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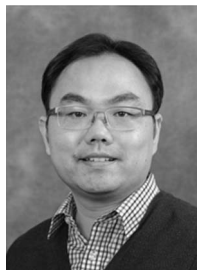
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Iterative Blade Element Momentum Theory for Predicting Coaxial Rotor Performance in Hover



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This paper presents a new blade element momentum theory (BEMT) for a coaxial rotor in hover. The new BEMT iteratively solves the upper and lower rotor induced velocities to account for the mutual rotor-to-rotor interaction. The upper rotor induced velocity is affected by the lower rotor thrust and induced velocity, whereas the lower rotor induced velocity is affected by the upper rotor thrust and induced velocity. Two empirical constants are included in each rotor calculation. This new BEMT provides the performance of each rotor as a function of the rotor separation distance. The new BEMT is validated with measurement data for two coaxial rotor experiments. The first experiment validates the thrust to power coefficients at a given separation distance. The second experiment validates each rotor's figure of merit, thrust, power, interference loss factors, etc. as a function of the rotor separation distance. It is shown that the BEMT captures the trends and magnitudes of the performance as a function of the rotor separation distance compared to the measurement data. Detailed radial distributions of aerodynamic properties are also presented at several separation distances.

Nomenclature

A	rotor disk area, m^2 , or empirical constant
A_i	inverse of Jacobian matrix of $(i - 1)$ th Broyden trim iteration
B_l	upper to lower rotor influence empirical constant
B_u	lower to upper rotor influence empirical constant
C_P	power coefficient
C_Q	torque coefficient
C_T	thrust coefficient
C_{l_α}	lift curve slope
D	rotor diameter, m
dC_T	blade sectional thrust coefficient
d_0, d_1, d_2	drag coefficient empirical constants
F	Prandtl's tip loss function
F_1, F_2	residual functions for Broyden trim iteration
$\vec{F}$	vector of residual functions
J	Jacobian matrix
K_t, K_u, K_l	wake contraction ratios from the upper rotor to the infinity, from the upper rotor to the lower rotor, and from the lower rotor to the infinity, respectively
$K_{u l}$	upper/lower rotor power to isolated rotor power ratio
N_b	number of blades
P	pressure, Pa, or rotor power, W
P_d	profile power, W

P_i	induced power, W
R	rotor blade radius, m
r	nondimensional blade sectional distance, y/R
$\vec{s}_i$	collective pitch angle error at i th Broyden trim iteration, deg
T	thrust, N
V_c	climb velocity, m s^{-1}
$v_{i,u}, v_{i,l}$	induced velocities on the upper and lower rotors, respectively, m s^{-1}
y	blade sectional radial distance, m
$\vec{y}_i$	difference in residuals between i th and $(i - 1)$ th Broyden trim iterations
Z	rotor-to-rotor separation distance, m
α_0	zero-lift angle of attack, rad
θ	collective pitch angle, deg
$\vec{\theta}^{(i)}$	collective pitch angle vector for i th Broyden trim iteration, deg
κ	loss factor
κ_{int}	interference loss factor
κ_{sep}	separation loss factor
$\kappa_{u l}$	rotor-on-rotor loss factor
λ	inflow ratio
ρ	density, kg m^{-3}
ψ	wake age, rad
Ω	rotor angular velocity, rad/s
ω	flow angular velocity, rad/s
FM	figure of merit

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Subscripts

c	climb
i	induced flow
l	lower rotor
tip	blade tip
u	upper rotor
0	far upstream
1	above upper rotor disk
2	below upper rotor disk
3	above lower rotor disk
4	below lower rotor disk
∞	far downstream

Introduction

A coaxial rotor has distinct advantages over a single rotor. A counterrotating rotor cancels out the yaw moment generated by each rotor so that the torque balance is met without a tail rotor. A removal of a tail rotor enables the design of the vehicle system to be simpler and provides additional space in the tail; additionally, tail rotors use 10–20% of the total engine power, which is not used for lift or thrust. Removing it saves power, and thus engine weight. Auxiliary propellers can often be installed in the tail of the coaxial rotorcraft, which provides an additional propulsive force and enables the high-speed flight of rotorcraft (Ref. 1). Coaxial rotorcraft are also more compact, improving maneuverability and allowing them to land in cramped conditions, such as flight decks on ships and aircraft carriers.

Recently, a lift-offset coaxial rotor has gained renewed interest in military rotorcraft operation. A lift-offset coaxial rotor is designed to provide additional lift on the advancing side of each rotor (Refs. 2, 3). This concept provides higher lift on the advancing side and lower lift on the retreating side. A reduced angle of attack on the retreating side helps mitigate dynamic stall and hence allows the rotorcraft to fly at a much faster speed compared to conventional helicopters without the issues of dynamic stall. Numerical studies on a lift-offset coaxial rotor can be found from Refs. 4–8.

Ramasamy (Ref. 9) recently investigated the performance of a model-scaled coaxial rotor for untwisted and twisted rotor blades. He presented the effect of axial separation distance on the rotor performance such as the thrust, torque, and figure of merit. This experimental research provides benchmark cases for the validation of numerical studies of a coaxial rotor in hover (Ref. 10). Bhagwat (Ref. 11) analyzed the comparison between a corotating coaxial rotor with a counterrotating coaxial rotor. He found that the swirl loss and recovery play a small role in the performance of a coaxial rotor.

Blade element momentum theory (BEMT) is one of the simplest numerical methods to analyze rotorcraft aerodynamics and performance in hover and axial flights. BEMT has been well validated for rotorcraft performance (Ref. 12), and it has been widely used as a preliminary design tool in the rotorcraft industry.

Breguet (Ref. 13) was among the first to analyze coaxial rotors using experimental data as early as 1936. Taylor (Ref. 14) conducted tests using balsa wood dust to visualize the coaxial rotor wake, noting that the tip vortices do not merge or cancel out. This information is useful in coaxial rotor prediction methods, especially free-vortex methods. In 1951, Harrington (Ref. 15) used Gessow's (Ref. 16) BEMT method to compare the existing theory with his experimental results on the Harrington 1 and Harrington 2 rotors. In his BEMT calculations, he modeled the coaxial rotor as a rotor with twice the rotor solidity of the single rotor and complete tip loss over the 3% tip portion of the blade. The method showed good results because the rotor separation was small (8–9.3%

Z/D). Dingeldein (Ref. 17) also used Gessow's BEMT method using the same assumptions for his experimental results, also showing good agreement with experimental results conducted with a similar separation distance of 9.5% Z/D .

Payne (Ref. 18) and Ginzel (Ref. 19) considered two coaxial rotors assumed to be on the same plane; Payne used momentum theory, and Ginzel used a modified blade element theory. Payne assumed that each rotor generates half of the total thrust. His approach resulted in significant overpredictions of rotor-induced power since the method did not consider the axial finite separation distance between two rotors. Leishman (Ref. 20) proposed a new BEMT approach for a coaxial rotor. The model accounts for the separation distance between two rotors and the influence from the upper rotor on the lower rotor. The induced inflow on the lower rotor disk is adjusted with the inflow of the upper rotor wake within the boundary in which the upper rotor tip vortex contraction meets on the lower rotor disk plane. The two main assumptions that Leishman used are (1) the contracted wake area of the upper rotor is known, such as in the ideal case in which the distance between the upper rotor and the lower rotor is infinite, and (2) the influence from the lower rotor on the upper rotor is neglected. Although these two assumptions make the analysis simpler, they do not properly represent a realistic coaxial rotor system. In particular, the rotor separation distance is relatively short for most coaxial rotors, so the effect of the separation distance should be considered in the rotor wake contraction, and the effect of the lower rotor on the upper rotor cannot be ignored as shown in Ramasamy's experimental study (Ref. 9).

Giovanetti and Hall (Ref. 21) included the influence of the lower rotor on the upper rotor using the Bio-Savart law and an empirical parameter. However, they assumed a uniform axial inflow induced by the lower rotor acting on the upper rotor. They approximated the upper rotor wake contraction as a function of vertical separation using this Biot-Savart law and the empirical constant. However, this equation provides the induced velocity measured at $r = 0$, which does not vary radially.

Juhasz et al. (Ref. 22) compared results for BEMT, free-vortex method (FVM), and computational fluid dynamics (CFD) for a coaxial rotor case. The BEMT method they used assumed single-rotor BEMT equations for climb on the upper rotor and in hover for the lower rotor. Like Giovanetti and Hall (Ref. 21), the interrotor influence was accounted for using a semiempirical Biot-Savart law, and the induced velocities on the opposite rotor were assumed to be uniform. The FVM used was the Maryland free wake method, and the CFD code used was OVERFLOW 2 using an overset grid. The BEMT was found to be the best predictor of thrust, and CFD was found to be the best predictor of torque. The error in the CFD results may have been due to the fact that the CFD was set up with the experimental collective pitch angles but not the experimental thrust. However, these results provide good validation for BEMT methods compared to other higher fidelity methods. The FVM had already been well validated for coaxial rotors by Lim et al. (Ref. 23).

In the current paper, we propose a new BEMT for a coaxial rotor. We remove the assumptions used in earlier models. We account for the actual finite axial separation distance in a coaxial rotor, include the mutual interaction between the upper rotor and the lower rotor, handle a nonuniform induced velocity, and derive new coaxial momentum equations. This new BEMT provides predictions of coaxial rotor performance as a function of axial separation distance. The new BEMT code is validated with two experiments, especially examining the aerodynamic performance as a function of the separation distance of a coaxial rotor in hover.

Iterative Blade Element Momentum Theory

In this section, an iterative BEMT approach that solves the upper and lower rotors simultaneously will be derived. The upper rotor solution

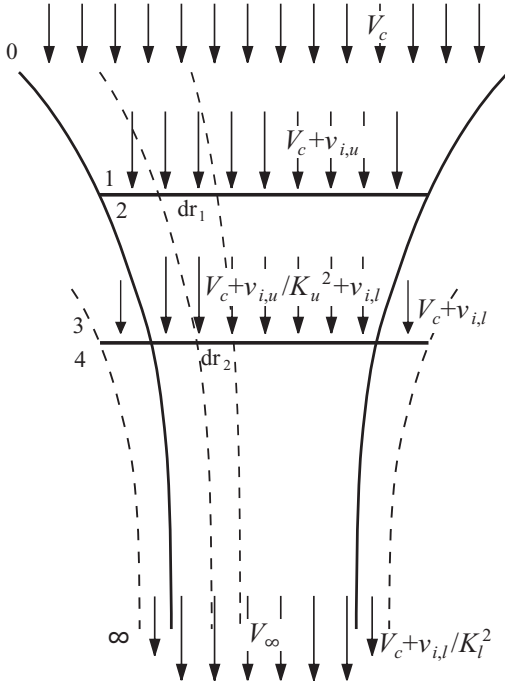


Fig. 1. Wake contraction for a coaxial rotor.

includes the influence of the lower rotor, and the lower rotor solution includes the influence of the upper rotor. The axial separation distance plays an important role in the rotor-to-rotor interaction. The BEMT is iterated until the performance of each rotor is converged. Two trim conditions are applied: torque balance and required thrust.

Figure 1 shows the control volume with two rotors. The climb velocity, V_c , is included for axial flight. For hover, V_c is zero. The Bernoulli equations can be written in each section:

$$P_0 + \frac{1}{2}\rho V_c^2 = P_1 + \frac{1}{2}\rho V_1^2 \quad (1)$$

$$P_2 + \frac{1}{2}\rho V_2^2 = P_3 + \frac{1}{2}\rho V_3^2 \quad (2)$$

$$P_4 + \frac{1}{2}\rho V_4^2 = P_\infty + \frac{1}{2}\rho V_\infty^2 \quad (3)$$

Note that $V_1 = V_2$, $V_3 = V_4$, and $P_0 = P_\infty$.

The sectional thrust in the streamtube for the upper rotor is written as

$$\frac{dT_u}{dA_u} = P_2 - P_1 = P_3 - P_0 + \frac{1}{2}\rho V_3^2 - \frac{1}{2}\rho V_c^2 \quad (4)$$

Using the following equation:

$$P_3 = P_4 - \frac{dT_l}{dA_l} = P_\infty + \frac{1}{2}\rho V_\infty^2 - \frac{1}{2}\rho V_4^2 - \frac{dT_l}{dA_l} \quad (5)$$

the sectional thrust becomes

$$\frac{dT_u}{dA_u} = -\frac{dT_l}{dA_l} + \frac{1}{2}\rho V_\infty^2 - \frac{1}{2}\rho V_c^2 \quad (6)$$

or

$$dT_u = dA_u \left[-\frac{dT_l}{dA_l} + \frac{1}{2}\rho V_\infty^2 - \frac{1}{2}\rho V_c^2 \right] \quad (7)$$

Finally, we can find that

$$\frac{dT_u}{dA_u} + \frac{dT_l}{dA_l} = \frac{\rho}{2} [V_\infty^2 - V_c^2] \quad (8)$$

Equation (8) simply represents that the change in the dynamic pressure between the outlet and inlet is equivalent to the total force per unit area acting on the streamtube from two rotors.

Let us define y as the radial distance of a streamtube at the tip, and $K_t = y_\infty/y_1$ is the contraction ratio from 1 to ∞ in the presence of the lower rotor, $K_u = y_3/y_1$ is the contraction ratio from the upper rotor to the lower rotor, and $K_l = y_\infty/y_3$ is the contraction ratio from 3 to ∞ .

The far downstream velocity within the slipstream of the upper rotor becomes

$$V_\infty = V_c + \frac{v_{i,u}}{K_t^2} + \frac{v_{i,l}}{K_l^2} \quad (9)$$

where $v_{i,u}$ and $v_{i,l}$ are the induced velocities by the upper and lower rotors at each rotor plane, as shown in Fig. 1. Outside the slipstream of the upper rotor, $v_{i,u}$ becomes zero.

Then, Eq. (8) becomes

$$\frac{dT_u}{dA_u} + \frac{dT_l}{dA_l} = \frac{\rho}{2} \left[\left(V_c + \frac{v_{i,u}}{K_t^2} + \frac{v_{i,l}}{K_l^2} \right)^2 - V_c^2 \right] \quad (10)$$

or

$$\begin{aligned} \frac{dC_{T_u}}{2\pi y_u dy_u} + \frac{dC_{T_l}}{2\pi y_l dy_l K_u^2} \\ = \frac{\rho}{2\rho(\pi R^2)(\Omega R)^2} \left[\left(V_c + \frac{v_{i,u}}{K_t^2} + \frac{v_{i,l}}{K_l^2} \right)^2 - V_c^2 \right] \end{aligned} \quad (11)$$

Note that $y_l dy_l = y_u dy_u K_u^2$. Rearranging the above equation yields

$$V_c^2 + \frac{\Omega^2 R^4}{y_u dy_u} \left(dC_{T_u} + \frac{dC_{T_l}}{K_u^2} \right) = \left(V_c + \frac{v_{i,u}}{K_t^2} + \frac{v_{i,l}}{K_l^2} \right)^2 \quad (12)$$

The solution for $v_{i,u}$ can be found by

$$v_{i,u} = K_t^2 \left[\sqrt{V_c^2 + \frac{\Omega^2 R^4}{y_u dy_u} \left(dC_{T_u} + \frac{dC_{T_l}}{K_u^2} \right)} - V_c - \frac{v_{i,l}}{K_l^2} \right] \quad (13)$$

or

$$v_{i,u} = K_t^2 \sqrt{V_c^2 + \frac{\Omega^2 R^4}{y_u dy_u} \left(dC_{T_u} + \frac{dC_{T_l}}{K_u^2} \right)} - K_t^2 V_c - K_u^2 v_{i,l} \quad (14)$$

The nondimensional induced velocity, or inflow ratio for the upper rotor, can be written as

$$\lambda_{i,u} = K_t^2 \sqrt{\lambda_c^2 + \frac{1}{r_u dr_u} \left(dC_{T_u} + \frac{dC_{T_l}}{K_u^2} \right)} - K_t^2 \lambda_c - K_u^2 \lambda_{i,l} \quad (15)$$

where r represents the nondimensionalized radial distance.

The nondimensional induced velocity on the lower rotor can be similarly obtained by

$$\lambda_{i,l} = K_l^2 \sqrt{\lambda_c^2 + \frac{1}{r_l dr_l} (K_u^2 dC_{T_u} + dC_{T_l})} - K_l^2 \lambda_c - \frac{\lambda_{i,u}}{K_u^2} \quad (16)$$

The total inflow ratios can be defined as

$$\begin{aligned} \lambda_u &= \lambda_c + \lambda_{i,u} \\ \lambda_l &= \begin{cases} \lambda_c + \lambda_{i,l} + \lambda_{i,u}/K_u^2 & \text{for } r_l \leq K_u r_{u,\text{tip}} \\ \lambda_c + \lambda_{i,l} & \text{for } r_l > K_u r_{u,\text{tip}} \end{cases} \end{aligned} \quad (17)$$

where $r_{u,\text{tip}}$ is the nondimensionalized tip radius of the upper rotor.

Finally, adding the Prandtl's tip loss effect with the F function (Ref. 12) and two empirical constants, B_u and B_l , results in

$$\lambda_u = K_t^2 \sqrt{\lambda_c^2 + \frac{1}{r_u dr_u} \left(\frac{dC_{T_u}}{F} + \frac{dC_{T_l}}{K_u^2} \right)} + (1 - K_t^2) \lambda_c - K_u^2 \lambda_{i,l} + B_u \lambda_{i,l} \quad (18)$$

and

$$\lambda_l = \begin{cases} K_l^2 \sqrt{\lambda_c^2 + \frac{1}{r_l dr_l} \left(K_u^2 dC_{T_u} + \frac{dC_{T_l}}{F} \right)} + (1 - K_l^2) \lambda_c - B_l \lambda_{i,u} & \text{for } r_l \leq K_u r_{u,\text{tip}} \\ K_l^2 \sqrt{\lambda_c^2 + \frac{1}{r_l dr_l} \left(\frac{dC_{T_l}}{F} \right)} + (1 - K_l^2) \lambda_c & \text{for } r_l > K_u r_{u,\text{tip}} \end{cases} \quad (19)$$

Empirical constant, B_u , in Eq. (18) has been added to include the effect of the lower rotor on the upper rotor. This constant is multiplied by $\lambda_{i,l}$ so that the effect of the lower rotor depends on the magnitude of the induced velocity of the lower rotor. B_u was added for two reasons: to account for the portion of the lower rotor induced velocity that goes through the upper rotor and to relax the equations in order for the iteration to converge to a sensible solution. Empirical constant B_l in Eq. (19) has been added to augment the effect of the upper rotor on the lower rotor when the area is within the influence zone of the upper rotor wake. This empirical constant is multiplied by the induced velocity of the upper rotor. Since the induced velocity of the upper rotor is a function of the radial and separation distances, the effect of the upper rotor on the lower rotor is also a function of the radial and separation distances. B_l was added to improve the trends of the results. Later, it is found that B_u of 0.5 and B_l of 0.12 provide the closest agreement with the measurement data. These values are used in all subsequent figures with coaxial BEMT curves. Our single-rotor BEMT calculations are performed using standard equations, which do not depend on these coefficients.

Equations (18) and (19) can be solved by the BEMT for each rotor. However, it is worthwhile noting that the upper rotor inflow solution is influenced by the lower rotor thrust and lower rotor inflow in Eq. (18). Similarly, the lower rotor inflow is affected by the upper rotor thrust and upper rotor inflow in Eq. (19). Therefore, an iterative approach is required to solve these coupled equations. For example, for the upper rotor BEMT calculation, the lower rotor thrust and the lower rotor inflow ratio are known a priori from an earlier iteration or the initial assumption of zero for both. Then, the upper rotor BEMT determines the upper rotor thrust and the upper rotor inflow ratio. For the lower rotor BEMT calculation, the upper rotor thrust and the upper rotor inflow ratio are given from an earlier iteration or the initial assumption of zero for both. Then, the lower rotor BEMT determines the lower rotor thrust and the lower rotor inflow ratio. The rotor-to-rotor iteration is continued until the thrusts and the inflow ratios for both rotors fully converge.

Since a streamtube contracts from the upper rotor to the lower rotor, the radial locations following the streamtube are not the same for both rotors. Furthermore, the influenced radial locations may not be on the designated control points, at which the aerodynamic data are defined. Therefore, a numerical interpolation is required on each rotor blade when coupling the upper rotor BEMT and lower rotor BEMT.

Blade element theory

The momentum equations above were solved by coupling them with the conventional blade element theory (BET) equation, given below in Eq. (20) (Ref. 12):

$$dC_T = \frac{1}{2} \sigma c_l r^2 dr \quad (20)$$

A similar form of Eq. (20) can be used to find the profile power along the blade:

$$dC_{P_d} = \frac{1}{2} \sigma c_d r^3 dr \quad (21)$$

BET equations (20) and (21) above are the same used in conventional BEMT methods. This is because the airfoil equations are the same in coaxial and single-rotor cases. The lift and drag coefficients above were found using the following empirical equations:

$$c_l = C_{l_\alpha} (\alpha - \alpha_0) \quad (22)$$

$$c_d = d_0 + d_1 \alpha + d_2 \alpha^2 \quad (23)$$

Since all the rotors we investigated had symmetric airfoils, α_0 is zero. C_{l_α} is equal to 2π according to thin airfoil theory; in reality, it is slightly lower. Since the results were not very sensitive to C_{l_α} , a value of 5.8 rad^{-1} was chosen for all rotors. Empirical coefficients d_0 , d_1 , and d_2 were fit to single-rotor experimental results. Since the drag coefficient is heavily dependent on factors such as Reynold's number and surface roughness, d_0 and d_2 were found to fit the single-rotor experimental results. Since the airfoils were symmetric, d_1 is zero and d_0 is the zero-lift drag coefficient, found using the zero-thrust results. The same coefficients were then used for the coaxial cases. For simplicity, the airfoil shape was assumed to be constant along the blade. Taper effects were taken into account.

The effects of swirl were also included for each individual rotor. However, the effect of the upper rotor swirl on the lower rotor was not included since it was found to have very little effect. Equation (24) gives the flow rotational velocity across the blade for each rotor. The flow rotation would slightly decrease the horizontal flow component across the airfoil, thus slightly decreasing the thrust.

$$\omega = \Omega - \sqrt{\Omega^2 - \frac{4\Omega^2 \lambda \lambda_i}{r^2}} \quad (24)$$

Prescribed wake contraction

The contraction ratio from the upper rotor to the lower rotor, K_u , depends on the axial separation distance. To include the effect of the axial separation distance, we use a prescribed wake contraction. This assumes that the coaxial wake contraction is equal to the single-rotor wake contraction; although this is a large assumption, it is a necessary one to include the rotor separation distance in the model. Empirical constants B_u and B_l were fit to experimental data and thus also correct error caused by this assumption. Gray (Ref. 24) first recognized the fundamental shape of the wake of a hovering rotor and developed the form of its equation. Landgrebe (Ref. 25) developed its parameters, proposing the following prescribed wake model:

$$\frac{z_{\text{tip}}}{R} = \begin{cases} k_1 \psi_\omega & \text{for } 0 \leq \psi_\omega \leq 2\pi/N_b \\ \left(\frac{z_{\text{tip}}}{R} \right)_{\psi_\omega=2\pi/N_b} + k_2 \left(\psi_\omega - \frac{2\pi}{N_b} \right) & \text{for } \psi_\omega \geq 2\pi/N_b \end{cases} \quad (25)$$

$$\frac{y_{\text{tip}}}{R} = r_{\text{tip}} = A + (1 - A) \exp(-\Lambda \psi_\omega) \quad (26)$$

where ψ_ω is the wake age, $k_1 = -0.25(C_T/\sigma + 0.001\theta_{tw})$, $k_2 = -(1.41 + 0.0141\theta_{tw})\sqrt{C_T/2}$, $A = 0.78$, and $\Lambda = 0.145 + 27C_T$. For our purpose, the wake age is determined from a fixed axial separation distance, z_{tip} , of interest from Eq. (25). From the determined wake age, the contracted radius, y_{tip} , is determined from Eq. (26). It should be noted that C_T is included in the parameter k_1 so that the wake contraction is a function of the upper rotor thrust. Note that the upper thrust is also affected by the wake contraction in Eq. (18). Therefore, the computation of the prescribed wake contraction is fully included in the iterative BEMT process. This prescribed wake model provides the contraction ratio of

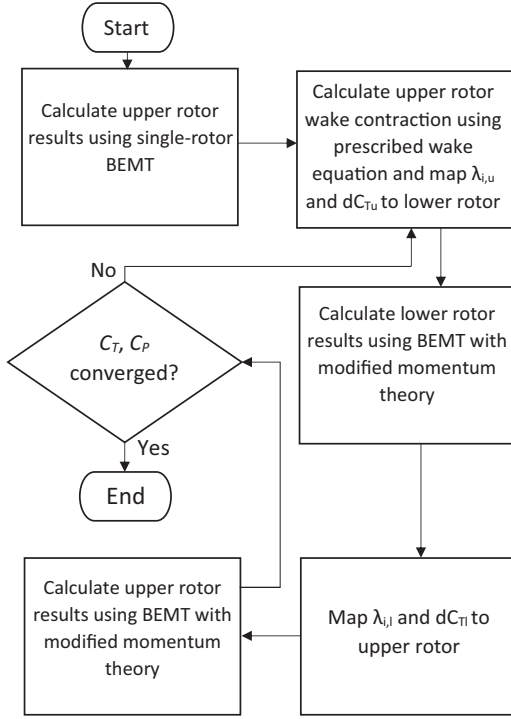


Fig. 2. Flow chart of the iterative BEMT method

0.78 at the far downstream, which is larger than for the ideal case, which is 0.707, and consistent with the measured one. For this reason, we use K_I of 0.78 as well. However, K_I includes the double wake contraction effects: the wake contraction from the upper rotor to the lower rotor and then from the lower rotor to the far downstream so that $K_I = K_u \times K_l$. In this paper, we assume the same wake contraction from the root to the tip.

A flow chart of the iterative BEMT incorporating the prescribed wake is shown in Fig. 2.

Trim process

In the coaxial rotor, two trim conditions are considered. The first one is the balanced torque or $C_{Qu} = C_{Ql}$, where C_{Qu} is the torque of the upper rotor and C_{Ql} is the torque of the lower rotor. The second trim condition is the required total thrust or $C_{Tu} + C_{Tl} = C_{Treq}$, where C_{Tu} and C_{Tl} are the thrust of the upper and lower rotors, respectively, and C_{Treq} is the required thrust. We use Broyden's method to find the collective pitch angles (θ_u and θ_l) of the upper and lower rotors to meet these trim conditions.

The trim solution process is described as follows.

- (1) Let $\vec{\theta}^{(0)} = [\theta_u^{(0)}, \theta_l^{(0)}]$, $F_1(\theta_u, \theta_l) = C_{Qu} - C_{Ql}$, $F_2(\theta_u, \theta_l) = C_{Tu} + C_{Tl} - C_{Treq}$, and $\vec{F} = [F_1, F_2]$. The method finds the roots of $\vec{F}$.
- (2) Calculate $F_1(\vec{\theta}^{(0)})$ and $F_2(\vec{\theta}^{(0)})$ by solving the iterative BEMT.
- (3) Calculate $A_0^{-1} = J(\vec{\theta}^{(0)})^{-1}$, where J is the Jacobian matrix as follows:

$$\begin{bmatrix} \frac{\partial F_1(\vec{\theta}^{(0)})}{\partial \theta_u} & \frac{\partial F_1(\vec{\theta}^{(0)})}{\partial \theta_l} \\ \frac{\partial F_2(\vec{\theta}^{(0)})}{\partial \theta_u} & \frac{\partial F_2(\vec{\theta}^{(0)})}{\partial \theta_l} \end{bmatrix} \quad (27)$$

The numerical differentiation is performed to find the Jacobian matrix.

- (4) Calculate $\vec{\theta}^{(1)} = \vec{\theta}^{(0)} - A_0^{-1} F(\vec{\theta}^{(0)})$.
- (5) Calculate the iterative BEMT to find $F(\vec{\theta}^{(1)})$.
- (6) Calculate $\vec{y}_1 = F(\vec{\theta}^{(1)}) - F(\vec{\theta}^{(0)})$.
- (7) Calculate $\vec{s}_1 = \vec{\theta}^{(1)} - \vec{\theta}^{(0)}$.

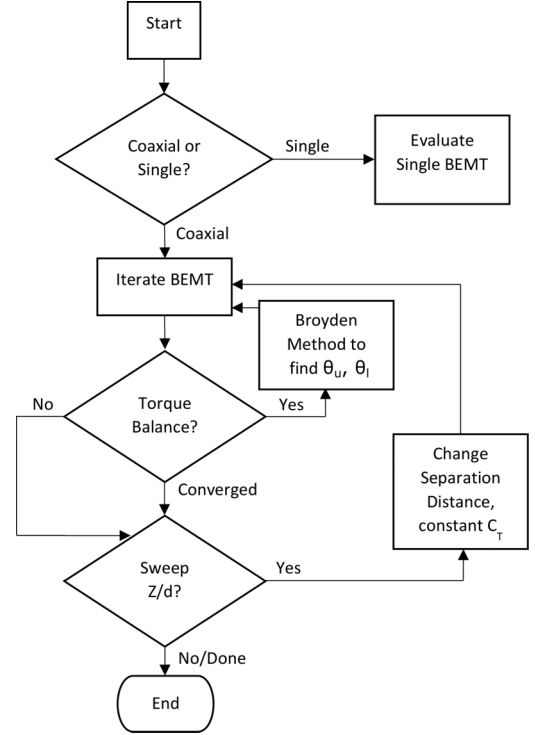


Fig. 3. Flow chart of the method

- (8) Calculate $A_1^{-1} = A_0^{-1} + \frac{1}{\vec{s}_1^T A_0^{-1} \vec{y}_1} [(\vec{s}_1 - A_0^{-1} \vec{y}_1) \vec{s}_1^T A_0^{-1}]$.
- (9) Calculate $\vec{\theta}^{(2)} = \vec{\theta}^{(1)} - A_1^{-1} F(\vec{\theta}^{(1)})$.

- (10) Repeat steps 4–9 until the L2 norm error is below the convergence criteria. The resulting θ_u and θ_l are in a torque-balanced condition.

Figure 3 shows a flowchart of the overall numerical process, which includes the iterative BEMT, torque balance, and sweep of the separation distance. The *iterative BEMT* in the flow chart indicates the flow chart shown in Fig. 2.

Results

In this section, a standard single-rotor BEMT is first validated for an isolated rotor to find the lift slope and drag polar coefficients for a selected rotor, and then the proposed iterative BEMT is validated against the measurement data of the coaxial rotor performance (Refs. 9, 15). In particular, the effect of the axial separation distance on the upper rotor, lower rotor, and coaxial rotor performance is predicted by the BEMT for the first time. An untwisted rotor blade and two thrust conditions are considered for these detailed analyses. The separation loss, interference loss, and rotor-to-rotor influence loss in a coaxial system are also studied in comparison with a single rotor. The local distributions of the thrust, inflow, and angle of attack are analyzed at different axial distances. For all subsequent coaxial results, empirical values of $B_u = 0.5$ and $B_l = 0.12$ are used.

Validation case 1

In this subsection, the iterative BEMT predictions are validated with experimental data from Harrington's two rotor cases (Ref. 15). The rotor has two blades, and the rotor radius is 12.5 ft. For rotor 1 the rotor blade is tapered with a taper ratio of 0.39, and for rotor 2 the blade is not tapered. The rotor-to-rotor separation distances for both rotors are 8% and 9.3% of the rotor diameter, respectively. The lift and drag polar coefficients are

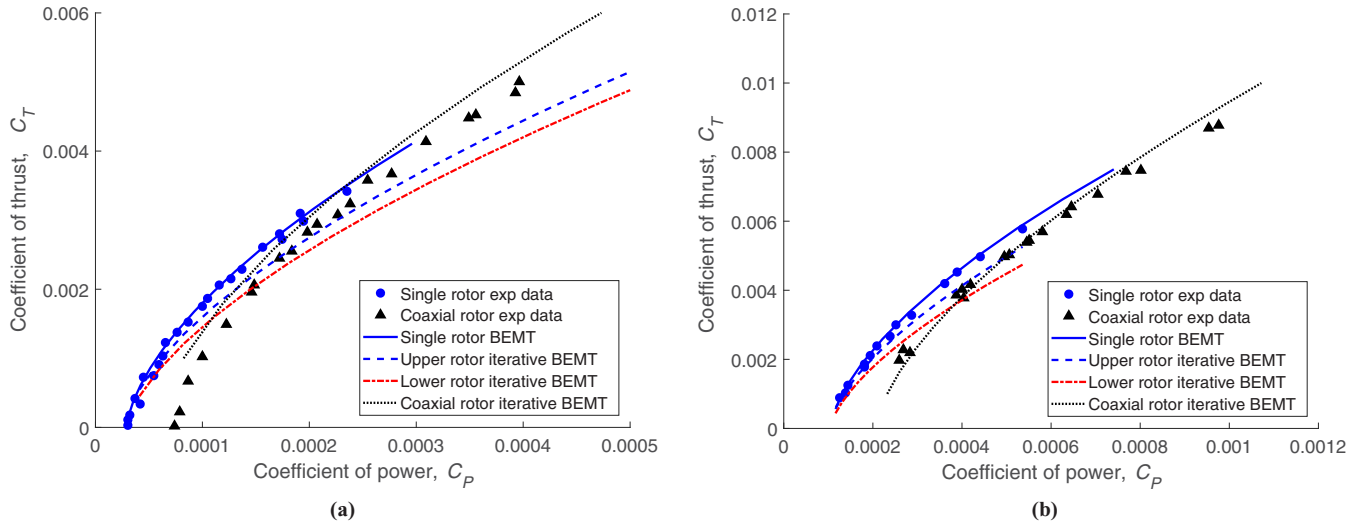


Fig. 4. Coaxial rotor BEMT validation for the power and thrust coefficients: (a) Harrington rotor 1 and (b) Harrington rotor 2 (data source: Ref. 15).

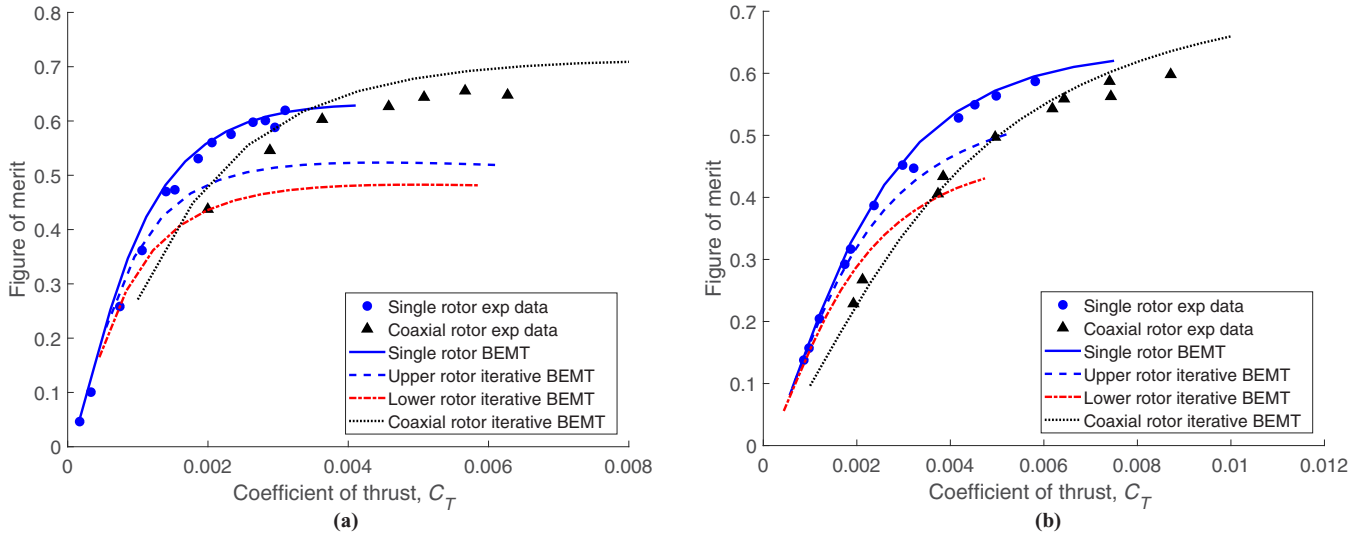


Fig. 5. Coaxial rotor BEMT validation for figure of merit: (a) Harrington rotor 1 and (b) Harrington rotor 2 (data source: Ref. 15).

obtained from the single-rotor experiment validation, which is not shown here. The coefficients are $d_0 = 0.0075$ and $d_2 = 0.8$ for Harrington 1 and $d_0 = 0.0110$ and $d_2 = 0.65$ for Harrington 2.

Figure 4 shows the thrust to power coefficients for the two rotor cases. The predictions are compared to the measured data for the single rotor and the coaxial rotor. It is shown that the predictions are in good agreement with the data for both cases. The coaxial rotor predictions are more accurate for rotor 2 (straight blade) than for rotor 1 (tapered blade). The upper and lower rotor predictions, which were obtained from the iterative BEMT for a coaxial rotor, are also included in the plot. For each rotor case, it is shown that the upper rotor generates more thrust than the lower rotor at the same power. In addition, the coaxial rotor generates more thrust than each individual rotor at the same power.

Figure 5 shows the figure of merit for both coaxial rotors. The traditional coaxial rotor figure of merit used in subsequent figures is calculated as follows (Ref. 23):

$$FM_{\text{coax}} = \frac{(C_{T,u} + C_{T,l})^{3/2}}{\sqrt{2}(C_{P,u} + C_{P,l})} \quad (28)$$

It is found that the single-rotor BEMT accurately predicts the figure of merit of the single rotor. In coaxial rotor configurations, the predicted figure of merit for both the upper and lower rotors obtained by the iterative BEMT is lower than that of the single rotor. However, the upper rotor shows a higher figure of merit than the lower rotor. This is consistent with Fig. 4. The figure of merit for the coaxial rotor, as calculated in Eq. (28), shows good predictions compared to the measurement data. The prediction is slightly more accurate for Harrington rotor 2 (straight blade) than for Harrington rotor 1 (tapered blade), as shown in Fig. 5.

Validation case 2

Although the Harrington's rotor cases were useful to validate the proposed iterative BEMT at a fixed separation distance, there were not enough data to predict performance behaviors as a function of the axial separation distance. In this subsection, Ramasamy's coaxial rotor experimental data (Ref. 9) are used to validate the new iterative BEMT as a function of the axial separation distance. Ramasamy's experiment

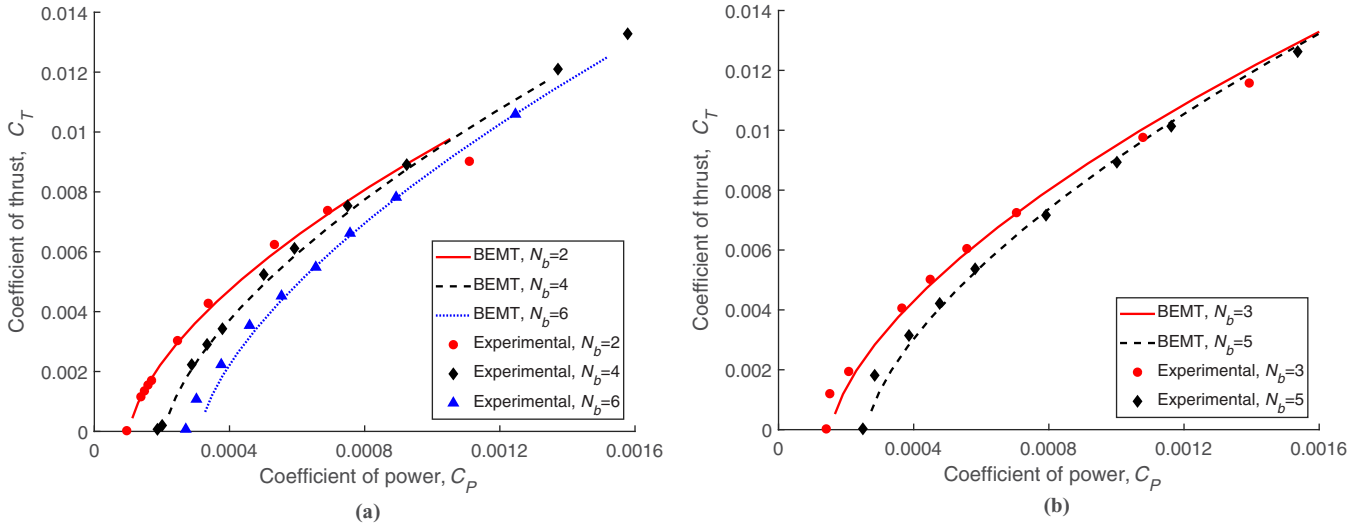


Fig. 6. Ramasamy untwisted single-rotor BEMT validation for the power and thrust coefficients: (a) the number of blades: 2, 4, 6 and (b) the number of blades: 3, 5 (data source: Ref. 9).

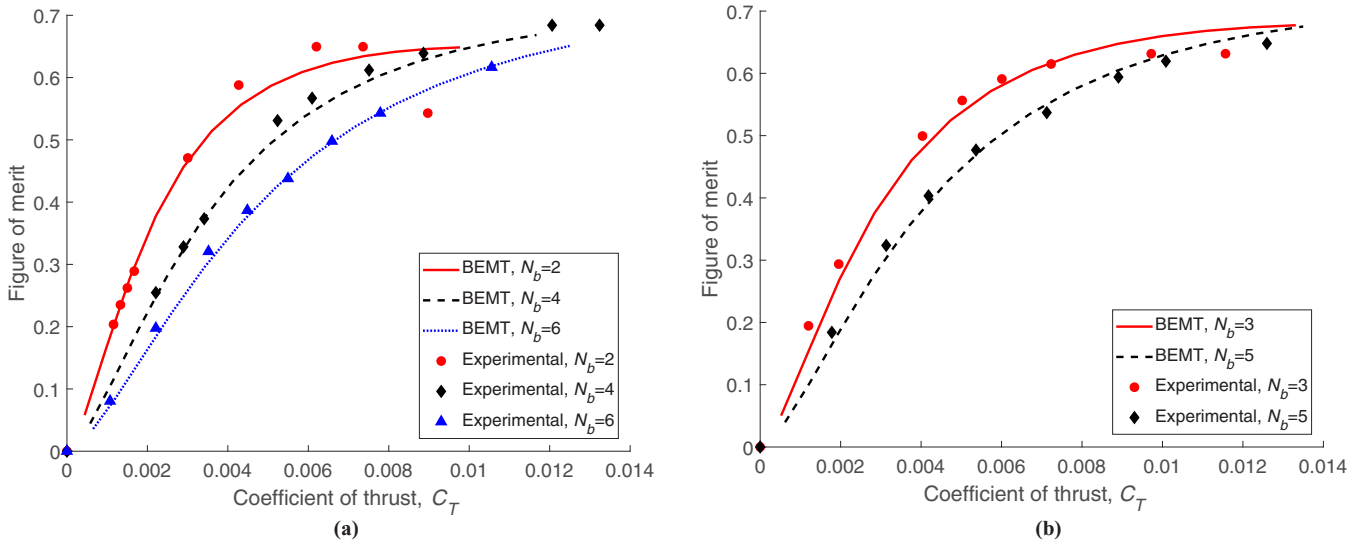


Fig. 7. Ramasamy untwisted single-rotor BEMT validation for figure of merit: (a) the number of blades: 2, 4, 6 and (b) the number of blades: 3, 5 (data source: Ref. 9).

includes comprehensive and detailed performance results for a model-scaled coaxial rotor in hover.

The coaxial rotor of our choice for the validation has three blades on each rotor. The blade uses a NACA0012 untwisted airfoil. The blade radius is 2.17 ft, and the chord length is 2.29 inches. The root cutout is 19.1%. The rotor RPM is 800, and the tip Mach number is 0.25. The Reynolds number at the tip is 325,000.

Figures 6 and 7 show comparisons of BEMT predictions and measurement data for the power-thrust coefficients and the figure of merit for an isolated rotor with various number of blades. The coefficients for the drag polar are $d_0 = 0.015$ and $d_2 = 0.4$.

It is shown that the BEMT agrees very well with the measurement data, except for two blades at high thrust, in which the figure of merit decreases rapidly due to blade stall in the measurement while the prediction does not capture this stall behavior. This single-rotor validation provides a slope of the lift coefficient and drag polar coefficients as a function of angle of attack, which are used for the coaxial rotor BEMT.

Figure 8 shows a convergence history of the thrust for the upper and lower rotors for two total thrust values ($C_T = 0.007$ and 0.014) at two separation distances ($Z/D = 0.08$ and 1.0). The L2 norm of $1E-4$ is used for the convergence criterion of the trim solution. The maximum errors of $7E-10$ and $7E-11$ are used for the convergence of C_T and C_P , respectively, for the iterative BEMT. In other words, the iterative BEMT is performed until both the error thresholds of C_T and C_P are met. In Fig. 8, a large change in the mean value initially occurs due to the adjustment of the trim solutions. Within one setting of the trim solution, the solution is converged through the iterative BEMT. For a larger thrust value, the convergence takes more iterations than for a smaller thrust value. The trim solution is converged in about five iterations, and the iterative BEMT is converged in about 10–20 iterations. For a smaller separation distance, the convergence takes more iterations than for a larger separation distance. For the number of iterations of 100 or 150, it only takes a couple of seconds on a desktop computer.

Figure 9 shows the sensitivity of the figure of merit of the empirical constants B_u and B_l . Measured at the point of most change, a 20% change

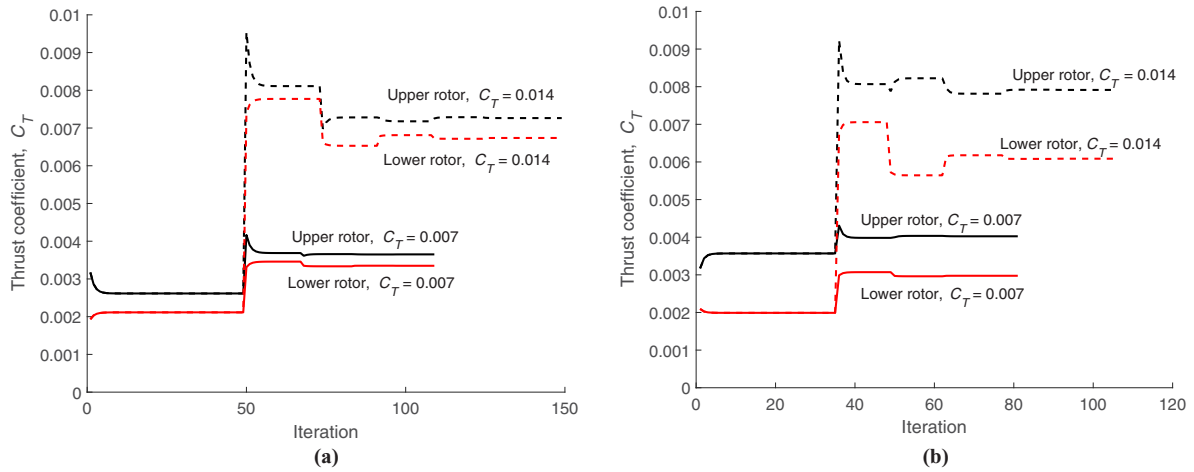


Fig. 8. Convergence history of individual rotor thrust for iterative BEMT and trim solution at high and low total thrust conditions of $C_T = 0.007$ and $C_T = 0.014$: (a) $Z/D = 0.08$ and (b) $Z/D = 1.0$.

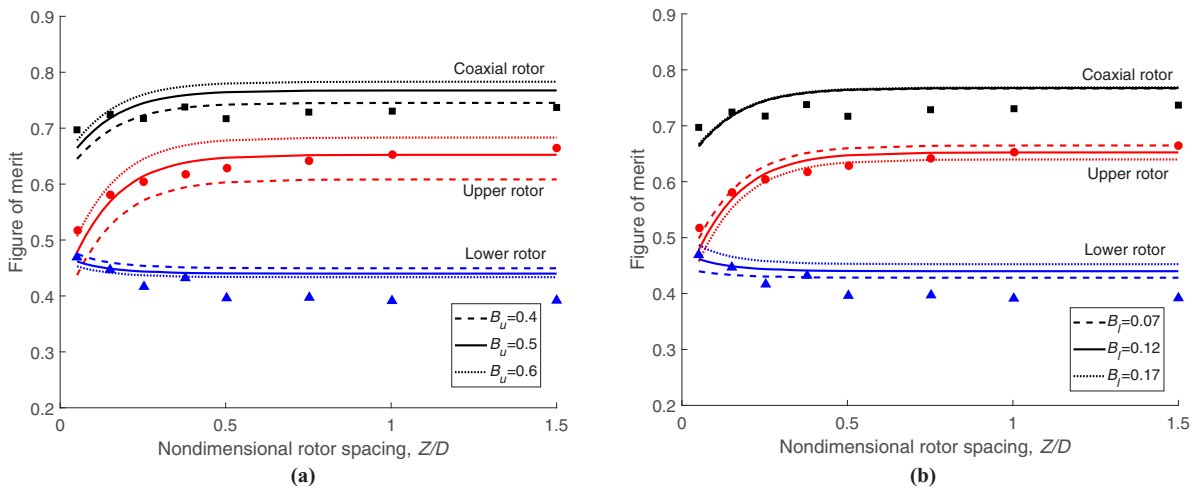


Fig. 9. Sensitivity of figure of merit to empirical constants B_u and B_l compared to Ramasamy untwisted coaxial rotor data (data source: Ref. 9).

in B_u causes a 5% change in the figure of merit, and a 41.7% change in B_l causes a 6.5% change in the figure of merit. B_u affects the upper rotor the most, and B_l changes the shape of the FM curves at small separation distances. The BEMT curves were fit to the experimental results by tuning the empirical coefficients to find the best values.

Figure 10 shows the effect of the axial separation distance on the figure of merit for the total coaxial thrust coefficient of 0.014. Both the prediction and the measurement data are included in the figure. It should be noted that Ramasamy (Ref. 9) used a modified definition of figure of merit in his paper that was presented by Leishman and Syal (Ref. 26). We have converted the figure of merit to the traditional definition for consistency with the earlier validation of the Harrington rotor. It is found that the upper rotor shows a decrease in the figure of merit as the separation distance approaches zero. For the lower rotor, the figure of merit is much lower than that of the upper rotor, and it remains constant for most of the separation distances except near zero separation distance in which it slightly increases. The iterative BEMT prediction captures the trend of the figure of merit for the upper rotor and lower rotor as a function of the separation distance. For the upper rotor the prediction is in good agreement, but for the lower rotor the prediction is overpredicted by about 10%. Both the measured and predicted figures

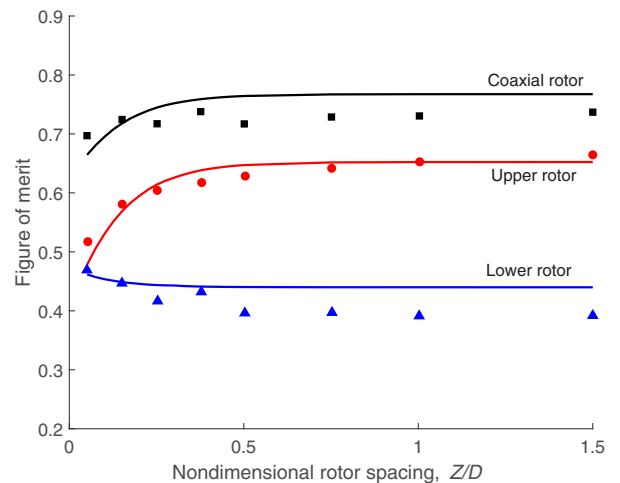


Fig. 10. Ramasamy untwisted coaxial rotor figure of merit as a function of the coaxial rotor spacing for $C_T = 0.014$ (data source: Ref. 9).

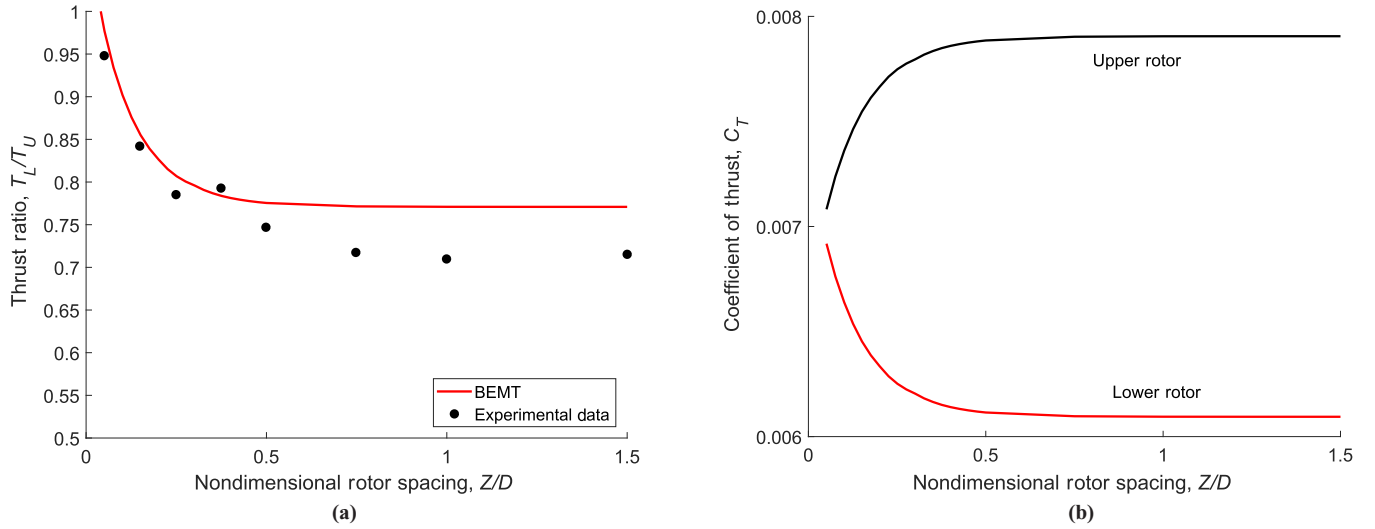


Fig. 11. Ramasamy untwisted coaxial rotor thrust as a function of the rotor spacing for $C_T = 0.014$: (a) thrust ratio and (b) thrust coefficient for the upper and lower rotors (data source: Ref. 9).

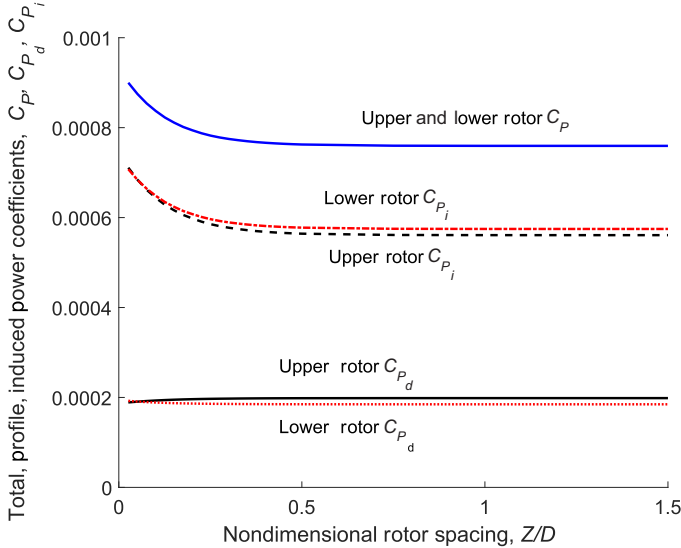


Fig. 12. Ramasamy untwisted coaxial rotor power coefficient as a function of the rotor spacing for $C_T = 0.014$.

of merit for the upper rotor converges for the isolated rotor for a large separation distance, or Z/D greater than 0.5.

Figure 11 shows the rotor thrust as a function of the rotor separation distance. For the separation distance larger than 0.5, the lower rotor carries only about 70% of the upper rotor thrust. As the separation distance decreases, the lower rotor thrust increases and approaches 100% at near zero distance. The prediction shows about 10% higher thrust ratio for the lower rotor at a large separation distance. However, overall the iterative BEMT accurately predicts this thrust ratio trend.

Figure 12 shows the power coefficients as a function of the rotor separation distance. It is shown that the profile powers remain constant for both rotors over the entire axial distance range. However, the induced powers increase as the rotor separation distance decreases since the rotor-to-rotor interference increases. The total power is the same for the upper and lower rotors due to the imposed torque-balanced trim condition.

Figure 13 shows the collective pitch angles for the upper and lower

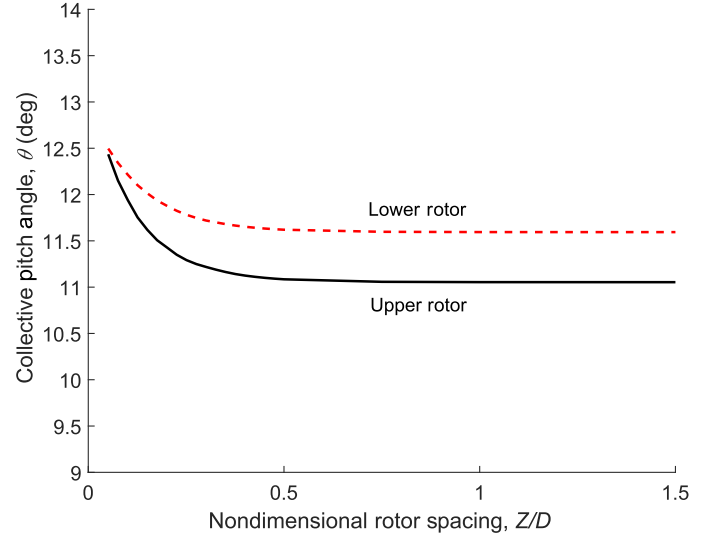


Fig. 13. Ramasamy untwisted coaxial rotor collective pitch angle as a function of the rotor spacing for $C_T = 0.014$.

rotors as a function of the separation distance. It is shown that the lower rotor needs about a one degree higher collective pitch angle than the upper rotor beyond Z/D greater than 0.5 to meet the trim condition. This is attributed to the fact that the lower rotor experiences the higher induced velocity due to the influence of the upper rotor wake. As the separation distance decreases, two collective pitch angles increase and reach the same value to compensate the loss of the thrust due to higher induced velocities on the both rotors at a short separation distance.

Figure 14 shows two coaxial rotor loss factors. The first loss is referred to as the separation loss factor, which is denoted as κ_{sep} . This loss is the ratio of the coaxial rotor induced power to the single-rotor induced power with the same solidity as the coaxial rotor for the same thrust, or a single rotor with six blades in this example: $\kappa_{sep} = \frac{P_{L,coax}}{P_{L,single,N_b=6}}$. As can be seen in Fig. 14(a), this loss factor is less than 1.0, or the coaxial rotor is more efficient than the single rotor with the same solidity at a large separation distance. Ramasamy (Ref. 9) attributed the higher performance of the coaxial rotor to the swirl recovery and the axial

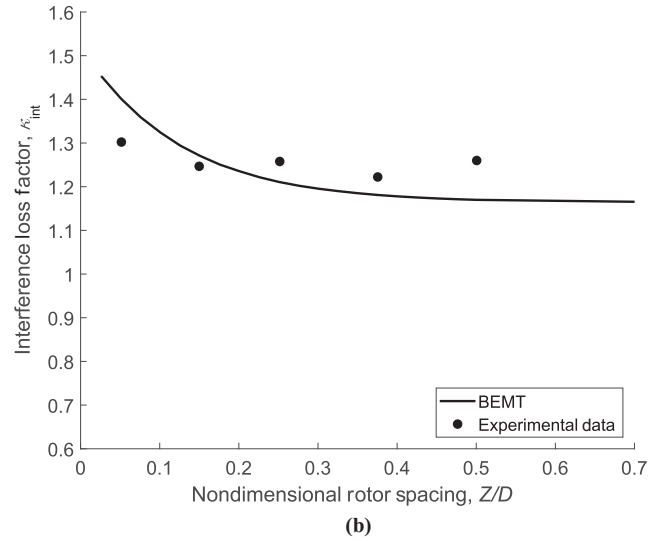
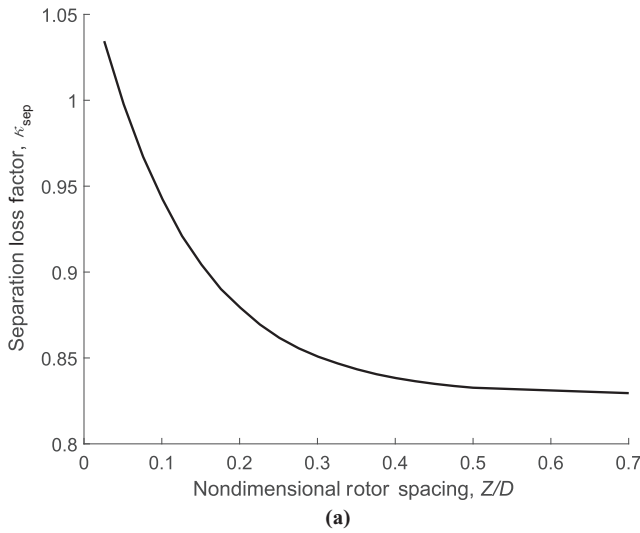


Fig. 14. Ramasamy untwisted coaxial rotor loss factors as a function of the rotor spacing for $C_T = 0.014$: (a) separation loss factor and (b) interference loss factor (data source: Ref. 9).

separation effects. However, Bhagwat (Ref. 11) discussed that the swirl recovery is negligible in the coaxial performance gain. Leishman and Syal (Ref. 26) used a simple momentum theory to show that a six-bladed rotor requires more power than two three-bladed coaxial rotors with the lower rotor operating in the fully contracted wake of the upper rotor. Our results are consistent with the earlier research. As the separation distance decreases, κ_{sep} approaches one. At the rotor plane, this value should be exactly 1.0, but Fig. 14(a) shows a slightly higher value than 1.0 in the rotor plane. This is because the single-rotor BEMT was conducted with an analytical wake contraction of 0.707, whereas the coaxial BEMT was conducted with a prescribed wake contraction of 0.78. The interference loss, which is denoted as κ_{int} , is shown in Fig. 14(b). This loss represents the increase in the coaxial rotor power with separation compared to two three-bladed rotors in one plane: $\kappa_{int} = \frac{P_{i,coax}}{2P_{i,single,N_b=3}}$. It is shown that 20–30% higher power is needed for a coaxial rotor. As the rotor distance decreases, the interference loss increases since the rotor-to-rotor effect increases. The prediction matches well with the measurement data.

Figure 15 shows the rotor-to-rotor interference factors as a function of the rotor separation distance. The upper or lower rotor power can be written as

$$P_{u/l} = P_d + \kappa_{u/l} \frac{T_{u/l}^{3/2}}{\sqrt{2\rho A}} \quad (29)$$

where P_d is the profile power, calculated by integrating the total drag along the blade. Although P_d is commonly used as a constant, defined by the zero-collective pitch angle power, in this case it is recalculated every time to more accurately separate profile and induced power. $T_{u/l}$ denotes the thrust on the upper or lower rotor, κ represents the single-rotor induced power loss factor, and $\kappa_{u/l}$ represents the rotor-to-rotor interference factor for the upper or lower rotor. In other words, κ_u represents the induced power loss on the upper rotor due to the effect of the lower rotor and κ_l represents the induced power loss on the lower rotor due to the effect of the upper rotor. These loss factors can be obtained by

$$\kappa_{u/l} = K_{u/l} \frac{(C_{Treq}/2)^{3/2}}{(C_{T_{u/l}})^{3/2}} \quad (30)$$

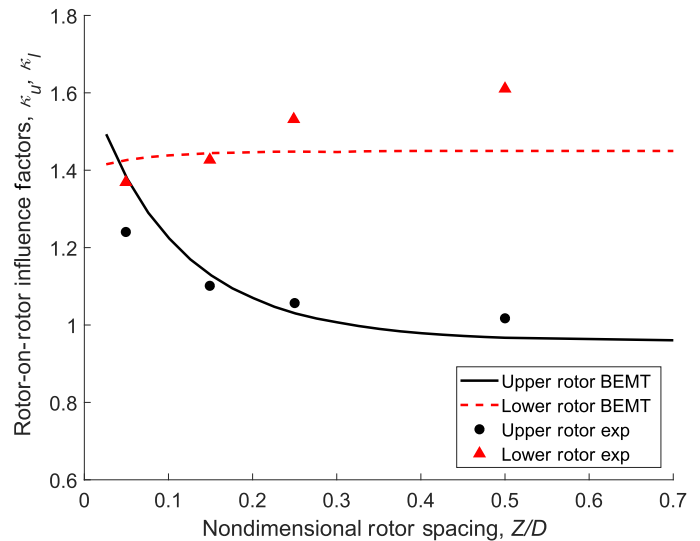


Fig. 15. Ramasamy untwisted coaxial rotor-to-rotor influence factors as a function of the rotor spacing for $C_T = 0.014$ (data source: Ref. 9).

where C_{Treq} is the required total thrust and $K_{u/l}$ is given as

$$K_{u/l} = \frac{P_{i,u/l}}{P_{i,single}} \quad (31)$$

where $P_{i,u/l}$ is the induced power on the upper or lower rotor and $P_{i,single}$ is the induced power of a single rotor at $C_T = \frac{C_{Treq}}{2}$.

As can be seen in Fig. 15, both loss factors are greater than 1.0. The lower rotor influence loss factor is 60% higher than the upper rotor loss factor at a large axial separation distance. This indicates that the effect of the upper rotor on the lower rotor is much more significant than the opposite case or the lower rotor performs poorly due to the wake of the upper rotor. As the axial distance decreases, the upper rotor influence factor rapidly increases, or the upper rotor loses the performance as the effect of the lower rotor increases at short axial distances. This raised loss factor is also magnified with the reduced thrust on the upper rotor at short separation distances as shown in Fig. 11 since the thrust coefficient appears in the denominator of Eq. (30). The lower rotor influence factor

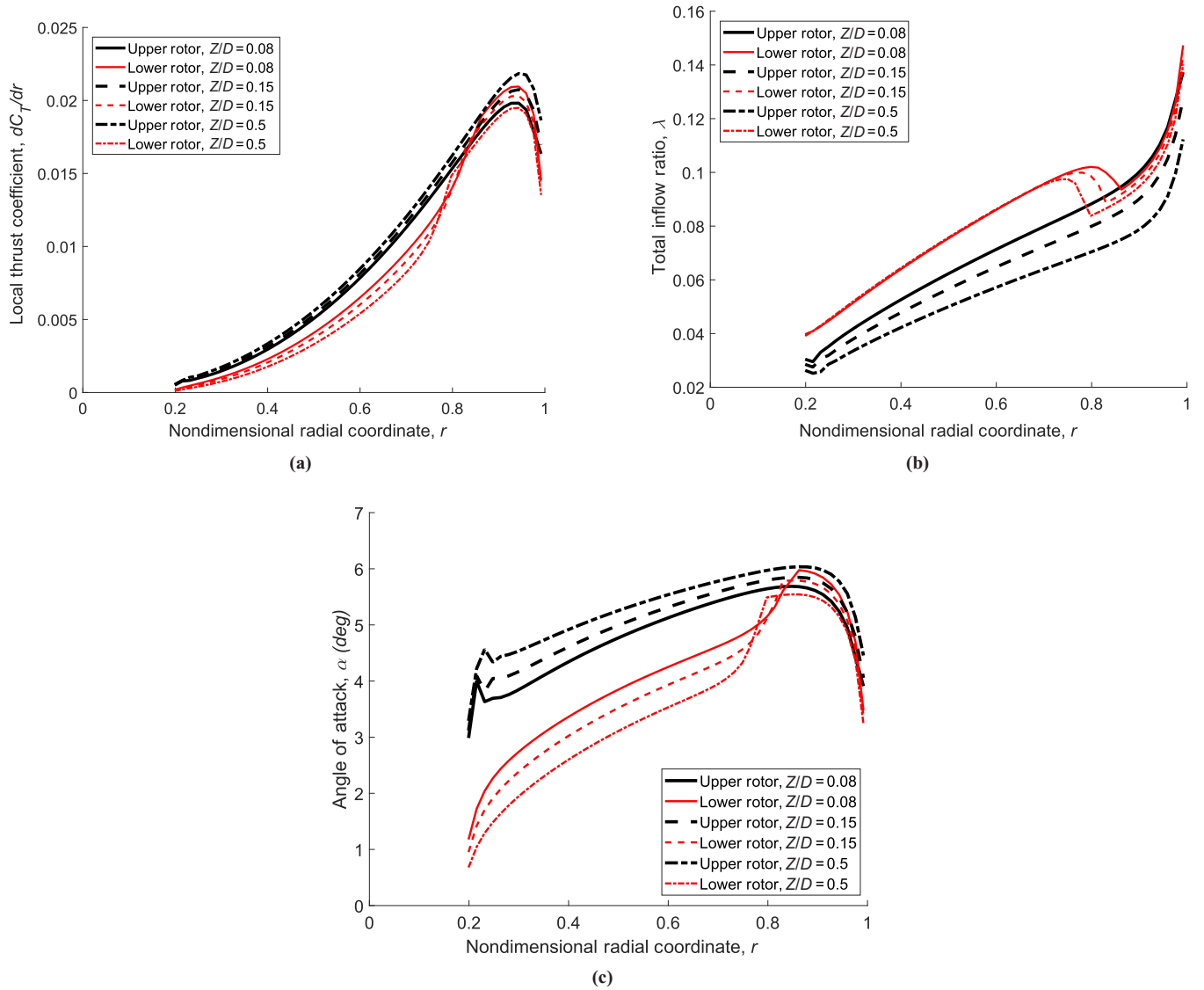


Fig. 16. Ramasamy untwisted coaxial rotor sectional aerodynamics for the upper and lower rotors at different spacing distances for $C_T = 0.014$.: (a) local thrust coefficient, (b) total inflow ratio, and (c) local angle of attack.

slightly decreases as the axial distance decreases. Even though the rotor-induced velocity and the induced power increases on the lower rotor as the separation distance decreases, the thrust is also significantly increased, as shown in Fig. 11, to meet the desired, overall coaxial rotor thrust requirement. This increase in the thrust results in slightly lowering the rotor-on-rotor influence factor on the lower rotor at short separation distances. Overall, the prediction captures the trend well for both the upper and lower rotors.

Figure 16 shows the local thrust, total inflow ratio, and angle of attack at three separation locations as a function of the nondimensionalized radial location. At $Z/D = 0.08$, the upper rotor generates the higher thrust up to the nondimensionalized radial position of about 0.8, but the lower rotor generates the higher thrust beyond this radial position. At $Z/D = 0.15$ and 0.5 , the upper rotor generates a higher thrust over the entire blade radius than the lower rotor. As the separation distance decreases, the upper rotor thrust decreases and the lower rotor thrust increases. This is consistent with Fig. 11. The total inflow ratio shows that the lower rotor has a higher induced velocity than the upper rotor up to the nondimensionalized radial position of 0.8. As the separation distance

increases, the inflow ratio on the upper rotor is noticeably reduced on the entire blade section, but the inflow ratio on the lower rotor remains the same inside and outside the upper rotor wake influence, or contraction zone. However, the transition location, or the boundary of the contraction zone, changes with the separation distance. This transition location moves further toward the inboard as the separation distance increases because of the upper rotor wake contraction. Note that the angle of attack is greatly affected by the inflow and the collective pitch angle. For the outboard section ($r > 0.8$) at a short separation distance ($Z/D = 0.08$), the induced velocities are similar for the upper and lower rotors. As a result, the angle of attack of the lower rotor at the outboard section ($r > 0.8$) is slightly higher than that of the upper rotor due to a higher collective pitch angle on the lower rotor, as shown in Fig. 13. With the exception of the outboard section, the upper rotor has higher angles of attack than the lower rotor due to smaller inflow on the upper rotor, which results in the higher thrust for the upper rotor up to nondimensionalized radial position equal to 0.8. At $Z/D = 0.5$, the lower rotor has the higher induced velocities or lower angles of attack or lower local thrusts compared to the upper rotor at the entire blade section.

Conclusions

An iterative BEMT model was developed to predict coaxial rotor performance and aerodynamics. In this method, the upper rotor solution is influenced by the lower rotor thrust and inflow and the lower rotor solution is affected by the upper rotor thrust and inflow. The mutual interaction is solved iteratively. Two empirical constants were added to match the measurement data. It is worthwhile noting that these empirical constants may differ for a different rotor blade planform, such as a twisted rotor.

The newly developed BEMT was validated against two coaxial rotor experiments. For Harrington's rotors, the thrust and power coefficients were well predicted for one specific axial separation distance. It was shown that the coaxial rotor performance is higher than each individual rotor at the same power. For Ramasamy's rotor experiment, the BEMT predicted the figure of merit, thrust, power, interference losses, etc. compared to the measurement data for two coaxial rotor C_T cases, one of which was shown. It was found that the predictions are in good agreement with the data. The trends and magnitudes of the performance and losses were well captured in terms of the axial separation distance.

The detailed local aerodynamics were also predicted. It was found that at relatively short separation distances, the upper rotor generates more thrust at the inboard and the lower rotor generates more thrust at the outboard. At large separation distances, the upper rotor generates more thrust at the entire blade section. The induced velocity and angle of attack supported this change of the trend.

The newly developed BEMT code runs fast. It only takes a couple of minutes on a single desktop computer. This simple and fast computation allows for the preliminary design of a coaxial rotor. However, further validations are recommended to have a more confidence in this simple prediction method. For example, high-fidelity CFD can be used to validate the BEMT results for local aerodynamic quantities.

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