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A Modified Mean Curvature Flow of Entire Locally Lipschitz Star-Shaped Hypersurfaces in Hyperbolic Space

Author

Allmann, Patrick A

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A MODIFIED MEAN CURVATURE FLOW OF ENTIRE LOCALLY LIPSCHITZ STAR-SHAPED HYPERSURFACES IN HYPERBOLIC SPACE

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Patrick Allen Allmann

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The Dissertation of Patrick Allen Allmann
is approved:

Professor Longzhi Lin, Chair

Professor Jie Qing

Professor Richard Montgomery

Quentin Williams
Interim Vice Provost and Dean of Graduate Studies

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Abstract

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by

Patrick Allen Allmann

In [GS00], B. Guan and J. Spruck showed the existence of smooth radial graphs of constant mean curvature with prescribed C^0, star-shaped boundary at infinity using elliptic PDE methods and the maximum principle. Surfaces of constant mean curvature are critical points of an area functional with a volume constraint. In [DSS09], D. De Silva and J. Spruck showed the same result mentioned above in [GS00] using variational methods. It is a natural question then to ask whether we can approach this problem using the negative gradient flow of that area-volume functional. Such a flow, called **modified mean curvature flow**, was first introduced by L. Lin and L. Xiao in [LX12]. There they showed, starting with a star-shaped hypersurface with a global C^1 bound, the longtime existence of the modified mean curvature flow. Moreover, they recovered the previous results by showing the flow converges to a stationary solution.

This work is inspired by these three works. Here, we show the longtime existence of a smooth modified mean curvature flow of hypersurfaces in hyperbolic space if the initial hypersurface is locally Lipschitz and star-shaped. This result can be considered as a generalization of the main theorem of [Unt03] by P. Unterberger, in which they show a longtime existence result of **mean curvature flow** in the same ambient and initial setting. It's also a hyperbolic version of the nonparametric mean curvature flow in Euclidean space studied by K. Ecker and

G. Huisken in [EH91a]. There they found a locally Lipschitz vertical graph moving by its mean curvature becomes a smooth vertical graph for all time.

To John and Melissa.

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Chapter 1

Introduction

In [Dou31], it was shown, given a Jordan curve in a Euclidean space, there exists a minimal surface with that Jordan curve as boundary. This is Plateau's Problem. The existence of surfaces of constant mean curvature with prescribed Jordan curve as boundary was studied in [Hil70]. In [Str88], constant mean curvature surfaces with free boundary were studied using variational techniques. More recently, in [ZZ19], a min-max theory was developed to show the existence of smooth, closed, constant mean curvature hypersurfaces in any closed Riemannian manifold of low dimension. The asymptotic plateau problem, in contrast, asks under which conditions are there complete hypersurfaces of constant mean curvature with prescribed boundary at infinity. We refer to [Cos13] for a survey of the asymptotic Plateau problem.

Throughout, n is a nonnegative integer and $\mathbb{H}$ denotes $n+1$ dimensional hyperbolic space, while $\partial_\infty\mathbb{H}$ denotes its asymptotic boundary at infinity. Also, σ denotes a real number strictly in between $-n$ and n, and Γ denotes a closed, codimension 1 submanifold of $\partial_\infty\mathbb{H}$. With these notations, we have the following formulation of the asymptotic Plateau problem.

Question 1.0.1. Does there exist Σ, a hypersurface of $\mathbb{H}$, such that Σ has constant mean curvature (CMC) σ and such that the asymptotic boundary of Σ is Γ?

The answer to this problem is affirmative if we allow "hypersurface" to mean "reduced boundary of a set of locally finite perimeter", c.f. [Ton96, Theorem 1.4]. The case $\sigma = 0$ is answered in [And82, Theorem 3], where such Σ's take the form of locally integral currents. These two approaches, from the geometric measure theory point of view, are weak formulations of the asymptotic plateau problem, c.f. [Fed69, 4.1.24, 4.5.1].

We may then ask about the regularity of these solutions. If $\Gamma \in C^{1,\alpha}$, then the main result in [HL87] implies $M \cup \Gamma$ is a finite union of $C^{1,\alpha}$ manifolds near Γ, if M is the support of Σ. In [Lin89], it's shown that Σ is as smooth as Γ, as long Σ is a vertical graph above the region which Γ bounds. [Ton96, Theorem 1.5] implies lower order regularity of solutions near the boundary. In particular, if $k \leq n$ and Γ is C^k, then Σ is C^k near Γ. When $\sigma \neq 0$, we can't expect Σ to have higher order regularity, even if Γ is smooth and embedded, as seen by [Ton96, Theorem 1.8, c.f. Theorem 6.1]. Higher order boundary regularity of nonparametric solutions of the asymptotic plateau problem is studied in [HW16].

Stronger regularity results are obtained if more is known about the geometry of Γ. For example, by [NS96, Theorem 1.1], if Γ is of class $C^{2,\alpha}$ and mean convex, then Σ can be written uniquely as the graph of a function in $C^{\infty}(\Omega) \cap C^{2,\alpha}(\overline{\Omega})$, where $\Gamma = \partial\Omega$. We also recall [GS00, Theorem 1.1].

Theorem 1.0.2. [GS00, Theorem 1.1] If Γ is star-shaped and $C^{1,1}$, then Σ is unique, star-shaped, smooth and continuous up to the boundary.

This theorem is proved using elliptic PDE theory and the maximum principle. A hypersurface is star-shaped if it bounds a star domain, which is a set containing a point such that any point in the set can be connected to the point by a line. Similarly, we recall

Theorem 1.0.3. [DSS09, Theorem 1.4] If Γ is star-shaped and C^0, then Σ is unique, star-shaped, smooth and continuous up to the boundary.

This theorem is proved using techniques from the calculus of variations. C^2 hypersurfaces of constant mean curvature in hyperbolic space locally satisfy the Euler-Lagrange equation of a certain energy functional, I, c.f. Lemma A.2.4 . I is an area functional with a volume constraint, and can be written as $I = A + n\sigma V$, where A is the hyperbolic area of the hypersurface and V is the hyperbolic volume enclosed by the hypersurface and a hyperbolic cylinder. [DSS09, Theorem 1.3] roughly states a star-shaped local minimizer of I is necessarily smooth.

Just as mean curvature flow is considered the negative gradient flow of the area functional, the negative gradient flow of I is modified mean curvature flow, introduced in [LX12], whose work is a parabolic analogue of [GS00]. Stationary solutions of modified mean curvature flow are constant mean curvature hypersurfaces. Therefore, showing longtime existence and convergence of an initial-boundary value problem for modified mean curvature flow implies the existence of a solution of the asymptotic plateau problem. The main theorem of [LX12] implies just this if the initial hypersurface is star-shaped and satisfies a uniform equidistant sphere condition, called the uniform local ball condition (ULBC) (1.1.7). That is,

Theorem 1.0.4. [LX12, Theorem 1.1] Let Γ be the boundary of a star-shaped C^{1+1} domain in $\partial_\infty \mathbb{H}$ and Γ_ε be its vertical lift for $\varepsilon > 0$ sufficiently small. Let $\Sigma_0 = \lim_{\varepsilon \to 0} \Sigma_0^\varepsilon$ be the limiting hypersurface of radial graphs $\Sigma_0^\varepsilon \in C^{1+1}$ with $\partial\Sigma_0^\varepsilon = \Gamma_\varepsilon$. Suppose Σ_0^ε has a uniform Lipschitz bound and satisfies the uniform local ball condition. Then

(i) There exists a unique solution $\mathbf{F}(\mathbf{z},t) \in C^\infty(\mathbb{S}_+ \times (0,\infty)) \cap C^{1+1,\frac{1}{2}+\frac{1}{2}}(\overline{\mathbb{S}_+} \times (0,\infty)) \cap C^0(\overline{\mathbb{S}_+} \times [0,\infty))$ to the modified mean curvature flow.

(ii) There exist $t_i \to \infty$ such that $\Sigma_{t_i} = \mathbf{F}(\mathbb{S}_+, t_i)$ converges to a unique stationary smooth complete hypersurface $\Sigma_\infty \in C^\infty(\mathbb{S}_+) \cap C^{1+1}(\overline{\mathbb{S}_+})$ (as a radial graph over $\mathbb{S}_+$) which has constant hyperbolic mean curvature σ and $\partial_\infty\Sigma_\infty = \Gamma$ asymptotically. Also, each Σ_t is a complete radial graph over $\mathbb{S}_+$.

(iii) If additionally Σ_0^ε has mean curvature $H^\varepsilon \geq \sigma$ for all $\varepsilon > 0$ sufficiently small, then Σ_t converges uniformly to Σ_∞ for all t.

Mean curvature flow ($\sigma = 0$) was first weakly formulated in [Bra78] using geometric measure theory. Classically, if a closed convex codimension 1 submanifold is embedded in Euclidean space and flows by its mean curvature, then it converges to a point in finite time. This is the main result in [Hui84]. Further study on the singularities formed by mean curvature flow in Euclidean space is found in the series [CM12], [CM13] and [CM14], where the entropy functional is introduced. An application of mean curvature flow classifying the immersion of spheres in Riemannian manifolds is given in [Hui86]. Standard notes on the classical mean curvature flow in Euclidean space are given in [Man11], [CMP15], and [Whi02].

Mean curvature flow of spacelike hypersurfaces in pseudo-Euclidean space are studied in [LL19]. Moreover, in [EH91b], the motion of surfaces moving by an evolution equation with a prescribed mean curvature function is applied to show the existence of spacelike constant mean curvature in cosmological spacetimes. Mean curvature flow of star-shaped, locally Lipschitz initial hypersurfaces in $\mathbb{H}$ are studied in [Unt03], whose methods we draw from, which draws from the methods in [EH91a]. In [Unt03], longtime existence is established. Convergence is obtained if, in addition, the initial hypersurface's asymptotic boundary is a circle, an assumption subsumed by the uniform local ball condition as established by [LX12], c.f. Remark 1.1.7. In [EH89], the longtime existence of complete vertical graphs over n dimensional Euclidean space with a uniform gradient bound is demonstrated. This result is improved in [EH91a]. Again, longtime existence is established without any hypotheses of growth at infinity. We remark that the modified mean curvature flow behaves differently than the mean curvature flow in hyperbolic space, c.f. Remark 2.2.10.

In this work, we prove Theorem 1.1.5. That is, we relax the uniform local ball condition and prove longtime existence of the initial-boundary value problem of modified mean curvature flow if the initial data are star-shaped and locally Lipschitz.

Theorem 1.1.5. If $\mathbf{F}_0 : \mathbb{S}^n_+ \to \mathbb{H}$ is a map such that $\Sigma_0 = \mathbf{F}_0(\mathbb{S}^n_+)$ is a locally Lipschitz continuous radial graph over $\mathbb{S}^n_+$, then the Cauchy initial-boundary value problem for the modified mean curvature flow (1.1.1) has a solution $\mathbf{F} \in C^\infty(\mathbb{S}^n_+ \times (0,\infty)) \cap C^{0,1\times 0,1/2}_{loc}(\mathbb{S}^n_+ \times [0,\infty))$ and $\mathbf{F}(\mathbb{S}^n_+, t)$ is a complete radial graph over $\mathbb{S}^n_+$ for any $t \geq 0$.

A sufficient condition for convergence, but weaker than the uniform local ball condition, is not provided here. This question is interesting and we hope to address it in a later work.

By choosing to use star-shaped initial surfaces, we study a nonparametic version of modified mean curvature flow. The resulting quasilinear parabolic PDE is degenerate at the asymptotic boundary and wherever the gradient becomes unbounded. Refer to equation (1.1.3). Nevertheless, we obtain a priori interior gradient bounds (Theorem 2.2.13) and subsequently all higher order bounds (Theorem 2.3.5, Theorem 2.3.7), which allows us to make an approximation argument.

There are a number of directions this line of questioning can point. There is more to consider when star-shaped hypersurfaces move by modified mean curvature flow in $\mathbb{H}$. Of course, we would like to obtain a sharp regularity or geometric condition on the initial star-shaped hypersurface, weaker than the uniform local ball condition, which guarantees convergence of the flow. In [LX12], L. Lin and L. Xiao showed an initial star-shaped hypersurface with a global C^1 bound moving under MMCF converges to a smooth, stationary star-shaped solution of constant mean curvature. What happens if we drop the uniform control on the gradient but still retain some information of it near the boundary? For example, we may consider when the gradient of the initial radial graph is O of some power of the reciprocal of the Euclidean height above $\partial_\infty \mathbb{H}$. In any case, what examples are there of convergence or nonconvergence? Our result includes the longtime existence of MMCF starting with a horosphere (constant mean curvature $\pm n$), whose asymptotic boundary is degenerate in the sense that it's a single point. There we don't have convergence but a translating self-similar solution for all time.

Also, is it worthwhile to consider more general ambient spaces other than $\mathbb{H}$, to possibly consider nonpositively curved ambient manifolds or even just hyperbolic ones, as in [HLZ16]? What can we say if $H-\sigma$ is instead a symmetric function of the initial hypersurface's principle curvatures, as in [GSS09] and [JX19]? We may consider the case of higher codimension, as in [Wan02]. In all of these cases, is there a corresponding energy functional which the flow is the negative gradient flow of?

1.1 The main theorem

Now, we let $\mathbf{F} : \mathbb{S}^n_+ \times [0,\infty) \to \mathbb{H}$ be a one-parameter family of complete embedded star-shaped hypersurfaces moving by the modified mean curvature flow in hyperbolic space with fixed parabolic boundary data. That is, $\mathbf{F}(\cdot,t)$ is a smooth one-parameter family of smooth embeddings with images $\Sigma_t = \mathbf{F}(\mathbb{S}^n_+,t)$, satisfying the evolution equation

$$\begin{cases} \dfrac{\partial}{\partial t}\mathbf{F}(\mathbf{z},t) = (H-\sigma)\nu_H\,, \quad (\mathbf{z},t) \in \mathbb{S}^n_+ \times (0,\infty)\,, \\ \mathbf{F}(\mathbb{S}^n_+,0) = \Sigma_0 \end{cases} \tag{1.1.1}$$

More precisely, we suppose the solution $\mathbf{F}(\mathbf{z},t)$ to the modified mean curvature flow (1.1.1) can be represented as a complete radial graph over $\mathbb{S}^n_+$. That is,

$$\mathbf{F}(\mathbf{z},t) \,=\, e^{v(\mathbf{z},t)}\mathbf{z}\,, \quad (\mathbf{z},t) \in \mathbb{S}^n_+ \times [0,\infty). \tag{1.1.2}$$

We call such a function $v(\mathbf{z},t)$ the radial height of $\Sigma_t = \mathbf{F}(\mathbb{S}^n_+,t)$. Then one observes that the Cauchy initial-boundary value problem for the modified mean curvature flow (1.1.1) is equivalent (Lemma A.2.2) to the following degenerate parabolic PDE with initial and boundary conditions:

$$\begin{cases} \dfrac{\partial v(\mathbf{z},t)}{\partial t} = y^2\alpha^{ij}v_{ij} - ny\langle \mathbf{e}, \nabla^S v\rangle_E - \sigma y w\,, \quad (\mathbf{z},t) \in \mathbb{S}^n_+ \times (0,\infty)\,, \\ v(\mathbf{z},0) = v_0(\mathbf{z})\,, \quad \mathbf{z} \in \mathbb{S}^n_+\,, \end{cases} \tag{1.1.3}$$

where we represent Σ_0 as the radial graph of the function e^{v_0} over $\mathbb{S}^n_+$. Here $y = \langle \mathbf{e}, \mathbf{z}\rangle_E$. Also, $\alpha^{ij} = \gamma^{ij} - \frac{\gamma^{ik}\gamma^{jl}v_k v_l}{w^2}\,, 1 \leq i,j \leq n$, $w = (1+|\nabla^S v|^2)^{1/2}$ and we denote by γ_{ij} the standard metric of $\mathbb{S}^n_+$ and γ^{ij} its inverse. By Lemma A.2.1, Σ_t remains a radial graph as long as the support function $\langle \nu_E, x\rangle_E$ satisfies

$$\langle \nu_E, x\rangle_E \,>\, 0 \tag{1.1.4}$$

for all $x \in \Sigma_t$, where ν_E is the Euclidean outward unit normal vector of Σ_t.

In this work, we would like to show the longtime existence of the modified mean curvature flow (MMCF) without the uniform local ball condition at the infinity of the initial hypersurface. To this end, we consider the modified mean curvature flow starting from a locally Lipschitz continuous radial graph $\Sigma_0 \subset \mathbb{H}$ and show the longtime existence of the flow. More precisely, we prove

Theorem 1.1.5. If $\mathbf{F}_0 : \mathbb{S}^n_+ \to \mathbb{H}$ is a map such that $\Sigma_0 = \mathbf{F}_0(\mathbb{S}^n_+)$ is a locally Lipschitz continuous radial graph over $\mathbb{S}^n_+$, then the Cauchy initial-boundary value problem for the modified mean curvature flow (1.1.1) has a solution $\mathbf{F} \in C^\infty(\mathbb{S}^n_+ \times (0,\infty)) \cap C^{0,1\times 0,1/2}_{loc}(\mathbb{S}^n_+ \times [0,\infty))$ and $\mathbf{F}(\mathbb{S}^n_+, t)$ is a complete radial graph over $\mathbb{S}^n_+$ for any $t \geq 0$.

Here, $\mathbf{F} \in C^\infty(\mathbb{S}^n_+ \times (0,\infty))$ means the components of $\mathbf{F}$ have continuous partial derivatives of every order. $\mathbf{F} \in C^{0,1\times 0,1/2}_{loc}(\mathbb{S}^n_+ \times [0,\infty))$ means $\mathbf{F}$ is locally Lipschitz continuous in $\mathbb{S}^n_+ \times \{t\}$ for all $t \geq 0$ and locally Hölder continuous with exponent $\frac{1}{2}$ in $\{\mathbf{z}\} \times [0,\infty)$ for all $\mathbf{z} \in \mathbb{S}^n_+$.

Remark 1.1.6. By the work of Guan, Spruck [GS00], Xiao and Lin, [LX12], given a $C^{1,1}$ star-shaped $n-1$ dimensional closed submanifold at the infinity $\partial_\infty \mathbb{H}$, we can find a suitable initial hypersurface such that the modified mean curvature flow exists for all time and converges to a hypersurface of constant mean curvature which has the given submanifold as the asymptotic boundary. On the other hand, modified mean curvature flow, starting from a horosphere $\{x \in \mathbb{H} \mid x^{n+1} = c\}, c > 0$ (whose infinity is degenerate, a point in $\partial_\infty \mathbb{H}$), exists for all time but never converges. Such an example shows convergence of the flow depends on the behavior of the initial asymptotic boundary. We expect some intermediate geometric condition (i.e., if some degeneracy of the initial asymptotic boundary is allowed) that is weaker than the uniform local ball condition in [LX12] will guarantee the convergence of the flow.

Remark 1.1.7. We shall digress in order to state the uniform local ball condition (ULBC). If $(x_0)^{n+1} = 0$, $r > 0$, an equidistant sphere is a cap

$$S^{\pm\sigma}_{x_0,r} = \partial B_r\left(x_0 \pm \frac{\sigma r}{n}\mathbf{e}\right) \cap \mathbb{R}^{n+1}_{+}$$

with hyperbolic CMC σ with respect to its outward pointing unit normal (as radial graphs), as computed using Lemma 2.1.3, where $B_r\left(x_0 \pm \frac{\sigma r}{n}\mathbf{e}\right)$ is the ball of Euclidean radius r and center $x_0 \pm \frac{\sigma r}{n}\mathbf{e}$. For any $x \in \mathbb{H}$, $r(x)$ is the hyperbolic distance from x to the x^{n+1}-axis. We define $\mathbf{C}_\varepsilon = \{x \in \mathbb{H} \mid \cosh(r(x)) \le \frac{1}{\varepsilon}\}$. If $\mathbf{F}_0 : \mathbb{S}^n_+ \to \mathbb{H}$ is an immersion, $\Sigma_0 = \mathbf{F}_0(\mathbb{S}^n_+)$, $\Omega_\varepsilon = \mathbf{F}_0^{-1}(\mathbf{C}_\varepsilon \cap \Sigma_0)$, $\Sigma_0^\varepsilon = \mathbf{F}_0(\Omega_\varepsilon)$ and $\Gamma_\varepsilon = \mathbf{F}_0(\partial\Omega_\varepsilon)$, then we say Σ_0 satisfies the ULBC if there exist four numbers $R_\pm > 0$ and $\delta > 0$ such that, for all $\varepsilon > 0$, for all $p \in \Gamma_\varepsilon$, there are $a'_\pm \in \mathbb{R}^{n+1}$ with $(a'_\pm)^{n+1} = 0$ such that

$$\Sigma_0^\varepsilon \cap B_\delta(p) \cap S^{\pm\sigma}_{a'_\pm, R_\pm} = \{p\}.$$

The ULBC is a geometric condition on the initial hypersurface which guarantees a uniform gradient bound on the parabolic boundary of the flow. A complete hypersurface with the ULBC resembles a CMC hypersurface near its asymptotic boundary. An example is found in Figure 1.1. From Lemma A.1.3, each Σ_0^ε satisfies the ULBC if it's smooth enough.

The remainder of this work is organized as follows. In Section 2.1, we establish some known results from differential geometry adapted to our setting. In Section 2.2, we use the evolution equation of the support function $\langle \nu_E, x\rangle_E$ (see Proposition 2.2.8) and an appropriate space-time cut-off function together with a conventional maximum principle argument to show

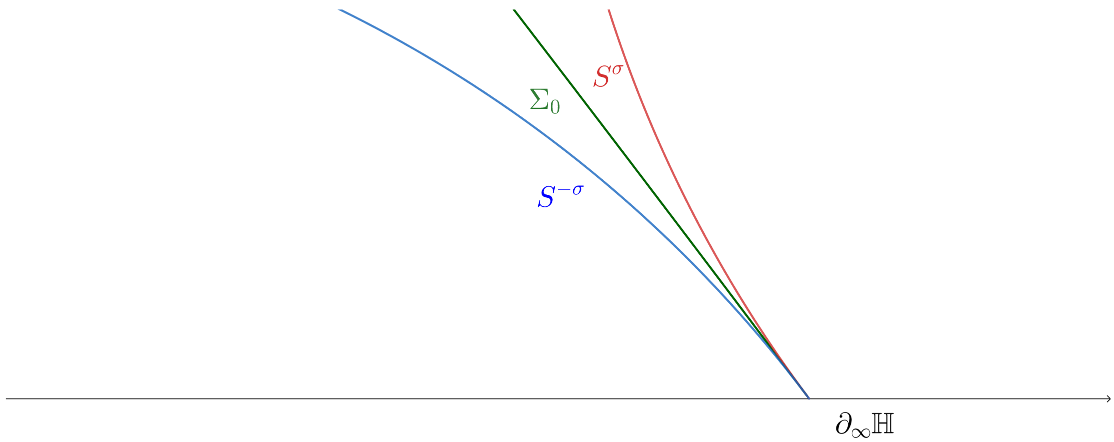

Figure 1.1: ULBC

a uniform interior gradient estimate for the modified mean curvature flow (see Theorem 2.2.13). In Section 2.3, we show the interior estimates on all other higher order derivatives for the modified mean curvature flow (see Theorem 2.3.5 and Theorem 2.3.7). We prove the main Theorem 1.1.5 in Section 2.4. Most of this work can be found in [ALZ20].

Chapter 2

Evolution Equations and Interior Estimates

2.1 Some differential geometry

$\mathbb{R}^{n+1}$ denotes an $n+1$ dimensional real inner product space with a fixed orthonormal basis $e_1,\ldots,e_{n+1}$, unconventionally called Euclidean space. $\mathbb{S}^n$ denotes the n-sphere equipped with the pullback metric from its embedding in $\mathbb{R}^{n+1}$, sometimes called the round sphere. The super or subscripts E,H,S are used on operators to distinguish between their definitions in Euclidean, hyperbolic, or spherical space, respectively. For example, $\langle,\rangle_E$ is the inner product on $\mathbb{R}^{n+1}$, ∇^H denotes the Levi-Civita connection on $\mathbb{H}$, and so on. Operators without subscripts or superscripts are operators on a hypersurface of $\mathbb{H}$. Greek indices will range from 1 to $n+1$, while Latin indices will range from 1 to n.

We use the upper-half plane model of $\mathbb{H}$. We denote e_{n+1} by $\mathbf{e}$, and, for any $x\in\mathbb{R}^{n+1}$, $x^{n+1}=\langle x,\mathbf{e}\rangle_E$, $|x|_E=\sqrt{\langle x,x\rangle_E}$. $\mathbb{H}$ is identified with $(\mathbb{R}^{n+1}_+,ds^2_H)$, where $\mathbb{R}^{n+1}_+=\{x\in\mathbb{R}^{n+1}\mid x^{n+1}>0\}$ is the upper-half plane of Euclidean space, and ds^2_H is the standard hyperbolic metric on the upper-half plane. That is, for any vector fields u, v defined locally on $\mathbb{R}^{n+1}_+$, for any x in the intersection of their domains,

$$ds^2_H(u,v)(x) = \frac{\langle u,v\rangle_E(x)}{(x^{n+1})^2}.$$

$ds^2_H(u,v)$ is also denoted by $\langle u,v\rangle_H$.

$\partial_\infty \mathbb{H}$ is identified with $\partial\mathbb{R}^{n+1}_+ \cup \{\infty\}$, which is homeomorphic to $\mathbb{S}^n$, where $\partial\mathbb{R}^n_+$ denotes the topological boundary of $\mathbb{R}^{n+1}_+$, $\{x \in \mathbb{R}^{n+1} \mid x^{n+1} = 0\}$. We denote the upper unit hemisphere, $\mathbb{S}^n \cap \mathbb{R}^{n+1}_+$, by $\mathbb{S}^n_+$ so that $\partial\mathbb{S}^n_+ \subset \{x \in \mathbb{R}^{n+1} \mid x^{n+1} = 0\}$.

The ambient Riemann curvature tensor with respect to the hyperbolic metric used here is

$$(R^H)(X,Y)Z = \nabla^H_Y \nabla^H_X Z - \nabla^H_X \nabla^H_Y Z + \nabla^H_{[X,Y]} Z.$$

We define $(R^H)_{\alpha\beta\gamma\delta} = \langle (R^H)(e_\alpha, e_\beta)e_\gamma, e_\delta\rangle_H$, the components of the hyperbolic Riemann curvature tensor, and

$$(\mathrm{Ric}^H)_{\alpha\gamma} = (ds^2_H)^{\beta\delta}(R^H)_{\alpha\beta\gamma\delta}, \tag{2.1.1}$$

the components of the hyperbolic Ricci tensor, where $(ds^2_H)^{\alpha\gamma}$ is the inverse of ds^2_H.

Since the upper-half space model of hyperbolic space $\mathbb{H}$ and $\mathbb{R}^{n+1}_+$ are conformal, we have

Proposition 2.1.2. For any two vector fields X, Y on $\mathbb{H}$,

$$\nabla^H_X Y = \nabla^E_X Y + \frac{1}{x^{n+1}}(\langle X,Y\rangle_E \mathbf{e} - \langle X,\mathbf{e}\rangle_E Y - \langle Y,\mathbf{e}\rangle_E X).$$

For a class 2 hypersurface $\Sigma \subset \mathbb{H}$, for any $p \in \Sigma$, we let $\{\mathbf{v}_i\}_{i=1}^n$ be a basis of $T_p\Sigma$, denote the induced metric on Σ by

$$g_{ij} = \langle \mathbf{v}_i, \mathbf{v}_j\rangle_H,$$

and let ν_H be an unit normal vector of $T_p\Sigma$ with respect to ds_H^2. We denote the second fundamental form on Σ by

$$a_{ij} = \langle \nabla^H_{\mathbf{v}_i} \mathbf{v}_j, \nu_H \rangle_H,$$

so that the mean curvature of Σ with respect to the hyperbolic metric is

$$H = g^{ij} a_{ij},$$

where g^{ij} is the inverse of g_{ij}. With this we have

Lemma 2.1.3.

$$\kappa_i^H = x^{n+1} \kappa_i^E + \nu^{n+1},$$

where κ_i^H and κ_i^E are hyperbolic and Euclidean principle curvatures of Σ, respectively, and $\nu^{n+1} = \langle \nu_E, \mathbf{e} \rangle_E$. Therefore,

$$H = x^{n+1} H^E + n\nu^{n+1}.$$

Proof. Note that the hyperbolic principle curvatures κ_i^H's are the roots of

$$\begin{aligned} \det\left(a_{ij} - \kappa^H g_{ij}\right) &= \det\left(\frac{a_{ij}^E}{x^{n+1}} - \frac{\nu^{n+1}}{(x^{n+1})^2} g_{ij}^E - \kappa^H \frac{g_{ij}^E}{(x^{n+1})^2}\right) \\ &= (x^{n+1})^{-n} \det\left(a_{ij}^E - \frac{\kappa^H - \nu^{n+1}}{x^{n+1}} g_{ij}^E\right), \end{aligned}$$

so that the proposition follows from

$$\kappa_i^E = \frac{1}{x^{n+1}} \left(\kappa_i^H - \nu^{n+1}\right).$$

□

We note that the Riemann curvature tensor is

$$(R^H)_{\alpha\beta\gamma\delta} = \langle (R^H)(\mathbf{e}_\alpha, \mathbf{e}_\beta)\mathbf{e}_\gamma, \mathbf{e}_\delta \rangle_H = \delta_{\alpha\delta}\delta_{\beta\gamma} - \delta_{\alpha\gamma}\delta_{\beta\delta},$$

since $\mathbb{H}$ has constant sectional curvature -1. In particular, $\nabla R^H = 0$. Also, the Gauss equation in this setting reads as

$$\text{Gauss: } R_{ijkl} = a_{ik}a_{jl} - a_{il}a_{jk} + (R^H)_{ijkl},$$

where the index 0 denotes the ν_H direction. We note also that we have the interchange of two covariant derivatives on a two tensor:

$$\nabla_j \nabla_i a_{kl} = \nabla_i \nabla_j a_{kl} + a_{km} R_{jil}{}^m + a_{lm} R_{jik}{}^m,$$

where $R_{ijk}{}^m = g^{ml} R_{ijkl}$. Using these equations one can derive the following well-known Simons' identity.

Lemma 2.1.4. On a class 2 hypersurface $\Sigma \subset \mathbb{H}$, we have

(i) (Simons' identity)

$$\Delta a_{ij} = \nabla_i \nabla_j H + H a_{mi} a_j^m - |A|^2 a_{ij} - n a_{ij} + H \delta_{ij},$$

where Δ is the Laplacian for tensors on Σ, ∇ the covariant derivative on Σ, and $A = (a_{ij})$ the second fundamental form on Σ, all with respect to the induced hyperbolic metric.

(ii) $\quad \Delta |A|^2 = 2a^{ij} \nabla_i \nabla_j H + 2H \mathrm{Tr}(A^3) - 2|A|^4 - 2n|A|^2 + 2H^2 + 2|\nabla A|^2.$

Proof. We include a proof for the sake of completeness; we refer to [Hui86] for general ambient manifolds. Fix a point on Σ. In normal coordinates, for (i), we have

$$\begin{aligned}
\Delta a_{ij} &= \nabla_k \nabla_k a_{ij} = \nabla_k \nabla_j a_{ik} \\
&= \nabla_i \nabla_k a_{jk} + a_{jl} R_{kik}{}^{l} + a_{kl} R_{kij}{}^{l} \\
&= \nabla_i \nabla_j H + a_j^l (a_{kk} a_{il} - a_{kl} a_{ik} + (R^H)_{kik}{}^{l}) + a_{kl} (a_{kj} a_{il} - a_{kl} a_{ij} + (R^H)_{kij}{}^{l}) \\
&= \nabla_i \nabla_j H + H a_{il} a_j^l + a_{jl} (\delta_{kl} \delta_{ik} - \delta_{kk} \delta_{il}) - |A|^2 a_{ij} + a_{kl} (\delta_{kl} \delta_{ij} - \delta_{jk} \delta_{il}) \\
&= \nabla_i \nabla_j H + H a_{il} a_j^l - |A|^2 a_{ij} - n a_{ij} + H \delta_{ij}.
\end{aligned}$$

For (ii), we have

$$\begin{aligned}
\Delta |A|^2 &= 2a^{ij} \Delta a_{ij} + 2 |\nabla A|^2 \\
&= 2a^{ij} \nabla_i \nabla_j H + 2H \operatorname{Tr}(A^3) - 2|A|^4 - 2n|A|^2 + 2H^2 + 2|\nabla A|^2 . \qquad \square
\end{aligned}$$

2.2 Interior gradient estimates

Proposition 2.2.1. For a function $f : \Sigma_t \to \mathbb{R}$, where Σ_t moves by (1.1.1), we have

$$\begin{aligned}
\left(\frac{\partial}{\partial t} - \Delta \right) f = &- (x^{n+1})^2 (\Delta_E f - \langle \nabla^E_{\nu_E} \nabla^E f, \nu_E \rangle_E) \\
&+ x^{n+1} ((n-2) \langle \nabla^E f, \mathbf{e} \rangle_E + 2 \langle \nabla^E f, \nu_E \rangle_E \langle \nu_E, \mathbf{e} \rangle_E - \sigma \langle \nabla^E f, \nu_E \rangle_E),
\end{aligned}$$

where Δ is the Laplace-Beltrami operator on Σ_t, $\frac{\partial}{\partial t} = F_*(\partial/\partial t) = (H - \sigma)\nu_H$, Δ_E is the standard Euclidean Laplacian, and $\nabla^E f$ is the Euclidean gradient of f.

Proof. We first note

$$\nabla f = \nabla^H f - \langle \nabla^H f, \nu_H \rangle_H \nu_H \,,$$

$$\mathrm{div} = \mathrm{div}_H - \langle \nabla^H_{\nu_H} \cdot, \nu_H \rangle_H \,,$$

$$\nabla^H f = (x^{n+1})^2 \nabla^E f \,,$$

$$\mathrm{div}_H = \mathrm{div}_E - \frac{n+1}{x^{n+1}} \langle \cdot, \mathbf{e} \rangle_E \,.$$

Along with Proposition 2.1.2, these give

$$\begin{aligned}
\Delta f =& \mathrm{div}\, \nabla f \\
=& \mathrm{div}_H (\nabla^H f - \langle \nabla^H f, \nu_H \rangle_H \nu_H) - \langle \nabla^H_{\nu_H} (\nabla^H f - \langle \nabla^H f, \nu_H \rangle_H \nu_H), \nu_H \rangle_H \\
=& \mathrm{div}_H \nabla^H f - \langle \nabla^H f, \nu_H \rangle_H \,\mathrm{div}_H \nu_H - \nu_H \langle \nabla^H f, \nu_H \rangle_H \\
& \langle \nabla^H_{\nu_E} \nabla^H f, \nu_E \rangle_E + \nu_H \langle \nabla^H f, \nu_H \rangle_H \\
=& \mathrm{div}_H \nabla^H f - \langle \nabla^H_{\nu_E} \nabla^H f, \nu_E \rangle_E + H \langle \nabla^H f, \nu_H \rangle_H \\
=& \mathrm{div}_E ((x^{n+1})^2 \nabla^E f) - (n+1) x^{n+1} \langle \nabla^E f, \mathbf{e} \rangle_E \\
& - \langle \nabla^E_{\nu_E} ((x^{n+1})^2 \nabla^E f), \nu_E \rangle_E - x^{n+1} \langle \nu_E, \nabla^E f \rangle_E \langle \nu_E, \mathbf{e} \rangle_E \\
& + x^{n+1} \langle \nu_E, \mathbf{e} \rangle_E \langle \nabla^E f, \nu_E \rangle_E + x^{n+1} \langle \nabla^E f, \mathbf{e} \rangle_E + H \langle \nabla^E f, \nu_H \rangle \\
=& (x^{n+1})^2 \,\mathrm{div}_E \nabla^E f + 2 x^{n+1} \langle \nabla^E f, \mathbf{e} \rangle_E - (n+1) x^{n+1} \langle \nabla^E f, \mathbf{e} \rangle_E \\
& - (x^{n+1})^2 \langle \nabla^E_{\nu_E} \nabla^E f, \nu_E \rangle_E - 2 x^{n+1} \langle \nu_E, \mathbf{e} \rangle_E \langle \nabla^E f, \nu_E \rangle_E \\
& + x^{n+1} \langle \nabla^E f, \mathbf{e} \rangle_E + H \langle \nabla^E f, \nu_H \rangle_E \\
=& (x^{n+1})^2 (\Delta_E f - \langle \nabla^E_{\nu_E} \nabla^E f, \nu_E \rangle_E) - x^{n+1} ((n-2) \langle \nabla^E f, \mathbf{e} \rangle_E \\
& - 2 \langle \nu_E, \mathbf{e} \rangle_E \langle \nabla^E f, \nu_E \rangle_E) + H \langle \nabla^E f, \nu_H \rangle_E .
\end{aligned}$$

Combining this with

$$\frac{\partial}{\partial t}f = (H-\sigma)\nu_H f = H\langle \nabla^E f, \nu_H\rangle_E - x^{n+1}\sigma\langle \nabla^E f, \nu_E\rangle_E$$

gives the desired result. □

We note that there is a C^0-estimate that comes for free.

Remark 2.2.2. Notice $|x|_E$ is bounded above on any compact region of Σ_t, by the same constant, for all time. To see this, there exist, for any $r > 0$, caps $\{(x_1,\dots,x^{n+1}) \in \mathbb{H} : (x_1)^2 + \cdots (x_n)^2 + (x^{n+1} + \sigma r/n)^2 = r^2\}$, with constant hyperbolic mean curvature σ. These caps have bounded $|x|_E$. The result follows from a comparison principle for MMCF, found in Lemma A.3.1. That is, two initially disjoint hypersurfaces moving by MMCF in hyperbolic space remain disjoint as long as the flow exists.

The MMCF (1.1.1) for complete radial graphs is a (degenerate) quasi-linear parabolic PDE, see (1.1.3). We would like to use the conventional maximum principle techniques to obtain interior estimates. Similar interior estimates were obtained in [LX12, Section 9] using the same techniques. However, the estimate there is not uniform in ε and therefore it is not sufficient in our current case. In order to overcome the degeneracy at infinity of the PDE and achieve the uniform interior estimate, we first need to find an appropriate space-time cut-off function. To do so, we let $r(x)$ be the hyperbolic distance from a point $x \in \mathbb{H}$ to the x^{n+1}-axis. Then

$$\cosh r = \frac{|x|_E}{x^{n+1}},$$

where $|x|_E = \sqrt{\langle x, x\rangle_E}$, see e.g. [BP92, Cor. A.5.8.]. In the following, we let $\mathbf{z} = \frac{x}{|x|_E}$.

Proposition 2.2.3.

$$\left(\frac{\partial}{\partial t}-\Delta\right)\cosh r=\frac{1}{\cosh r}(1-\langle\nu_E,\mathbf{z}\rangle_E^2)-(n-\sigma\langle\nu_E,\mathbf{e}\rangle_E)\cosh r-\sigma\langle\nu_E,\mathbf{z}\rangle_E.$$

Proof. We notice

$$\begin{aligned}
\nabla^E|x|_E &= \mathbf{z},\\
\nabla^E_{\nu_E}\nabla^E|x|_E &= \nabla^E_{\nu_E}\mathbf{z} = \nu_E|x|_E^{-1}x+|x|_E^{-1}\nu_E = -|x|_E^{-1}\langle\mathbf{z},\nu_E\rangle_E\mathbf{z}+|x|_E^{-1}\nu_E\,,\\
\Delta_E|x|_E &= \operatorname{div}_E\mathbf{z} = -|x|_E^{-1}+|x|_E^{-1}(n+1)=n|x|_E^{-1}\,.
\end{aligned}$$

Moreover, we have

$$\begin{aligned}
\nabla^E(x^{n+1})^{-1} &= -(x^{n+1})^{-2}\mathbf{e}\,,\\
\nabla^E_{\nu_E}\nabla^E(x^{n+1})^{-1} &= 2(x^{n+1})^{-3}\langle\mathbf{e},\nu_E\rangle_E\mathbf{e}\,,\\
\Delta_E(x^{n+1})^{-1} &= 2(x^{n+1})^{-3}\,,\\
\nabla^E\cosh r &= (x^{n+1})^{-1}\mathbf{z}-(x^{n+1})^{-2}|x|_E\mathbf{e} = (x^{n+1})^{-1}\mathbf{z}-(x^{n+1})^{-1}(\cosh r)\mathbf{e}\,,\\
x^{n+1}\nabla^E\cosh r &= \mathbf{z}-(\cosh r)\mathbf{e}\,,
\end{aligned}$$

and

$$\begin{aligned}
&\nabla^E_{\nu_E}\nabla^E\cosh r\\
=&\nabla^E_{\nu_E}((x^{n+1})^{-1}\mathbf{z}-(x^{n+1})^{-1}(\cosh r)\mathbf{e})\\
=&-(x^{n+1})^{-2}\langle\nu_E,\mathbf{e}\rangle_E\mathbf{z}+(x^{n+1})^{-1}(-|x|_E^{-1}\langle\mathbf{z},\nu_E\rangle_E\mathbf{z}+|x|_E^{-1}\nu_E)+(x^{n+1})^{-2}\langle\nu_E,\mathbf{e}\rangle_E(\cosh r)\mathbf{e}\\
&-(x^{n+1})^{-1}\langle(x^{n+1})^{-1}\mathbf{z}-(x^{n+1})^{-1}(\cosh r)\mathbf{e},\nu_E\rangle_E\\
=&(x^{n+1})^{-2}\left(-\langle\mathbf{e},\nu_E\rangle_E\mathbf{z}-\frac{1}{\cosh r}\langle\mathbf{z},\nu_E\rangle_E\mathbf{z}+\frac{1}{\cosh r}\nu_E-\langle\mathbf{z},\nu_E\rangle_E\mathbf{e}+2\cosh r\langle\mathbf{e},\nu_E\rangle_E\mathbf{e}\right).
\end{aligned}$$

Now, since $\langle \mathbf{z}, \mathbf{e} \rangle_E = \frac{1}{\cosh r}$, we have

$$\begin{aligned}
\Delta_E \cosh r &= \Delta_E (x^{n+1})^{-1} |x|_E \\
&= 2\langle \nabla^E (x^{n+1})^{-1}, \nabla^E |x|_E \rangle_E + (x^{n+1})^{-1} \Delta_E |x|_E + |x|_E \Delta_E (x^{n+1})^{-1} \\
&= (x^{n+1})^{-2} \left((n-2) \frac{1}{\cosh r} + 2\cosh r \right).
\end{aligned}$$

Therefore, we finally arrive at

$$\begin{aligned}
\left(\frac{\partial}{\partial t} - \Delta \right) \cosh r = &- (x^{n+1})^2 (\Delta_E \cosh r - \langle \nabla^E_{\nu_E} \nabla^E \cosh r, \nu_E \rangle_E) \\
&+ x^{n+1} [(n-2)\langle \nabla^E \cosh r, \mathbf{e} \rangle_E + 2\langle \nabla^E \cosh r, \nu_E \rangle_E \langle \mathbf{e}, \nu_E \rangle_E \\
&- \sigma \langle \nabla^E \cosh r, \nu_E \rangle_E] \\
= &(2-n)\langle \mathbf{z}, \mathbf{e} \rangle_E - 2\cosh r - \frac{1}{\cosh r} \langle \mathbf{z}, \nu_E \rangle_E^2 + \frac{1}{\cosh r} \\
&- 2\langle \mathbf{z}, \nu_E \rangle_E \langle \mathbf{e}, \nu_E \rangle_E + 2\cosh r \langle \mathbf{e}, \nu_E \rangle_E^2 \\
&+ (n-2)\langle \mathbf{z}, \mathbf{e} \rangle_E - (n-2)\cosh r + 2\langle \mathbf{z}, \nu_E \rangle_E \langle \mathbf{e}, \nu_E \rangle_E \\
&- 2\cosh r \langle \mathbf{e}, \nu_E \rangle_E^2 - \sigma \langle \mathbf{z}, \nu_E \rangle_E + \sigma \cosh r \langle \mathbf{e}, \nu_E \rangle_E \\
= &\frac{1}{\cosh r}(1 - \langle \nu_E, \mathbf{z} \rangle_E^2) - (n - \sigma \langle \mathbf{e}, \nu_E \rangle_E) \cosh r - \sigma \langle \mathbf{z}, \nu_E \rangle_E. \qquad \square
\end{aligned}$$

Now, for any $R > 0$, we define a space-time cut-off function (c.f. [Unt03])

$$\eta = \cosh R - e^{(n+\sigma)t} \left(\cosh r + \frac{\sigma}{n+\sigma} \right).$$

Then, for $\sigma \geq 0$ we have

$$\begin{aligned}\left(\frac{\partial}{\partial t} - \Delta\right)\eta = & -e^{(n+\sigma)t}\left((n+\sigma)\cosh r + \sigma + \left(\frac{\partial}{\partial t} - \Delta\right)\cosh r\right) \\ = & -e^{(n+\sigma)t}\Big[(n+\sigma)\cosh r + \sigma + \frac{1}{\cosh r}(1 - \langle \nu_E, \mathbf{z}\rangle_E^2) \\ & -(n - \sigma\langle \mathbf{e}, \nu_E\rangle_E)\cosh r - \sigma\langle \mathbf{z}, \nu_E\rangle_E\Big] \\ = & -e^{(n+\sigma)t}\Big[\frac{1}{\cosh r}(1 - \langle \nu_E, \mathbf{z}\rangle_E^2) + \sigma(1 - \langle \mathbf{z}, \nu_E\rangle_E \\ & + \cosh r(1 + \langle \mathbf{e}, \nu_E\rangle_E))\Big] \leq 0\,.\end{aligned}$$

Remark 2.2.4. We will only deal with the case of $\sigma \geq 0$. The case of $\sigma < 0$ can be handled using the hyperbolic isometric reflection $x^* = \frac{x}{|x|_E^2}$ with respect to $\mathbb{S}^n_+$, c.f. Lemma A.2.3 .

Remark 2.2.5. We notice that

$$\nu_E = \frac{\mathbf{z} - \nabla^S v}{\sqrt{1 + |\nabla^S v|^2}} \quad \text{and} \quad \langle \nu_E, \mathbf{z}\rangle_E = \frac{1}{|x|_E}\langle \nu_E, x\rangle_E = \frac{1}{\sqrt{1 + |\nabla^S v|^2}}\,.$$

Therefore, in order to obtain the interior gradient estimate on $|\nabla^S v|$, it's enough to obtain a positive lower bound on $\langle \nu_E, \mathbf{z}\rangle_E$, which is (almost) equivalent to $\langle \nu_E, x\rangle_E = x^{n+1}\langle \nu_H, x\rangle_H$, thanks to the C^0-estimate on $|x|_E$ using appropriate barriers (see Remark 2.2.2). Thus, in what follows, we will first look at the evolution equation of $\langle \nu_H, x\rangle_H$ and finally arrive at the evolution equation of $\langle \nu_E, x\rangle_E$ (see Proposition 2.2.8). Then the cut-off function and maximum principle techniques apply conventionally.

From here on we suppose the $\mathbf{v}_i$'s are in fact a normal coordinate basis of $T_p\Sigma_t$ with respect to the hyperbolic metric. We may extend the vector fields $\mathbf{v}_i$ and ν_H on Σ_t to a neighborhood of $\mathbb{H}$ by requiring that $\mathbf{v}_i$ is constant along the integral curves of x, so that $[\mathbf{v}_i, x] = [\nu_H, x] = 0$, where, e.g., $[\mathbf{v}_i, x]$ is the Lie bracket of $\mathbf{v}_i$ and x, c.f. [Bar84]. We note that the Codazzi equation becomes, since $\mathbb{H}$ has constant sectional curvature,

$$a_{ij,k} = a_{ik,j}. \tag{2.2.6}$$

Proposition 2.2.7. For radial graphs moving by MMCF,

$$\left(\frac{\partial}{\partial t} - \Delta\right)\langle \nu_H, x\rangle_H = (|A|^2 - n)\langle \nu_H, x\rangle_H,$$

where $|A|^2 = g^{ij}g^{kl}a_{ik}a_{jl}$ is the norm squared of the second fundamental form on Σ_t.

Proof. We have, using $[\mathbf{v}_i, x] = 0$, (2.1.1), and Codazzi equation (2.2.6), and summing over repeated indices,

$$\begin{aligned}
\Delta\langle \nu_H, x\rangle_H =& \mathbf{v}_i\mathbf{v}_i\langle \nu_H, x\rangle_H = \mathbf{v}_i\langle \nabla^H_{\mathbf{v}_i}\nu_H, x\rangle_H + \mathbf{v}_i\langle \nu_H, \nabla^H_{\mathbf{v}_i}x\rangle_H \\
=& -\langle \nabla^H_{\mathbf{v}_i} a_{ij}\mathbf{v}_j, x\rangle_H - |A|^2\langle \nu_H, x\rangle_H - 2\langle a_{ij}\mathbf{v}_j, \nabla^H_{\mathbf{v}_i}x\rangle_H \\
&+ \langle \nu_H, (R^H)(x, \mathbf{v}_i)\mathbf{v}_i\rangle_H + \langle \nu_H, \nabla^H_x \nabla^H_{\mathbf{v}_i}\mathbf{v}_i\rangle_H \\
=& -\mathbf{v}_j(H)\langle \mathbf{v}_j, x\rangle_H + \langle (R^H)(x, \mathbf{v}_i)\mathbf{v}_i, \nu_H\rangle_H - |A|^2\langle \nu_H, x\rangle_H + a_{ij}xg^{ij} + xa_{ii} \\
=& -\langle \nabla H, x\rangle_H - \mathrm{Ric}^H(\nu_H, \nu_H)\langle \nu_H, x\rangle_H - |A|^2\langle \nu_H, x\rangle_H + x(H) \\
=& (n - |A|^2)\langle \nu_H, x\rangle_H - \langle \nabla H, x\rangle_H + x(H).
\end{aligned}$$

Notice $\nabla^H_{\frac{\partial}{\partial t}}\nu_H$ is tangential, and $[\frac{\partial}{\partial t}, \mathbf{v}_i] = 0$ from the naturality of the Lie bracket. So,

$$\langle \nabla^H_{\frac{\partial}{\partial t}}\nu_H, \mathbf{v}_i\rangle_H = -\langle \nu_H, \nabla^H_{\mathbf{v}_i}\frac{\partial}{\partial t}\rangle_H = -\mathbf{v}_i(H-\sigma) - (H-\sigma)\langle \nu_H, \nabla^H_{\mathbf{v}_i}\nu_H\rangle_H = -\mathbf{v}_iH,$$

which implies

$$\nabla^H_{\frac{\partial}{\partial t}}\nu_H = -\nabla H.$$

Also,

$$\langle \nu_H, \nabla^H_{\nu_H} x\rangle_H = \langle \nu_E, \nabla^E_{\nu_E} x + \frac{1}{x^{n+1}}(\langle \nu_E, x\rangle_E\, \mathbf{e} - \langle \nu_E, \mathbf{e}\rangle_E x - \langle x, \mathbf{e}\rangle_E \nu_E\rangle_E = 0$$

since $\nabla^E_{\nu_E} x = \nu_E$ and $\langle x, \mathbf{e}\rangle_E = x^{n+1}$. Hence,

$$\begin{aligned}\frac{\partial}{\partial t}\langle \nu_H, x\rangle_H &= \langle \nabla^H_{\frac{\partial}{\partial t}} \nu_H, x\rangle_H + (H-\sigma)\langle \nu_H, \nabla^H_{\nu_H} x\rangle_H \\ &= -\langle \nabla H, x\rangle_H.\end{aligned}$$

Finally, notice that $x(H) = 0$ since x is a Killing vector field in $\mathbb{H}$, c.f. [HLZ16, Appendix]. □

Proposition 2.2.8. For radial graphs moving by MMCF,

$$\left(\frac{\partial}{\partial t} - \Delta\right)\langle \nu_E, x\rangle_E = (|A|^2 - \sigma\langle \nu_E, \mathbf{e}\rangle_E)\langle \nu_E, x\rangle_E - 2\langle \nabla\langle \nu_E, x\rangle_E, x^{n+1}\mathbf{e}\rangle_H. \tag{2.2.9}$$

Remark 2.2.10. In the case of MCF, i.e., $\sigma = 0$, equation (2.2.9) and the maximum principle yield immediately a global gradient bound for the approximate MCF (starting from the compact hypersurface Σ_0^ε), which ensures the global existence of the approximate MCF, see [Unt03]. On the other hand, in the case $\sigma \neq 0$, the maximum principle is not applicable directly, but thanks to the existence result from [LX12] for the approximate MMCF we are able to get around with this, see Section 2.4.

Proof. We have, using $\nabla x^{n+1} = \nabla^H x^{n+1} - \langle \nabla^H x^{n+1}, \nu_H\rangle_H \nu_H = (x^{n+1})^2(\mathbf{e} - \langle \nu_E, \mathbf{e}\rangle_E \nu_E)$, that

$$|\nabla x^{n+1}|_H^2 = (x^{n+1})^2(1 - \langle \nu_E, \mathbf{e}\rangle_E^2).$$

Hence, using Proposition 2.2.1, we have

$$\begin{aligned}\left(\frac{\partial}{\partial t}-\Delta\right)\langle\nu_E,x\rangle_E &= \left(\frac{\partial}{\partial t}-\Delta\right)\left(x^{n+1}\langle\nu_H,x\rangle_H\right)\\
&= x^{n+1}\left(\frac{\partial}{\partial t}-\Delta\right)\langle\nu_H,x\rangle_H+\langle\nu_H,x\rangle_H\left(\frac{\partial}{\partial t}-\Delta\right)x^{n+1}\\
&\quad -2\langle\nabla x^{n+1},\nabla\langle\nu_H,x\rangle_H\rangle_H\\
&= (|A|^2-n)\langle\nu_E,x\rangle_E+\langle\nu_E,x\rangle_E(n-2+2\langle\nu_E,\mathbf{e}\rangle_E^2-\sigma\langle\nu_E,\mathbf{e}\rangle_E)\\
&\quad -2\left\langle\nabla x^{n+1},\frac{1}{x^{n+1}}\nabla\langle\nu_E,x\rangle_E\right\rangle_H-2\left\langle\nabla x^{n+1},\langle\nu_E,x\rangle_E\nabla\frac{1}{x^{n+1}}\right\rangle_H\\
&= (|A|^2-2+2\langle\nu_E,\mathbf{e}\rangle_E^2-\sigma\langle\nu_E,\mathbf{e}\rangle_E)\langle\nu_E,x\rangle_E\\
&\quad -2\left\langle x^{n+1}\mathbf{e},\nabla\langle\nu_E,x\rangle_E\right\rangle_H+2\langle\nu_E,x\rangle_E(1-\langle\nu_E,\mathbf{e}\rangle_E^2)\\
&= (|A|^2-\sigma\langle\nu_E,\mathbf{e}\rangle_E)\langle\nu_E,x\rangle_E-2\langle\nabla\langle\nu_E,x\rangle_E,x^{n+1}\mathbf{e}\rangle_H. \qquad\square\end{aligned}$$

Now, in order to obtain the interior estimate using maximum principle techniques, we multiply $\langle\nu_E,x\rangle_E^{-1}$ by the space-time cut-off function and let

$$\xi=\eta^3\langle\nu_E,x\rangle_E^{-1}=\left(\cosh R-e^{(n+\sigma)t}\left(\cosh r+\frac{\sigma}{n+\sigma}\right)\right)^3\langle\nu_E,x\rangle_E^{-1}. \tag{2.2.11}$$

Proposition 2.2.12. For radial graphs moving by MMCF with $\sigma\in[0,n)$,

$$\left(\frac{\partial}{\partial t}-\Delta\right)\xi\le(n+2)\xi.$$

Proof. This is a straight-forward calculation.

$$
\begin{aligned}
\left(\frac{\partial}{\partial t}-\Delta\right)\xi =& \langle\nu_E,x\rangle_E^{-1}\left(\frac{\partial}{\partial t}-\Delta\right)\eta^3+\eta^3\left(\frac{\partial}{\partial t}-\Delta\right)\langle\nu_E,x\rangle_E^{-1}-2\left\langle\nabla\eta^3,\nabla\langle\nu_E,x\rangle_E^{-1}\right\rangle_H \\
=& 3\eta^2\langle\nu_E,x\rangle_E^{-1}\left(\frac{\partial}{\partial t}-\Delta\right)\eta-6\eta\langle\nu_E,x\rangle_E^{-1}|\nabla\eta|_H^2-\eta^3\langle\nu_E,x\rangle_E^{-2}\left(\frac{\partial}{\partial t}-\Delta\right)\langle\nu_E,x\rangle_E \\
& -2\eta^3\langle\nu_E,x\rangle_E^{-3}|\nabla\langle\nu_E,x\rangle_E|_H^2+6\eta^2\langle\nu_E,x\rangle_E^{-2}\left\langle\nabla\eta,\nabla\langle\nu_E,x\rangle_E\right\rangle_H \\
\leq& -\eta^3\langle\nu_E,x\rangle_E^{-2}\left((|A|^2-\sigma\langle\nu_E,\mathbf{e}\rangle_E)\langle\nu_E,x\rangle_E-2\langle\nabla\langle\nu_E,x\rangle_E,x^{n+1}\mathbf{e}\rangle_H\right) \\
& -\frac{1}{2}\eta^3\langle\nu_E,x\rangle_E^{-3}|\nabla\langle\nu_E,x\rangle_E|_H^2 \\
\leq& \eta^3\langle\nu_E,x\rangle_E^{-1}\left(\langle\nu_E,\mathbf{e}\rangle_E\sigma-|A|^2+2\right)\leq(n+2)\xi,
\end{aligned}
$$

where we have used

$$
2\eta^3\langle\nu_E,x\rangle_E^{-2}\langle\nabla\langle\nu_E,x\rangle_E,x^{n+1}\mathbf{e}\rangle_H\leq\frac{1}{2}\eta^3\langle\nu_E,x\rangle_E^{-3}|\nabla\langle\nu_E,x\rangle_E|_H^2+2\eta^3\langle\nu_E,x\rangle_E^{-1},
$$

and

$$
6\eta^2\langle\nu_E,x\rangle_E^{-2}\left\langle\nabla\eta,\nabla\langle\nu_E,x\rangle_E\right\rangle_H\leq6\eta\langle\nu_E,x\rangle_E^{-1}|\nabla\eta|_H^2+\frac{3}{2}\eta^3\langle\nu_E,x\rangle_E^{-3}|\nabla\langle\nu_E,x\rangle_E|_H^2,
$$

from Young's inequality. □

The following theorem is the main technical interior gradient estimate.

Theorem 2.2.13. For any $R\geq\cosh^{-1}\left(\frac{\sigma}{n+\sigma}e^{(n+\sigma)T}\right)$ and $\theta\in\left(\frac{\sigma}{(n+\sigma)\cosh R}e^{(n+\sigma)T},1\right)$ such that $\{x\in\Sigma_t\mid r\leq R\}$ is a compact radial graph for all $t\in[0,T]$, we have

$$
\sup_{\{x\in\Sigma_t|e^{(n+\sigma)t}(\cosh r+\frac{\sigma}{n+\sigma})\leq\theta\cosh R\}}\langle\nu_E,\mathbf{z}\rangle_E^{-1}\leq e^{(n+2)T+v_{\mathrm{osc}}}(1-\theta)^{-3}\sup_{\{x\in\Sigma_0|r\leq R\}}\langle\nu_E,\mathbf{z}\rangle_E^{-1},
$$

where $v_{\mathrm{osc}}=\max_{t\in[0,T]}\max_{\{x\in\Sigma_t|r\leq R\}}v-\min_{t\in[0,T]}\min_{\{x\in\Sigma_t|r\leq R\}}v$ is the oscillation of the radial height of x (see (1.1.2)) in $\bigcup_{t\in[0,T]}\{x\in\Sigma_t\mid r\leq R\}$.

Proof. The previous proposition and Hamilton's trick imply, for almost all $t \in (0,T)$,

$$\frac{d}{dt} \sup_{\{x\in\Sigma_t | r\le R\}} \xi \le (n+2) \sup_{\{x\in\Sigma_t | r\le R\}} \xi,$$

so we may integrate from 0 to T to obtain

$$\sup_{\{x\in\Sigma_T | r\le R\}} \eta^3 \langle \nu_E, x\rangle_E^{-1} \le e^{(n+2)T} \sup_{\{x\in\Sigma_0 | r\le R\}} \eta^3 \langle \nu_E, x\rangle_E^{-1}.$$

Now notice $e^{v_{\min}} \le |x|_E$ implies

$$e^{(n+2)T - v_{\min}} \sup_{\{x\in\Sigma_0 | r\le R\}} \eta^3 \langle \nu_E, \mathbf{z}\rangle_E^{-1} \ge e^{(n+2)T} \sup_{\{x\in\Sigma_0 | r\le R\}} \eta^3 \langle \nu_E, x\rangle_E^{-1}.$$

Similarly, $e^{v_{\max}} \ge |x|_E$ implies

$$e^{-v_{\max}} \sup_{\{x\in\Sigma_T | r\le R\}} \eta^3 \langle \nu_E, \mathbf{z}\rangle_E^{-1} \le \sup_{\{x\in\Sigma_T | r\le R\}} \eta^3 \langle \nu_E, x\rangle_E^{-1}.$$

These two inequalities imply then

$$\sup_{\{x\in\Sigma_T | r\le R\}} \eta^3 \langle \nu_E, \mathbf{z}\rangle_E^{-1} \le e^{(n+2)T + v_{\max} - v_{\min}} \sup_{\{x\in\Sigma_0 | r\le R\}} \eta^3 \langle \nu_E, \mathbf{z}\rangle_E^{-1}.$$

We also have

$$\sup_{\{x\in\Sigma_T | e^{(n+\sigma)t}(\cosh r + \frac{\sigma}{n+\sigma}) \le \theta \cosh R\}} \eta^3 \langle \nu_E, \mathbf{z}\rangle_E^{-1} \le \sup_{\{x\in\Sigma_T | r\le R\}} \eta^3 \langle \nu_E, \mathbf{z}\rangle_E^{-1},$$

and $\eta^3 \ge (1-\theta)^3 \cosh^3 R$ in $\{x \in \Sigma_t \mid e^{(n+\sigma)t}(\cosh r + \frac{\sigma}{n+\sigma}) \le \theta \cosh R\}$ since $\theta \cosh R + \eta \ge \cosh R$ there. We also have $\eta^3 \le \cosh^3 R$ everywhere. These facts, along with replacing T with any $t \in [0,T)$, imply the result. □

The above theorem allows us to prove Theorem 1.1.5. We stress it provides an interior bound, making no assumptions on the asymptotic boundary of the initial surface.

2.3 Interior estimates on higher order derivatives

In order to obtain the estimates on higher order derivatives, we also need the evolution equation for the second fundamental form.

Lemma 2.3.1. On $\Sigma_t \subset \mathbb{H}$, we have

$$
\begin{aligned}
&(i) \quad \frac{\partial}{\partial t} a_{ij} = \nabla_i \nabla_j H - (H-\sigma) a_i^k a_{jk} + (H-\sigma)(R^H)_{i0j0}\,, \\
&(ii) \quad \frac{\partial}{\partial t} |A|^2 = 2a^{ij} \nabla_i \nabla_j H + 2(H-\sigma)\mathrm{Tr}(A^3) - 2H(H-\sigma)\,, \\
&(iii) \quad \left(\frac{\partial}{\partial t} - \Delta\right)|A|^2 = 2|A|^4 + 2n|A|^2 - 2|\nabla A|^2 - 4H^2 + 2\sigma(H - \mathrm{Tr}(A^3))\,.
\end{aligned}
$$

Proof. (i) Note that $\nabla^H_{\mathbf{v}_i} \mathbf{v}_j = a_{ij} \nu_H$, we compute

$$
\begin{aligned}
\frac{\partial}{\partial t} a_{ij} &= \langle \nabla^H_{\frac{\partial}{\partial t}} \nabla^H_{\mathbf{v}_i} \mathbf{v}_j, \nu_H \rangle_H \\
&= \langle \nabla^H_{\mathbf{v}_i} \nabla^H_{\mathbf{v}_j} \frac{\partial}{\partial t}, \nu_H \rangle_H + \langle (R^H)(\mathbf{v}_i, \partial/\partial t)\mathbf{v}_j, \nu_H \rangle_H \\
&= \langle \nabla^H_{\mathbf{v}_i} \nabla^H_{\mathbf{v}_j} ((H-\sigma)\nu_H), \nu_H \rangle_H + (H-\sigma)(R^H)_{i0j0} \\
&= \langle \nabla^H_{\mathbf{v}_i} (\nabla^H_{\mathbf{v}_j} H \nu_H) - \nabla^H_{\mathbf{v}_i}((H-\sigma) a_j^k \mathbf{v}_k), \nu_H \rangle_H + (H-\sigma)(R^H)_{i0j0} \\
&= \nabla_i \nabla_j H - (H-\sigma) a_i^k a_{jk} + (H-\sigma)(R^H)_{i0j0}\,.
\end{aligned}
$$

(ii) Notice $\frac{\partial}{\partial t} g^{ij} = 2(H-\sigma) g^{ik} g^{jl} a_{kl}$, so that

$$
\begin{aligned}
\frac{\partial}{\partial t} |A|^2 &= \frac{\partial}{\partial t}\left(g^{ij} g^{kl} a_{ik} a_{jl}\right) \\
&= 4(H-\sigma) a^{ij} a_{ik} a_j^k + 2a^{ij}\left(\nabla_i \nabla_j H - (H-\sigma) a_{ik} a_j^k + (H-\sigma)(R^H)_{i0j0}\right) \\
&= 2a^{ij} \nabla_i \nabla_j H + 2(H-\sigma)\mathrm{Tr}(A^3) - 2H(H-\sigma).
\end{aligned}
$$

(iii) Combining (ii) with Simons' identity. □

2.3.1 Estimates on the second derivatives

Now let $u = \langle \nu_E, x\rangle_E^{-1}$ and define

$$\varphi = \varphi(u^2) = \frac{u^2}{1-ku^2}$$

where

$$k = \left(2 \sup_{t\in[0,T]} \sup_{\{x\in\Sigma_t | r\le R\}} u^2\right)^{-1}.$$

Let φ' denote differentiation of φ with respect to u^2. From Remark 2.2.2, we know that

$$c_0 \le |x|_E^{-2} \le \varphi$$

for some constant c_0 depending on Σ_0.

Combining Proposition 2.2.8 with (iii) of Lemma 2.3.1, we obtain:

Lemma 2.3.2. On $\{x\in\Sigma_t | r\le R\}$ and Σ_t moves by MMCF, we have

$$\left(\frac{\partial}{\partial t} - \Delta\right)\left(|A|^2\varphi\right) \le -k|A|^4\varphi^2 + \left(\frac{c(n,c_0)}{k} - k\varphi'|\nabla v|^2\right)|A|^2\varphi$$
$$-\varphi^{-1}\langle\nabla\varphi, \nabla(|A|^2\varphi)\rangle_H + \sigma^2\varphi\,.$$

Proof. We have

$$\begin{aligned}&\left(\frac{\partial}{\partial t} - \Delta\right)\left(|A|^2\varphi\right)\\ =\ &\varphi\left(\frac{\partial}{\partial t} - \Delta\right)|A|^2 + |A|^2\left(\frac{\partial}{\partial t} - \Delta\right)\varphi - 2\langle\nabla|A|^2, \nabla\varphi\rangle_H\\ :=\ &\mathrm{I} + \mathrm{II} + \mathrm{III}\,.\end{aligned}$$

By (iii) of Lemma 2.3.1, we have

$$\begin{aligned}
\mathrm{I} &= \varphi\left(2|A|^4+2n|A|^2-2|\nabla A|^2-4H^2+2\sigma(H-\mathrm{Tr}(A^3))\right)\\
&\leq \varphi\left(2|A|^4+2n|A|^2-2|\nabla A|^2-4H^2+\sigma\left(H^2c_2+\frac{1}{c_2}+\frac{|A|^2}{c_1}+c_1|A|^4\right)\right)\\
&\leq \varphi(2+c_1\sigma)|A|^4+\varphi\left(2n+\frac{\sigma}{c_1}\right)|A|^2-2\,\varphi\,|\nabla A|^2+\frac{\sigma}{c_2}\,\varphi
\end{aligned}$$

where we used Young's inequality and the fact that $|\mathrm{Tr}(A^3)|\leq|A|^3$. We also chose constants c_1,c_2 such that $c_1\sigma\leq c_0k$ and $c_2\sigma\leq 4$, where $c_0\leq\varphi$.

For the second term II, by Proposition 2.2.8 we have

$$\begin{aligned}
\left(\frac{\partial}{\partial t}-\Delta\right)\varphi &= -2\,\varphi'\,u^3\left(\frac{\partial}{\partial t}-\Delta\right)\langle\nu_E,x\rangle_E-6\,\varphi'\,|\nabla u|^2-4\,\varphi''\,u^2|\nabla u|^2\\
&= -2\,\varphi'\,u^2(|A|^2-\sigma\langle\nu_E,\mathbf{e}\rangle_E)-4\,\varphi'\,u\langle\nabla u,x^{n+1}\mathbf{e}\rangle_H-(6+8k\,\varphi)\,\varphi'\,|\nabla u|^2
\end{aligned}$$

since $\varphi''\,u^2=2k\,\varphi\,\varphi'$.

Therefore, using Young's inequality again we get

$$\mathrm{II}\leq -2u^2\,\varphi'\,|A|^4-(6+8k\,\varphi)\,\varphi'\,|A|^2|\nabla u|^2+k\,\varphi\,\varphi'\,|A|^2|\nabla u|^2+\frac{4}{c_0k}|A|^2\,\varphi+4n|A|^2\,\varphi\,,$$

since $\sigma<n,\varphi'\,u^2\leq 2\,\varphi$ and $\frac{\varphi}{c_0}\geq 1$.

For the third term III, we compute:

$$\begin{aligned}
\mathrm{III} &= -\varphi^{-1}\langle\nabla\varphi,\nabla(|A|^2\,\varphi)\rangle_H+\varphi^{-1}\,|A|^2|\nabla\varphi|^2-\langle\nabla|A|^2,\nabla\varphi\rangle_H\\
&= -\varphi^{-1}\langle\nabla\varphi,\nabla(|A|^2\,\varphi)\rangle_H+4\,\varphi^{-1}(\varphi'\,u)^2|A|^2|\nabla u|^2-4\,\varphi'\,u|A|\langle\nabla|A|,\nabla u\rangle_H\\
&\leq -\varphi^{-1}\langle\nabla\varphi,\nabla(|A|^2\,\varphi)\rangle_H+6\,\varphi^{-1}(\varphi'\,u)^2|A|^2|\nabla u|^2+2|\nabla|A||^2\,\varphi\,.
\end{aligned}$$

From Kato's inequality, $|\nabla|A||^2 \le |\nabla A|^2$, so that

$$\mathrm{I}+\mathrm{II}+\mathrm{III} \le (\varphi(2+c_1\sigma)-2u^2\varphi')|A|^4+\left(6n+\frac{\sigma}{c_1}+\frac{4}{c_0k}\right)|A|^2\varphi+\frac{\sigma}{c_2}\varphi$$
$$+(6\varphi^{-1}(\varphi' u)^2-(6+7k\varphi)\varphi')|A|^2|\nabla u|^2-\varphi^{-1}\langle\nabla\varphi,\nabla(|A|^2\varphi)\rangle_H\,.$$

Note that since $c_1\sigma \le c_0k$ and $\varphi-u^2\varphi' = -k\varphi^2$, we have $\varphi(2+c_1\sigma)-2u^2\varphi' \le -k\varphi^2$. Moreover,

$$6\varphi^{-1}(\varphi' u)^2-(6+7k\varphi)\varphi' \,=\, -k\varphi\varphi'\,.$$

Now let $c_1 = \frac{c_0k}{\sigma}$ and $c_2 = \frac{1}{\sigma}$, then $6n+\frac{\sigma}{c_1}+\frac{4}{c_0k} \le \frac{c(n,c_0)}{k}$ and on $\{x\in\Sigma_t | r\le R\}\cap\{|A|^2\ge 1\}$, we have

$$\mathrm{I}+\mathrm{II}+\mathrm{III} \le -k|A|^4\varphi^2+\left(\frac{c(n,c_0)}{k}-k\varphi'|\nabla u|^2\right)|A|^2\varphi-\varphi^{-1}\langle\nabla\varphi,\nabla(|A|^2\varphi)\rangle_H+\sigma^2\varphi\,.$$

This proves the lemma. □

Now we are ready to show the interior estimates on the second fundamental form $|A|$ (i.e., $|\nabla^2 v|$). For simplicity, let

$$g = |A|^2\varphi\,.$$

Then the previous lemma says

$$\left(\frac{\partial}{\partial t}-\Delta\right)g \le -kg^2+\left(\frac{c(n,c_0)}{k}-k\varphi'|\nabla u|^2\right)g-\varphi^{-1}\langle\nabla\varphi,\nabla g\rangle_H+\sigma^2\varphi\,.$$

Now let

$$\eta = (\cosh R-\cosh r)^2$$

be the spacial cut-off function, and let η' denote the differentiation with respect to $\cosh r$. Then, from Proposition 2.2.3, we have

$$\left(\frac{\partial}{\partial t}-\Delta\right)(-\cosh r) = -\left[\frac{1}{\cosh r}(1-\langle \nu_E, \mathbf{z}\rangle_E^2) - (n-\sigma\langle \nu_E, \mathbf{e}\rangle_E)\cosh r - \sigma\langle \nu_E, \mathbf{z}\rangle_E\right]$$

$$\leq (\sigma+n)\cosh r+\sigma\,.$$

So that

$$\left(\frac{\partial}{\partial t}-\Delta\right)\eta = 2(\cosh R-\cosh r)\left(\frac{\partial}{\partial t}-\Delta\right)(-\cosh r) - 2|\nabla \cosh r|^2$$

$$\leq 2(\sigma+n)\cosh^2 R + 2\sigma\cosh R - 2|\nabla\cosh r|^2$$

$$\leq 2(2\sigma+n)\cosh^2 R - 2|\nabla\cosh r|^2\,,$$

if $\sigma \leq \cosh R$, namely, R is sufficiently large, e.g., $\cosh R \geq n$.

Therefore, we compute:

$$\begin{aligned}
\left(\frac{\partial}{\partial t}-\Delta\right)(g\eta) \leq& \left[-kg^2+\left(\frac{c(n,c_0)}{k}-k\varphi'\,|\nabla u|^2\right)g-\varphi^{-1}\langle\nabla\varphi,\nabla g\rangle_H+\sigma^2\varphi\right]\eta \\
&+g\left(\frac{\partial}{\partial t}-\Delta\right)\eta-2\langle\nabla g,\nabla\eta\rangle \\
\leq& -kg^2\eta+\left(\frac{c(n,c_0)}{k}\right)g\eta-\varphi^{-1}\langle\nabla\varphi,\nabla(g\eta)\rangle_H+\frac{|\eta'|^2 g}{k\eta u^2}|\nabla\cosh r|^2 \\
&+\sigma^2\varphi\eta+g\left(\frac{\partial}{\partial t}-\Delta\right)\eta-2\eta^{-1}\langle\nabla(g\eta),\nabla\eta\rangle+2\eta^{-1}g|\nabla\eta|^2 \\
\leq& -kg^2\eta+\left(\frac{c(n,c_0)}{k}\right)g\eta-\langle\varphi^{-1}\nabla\varphi+2\eta^{-1}\nabla\eta,\nabla(g\eta)\rangle_H \\
&+\sigma^2\varphi\eta+g\left(2(2\sigma+n)\cosh^2 R-2|\nabla\cosh r|^2\right)+g|\nabla\cosh r|^2\left(\frac{4}{ku^2}+8\right) \\
\leq& -kg^2\eta+\left(\frac{c(n,c_0)}{k}\right)g\eta-\langle\varphi^{-1}\nabla\varphi+2\eta^{-1}\nabla\eta,\nabla(g\eta)\rangle_H \qquad (2.3.3)\\
&+30ng\left(1+\frac{|x|_E^2}{k}\right)\cosh^2 R+\sigma^2\varphi\eta\,,
\end{aligned}$$

where we used Young's inequality and the facts that $\varphi^{-1}\nabla\varphi = 2\varphi u^{-3}\nabla u$ and $\varphi' = \varphi^2 u^{-4}$ and $\eta^{-1}|\nabla\eta|^2 = \eta^{-1}|\eta'|^2|\nabla\cosh r|^2 = 4|\nabla\cosh r|^2 \leq 4(1+\cosh r)^2$. Therefore, we have

$$\left(\frac{\partial}{\partial t} - \Delta\right)(g\eta t) \leq -kg^2\eta t + \left(\frac{c(n,c_0)}{k}t+1\right)g\eta - \langle \varphi^{-1}\nabla\varphi + 2\eta^{-1}\nabla\eta, \nabla(g\eta t)\rangle_H$$
$$+30ng\left(1+\frac{1}{c_0 k}\right)(\cosh^2 R)t + \sigma^2\varphi\eta t\,. \tag{2.3.4}$$

Now at a point (x_0,t_0) where $\sup_{[0,T]}\sup_{\{x\in\Sigma_t|r\leq R\}}(g\eta t) \neq 0$ is attained for $t_0 > 0$, we have

$$kg^2\eta t_0 \leq \left(\frac{c(n,c_0)}{k}t_0+1\right)g\eta + 30ng\left(1+\frac{1}{c_0 k}\right)(\cosh^2 R)t_0 + \sigma^2\varphi\eta t_0,$$

which implies (dividing by $kg = k|A|^2\varphi$ on both sides) at (x_0,t_0) we have

$$\begin{aligned}
&g(x_0,t_0)\eta(x_0,t_0)t_0 \\
&\leq \frac{1}{k}\left(\frac{c(n,c_0)}{k}t_0+1\right)\cosh^2 R + \frac{30n}{k}\left(1+\frac{1}{c_0 k}\right)(\cosh^2 R)t_0 + \frac{\sigma^2}{k|A|^2}(\cosh^2 R)t_0 \\
&\leq \frac{c(n,c_0)}{k^2}(1+T)\cosh^2 R + \frac{30n}{k}\left(1+T+\frac{\sigma^2 T}{|A|^2(x_0,t_0)}\right)\cosh^2 R\,.
\end{aligned}$$

Note that for any $(x,t) \in \{x\in\Sigma_t|\cosh r \leq \theta\cosh R\}\times[0,T]$ we have

$$g(x,t)\eta(x,t)t \leq g(x_0,t_0)\eta(x_0,t_0)t_0 \quad \text{and} \quad \eta \geq (1-\theta)^2\cosh^2 R\,.$$

If $|A|^2(x_0,t_0) \leq 1$, then

$$\begin{aligned}
c_0|A|^2(x,T) &\leq \frac{1}{T}\eta^{-1}(x,T)\,\varphi(x_0,t_0)\eta(x_0,t_0)t_0 \\
&\leq 4(1-\theta)^{-2}\sup_{t\in[0,T]}\sup_{\{x\in\Sigma_t|r\leq R\}} u^2 \\
&\leq \frac{8}{c_0}(1-\theta)^{-2}\sup_{t\in[0,T]}\sup_{\{x\in\Sigma_t|r\leq R\}} u^4\,,
\end{aligned}$$

where we used $c_0 \leq \varphi \leq 2u^2$ and $\eta \leq 2\cosh^2 R$. Otherwise, if $|A|^2(x_0,t_0) > 1$ then we have

$$c_0|A|^2(x,T) \leq g(x,T) \leq \left[\frac{c(n,c_0)}{k^2}\left(1+\frac{1}{T}\right)+\frac{30n}{k}\left(1+\frac{1}{T}+\sigma^2\right)\right](1-\theta)^{-2}$$
$$\leq c(n,c_0)\left(1+\frac{1}{T}\right)(1-\theta)^{-2}\sup_{t\in[0,T]}\sup_{\{x\in\Sigma_t|r\leq R\}} u^4 .$$

Since $T > 0$ was arbitrary, we have just proved

Theorem 2.3.5. For all $t \in [0,T]$, any $R \geq \cosh^{-1}(n)$ and any $\theta \in (0,1)$ we have

$$\sup_{\{x\in\Sigma_t\,|\,\cosh r\leq\theta\cosh R\}} |A|^2 \leq c(n,c_0)\left(1+\frac{1}{t}\right)(1-\theta)^{-2}\sup_{s\in[0,t]}\sup_{\{x\in\Sigma_s|r\leq R\}} u^4 .$$

2.3.2 Estimates on all the higher order derivatives

The estimates on all the higher order derivatives can be obtained analogously by looking at the evolution equations of the higher derivatives of the second fundamental form. We recall, for example, [EH91a] and [Unt03]. For this, we have

Lemma 2.3.6. For hypersurfaces Σ_t moving by MMCF in $\mathbb{H}$ which can be written locally as radial graphs, we have

(i)

$$\left(\frac{\partial}{\partial t}-\Delta\right)\nabla^m A = \sum_{i+j+k=m}\nabla^i A*\nabla^j A*\nabla^k A+\sigma\sum_{i+j=m}\nabla^i A*\nabla^j A$$
$$+\sum_{i+j=m}\nabla^i A*\nabla^j R^H+\sigma*\nabla^m R^H.$$

where $S*T$ is a tensor formed by contraction of tensors S and T by the metric g on Σ_t or its inverse ;

(ii)

$$\left(\frac{\partial}{\partial t}-\Delta\right)|\nabla^m A|^2 \le -2|\nabla^{m+1}A|^2 + c\left(\sum_{i+j+k=m}|\nabla^i A||\nabla^j A||\nabla^k A||\nabla^m A|\right.$$
$$\left. + \sigma\sum_{i+j=m}|\nabla^i A||\nabla^j A||\nabla^m A| + |\nabla^m A|^2 + \sigma|\nabla^m A|^2\right).$$

Theorem 2.3.7. For all $t\in[0,T]$, any $R\ge\cosh^{-1}(n)$ and any $\theta\in(0,1)$ we have

$$\sup_{\{x\in\Sigma_t\,|\,\cosh r\le\theta\cosh R\}}|\nabla^m A|^2 \le c\left(n, c_0, \sup_{s\in[0,t]}\sup_{\{x\in\Sigma_s|r\le R\}} u\right)\left(1+\frac{1}{t}\right)(1-\theta)^{-2}\left(1+\frac{1}{t}\right)^{m+1}.$$

Proof. Similar to the proof of Theorem 2.3.5, c.f. [EH91a, Theorem 3.4]. □

2.4 Proof of Theorem 1.1.5

Our goal in this section is to prove the main Theorem 1.1.5. We restate it for convenience.

Theorem 1.1.5. If $\mathbf{F}_0 : \mathbb{S}^n_+ \to \mathbb{H}$ is a map such that $\Sigma_0 = \mathbf{F}_0(\mathbb{S}^n_+)$ is a locally Lipschitz continuous radial graph over $\mathbb{S}^n_+$, then the Cauchy initial-boundary value problem for the modified mean curvature flow (1.1.1) has a solution $\mathbf{F}\in C^\infty(\mathbb{S}^n_+\times(0,\infty))\cap C^{0,1\times 0,1/2}_{loc}(\mathbb{S}^n_+\times[0,\infty))$ and $\mathbf{F}(\mathbb{S}^n_+,t)$ is a complete radial graph over $\mathbb{S}^n_+$ for any $t\ge 0$.

Proof. First we assume Σ_0 (or equivalently v_0) is smooth. For any $\varepsilon>0$, we define the solid cylinder

$$\mathbf{C}_\varepsilon = \left\{x\in\mathbb{H} : \frac{|x|_E}{x^{n+1}} \le \frac{1}{\varepsilon}\right\},$$

and let $\Sigma_0^\varepsilon = \Sigma_0\cap\mathbf{C}_\varepsilon$ and $\Omega_\varepsilon = \mathbf{F}_0^{-1}(\Sigma_0\cap\mathbf{C}_\varepsilon)$. Then Ω_ε is compact and $\Gamma_\varepsilon = \mathbf{F}_0(\partial\Omega_\varepsilon)$ is a smooth radial graph over $\partial\Omega_\varepsilon$.

From the existence result in [LX12] for the approximate MMCF, and from Lemma (A.1.3), we know that the initial-boundary value problem

$$\begin{cases} \dfrac{\partial}{\partial t}\mathbf{F}(\mathbf{z},t) = (H-\sigma)\nu_H\,, \quad (\mathbf{z},t)\in\Omega_\varepsilon\times(0,\infty)\,, \\ \quad \mathbf{F}(\mathbf{z},0) = \mathbf{F}_0(\mathbf{z})\,, \quad \mathbf{z}\in\Omega_\varepsilon\,, \\ \mathbf{F}(\partial\Omega_\varepsilon,t) = \Gamma_\varepsilon\,, \quad t\in[0,\infty) \end{cases} \tag{2.4.1}$$

has a unique radial graph solution $\mathbf{F}^\varepsilon_t(\mathbf{z}) = \mathbf{F}^\varepsilon(\mathbf{z},t)\in C^\infty(\Omega_\varepsilon\times(0,\infty))\cap C^{0,1\times 0,\frac{1}{2}}(\overline{\Omega_\varepsilon}\times(0,\infty))\cap C^0(\overline{\Omega_\varepsilon}\times[0,\infty))$, and we denote $\Sigma^\varepsilon_t = \mathbf{F}^\varepsilon(\Omega_\varepsilon,t)$.

Now, for every $\varepsilon\in(0,1)$, we let $v^\varepsilon(\mathbf{z},t)$ be the solution to (2.4.1) (c.f. (1.1.3)), namely,

$$\begin{cases} \dfrac{\partial v^\varepsilon(\mathbf{z},t)}{\partial t} = y^2\dfrac{\alpha^{ij}v^\varepsilon_{ij}}{n} - y\mathbf{e}\cdot\nabla^S v^\varepsilon - \sigma y w^\varepsilon\,, \quad (\mathbf{z},t)\in\Omega_\varepsilon\times(0,\infty)\,, \\ v^\varepsilon(\mathbf{z},0) = v_0(\mathbf{z})\,, \quad \mathbf{z}\in\Omega_\varepsilon\,, \\ v^\varepsilon(\mathbf{z},t) = \phi^\varepsilon(\mathbf{z})\,, \quad (\mathbf{z},t)\in\partial\Omega_\varepsilon\times[0,\infty)\,. \end{cases} \tag{2.4.2}$$

For a fixed $\delta_0 > 0$ sufficiently small, we let

$$E_{t,\varepsilon,\delta_0} := \Sigma^\varepsilon_t\cap\left\{x\in\mathbb{H}\mid r(x)\le\cosh^{-1}\left(\frac{1}{\delta_0}\right)\right\} = \Sigma^\varepsilon_t\cap\mathbf{C}_{\delta_0}\,,$$

where $r(x)$ is the hyperbolic distance from $x\in\mathbb{H}$ to the x^{n+1}-axis and $\cosh r(x) = \frac{|x|_E}{x^{n+1}}$. Then $E_{t,\varepsilon,\delta_0}$ is a compact radial graph and we have $E_{0,\varepsilon,\delta_0} = E_{0,\delta_0,\delta_0}$ for all $\varepsilon\le\delta_0$. By compactness, there exist caps S_1,S_2 with constant mean curvature σ such that the Euclidean norms satisfy $c^{-1}(\Sigma^{\delta_0}_0)\le|x_1|_E\le|\mathbf{F}^\varepsilon_0(\mathbf{z})|\le|x_2|_E\le c(\Sigma^{\delta_0}_0)$ for all $x_i\in S_i$, $i=1,2$, any $\mathbf{z}\in(\mathbf{F}^\varepsilon_0)^{-1}(E_{0,\varepsilon,\delta_0})$, and any $\varepsilon\le\delta_0$. This implies, by the comparison principle for MMCF, that for all $\varepsilon\le\delta_0$ we have

$$\sup_{t\in(0,\infty)}\ \sup_{\mathbf{z}\in(\mathbf{F}^\varepsilon_t)^{-1}(E_{t,\varepsilon,\delta_0})}|v^\varepsilon(\mathbf{z},t)|\le c_0\left(n,\delta_0,\sup_{\mathbf{z}\in\mathbf{F}_0^{-1}(E_{0,\delta_0,\delta_0})}|v_0(\mathbf{z})|\right)\,.$$

For $\theta \in (0,1)$, let

$$G_{t,\varepsilon,\delta_0,\theta} := \left\{ x \in E_{t,\varepsilon,\delta_0} \mid e^{(n+\sigma)t} \left(\cosh r(x) + \frac{\sigma}{n+\sigma} \right) \leq \frac{\theta}{\delta_0} \right\}.$$

We note that by Theorem 2.2.13, for all $\varepsilon \leq \delta_0$ and any $T_0 > 0$ we have

$$\sup_{t\in[0,T_0]} \sup_{\mathbf{z}\in(\mathbf{F}_t^\varepsilon)^{-1}(G_{t,\varepsilon,\delta_0,\frac{1}{2}})} |\nabla^S v^\varepsilon(\mathbf{z},t)| \leq e^{(n+2)T_0} c_1 \left(n, \delta_0, c_0, \sup_{\mathbf{z}\in\mathbf{F}_0^{-1}(E_{0,\delta_0,\delta_0})} |\nabla^S v_0(\mathbf{z})| \right).$$

For $\varepsilon_0 > 0$ and $\theta \in (0,1)$, we let

$$K_{t,\varepsilon,\varepsilon_0,\theta} := \left\{ x \in E_{t,\varepsilon,\delta_0} \mid \cosh r(x) \leq \frac{\theta}{\varepsilon_0} \right\}.$$

We choose $\delta_0 > 0$ sufficiently small such that $\frac{1}{\delta_0^{1/2}} - \frac{\sigma}{n+\sigma} \geq 2$, and let $T_0 = -\frac{1}{2(n+\sigma)} \log \delta_0$ and $\varepsilon_0 = \left(\frac{1}{\delta_0^{1/2}} - \frac{\sigma}{n+\sigma} \right)^{-1}$. Then, for our choices of δ_0, T_0, ε_0 we know that for any $\varepsilon \leq \delta_0$,

$$G_{T_0,\varepsilon,\delta_0,\frac{1}{2}} = K_{T_0,\varepsilon,\varepsilon_0,\frac{1}{2}}.$$

Hence, for all $\varepsilon \leq \delta_0$, we have

$$\sup_{t\in[0,T_0]} \sup_{\mathbf{z}\in(\mathbf{F}_t^\varepsilon)^{-1}(K_{t,\varepsilon,\varepsilon_0,\frac{1}{2}})} |\nabla^S v^\varepsilon(\mathbf{z},t)| \leq e^{(n+2)T_0} c_1 \left(n, \delta_0, c_0, \sup_{\mathbf{z}\in\mathbf{F}_0^{-1}(E_{0,\delta_0,\delta_0})} |\nabla^S v_0(\mathbf{z})| \right).$$

Therefore, by Theorem 2.3.7 , for any integer $m \geq 2$ and any $\varepsilon \leq \delta_0$, we have

$$\sup_{t\in[0,T_0]} \sup_{\mathbf{z}\in(\mathbf{F}_t^\varepsilon)^{-1}(K_{t,\varepsilon,\varepsilon_0,\frac{1}{2}})} |(\nabla^S)^m v^\varepsilon(\mathbf{z},t)| \leq c_m(n, \delta_0, c_1).$$

Hence, for such fixed $\delta_0 > 0$, by the Arzelà-Ascoli Theorem, there exists some sequence $\{\varepsilon_{i,0}\}_{i=1}^\infty$ such that $\varepsilon_{i,0} \to 0$ as $i \to \infty$ and such that $v^{\varepsilon_{i,0}}$ converges in $C^\infty(\Omega_{2\varepsilon_0} \times [0,T_0])$ to some $v^{\varepsilon_0,T_0} \in C^\infty(\Omega_{2\varepsilon_0} \times [0,T_0])$ as $i \to \infty$ which solves (2.4.2). Now we fix a descending sequence of positive real numbers $\{\delta_k\}_{k=0}^\infty$ such that $\delta_k \to 0$ as $k \to \infty$. Then define $T_k = -\frac{1}{2(n+\sigma)} \log \delta_k$, and $\frac{1}{\varepsilon_k} = \frac{1}{\delta_k^{1/2}} - \frac{\sigma}{n+\sigma}$. Then $T_k \to \infty$ and $\varepsilon_k \to 0$ as $k \to \infty$.

For non-negative integers k, there is a function $v^{\varepsilon_k,T_k} \in C^\infty(\Omega_{2\varepsilon_k} \times [0,T_k])$ satisfying the conditions of (2.4.2) such that v^{ε_k,T_k} is the uniform limit of some sequence $\{v^{\varepsilon_{i,k}}\}_{i=1}^\infty$ and

$$v^{\varepsilon_k,T_k}|_{\Omega_{2\varepsilon_l}\times[0,T_l]} = v^{\varepsilon_l,T_l}$$

for all non-negative integers $l \leq k$. We can prove this by induction. The base case $k=0$ was done above. Our interior estimates imply we have uniform bounds of v^ε and its derivatives on $\Omega_{2\varepsilon_{k+1}} \times [0,T_{k+1}]$ for $\varepsilon \leq \delta_{k+1}$. So, again by the Arzelà-Ascoli Theorem, there exists a subsequence $\{v^{\varepsilon_{i,k+1}}\}_{i=1}^\infty$ of $\{v^{\varepsilon_{i,k}}\}_{i=1}^\infty$ such that $v^{\varepsilon_{i,k+1}}$ converges in $C^\infty(\Omega_{2\varepsilon_{k+1}} \times [0,T_{k+1}])$ to some $v^{\varepsilon_{k+1},T_{k+1}} \in C^\infty(\Omega_{2\varepsilon_{k+1}} \times [0,T_{k+1}])$ as $i \to \infty$. Since $\Omega_{2\varepsilon_k} \times [0,T_k] \subset \Omega_{2\varepsilon_{k+1}} \times [0,T_{k+1}]$ and $\{v^{\varepsilon_{i,k+1}}\}_{i=1}^\infty$ is a subsequence of $\{v^{\varepsilon_{i,k}}\}_{i=1}^\infty$, we must have $v^{\varepsilon_{k+1},T_{k+1}}|_{\Omega_{2\varepsilon_k}\times[0,T_k]} = v^{\varepsilon_k,T_k}$.

If $(\mathbf{z},t) \in \mathbb{S}^n_+ \times [0,\infty)$, then there exists some non-negative integer k such that $(\mathbf{z},t) \in \Omega_{2\varepsilon_k} \times [0,T_k]$. We define $v(\mathbf{z},t) = v^{\varepsilon_k,T_k}(\mathbf{z},t)$. Then our construction of the sequence v^{ε_k,T_k} shows v is well-defined. Moreover, if we define $\mathbf{F}(\mathbf{z},t) = e^{v(\mathbf{z},t)}\mathbf{z}$ on $\mathbb{S}^n_+ \times [0,\infty)$, then $\mathbf{F} \in C^\infty(\mathbb{S}^n_+ \times [0,\infty))$ solves (1.1.1) up to a reparameterization of $\mathbb{S}_+$ which leaves Σ_0 fixed by Lemma (A.2.2).

Now if Σ_0 is merely locally Lipschitz continuous, then for any fixed compact subset $\Omega \subset \mathbb{S}^n_+$, we can approximate v_0 by smooth functions v_0^s with the same Lipschitz bound as the Lipschitz bound of v_0 on Ω. By the above arguments, for every s, there is a smooth one parameter family of functions v_t^s solving (2.4.2) with initial data v_0^s. Now our interior estimates imply v_t^s and all its derivatives are uniformly bounded in any compact set $K \subset \Omega$, which again implies the existence of a uniform limit $v \in C^\infty(K \times (0,T]) \cap C^{0,1\times 0,1/2}(K \times [0,T])$. Since Ω and T are arbitrary, this establishes the existence of a function $v \in C^\infty(\mathbb{S}^n_+ \times (0,\infty)) \cap C^{0,1\times 0,1/2}_{loc}(\mathbb{S}^n_+ \times [0,\infty))$ which solves (1.1.3). $\square$

Appendix A

Some Auxiliary Facts

A.1 Some hyperbolic geometry

Lemma A.1.1. a reasonable choice of σ. For any embedded $\Gamma \subset \partial_\infty \mathbb{H}$, there does not exist any immersed class 2 hypersurface $\Sigma \subset \mathbb{H}$ with $\partial_\infty \Sigma = \Gamma$ and with CMC $\sigma, |\sigma| \geq n$.

Proof. This follows directly from the comparison principle in [GT01, Theorem 10.1]. We suppose Σ has $|\text{CMC}| \geq n$. We fix a horosphere, S, such that there is some open $U \subseteq \mathbb{H}$ with $S \cap \Sigma \cap U = \{p\}$ for some $p \in \mathbb{H}$. We know $|H(S)| = n$. There exists a cap, S', such that $H(S')$ is close to $H(S)$, $|H(S')| < n$, and such that there is some open $U' \subseteq \mathbb{H}$ with $S' \cap \Sigma \cap U' = \{p\}$. We can locally write Σ and S' as graphs of real-valued functions, f_Σ, $f_{S'}$, respectively, over the same hyperplane around p. Then $|H(f_\Sigma)| > |H(f_{S'})|$ implies $|f_\Sigma| < |f_{S'}|$ everywhere by comparison. However, $f_\Sigma(p) = f_{S'}(p)$. Hence, $\sigma = |H(f_\Sigma)| \leq |H(f_{S'})| < n$.

□

Lemma A.1.2. balls are open in nontangential spheres.

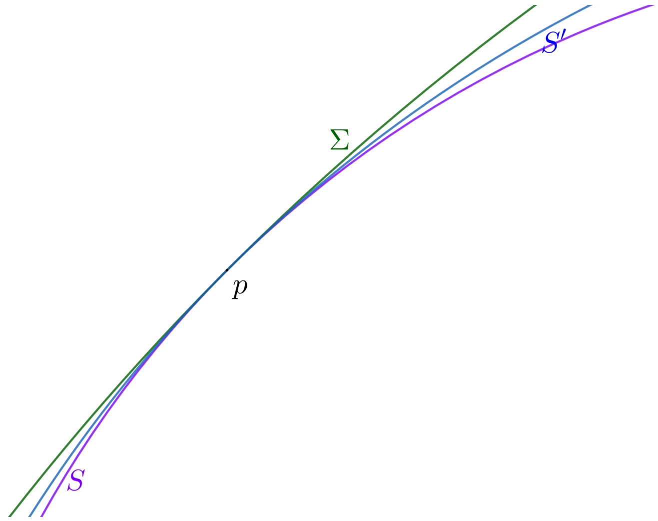

Figure A.1: $H(S) = -1$, $H(S') = -0.6$, $\Sigma = \mathbb{S}^n_+$

If $B_r(a), B_s(b) \subset \mathbb{R}^n$ are open balls, $p \in \partial B_r(a) \cap \partial B_s(b)$ and $T_p\partial B_r(a) + T_p\partial B_s(b) = \mathbb{R}^n$, then $B_r(a) \cap \partial B_s(b)$ is open and nonempty in $\partial B_s(b)$.

Proof. If $u = a - p$, we let $v = \mathrm{proj}_{T_p\partial B_s(b)}(u) - a$ and $w = u + v$. Then $|v| = kr$ for some $k < 1$, $w \in T_p\partial B_s(b)$ and $w \neq 0$. Hence, there exists a nonconstant $\gamma : [0,1] \to \partial B_s(b)$ such that $\gamma(0) = p$ and $\gamma'(0) = w$. Locally (for small enough t), $\gamma(t) = p + tw + o(t^2)y$ for some $y \in \mathbb{R}^n$. Then, for some fixed $c > 0$,

$$|\gamma(t) - a| \leq r(1 - (1-k)t + c/rt^2) < r$$

for small enough $t > 0$. □

Lemma A.1.3. local equidistant sphere condition. For any immersed $C^{1,1}$ hypersurface $\Sigma \subset \mathbb{H}$, for every point $p \in \Sigma$, there exists a hypersurface $S_\sigma \subset \mathbb{H}$ and an open set $U \subseteq \mathbb{H}$ such that S_σ has CMC σ and $\Sigma \cap S_\sigma \cap U = \{p\}$.

Proof. For a $C^{1,1}$ hypersurface Σ immersed in the upper-half space $\mathbb{R}^{n+1}_+$, for every point $p \in \Sigma$, by the smoothness of Σ, there exists a ball $B_r(a)$ interior to Σ such that $p \in \partial B_r(a)$ by [Bar09, Theorem 1.0.9]. Then there exist $a' \in \mathbb{R}^n$ and $R > 0$ such that $p \in \partial B_R((a', -\sigma R)) \cap \partial B_r(a)$ and $T_p \partial B_R((a', -\sigma R))$ and $T_p \partial B_r(a)$ are not tangential (we make sure $p - a$ and $p - (a', -\sigma R)$ are not scalar multiples of each other). Then $S_\sigma = \partial B_R((a', -\sigma R)) \cap B_r(a)$ is interior to Σ, S_σ is nonempty and open in $\partial B_R((a', -\sigma R))$ by Lemma A.1.2, and so has constant hyperbolic mean curvature σ with respect to its outward unit normal. The same can be done with exterior spheres by considering $\partial B_R((a', \sigma R))$. We refer to Figure 1.1. □

A.2 Radial graphs

Lemma A.2.1. a characterization of radial graphs. If $S \subseteq \mathbb{R}^{n+1}_+$ is C^1, open and bounded, then S is star-shaped with respect to the origin if and only if $\langle \nu_E, x \rangle_E > 0$ for every $x \in \partial S = \Sigma$, where ν_E is orthogonal to $T_x\Sigma$ and is outward pointing.

Proof. One direction is obvious, namely, if S is C^1 and star-shaped with respect to the origin, then Σ is a C^1 radial graph, i.e., there is some $v \in C^1(\mathbb{S}^n_+)$ such that $\Sigma = \{e^{v(\mathbf{z})}\mathbf{z} \mid \mathbf{z} \in \mathbb{S}^n_+\}$, from which it follows that then $\langle \nu_E, x \rangle_E > 0$ for every $x \in \Sigma$. This follows from the explicit computation of the outward unit normal to Σ: $\nu_E = (\mathbf{z} - \nabla^S v)/\sqrt{1 + |\nabla^S v|^2_S}$.

Conversely, we suppose $\langle \nu_E(x), x \rangle_E > 0$ for every $x \in \Sigma$. For the sake of deriving a contradiction, we suppose there is some $x \in \Sigma$ such that $tx \notin S$ for some $t \in (0,1)$. The set $K = \{t \in (0,1) \mid tx \notin S\}$ is nonempty and closed in $(0,1)$. Then $t_0 = \sup K \in K$ and we let $y = t_0 x$. We claim $y \in \Sigma$. Otherwise, there is some $r > 0$ such that $B_r(y) \subset \mathbb{R}^{n+1}_+ \setminus S$. In particular, $(1+t)y \notin S$ for small positive t. But $(1+t)t_0 > t_0$ if $t > 0$, contradicting the maximality of t_0. This implies $(1+t)y \in S$ for small positive t. Since $\nu_E(y)$ is outward pointing, $y + t\nu_E(y) \notin S$ for small positive t. Since $\nu_E(y)$ is normal to $T_y\Sigma$, it follows $\langle \nu_E(y), y\rangle_E \leq 0$, a contradiction. $\square$

Lemma A.2.2. invariance of MMCF under tangential perturbations. For any $\mathbf{F} : \mathbb{S}^n_+ \times [0,\infty) \to \mathbb{H}$, if there is some $v : \mathbb{S}^n_+ \times [0,\infty) \to \mathbb{R}$ such that $\mathbf{F}(\mathbf{z},t) = e^{v(\mathbf{z},t)}\mathbf{z}$ for all $(\mathbf{z},t) \in \mathbb{S}^n_+ \times [0,\infty)$, then $\mathbf{F}$ satisfies (1.1.1) if and only if v satisfies (1.1.3) up to a reparametrization of $\mathbb{S}^n_+ \times [0,\infty)$, keeping $\mathbb{S}^n_+ \times \{0\}$ fixed.

Proof. Suppose $\tau_1, \ldots, \tau_n$ is a local frame field on $\mathbb{S}^n_+$. For a function f on $\mathbb{S}^n_+$, denote $\nabla^S_i f = \tau_i f =: f_i$ and $\left(\left(\nabla^S\right)^2\right)_{ij} f =: f_{ij}$, where ∇^S is the Levi-Civitia connection on $\mathbb{S}^n$ with respect to the standard round metric γ_{ij}.

If r^i are the standard coordinate functions on $\mathbb{R}^n$, and if we define

$$R(r^1,\ldots,r^n) = \left(r^1,\ldots,r^n, \sqrt{1-\sum_{i=1}^n (r^i)^2}\right) \in \mathbb{S}^n_+$$

with $(r^1,\ldots,r^n) \in B^n_1(0)$, the unit ball in $\mathbb{R}^n$ center the origin, then $\mathbf{F}(R,t) : B^n_1(0) \times \{t\} \to \Sigma_t$ is a local parametrization of Σ_t for each $t \geq 0$. It follows, for each $p \in \Sigma_t$, if $r = (\mathbf{F}(R,t))^{-1}(p) \in B^n_1(0)$ and $\mathbf{z} = R(r) = \mathbf{F}^{-1}(p,t) \in \mathbb{S}^n_+$, then

$$\mathbf{v}_i(p) := \mathbf{F}(\cdot,t)_*(\tau_i)(p) = \frac{\partial \mathbf{F}(R,t)}{\partial r^i}(r) = e^v(\tau_i + v_j\mathbf{z})(\mathbf{z}),$$

the pushforward, or differential, of τ_i under $\mathbf{F}(\cdot,t)$. From this it follows

$$\nabla^E_{\mathbf{v}_i}\mathbf{v}_j(p) = \frac{\partial^2 \mathbf{F}(R,t)}{\partial r^i \partial r^j}(r) = e^v\left(v_i\tau_j + v_j\tau_i + {}^S\Gamma^k_{ij}\tau_k + (v_iv_j + \tau_iv_j - \gamma_{ij})\mathbf{z}\right)(\mathbf{z}),$$

where ${}^S\Gamma^k_{ij}$ are the Christoffel symbols of the round metric with respect to γ_{ij}. We deduce the

Euclidean outward unit normal to Σ_t is

$$\nu_E = \frac{\mathbf{z} - \nabla_S v}{w},$$

where $w^2 = 1 + v^q v_q$ and $v^q = \gamma^{qp} v_p$. Therefore,

$$\begin{aligned} a^E_{ij} &= \langle \nabla^E_{\mathbf{v}_i} \mathbf{v}_j, \nu_E \rangle_E \\ &= \frac{e^v}{w}(v_i v_j + \tau_i v_j - \gamma_{ij} - v_i v_j - v_i v_j - {}^S\Gamma^k_{ij} v_k) \\ &= \frac{e^v}{w}(v_{ij} - v_i v_j - \gamma_{ij}), \end{aligned}$$

which is the second fundamental form on Σ_t with respect to the Euclidean metric. Using Proposition 2.1.2, the second fundamental form on Σ_t with respect to the hyperbolic metric is

$$a_{ij} = \frac{1}{yw}(v_{ij} - v_i v_j - \gamma_{ij}) + \frac{g_{ij}}{w}\left(y - \langle \mathbf{e}, \nabla^S v \rangle_E\right),$$

where $y = \langle \mathbf{e}, \mathbf{z} \rangle_E$.

Also, the components of the induced Euclidean metric on Σ_t with respect to the frame $\mathbf{v}_i$ are $g^E_{ij}(p) = e^{2v}(\gamma_{ij} + v_i v_j)(\mathbf{z})$, so that its inverse is given by $g^{ij}_E(p) = e^{-2v}(\gamma^{ij} - \frac{v^i v^j}{w^2})(\mathbf{z})$, so that $g_{ij}(p) = \frac{1}{y^2}(\gamma_{ij} + v_i v_j)(\mathbf{z})$ and $g^{ij}(p) = y^2(\gamma^{ij} - \frac{v^i v^j}{w^2})(\mathbf{z})$.

So, the mean curvature of Σ_t with respect to the hyperbolic metric is

$$H = g^{ij} a_{ij} = \frac{1}{yw} g^{ij} v_{ij} - \frac{n}{w}\langle \mathbf{e}, \nabla^S v \rangle_E.$$

Finally, if $\mathbf{F}(\mathbf{z},t) = e^{v(\mathbf{z},t)}\mathbf{z}$ satisfies $\frac{\partial}{\partial t}\mathbf{F}(\mathbf{z},t) = (H - \sigma)\nu_H$, then

$$\frac{\partial v(\mathbf{z},t)}{\partial t} \cdot \frac{1}{yw} = \left\langle \frac{\partial}{\partial t}\mathbf{F}(\mathbf{z},t), \nu_H \right\rangle_H = \frac{1}{yw} g^{ij} v_{ij} - \frac{n}{w}\langle \mathbf{e}, \nabla^S v \rangle_E - \sigma.$$

Conversely, suppose v satisfies (1.1.3). Then

$$\left\langle \frac{\partial}{\partial t}\mathbf{F}(\mathbf{z},t), \nu_H \right\rangle_H = H - \sigma.$$

We now follow the proof of [Man11, Proposition 1.3.4]. From the above equation, there is some smooth vector field, X, on $\mathbb{S}^n_+ \times (0,\infty)$ such that $X(\mathbf{z},t) \in T_{\mathbf{F}(\mathbf{z},\mathbf{t})}\Sigma_t$, the tangent plane of $\Sigma_t = \mathbf{F}(\mathbb{S}^n_+,t)$ at $\mathbf{F}(\mathbf{z},t)$, and

$$\frac{\partial}{\partial t}\mathbf{F}(\mathbf{z},t) = (H-\sigma)\nu_H(\mathbf{z},t) + X(\mathbf{z},t)$$

for all $(\mathbf{z},t) \in \mathbb{S}^n_+ \times (0,\infty)$. We define, for a compact subset Ω of $\mathbb{S}^n_+$,

$$Y(\mathbf{z},t) = -(\mathbf{F}_*)^{-1}(X(\mathbf{z},t))$$

for all $(\mathbf{z},t) \in \Omega \times (0,\infty)$. Here, $\mathbf{F}_*$ is the pushforward of $\mathbf{F}$. Then Y is well-defined and smooth. Hence, by the existence and uniqueness theory for ODE's on a compact manifold, there is some smooth family of diffeomorphisms, Ψ, of Ω such that $\Psi(\mathbf{z},0) = \mathbf{z}$ for every $\mathbf{z} \in \Omega$ and

$$\frac{d}{dt}\Psi(\mathbf{z},t) = Y(\Psi(\mathbf{z},t),t)$$

for every $(\mathbf{z},t) \in \Omega \times [0,\infty)$. If $\tilde{\mathbf{F}}(\mathbf{z},t) = \mathbf{F}(\Psi(\mathbf{z},t),t)$ on $\Omega \times [0,\infty)$, then, certainly,

$$\frac{\partial}{\partial t}\tilde{\mathbf{F}}(\mathbf{z},t) = \tilde{H}(\mathbf{z},t)\tilde{\nu}_H(\mathbf{z},t)$$

on $\Omega \times (0,\infty)$ and $\tilde{\mathbf{F}}(\mathbf{z},0) = \mathbf{F}(\mathbf{z},0)$ on Ω. Since Ω is arbitrary, we may cover $\mathbb{S}^n_+$ with a compact exhaustion, for example $\Omega_\varepsilon = \{\mathbf{z} \in \mathbb{S}_+ \mid \langle \mathbf{e},\mathbf{z}\rangle_E \geq \varepsilon\}$ for all $\varepsilon \in (0,1)$. If Ψ_ε is the corresponding family of diffeomorphisms defined on $\Omega_\varepsilon \times [0,\infty)$ then $\Psi_\varepsilon|_{\Omega_\rho \times [0,\infty)}$ is the corresponding diffeomorphism defined on $\Omega_\rho \times [0,\infty)$ for all $\rho \geq \varepsilon$. Hence, by the uniqueness of the flow, $\Psi_\varepsilon|_{\Omega_\rho \times [0,\infty)} = \Psi_\rho$ for all $\rho \geq \varepsilon$.

So, since $(\mathbf{z},t) \in \mathbb{S}_+ \times [0,\infty)$ implies $(\mathbf{z},t) \in \Omega_\varepsilon \times [0,\infty)$ for some $\varepsilon > 0$, if we define $\Psi(\mathbf{z},t) = \Psi_\varepsilon(\mathbf{z},t)$, then Ψ is well-defined, Ψ is the identity on $\mathbb{S}^n_+ \times \{0\}$, and

$$\frac{d}{dt}\Psi(\mathbf{z},t) = Y(\Psi(\mathbf{z},t),t)$$

for every $(\mathbf{z},t) \in \mathbb{S}_+ \times [0,\infty)$. If we define $\tilde{\mathbf{F}}(\mathbf{z},t) = \mathbf{F}(\Psi(\mathbf{z},t),t)$ on $\mathbb{S}^n_+ \times [0,\infty)$, then $\tilde{\mathbf{F}}$ solves (1.1.1). □

Most of the evolution equations found in chapter 3 only work when $\sigma \geq 0$.

Lemma A.2.3. when $\sigma < 0$ and Σ_0 is a radial graph. The conclusion of Theorem 1.1.5 holds when $-n < \sigma < 0$.

Proof. The notation used here is similar to the notation used in the proof of Lemma A.2.2. We fix $\sigma \in (-n,0)$ and let $q(x) = x^* = \frac{x}{|x|_E^2}$ for any nonzero $x \in \mathbb{R}^{n+1}$. We note q restricted to $\mathbb{R}^{n+1}_+$ is an isometry of the upper-half plane model of $\mathbb{H}$ after computing its pushforward, $q_{*,x}$, at any $x \in \mathbb{R}^{n+1}_+$.

$$q_{*,x}(\vec{v}) = \frac{\vec{v}}{|x|_E^2} - \frac{2\langle \vec{v}, x\rangle_E}{|x|_E^4} \cdot x,$$

for any $\vec{v} \in T_x\mathbb{H}$. If Σ_0 is a complete, locally Lipschitz radial graph over $\mathbb{S}^n_+$, then $q^{-1}(\Sigma_0)$ is as well. Then, by Theorem 1.1.5, there is some $v : \mathbb{S}^n_+ \times [0,\infty) \to \mathbb{R}$ such that, if $\mathbf{F}'(\mathbf{z},t) = e^{-v(\mathbf{z},t)}\mathbf{z} \in \mathbb{H}$ for all $(\mathbf{z},t) \in \mathbb{S}^n_+ \times [0,\infty)$, then

$$\begin{cases} \dfrac{\partial}{\partial t}\mathbf{F}'(\mathbf{z},t) = (H(e^{-v(\mathbf{z},t)}\mathbf{z}) + \sigma)\,\nu_{H,e^{-v(\mathbf{z},t)}\mathbf{z}}, \quad (\mathbf{z},t) \in \mathbb{S}^n_+ \times [0,\infty), \\ \mathbf{F}'(\mathbb{S}^n_+, 0) = q^{-1}(\Sigma_0). \end{cases}$$

Here, $\nu_{H,e^{-v(\mathbf{z},t)}\mathbf{z}} = e^{-v(\mathbf{z},t)}y \cdot \frac{\mathbf{z} - \nabla_S - v}{\sqrt{1+(-v)^q(-v)_q}}$ is the outward pointing unit normal of $\Sigma'_t = \mathbf{F}'(\mathbb{S}^n_+, t)$ at $e^{-v(\mathbf{z},t)}\mathbf{z}$ and $H(e^{-v(\mathbf{z},t)}\mathbf{z}) = -\operatorname{div}_H(\nu_{H,e^{-v(\mathbf{z},t)}\mathbf{z}})$ is the hyperbolic mean curvature of Σ'_t at $e^{-v(\mathbf{z},t)}\mathbf{z}$, computed with respect to $\nu_{H,e^{-v(\mathbf{z},t)}\mathbf{z}}$. Then

$$\begin{aligned} q_{*,e^{-v(\mathbf{z},t)}\mathbf{z}}(\nu_{H,e^{-v(\mathbf{z},t)}\mathbf{z}}) &= \frac{\nu_{H,e^{-v(\mathbf{z},t)}\mathbf{z}}}{e^{-2v(\mathbf{z},t)}} - \frac{2\langle \nu_{H,e^{-v(\mathbf{z},t)}\mathbf{z}}, e^{-v(\mathbf{z},t)}\mathbf{z}\rangle_E}{e^{-4v(\mathbf{z},t)}} \cdot e^{-v(\mathbf{z},t)}\mathbf{z} \\ &= e^{v(\mathbf{z},t)}y \cdot \frac{\mathbf{z} + \nabla_S v}{w} - \frac{2e^{v(\mathbf{z},t)}y}{w}\mathbf{z} = -\nu_{H,e^{v(\mathbf{z},t)}\mathbf{z}}. \end{aligned}$$

Since $q : \mathbb{H} \to \mathbb{H}$ is an isometry,

$$H(e^{-v(\mathbf{z},t)}\mathbf{z}) = -\operatorname{div}_H(\nu_{H,e^{-v(\mathbf{z},t)}\mathbf{z}}) = -\operatorname{div}_H(q_{*,e^{-v(\mathbf{z},t)}\mathbf{z}}(\nu_{H,e^{-v(\mathbf{z},t)}\mathbf{z}}))$$

$$= \operatorname{div}_H(\nu_{H,e^{v(\mathbf{z},t)}\mathbf{z}})$$

$$= -H(e^{v(\mathbf{z},t)}\mathbf{z}).$$

So, if we define $\mathbf{F} = q \circ \mathbf{F}'$, then

$$\frac{\partial}{\partial t}\mathbf{F}(\mathbf{z},t) = -(H(e^{-v(\mathbf{z},t)}\mathbf{z}) + \sigma)\nu_{H,e^{v(\mathbf{z},t)}\mathbf{z}} = (H(e^{v(\mathbf{z},t)}\mathbf{z}) - \sigma)\nu_{H,e^{v(\mathbf{z},t)}\mathbf{z}}$$

for all $(\mathbf{z},t) \in \mathbb{S}^n_+ \times [0,\infty)$. Hence,

$$\begin{cases} \dfrac{\partial}{\partial t}\mathbf{F}(\mathbf{z},t) = (H(e^{v(\mathbf{z},t)}\mathbf{z}) - \sigma)\nu_{H,e^{v(\mathbf{z},t)}\mathbf{z}}, & (\mathbf{z},t) \in \mathbb{S}^n_+ \times (0,\infty), \\ \mathbf{F}(\mathbb{S}^n_+, 0) = \Sigma_0. \end{cases}$$

□

Lemma A.2.4. first variation of the energy functional I**.** We again use the notation used in the proof of Lemma A.2.2. If $\Omega \subset \mathbb{S}^n_+$ is open and relatively compact in $\mathbb{S}^n_+$, we define, for $\Sigma = \{e^{v(\mathbf{z})}\mathbf{z} \mid \mathbf{z} \in \Omega\}$,

$$I(\Sigma) = \int_\Omega wy^{-n}d\mathbf{z} + n\sigma\int_\Omega vy^{-(n+1)}d\mathbf{z}.$$

If $\mathbf{F}(\mathbf{z},t) = e^{v(\mathbf{z},t)}\mathbf{z}$ is defined on $\overline{\Omega} \times [0,\infty)$ such that $\frac{\partial \mathbf{F}}{\partial t} = 0$ on $\partial\Omega \times [0,\infty)$ and if $\Sigma_t = F(\Omega,t)$, then

$$\frac{\partial I(\Sigma_t)}{\partial t} = -n\int_\Omega \left\langle \frac{\partial \mathbf{F}}{\partial t}, (H-\sigma)\nu_H \right\rangle_H wy^{-n}d\mathbf{z}.$$

Proof.

$$\begin{aligned}
\frac{\partial I(\Sigma_t)}{\partial t} &= \int_\Omega \frac{\left\langle \nabla^S v, \nabla^S \frac{\partial v}{\partial t} \right\rangle_S y^{-n}}{w} d\mathbf{z} + n\sigma \int_\Omega \frac{\partial v}{\partial t} y^{-(n+1)} d\mathbf{z} \\
&= -\int_\Omega \mathrm{div}_S \left(\frac{y^{-n} \nabla^S v}{w} \right) \frac{\partial v}{\partial t} d\mathbf{z} + n\sigma \int_\Omega \frac{\partial v}{\partial t} y^{-(n+1)} d\mathbf{z} \\
&= -n \int_\Omega \frac{\partial v}{\partial t} H y^{-(n+1)} d\mathbf{z} + n\sigma \int_\Omega \frac{\partial v}{\partial t} y^{-(n+1)} d\mathbf{z} \\
&= -n \int_\Omega \left\langle \frac{\partial \mathbf{F}}{\partial t}, (H-\sigma)\nu_H \right\rangle_H w y^{-n} d\mathbf{z}.
\end{aligned}$$

The second equality follows from integration by parts, while the third follows from a well-known formula, found in [DSS09, Equation 1.2]. □

The above lemma shows MMCF is the negative gradient flow of the energy function I, a kind of area functional with a volume constraint. Hence, if $\mathbf{F}$ moves by MMCF and converges as $t \to \infty$, then the asymptotic limit has CMC σ.

A.3 A comparison principle for MMCF

Lemma A.3.1. A **comparison principle** holds for two smooth one-parameter families of hypersurfaces, $\Sigma_{1,t}$ and $\Sigma_{2,t}$, one compact, moving by their modified mean curvature in $\mathbb{H}$. That is, if they are initially disjoint, then they stay disjoint as long as they are moving by modified mean curvature flow.

Proof. We suppose, for the sake of deriving a contradiction, that $\Sigma_{1,t}$, $\Sigma_{2,t}$ intersect at some positive time. There is some minimal time, $t_0 > 0$, such that $\Sigma_{1,t_0} \cap \Sigma_{2,t_0} = \{x\}$ for some $x \in \mathbb{H}$, and, by minimality, $T_x\Sigma_{1,t_0}$ and $T_x\Sigma_{2,t_0}$ are parallel. We write Σ_{1,t_0} and Σ_{2,t_0} locally around x as radial graphs as follows. We let L be the normal line to $T_x\Sigma_{1,t_0}$ and $T_x\Sigma_{2,t_0}$. L intersects $\partial_\infty \mathbb{H}$ at some x_0, possibly after an isometry. Then there is some open $U \subseteq \mathbb{H} \cap \partial B_1(x_0)$ with $\frac{x-x_0}{|x-x_0|} \in U$, $\varepsilon > 0$, and $v^1, v^2 : U \times (t_0 - \varepsilon, t_0 + \varepsilon) \to \mathbb{R}$ such that $\{e^{v^i(\mathbf{z},t)}(\mathbf{z} - x_0) \mid \mathbf{z} \in U\} \subseteq \Sigma_{i,t}$ for all $i = 1, 2$, for all $t \in (t_0 - \varepsilon, t_0 + \varepsilon)$. On $U \times (t_0 - \varepsilon, t_0 + \varepsilon)$, MMCF becomes

$$\frac{\partial v^i(\mathbf{z},t)}{\partial t} = y^2 \alpha^{kl} v^i_{kl} - ny\langle \mathbf{e}, \nabla^S v^i \rangle_E - \sigma y w,$$

as in (1.1.3), for all $i = 1, 2$. Now, by construction, $\nabla^S v^i \left(\frac{x}{|x|} \right) = 0$ for all $i = 1, 2$. So, the equation becomes linear and uniformly parabolic near v^i and $\left(\frac{x}{|x|}, t_0 \right)$ for all $i = 1, 2$. Hence, if, say, $v^1(\mathbf{z},t) < v^2(\mathbf{z},t)$ for all $(\mathbf{z},t) \in U \times (t_0 - \varepsilon, t_0)$, then the assumption $v^1 \left(\frac{x}{|x|}, t_0 \right) = v^2 \left(\frac{x}{|x|}, t_0 \right)$ contradicts the maximum principle given in [PW67, Theorem 3.3.7]. □

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