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ERROR ANALYSIS FOR REMOTE REFERENCE MAGNETOTELLURICS

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DISCLAIMER

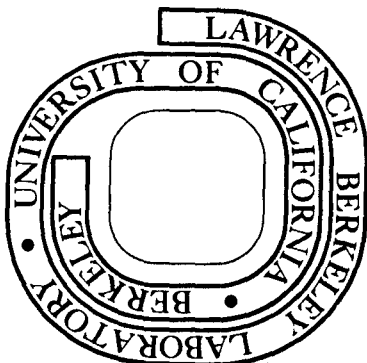ERROR ANALYSIS FOR REMOTE REFERENCE MAGNETOTELLURICS

by

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PREFACE

The magnetotelluric method has been under study at the University of California, Berkeley and the Lawrence Berkeley Laboratory for several years, most recently in connection with applications in geothermal exploration. Based on theoretical calculations and field tests, it was learned by the authors that uncorrelated noise which biases and scatters impedance estimates could be eliminated to a high degree by means of the remote magnetometer magnetotelluric technique (Gamble, et al., 1978; Goubau, et al., 1978). Although the method requires more instrumentation than conventional tensor MT (e.g., a second magnetometer and telemetry equipment are needed), the potential advantages over conventional MT are significant in terms of enabling one to perform faster and more accurate MT surveys. In a practical sense, this should translate into a greater cost-effectiveness for commercial MT surveys utilizing the new technology; it should provide better and less costly survey data for geothermal resource developers, particularly in areas contaminated by man-made electromagnetic noise.

This report, a companion to Gamble, et al., (1978), describes in detail the theory for making a reference magnetometer error analysis. Probable errors are illustrated for apparent resistivity, rotation angle, skewness and phase angles calculated from MT data obtained near Hollister, California.

Abstract

An error analysis is presented for the remote reference magneto-telluric method. The variances in the elements of the remote reference impedance tensor, $\underline{\underline{Z}}^R$, are calculated, and general expressions are found for the variances in an arbitrary function of $\underline{\underline{Z}}^R$. The variances in the apparent resistivities, the phases of the elements of $\underline{\underline{Z}}^R$, and the skewness are derived for a fixed coordinate frame. The variance in the rotation angle of the coordinate frame required to maximize the sum of the squares of the off-diagonal tensor elements is calculated, and expressions are presented for the variance in the apparent resistivities and phase angles in the rotated frames. The distribution of errors and the estimation of confidence limits are discussed. Expressions are given for the signal and noise power spectra of the electric, magnetic, and remote reference fields. Throughout the calculations, emphasis is placed on the physical assumptions made, and possible circumstances under which the assumptions may be violated are discussed. The calculations are illustrated with magnetotelluric data obtained near Hollister, California.

INTRODUCTION

The magnetotelluric (MT) method requires a determination of the electro-magnetic impedance of the earth's surface for normally incident plane waves. Let $\vec{E}_s(t)$ and $\vec{H}_s(t)$ be the horizontal electric and magnetic fields (signals) at the surface of the earth generated by such waves. The impedance, $\underline{\underline{Z}}(\omega)$, at an angular frequency ω is related to the Fourier components of the fields by the relation

$$\vec{E}_s(\omega) = \underline{\underline{Z}}(\omega)\vec{H}_s(\omega). \tag{1}$$

If one approximates $\vec{E}_s(\omega)$ and $\vec{H}_s(\omega)$ by the fields $\vec{E}(\omega)$ and $\vec{H}(\omega)$ obtained from experimental data, one inevitably introduces errors into Equation (1). The relationship between $\vec{E}(\omega)$ and $\vec{H}(\omega)$ is

$$\vec{\eta} = \vec{E} - \underline{\underline{Z}}\vec{H}, \tag{2}$$

where $\vec{\eta}$ is the total error caused by the noises in all of the fields. The errors in measuring $\vec{E}_s$ and $\vec{H}_s$ may be caused by inhomogeneous (non-plane-wave) sources, measurement noise, or errors in signal processing. To estimate $\underline{\underline{Z}}$, a statistical analysis is required.

Until recently, the most commonly used estimator for $\underline{\underline{Z}}$ was $\underline{\underline{Z}}^H$, obtained by minimizing the mean value of $|\eta_x|^2$ and $|\eta_y|^2$ (Vozoff, 1972). The estimator $\underline{\underline{Z}}^H$ for this least-squares analysis can be written concisely as

$$\underline{\underline{Z}}^H = [EH][HH]^{-1}, \tag{3}$$

where [AB] is the spectral density matrix for the fields A and B defined by

$$[AB] = \begin{bmatrix} \overline{A_x B_x^*} & \overline{A_x B_y^*} \\ \overline{A_y B_x^*} & \overline{A_y B_y^*} \end{bmatrix},$$

(4)

and the bars denote averages over Fourier harmonics in a narrow band of frequencies, and, usually, over data obtained at different times. In equation (4) the inverse of [AB] is

$$[AB]^{-1} = \frac{\begin{bmatrix} \overline{A_y B_y^*} & - \overline{A_x B_y^*} \\ - \overline{A_y B_x^*} & \overline{A_x B_x^*} \end{bmatrix}}{\overline{A_x B_x^*}\ \overline{A_y B_y^*} - \overline{A_y B_x^*}\ \overline{A_x B_y^*}}.$$

(5)

The magnitudes of the elements of $\underline{\underline{Z}}^H$ are biased downward by the noise powers in the magnetic channels (Sims, et al., 1971). If a reference field, $\vec{R}$, is measured simultaneously with $\vec{E}$ and $\vec{H}$, an estimate of $\underline{\underline{Z}}$ can be made that is not biased by noise in any field, provided the noise in the reference is uncorrelated with $\vec{E}$ or $\vec{H}$ (Clarke, et al., 1978; Gamble, et al., 1978; Goubau, et al., 1978). The impedance $\underline{\underline{Z}}^R$ estimated using the reference field is

$$\underline{\underline{Z}}^R = [ER][HR]^{-1}.$$

(6)

In this paper we derive expressions for estimating the random errors in $\underline{\underline{Z}}^R$ and any function calculated from $\underline{\underline{Z}}^R$. We begin by finding an exact expression that relates $\underline{\underline{Z}}^R - \underline{\underline{Z}}$ to the spectral

density matrix $[\eta R]$. We then compute variances for $\underline{\underline{Z}}^R$, and for any function of $\underline{\underline{Z}}^R$ in terms of measured cross-and autopowers, using the assumption that the signals and noises are statistically independent and that the noises are stationary. Expressions are given for the variances in the apparent resistivity, phase angles of the impedance tensor elements, and the skewness in a fixed coordinate system. Since it is a standard practice in MT to rotate the coordinate axes to minimize $|Z_{xx} - Z_{yy}|^2$, we give expressions for the variances in the rotation angle and in the apparent resistivities and phase angles in this rotated frame. The distribution of errors in both $\underline{\underline{Z}}^H$ and $\underline{\underline{Z}}^R$ and the calculation of confidence limits are discussed.

We show that the remote reference enables one to calculate the signal and noise power spectra of each field component and coherence between the noises in the two components of each field. This calculation also provides checks on the assumptions of the statistical independence of the noises.

Finally, examples of the error analysis and of the calculation of the signal and noise powers are given for real MT data taken near Hollister, California.

$$\text{CALCULATION OF } \underline{\underline{Z}}^R - \underline{\underline{Z}}$$

To compute $\underline{\underline{Z}}^R - \underline{\underline{Z}}$ it is convenient to introduce the error $\vec{\eta}^P$ predicted when $\underline{\underline{Z}}^R$ is substituted for $\underline{\underline{Z}}$ in equation (2):

$$\vec{\eta}^P = \vec{E} - \underline{\underline{Z}}^R\vec{H}. \tag{7}$$

On eliminating $\vec{E}$ between equations (2) and (7) one finds

$$\vec{\eta} = \vec{\eta}^P + \underline{\underline{\Delta}}\,\vec{H}, \tag{8}$$

where $\underline{\underline{\Delta}} = \underline{\underline{Z}}^R - \underline{\underline{Z}}$. The spectral density matrix $[\eta R]$ obtained by multiplying the components of $\vec{\eta}$ in equation (8) by the components of $\vec{R}^*$ and averaging is given by

$$[\eta R] = [\eta^P R] + \underline{\underline{\Delta}}[HR]. \tag{9}$$

Since $[\eta^P R] = 0$ from the definition of $\underline{\underline{Z}}^R$, [equation (6)], we find

$$\underline{\underline{\Delta}} = [\eta R][HR]^{-1}. \tag{10}$$

From equation (10), the $(ij)^{th}$ element of $\underline{\underline{\Delta}}$ is given by

$$\Delta_{ij} = \overline{\eta_i A_j^*} \bigg/ D \quad (i = x,y, \; j = x,y), \tag{11}$$

where

$$A_x = R_x \overline{H_y^* R_y} - R_y \overline{H_y^* R_x},$$

$$A_y = R_y \overline{H_x^* R_x} - R_x \overline{H_x^* R_y},$$

and $$D = \overline{H_x R_x^*} \; \overline{H_y R_y^*} - \overline{H_x R_y^*} \; \overline{H_y R_x^*}.$$

Equation (11) is exact for any level of noise in any field.

ESTIMATION OF VARIANCES IN $\underline{\underline{Z}}^R$

To compute the expected variances in the elements Z^R_{ij}, we assume that we have an ensemble of estimates for $\underline{\underline{Z}}^R$, and that each value of $\underline{\underline{Z}}^R$ was computed from identical sets of signals and stationary random noises. We wish to find the variance $\mathrm{Var}(Z^R_{ij})$ defined by

$$\text{Var}(Z_{ij}^R) = \langle |\Delta_{ij}|^2 \rangle - |\langle \Delta_{ij} \rangle|^2, \tag{12}$$

where $\langle \; \rangle$ is the ensemble average. This variance is the sum of the variances of the real and imaginary parts of Z_{ij}^R. We assume that $\vec{R}$ is uncorrelated with $\vec{\eta}$ so that $\langle \Delta_{ij} \rangle = 0$. Then, from equation (11),

$$\text{Var}(Z_{ij}^R) = \left\langle \frac{|\overline{\eta_i A_j}|^2}{|D|^2} \right\rangle. \tag{13}$$

If we substitute the measured value of $|D|^2$ in equation (13), $\text{Var}(Z_{ij}^R)$ can be written in expanded form as

$$\text{Var}(Z_{ij}^R) = \frac{1}{N^2 |D|^2} \left\langle \sum_{n=1}^{N} \sum_{m=1}^{N} \eta_{i,m} \eta_{i,n}^* A_{j,m} A_{j,n}^* \right\rangle, \tag{14}$$

where N is the number of independent determinations of each field. This approximation introduces an error into the variance of order $1/N$. For $m \neq n$, $\eta_{i,m}$ and $A_{j,m}$ are statistically independent of $\eta_{i,n}$ and $A_{j,n}$ (assuming that the analog filtering and Fourier transforming are performed appropriately). Thus, equation (14) reduces to

$$\text{Var}(Z_{ij}^R) = \frac{1}{N^2 |D|^2} \sum_{m=1}^{N} \langle |\eta_{i,m}|^2 \, |A_{j,m}|^2 \rangle. \tag{15}$$

If $\eta_{i,m}$ and $A_{j,m}$ are statistically independent, $\langle |\eta_{i,m}|^2 \, |A_{j,m}|^2 \rangle = \langle |\eta_{i,m}|^2 \rangle \langle |A_{j,m}|^2 \rangle$. Since the crosspowers in the $A_{j,m}$ are in fact not independent of the $\eta_{i,m}$, the equality is not exact. However, provided $\vec{R}$ is independent of $\vec{\eta}$, the error introduced is of order $1/N$, and can be neglected for large N. If the noises

are stationary, $\langle |n_{i,m}|^2 \rangle$ is independent of m, and equal to $\overline{|n_i|^2}$. Under these conditions, equation (15) simplifies to

$$\text{Var}(Z_{ij}^R) = \overline{|n_i|^2} \; \overline{|A_j|^2} \Big/ N |D|^2. \tag{16}$$

It is easy to show, using equations (8), (11), and (16), that $\overline{|n_i|^2} = \overline{|n_i^{\,P}|^2} \, [1 + \mathcal{O}\,(1/N)]$. Thus, for large N, one can replace $\overline{|n_i|^2}$ in equation (16) with

$$\overline{|n_i^{\,P}|^2} = \overline{|E_i|^2} - 2\text{Re}\left[Z_{ix}^R \, \overline{H_x E_i^*} + Z_{iy}^R \, \overline{H_y E_i^*} - Z_{ix}^R Z_{iy}^{R*} \, \overline{H_x H_y^*} \right]$$
$$+ |Z_{ix}^R|^2 \, \overline{|H_x|^2} + |Z_{iy}^R|^2 \, \overline{|H_y|^2}, \tag{17}$$

where $\text{Re}(x)$ is the real part of x.

We emphasize that the variance of Z_{ij}^R is correctly given by equation (16) only if: (1) $\vec{R}$ is uncorrelated with the noises in $\vec{E}$ and $\vec{H}$, (2) in $\vec{E}$ and $\vec{H}$ the noises are independent of the signals, and (3) the noises are stationary. The purpose of the remote reference technique is to ensure that the first condition is satisfied. The second assumption is likely to be well satisfied if the noises are generated locally. On the other hand, if the noises arise from inhomogeneous atmospheric sources, both assumptions 1 and 2 may be violated. Assumption 2 could also be violated if the measuring equipment produces errors that are proportional to the signals. The requirement of stationarity is not particularly restrictive. We require only that the ensemble average and measured time average of the noise powers be equal. Stationarity does not require that the noise in short segments of our data be the same for all segments. For example, magnetic fields from passing vehicles might introduce much more noise into some data segments than others,

yet the ensemble and time averages of the noise power will still be equal, provided the times at which vehicles pass by in each experiment in the ensemble are random. It should be noted that we do not need to assume that the signals are stationary. $\underline{\underline{Z}}^R$ and the errors in $\underline{\underline{Z}}^R$ involve only the ratios of average crosspowers, and, since $\vec{E}$, $\vec{H}$, and $\vec{R}$ are causally related, these ratios do not depend on the statistics of the fields.

It is important to note from equation (16) that $\text{Var}(Z_{ij}^R) = 0$ when there is no noise in $\vec{E}$ and $\vec{H}$, regardless of the noise power in $\vec{R}$. Also, when the noise power in $\vec{R}$ is negligible and the crosspowers in A_j and D can be approximated by their noise-free values, it can be shown that $\text{Var}(Z_{ij}^R)$ is independent of the tensor relating $\vec{R}$ and $\vec{H}$. Under these conditions, for given $\overline{|n|^2}$, $\text{Var}(Z_{ij}^R)$ diverges as $\overline{|H_{sx}|^2}\,\overline{|H_{sy}|^2} - \overline{|H_{sx}H_{sy}^*|}^2 \to 0$, that is, as the polarization of the signal, $\vec{H}_s$, increases. When there is noise in the reference, one can easily verify that the contribution of the noise power to $\text{Var}(Z_{ij}^R)$ increases as the polarization of $\vec{R}$ increases. Thus, the electric field from a telluric array in a location with a highly anisotropic apparent resistivity may not be a suitable reference.

It is interesting to realize that, if $\vec{H}$ is noise-free, and if one replaces $\vec{R}$ by $\vec{H}$ in equation (10), equation (16) gives the expected variance* in the least-squares estimate of $\underline{\underline{Z}}^H$. Since $\underline{\Delta}$ is independent of the orientation of $\vec{R}$ relative to $\vec{H}$, the variance in $\underline{\underline{Z}}^R$ is identical with the variance in $\underline{\underline{Z}}^H$ for any noise-free $\vec{R}$ if $\vec{H}$ is also noise-free. Thus, because $\underline{\underline{Z}}^H$ is obtained by minimizing the mean square error in equation (1), $\underline{\underline{Z}}^R$ also minimizes the mean square error. On the other hand, if there is noise in $\vec{H}$,

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* In arriving at equation (16), we neglected terms of order $1/N$ in estimating $\overline{|n_i|^2}$. If the only noise is in the electric field, it is easy to show that the unbiased estimator of $\overline{|n_i|^2}$ is $[N/(N-2)]\,\overline{|n_i^P|^2}$ for all N and for any noise-free $\vec{R}$.

for large N the bias errors in $\underline{\underline{Z}}^H$ are large compared to the random errors

in either $\underline{\underline{Z}}^H$ or $\underline{\underline{Z}}^R$. Therefore, when there is noise in $\vec{H}$, $\underline{\underline{Z}}^H$ is not a good

estimate of $\underline{\underline{Z}}$, and the question of the relative random errors in $\underline{\underline{Z}}^R$ and

$\underline{\underline{Z}}^H$ becomes academic.

There are two earlier attempts in the literature to calculate the

expected errors in   estimates of the individual elements of the impedance

tensor. Bentley (1973) attempted such a calculation for $\underline{\underline{Z}}^H$. His calculation

assumes that there is no noise in the measured fields,   that the signals

have stationary power spectra, and that the only source of error is the sampling

error. In fact, only the ratios of power spectra enter into the estimate of $\underline{\underline{Z}}$,

and these ratios are not affected by sampling errors. Thus, Bentley should have

obtained a null estimate for the errors, but did not because he neglected the

correlations between the errors in the estimates of the power spectra.

Reddy et al. (1976) have estimated the random errors in the individual elements

of $\underline{\underline{Z}}^H$ using an expression derived from the error in a combination of the

elements (Jenkins and Watts, 1968; Bendat and Piersol, 1971; and Goodman, 1965)

via a very rough approximation. The approximation is necessarily very rough

because the error distribution of the combination does not contain enough

information to specify the individual errors, and, in addition, the expression

for the joint errors is valid only for a noise-free magnetic field. Thus,

neither approach appears to be appropriate for real MT data.

ESTIMATION OF VARIANCES IN FUNCTIONS OF $\underline{\underline{Z}}^R$

However large the noise, the errors Δ_{ij} in equation (11) can be made

arbitrarily small by making N sufficiently large. For small errors, any

function ξ of $\underline{\underline{Z}}^R$ can be expanded to first order in Δ_{ij} and Δ_{ij}^*. In these expansions, it is convenient to shorten the notation as follows:

$$\Delta_{ij} \ (i=x,y, \ j=x,y) \leftrightarrow \Delta_k \ (k=1,2,3,4), \tag{18}$$

where $1 = xx$, $2 = xy$, $3 = yx$, $4 = yy$. We also drop the supercript R from $\underline{\underline{Z}}^R$. In terms of Δ_k and Δ_k^*, the error, $\delta\xi$, in ξ is given by

$$\delta\xi = \sum_{k=1}^{4} \left(\frac{d\xi}{dZ_k} \Delta_k + \frac{d\xi}{dZ_k^*} \Delta_k^* \right). \tag{19}$$

Since $\langle\Delta_k\rangle = 0$, the variance in ξ is $\mathrm{Var}(\xi) = \langle|\delta\xi|^2\rangle$. If $\vec{\eta}$ and $\vec{R}$ are uncorrelated, $\langle\Delta_k\Delta_\ell\rangle = 0$ for all k and ℓ, since the signals and noises are complex numbers of random phase. Thus $\mathrm{Var}(\xi)$ has the form

$$\mathrm{Var}(\xi) = \sum_{k=1}^{4} \sum_{\ell=1}^{4} \left[\frac{d\xi}{dZ_k} \frac{d\xi^*}{dZ_\ell^*} \langle\Delta_k\Delta_\ell^*\rangle \right.$$
$$\left. + \frac{d\xi}{dZ_k^*} \frac{d\xi^*}{dZ_\ell} \langle\Delta_k^*\Delta_\ell\rangle \right]. \tag{20}$$

which simplifies to

$$\mathrm{Var}(\xi) = \sum_{k=1}^{4} G_{kk} \langle|\Delta_k|^2\rangle + 2\mathrm{Re}\left[\sum_{k=1}^{3} \sum_{\ell=k+1}^{4} G_{k\ell} \langle\Delta_k\Delta_\ell^*\rangle \right]. \tag{21}$$

Here,

$$G_{k\ell} = G_{\ell k}^* = \frac{d\xi}{dZ_k} \frac{d\xi^*}{dZ_\ell^*} + \frac{d\xi}{dZ_k^*} \frac{d\xi^*}{dZ_\ell} , \tag{22}$$

If ξ is real, $G_{k\ell} = 2\mathrm{Re}\left[\frac{d\xi}{dZ_k} \frac{d\xi^*}{dZ_\ell^*} \right]$. We evaluate the ensemble average $\langle\Delta_k\Delta_\ell^*\rangle$, as discussed in the previous section,

using equation (11). Using our original notation we replace $\langle \Delta_k \Delta_\ell^* \rangle$ with $\langle \Delta_{ij} \Delta_{nm}^* \rangle$, and find

$$\langle \Delta_{ij} \Delta_{nm}^* \rangle = \frac{\overline{\eta_i^P \eta_n^{P*}} \; \overline{A_j A_m^*}}{ND} \qquad (23)$$

where we have approximated $\langle \eta_i^* \eta_n^* \rangle$ by $\overline{\eta_i^P \eta_n^P}^*$. In equation (23) $\overline{\eta_i^P \eta_n^{P*}}$ and $\overline{A_j A_m^*}$ can be expressed in terms of measured crosspowers and autopowers as follows:

$$\overline{\eta_i^P \eta_n^{P*}} = \overline{E_i E_n^*} - Z_{ix} \overline{H_x E_n^*} - Z_{iy} \overline{H_y E_n^*} - Z_{nx}^* \overline{H_x^* E_i} - Z_{ny}^* \overline{H_y^* E_i} + Z_{ix} Z_{nx}^* \overline{|H_x|^2}$$

$$+ Z_{iy} Z_{nx}^* \overline{H_y H_x^*} + Z_{ix} Z_{ny}^* \overline{H_x H_y^*} + Z_{iy} Z_{ny}^* \overline{|H_y|^2}, \qquad (24)$$

and

$$\overline{A_j A_m^*} = \overline{R_j R_m^*} \; \overline{H_k R_k^*} \; \overline{H_\ell R_\ell^*} + \overline{R_k R_\ell^*} \; \overline{H_k R_j^*} \; \overline{H_\ell R_m^*} - \overline{R_j R_\ell^*} \; \overline{H_k R_k^*} \; \overline{H_\ell R_m^*}$$

$$- \overline{R_k R_m^*} \; \overline{H_k R_j^*} \; \overline{H_\ell R_\ell^*}, \qquad (25)$$

where $k = x,y$, $\ell = x,y$, and $k \neq j$ and $\ell \neq m$. It is apparent that $\mathrm{Var}(\xi)$ will, in general, depend on all 15 crosspowers and 6 autopowers of the components of the fields.

To illustrate the use of equation (21) we compute the variance in $\mathrm{Re}(Z_\mu)$, where $\mu = 1,2,3$, or 4. Substituting $\xi = \xi^* = (Z_\mu + Z_\mu^*)/2$ and $d\xi/dZ_k = d\xi^*/dZ_k^* = 1/2 \, \delta_{\mu k}$ (δ is the Kronecker delta), one finds $\mathrm{Var}[\mathrm{Re}(Z_\mu)] = 1/2 \langle |\Delta_\mu|^2 \rangle = \tfrac{1}{2} \mathrm{Var}(Z_\mu)$. Since $\mathrm{Var}(Z_\mu) = \mathrm{Var}[\mathrm{Re}(Z_\mu)] + \mathrm{Var}[\mathrm{Im}(Z_\mu)]$ [$\mathrm{Im}(x)$ is the imaginary part of x], this example proves that $\mathrm{Var}[\mathrm{Re}(Z_\mu)] = \mathrm{Var}[\mathrm{Im}(Z_\mu)]$.

The elements of the apparent resistivity matrix $\underline{\underline{\rho}}$ associated with $\underline{\underline{Z}}$ are defined by $\underline{\rho}_k = 0.2T |\underline{Z}_k|^2$, where T is the period in seconds and $\underline{\underline{Z}}$ has dimensions of mV/(kmγ). If we choose $\xi = \xi^* = Z_\mu Z_\mu^*$ in equation (21) then $d\xi/dZ_k$

$= Z_\mu^* \delta_{\mu k}$, $G_{k\ell} = 2 |Z_\mu|^2 \delta_{\mu k} \delta_{\mu \ell}$, and $\text{Var}(\xi) = 2 |Z_\mu|^2 <|\Delta_\mu|^2>$. Thus the variance of the element ρ_μ is given by

$$\text{Var}(\rho_\mu) = (0.2T)^2 \text{Var}(\xi) = 0.4T\rho_\mu <|\Delta_\mu|^2>. \tag{26}$$

The phase, ϕ_μ of Z_μ^* is defined by

$$\tan \phi_\mu = (Z_\mu - Z_\mu^*)/i(Z_\mu + Z_\mu^*), \tag{27}$$

where $i = \sqrt{-1}$. If $\xi = \tan\phi_\mu$, $\text{Var}(\phi_\mu) = \cos^4\phi_\mu \text{Var}(\xi)$. In equation (22)

$$d\xi/dZ_k = \delta_{\mu k} 2Z_\mu^*/i(Z_\mu + Z_\mu^*)^2,$$

and

$$G_{k\ell} = 8 |Z_\mu|^2 \delta_{\mu k} \delta_{\mu \ell}/|Z_\mu + Z_\mu^*|^4.$$

Thus

$$\text{Var}(\phi_\mu) = 8\cos^4\phi_\mu <|\Delta_\mu|^2> |Z_\mu|^2 \Big/ |Z_\mu + Z_\mu^*|^4. \tag{28}$$

To find the variance in the skewness, S, we define $\xi = S^2$ $= |Z_{xx} + Z_{yy}|^2/ |Z_{xy} - Z_{yx}|^2$. One obtains

$$\text{Var}(S) = \text{Var}(\xi)/4S , \tag{29}$$

where $\text{Var}(\xi)$ is given by equation (21) with the following values of $G_{k\ell}$:

$$G_{11} = G_{14} = G_{44} = 2S^2/|Z_{xy} - Z_{yx}|^2, \tag{30}$$

$$G_{22} = G_{33} = -G_{23} = S^2 G_{11}, \tag{31}$$

and

$$G_{12} = G_{24} = -G_{13} = -G_{34} = -G_{11} \frac{Re\left[(Z_{xx} + Z_{yy})\ (Z_{xy}^* - Z_{yx}^*)\right]}{|Z_{xy} - Z_{yx}|^2} \ . \tag{32}$$

The rotation angle, Θ, that minimizes $|Z_{xx} - Z_{yy}|^2$ also maximizes $|Z_{xy}|^2 + |Z_{yx}|^2$ (Sims and Bostick, 1969). Θ satisfies the equation

$$\tan 4\Theta = \frac{2Re\left[(Z_{yy} - Z_{xx})(Z_{xy}^* + Z_{yx}^*)\right]}{|Z_{xy} + Z_{yx}|^2 - |Z_{yy} - Z_{xx}|^2} \ . \tag{33}$$

For any integer, m, $\Theta \pm m\pi/4$ also satisfies equation (33), but the solutions with odd m maximize $|Z_{xx} - Z_{yy}|^2$. If we choose $\xi = \xi^* = \tan 4\Theta$, then $Var(\Theta) = \cos^4 4\Theta\ Var(\xi)/16$. $Var(\xi)$ is given by equation (21) with

$$G_{11} = G_{44} = -G_{14} = 2|\alpha|^2\ |Z_{xy} + Z_{yx}|^2,$$

$$G_{22} = G_{33} = G_{23} = 2|\alpha|^2\ |Z_{xx} - Z_{yy}|^2, \tag{34}$$

and

$$G_{12} = G_{13} = -G_{24} = -G_{34} = 2|\alpha|^2 Re\left[(Z_{xy}^* + Z_{yx}^*)(Z_{yy} - Z_{xx})\right], \tag{35}$$

where

$$\alpha = \frac{(Z_{xy} + Z_{yx})^2 + (Z_{yy} - Z_{xx})^2}{\left[|Z_{xy} + Z_{yx}|^2 - |Z_{yy} - Z_{xx}|^2\right]^2} \ . \tag{36}$$

The expressions for the variances in the apparent resistivities and the phases of the impedance tensor in this rotated coordinate system are given in the Appendix.

CONFIDENCE LIMITS

Although least squares linear regression is not the best method of determining $\underline{\underline{Z}}$, the least squares principle is appropriate for the comparison of different estimates of $\underline{\underline{Z}}$. For example, the best model of the ground in a statistical sense minimizes the mean square of the magnitudes of the differences between the modeled and measured values of $\underline{\underline{Z}}$, weighted in inverse proportion to the variances. However, to determine the statistical significance of this discrepancy, one requires the distribution of the errors of the estimates, not just the variances.

In the expression for the error, Δ_{ij} [equation (11)], D can be approximated by its noise-free value for large N. In this approximation, Δ_{ij} is just the sum of N complex random errors (one for each k, $1 \leqslant k \leqslant N$), and, by the central limit theorem (Jenkins and Watts, 1968), its real and imaginary parts are normally distributed. Since the error Δ_{ij} is of random phase Thus, $<\mathrm{Re}(\Delta_{ij})\mathrm{Im}(\Delta_{ij})> = 0./\mathrm{Re}(\Delta_{ij})$ and $\mathrm{Im}(\Delta_{ij})$ are also statistically independent. The sums of the squares of n independent normally distributed random variables with unity variance and zero mean has a X_n^2 distribution. Thus, if $\vec{H}$ and $\vec{R}$ are noise-free, $|\Delta_{ij}|^2/\mathrm{Var}[\mathrm{Re}(\Delta_{ij})] = 2|\Delta_{ij}|^2 \, N|D|^2/<|n_i|^2>\overline{|A_j|^2}$ has a X_2^2 distribution. In this expression, the unknown quantity $<|n_i|^2>$ is best approximated by $\overline{|n_i^P|^2}$. The errors introduced by this approximation must be included to obtain an unbiased estimate of the confidence limits. Since $(2N-4)|n_i^P|^2/<|n_i|^2>$ has a X_{2N-4}^2 distribution (Jenkins and Watts, 1968) the quantity $|\Delta_{ij}|^2 \, N|D|^2/\overline{|n_i^P|^2} \; \overline{|A_j|^2}$ has a Fisher F distribution $F_{2,2N-4}$. For large N, the modification introduced by

$\overline{|n_i^P|^2}$ is small. For example, for N>25, the confidence limits for the

$F_{2,2N-4}$ distribution are less than 6%   larger than those for the χ_2^2

distribution up to the 95% confidence level. If the signal-to-noise ratios of

$\vec{R}$ and $\vec{H}$ are much greater than the signal-to-noise ratio of $\vec{E}$, this small correction

to the confidence limits may be significant. If the noise is not predominantly

in $\vec{E}$, the other corrections of order 1/N that we have neglected will cause mod-

ifications of the distribution comparable with the difference between the χ^2

and F   distributions. These modifications cannot be described in terms of

elementary distribution functions. Thus, for most applications, the χ^2

distribution should be adequate, and as accurate as can be obtained without

extraordinary effort.

Errors estimated from the first-order Taylor expansion, equation (19),

for example errors in the apparent resistivity, are linear functions of the

errors in the real and imaginary parts of $\underline{\underline{Z}}^R$. Within the limits of accuracy

of the Taylor expansion, these errors are also normally distributed. The

confidence limits of these quantities are again modified by the estimation

of $<|n_i|^2>$ by $\overline{|n_i^P|^2}$ so that the proper distribution is that of the ratio of a

normally distributed to a χ^2 - distributed variable, or a Student t distribution.

However, the corrections to a normal distribution will be significant only

when the confidence intervals are so small that the Taylor expansion introduces

a negligible error, and, as before, the noise is predominantly in $\vec{E}$. For most

purposes, a normal distribution should be entirely adequate.

As an example, consider two independent sets, a and b, of M estimates of

$Z_{ij}(\omega)$, $Z_{ij}^a(\omega_k)$ and $Z_{ij}^b(\omega_k)$, where $1 \leq k \leq M$. We calculate the probability that the

disagreement between the sets arose from random errors alone assuming that the

errors in set b are negligible compared to the errors in set a. Such a calculation

would be required if one wanted to determine the significance of the difference between a model of the ground (set b) and a sounding (set a) or if one considered rejecting a small subset of the data (set a) because of its disagreement with the rest of the data (set b). If the quantity

$$\delta_{ij}(\omega_k) \equiv 2|\Delta_{ij}(\omega_k)|^2 N|D|^2 / \left(\overline{|n_i^P|^2} \ \overline{|A_j|^2} \right) \tag{37}$$

has a χ_2^2 distribution then the total discrepancy, $\sum_{k=1}^{M} \delta_{ij}^a(\omega_k)$, has a χ_{2M}^2 distribution. Neglecting the errors in $Z_{ij}^b(\omega_k)$, we find

$$\sum_{k=1}^{M} \delta_{ij}^a(\omega_k) \equiv \zeta = \sum_{k=1}^{M} \frac{2|Z_{ij}^a(\omega_k) - Z_{ij}^b(\omega_k)|^2 N^a |D^a|^2}{\overline{|n_i^{Pa}|^2} \ \overline{|A_j^a|^2}} \tag{38}$$

Thus the probability that $\zeta > \alpha$ through random errors alone is $1 - \chi_{2M}^2(\alpha)$.

DETERMINATION OF SIGNAL AND NOISE POWERS

The random errors in $\underline{\underline{Z}}^R$ depend only on the combined noise, $\vec{n}$, rather than on individual noises in $\vec{E}$ and $\vec{H}$. Nevertheless, the determination of the noises in the individual fields is obviously of practical interest. With a remote reference the signal and noise power spectral densities can be evaluated as follows.

The value of $\underline{\underline{Z}}^R$ obtained from equation (6) and the measured magnetic field, $\vec{H}$, predict an electric field $\vec{E}^P$, where

$$\vec{E}^P = \underline{\underline{Z}}^R \vec{H}. \tag{39}$$

$\vec{E}^P$ contains contributions from the signal $\vec{H}_s$ and the noise $\vec{H}_n = \vec{H} - \vec{H}_s$. If the noises are uncorrelated with each other and with the signals, the spectral density matrix

$$[E^P E] = \underline{\underline{Z}}^R [HE] = [ER][HR]^{-1}[HE] \tag{40}$$

has the expectation value of the spectral density matrix $[E_s E_s]$, where

$$[E_s E_s] = \begin{bmatrix} \overline{|E_{sx}|^2} & \overline{E_{sx} E_{sy}^*} \\ \\ \overline{E_{sy} E_{sx}^*} & \overline{|E_{sx}|^2} \end{bmatrix} .$$

The matrix $[E_s E_s]$ is Hermitian: The diagonal elements are real, and the off-diagonal elements are the complex conjugates of each other. On the other hand, $[E_p E]$ is, in general, not Hermitian because of the noises $\vec{E}_n, \vec{H}_n$, and $\vec{R}_n$. From Figure 1, it can be seen that if the phases of the noises are unknown, the best estimate for $[E_s E_s]$ is the Hermitian part of $[E^P E]$, $[E_s E_s]^P$ given by

$$[E_s E_s]^P = \tfrac{1}{2}\{[E^P E] + [E^P E]^\dagger\} = ([E^P E] + [_{EE}^P])/2, \tag{41}$$

where $\dagger$ denotes the Hermitian conjugate.

The spectral density matrixes for the other fields can be estimated similarly:

and

$$[H_s H_s]^P = ([H^P H] + [HH^P])/2, \tag{42}$$

where

$$[R_s R_s]^P = ([R^P R] + [RR^P])/2, \tag{43}$$

and

$$[H^P H] = [HR][ER^{-1}][EH], \tag{44}$$

$$[R^P R] = [RE][HE]^{-1}[HR]. \tag{45}$$

One can calculate the spectral density matrices for the noises by subtracting the estimated signal density matrices from the measured spectral density matrices, for example

$$[E_n E_n] = [EE] - [E_s E_s]^P. \tag{46}$$

The noise matrixes contain the crosspowers $\overline{E_{nx} E_{ny}^*}$, $\overline{H_{nx} H_{ny}^*}$, and $\overline{R_{nx} R_{ny}^*}$. Thus, one can determine whether there are significant correlations between the noises in the two components of each field. Such correlations may be indicative of measurement errors, and could be generated, for example, by noise from a common electrode, or by a moving magnetic object.

The remote reference method requires the measurement of the three fields $\vec{E}$, $\vec{H}$, and $\vec{R}$, each with two components. Correlations between the noises in the two components of each field do not bias the estimates of $\underline{\underline{Z}}^R$, the errors in $\underline{\underline{Z}}^R$, or the signal and noise power spectral density matrices. However, any correlation between a measured field and the noise in another field will bias the estimates of the signal and noise power spectra. Such correlations would usually cause a significant non-Hermitian part in the matrices $[E^P E]$, $[H^P H]$, and $[R^P R]$.

$\underline{\underline{Z}}^R$ will be biased only by correlations between $\vec{n}$ and $\vec{R}$. Thus, under most circumstances, the requirement that $[E^P E]$, $[H^P H]$, and $[R^P R]$ be Hermitian provides a sufficient but not necessary check on the correlations that would bias $\underline{\underline{Z}}^R$. However, if the ionospheric signal is from a fixed inhomogeneous source, these matrices would/be Hermitian, but $\underline{\underline{Z}}^R$ would be biased.
still

APPLICATION TO MAGNETOTELLURIC DATA

We illustrate the calculations of signal and noise powers and the error
analysis with magnetotelluric data obtained at the Upper La Gloria site in
Bear Valley, near Hollister, California (Clarke et al., 1978, Gamble et al.,
1978, Goubau et al., 1978). The signal powers, equations (41)/and noise powers,
to (43),
equation (46), for the components of $\vec{E}$, $\vec{H}$, and $\vec{R}$ are shown in figures 2, 3, and
4, respectively. The non-Hermitian parts of the predicted autopower spectral
density matrices were very small. For example, the imaginary parts of the
predicted autopowers (such as $\overline{E_x^P E_x^*}$) were always less than 10% of the real
parts, and averaged about 1%. For periods shorter than 3 seconds, where we
had the most data, they were always less than 2%. Thus, even if the noise
coherencies were statistically significant, they were too small to have any
practical importance in these calculations.

Because of random errors in the calculated signal spectral density matrices,
it is possible for the calculated noise powers to be negative. This behavior was
never observed in the data presented, even though the signal-to-noise ratios
varied from 100:1 for E_y at 0.1 second period and 200:1 for H_y at 85 second
period to 1:7 for R_x at 9 second period. The signal power spectra for these
data are particularly steep; for example, around 15 second period they increase
roughly as the 8th power of the period. Nonetheless, the calculated noise
spectra are comparatively smooth, indicating that the random errors are small.

In Figure 5 we plot the rotated apparent resistivities and their
associated probable errors as functions of period. In Figure 6 we plot θ_{nx}
(the angle between magnetic north and the rotated x-axis) and the skewness

with their probable errors. Similarly, the phases of $Z_{xy}(\theta)$ and $Z_{yx}(\theta)$ are plotted in Figure 7.

The results are reproducible within the calculated random error. Where bands overlap the values usually agree within the probable errors. One can draw very smooth curves through at least 50 percent of the probable error ranges. For periods shorter than 3 seconds we made a crude estimate of the expected standard deviations of the resistivities by comparing the results calculated from subsets of the data gathered at different times (Gamble, et al., 1978). For the 25 apparent resistivities with the smallest calculated probable error the rms expected standard deviation of the means from that estimate was 88% of the rms of the standard deviations calculated in this paper.

CONCLUDING REMARKS

The remote reference method enables one to obtain unbiased estimates of the impedance tensor, and to calculate accurate confidence limits for each element. The absence of bias enables one to perform fast, accurate MT surveys. For example, the smallest random error in apparent resistivity reported in this paper, ±0.4% for ρ_{xy} at a period of 0.062 second, was obtained from only 30 minutes of data. Furthermore, the method can be used to determine the impedance tensor accurately even in areas contaminated by high levels of noise.

The separation of signal from noise using a remote reference may make possible new types of geophysical measurements. For example, the separation of ionospherically generated magnetotelluric signals from local natural sources of electric and magnetic field fluctuations may enable one to study the electrical and magnetic effects of seismic waves or tidal stress. One could also subtract

the magnetotelluric signals from the measurements of active electric
or electromagnetic surveys, and thus extend the accuracy and efficiency
of the survey and the effective range of the transmitter (Morrison, 1978).

ACKNOWLEDGMENTS

APPENDIX

The equations obtained in the body of this paper can be used to calculate variances in any coordinate system provided one first rotates the measured spectral density matrixes to the desired orientation. However, if the rotation angle of the coordinate system is itself determined from the data, additional errors will be introduced in the calculated quantities because of the uncertainty in the rotation angle. This appendix contains expressions for the variances of the apparent resistivities and of the phases of the elements of the impedance tensor in the coordinate system rotated by the angle θ obtained from equation (33).

A rotation of a right-handed coordinate system about the $\hat{z}$ axis by an angle θ will change the matrix representation of any tensor from $\underset{\approx}{T}$ to

$$\underset{\approx}{T}' = \underset{\approx}{Q} \, \underset{\approx}{T} \, \underset{\approx}{Q}^{-1}, \tag{A-1}$$

where

$$\underset{\approx}{Q} = (\underset{\approx}{Q}^{-1})^{\dagger} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}. \tag{A-2}$$

Rotated Apparent Resistivities

Define the rotated apparent resistivity matrix (which is not a tensor) by $\rho'_{\mu} = 0.2T|Z'_{\mu}|^2$, where $\underset{\approx}{Z}'$ is the impedance tensor in the rotated coordinate system. Then, if $\xi_{\mu} = |Z'_{\mu}|^2$, $\mathrm{Var}(\rho'_{\mu}) = (0.2T)^2 \, \mathrm{Var}(\xi_{\mu})$. $\mathrm{Var}(\xi_{\mu})$ is given by equation (21) with

$$G_{k\ell} = 2\mathrm{Re}\left[\frac{d\xi_{\mu}}{dZ_k}\left(\frac{d\xi_{\mu}}{dZ_\ell}\right)^*\right], \tag{A-3}$$

and

$$\frac{d\xi_{\mu}}{dZ_k} = \frac{\partial\xi_{\mu}}{\partial Z_k} + \frac{\partial\xi_{\mu}}{\partial\theta}\frac{\partial\theta}{\partial Z_k}. \tag{A-4}$$

Using equations (A-1) and (A-4) we find

$$\frac{d\xi_\mu}{dZ_k} = Z_\mu'^* U_\mu(k) + 2\text{Re}[Z_\mu'^* V_\mu] \frac{\partial\theta}{\partial Z_k},$$ (A-5)

where

$$\underset{\approx}{U}(k) = \underset{\approx}{Q}\left(\frac{\partial \underset{\approx}{Z}}{\partial Z_k}\right)\underset{\approx}{Q}^{-1},$$ (A-6)

and

$$V = \frac{\partial \underset{\approx}{O}}{\partial\theta} \underset{\approx}{Z} \underset{\approx}{Q}^{-1} + \underset{\approx}{Q} \underset{\approx}{Z} \frac{\partial \underset{\approx}{O}^{-1}}{\partial\theta} .$$ (A-7)

The elements of equation (A-6) are

$$\underset{\approx}{U}(xx) = \begin{bmatrix} \cos^2\theta & -\sin\theta\cos\theta \\ -\sin\theta\cos\theta & \sin^2\theta \end{bmatrix},$$ (A-8)

$$\underset{\approx}{U}(xy) = \begin{bmatrix} \sin\theta\cos\theta & \cos^2\theta \\ -\sin^2\theta & -\sin\theta\cos\theta \end{bmatrix},$$ (A-9)

$$\underset{\approx}{U}(yx) = \begin{bmatrix} \sin\theta\cos\theta & -\sin^2\theta \\ \cos^2\theta & -\sin\theta\cos\theta \end{bmatrix},$$ (A-10)

and $$\underset{\approx}{U}(yy) = \begin{bmatrix} \sin^2\theta & \sin\theta\cos\theta \\ \sin\theta\cos\theta & \cos^2\theta \end{bmatrix}.$$ (A-11)

$\underset{\approx}{V}$ can be written as

$$\underset{\approx}{V} = \begin{bmatrix} -\sin2\theta & \cos2\theta \\ -\cos2\theta & -\sin2\theta \end{bmatrix} \begin{bmatrix} Z_{xx} - Z_{yy} & Z_{xy} + Z_{yx} \\ Z_{xy} + Z_{yx} & Z_{yy} - Z_{xx} \end{bmatrix}.$$ (A-12)

From equation (33),

$$\partial\theta/\partial Z_{xx} = -\partial\theta/\partial Z_{yy} = -\alpha^*\cos^24\theta(Z_{xy} + Z_{yx})/4,$$ (A-13)

and

$$\partial\theta/Z_{yx} = \partial\theta/\partial Z_{xy} = \alpha^*\cos^24\theta(Z_{xx} - Z_{yy})/4,$$ (A-14)

where α is defined in equation (36).

Phase of Z'_μ

Let ϕ'_μ be the phase of Z'_μ, and define

$$\xi_\mu = \tan\phi'_\mu = \frac{Z'_\mu - Z'^*_\mu}{i(Z'_\mu + Z'^*_\mu)} \quad . \tag{A-15}$$

Then, $\text{Var}(\phi'_\mu) = \cos^4\phi'_\mu \, \text{Var}(\xi_\mu)$, and $\text{Var}(\xi_\mu)$ is given by equation (21). From equations (A-15) and (A-4), we find

$$\frac{d\xi_\mu}{dZ_k} = -2i\left(\frac{Z'^*_\mu \dfrac{dZ'_\mu}{dZ_k} - Z'_\mu \dfrac{dZ'^*_\mu}{dZ_k}}{(Z'_\mu + Z'^*_\mu)^2}\right) \tag{A-16}$$

$$= -2i\left(\frac{Z'^*_\mu U_\mu(k) + 2i\text{Im}[Z'^*_\mu V_\mu]\partial\theta/\partial Z_k}{(Z'_\mu + Z'^*_\mu)^2}\right),$$

where $\underline{\underline{U}}(k)$, $\underline{\underline{V}}$, and $\partial\theta/\partial Z_k$ have been defined in equations (A-8) through (A-14).

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CAPTIONS

Fig. 1.   Graph in the complex plane of an autopower estimate, $\overline{E_x^P E_x^*}$, which

is the sum of the signal autopower $\overline{E_{sx} E_{sx}^*}$ and an error r of unknown

phase.   XBL 783-4695

Fig. 2.   Electric field signal and noise power spectral densities vs. period.
XBL 783-4697.

Fig. 3.   Magnetic field signal and noise power spectral densities vs. period.
XBL 783-4696.

Fig. 4.   Remote magnetic reference signal and noise power spectral densities

vs. period.   XBL 783-4698.

Fig. 5.   Apparent resistivities in the rotated coordinate system and their

probable error vs. period.   XBL 783-4692.

Fig. 6.   Angle between magnetic north and rotated x-axis and skewness with

their probable errors vs. period.   XBL 783-4693

Fig. 7.   Phases of Z_{xy} and Z_{yx} in rotated coordinate system and their probable

errors vs. period.   XBL 783-4694
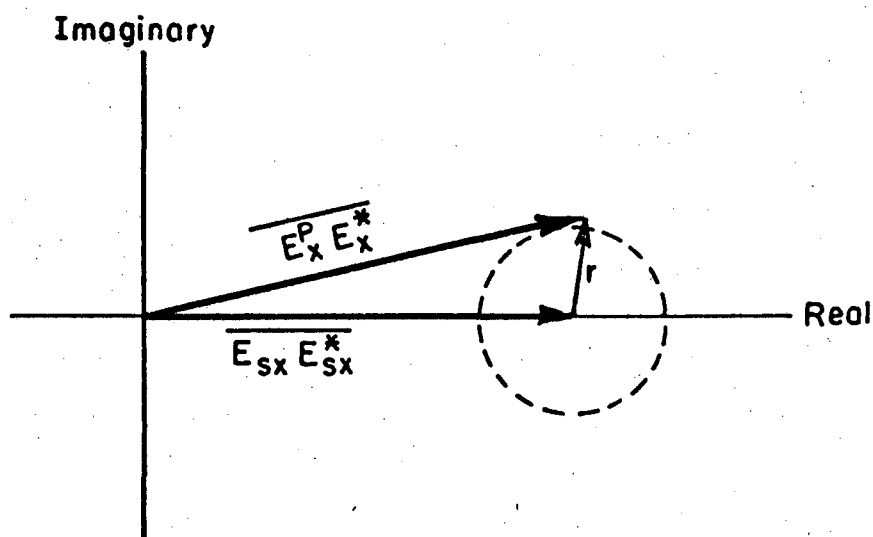
Fig. 1
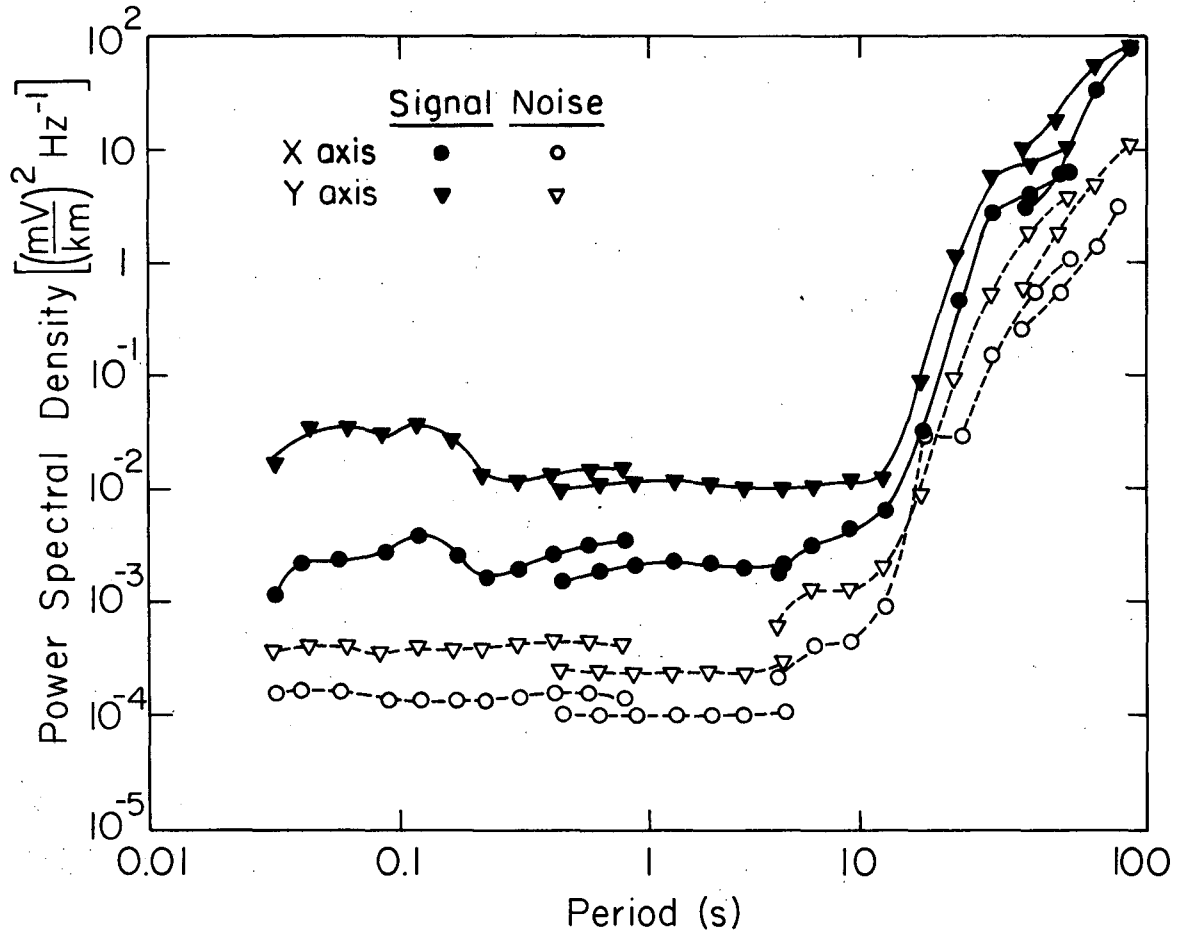
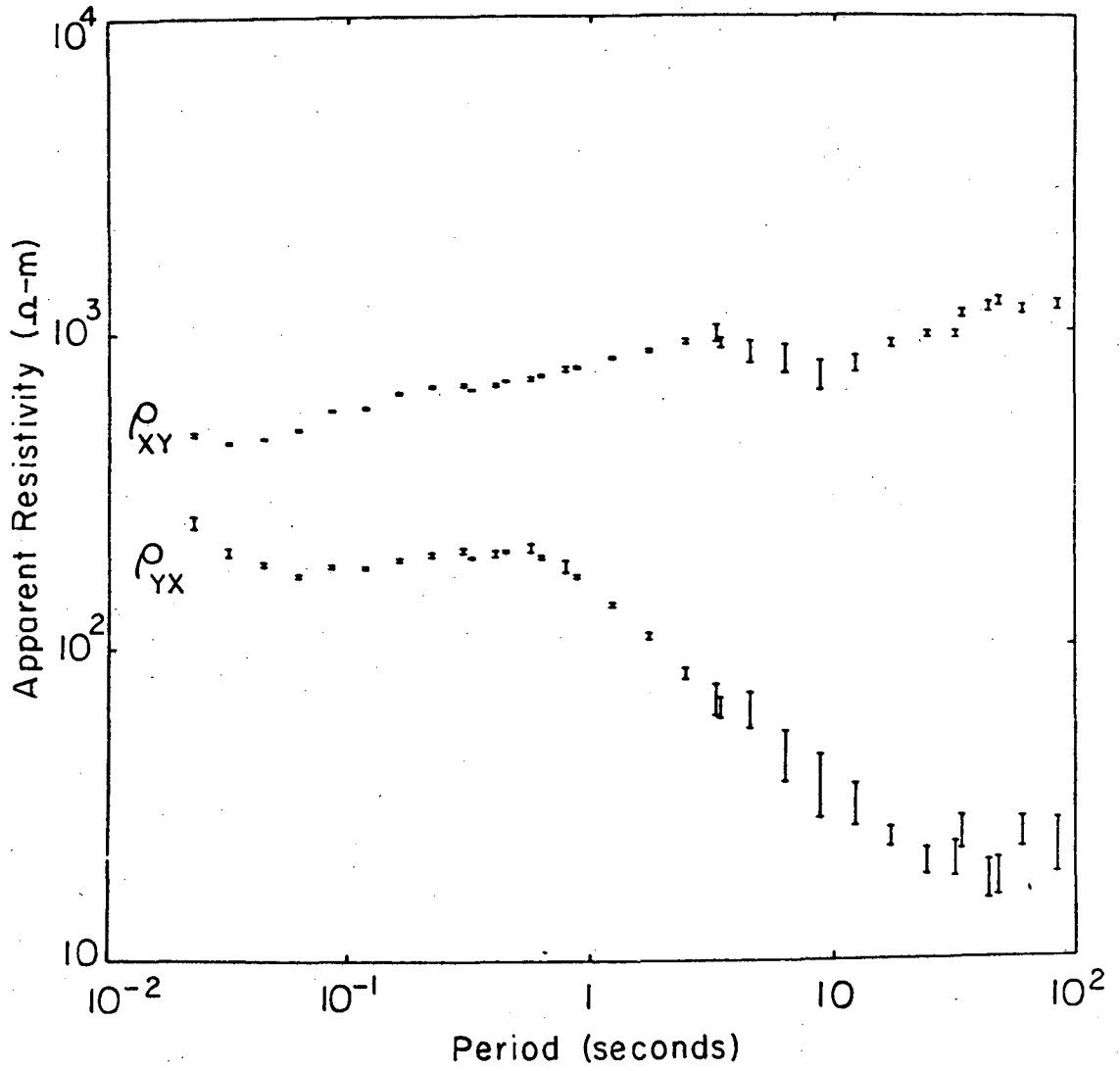XBL 783-4697

Fig. 2
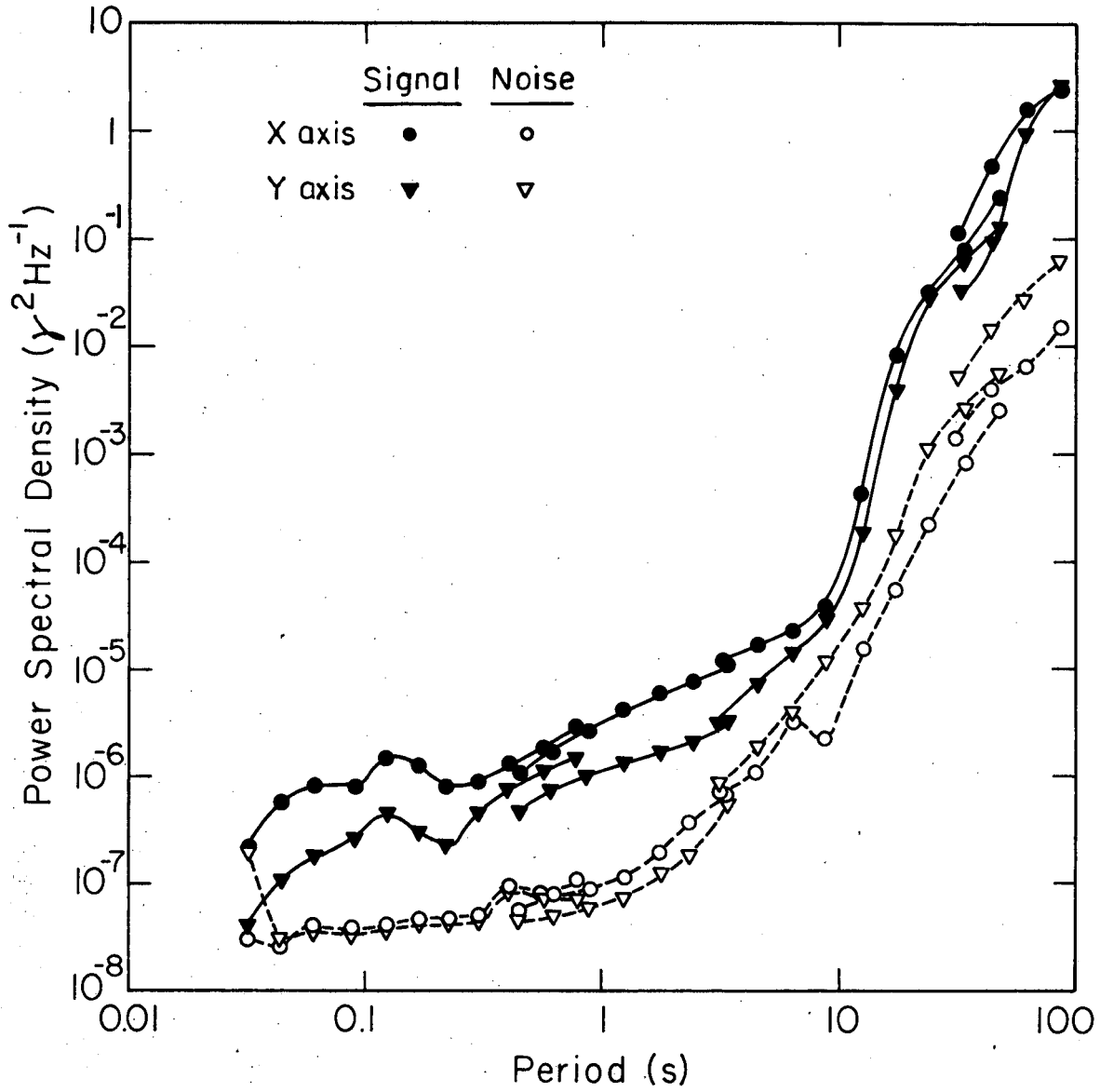
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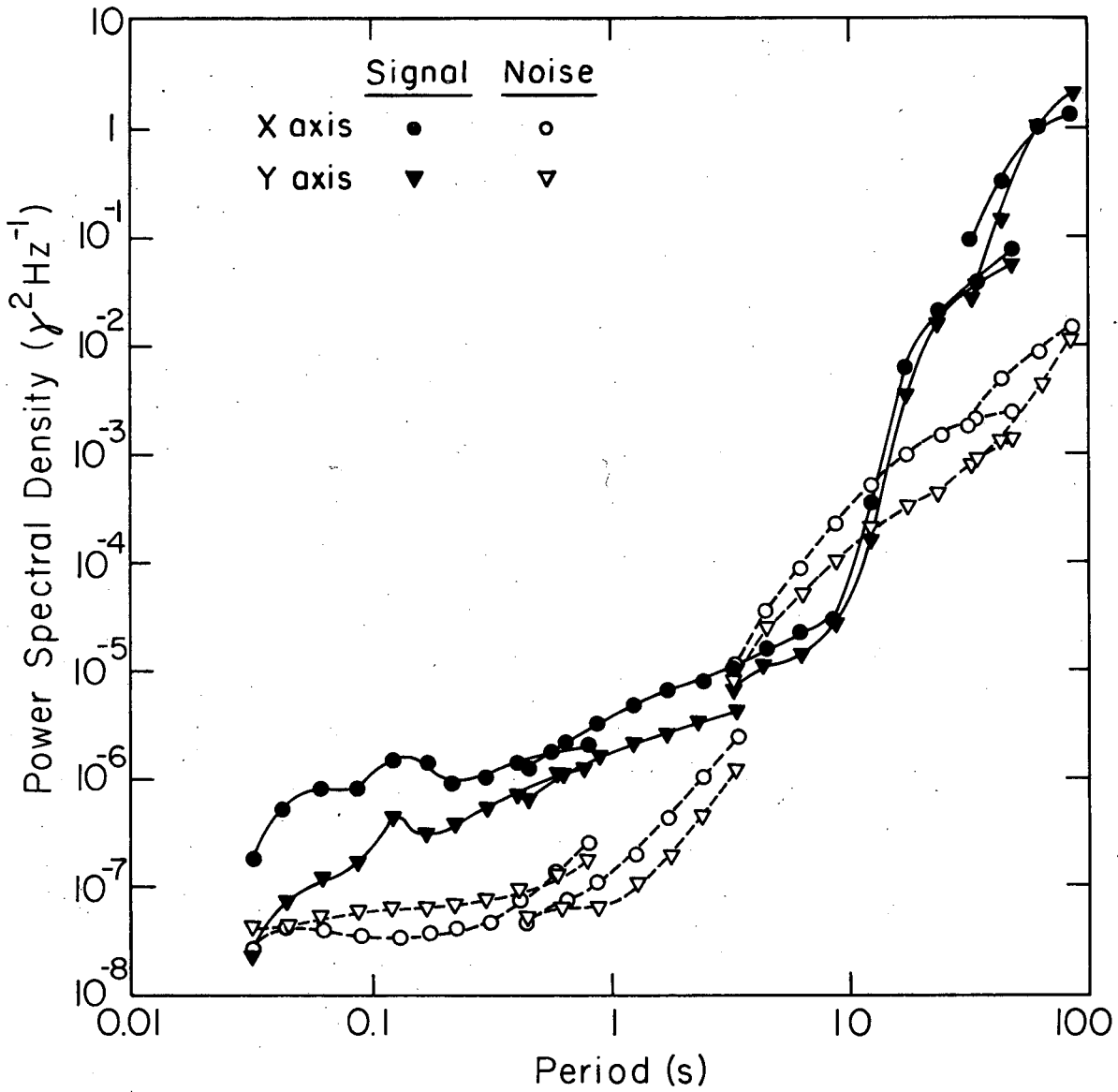
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Fig. 4

Fig. 5

XBL 783-4692
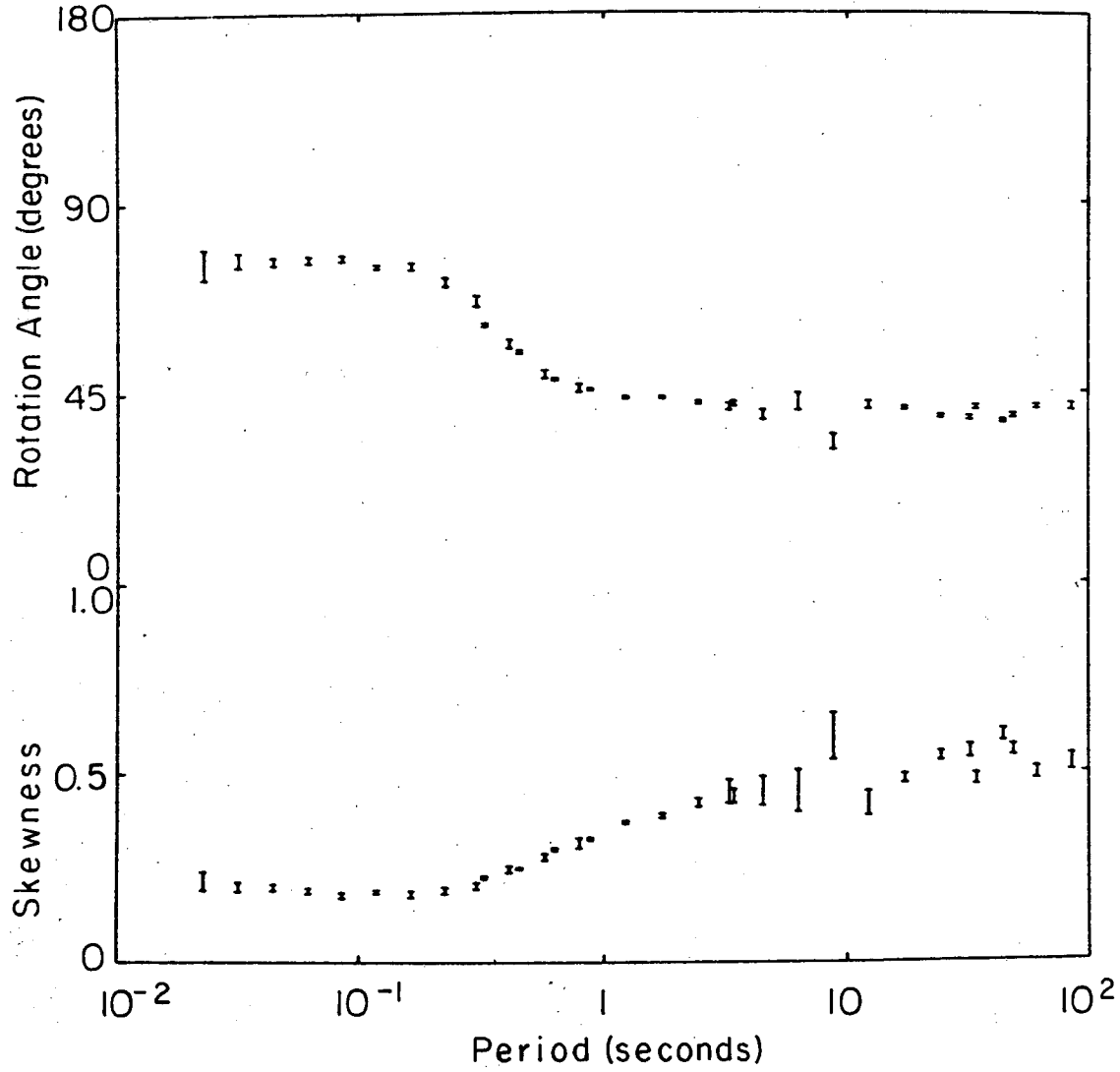
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Fig. 6

Fig. 7

XBL 783-4694