$ and $\langle CDEF \rangle$ to zero accidentally also sets the third propagator to zero i.e. a double pole. In the parametrization (5.9), it is clear that there are only four types of such zeros: $x_i = x, y_i = y, z_i = z$

or $w_i = w$. Interestingly, this fact continues to all L -loop, giving us the four types of double pole cuts **DPL** that can now be classified into four types: $\langle AB_i 12 \rangle = 0, \langle AB_i 23 \rangle = 0, \langle AB_i 34 \rangle = 0$ or $\langle AB_i 14 \rangle = 0$.

To see that the L -loop result (6.77) vanishes on a **DPL** cut, we note that each of the numerators simplifies significantly on these cuts. For example, in the parametrization (5.9), let us consider the building blocks N on the double pole cut where $\langle AB_i 14 \rangle = 0$, i.e. $x_i = 0, y_i = y$. We discuss here this particular cut, but the others follow similarly.

$$(B.3) \quad \begin{aligned} N_{ij}^+ &= N_{ij}^- = y(z_i + z_j), & N_i^b &= N_{i_1 i_2 \dots i_{2n}}^b = 0, \\ N_i^a &= N_i^c = y z_i, & N_{i_1 i_2 \dots i_{2n}}^a &= N_{i_1 i_2 \dots i_{2n}}^c = y^n \left(\prod_{j \in \text{even}} z_{i_j} + \prod_{j \in \text{odd}} z_{i_j} \right). \end{aligned}$$

On this cut, the full numerator (6.82) simplifies:

$$(B.4) \quad y^L \prod_{i=1}^L (z_i + z_{i+1}) + y^L (2^L - 4) \prod_{i=1}^L z_i - y^L \sum_{p \in \text{even}} \sum_i \left(\prod_{\substack{j \in \text{even} \\ j < p}} z_{i_j} + \prod_{\substack{j \in \text{odd} \\ j < p}} z_{i_j} \right)^2 \prod_{j > p} z_{i_j}.$$

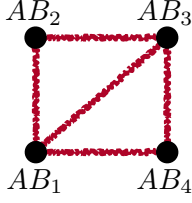
We can now consider the coefficient of the two different types of z_i monomials:

$$(B.5) \quad \begin{aligned} \prod_{i=1}^L z_i : 2 + 2^L - 4 - \sum_{p \in \text{even}} \sum_i 2 &= 2 + 2^L - 4 - \sum_{p \in \text{even}} 2 \frac{L!}{(L-p)!p!} \\ &= 2 + 2^L - 4 - 2^L + 2 = 0, \\ \prod_{\{i_1, \dots, i_{\frac{L}{2}}\}} z_{i_j}^2 \prod_{\{i_{p+1}, \dots, i_L\}} z_{i_j} : 1 - 1 &= 0. \end{aligned}$$

This concludes the proof that our result vanishes on *all* illegal cuts and thus the full numerator (6.83) is indeed the correct one-cycle integrand for planar $\mathcal{N} = 4$ SYM at L -loop.

B.2 Two-loop invariant basis

Let us determine the two-loop invariant $R_{1234,13}^{2\text{-loop}}$ of the graph,

(B.6) 

It is useful to organize the basis of terms for the ansatz into three groups

(B.7)
$$\mathcal{B} = \{\mathcal{B}_a, \mathcal{B}_b, \mathcal{B}_{ab}\},$$

where $\mathcal{B}_a$ only contains $N^{(a)}$ and $N^{(c)}$ terms, $\mathcal{B}_b$ only contains $N^{(b)}$ and $N^{(c)}$ terms and $\mathcal{B}_{ab}$ contains terms with a – b mixing, ie. both $N^{(a)}$ and $N^{(b)}$ terms. Obviously $\mathcal{B}_b$ can be obtained from $\mathcal{B}_a$ by relabeling $a \rightarrow b$.

In the construction of the basis, we follow the following logic: we start with terms that include only one-line invariants $N_k^{(a,b)}$, then also two-line invariants etc. The first term to consider is

(B.8)
$$N_1^{(a)} N_2^{(a)} N_3^{(a)} N_4^{(a)}.$$

This already satisfies all singlet and also doublet conditions for any pair of lines (even the link $\langle AB_2 AB_4 \rangle$ is covered which we do not need). Based on the little group weights of the numerator, the only term left we can add is $N_{13}^{(a,b,c)}$, hence we add to our basis sets,

(B.9)
$$\mathcal{B}_a^{(1)} = N_1^{(a)} N_2^{(a)} N_3^{(a)} N_4^{(a)} N_{13}^{(a)}, \quad \mathcal{B}_a^{(2)} = N_1^{(a)} N_2^{(a)} N_3^{(a)} N_4^{(a)} N_{13}^{(c)},$$

(B.10)
$$\mathcal{B}_{ab}^{(1)} = N_1^{(a)} N_2^{(a)} N_3^{(a)} N_4^{(a)} N_{13}^{(b)}.$$

We can also have a mix of $N_j^{(a)}$ and $N_k^{(b)}$ terms. The inequivalent possibilities have ‘skeletons’ built from one-line invariants and then a completion using two-line invariants to satisfy all singlet and double conditions.

- | | | |
|---|----------------------------|----------------|
| (1) $N_1^{(a)} N_2^{(a)} N_3^{(a)} N_4^{(b)}$ | one option for completion: | $N_{13}^{(b)}$ |
| (2) $N_1^{(a)} N_2^{(a)} N_3^{(b)} N_4^{(a)}$ | one option for completion: | $N_{13}^{(a)}$ |
| (3) $N_1^{(a)} N_2^{(b)} N_3^{(a)} N_4^{(b)}$ | one option for completion: | $N_{13}^{(b)}$ |

- (4) $N_1^{(a)} N_2^{(a)} N_3^{(b)} N_4^{(b)}$ no option for completion
(5) $N_1^{(b)} N_2^{(a)} N_3^{(b)} N_4^{(a)}$ one option for completion: $N_{13}^{(a)}$

As a result, we get four new terms that belong to $\mathcal{B}_{ab}$,

$$(B.11) \quad \mathcal{B}_{ab}^{(2)} = N_1^{(a)} N_2^{(a)} N_3^{(a)} N_4^{(b)} N_{13}^{(b)}, \quad \mathcal{B}_{ab}^{(3)} = N_1^{(a)} N_2^{(a)} N_3^{(b)} N_4^{(a)} N_{13}^{(a)}$$

$$(B.12) \quad \mathcal{B}_{ab}^{(4)} = N_1^{(a)} N_2^{(b)} N_3^{(a)} N_4^{(b)} N_{13}^{(b)}, \quad \mathcal{B}_{ab}^{(5)} = N_1^{(b)} N_2^{(a)} N_3^{(b)} N_4^{(a)} N_{13}^{(a)}.$$

The next case is when we have only three one-line invariants. There are two skeletons which involve only $N_j^{(a)}$,

- (1) $N_1^{(a)} N_2^{(a)} N_3^{(a)}$ one option for completion: $N_{14}^{(a)} N_{34}^{(c)}$
(2) $N_1^{(a)} N_2^{(a)} N_4^{(a)}$ no option for completion

and four skeletons with the mix of $N_j^{(a)}$ and $N_k^{(b)}$ but none of them can be completed to a valid term,

- (1) $N_1^{(a)} N_2^{(a)} N_3^{(b)}$ no option for completion
(2) $N_1^{(a)} N_2^{(b)} N_3^{(a)}$ no option for completion
(3) $N_1^{(a)} N_2^{(a)} N_4^{(b)}$ no option for completion
(4) $N_1^{(b)} N_2^{(a)} N_4^{(a)}$ no option for completion

As a result we get only one new term in the a -sector,

$$(B.13) \quad \mathcal{B}_a^{(3)} = N_1^{(a)} N_2^{(a)} N_3^{(a)} N_{14}^{(a)} N_{34}^{(c)}.$$

Next, we only consider the presence of two one-line invariants, there are three skeletons with only $N_j^{(a)}$,

- (1) $N_1^{(a)} N_2^{(a)}$ three options for completion: $N_{34}^{(a)} N_{34}^{(c)} N_{13}^{(a,b,c)}$
(2) $N_1^{(a)} N_3^{(a)}$ five options for completion: $N_{24}^{(a)} N_{24}^{(c)} N_{13}^{(a,b,c)}$, $N_{12}^{(a)} N_{24}^{(c)} N_{34}^{(a)}$, $N_{12}^{(c)} N_{24}^{(a)} N_{34}^{(c)}$
(3) $N_2^{(a)} N_4^{(a)}$ three options for completion: $N_{13}^{(a)} N_{13}^{(c)} N_{13}^{(a,b,c)}$

There are two extra terms for the $N_1^{(a)} N_3^{(a)}$ skeleton which include a four-line invariant, namely we can have

$$(B.14) \quad N_1^{(a)} N_3^{(a)} \quad \text{completed by } N_{1234}^{(a)} N_{24}^{(c)} \text{ or } N_{1234}^{(c)} N_{24}^{(a)}$$

In the mixed sector we also have three skeletons,

- (1) $N_1^{(a)} N_2^{(b)}$ one option for completion: $N_{34}^{(a)} N_{34}^{(c)} N_{13}^{(b)}$
(2) $N_1^{(a)} N_3^{(b)}$ one option for completion: $N_{24}^{(a)} N_{24}^{(c)} N_{13}^{(b)}$
(3) $N_2^{(a)} N_4^{(b)}$ one option for completion: $N_{13}^{(a)} N_{13}^{(c)} N_{13}^{(b)}$

As a result we get new basis terms, in particular in the a -sector

$$(B.15) \quad \mathcal{B}_a^{(4)} = N_1^{(a)} N_2^{(a)} N_{34}^{(a)} N_{34}^{(c)} N_{13}^{(a)}, \quad \mathcal{B}_a^{(5)} = N_1^{(a)} N_2^{(a)} N_{34}^{(a)} N_{34}^{(c)} N_{13}^{(c)},$$

$$(B.16) \quad \mathcal{B}_a^{(6)} = N_1^{(a)} N_3^{(a)} N_{24}^{(a)} N_{24}^{(c)} N_{13}^{(a)}, \quad \mathcal{B}_a^{(7)} = N_1^{(a)} N_3^{(a)} N_{24}^{(a)} N_{24}^{(c)} N_{13}^{(c)},$$

$$(B.17) \quad \mathcal{B}_a^{(8)} = N_1^{(a)} N_3^{(a)} N_{12}^{(a)} N_{24}^{(c)} N_{34}^{(a)}, \quad \mathcal{B}_a^{(9)} = N_1^{(a)} N_3^{(a)} N_{12}^{(c)} N_{24}^{(a)} N_{34}^{(c)},$$

$$(B.18) \quad \mathcal{B}_a^{(10)} = N_2^{(a)} N_4^{(a)} N_{13}^{(a)} N_{13}^{(c)} N_{13}^{(a)}, \quad \mathcal{B}_a^{(11)} = N_2^{(a)} N_4^{(a)} N_{13}^{(a)} N_{13}^{(c)} N_{13}^{(c)},$$

$$(B.19) \quad \mathcal{B}_a^{(12)} = N_1^{(a)} N_3^{(a)} N_{1234}^{(a)} N_{24}^{(c)}, \quad \mathcal{B}_a^{(13)} = N_1^{(a)} N_3^{(a)} N_{1234}^{(c)} N_{24}^{(a)}.$$

and in the mixed $a - b$ sector,

$$(B.20) \quad \mathcal{B}_{ab}^{(6)} = N_1^{(a)} N_3^{(a)} N_{24}^{(a)} N_{24}^{(c)} N_{13}^{(b)}, \quad \mathcal{B}_{ab}^{(7)} = N_1^{(a)} N_2^{(a)} N_{34}^{(a)} N_{34}^{(c)} N_{13}^{(b)},$$

$$(B.21) \quad \mathcal{B}_{ab}^{(8)} = N_2^{(a)} N_4^{(a)} N_{13}^{(a)} N_{13}^{(c)} N_{13}^{(b)}, \quad \mathcal{B}_{ab}^{(9)} = N_1^{(a)} N_2^{(b)} N_{34}^{(a)} N_{34}^{(c)} N_{13}^{(b)},$$

$$(B.22) \quad \mathcal{B}_{ab}^{(10)} = N_1^{(a)} N_3^{(b)} N_{24}^{(c)} N_{12}^{(b)} N_{34}^{(a)}, \quad \mathcal{B}_{ab}^{(11)} = N_2^{(a)} N_4^{(b)} N_{13}^{(a)} N_{13}^{(c)} N_{13}^{(b)},$$

$$(B.23) \quad \mathcal{B}_{ab}^{(12)} = N_1^{(a)} N_3^{(a)} N_{1234}^{(b)} N_{24}^{(c)}.$$

There are only two possible skeleton terms with one one-line invariant each – either $N_1^{(a)}$ or $N_2^{(a)}$, but none of them can be completed into a valid term with two-line invariants $N_{ij}^{(a,b,c)}$. For $N_1^{(a)}$ we can also use four-line invariants, and we get two more terms in the basis,

$$(B.24) \quad \mathcal{B}_a^{(14)} = N_1^{(a)} N_{1234}^{(a)} N_{23}^{(c)} N_{34}^{(c)}, \quad \mathcal{B}_a^{(15)} = N_1^{(a)} N_{1234}^{(c)} N_{23}^{(a)} N_{34}^{(a)}.$$

There are no terms with no one-line invariants which would match all constraints. The best candidate would be $N_{1234}^{(a)} N_{1234}^{(c)}$ but we are left with only one more term to add to take care of both positive and negative conditions for the $\langle AB_1 AB_3 \rangle$ link, and that is not possible. Our final ansatz is then

$$(B.25) \quad R_{1234,13}^{2\text{-loop}} = \sum_{j=1}^{15} c_j \left(\mathcal{B}_a^{(j)} + \mathcal{B}_b^{(j)} \right) + \sum_{k=1}^{12} d_k \mathcal{B}_{ab}^{(k)}.$$

Note that the basis elements are symmetrized according to the graph. For example, two of the properly symmetrized terms are

$$(B.26) \quad \mathcal{B}_a^{(3)} = N_1^{(a)} N_2^{(a)} N_3^{(a)} \left(N_{14}^{(a)} N_{34}^{(c)} + N_{14}^{(c)} N_{34}^{(a)} \right),$$

$$(B.27) \quad \begin{aligned} \mathcal{B}_{ab}^{(8)} = & N_1^{(a)} \left(N_2^{(b)} N_{34}^{(a)} N_{34}^{(c)} N_{13}^{(b)} + N_4^{(b)} N_{23}^{(a)} N_{23}^{(c)} N_{13}^{(b)} \right) \\ & + N_3^{(a)} \left(N_2^{(b)} N_{14}^{(a)} N_{14}^{(c)} N_{13}^{(b)} + N_4^{(b)} N_{12}^{(a)} N_{12}^{(c)} N_{13}^{(b)} \right). \end{aligned}$$

and similarly for all other basis terms – we need to symmetrize in $AB_1 \leftrightarrow AB_3$, $AB_2 \leftrightarrow AB_4$.

Fixing on cuts

Now we impose cancellation of all illegal cuts, and the cancellation of the double pole for the whole $\Omega_G^{2-\text{loop}}$ form and get a unique solution,

$$(B.28) \quad c_1 = c_2 = -8, \quad c_3 = -2, \quad c_8 = c_9 = c_{12} = c_{13} = 0,$$

$$c_4 = c_5 = c_6 = c_7 = c_{10} = c_{11} = c_{14} = c_{15} = 1,$$

$$d_1 = d_3 = d_4 = d_5 = 0, \quad d_2 = -4, \quad d_6 = d_7 = d_8 = 0, \quad d_9 = d_{10} = d_{11} = 1, \quad d_{12} = -2.$$

This gives us $R_{1234,13}^{2-\text{loop}}$ which completes the calculation of the two-cycle negative geometry. Plugging back the coefficients we find that the structure of the result is very suggestive and gives a hope that the closed form exists also for graphs with more cycles.

APPENDIX C

Appendix for Chapter 7

This appendix contains some detailed computations done by the author to support the results of chapter 7.

C.1 Results for the Γ_{cusp} Contribution of Infinite Classes

In this section, we provide detailed results for how the ladder class attached to the triangle and box diagrams contributes to Γ_{cusp} after resummation. Recall that the $\mathcal{I}$ operation is the integral (7.81), and we can perform it in our case explicitly. We can use the shuffle identities to put any HPL that appear in the $\mathcal{F}_3(z)$ in a form

$$(C.1) \quad \log^a(z) H_{-m_n, \dots, -m_1}(z),$$

where m_i are non-zero. Then the integrals we have to perform to get the Γ_{cusp} contribution are of the form

$$(C.2) \quad \frac{1}{2\pi} \int_{-\pi}^{\pi} H_{-m_2, -m_1}(e^{i\phi}) (i\phi)^a \frac{\cosh(\sqrt{2}g\phi)}{\cosh \sqrt{2}g\pi} d\phi,$$

which result in the following types of sums

$$(C.3) \quad \sum_{j_2 > j_1 > 0}^{\infty} \frac{j_2^a}{j_1^b (j_2^2 + 2g^2)^3}, \quad -3 \leq a \leq 4, \quad 1 \leq b \leq 3.$$

The precise list of a 's and b 's needed is

a	-3	-2	-1	-1	0	1	1	2	3	3	4
b	1	2	1	3	2	1	3	2	1	3	2

All of these sums can be performed analytically and be written in terms of Gamma functions and their derivatives (as explained in Appendix (C.2.2)). Note that the $\frac{\Gamma'(2-i\sqrt{2}g)}{\Gamma(2-i\sqrt{2}g)} = \psi^0(2-i\sqrt{2}g)$ terms that appear go like $\log g$ at strong coupling, but these terms end up canceling out between each other. While the leading $\log g$ terms that come from the HPLs with -1 indices cancel out between $H_{-1,-1}(z)$, $H_{-1,-1,0,0}(z)$, and $H_{-1}(z)$, we find that the lower order terms that have $\log(g)$ also

cancel or come in pairs that go like $i\pi$ (such as $\log(-g) - \log(g) = -i\pi$). The full result is too long to include here, but at $g \gg 1$ it becomes a polynomial in g as seen below

$$(C.4) \quad \mathcal{I}[g^4 \mathcal{F}_3(g, z) \mathcal{F}_{\text{ladder}}(g, z)]_{g \gg 1} \sim \frac{11}{45} \pi^3 \sqrt{2} g^5 - 8g^4 + \frac{\sqrt{2}(1 + \pi^2)}{\pi} g^3 - \frac{19}{9} g^2 + \frac{9\sqrt{2}}{8\pi} g.$$

Using $g \partial_g \Gamma_{\text{cusp}} = -8\mathcal{I}[\mathcal{F}(g, z)]$, we can extract the contribution to Γ_{cusp} as

$$(C.5) \quad \Gamma_{\text{cusp}} \sim -\frac{88}{225} \pi^3 \sqrt{2} g^5 + 16g^4 - \frac{8\sqrt{2}(1 + \pi^2)}{3\pi} g^3 + \frac{76}{9} g^2 - \frac{9\sqrt{2}}{\pi} g + \mathcal{O}(1).$$

We see that the leading order scales like $\Gamma_{\text{cusp}} \sim g^5$ which is in contrast with the correct behavior $\Gamma_{\text{cusp}} \sim g$ and it is also faster than the resummed Wilson loop function $\mathcal{F}_{rs}(g, z) \sim g^4$. Note that this is similar to what happens with the resummation of all tree diagrams where $\mathcal{F}_{\text{tree}}(g, z) \sim \mathcal{O}(\frac{1}{g^2})$ for $g \gg 1$ but the contribution to Gamma cusp is $\Gamma_{\text{tree}} \sim g$. On the other hand, even with the behavior $\sim g^5$, this is better than the leading $\sim g^6$ behavior of a triangle diagram at weak coupling. Hence we can refer to this sum also a partially summable. Note that the appearance of Euler constant γ in (C.6) is spurious from the point of view of the full result where only the single zeta values appear.

Similar result can be obtained for the box and ladder series,

$$(C.6) \quad \Gamma_{\text{cusp}} \left(g^6 \begin{array}{c} \bullet \text{---} \bullet \\ | \quad | \\ \bullet \text{---} \otimes \end{array} - g^8 \begin{array}{c} \bullet \text{---} \bullet \\ | \quad | \\ \bullet \text{---} \bullet \text{---} \otimes \end{array} + g^{10} \begin{array}{c} \bullet \text{---} \bullet \\ | \quad | \\ \bullet \text{---} \bullet \text{---} \bullet \text{---} \otimes \end{array} + \dots \right) = \\ \Gamma_{\text{cusp}} \left(g^6 \begin{array}{c} \bullet \text{---} \bullet \\ | \quad | \\ \bullet \text{---} \otimes \end{array} \times \otimes \text{Sum of ladders} \right)$$

and

$$(C.7) \quad \mathcal{I}[g^6 \mathcal{F}_4(z) \mathcal{F}_{\text{ladder}}(g, z)]_{g \gg 1} \sim \frac{394}{945} \pi^5 \sqrt{2} \tanh(\sqrt{2}\pi g) g^7 + \left(\frac{4\pi^4}{15} - 192\zeta_3 \right) g^6 - \frac{(2\sqrt{2}(990\zeta_3 + 13\pi^4 - 255\pi^2)) \tanh(\sqrt{2}\pi g)}{15\pi} g^5 + \mathcal{O}(g^4),$$

$$(C.8) \quad \Gamma_{\text{cusp}} \sim -\frac{3152\sqrt{2}\pi^5}{6615}g^7 - \frac{4}{3}\left(\frac{4\pi^4}{15} - 192\zeta_3\right)g^6 - \frac{16\sqrt{2}(990\zeta_3 + 13\pi^4 - 255\pi^2)}{75\pi}g^5 + \mathcal{O}(g^4),$$

where we don't include the powers of g lower than g^5 . Note that this sum is also partially summable (the leading term contributes $\sim g^8$ in weak coupling).

Note that just like the three-loop case, the Γ_{cusp} contribution from the resummed result is the same as the leading diagram in the weak coupling expansion suppressed by $\frac{1}{g}$. Furthermore, even though individual terms give $\log(g)$ contributions, they again end up canceling out. We can actually see the same phenomenon even for tree diagrams, we consider

$$(C.9) \quad g^6 \text{ (triangle)} - g^8 \text{ (triangle + line)} + g^{10} \text{ (triangle + 2 lines)} + \dots$$

$$(C.10) \quad \mathcal{I}[\mathcal{F}_{3l,\text{ladder}}] = \sqrt{8}\pi g^3 \tanh(\sqrt{2}\pi g) - 6g^2 + \frac{3\sqrt{2}g \tanh(\sqrt{2}\pi g)}{\pi}.$$

In this case, the leading behavior is $\sim g^3$, with even a larger suppression from the first term in the sum that behaves like $\sim g^6$, still short of the correct $\Gamma_{\text{cusp}} \sim g$ behavior.

We can also attach the whole series of tree graphs to the triangle core and compute the contribution to the Γ_{cusp} of the following series,

$$(C.11) \quad \Gamma_{\text{cusp}} \left(g^4 \text{ (triangle)} \times \left(\text{triangle} - g^2 \text{ (triangle + line)} + g^4 \text{ (triangle + 2 lines)} \right. \right.$$

$$(C.12) \quad \left. + \frac{g^4}{2!} \text{ (triangle)} - g^6 \text{ (triangle + 3 lines)} - g^6 \text{ (triangle + 2 lines)} - \frac{g^6}{2!} \text{ (triangle + line)} \dots \right)$$

$$= \Gamma_{\text{cusp}} \left(g^4 \text{ (triangle)} \times \text{Sum of trees} \right)$$

We cannot solve the full result analytically, but the leading contribution for $g \gg 1$ is again $\Gamma_{\text{cusp}} \sim g^5$ as shown in Appendix (C.2.4). This all demonstrates that we have to consider a larger set of one-cycle negative geometries to get the right behavior for $g \gg 1$. In fact, it is an important open question if the sum of *all* one-cycle geometries actually goes like $\sim g$ for large g or if we have to combine it with higher cycle geometries.

C.2 Details on the triangle-ladder computation

C.2.1 Differential Equation for $\mathcal{F}_{\text{tri-lad}}(g, z)$

We can solve the differential equation for $\mathcal{F}_{\text{tri-lad}}(g, z)$ in terms of infinite series, by taking the Mellin transform of both sides. Since the source $\mathcal{F}_3(z)$ is linear in HPLs, the solution $\mathcal{F}_{\text{tri-lad}}(g, z)$ will be a linear combination of functions $f_i(g, z)$ that solve this differential equation for each source term. Taking the Mellin Transform, we have

$$(C.13) \quad \tilde{\mathcal{F}}(\nu, g) = \int_0^1 z^{\nu-1} \mathcal{F}(g, z) dz .$$

Note that the integration domain is reduced from $(0, \infty)$ to $(0, 1)$ since we are working with harmonic polylogarithms which remain invariant under $z \rightarrow \frac{1}{z}$. Then in Mellin space, the action of the differential box operator is given by

$$(C.14) \quad \frac{1}{2} \int_0^1 (z \partial_z)^2 z^{\nu-1} f(z) dz = \frac{\nu^2}{2} \tilde{f}(\nu) .$$

Using this for the full differential equation, the Laplace equation from before (7.71) conveniently becomes

$$(C.15) \quad \tilde{\mathcal{F}}(\nu, g) = \frac{g^4 \nu^2}{2g^2 + \nu^2} \tilde{\mathcal{F}}_3(\nu, g) ,$$

where $\tilde{\mathcal{F}}(\nu, g)$ is the Mellin transform of $\mathcal{F}_{\text{tri-lad}}(g, z)$ and $\tilde{\mathcal{F}}_3(\nu)$ is the Mellin transform of $\mathcal{F}_3(z)$. The RHS is a sum over different harmonic sums, so the task at hand is to take the inverse Mellin transform of terms of the form $\frac{\nu^2}{\nu^2 + g^2} \sum_{i=\nu}^{\infty} \frac{-1}{i^p}$. Some pieces are easier than others, so let's start with the easiest term to see how it works for the general case. In Mellin space, $H_{0,0,0,0}(z) = \frac{\log^4(z)}{24}$ is given by

$$(C.16) \quad \mathcal{M}(H_{0,0,0,0}(z), \nu) = \frac{1}{\nu^5} .$$

Then the inverse transform of $\frac{8\nu^2}{2g^2+\nu^2}\mathcal{M}(H_{0,0,0,0}(z),\nu)$ is given by

$$(C.17) \quad f_1(g, z) = \int_C z^{-\nu} \frac{8}{(\nu^2 + 2g^2)\nu^3} d\nu.$$

If we choose a contour that only has real poles, we find

$$(C.18) \quad f_1(g, z) = \frac{2(g^2 \log^2(z) - 1)}{g^4}.$$

This indeed satisfies the equation

$$(C.19) \quad \frac{1}{2}(z\partial_z)^2 f_1(g, z) + g^2 f_1(g, z) = \frac{1}{2}(z\partial_z)^2 (8H_{0,0,0,0}(z)) = 2\log^2(z).$$

where we have suppressed the g^4 on the RHS for convenience and will restore it at the end after all terms are gathered. Note that we can also add terms of the form $C(g) \cos(\sqrt{2}g \log z)$ and $D(g) \sin(\sqrt{2}g \log z)$ which vanish under the operator $\frac{1}{2}(z\partial_z)^2 + g^2$. This corresponds to the homogeneous solution of the differential equation. We will discuss how to fix the coefficients of the homogeneous solution later and focus first on the particular solution, which we write as $\mathcal{F}_p(g, z)$.

Using the same logic as above, we can similarly solve rest of the terms in $\square \mathcal{F}_p(g, z) + g^2 \mathcal{F}_p(g, z) = g^4 \square \mathcal{F}_3(z)$ for each HPL that the $\square$ on the RHS acts on. Recalling that

$$(C.20) \quad \begin{aligned} \mathcal{F}_3(z) &= 8H_{0,0,0,0} + 8H_{-1,0,0,0} - 16H_{-1,-1,0,0} + 8H_{-2,0,0} - 8\zeta_3(2H_{-1} - H_0) \\ &+ 4\pi^2(H_{-1,0} - 2H_{-1,-1} + H_{-2}) + \frac{13\pi^4}{45}. \end{aligned}$$

We denote the partial solutions $f_i(g, z)$ such that $\mathcal{F}_p(g, z) = g^4 \sum_i^9 f_i(g, z)$ and each solves

$$\begin{aligned} \square f_1(g, z) + g^2 f_1(g, z) &= \square(8H_{0,0,0,0}(z)) \\ \square f_2(g, z) + g^2 f_2(g, z) &= \square(8H_{-1,0,0,0}(z)) \\ \square f_3(g, z) + g^2 f_3(g, z) &= \square(-16H_{-1,-1,0,0}(z)) \\ \square f_4(g, z) + g^2 f_4(g, z) &= \square(8H_{-2,0,0}(z)) \\ \square f_5(g, z) + g^2 f_5(g, z) &= \square(-16\zeta_3 H_{-1}(z)) \\ \square f_6(g, z) + g^2 f_6(g, z) &= \square(8\zeta_3 H_0(z)) \\ \square f_7(g, z) + g^2 f_7(g, z) &= \square(4\pi^2 H_{-1,0}(z)) \end{aligned}$$

$$\square f_8(g, z) + g^2 f_8(g, z) = \square(-8\pi^2 H_{-1,-1}(z))$$

$$\square f_9(g, z) + g^2 f_9(g, z) = \square(4\pi^2 H_{-2}(z))$$

(C.21)

These functions are written below

$$(C.22) \quad f_2(g, z) = \sum_{n=1}^{\infty} -\frac{4e^{i\pi n} n z^n \log^3(z)}{6g^2 + 3n^2} + \frac{4e^{i\pi n} (n^2 - 2g^2) z^n \log^2(z)}{(2g^2 + n^2)^2} \\ - \frac{8e^{i\pi n} n (n^2 - 6g^2) z^n \log(z)}{(2g^2 + n^2)^3} + \frac{8e^{i\pi n} (4g^4 - 12g^2 n^2 + n^4) z^n}{(2g^2 + n^2)^4}$$

$$(C.23) \quad f_3(g, z) = \sum_{i,j=1}^{\infty} -\frac{8e^{i\pi(i+j)}(i+j) \log^2(z) z^{i+j}}{j(2g^2 + (i+j)^2)} + \frac{16e^{i\pi(i+j)} \log(z) (2g^2 i + (i+j)^2(i+2j)) z^{i+j}}{j^2(2g^2 + (i+j)^2)^2} \\ - \frac{16e^{i\pi(i+j)} (4g^4 i + 2g^2(i+j) (2i^2 + 4ij - j^2) + (i+j)^3 (i^2 + 3ij + 3j^2)) z^{i+j}}{j^3(2g^2 + (i+j)^2)^3}$$

$$(C.24) \quad f_4(g, z) = \sum_{n=1}^{\infty} \frac{8e^{i\pi n} (2g^2 - 3n^2) z^n}{(2g^2 + n^2)^3} - \frac{4e^{i\pi n} z^n \log^2(z)}{2g^2 + n^2} + \frac{16e^{i\pi n} n z^n \log(z)}{(2g^2 + n^2)^2}$$

$$(C.25) \quad f_5(g, z) = 16\zeta_3 \sum_{n=1}^{\infty} \frac{e^{i\pi n} n z^n}{2g^2 + n^2},$$

$$(C.26) \quad f_6(g, z) = f_h,$$

$$(C.27) \quad f_7(g, z) = -4\pi^2 \sum_{n=1}^{\infty} \frac{z^n e^{i\pi n} (g^2 n \log(z) + g^2 + n^3 \log(z) - n^2)}{(2g^2 + n^2)^2},$$

$$(C.28) \quad f_8(g, z) = - \sum_{n=1}^{\infty} \frac{4\pi^2 e^{i\pi n} z^n (2g^2 n \log(z) + 2g^2 + n^3 \log(z) - n^2)}{(2g^2 + n^2)^2},$$

$$(C.29) \quad f_9(g, z) = -4\pi^2 \sum_{n=1}^{\infty} \frac{e^{i\pi n} z^n}{2g^2 + n^2},$$

and thus

$$(C.30) \quad \mathcal{F}_p(g, z) = g^4 \sum_{i=1}^9 f_i(g, z).$$

where f_h denotes terms that go into the homogenous solution. Most of the sums here can be analytically expressed in terms of hypergeometric functions (put in the ancillary file `Ftriangleladder.m`), but the sums $f_3(g, z)$ and $f_8(g, z)$ don't have a nice form. These terms however have an integral representation, which we can extract the large g asymptotics of using integration by parts. For $g \gg 1$, these integral terms go like (to leading order in g)

$$(C.31) \quad \mathcal{F}_{IT}(g, z)_{g \gg 1} \sim \frac{4zg^2}{(z+1)^2} (2\text{Li}_3(-z) - 2\text{Li}_2(-z)\log(z) - \log(z+1)\log^2(z)) + \mathcal{O}(g)$$

We can then expand the full result for the particular solution at strong coupling and get

$$(C.32) \quad \mathcal{F}_{3,p}(g, z)_{g \gg 1} \sim \frac{g^2 (4(z(z+4)+1)\log^2(z) + 4\pi^2 z \log(z) - 8\pi^2 z \log(z+1))}{2(z+1)^2} \\ + \frac{2g^2 z (12\text{Li}_3(-z) - 12\text{Li}_2(-z)\log(z) + \log^3(z) - 6\log(z+1)\log^2(z) - 12\zeta_3 + 6\pi^2)}{3(z+1)^2} \\ + \mathcal{O}(g).$$

Note that this is only the particular solution to the equation

$$(C.33) \quad \square \mathcal{F}_{\text{tri-lad}}(g, z) + g^2 \mathcal{F}_{\text{tri-lad}}(g, z) = g^4 \square \mathcal{F}_3(z),$$

but we also have to add the homogeneous solution that solves

$$(C.34) \quad \square \mathcal{F}_h(g, z) + g^2 \mathcal{F}_h(g, z) = 0,$$

namely $\mathcal{F}_h(g, z) = c_1 \sin(\sqrt{2}g \log(z)) + c_2 \cos(\sqrt{2}g \log(z))$. While this looks identical to the ladder example, this time the boundary conditions are slightly different. The full solution is then $\mathcal{F}_{rs}(g, z) = \mathcal{F}_p(g, z) + \mathcal{F}_h(g, z)$ and the coefficients c_1, c_2 must be fixed using the boundary conditions. The first

boundary condition, namely $\frac{d}{dz}\mathcal{F}_{\text{tri-lad}}(g, z)|_{z=-1} = 0$, sets c_1 to 0. Fixing c_2 is more subtle. For this, we need the value of the full solution at $z = -1$, which we know must equal $g^4\mathcal{F}_3(-1)$. However, the source term of the differential equation has a branch point at $z = 0$ and the solution is only analytic for $z > 0$. So we can essentially treat our $\mathcal{F}_{\text{tri-lad}}(g, z)$ as being defined on positive real z , and analytically continue it through the upper half plane and evaluate it at $z = -1$. Then we have the following condition

$$(C.35) \quad \mathcal{F}_{\text{tri-lad}}(z = e^{i\pi}, g) = g^4\mathcal{F}_3(z = e^{i\pi}) = \frac{22\pi^4 g^4}{45}$$

This fixes the c_2 term and gives us the homogeneous solution

$$(C.36) \quad \mathcal{F}_h(g, z)|_{g \gg 1} \sim \frac{1}{45}g^2 \cos\left(\sqrt{2}g \log(z)\right) \left((22\pi^4 g^2 + 90\pi^2 + 90) \operatorname{sech}\left(\sqrt{2}\pi g\right) + \mathcal{O}(ge^{-\sqrt{2}g\pi}) \right).$$

Then at strong g and $z > 0$, the full solution $F_{\text{tri-lad}}(g, z) = F_p(g, z) + F_h(g, z)$ to leading order is

$$F_{\text{tri-lad}}(g, z)_{g \gg 1} \sim -\frac{g^4 (2(z(z+4)+1)\log^2(z) + 2\pi^2 z \log(z) - 4\pi^2 z \log(z+1))}{(z+1)^2} - \frac{2g^4 z (12\operatorname{Li}_3(-z) - 12\operatorname{Li}_2(-z)\log(z) + \log^3(z) - 6\log(z+1)\log^2(z) - 12\zeta_3 + 6\pi^2)}{3(z+1)^2} + \mathcal{O}(g^3).$$

where we have written $F(g, z)$ instead of $\mathcal{F}(g, z) = -\frac{1}{g^2}F(g, z)$. Note that this is simply the source term divided by g^2 , which amounts to ignoring the differential operator to leading order when g is large. This is only true for $z > 0$, where the negative geometries live, because the homogeneous solution is exponentially suppressed in this region. In particular, the leading behavior of $\mathcal{F}_h(z > 0, g)$ is $g^4 e^{-\sqrt{2}g\pi}$, which is exponentially suppressed for large g . However, the numerator has a $\cos(\sqrt{2}g \log(z))$ factor, which contributes a $e^{\sqrt{2}g\pi}$ that cancels this suppression upon analytical continuation to $z = -1$. This explains how we can match the triangle diagram's g^4 power at $z = -1$ despite having the full $\mathcal{F}_{\text{tri-lad}}(g, z)$ behave like g^2 for large g in the region we are interested in. Indeed, the Γ_{cusp} contribution of $\mathcal{F}_{\text{tri-lad}}(g, z)$ is not given by the leading order term we quote because the homogeneous solution $\mathcal{F}_h$ picks up a $\cosh \sqrt{2}g\pi$ along the contour we integrate. Thus, the part that was sub-leading for positive and real z becomes relevant.

For well behaved source terms (e.g. for $\square\mathcal{F}_4(z)$), we should expect to find that the $\frac{1}{g^2}$ drop in $F(g, z)$ after resumming ladder attachments is a general feature. Say we have a generic “core”

diagram with $\square \mathcal{F}_c(z) = h(z)$. If the core has L loops, the differential equation at hand reads

$$(C.37) \quad \square F_{c,l}(g, z) + g^2 F_{c,l}(g, z) = g^{2L} h(z)$$

where $\mathcal{F}_{c,l}(g, z) = -\frac{1}{g^2} F_{c,l}(g, z)$ corresponds to resumming all ladder attachments to the core diagram. If $h(z)$ is non-singular for $z > 0$, then $F_{p,cl}(g, z) \sim g^{2L-2}$ at most in large g . The homogeneous solution should then be of the form $F_{h,cl}(g, z) = c(g) \cos(\sqrt{2}g \log(z))$, where $c(g)$ must be solved by $F_{c,l}(g, z = e^{i\pi}) = g^{2L} h(z = e^{i\pi})$ and the sine term's coefficient is set to 0 by $\frac{d}{dz} F_{c,l}(z)|_{z=1} = 0$. If we find a branch cut at $z = 0$ again, we can repeat the procedure from above, in which case $c(g)$ picks up $\frac{1}{\cosh(\sqrt{2}g\pi)}$ in order to match the leading term. However, for $z > 0$, this exponentially suppresses the homogeneous solution for large g and kills any polynomial dependence for g we have the numerator in $F_h(g, z)$. Hence, the particular solution should control the asymptotics and give $F_{c,l}(g \gg 1, z > 0) \sim g^{2L-2}$ at most.

C.2.2 Necessary Sums for $I[\mathcal{F}_{\text{ladder}}(g, z)\mathcal{F}_3(z)]$

For $|z| < 1$, the harmonic polylogarithms that appear in our $\mathcal{F}(z)$ functions have the series expansion [223]

$$(C.38) \quad H_{-m_n, \dots, -m_1}(z) \log(z)^a = (-1)^n \sum_{j_n > \dots > j_1} \frac{\log(z)^a (-z)^{j_n}}{j_n^{m_n} \dots j_1^{m_1}}.$$

So in order to compute $\mathcal{I}[\mathcal{F}_3(z)\mathcal{F}_{\text{ladder}}(g, z)]$, we must compute integrals of the form

$$(C.39) \quad (-1)^n \sum_{j_n > \dots > j_1} \int_{-\pi}^{\pi} \frac{\cosh \sqrt{2}g\phi(i\phi)^a (-e^{i\phi j_n})}{j_n^{m_n} \dots j_1^{m_1}}.$$

While the integration itself is easy, the nested sums that appear are non-trivial to sum into analytical functions. We are able to perform these nested sums for depth up to 2. Here is a list of sums we will find useful:

$$\begin{aligned} \sum_{j_1=1}^{\infty} \frac{1}{j_1^2(a+j_1)} &= \frac{\zeta_2}{a} - \frac{1}{a^2} H_a \\ \sum_{j_2 > j_1=1}^{\infty} \frac{1}{j_1 j_2(a+j_2)} &= \frac{1}{2a} H_a - \frac{1}{2a} H_{a,2} \\ \sum_{j_1=1}^{\infty} \frac{1}{j_1(a+j_1)} &= \frac{1}{a} H_a \end{aligned}$$

$$\begin{aligned}
\sum_{j_2 > j_1 = 1}^{\infty} \frac{1}{j_1(a + j_2)} &= \zeta_{1,1} - \frac{1}{2}H_a^2 - \frac{1}{2}H_{a,2} \\
\sum_{j_1=1}^{\infty} \frac{1}{(a + j_1)} &= \zeta_1 - H_a \\
\sum_{j_1=1}^{\infty} \frac{1}{j_1^3(a + j_1)} &= \frac{\zeta_3}{a} - \frac{\zeta_2}{a^2} + \frac{H_b}{b^3} \frac{1}{a} H_a \\
\sum_{j_1=1}^{\infty} \frac{j_1^2}{(a + j_1)^4} &= \zeta_2 - 2a\zeta_3 + a^2\zeta_4 - H_{a,2} + 2aH_{a,3} - a^2H_{a,4} \\
\sum_{j_1=1}^{\infty} \frac{j_1}{(a + j_1)^4} &= \zeta_3 - a\zeta_4 - H_{a,3} + aH_{a,4} \\
\sum_{j_1=1}^{\infty} \frac{1}{(a + j_1)^4} &= \zeta_4 - H_{a,4} \\
\sum_{j_1=1}^{\infty} \frac{1}{j_1(a + j_1)^4} &= \zeta_2 - 2a\zeta_3 + a^2\zeta_4 - H_{a,2} + 2aH_{a,3} - a^2H_{a,4} \\
\sum_{j_1=1}^{\infty} \frac{1}{j_1^2(a + j_1)^4} &= \frac{4\zeta_2}{a^4} + \frac{2\zeta_3}{a^3} + \frac{\zeta_4}{a^2} - \frac{4H_a}{a^5} - \frac{3H_{a,2}}{a^4} - \frac{2H_{a,3}}{a^3} - \frac{H_{a,4}}{a^2} \\
\sum_{j_1=1}^{\infty} \frac{1}{j_1^2(a + j_1)^3} &= \frac{3\zeta_2}{a^3} + \frac{\zeta_3}{a^2} - \frac{3H_a}{a^4} - \frac{2H_{a,2}}{a^3} - \frac{H_{a,3}}{a^2}
\end{aligned}$$

For the 1-cycle geometries we have, namely one that have the triangle or the box as a core, the most complicated nested sums have the form

$$(C.40) \quad \sum_{j_2 > j_1 > 0}^{\infty} \frac{j_2^a}{j_1^b(j_2^2 + 2g^2)^c}.$$

We can express these sums in terms of Polygamma functions and their derivatives. Let's start with $a = -1, b = 2, c = 2$ as an illustrative example. First, we can re-write the double sum in the simpler form

$$(C.41) \quad \sum_{j_2 > j_1 > 0}^{\infty} \frac{1}{j_1^2 j_2 (j_2^2 + 2g^2)^2} = \sum_{j_2=1}^{\infty} \sum_{j_1=0}^{\infty} \frac{1}{j_1^2 (j_2 + j_1) ((j_2 + j_1)^2 + 2g^2)^2}.$$

Doing a partial decomposition, the summand can be written as

$$\frac{1}{j_1^2 j_2 (j_2^2 + 2g^2)^2} = \frac{1}{4g^4 j_1^2 (j_1 + j_2)} - \frac{1}{8g^4 j_1^2 (-i\sqrt{2}g + j_1 + j_2)} - \frac{1}{8g^4 j_1^2 (i\sqrt{2}g + j_1 + j_2)}$$

$$(C.42) \quad + \frac{i}{8\sqrt{2}g^3j_1^2(-i\sqrt{2}g+j_1+j_2)^2} - \frac{i}{8\sqrt{2}g^3j_1^2(i\sqrt{2}g+j_1+j_2)^2}.$$

These sums can be written in terms of polygamma functions, which have the following series representation

$$(C.43) \quad \psi^{(m)}(z) = (-1)^{m+1}m! \sum_{k=0}^{\infty} \frac{1}{(z+k)^{m+1}}.$$

The sum over j_2 then becomes

$$(C.44) \quad \frac{2\psi^{(0)}(ig\sqrt{2}+j_1+1) + 2\psi^{(0)}(-ig\sqrt{2}+j_1+1) - i\sqrt{2}g\psi^{(1)}(ig\sqrt{2}+j_1+1)}{16g^4j_1^2} \\ + \frac{i\sqrt{2}g\psi^{(1)}(-ig\sqrt{2}+j_1+1) - 4\psi^{(0)}(j_1+1)}{16g^4j_1^2}.$$

The $\frac{\psi^{(0)}(-ig\sqrt{2}+j_1+1)}{16g^4j_1^2}$ sums can be evaluated using the identity $\psi^{(0)}(1+z) = \psi^{(0)}(z) + \frac{1}{z}$. More specifically, $\psi^{(0)}(-ig\sqrt{2}+j_1+1) = \psi(1-ig\sqrt{2}) + \sum_{k=0}^{j_1-1} \frac{1}{1-ig\sqrt{2}+k}$ and the sum becomes

$$(C.45) \quad \sum_{j_2>j_1>0}^{\infty} \frac{1}{j_1^2j_2(j_2^2+2g^2)^2} = -\frac{12 {}_4F_3^{(b_3,1)}(\{1,1,1,2-i\sqrt{2}g\},\{2,2,2-i\sqrt{2}g\},1)}{96g^4} \\ - \frac{6 {}_4F_3^{(a_4,1)}(\{1,1,1,2-i\sqrt{2}g\},\{2,2,2-i\sqrt{2}g\},1) + \pi^2\psi^{(0)}(2-i\sqrt{2}g)}{96g^4},$$

where the notation ${}_pF_q^{(b_3,0)}$ means the generalized hypergeometric function

$$(C.46) \quad {}_pF_q(\{a_1, \dots, a_p\}, \{b_1, \dots, b_q\}, z),$$

is being differentiated with respect to the index b_3 a single time. While these hypergeometric terms frequently appear in sums of our interest, they always do so in a sub-leading way in g and fall off in the strong g limit. The sums involving higher order polygamma functions can be evaluated by taking appropriate derivatives in g of sums of the form above, using the identity $\frac{d^m}{dz^m}\psi^{(m)}(1-ig\sqrt{2}+j_1) = (-ig\sqrt{2})^m\psi^{(0)}(1-ig\sqrt{2}+j_1)$.

C.2.3 Coefficients in our single cycle+ladder results

We have the general $\mathcal{F}$ function for n -ladders attached to a triangle as follows

$$\mathcal{F}_{3,n}(z) = (-2)^n \left[8H_{\vec{0}_{2n},0,0,0,0}(z) + 8H_{\vec{0}_{2n},-1,0,0,0}(z) - 16H_{\vec{0}_{2n},-1,-1,0,0}(z) + 8H_{\vec{0}_{2n},0,-1,0,0}(z) \right]$$

$$\begin{aligned}
& -16\zeta_3 H_{\vec{0}_{2n,-1}}(z) + 4\pi^2 \left(H_{\vec{0}_{2n,-1,0}}(z) - 2H_{\vec{0}_{2n,-1,-1}}(z) + H_{\vec{0}_{2n,-2}}(z) \right) \\
& + \frac{13\pi^4}{45} \frac{1}{(2n)!} \log^{2n}(z) + 8\zeta_3 \frac{1}{(2n+1)!} \log^{2n+1}(z) \\
& + \sum_{k=1}^n (-1)^k \left(\frac{C_k}{[2(n-k)+1]!} \log z^{2(n-k)+1} + \frac{D_k}{[2(n-k)]!} \log z^{2(n-k)} \right) \Bigg].
\end{aligned}
\tag{C.47}$$

Now, let's define the sequence η_n recursively by $\eta_0 = 8$ and for $n > 0$,

$$\eta_n \equiv -\frac{8}{(2n+1)!} \left[1 + (2n-1)(2n+1)2^{2n}B_{2n} \right] - \sum_{k=1}^{n-1} \frac{(-1)^k \eta_k}{(2n-2k+1)!}
\tag{C.48}$$

$$= 4, \frac{17}{15}, \frac{457}{1890}, \frac{3287}{75600}, \dots
\tag{C.49}$$

Then the C_n coefficients of $\log z$ (which are defined with an alternating sign with respect to $I_{3,n}$ above) are given for all n by

$$C_n = (-1)^n \sum_{k=0}^n \eta_k \pi^{2k} \zeta(2n-2k+3).
\tag{C.50}$$

The Bernoulli numbers B_{2N} appear because powers of π^2 are generated both by analytic continuation of $\log z$ and from Riemann zeta values ζ_{2k} .

The first few C_n look simpler using even zeta values instead of powers of π^2 :

$$C_0 = 8f_3,
\tag{C.51}$$

$$C_1 = -(8f_5 + 24\zeta_2 f_3),
\tag{C.52}$$

$$C_2 = 8f_7 + 24\zeta_2 f_5 + 102\zeta_4 f_3,
\tag{C.53}$$

$$C_3 = -\left(8f_9 + 24\zeta_2 f_7 + 102\zeta_4 f_5 + \frac{457}{2}\zeta_6 f_3 \right).
\tag{C.54}$$

Here we also introduced the f alphabet arising in a motivic decomposition of MZVs [237], as encoded in the package HYPERLOGPROCEDURES by Oliver Schnetz [238]. For odd Riemann zeta values, the f -alphabet is rather trivial,

$$f_{2k+1} \equiv \zeta(2k+1).
\tag{C.55}$$

The constants D_n can also be written in an all-orders recursive form:

$$(C.56) \quad D_n = (-1)^n \left[\rho_n \pi^{2n+4} - \sum_{j=0}^n \xi_{j+1} \pi^{2j} \sum_{k=1}^{n-j} k(2k-1) f_{2(n-j-k)+3, 2k+1} \right],$$

where the pure- π coefficients ρ_n are given recursively by $\rho_0 = 13/45$, $\rho_1 = 7/45$, and

$$(C.57) \quad \rho_n = (-1)^{n+1} \left[\frac{8}{(2n+4)!} \left(1 - 2^{2n+3} B_{2n+4} \right) + \frac{13}{45} \frac{1}{(2n)!} + \sum_{k=1}^{n-1} (-1)^k \frac{\rho_k}{[2(n-k)]!} \right]$$

$$(C.58) \quad = \frac{7}{45}, \frac{4891}{75600}, \frac{97987}{3742200}, \frac{866066051}{81729648000}, \dots$$

The seed sequence for the depth-two MZV terms, ξ_n , is given by $\xi_1 = 16$

$$(C.59) \quad \xi_n = \sum_{k=1}^{n-1} (-1)^{n+k+1} \frac{\xi_k}{[2(n-k)]!}$$

$$(C.60) \quad = 16, 8, \frac{10}{3}, \frac{61}{45}, \frac{277}{504}, \dots$$

The depth-two MZV's are encoded in the f -alphabet.

The first few D_n 's are given by:

$$(C.61) \quad D_0 = 26\zeta_4,$$

$$(C.62) \quad D_1 = 16f_{3,3} - 147\zeta_6,$$

$$(C.63) \quad D_2 = - \left(16f_{5,3} + 96f_{3,5} + 48\zeta_2 f_{3,3} - \frac{4891}{8}\zeta_8 \right),$$

$$(C.64) \quad D_3 = 16f_{7,3} + 96f_{5,5} + 240f_{3,7} + \zeta_2(48f_{5,3} + 288f_{3,5}) + 300\zeta_4 f_{3,3} - \frac{97987}{40}\zeta_{10}.$$

Once we have the $\mathcal{F}$ for a diagram, we can compute the Γ_{cusp} contributions from the prescription in [77] given by

$$(C.65) \quad g\partial_g \Gamma_{\text{cusp}}(g) = -8\mathcal{I}[\mathcal{F}(g, z)],$$

where $\mathcal{I}$ integrates F through a contour along the unit circle, such that

$$(C.66) \quad \mathcal{I}[f(z)] = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{i\phi}) d\phi.$$

We can use this to work out

$$(C.67) \quad \mathcal{I}[z^p] = \frac{\sin \pi p}{\pi p},$$

and similarly

$$(C.68) \quad \mathcal{I}[\log z^n] = \frac{\pi^n \cos\left(\frac{\pi n}{2}\right)}{n+1}.$$

The integrated answer $\mathcal{I}_n \equiv I[\mathcal{F}_{3,n}(z)]$ is then given by

$$(C.69) \quad \begin{aligned} \mathcal{I}_n = & \alpha_n + 32 (-1)^{n+1} \sum_{k=1}^n k f_{2(n-k)+3, 2k+1} \\ & + (-1)^{n+1} \sum_{j=0}^n \xi_{j+1} \pi^{2j} \sum_{k=1}^{n-j} k(2k-1) f_{2(n-j-k)+3, 2k+1}, \end{aligned}$$

and the Γ_{cusp} contribution of each diagram is

$$(C.70) \quad \begin{aligned} \Gamma_{\text{cusp}}(\mathcal{F}_{3,n}(z)) = & -\frac{4g^{2(n+3)}}{(n+3)} \left(\alpha_n + 32 (-1)^{n+1} \sum_{k=1}^n k f_{2(n-k)+3, 2k+1} \right. \\ & \left. + (-1)^{n+1} \sum_{j=0}^n \xi_{j+1} \pi^{2j} \sum_{k=1}^{n-j} k(2k-1) f_{2(n-j-k)+3, 2k+1} \right), \end{aligned}$$

where the pure- π terms are given by $\alpha_0 = 2/15 \times \pi^4$, $\alpha_1 = -176/2835 \times \pi^6$, and for $n > 1$,

$$(C.71) \quad \begin{aligned} \alpha_n = & (-1)^{n+1} \left[\frac{4}{3} \frac{(2n+5)(2n^2+n+3)}{2n+1} \zeta(2n+4) + \frac{8}{3} \pi^2 \zeta(2n+2) \right] \\ & + \pi^{2n+4} \left[\frac{8}{(2n+5)!} + \frac{13}{45(2n+1)!} + \sum_{k=1}^n (-1)^k \frac{\rho_k}{[2(n-k)+1]!} \right]. \end{aligned}$$

We could also replace the even zeta values with Bernoulli numbers in this formula.

Here is an alternate version of $\mathcal{I}_n$ which uses conventional MZVs, namely $\zeta_{m,n} = \sum_{j>i>0} \frac{1}{j^m i^n}$, instead of the f -alphabet [238]:

$$(C.72) \quad \begin{aligned} \mathcal{I}_n = & \beta_n + (-1)^n \left[16 \zeta_{2n+2,2} - 32 \sum_{k=1}^n k \zeta_{2k+1} \zeta_{2(n-k)+3} \right. \\ & + \sum_{j=0}^n \xi_{j+1} \pi^{2j} \left(-\zeta_{3,2(n-j)+1} + [2(n-j)+1] \zeta_{2(n-j)+2,2} \right. \\ & \left. \left. + (n-j+1)[2(n-j)+1] \zeta_{2(n-j)+3,1} \right) \right], \end{aligned}$$

where the pure- π terms are now given by:

$$\begin{aligned}
\beta_n = (-1)^{n+1} & \left[-8\zeta_{2n+4} + \frac{8}{3}\pi^2\zeta_{2n+2} \right. \\
& + \sum_{k=0}^n \xi_{k+1} \pi^{2k} \zeta_{2(n-k)+4} \left(\frac{1}{6}[2(n-k)+1][2(n-k)^2 + 2(n-k) - 3] \right. \\
& \quad \left. \left. + \tilde{\delta}_{2(n-k)+1} \right) \right] \\
& + \pi^{2n+4} \left[\frac{8}{(2n+5)!} + \frac{13}{45(2n+1)!} + \sum_{k=1}^n (-1)^k \frac{\rho_k}{[2(n-k)+1]!} \right].
\end{aligned}
\tag{C.73}$$

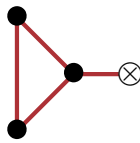
and

$$\begin{aligned}
\tilde{\delta}_n &= \frac{5}{4}, \quad n = 1, \\
&= \frac{1}{2}, \quad n = 3, \\
&= 0, \quad \text{otherwise.}
\end{aligned}
\tag{C.74}$$

As an example, we show the Γ_{cusp} contributions of the triangle with one, two and three ladders attached on one end below. To start off, the triangle with no ladders attached contributes

$$|\Gamma_{\text{triangle}}| = \frac{16g^6}{6} \left(\frac{2\pi^4}{15} \right) = \frac{8g^6\pi^4}{45}.
\tag{C.75}$$

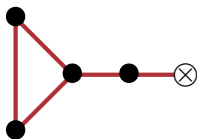
Next, the triangle with a single ladder attached, namely the following diagram

$$\tag{C.76}$$


contributes

$$|\Gamma_{\text{triangle},1}| = g^8 \left(\frac{352\pi^6}{2835} - 48\zeta_3^2 \right).
\tag{C.77}$$

For double ladder configuration

$$\tag{C.78}$$


we have

$$(C.79) \quad \Gamma_{\text{triangle},2} = -\frac{4g^{10}}{5} \left(64\zeta_{5,3} + 384\zeta_{6,2} - \frac{1}{3}32\pi^2\zeta_3^2 - 1408\zeta_5\zeta_3 + \frac{136\pi^8}{567} \right)$$

Note that we can't reduce this expression to single zetas, so we get our first true multiple-zeta value at 5-loops. We can re-write $\zeta_{5,3}$ in terms of $\zeta_{6,2}$ for a cleaner expression and get

$$(C.80) \quad \Gamma_{\text{triangle},2} = -\frac{4g^{10}}{5} \left(224\zeta_{6,2} - \frac{1}{3}32\pi^2\zeta_3^2 - 1088\zeta_5\zeta_3 + \frac{2812\pi^8}{14175} \right)$$

Whether we choose to express everything in terms of $\zeta_{5,3}$ or $\zeta_{6,2}$, we get an answer we can't reduce to only single zetas. For the triangle + 3-ladder, we have

$$(C.81) \quad \Gamma_{\text{triangle},3} = \frac{2g^{12}}{3} \left(\frac{128}{15} (5\pi^2\zeta_{5,3} + \pi^4\zeta_3^2 + 30\pi^2\zeta_5\zeta_3 + 555\zeta_7\zeta_3 + 270\zeta_5^2) - 128\zeta_{7,3} - 1024\zeta_{8,2} - \frac{25952\pi^{10}}{155925} \right).$$

Again, we can't reduce this expression to only single zetas. We can do the same type of analysis for the box. The inverted Laplacian acting on the box n -times gives:

$$(C.82) \quad \begin{aligned} \mathcal{F}_{4,n}(z) = & (-2)^{n+4} \left[3H_{\vec{0}_{2n+6}}(z) + 2H_{\vec{0}_{2n},-1,\vec{0}_5}(z) + 3H_{\vec{0}_{2n},-1,-1,\vec{0}_4}(z) - \frac{1}{2}H_{\vec{0}_{2n+1},-1,\vec{0}_4}(z) \right. \\ & - 4H_{\vec{0}_{2n+1},-1,-1,\vec{0}_3}(z) + 3H_{\vec{0}_{2n+2},-1,\vec{0}_3}(z) + \frac{3}{2}H_{\vec{0}_{2n+3},-1,0,0}(z) - 2H_{\vec{0}_{2n},-1,-2,\vec{0}_3}(z) \\ & + 2H_{\vec{0}_{2n+1},-1,-2,0,0}(z) + \frac{1}{6}\pi^2 H_{\vec{0}_{2n+4}}(z) + \frac{7}{6}\pi^2 H_{\vec{0}_{2n},-1,\vec{0}_3}(z) + \frac{1}{2}\pi^2 H_{\vec{0}_{2n},-1,-1,0,0}(z) \\ & - 5H_{\vec{0}_{2n},-1,-3,0,0}(z) - \frac{5}{12}\pi^2 H_{\vec{0}_{2n+1},-1,0,0}(z) - \frac{2}{3}\pi^2 H_{\vec{0}_{2n+1},-1,-1,0}(z) \\ & + \frac{1}{2}\pi^2 H_{\vec{0}_{2n+2},-1,0}(z) + \frac{3}{4}\pi^2 H_{\vec{0}_{2n+3},-1}(z) + 2\zeta_3 H_{\vec{0}_{2n+3}}(z) + 4\zeta_3 H_{\vec{0}_{2n},-1,0,0}(z) \\ & - 8\zeta_3 H_{\vec{0}_{2n+1},-1,0}(z) - \frac{1}{3}\pi^2 H_{\vec{0}_{2n},-1,-2,0}(z) + \pi^2 H_{\vec{0}_{2n+1},-1,-2}(z) + \frac{5}{72}\pi^4 H_{\vec{0}_{2n+2}}(z) \\ & + \frac{11}{45}\pi^4 H_{\vec{0}_{2n},-1,0}(z) + \frac{1}{8}\pi^4 H_{\vec{0}_{2n},-1,-1}(z) - \frac{5}{2}\pi^2 H_{\vec{0}_{2n},-1,-3}(z) - \frac{43}{144}\pi^4 H_{\vec{0}_{2n+1},-1}(z) \\ & + \frac{31}{2}\zeta_5 H_{\vec{0}_{2n+1}}(z) - 19\zeta_5 H_{\vec{0}_{2n},-1}(z) + \frac{3}{2}\pi^2 \zeta_3 H_{\vec{0}_{2n+1}}(z) - \frac{1}{3}\pi^2 \zeta_3 H_{\vec{0}_{2n},-1}(z) \\ & \left. + 11\zeta_3^2 H_{\vec{0}_{2n}}(z) + \frac{505\pi^6 H_{\vec{0}_{2n}}(z)}{9072} + \sum_{k=1}^n \hat{C}_k \frac{\log(z)^{2(n-k)+1}}{(2(n-k)+1)!} + \sum_{k=1}^n \hat{D}_k \frac{\log(z)^{2(n-k)}}{(2(n-k))!} \right]. \end{aligned}$$

The coefficients $\hat{C}_n$ and $\hat{D}_n$ are determined recursively using the boundary condition $I_{4,n}(-1) = 0$. The imaginary and real parts of the boundary condition fix $\hat{C}_n$ and $\hat{D}_n$ respectively. Let us start

with

$$(C.83) \quad \hat{C}_n = 2i \sum_{a=0}^{n-1} \frac{1}{(2n)!} |2^{2n-1} - 1| \pi^{2n-1} B_{2n} \hat{f}_{a-n},$$

where B_n is the n^{th} Bernoulli number and $\hat{f}_n$ is the imaginary part of the other functions in $I_{4,n}$ evaluated at -1. The first few coefficients $\hat{C}_n$ are then

$$(C.84) \quad \begin{aligned} \hat{C}_1 &= i \frac{1}{\pi} \hat{f}_1, \\ \hat{C}_2 &= \frac{1}{6} i \pi \hat{f}_1 + i \frac{1}{\pi} \hat{f}_2, \\ \hat{C}_3 &= \frac{7}{360} i \pi^3 \hat{f}_1 + \frac{1}{6} i \pi \hat{f}_2 + i \frac{1}{\pi} \hat{f}_3, \\ \hat{C}_4 &= \frac{31}{15120} i \pi^5 \hat{f}_1 + \frac{7}{360} i \pi^3 \hat{f}_2 + \frac{1}{6} i \pi \hat{f}_3 + i \frac{1}{\pi} \hat{f}_4. \end{aligned}$$

The function $\hat{f}_n$ can be expressed purely in terms of zeta and multiple zeta values by using the shuffle identities satisfied by polylogarithms. Simplified, this gives

$$(C.85) \quad \begin{aligned} \hat{f}_n &= -\frac{1}{30} i \pi \left[5(-12(2n+5)\zeta_{2n+2,3} - 6(n+1)((10n+17)\zeta_{2n+3,2} + (2n+3)(2n+5)\zeta_{2n+4,1}) \right. \\ &\quad + 24(2n+3)\zeta_3\zeta_{2n+2} - \pi^2(n+1)(10n+9)\zeta_{2n+3} - 4(n+1)(n+2)(n(2n+5)+12)\zeta_{2n+5}) \\ &\quad \left. + 180\zeta_{2n+1,4} + (-\pi^2)^n \left[\frac{30\pi^2(3n(2n+5)+7)\zeta_3}{\Gamma(2n+4)} - \pi^4 4^{n+1} \zeta_{-2n} \Gamma(-2n) + \frac{465\zeta_5}{\Gamma(2n+2)} \right] \right]. \end{aligned}$$

Putting this together, gives the first few $\hat{C}_i$:

$$(C.86) \quad \begin{aligned} \hat{C}_1 &= -6\zeta_{3,4} + 14\zeta_{4,3} + 54\zeta_{5,2} - 63\zeta_4\zeta_3 - \frac{33}{2}\zeta_2\zeta_5 + 286\zeta_7, \\ \hat{C}_2 &= -\pi^2\zeta_{3,4} + 14\zeta_2\zeta_{4,3} + 54\zeta_2\zeta_{5,2} - 6\zeta_{5,4} + 18\zeta_{6,3} + 111\zeta_{7,2} - \frac{5419}{16}\zeta_6\zeta_3 - \frac{1887}{8}\zeta_4\zeta_5 \\ &\quad + 184\zeta_2\zeta_7 + 996\zeta_9, \\ \hat{C}_3 &= -\frac{21}{2}\zeta_4\zeta_{3,4} + \frac{49}{2}\zeta_4\zeta_{4,3} + \frac{189}{2}\zeta_4\zeta_{5,2} - 6\zeta_2\zeta_{5,4} + 18\zeta_2\zeta_{6,3} + \frac{37}{2}\pi^2\zeta_{7,2} - 6\zeta_{7,4} + 22\zeta_{8,3} \\ (C.87) \quad &\quad + 188\zeta_{9,2} - \frac{30035}{32}\zeta_8\zeta_3 - \frac{25095}{32}\zeta_6\zeta_5 - \frac{289}{2}\zeta_4\zeta_7 + 756\zeta_2\zeta_9 + 2580\zeta_{11}. \end{aligned}$$

The coefficients $\hat{D}_n$ can similarly be written now using Euler numbers E_n ,

$$(C.88) \quad \hat{D}_n = \sum_{a=0}^{n-1} \frac{1}{(2n)!} \pi^{2n} (-1)^n E_{2n} \hat{g}_{a-n}.$$

As before, the function $\hat{g}_n$ is the value of the other polylogarithms in $I_{4,n}(z)$ evaluated at $z = -1$. The first few coefficients are

$$\begin{aligned}
\hat{D}_1 &= \hat{g}_1 \\
\hat{D}_2 &= \frac{1}{2}\pi^2 \hat{g}_1 + \hat{g}_2 \\
\hat{D}_3 &= \frac{5}{24}\pi^4 \hat{g}_1 + \frac{1}{2}\pi^2 \hat{g}_2 + \hat{g}_3 \\
\hat{D}_4 &= \frac{61}{720}\pi^6 \hat{g}_1 + \frac{5}{24}\pi^4 \hat{g}_2 + \frac{1}{2}\pi^2 \hat{g}_3 + \hat{g}_4 .
\end{aligned}
\tag{C.89}$$

Using the shuffle identities to write these in terms of zeta and multiple zeta values, we can simplify the result to

$$\begin{aligned}
\hat{g}_n &= -\frac{7}{3}\pi^2 \zeta_{2n+1,3} - 19\zeta_{2n+1,5} - \frac{1}{3}\pi^2(10n+9)\zeta_{2n+2,2} + \frac{1}{6}\left((24-72n)\zeta_{2n+2,4} \right. \\
&\quad \left. + (n+1)(-2\pi^2(6n+11)\zeta_{2n+3,1} + 12(2n+9)\zeta_{2n+3,3} + (2n+3)((20n+46)\zeta_{2n+4,2} \right. \\
&\quad \left. + (n+2)(6n+19)\zeta_{2n+5,1})) - 4\zeta_3(n+1)(2n+5)\zeta_{2n+3} - \frac{1}{90}\pi^4(46n+13)\zeta_{2n+2} \right. \\
&\quad \left. + \frac{1}{9}\pi^2(2n+3)(n(n+3)-7)\zeta_{2n+4} + \frac{1}{60}(n+2)(2n+5)(n(2n(8n+29)+189)+117)\zeta_{2n+6} \right. \\
&\quad \left. + \frac{11\zeta_3^2(-1)^n\pi^{2n}}{\Gamma(2n+1)} + \frac{1}{3}\left(7\pi^2\zeta_3+57\zeta_5\right)(-1)^n2^{2n+1}\pi^{2n}\zeta_{-2n}\Gamma(-2n) \right. \\
&\quad \left. + \frac{(-1)^n(n(n(5n(2n(202n(2n+21)+17549)+71967)+373331)+172998)+19368)\pi^{2n+6}}{1134\Gamma(2n+7)} \right) .
\end{aligned}
\tag{C.90}$$

Putting this together gives the first few D_i :

$$\begin{aligned}
\hat{D}_1 &= 19\zeta_{3,5} - 44\zeta_{5,3} - 110\zeta_{6,2} + 5\pi^2\zeta_3^2 + 162\zeta_5\zeta_3 + \frac{13229}{576}\zeta_8 , \\
\hat{D}_2 &= 57\zeta_2\zeta_{3,5} - 118\zeta_2\zeta_{5,3} + 19\zeta_{5,5} - 272\zeta_2\zeta_{6,2} + 20\zeta_{6,4} - 78\zeta_{7,3} - 301\zeta_{8,2} + \frac{735}{4}\zeta_4\zeta_3^2 \\
&\quad + 334\zeta_2\zeta_5\zeta_3 + 542\zeta_7\zeta_3 + 198\zeta_5^2 - \frac{253457\zeta_{10}}{256} , \\
\hat{D}_3 &= \frac{1425}{4}\zeta_4\zeta_{3,5} - 720\zeta_4\zeta_{5,3} + 57\zeta_2\zeta_{5,5} - \frac{3255}{2}\zeta_4\zeta_{6,2} + 60\zeta_2\zeta_{6,4} - 220\zeta_2\zeta_{7,3} \\
&\quad + 19\zeta_{7,5} - 825\zeta_2\zeta_{8,2} + 32\zeta_{8,4} - 120\zeta_{9,3} - 636\zeta_{10,2} + \frac{3129}{4}\zeta_6\zeta_3^2 + \frac{3795}{2}\zeta_4\zeta_5\zeta_3 \\
&\quad + 460\zeta_2\zeta_7\zeta_3 + 1286\zeta_9\zeta_3 + 478\zeta_2\zeta_5^2 + 1091\zeta_5\zeta_7 - \frac{5793961333\zeta_{12}}{707584} .
\end{aligned}
\tag{C.91}$$

We see that all coefficients are products of zeta values times integers over powers of 2 with a two exceptions:

- (1) The coefficient of $H_{\tilde{0}_{2n}}$.
- (2) The coefficient of ζ_{6+2n} in D_n .

Both of these are related to the constant term in the box integral which cannot be written as a zeta value times an integer over a power of 2.

Now that we have the Wilson loop expectation value $\mathcal{F}(z)$ for this set of graphs, we can also calculate their contribution to Γ_{cusp} .

$$\begin{aligned}
& \sum_{a=1}^n \frac{(-1)^{a+1} \pi^{2a-2} \hat{D}_{-a+n+1}}{(2a-1)(2a-2)!} + \frac{11(-1)^n \pi^{2n} \zeta_3^2}{(2n+1)(2n)!} + \frac{505(-1)^n \pi^{2n+6}}{9072(2n+1)(2n)!} - \frac{5(-1)^n \pi^{2n+6}}{72(2n+3)(2n+2)!} \\
& + \frac{(-1)^n \pi^{2n+6}}{6(2n+5)(2n+4)!} - \frac{3(-1)^n \pi^{2n+6}}{(2n+7)(2n+6)!} - 4n^3 \zeta_{2n+5,1} - 10n^2 \zeta_{2n+4,2} - 26n^2 \zeta_{2n+5,1} \\
& - 4n \zeta_{2n+3,3} - 37n \zeta_{2n+4,2} - 54n \zeta_{2n+5,1} + 6\zeta_{2n+2,4} - 12\zeta_{2n+3,3} - 33\zeta_{2n+4,2} - 36\zeta_{2n+5,1} \\
& + \frac{1}{12} \pi^2 (8n+5) \zeta_{2n+4} - \frac{1}{2} (8n+5) \zeta_{2n+6} - \frac{4}{3} n^4 \zeta_{2n+6} - 10n^3 \zeta_{2n+6} - \frac{5}{3} \pi^2 n^2 \zeta_{2n+4} \\
& - \frac{101}{3} n^2 \zeta_{2n+6} + 8n \zeta_3 \zeta_{2n+3} - \frac{11}{2} \pi^2 n \zeta_{2n+4} - \frac{107}{2} n \zeta_{2n+6} - \frac{1}{15} \pi^4 \zeta_{2n+2} + 16 \zeta_3 \zeta_{2n+3} \\
(C.92) \quad & - \frac{47}{12} \pi^2 \zeta_{2n+4} - 35 \zeta_{2n+6},
\end{aligned}$$

where $\hat{D}_i$ are as defined above.

C.2.4 Γ_{cusp} Contribution of $\mathcal{F}_3(z) \mathcal{F}_{\text{tree}}(g, z)$

We can apply the same summation techniques for the $\mathcal{I}[\mathcal{F}_{\text{tree}}(z) \mathcal{F}_3(z)]$, which now gives us triple sums instead of double sums. First, we use

$$(C.93) \quad \mathcal{F}_{\text{tree}}(z) = \frac{A^2 z^A}{g^2 (1+z^A)^2} = \frac{A}{g^2} \sum_m (mA) (-1)^m z^{mA}.$$

Then the integrals of interest are of the form

$$(C.94) \quad \int_{-\pi}^{\pi} H_{-m_2, -m_1}(e^{i\phi}) (i\phi)^a (e^{i\phi mA}) d\phi.$$

Using the methods outlined in Appendix (C.2.2), we can analytically perform the sums over the HPL terms, but we still have leftover summation terms from writing $\mathcal{F}_{\text{tree}}(e^{i\phi})$ as an infinite sum.

More specifically, we get terms of the form

$$(C.95) \quad \sum_m \frac{8(-1)^m \cos(\pi Am) \psi^{(i)}(Am+1)}{A^j m^j},$$

where the integers i, j run between 1 and 4. We can check that each of these sums are convergent and we can also perform the sum for the term that gives the leading order contribution analytically, which gives

$$(C.96) \quad \mathcal{I}[g^4 \mathcal{F}_{\text{tree}} \mathcal{F}_3(z)]_{g \gg 1} \sim \frac{22\pi^3}{45} g^5 + \mathcal{O}(g^4 \log(g)).$$

Interestingly, while the leading g term is free of logarithms, like before, the sub-leading log terms don't cancel out. While we can't sum the sub-leading terms analytically, we can perform the summations after taking the asymptotic expansions of polylogarithms, i.e. $\psi^{(k)}(z \gg 1) \sim \frac{1}{z^k}$ or by approximating $1 + Am \sim 1 + m$ for large g . Doing this approximation gives us sub-leading log terms. This is confirmed by evaluating the integral numerically for different values of g . The full result for $g \gg 1$ is

$$(C.97) \quad \begin{aligned} \mathcal{I}[g^4 \mathcal{F}_{\text{tree}}(g, z) \mathcal{F}_3(z)] &\sim \frac{22\pi^3}{45} g^5 + 5.13g^4 - 16.g^4 \log(g) - 5.5g^3 - 13.8g^3 \log(g) - 8.5g^2 \log(g) \\ &- 4.8g^2 - 3.2g - 4.5g \log(g) - 2.1 \log(g) + \mathcal{O}\left(\frac{1}{g}\right). \end{aligned}$$

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