Title
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Using cognitive models to guide instructional design: The case of fraction division

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Abstract

Cognitive modeling can be used to compare alternative instructional strategies and to guide the design of curriculum materials. We modeled two alternative strategies for fraction division, and the models led to specific empirical predictions of the benefits and drawbacks of each strategy. These insights provided concrete suggestions for developing lessons on fraction division, including a new potential strategy that combines the benefits of the two strategies. This on-going work illustrates the potential of cognitive modeling for informing the design of better mathematics curricula.

Background

Although U.S. students are fairly proficient at performing routine calculations, they lack a conceptual understanding of mathematics and have difficulty solving non-routine problems (Lindquist, 1989; Jakwerth, 1999). These findings have spurred many educators to call for an increasing focus on building understanding and problem solving skill in mathematics instruction. The National Council for Teachers of Mathematics (NCTM) standards state the overarching learning goal as: "Students must learn mathematics with understanding, actively building new knowledge from experience and prior knowledge," (p. 16, NCTM, 2000). The standards proposed by NCTM, and curriculum and evaluations based on them, have met with opposition from advocates of "back-to-basics" approach (Mathematically Correct, 2000). Mathematicians, politicians, teachers and parents have raised concerns that students are not learning their arithmetic facts and basic computational skills and are lobbying to abandon these reform efforts. Further, many teachers have been resistant to changing their teaching practices and doubt the benefits of reform-based curricula.

Although there is agreement on the need to improve the mathematics curriculum in the U.S., there is considerable disagreement on how the curriculum should be changed. Controversy over how to teach fraction division helps to illustrate this fundamental conflict over whether the curriculum should focus on gaining conceptual understanding or on proficiency in retrieving facts and executing computational procedures. To solve fraction division problems, students are traditionally taught the computational procedure of inverting the divisor and changing the operation to multiplication (invert-and-multiply strategy). As an alternative, the NCTM 2000 standards proposed a picture division strategy where students draw a picture of the starting amount, repeatedly "cut off" groups of the size specified by the divisor and count the resulting number of groups. For example, to solve six divided by 3/4, students could draw a line six units long, divide each unit into fourths, and then start at six and mark off groups of three fourths to find how many 3/4 are in six. According to the standards: "Lacking an understanding of the underlying rationale [for invert and multiply], many students are therefore unable to repair their errors and clear up their confusions about division of fractions.. Carefully sequenced experiences with [picture division] problems such as these can help students build an understanding of division of fractions." (p.218 NCTM, 2000).

In principle, an ideal approach to informing the debate on how best to teach a particular mathematical topic is to conduct a multi-year, multi-site experimental study comparing a reform-based approach with a back-to-basics approach. In addition to the practical limitations of this approach, an empirical evaluation does not explain the reasons for the results or offer insights into how to apply these results to other topics.

We have begun to explore the role of cognitive models in helping to inform this debate. Developing cognitive models offers four key advantages. First, developing cognitive models requires precise and unambiguous specification of problem representations and action sequences and allows for detailed comparisons of problem solving strategies. Second, the specificity of the models leads to generation of specific hypotheses that can be tested through smaller, focused, empirical studies. Third, cognitive models can be used to understand and explain empirical results, allowing researchers to understand the mechanisms underlying the differences and to extrapolate the findings to other domains. Finally, inspection and evaluation of these models yield concrete suggestions for better content and methods of teaching a particular topic.

To illustrate the potential of cognitive modeling for informing the current debate in mathematics instruction, we describe our use of cognitive modeling to guide the design of lessons on fraction division (as part of a middle-school math curriculum we are developing). Rational number concepts and procedures are a cornerstone of middle-school mathematics, but U.S. students perform poorly on a range of rational number problems, including fraction division problems. (e.g. Lindquist, 1989; Lesh & Landau, 1983). Fraction division is a representative topic in mathematics for which the standards-based and back-to-basic movements have proposed alternative strategies.

In the current paper, we present cognitive models of both fraction division strategies, outline predictions for learning and transfer that are revealed by the models, and offer preliminary implications of the models for instructional design, including a new strategy for fraction division that was suggested by this work.

Cognitive Models

Our cognitive models are based on ACT-R theory, which breaks knowledge into two main categories – a declarative knowledge base of facts and a procedural knowledge base of production rules (Anderson, 1993). Declarative knowledge includes both prior domain knowledge and representation of the current problem situation. A production rule is a simple IF-THEN statement that manipulates declarative knowledge, and a series of production rules model actions for solving a problem.

Model of Picture Division Strategy

We collected informal verbal protocols from five sixth-grade students while they solved basic fraction division problems such as 15 ÷ 1 1/2. The students had received no formal instruction on fraction division. One student spontaneously used a picture division strategy, and the other students were provided with a picture and encouraged to try using the strategy.

The combination of the task analysis and students' think alouds revealed 4 main sub-goals for implementing this strategy: 1) identify the values in the problems and draw the appropriate picture, 2) mark the picture into the appropriate size groups, 3) count the number of groups, and 4) convert the remainder (if there is one) to a fractional value. These four sub-goals translate into 13 key steps or actions that the problem solver must take (see Table 1; a dotted line designates the beginning of a new sub-goal). These actions were instantiated as productions in an intelligent tutoring system that is based on ACT-R theory (Anderson, Corbett, Koedinger & Pelletier, 1995).

Table 1: Cognitive model of picture division strategy

Productions	Student Example
1. Identify-starting-amount	Here's her 8 foot long board
2. Draw-whole-starting-amount	[draws line with 8 sections]
OR	
3. Draw-mixed-starting-amount	
4. Identify-size-of-groups	And she wants each one [shelf] to be a half,
5. Identify-value-of-divisions	So, I'd split each one in half
6. Draw-divisions	[marks each whole in half]
7. Identify-step-size	[group size = 1; skip to 9]
8. Mark-first-group	
9. Mark-next-group	
10. Finished-marking-groups	
11. Count-whole-groups	And then that's how many shelves. [counts] 16.
12. Identify-remaining-divisions	NA
13. Step-size-as-denominator-of-remainder	NA

Note: Extra productions would be needed to solve problems where the denominators of the dividend and divisor are different and one is not a multiple of the other (e.g. 3/5 ÷ 1/3).

Table 2: Example declarative knowledge chunk: Representation of 2/3

```
QUANTITY2/3>
 isa number
 whole           0
 top-number      2
 bottom-number   3
 parts-per-whole 3  ;;Picture Division strategy only
 needed-parts    2  ;;Picture Division strategy only
```

The productions in the picture division model rely on meaningful representation of problem information in declarative memory. First, selection of this strategy comes from representing the meaning of division as finding the number of groups of a given size in the starting amount. Second, the productions rely on a quantity-based representation of fractions. Students need to represent fractions as parts of a whole (e.g. 2/3 is two out of three equal size parts) rather than only as a visual arrangement of numbers (e.g. 2 is the top number and 3 is the bottom number). Table 2 provides a sample declarative chunk used in representing the problem.

Model of Invert-and-Multiply-Strategy

The invert-and-multiply strategy can be broken into 4 main sub-goals: 1) identify the dividend and the divisor, 2) if needed, convert whole numbers and mixed numbers to fractions, 3) invert the divisor and multiply the two fractions, and 4) if needed, simplify the answer by

Table 3: Cognitive model for invert-and-multiply strategy

Productions	Student Example
1. Identify-dividend	12
2. Identify-divisor	1 1/2
3. Whole-dividend-to-fraction	12/1
4. Identify-mixed-dividend	NA
5. Identify-mixed-divisor	
6. Mixed-to-fraction	1 1/2 is 3/2
7. Invert-divisor	So, make it 12/1 * 2/3
8. Multiply-top-&-btm-#s	Equals 24/3
9. Improper-to-mixed	That is 8
10. ID-whole-#-answer	[Done]
11. ID-if-quotient-is-reducible	NA
12. Reduce Fraction	NA

converting an improper fraction to a mixed number and/or by putting the fraction in simplest terms. These four sub-goals translate into 11 key steps or actions that the problem solver must take (see Table 3; dotted lines designate the beginning of a new sub-goal).

The declarative knowledge used by the invert-and-multiply strategy differs from that used by the picture division strategy. In the invert and multiply strategy, division is represented as performing actions on numbers. Fractions are represented as a visual arrangement of digits, and quantity-based knowledge is not used (see Table 2).

Empirical Support for Our Models

We designed a brief intervention to validate and refine our cognitive models. The students had already been taught the invert-and-multiply strategy, and we gave them a brief lesson on the picture division strategy. Thirty-two ninth-grade students from two math classes for below-average math students participated in the study.

On the first day of our study, students received a 10-minute lesson on fraction division from their classroom teacher. The teacher discussed how to solve two types of fraction division problems using each strategy. Instruction on the picture division strategy focused on forming a quantity-based representation of the problem without detailed instruction on the actions (productions) for implementing the strategy. The teacher then reviewed the steps for using the invert-and-multiply. After this brief lesson, students were randomly assigned to use one of the two strategies to solve a set of problems. Students solved two problems using the assigned strategy and received feedback and help in finding the correct answer if needed. Students then solved a set of 10 problems without feedback or help – 4 instructed problems (problems with whole numbers and/or unit fractions), 5 transfer problems (problems with non-unit fractions and with mixed numbers), and a fraction multiplication problem. Students had approximately 20 minutes to solve the problems, and their compliance with the strategy instructions was high. In the invert-and-multiply group, there was no trace of students using a pictorial strategy, and in the picture division group, a picture was drawn on 83% of attempted problems. Four days later, students were asked to solve a parallel set of 10 problems using any strategy they wanted.

Students had difficulty learning the picture division strategy from our brief lesson. On Day 1, they solved 60% of the instructed problem types correctly, but only 9% of the transfer problems correctly. Many students got stuck and did not finish the assessment; students only attempted 57% of the problems (compared to students attempting 96% of problems in the invert-and-multiply group).

Not surprisingly, students who were assigned to use the familiar invert-and-multiply strategy solved more problems correctly on Day 1, compared to the Picture Division group (49% vs. 28% correct; $\underline{F}$ (1,30) = 16.96, $\underline{p}$ < .01). They solved 89% of the instructed problem types correctly, but only 31% of the transfer problems correctly (although they had previously received instruction on these problem types as well.) Student had particular difficulty when the problems involved mixed numbers. Only half of the students solved at least one problem of this type correctly.

When students were free to choose any strategy on Day 2, students used the more familiar and well practiced invert-and-multiply strategy on a majority of problems ($\underline{M}$ = 62% of trials with a mean accuracy of 60%). The picture division strategy was used on 10% of problems, and only by students who were assigned this strategy on Day 1.

To help explain the difficulties of each strategy, students' incorrect solutions were classified using the productions in the relevant cognitive model. We distinguished between failing to initiate a production and implementing a production incorrectly (an error). The ease of coding student solutions is an additional benefit of developing cognitive models. We report solution data from Day 1, but a similar pattern arises on Day 2.

Tables 4 and 5 show the distribution of failures and errors over the productions for each strategy. On the picture division strategy, students often did not know how to start the problem. When students attempted the problem, they often did not succeed on the first sub-goal – identifying values and setting up the picture. There were a surprising number of errors in identifying the dividend and in drawing the divisions correctly (e.g. students added 3 extra divisions per whole for 1/3, thus making fourths). Students' errors on identify-parts-per-whole varied by problem type, suggesting that an additional production was needed in our model. When the dividend was a fraction, students sometimes divided the fractional amount, rather than the whole, into the specified number of parts (e.g. for 1/2 ÷ 1/10, dividing the half into 10 sections). Students need an extra production for mapping the parts-per-whole to the parts-per-fraction (e.g. if 10 division in one whole, half as many (5) in 1/2). We have very little data on the difficulty of productions that occur later in the sequence because students often abandoned this strategy.

Students using the invert-and-multiply strategy were much more likely to attempt to solve a problem, and the majority of mistakes arose from failing to or incorrectly converting mixed numbers to fractions. However, this error did not cause students to abandon the strategy. Rather, students made illegitimate adaptations to the strategy, such as inverting the fractional portion of the divisor and then multiplying the whole number portions and the fraction portions separately (e.g. 8 2/3 ÷ 2 1/3 = 8 2/3 * 2 3/1 = 16 6/3). Further, students' errors on the fraction multiplication problem suggested that the conditions for firing the invert-divisor production were overly general for many students. Half of the students in the invert and multiply group inverted the second fraction before multiplying.

Table 4: Classification of students' incorrect solutions using the picture division strategy on day 1

Action/Production	No. of Errors	No. of Failures
<Start Problem>	NA	64
ID /Draw-whole-starting-amt	6	1
ID /Draw-mixed-starting-amt	7	-
Identify-size-of-groups	-	-
Identify-value-of-divisions	6	11
Draw-divisions	10	-
Identify-step-size	7	1
Mark-first-group	-	-
Mark-next-group	1	-
Finished-marking-groups	-	-
Count-whole-groups	2	1
Identify-remaining-divisions	NA	NA
Step-size-as-denominator-of-remainder	NA	NA

Table 5: Classification of students' incorrect solutions using the invert and multiply strategy on day 1

Action/Production	No. of Errors	No. of Failures
<Start problem>	NA	8
ID Dividend	-	-
ID Divisor	-	-
Whole-to-fraction	-	-
ID-mixed-dividend & Mixed-to-fraction	13	15
ID-mixed-divisor & Mixed-to-fraction	4	9
Invert-divisor	2	9
Multiply-top-& -btm-#s	4	8
Improper-to-mixed	3	9
ID-if-common-factor	NA	NA
Reduce Fraction	NA	NA

Implications of the results for the cognitive models

The empirical results revealed a necessary refinement to the picture division model and validated the other productions in the two models. Students' errors when using the picture division strategy indicated that an additional production was needed when the dividend was a fraction. Otherwise, the models captured students' behaviors quite well.

The empirical results also provide information on common buggy rules and on the frequency of correct productions "failing to fire". Students' buggy rules will be modeled as production rules, allowing us to identify the source of the differences in the correct and incorrect productions. This information can be used to target instruction at addressing or preventing these errors.

These results also highlight the importance of the declarative knowledge structures. The ninth-grade students in this study did not seem to form quantity-based representations of fractions or to represent division as finding the number of groups of a certain size in the starting amount. Without these declarative knowledge structures, students had great difficulty implementing the initial productions for the picture division strategy. In contrast, the invert-and-multiply strategy only relies on a superficial representation of division and of the position of the digits in fractions, although a quantity-based representation of the values could be used to recognize errors in its execution (e.g. that multiplying the whole numbers will lead to too large of an answer). After more than 5 years of instruction on the division operator and on fractions, these students did not seem to be forming meaningful representations of either.

Predictions from the models

Cognitive models of the picture division and invert-and-multiply strategies can lead to comparative predictions for 1) difficulty of learning each strategy, 2) efficiency of using each strategy once learned, 3) generality of each strategy to the range of fraction division problems, 4) retention of the strategies, and 5) transfer.

First, the ease of learning the two strategies depends on students' prior knowledge. In particular, learning difficulty should be predicted by two factors – how students represent fractions and division and how well they know symbol manipulation rules for working with fractions. If students form quantity-based representations of fractions and attach meaning to the division operation, learning the picture division strategy should be relatively straightforward since a majority of the productions are based on familiar and well practiced knowledge (e.g. marking sections and counting). However, if students only represent fractions as visual arrangements of digits and division as manipulating symbols, this representation is not compatible with the strategy, so the strategy will be difficult to learn. The invert-and-multiply is not dependent on a quantity-based representation of fractions. In contrast, the ease of learning this strategy depends on how well students already know productions for converting whole and mixed numbers to fractions and for converting improper fractions to mixed numbers.

Second, our models support the predictions that the two strategies will not be equally efficient once they are mastered. Although the total number of productions to learn is similar in the two strategies (13 vs. 12), the number of production firings is often higher for the picture division strategy because some productions must fire many times. For example, to solve $6 \div 3/4$, the draw-divisions production fires 18 times and the mark-next-group production fires 6 times. Thus, to solve this problem, the picture division strategy has 32 production firings whereas the invert-and-multiply strategy has 6 production firings. On a majority of problems, the invert-and-multiply strategy is more efficient than the picture division strategy once the strategy is mastered.

Third, the ease of applying the two strategies to the full range of fraction division problems is not equivalent. Once the full set of productions is mastered for the invert-and-multiply strategy, it can be applied to any fraction division problem. In contrast, the picture division strategy becomes very cumbersome if the dividend is large, the denominator of the divisor is large, or if the denominators of the dividend and divisor are not "friendly" (i.e. one denominator is not a factor of the other, such as 3 and 5). The first two constraints require an unmanageable number of firings of the draw-divisions and mark-next-group productions. The third constraint requires a new set of productions for finding equivalent fractions with a common denominator, thus necessitating extra productions that are not well grounded in the situation. Overall, the picture division and invert-and-multiply strategy can both be used to solve a majority of fraction division problems, but the invert-and-multiply strategy has the advantage of more uniform difficulty on all types of problems.

The fourth prediction concerns the retention of the two strategies and confers an advantage to the picture division strategy. In ACT-R, recall is based on spreading activation, so knowledge that is connected to a richer network of knowledge chunks is easier to recall (Anderson, 1993). The picture division strategy utilizes rich knowledge representations of quantities and operations, so this network of relations should facilitate recall. In contrast, the invert-and-multiply strategy utilizes sparse, visual-based representations that are not connected to a rich knowledge base, so this strategy should be harder to recall after a delay. Our results indicate that students have difficulty correctly retrieving all of the relevant productions for invert-and-multiply. In addition, both level-of-processing and dual-code theories of memory (Craik & Lockhart, 1972; Paivio, 1971) suggest that the richer problem representations utilized by the picture division strategy should lead to better recall of this strategy, compared to the invert-and-multiply strategy. Thus, we predict that recall of the picture division strategy will be more robust.

Fifth, the models lead to very different transfer predictions. Inspection of the models indicates no overlap in the productions that are used by each strategy, so learning one strategy will not aid learning of the other. The two strategies also transfer differently to other topics. When knowledge chunks are activated, their memory trace is strengthened (Anderson, 1993), so quantity-based representations of fractions and a meaning-based representation of division are strengthened (and possibly refined) when students use the picture division strategy. Thus, learning the picture division strategy should facilitate performance on tasks utilizing these representations. Representing fractions as part-whole quantities provides a powerful declarative knowledge structure for choosing and implementing a variety of strategies for tasks such as comparing magnitudes, estimating, or adding and subtracting fractions. The picture division strategy should also transfer to decimal division since it strengthens a meaningful representation of division and many of the productions can be used to solve division problems with decimals. In contrast, the invert-and-multiply strategy should facilitate performance on problems involving other fraction operations or algebraic simplification. This strategy strengthens productions that are also used for adding, subtracting and multiplying fractions, such as converting improper fractions to mixed numbers, reducing fractions, and multiplying fractions (although students may over-generalize the strategy and also invert the second fraction when multiplying fractions). Productions from this strategy can also be applied to simplifying algebraic expressions. Overall, the two strategies should aid performance on very different types of transfer problems.

Developing cognitive models of the two strategies leads to precise predictions of the benefits and drawbacks to each strategy. The picture division strategy should be easy to learn if students have quantity-based representations of fractions, should be recalled after a delay, and should transfer to tasks such as comparing fractions and dividing by a decimal. In contrast, the invert-and-multiply strategy should be easy to learn if students already know productions for manipulating fractions, should be efficient and broadly applicable once mastered, and should transfer to other fraction operations and to algebra.

Implications for Instructional Design

Comparing the benefits and drawbacks of each strategy allows for an informed decision on whether and how to teach each strategy. Neither of the strategies was strong along all five dimensions that we considered (difficulty of learning, efficiency, generality, retention and transfer). Instead, there were trade-offs for learning each strategy.

How the fraction division problems are represented in declarative memory helps to explain the benefits and drawbacks to each strategy. The picture division strategy supports a quantity-based representation of fractions as a specified number of parts of a whole. Quantity-based representations provide a unified representation that can be used when solving a large variety of rational number problems, such as modeling, estimating, comparing, and doing arithmetic with fractions. Thus, retention of the strategy should be high. In contrast, the invert-and-multiply strategy relies on a visual, position-based representation, and this representation requires different, special-purpose productions to solve a similar variety of rational number problems, and retention of the productions would be relatively low. However, these specialized productions lead to more efficient performance.

Ideally, instruction could bridge from the more meaningful and grounded strategy of picture division to the more abstract and efficient strategy of invert and multiply, while maintaining high retention. Unfortunately, there is no overlap in the problem representations or the productions used by these two strategies, making it difficult to build from one to the next. Because of this limitation, we developed a third strategy, labeled the common denominator strategy, which builds off the picture division strategy and leads to an efficient and general method for dividing fractions. Because this strategy builds on the picture division strategy, we first discuss suggestions for teaching the picture division strategy and then outline a model of this new strategy.

The cognitive model suggests a careful sequence of lessons for teaching the picture division strategy. Students should first learn to represent fractions as part-whole quantities. Next, students should be taught to use the picture division strategy on problems that rely on the fewest number of productions - dividing a whole number by a unit fraction. After students have learned this minimum set of five productions, they will need help identifying the group size of non-unit fractions and mixed numbers, identifying the number of smaller divisions in bigger divisions if both numbers contain fractions, and converting remainders to fractional values when needed.

After students have experience with the picture division strategy, the common denominator strategy can be introduced as a more general and efficient strategy. Initially, the common denominator strategy can be tightly grounded by the picture division strategy, and then it can be abstracted to a more efficient algorithm. Both the grounded and abstract versions of the common denominator strategy are illustrated in Table 6. The strategy has five main sub-goals: 1) identify the initial values, 2) find the total number of divisions in the starting amount (which may involve finding a common denominator for the dividend and divisor), 3) identify the size of each group (with this common denominator), 4) divide the total number of division by the group size, 5) simplify the answer. After identifying the initial values, students must figure out the total number of divisions in the starting amount, which is analogous to marking the divisions and counting the total number of divisions. To identify the group size, students must make sure the divisor is a fraction that has the same number of parts-per-whole (denominator) as the dividend. Next, the number of groups is found by dividing the total number of divisions by the group size (i.e. dividing the two numerators), which is analogous to marking the groups on the picture and counting the number of groups. This leads to an answer in appropriate fractional form, although the answer may need to be converted from an improper fraction to a mixed numbers. After linking this strategy to the picture division strategy, a more formal, symbol-based strategy can be abstracted, which relies on converting whole and mixed numbers to fractions and finding fractions with a common denominator and then dividing the numerators and denominators. This strategy retains the quantity-based representations of the picture division strategy while being more efficient and general than this strategy. We have used these analyses to design a set of lessons on fraction division that integrate all three strategies, and we are piloting these lessons with sixth grade students who have no prior experience with fraction division.

Table 6: Example of common denominator strategy for solving 1 1/2 ÷ 3/4

Grounded approach:
Because 1/2 = 2/4, and 1 whole = 4 fourths, 1 1/2 = 6/4.
now have: 6/4 ÷ 3/4
In 6/4, there are 2 groups of 3/4, so the answer is 2.
Abstract approach:
Equivalent fractions: 1/2 = 2/4
Mixed to fraction: have 1 2/4: 1 * 4 = 4; 4 + 2 = 6, so 6/4
Now have 6/4 ÷ 3/4
6 ÷ 3 = 2; 4 ÷ 4 = 1
Answer is 2/1, and because any number divided by 1 is that number, the answer is 2.

In summary, cognitive modeling is a promising tool for evaluating alternative strategies and techniques that can be leveraged in the development of better curriculum material and instructional approaches.

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