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by

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Summary

The generalized plasticity model, which has previously been discussed by the author primarily in a theoretical way, is made specific in the form of a simple version based on a single function of the state variables. After a review of the basic concepts of generalized plasticity the appropriate simplifications are introduced, leading to a one-dimensional model that can be used to generate, in closed form, stress-strain diagrams under arbitrary stress-controlled loading, including cyclic loading. This model is then extended into a multiaxial form that is used to solve the problem of the plastic expansion (with elastic deformations neglected) of a pressurized thick-walled tube.

1. Introduction

In the 1970s I first began writing about what I then called a "simple" theory of rate-independent plasticity [1-2]. I called it that because I thought it conceptually simpler than the classical theory; for one thing, it does not require the concept of a yield criterion, an essential ingredient of classical plasticity. I devoted further work to elaborating the axiomatic structure of the theory [3], with attention to the form taken by the maximum-dissipation postulate [4], the uniqueness theorem [5], and the propagation of acceleration waves [6]. In the course of this work it became apparent how classical plasticity may be precisely defined as a special case, and I have consequently come to call the theory in question *generalized plasticity*. With the exception of an illustrative application to concrete [7], this work has been largely abstract, dealing with the model in its most general form. The purpose of the present paper is to present a specific and quite simple version of the model that can be put to immediate computational use. By way of introduction, a brief summary of the essential features of general plasticity is given first.

There are two fundamental assumptions underlying the model. The first is that the local mechanical state† in a body described by the model is determined by the control variables (typically the components of stress or strain, though mixed control is possible as well) and a finite number of internal variables. The second is that the relation between stress and strain, as mediated by the internal variables, is rate-independent. For the sake of definiteness stress control is assumed, with the stress components assembled in the vector $\boldsymbol{\sigma}$, while the internal-variable vector is denoted $\boldsymbol{\xi}$.

A local process is defined as *elastic* if the internal-variable vector remains constant throughout. It follows from rate-independence that a process in which the control variables remain constant is necessarily elastic. The *elastic range* of a state defined by $(\boldsymbol{\sigma}, \boldsymbol{\xi})$, denoted $E(\boldsymbol{\sigma}, \boldsymbol{\xi})$, is defined as the set of stress vectors attainable from $\boldsymbol{\sigma}$ by means of an elastic process. Clearly, $\boldsymbol{\sigma}$ itself belongs to $E(\boldsymbol{\sigma}, \boldsymbol{\xi})$. Furthermore, if $\Sigma(\boldsymbol{\xi})$ denotes the set of all stresses $\boldsymbol{\sigma}$ such that $(\boldsymbol{\sigma}, \boldsymbol{\xi})$ is a possible state, then it is a quite reasonable assumption that for all $\boldsymbol{\sigma} \in \Sigma(\boldsymbol{\xi})$, $E(\boldsymbol{\sigma}, \boldsymbol{\xi})$ is a closed subset of $\Sigma(\boldsymbol{\xi})$; a similar assumption was made by Pipkin and Rivlin [8]. The assumption, in effect, limits the possible processes to "reasonable" ones. As a result, a given $\boldsymbol{\sigma} \in \Sigma(\boldsymbol{\xi})$ is either an interior point or a boundary point of $E(\boldsymbol{\sigma}, \boldsymbol{\xi})$. If it is an interior point, then all stresses in a sufficiently small neighborhood of $\boldsymbol{\sigma}$ are attainable elastically, and $(\boldsymbol{\sigma}, \boldsymbol{\xi})$ may be called an *elastic state*; the equations of evolution for $\boldsymbol{\xi}$ must be such that $\dot{\boldsymbol{\xi}} = 0$ at any elastic state. If, on the other hand, $(\boldsymbol{\sigma}, \boldsymbol{\xi})$ is a boundary point, then only the stresses located inward from the boundary of $E(\boldsymbol{\sigma}, \boldsymbol{\xi})$ at $\boldsymbol{\sigma}$ are attainable elastically, while those located outward can be attained only with a change in $\boldsymbol{\xi}$; $(\boldsymbol{\sigma}, \boldsymbol{\xi})$ may then be called a *plastic state*. If the boundary of $E(\boldsymbol{\sigma}, \boldsymbol{\xi})$ is formed by a surface in stress space, then this surface is

†Thermal effects are ignored here, though they can be incorporated with no difficulty.

equivalent to what was called the *loading surface* by Phillips and Sierakowski [9].

The set of all stresses $\boldsymbol{\sigma}$ such that $(\boldsymbol{\sigma}, \boldsymbol{\xi})$ is an elastic state is called the *elastic domain* at $\boldsymbol{\xi}$ and denoted $D(\boldsymbol{\xi})$. This set can be shown, under some none-too-stringent technical conditions, to be an open subset of $\Sigma(\boldsymbol{\xi})$, and its boundary, if formed by a surface, is equivalent to the *yield surface* of Phillips and Sierakowski [9]. In fact, the theory of plasticity with non-coincident yield and loading surfaces [9, 10] was the inspiration for my research. However, a closed set need not contain any interior points, so that in generalized plasticity *elastic states need not exist at all.* A case of some interest arises when an elastic domain takes the form of a shell of finite thickness, and the shell tends to zero thickness, so that two parts of the yield surface coalesce to form a *quasi-yield surface* [2]. In any case, if a yield surface does exist, plastic states may have stresses lying outside the yield surface, except in the special case represented by classical plasticity, which may be strictly defined as the case where the elastic range of $(\boldsymbol{\sigma}, \boldsymbol{\xi})$ is independent of $\boldsymbol{\sigma}$. All the features of classical plasticity (coincident yield and loading surfaces, impossibility of stresses outside the yield surface, and so on) follow from the definition [4].

2. Simplifications of the Model

Some simplifications are now introduced in order to reduce the model from an abstract to a specific form. To begin with, only a geometrically linear version of the model is discussed here, so that one is not concerned with Lagrangian or Eulerian components, objective rates, and the like.

If $(\boldsymbol{\sigma}, \boldsymbol{\xi})$ is a plastic state, and if the boundary of $E(\boldsymbol{\sigma}, \boldsymbol{\xi})$ is locally smooth at $\boldsymbol{\sigma}$, with the outward normal denoted $\mathbf{v}$, then both rate-independence and the defining property of a plastic state are satisfied if the equation of evolution for $\boldsymbol{\xi}$ at $(\boldsymbol{\sigma}, \boldsymbol{\xi})$ is given by

$$\boldsymbol{\xi} = \mathbf{g}(\boldsymbol{\sigma}, \boldsymbol{\xi})<\mathbf{v} \cdot \dot{\boldsymbol{\sigma}}>, \tag{1}$$

where $<\cdot>$ is the Macauley bracket, that is, $<x>=x$ for $x \geq 0$ and $<x>=0$ for $x<0$, and $\mathbf{g}$ is a function with values in the space of internal-variable vectors. For convenience, the outward normal vector $\mathbf{v}$ will be taken as a unit vector, that is, $|\mathbf{v}| = 1$, where $|\cdot|$ is an appropriate norm.

The internal-variable vector $\boldsymbol{\xi}$ will be assumed to be composed of the plastic strain vector $\boldsymbol{\varepsilon}^p$ and an additional internal-vector $\boldsymbol{\kappa}$, so that Equation (1) is replaced by the two equations

$$\dot{\boldsymbol{\varepsilon}}^p = h\boldsymbol{\lambda}<\mathbf{v}\cdot\dot{\boldsymbol{\sigma}}>, \tag{2}$$

$$\dot{\boldsymbol{\kappa}} = h\boldsymbol{\mu}<\mathbf{v}\cdot\dot{\boldsymbol{\sigma}}>, \tag{3}$$

where h is nonnegative, and zero at an elastic state, while, again for convenience, $|\boldsymbol{\lambda}| = 1$. The special case $\boldsymbol{\lambda} = \mathbf{v}$ corresponds to *normality* or *associated plasticity*.

A particularly simple associated form of Equation (2) is one that is defined by single dimensionless function $f(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}^p, \boldsymbol{\kappa})$ such that

$$h = \frac{1}{\beta}<f>, \tag{4}$$

where β is a constant having the dimension of stress, and

$$\boldsymbol{\lambda} = \mathbf{v} = \frac{\partial f/\partial\boldsymbol{\sigma}}{|\partial f/\partial\boldsymbol{\sigma}|}, \tag{5}$$

reflecting the assumption that loading surfaces are given *locally* by $f =$ constant. The resulting form of Equation (2) is therefore

$$\dot{\boldsymbol{\varepsilon}}^p = \frac{1}{\beta}<f>\mathbf{v}<\mathbf{v}\cdot\dot{\boldsymbol{\sigma}}>. \tag{6}$$

If there exists a region in stress space where $f<0$, then this region is just the elastic domain, and $f=0$ defines the yield surface. In that case the limit as $\beta \to 0$ represents classical plasticity: f

cannot be positive, while the limit of $<f>/\beta$ is determined by the classical consistency condition.† If, on the other hand, $f \geq 0$ everywhere, but there exist surfaces in stress space where $f = 0$, then these surfaces are the aforementioned quasi-yield surfaces.

Another interesting feature of the simple model is that it may be combined with the viscoplasticity model due to Perzyna [11] so that the plastic strain rate is given by

$$\dot{\boldsymbol{\varepsilon}}^p = <f>\mathbf{v}\left[\frac{1}{\beta}<\mathbf{v}\cdot\dot{\boldsymbol{\sigma}}> + \frac{1}{\gamma}\right], \tag{7}$$

where γ plays the role of a viscosity (though its dimensions are those of reciprocal time). A body described by (7) has both instantaneous plasticity and viscoplasticity, with the same yield criterion governing both. The combined model described by (7) will not be pursued here.

The function f may be given any form corresponding to the standard yield functions of plasticity theory—Mises, Tresca, etc.—with any hardening rule as reflected in the choice of the internal variables making up $\boldsymbol{\kappa}$ (the "hardening variables") and their evolution equations (3). A common choice is one in which $\boldsymbol{\kappa}$ consists of a single component κ, and $\boldsymbol{\mu}$ consists of a single component μ that is defined either as $|\boldsymbol{\lambda}|$ (corresponding to $\dot{\kappa} = |\dot{\boldsymbol{\varepsilon}}^p|$) or as $\boldsymbol{\sigma}\cdot\boldsymbol{\lambda}$ (corresponding to $\dot{\kappa} = \boldsymbol{\sigma}\cdot\dot{\boldsymbol{\varepsilon}}^p$). A combination of isotropic and one of the simple kinds of kinematic hardening may then be represented. More sophisticated hardening models require more components in $\boldsymbol{\kappa}$, exactly as in classical plasticity and viscoplasticity.

3. A Simple One-Dimensional Model

The behavior of the simple model introduced in the preceding section will be illustrated by a one-dimensional case, that is, one with only one independent stress component σ and its conjugate plastic strain ε^p. The hardening variable κ is defined by the evolution equation $\dot{\kappa} = |\dot{\varepsilon}^p|$; the hardening is assumed to be linear and to consist of both kinematic (with hardening coefficient α') and isotropic (with hardening coefficient α'') hardening. With $\alpha = \alpha' + \alpha''$, the yield function f is taken as

$$f = \frac{1}{\alpha}[|\sigma - \alpha'\varepsilon^p| - (\sigma_Y + \alpha''\kappa)], \tag{8}$$

where σ_Y is the initial yield stress. The stress–plastic strain diagrams for any stress-controlled process are then the result of integrating the differential equations

$$\frac{d\varepsilon^p}{d\sigma} = \frac{1}{\alpha\beta}<|\sigma - \alpha'\varepsilon^p| - (\sigma_Y + \alpha''\kappa)>, \tag{9}$$

$$d\kappa = |d\varepsilon^p|. \tag{10}$$

Since the equations are piecewise linear with constant coefficients, they can easily be integrated in closed form.

Initial loading curve. Upon initial loading, with stress and strain positive, we have $\kappa = \varepsilon^p$, and Equation (9) takes the form

$$\frac{d\varepsilon^p}{d\sigma} = \frac{1}{\alpha\beta}<\sigma - \sigma_Y - \alpha\varepsilon^p>, \tag{11}$$

the initial condition being $\varepsilon^p = 0$ at $\sigma = 0$. Thus the plastic strain remains at zero for $0 \leq \sigma \leq \sigma_Y$, while for $\sigma > \sigma_Y$, ε^p is the solution of

$$\frac{d\varepsilon^p}{d\sigma} + \frac{1}{\beta}\varepsilon^p = \frac{1}{\alpha\beta}(\sigma - \sigma_Y),$$

†In generalized plasticity there is, of course, no consistency condition, since there is no need to enforce a yield criterion; the major stumbling block of computational plasticity is thus eliminated.

whose solution satisfying $\varepsilon^p = 0$ at $\sigma = \sigma_Y$ is

$$\varepsilon^p = \frac{1}{\alpha}(\sigma - \sigma_Y - \beta) + \frac{\beta}{\alpha} e^{-(\sigma - \sigma_Y)/\beta}. \tag{12}$$

The curve is thus asymptotic to a straight line that is parallel to the line $\sigma = \sigma_Y + \alpha\varepsilon^p$ representing the yield surfaces, and displaced from it in the positive stress direction by the distance β. The quantity β is thus a measure of how far outside the yield surface stresses may lie, and it is obvious that classical plasticity is recovered when $\beta = 0$.

Reloading curves. If the initial loading process is interrupted by unloading that does not produce reverse plastic deformation and that is followed by reloading, the identity $\kappa = \varepsilon^p$ remains valid and the reloading is governed by Equation (11). If the plastic strain attained just before unloading is ε_1^p and if the stress is reduced to the level $\sigma_Y + \alpha\varepsilon_1^p$ or less (but greater than that required for reverse plastic deformation), then the reloading curve is given by

$$\varepsilon^p = \frac{1}{\alpha}(\sigma - \sigma_Y - \beta) + \frac{\beta}{\alpha} e^{-(\sigma - \sigma_Y - \alpha\varepsilon_1^p)/\beta}.$$

Note that this curve is asymptotic not only to the line $\sigma = \sigma_Y + \alpha\varepsilon^p$, but to the initial loading curve given by (12) as well.

The initial and reloading curves are shown in Figure 1.

Cyclic loading. As a final illustration we consider a loading program consisting of initial loading up to a maximum stress σ_m, unloading and reverse loading to $-\sigma_m$, reloading to σ_m, and so on. Let the index 1 designate the initial loading phase, 2 the first unloading and reverse loading phase, 3 the reloading, and so on, so that all the phases with odd-numbered index are characterized by algebraically increasing stress and nondecreasing plastic deformation, while all the phases with even-numbered index are characterized by algebraically decreasing stress and nonincreasing plastic deformation. Let ε_i^p and κ_i denote the values at the end of the ith phase of the plastic strain and the hardening variable, respectively. In the course of the ith phase, then, the hardening variable is given by

$$\kappa = \kappa_{i-1} + (-1)^{i-1}(\varepsilon^p - \varepsilon_{i-1}^p),$$

with $\kappa_0 = \varepsilon_0^p = 0$. Let c_i be defined by

$$c_i = 2\alpha'' \sum_{j=1}^{i} (-1)^{j-1} \varepsilon_j^p.$$

Then the solution of Equation (9) is as follows:

i odd:

$$\varepsilon^p = \varepsilon_{i-1}^p, \qquad \sigma < \sigma_Y + c_{i-1} + \alpha\varepsilon_{i-1}^p,$$

$$\varepsilon^p = \frac{1}{\alpha}\{\sigma - \sigma_Y - \beta - c_{i-1} + \beta\exp[-(\sigma - \sigma_Y - c_{i-1} - \alpha\varepsilon_{i-1}^p)/\beta]\},$$

$$\sigma_Y + c_{i-1} + \alpha\varepsilon_{i-1}^p \le \sigma \le \sigma_m$$

i even:

$$\varepsilon^p = \varepsilon_{i-1}^p, \qquad \sigma > -\sigma_Y - c_{i-1} + \alpha\varepsilon_{i-1}^p,$$

$$\varepsilon^p = \frac{1}{\alpha}\{\sigma + \sigma_Y + \beta + c_{i-1} - \beta\exp[(\sigma + \sigma_Y + c_{i-1} - \alpha\varepsilon_{i-1}^p)/\beta]\},$$

$$-\sigma_Y - c_{i-1} + \alpha\varepsilon_{i-1}^p \ge \sigma \ge -\sigma_m.$$

Calculated curves, using the assumed values $\sigma_Y = 1.5\beta$ and $\sigma_m = 2\beta$, are shown in Figures 2 and 3 for $\alpha'' = 0$ and $\alpha'' = \alpha'/3$, respectively. It is to be noted that the former value, representing purely kinematic hardening, leads to a symmetric limit cycle with numerically equal values of the extreme plastic strain. These values can be shown to be given by $\pm(\beta/\alpha)x$, x being the solution of the equation

$$x - e^{-z-x} = z - 1,$$

where $z = (\sigma_m - \sigma_Y)/\beta$; for the case at hand, $z = 0.5$ and $x = 0.067$. The presence of an isotropic-hardening component, on the other hand, leads to cycles with ever diminishing plastic deformation.

4. A Multiaxial Generalization

A generalization of the model of the preceding section to cover generalized states of stress requires only the determination of a form of the function $f(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}^P, \kappa)$ that reduces to (8) for the appropriate one-dimensional case. To be specific, let us assume that (8) applies to uniaxial tension and compression, and that initial multiaxial yielding is governed by either the Mises or the Tresca criterion. The general form of f is then

$$f(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}^P, \kappa) = \frac{1}{\alpha}(\tilde{\sigma} - \sigma_Y - \alpha''\kappa), \tag{13}$$

where $\tilde{\sigma}$ must be defined appropriately for each criterion, as must the norm in plastic-strain-rate space so that $|\mathbf{v}| = 1$ and $\dot{\kappa} = |\dot{\boldsymbol{\varepsilon}}^P|$. The definitions are as follows:

Mises criterion:

$$\tilde{\sigma} = \sqrt{\tfrac{3}{2}\tilde{s}_{ij}\tilde{s}_{ij}},$$

where $\tilde{s}_{ij} = \sigma_{ij} - \frac{1}{3}\sigma_{kk}\delta_{ij} - \frac{2}{3}\alpha'\varepsilon^P_{ij}$, and

$$|\dot{\boldsymbol{\varepsilon}}^P| = \sqrt{\tfrac{2}{3}\dot{\varepsilon}^P_{ij}\dot{\varepsilon}^P_{ij}}.$$

The normal vector $\mathbf{v}$ is thus given by the tensor with components $v_{ij} = (3/2\tilde{\sigma})(\tilde{s}_{ij}$, and the components of plastic strain rate are

$$\dot{\varepsilon}^P_{ij} = \frac{3}{2\alpha\tilde{\sigma}^2}\tilde{s}_{ij}<\tilde{\sigma} - \sigma_Y - \alpha''\kappa><\tilde{s}_{kl}\dot{\sigma}_{kl}>.$$

Tresca criterion:

$$\tilde{\sigma} = \frac{1}{2}(|\tilde{\sigma}_1 - \tilde{\sigma}_2| + |\tilde{\sigma}_2 - \tilde{\sigma}_3| + |\tilde{\sigma}_1 - \tilde{\sigma}_3|),$$

where $\tilde{\sigma}_i = \sigma_i - \frac{2}{3}\alpha'\varepsilon^P_i$, the subscripts referring to principal-axis components, and

$$|\dot{\boldsymbol{\varepsilon}}^P| = \frac{1}{2}(|\dot{\varepsilon}^P_1| + |\dot{\varepsilon}^P_2| + |\dot{\varepsilon}^P_3|).$$

For both criteria, uniaxial tension or compression corresponds to

$$\mathbf{v} = \operatorname{sgn}(\sigma_1 - \alpha'\varepsilon^P_1)(1, -\tfrac{1}{2}, -\tfrac{1}{2})$$

in principal-axis components. Since $\varepsilon^P_2 = \varepsilon^P_3 = -\frac{1}{2}\varepsilon^P_1$, it can easily be verified that f as given by (13) reduces to that given by (8) (with $\sigma = \sigma_1$ and $\varepsilon^P = \varepsilon^P_1$).

Plane plastic deformation. If it is assumed that $\varepsilon^P_2 = 0$ identically, then by analogy with classical plasticity based on the Tresca flow rule, $\tilde{\sigma}_2$ must be intermediate between $\tilde{\sigma}_1$ and $\tilde{\sigma}_3$, while the Mises flow rule requires more specifically that $\tilde{\sigma}_2 = \frac{1}{2}(\tilde{\sigma}_1 + \tilde{\sigma}_3)$ and therefore $\sigma_2 = \frac{1}{2}(\sigma_1 + \sigma_3)$, since

$\dot{\varepsilon}_3^p = -\dot{\varepsilon}_1^p$. Moreover, for the Mises rule $\dot{\kappa} = (2/\sqrt{3})|\dot{\varepsilon}_1^p|$ and $\mathbf{v} = (\sqrt{3}/2)(1, 0, -1)\,\mathrm{sgn}\,(\tilde{\sigma}_1 - \tilde{\sigma}_3)$, while for the Tresca rule $\dot{\kappa} = |\dot{\varepsilon}_1^p|$ and $\mathbf{v} = (1, 0, -1)\,\mathrm{sgn}\,(\tilde{\sigma}_1 - \tilde{\sigma}_3)$. For the sake of definiteness the Tresca formulation will be used, so that at a state where $\tilde{\sigma}_1 \geq \tilde{\sigma}_3$, f takes the form

$$f(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}^p, \kappa) = \frac{1}{\alpha}(\sigma_1 - \sigma_3 - \tfrac{4}{3}\alpha'\varepsilon_1^p - \sigma_Y - \alpha''\kappa).$$

If no unloading leading to reverse plastic deformation takes place, then $\kappa = \varepsilon_1^p$. With $\bar{\alpha} \overset{def}{=} \frac{4}{3}\alpha' + \alpha''$, the equation governing the plastic deformation is

$$\dot{\varepsilon}_1^p = \frac{1}{\alpha\beta}\langle\sigma_1 - \sigma_3 - \sigma_Y - \bar{\alpha}\varepsilon_1^p\rangle\langle\dot{\sigma}_1 - \dot{\sigma}_3\rangle,$$

since $\mathbf{v} = (1, 0, -1)$. For monotonic loading, this may be rewritten as

$$\frac{d\varepsilon_1^p}{ds} + \frac{1}{\bar{\beta}}\varepsilon_1^p = \frac{1}{\bar{\alpha}\bar{\beta}}s, \tag{14}$$

where $\bar{\beta} = \alpha\beta/\bar{\alpha}$ and $s = \sigma_1 - \sigma_3 - \sigma_Y$. Equation (14) can be solved in the form

$$\frac{\bar{\alpha}}{\bar{\beta}}\varepsilon_1^p = \frac{s}{\bar{\beta}} - 1 + e^{-s/\bar{\beta}}, \tag{15}$$

virtually the same as the one-dimensional result (12). If it is in turn necessary to solve Equation (15), then let this solution be written as $s = \bar{\beta}\phi(\bar{\alpha}\varepsilon_1^p/\bar{\beta})$.

Thick-walled tube under pressure. The preceding formulation can be readily applied to the problem of a thick-walled tube under internal pressure. The tube is assumed to be in a state of plane strain, and elastic deformations are neglected. Consequently, if u is the radial displacement, then as a result of incompressibility it must take the form $u = u_0 a/r$, where r is the radial coordinate and a is the inner radius, and $\varepsilon_1^p = u_0 a/r^2$. With σ_θ and σ_r denoting the hoop and radial stresses, respectively, and with b the outer radius, the internal pressure is given by

$$p = \int_a^b \frac{\sigma_\theta - \sigma_r}{r}\,dr,$$

and therefore, with $s = \sigma_\theta - \sigma_r - \sigma_Y$,

$$p = \sigma_Y \ln\frac{b}{a} + \bar{\beta}\int_1^{b/a} \frac{\phi(\bar{\alpha}u_0/a\bar{\beta}\rho^2)}{\rho}\,d\rho, \tag{16}$$

where ϕ is the aforementioned solution of Equation (15). This solution may be obtained by Newton's method, and the resulting values may be used in the numerical quadrature of the integral of Equation (16). A relation is thus obtained between $[p - \sigma_Y \ln(b/a)]/\bar{\beta}$ and $\bar{\alpha}u_0/\bar{\beta}a$ that depends parametrically on b/a. A calculated curve for $b/a = 2$ is shown in Figure 4, together with the straight-line asymptote that is obtained by assuming $s/\bar{\beta}$ large in Equation (15). With the exponential term neglected, ϕ can be obtained explicitly, and the integration leads to the asymptote

$$\frac{1}{\bar{\beta}}\left[p - \sigma_Y \ln\frac{b}{a}\right] \cong \ln\frac{b}{a} + \frac{1 - (a/b)^2}{2}\frac{\bar{\alpha}u_0}{\bar{\beta}a}.$$

An approximation for small values of u_0 may be obtained by noting that for $s/\bar{\beta}$ small, the right-hand side of (15) is approximately $(s/\bar{\beta})^2/2$ and hence ϕ can again be obtained explicitly. The result is

$$\frac{1}{\bar{\beta}}\left[p - \sigma_Y \ln\frac{b}{a}\right] \cong \left(1 - \frac{a}{b}\right)\sqrt{\frac{2\bar{\alpha}u_0}{\bar{\beta}a}}.$$

It can be seen that the pressure-displacement curve obtained is qualitatively quite similar to the uniaxial initial-loading curve of Figure 1.

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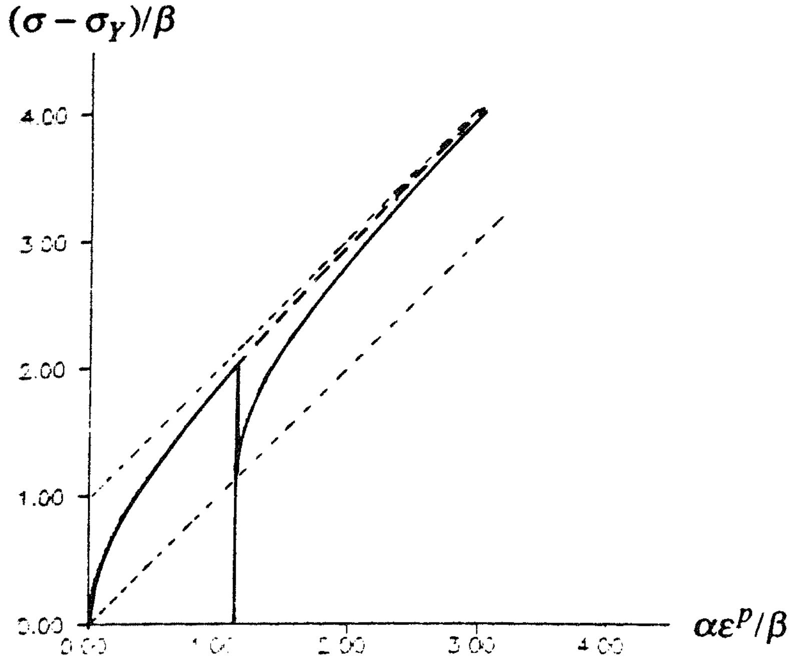

Figure 1

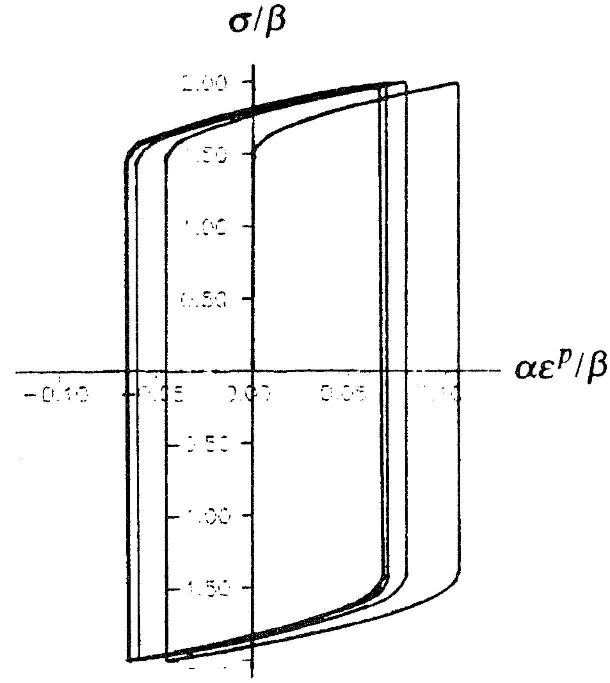

Figure 2

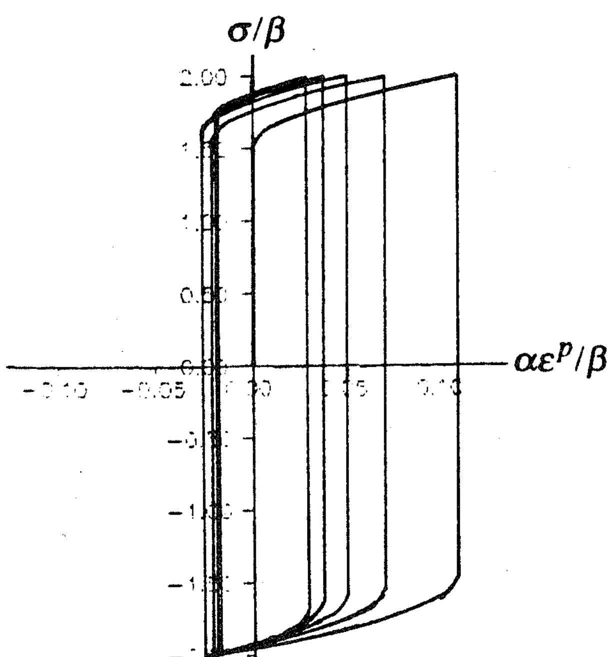

Figure 3

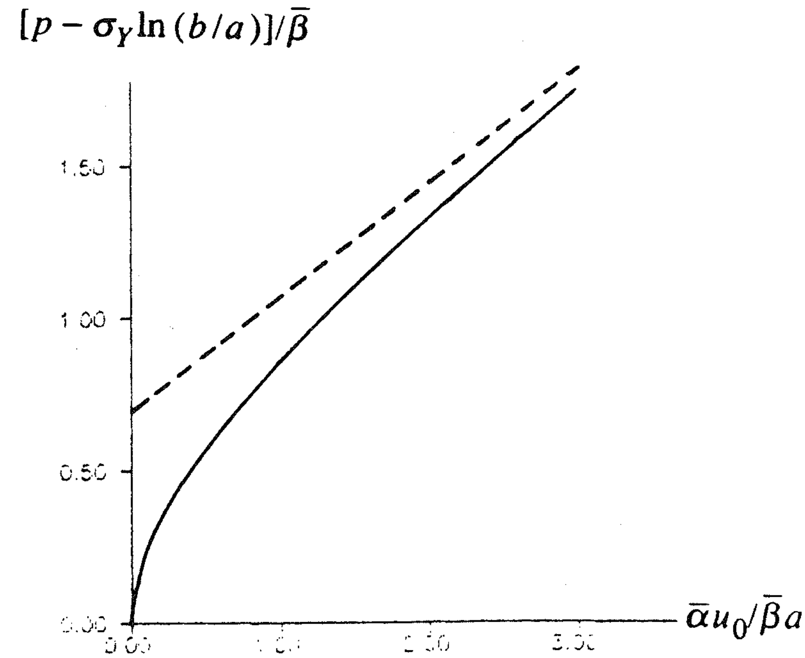

Figure 4