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Summary

Synchrotron light facilities are making ever increasing use of wigglers and undulators, to the extent that these devices are becoming a significant part of the beam optical system of the storage ring itself. This paper presents a theoretical formulation for investigating the effect of wigglers and undulators on beam dynamics in the approximation that the wiggler parameter, K, divided by γ is a small number and that the number of wiggler periods in one device is large. In addition to the linear forces which must be taken into account when tuning and matching the ring, non-linear stop bends are created, with even orders more serious than odd orders. Some numerical examples are given for devices similar to those proposed for the 1-2 GeV Synchrotron Radiation Source at Lawrence Berkeley Laboratory.

Introduction

For the next generation of synchrotron light storage rings, it is planned to make extensive use of wigglers and undulators; current designs consist primarily of long straight sections connected by bending and focusing sections. Thus the effects, linear and non-linear, of the insertion devices on beam dynamics become important questions. It has already been found[1] that the single (superconducting) wiggler inserted in the KEK Photon Factory introduces new non-linear resonances and seriously restricts the choice of working point in the tune diagram. This paper presents an analytic treatment of insertion devices consisting of many cells, for which the cell length is short enough to neglect variations of the β-functions in a cell length and for which the wiggler parameter, K, divided by γ is a small quantity. Radiation effects, if any, are not considered.

Equilibrium Orbit

The expressions used for the magnetic field are those suggested by K. Halbach:[2]

$$\begin{aligned} B_y &= B_0 \cosh k_x x \cosh k_y y \cos kz \\ B_x &= \frac{k_x}{k_y} B_0 \sinh k_x x \sinh k_y y \cos kz \qquad (1) \\ B_z &= -\frac{k}{k_y} B_0 \cosh k_x x \sinh k_y y \sin kz \end{aligned}$$

where

$$k_x^2 + k_y^2 = k^2 = \left(\frac{2\pi}{\ell}\right)^2 \qquad (2)$$

and ℓ is the cell length.

Here y is the vertical direction, x the horizontal and z the beam direction. As will be seen presently, B_y increasing with x for $k_x \neq 0$ provides horizontal focusing; the falling field occurring for a flat pole piece can be represented by the substitution $k_x \to ik_x$. A single harmonic in z is appropriate for closely spaced magnets or can be regarded as the lowest harmonic if the occupancy is less than unity.

The equation of motion in the horizontal plane, with $\cosh k_x x \sim 1$, is:

$$\frac{d^2x}{ds^2} = -\frac{1}{\rho}\frac{dz}{ds}\cos kz\ , \qquad (3)$$

where

ρ = radius of curvature in the field B_0, and

$$ds = vdt$$

The first integral of Eq. (3) is:

$$\frac{dx_e}{ds} = -\frac{1}{k\rho}\sin kz_e \qquad (4)$$

where the subscript (e) refers to the equilibrium orbit, which has zero slope at $s = z = 0$.

By using the relation,†

$$z_e'^2 = 1 - x_e'^2 = 1 - \frac{1}{k^2\rho^2}\sin^2 kz_e \qquad (5)$$

the second integration can be done exactly; however, to good approximation,

$$x_e = \frac{\rho}{k^2\rho^2}\cos kz_e \qquad (6)$$

The quantity, $1/k\rho$, is equal to the wiggler parameter, K, divided by γ and is taken to be small, typically 10^{-3} to 10^{-2} for 1-2 GeV electrons. x_e is measured in microns, but the modulation of x_e' combined with the modulation of the field components gives rise to significant first order focusing.

A Hamiltonian for Betatron Motion

Since mass and velocity are constant for motion in a magnetic field, a Hamiltonian giving the correct equations of motion can be written in what appears to be a non-relativistic form:

$$H = \frac{1}{2}\left[p_z^2 + (p_x - A_x \sin kz)^2 + (p_y - A_y \sin kz)^2\right] \qquad (7)$$

where

$$\begin{aligned} A_x &= \frac{1}{k\rho}\cosh k_x x \cosh k_y y \\ A_y &= -\frac{k_x}{k_y}\frac{\sinh k_x x \sinh k_y y}{k\rho} \end{aligned} \qquad (8)$$

H is dimensionless, with $p_z = z/v$, etc. and vt is the independent variable. It can be easily verified that the vector potential chosen yields the fields of Eqs. (1). Following the procedure given in Appendix B of Courant and Snyder,[3] a canonical transformation is required to change

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† Hereafter, a prime denotes differentiation with respect to s.

variables from (x,y,z) to (x_β, y_β, s) where s is distance along the equilibrium orbit, x_β is a displacement in the (x,z) plane perpendicular to the equilibrium orbit and $y_\beta = y$ is vertical displacement from the equilibrium orbit. The relations between variables are:

$$x = x_e + z'_e x_\beta \qquad p_x = z'_e\, p_{x_\beta} + \frac{x'_e}{1+\Omega x_\beta} p_s$$

$$z = z_e - x'_e x_\beta \qquad p_z = \frac{z'_e}{1+\Omega x_\beta} p_s - x'_e p_{x_\beta} \qquad (9)$$

$$y = y_\beta \qquad p_y = p_{y_\beta}$$

where x_e, z_e, x'_e, z'_e are regarded as functions of s, as described in the previous section, and $\Omega = (1/\rho)\cos kz_e$ is the local curvature of the equilibrium orbit.

The Hamiltonian becomes:

$$H = \frac{1}{2}\left[\left(\frac{p_s}{1+\Omega x} - x'_e A_x \sin kz\right)^2 + \left(p_x - z'_e A_x \sin kz\right)^2 \left(p_y - A_y \sin kz\right)^2\right] \qquad (10)$$

where the β-subscript has been dropped. Then, since H is a constant of the motion, the independent variable can be changed from time to distance, s, along the equilibrium orbit by solving (11) for $-p_s$, which is then the Hamiltonian describing x and y as functions of s:

$$H = -p_s \cong \frac{1}{2}\left[\left(p_x - z'_e A_x \sin kz\right)^2 + \left(p_y - A_y \sin kz\right)^2\right] - \Omega x - x'_e A_x \sin kz \qquad (11)$$

The next step will be to take averages over a wiggler period, keeping only terms of lowest order in $(k\rho)^{-1}$. Since A_x and A_y are of order $(k\rho)^{-1}, z'_e \sim .1$ from Eq. (5). The argument of the hyperbolic functions of x is

$$k_x(x_e + z'_e x) \cong k_x x + \frac{k_x}{k}\frac{\cos kz_e}{k\rho} \sim k_x x$$

The average of $x'_e \sin kz$ in the last term of the Hamiltonian is given by:

$$\overline{-x'_e \sin kz}^{\,s} =$$

$$= \frac{1}{k\rho}\frac{1}{\ell}\int_0^\ell dz_e \frac{\sin kz_e \sin k(z_e - x'_e x)}{z'_e}$$

$$\sim \frac{1}{k\rho}\frac{1}{2\pi}\int_0^{2\pi} d\theta \sin\theta \sin\left[\theta + \frac{x}{\rho}\sin\theta\right]$$

$$= \frac{1}{4\pi k\rho}\left(\int_0^{2\pi} d\theta \cos\frac{x}{\rho}\sin\theta - \int_0^{2\pi} d\theta \cos\left[2\theta + \frac{x}{\rho}\sin\theta\right]\right)$$

$$= \frac{1}{2k\rho}\left[J_0\left(\frac{x}{\rho}\right) - J_2\left(\frac{x}{\rho}\right)\right].$$

where J_0, J_2 are Bessel functions. Since $x/\rho \lesssim 10^{-2}$,

Similarly,

$$\overline{-x'_e \sin kz}^{\,s} \sim \overline{-x'_e \sin kz_e}^{\,z} = \frac{1}{2k\rho}$$

$$\overline{\sin^2 kz} \sim \overline{\sin^2 kz_e} \sim \frac{1}{2}$$

The Hamiltonian can then be written:

$$H = \frac{1}{2}\; p_x^2 + p_y^2 + \frac{1}{4k^2\rho^2}\left[\cosh^2 k_x x \cosh^2 k_y y + \frac{k_x^2}{k_y^2}\sinh^2 k_x x \sinh^2 k_y y\right] - \frac{\sin ks}{k\rho}\left[p_x(\cosh k_x x \cosh k_y y - 1) - \frac{k_x}{k_y} p_y \sinh k_x x \sinh k_y y\right] \qquad (12)$$

The term in $\sin ks$ is retained because it will give rise to odd-order resonances and therefore may not be negligible. The averaged potential leads only to even-order resonances, except for a feed-down from closed orbit errors.

Linear Motion

In linear approximation,

$$x'' = -\frac{1}{2\rho^2}\frac{k_x^2}{k^2}x, \qquad y'' = -\frac{1}{2\rho^2}\frac{k_y^2}{k^2}y \qquad (13)$$

If the pole piece is shaped to provide horizontal focusing, it would be reasonable to make $k_x^2 = k_y^2 = k^2/2$. In that case there would be a phase advance in both planes of $L/2\rho$ through the device (of length L) if the lattice were matched to the β function, equal to 2ρ. This phase advance could be as large as 60°-90° for a super-conducting wiggler. The tune shift due to a device being turned on would of course depend on what the lattice configuration might be with the device turned off. In any event, the lens action must be considered in lattice design, including dynamic changes as devices might be turned off and on during operation.

As mentioned earlier, if the vertical field decreases transversely from the center of the magnets, the sign of k_x^2 is reversed. In this case, an insertion is mildly defocusing in the horizontal plane, with

$$k_x^2 \sim -\frac{2}{a^2}\left(\frac{\Delta B}{B}\right),$$

where $\Delta B/B$ is the fractional drop in field at $|x| = a$.

Non-Linear Effects

The Hamiltonian for the entire ring is, to fourth order,

$$H = \frac{1}{2}\left[p_x^2 + p_y^2 + K_x x^2 + K_y y^2\right] + \left\{\frac{1}{12k^2\rho^2}\left[k_x^4 x^4 + k_y^4 y^4 + 3k_x^2 k^2 x^2 y^2\right] - \frac{\sin ks}{2k\rho}\left[p_x\left(k_x^2 x^2 + k_y^2 y^2\right) - 2k_x^2 p_y xy\right]\right\} S(s) \qquad (14)$$

where K_x and K_y are the linear focusing functions for the entire ring, including wigglers, and $S(s)$ is unity where wigglers exist and zero elsewhere. Resonances occur at integral values of $3\nu_x$, $\nu_x+2\nu_y$, $4\nu_x$, $4\nu_y$ and $2\nu_x+2\nu_y$. Following standard perturbation theory, angle and action variables, (J,ψ) and independent variable, $\phi = (1/\nu)\int(ds/\beta)$ are introduced and the nonlinear terms averaged in ϕ.

The average can be done analytically in two limiting cases. According to Eq. (14), an insertion device has a "natural" $\beta \approx \sqrt{2}\,\rho$. If it is required that β in the insertion region should not exceed a few meters, the lattice could be matched to this value for high field devices, whereas for low field devices, the focusing strength would be negligible compared to that of the adjacent quadrupoles and β would be "unmatched":

$$\beta = \beta^* + \frac{s^2}{\beta^*},$$

with β^* determined by other considerations.

At $3\nu_x = n$, the total stopband width is:

$$\text{Matched:} \qquad \Delta\nu_{SB} = \frac{3}{2\pi}\left[\frac{\varepsilon\rho^2}{\beta^3}\right]^{1/2}\left|\sin\frac{3L}{2\beta}\right|$$

$$\text{Unmatched:} \qquad \Delta\nu_{SB} = \frac{3}{4\pi}\,\frac{k_x^2}{k_\rho^2}\,\frac{L}{\beta^*}\,(\beta^*\varepsilon)^{1/2} \qquad (15)$$

The fourth order terms in (15) lead to a tune dependence on amplitude as well as a driving term for fourth order resonances. In both limiting cases, the magnitude of the driving term is less than the other. Therefore, resonance occurs only for $4\nu < n$ and the phase space pattern consists of islands surrounded by stable orbits. The tune dependence on emittance, which is suspected to play a significant role in determining dynamic aperture, is given by:

$$\Delta\nu_y = \frac{1}{8\pi}k_y^2 L\varepsilon \qquad \text{matched}$$

$$= \frac{1}{16\pi}\,\frac{k_y^4}{{k^2}_\rho{}^2}\,L\beta^{*2}\left[1 + \frac{2}{3}\left(\frac{L}{2\beta^*}\right)^2 + \frac{1}{5}\left(\frac{L}{2\beta^*}\right)^4\right]\varepsilon \qquad \text{unmatched} \qquad (16)$$

and similarly for ν_x with $k_y \to k_x$. The emittance contained inside the ring of islands at resonance is obtained by equating the expressions (16) to $(n/4)-\nu$. The half-width of the islands, relative to the emittance at which they occur is given by:

$$\frac{1}{2}\left(\frac{\Delta\varepsilon}{\varepsilon}\right)^2 = \frac{\beta}{6L}\left|\sin\frac{2L}{\beta}\right| \qquad \text{matched}$$

$$= \frac{1}{3}\,\frac{\left|1 - 2\left(\frac{L}{2\beta^*}\right)^2 + \frac{1}{5}\left(\frac{L}{2\beta^*}\right)^4\right|}{1 + \frac{2}{3}\left(\frac{L}{2\beta^*}\right)^2 + \frac{1}{5}\left(\frac{L}{2\beta^*}\right)^4} \qquad \text{unmatched} \qquad (17)$$

It is of interest to note that the islands disappear if $\beta = (2L/\pi)$ (matched) or $\beta^* \approx (L/\sqrt{2})$ (unmatched).

Table I gives numerical values for a typical undulator (weak field) and a super-conducting wiggler (strong field) with $\beta^* = 3$ meters. The emittance used corresponds to the admittance of the device itself; $k_x = k/\sqrt{2}$ for the 3rd order resonance and $k_y = k(k_x = 0)$ for the 4th order. If several devices are being used simultaneously, the tune dependences on amplitude add arithmetically and the resonance driving terms add weighted by suitable phase factors.

Table I. Equilibrium orbit amplitude, betatron amplitude tune dependence, stop-band widths and island half-widths for a typical undulator (unmatched) and a super-conducting wiggler (matched) at $E = 1.3$ GeV.

a) Parameters

	Peak Field (T)	Period (cm)	Length (m)	Half Aperture (cm)	$k\rho$
Undulator	.54	5.0	5	.88	1100
Wiggler	5.0	14.0	2	1.75	42.3

b) Beam Dynamic Properties

	x_e (μm)	$\frac{\partial\nu}{\partial\varepsilon}$ (m-rad)$^{-1}$	$\Delta\nu_{3x}\times10^4$	$\Delta\nu_{4y}\times10^3$	$\frac{\Delta\varepsilon_y}{\varepsilon}$
Undulator	7.2	295	2.5	5.9	.35
Wiggler	525	161	1.5	32.8	.17

Conclusion

The linear-optical effects appear to be large enough to be considered in the lattice design. Shaping the poles to make $k_x^2 > 0$ provides horizontal focusing, but further complicates the linear optics and introduces additional non-linear resonances; the gain may not be worth the cost. The perturbation analysis of non-linear effects indicates that tracking studies should be invoked to determine the true dynamic aperture.

References

1. Photon Factory Activity Report 1983/84, p. IV-34, National Laboratory for High Energy Physics KEK.

2. K. Halbach, private communication.

3. Courant and Snyder, Annals of Physics, 3, 1958, p. 45.