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Author
Xu, Hongquan

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HONGQUAN XU

Department of Statistics, University of California, Los Angeles, CA 90095-1554, U.S.A.

(hqxu@stat.ucla.edu)

Summary

The Nordstrom and Robinson code is a well-known nonlinear code in coding theory. This paper explores the statistical properties of this nonlinear code. Many nonregular designs with 32, 64, 128 and 256 runs and 7–16 factors are derived from it. It is shown that these nonregular designs are better than regular designs of the same size in terms of resolution, aberration and projectivity. Furthermore, many of these nonregular designs are shown to have generalized minimum aberration among all possible designs. Seven orthogonal arrays are shown to have unique wordlength pattern and four of them are shown to be unique up to isomorphism.

Some key words: Generalized minimum aberration; Generalized resolution; Generalized wordlength pattern; Linear programming; MacWilliams identity; Orthogonal array; Projectivity.

1 Introduction

Fractional factorial designs with factors at two levels are among the most widely used experimental designs. Regular fractional factorial designs are specified by some defining relations among the factors. They are typically chosen by the *minimum aberration* criterion (Fries & Hunter, 1980), which includes the *maximum resolution* criterion (Box & Hunter, 1961) as a special case. There are many recent results on the construction and properties of minimum aberration designs; see Wu & Hamada (2000, Ch. 4) for details and references.

There has been increasing interest in the study of *nonregular* designs because they enjoy some good projection properties; see Lin & Draper (1992), Wang & Wu (1995), Cheng (1995, 1998) and Box & Tyssedal (1996). The concepts of resolution and aberration have been extended to nonregular designs; see Deng & Tang (1999), Tang & Deng (1999), Ma & Fang (2001), Xu & Wu (2001) and Xu (2003). Nonregular designs from Hadamard matrices of order 16, 20 and 24 have been cataloged by Deng & Tang (2002) with a computer search. The construction of good nonregular designs remains challenging especially when the size is large.

This paper studies some nonregular designs derived from a well-known code in coding theory. A regular design is known as a linear code and a nonregular design is simply a nonlinear code. The connection between codes in coding theory and designs in statistics was first observed by Bose (1961). The Nordstrom and Robinson (NR) code, a well-known nonlinear code, was originally constructed by Nordstrom and Robinson (1967) and has been studied extensively in coding theory; see MacWilliams & Sloane (1977, Ch. 2 and 15). On the statistical side, the NR code is a nonlinear orthogonal array with 256 runs, 16 factors, two levels and strength 5 while a linear orthogonal array of the same size has strength at most 4 (Hedayat, Sloane & Stufken, 1999, Ch. 5 § 10). However, the statistical properties of the NR code have not been fully explored.

The NR code, as well as some background information on notation and definitions, is described in § 2. Many nonregular designs with 32, 64, 128 and 256 runs and 7–16 factors are derived from it and their statistical properties are studied in § 3. It is shown that these nonregular designs are better than regular designs of the same size in terms of resolution, aberration and projectivity. Some associated theoretic questions are addressed in § 4. With MacWilliams identities and linear programming, many of these nonregular designs are shown to have generalized minimum aberration among *all* possible designs. Furthermore, seven orthogonal arrays are shown to have unique wordlength pattern and four of them are shown to be unique up to isomorphism.

2 Background

2.1 Notation and definitions

A design D of N runs and n factors is represented by an $N \times n$ matrix where each row corresponds to a run and each column a factor. A two-level design takes on only two symbols, say -1 or $+1$. For $s = \{c_1, \dots, c_k\}$, a subset of k columns of D, define

$$J_k(s) = \left| \sum_{i=1}^{N} c_{i1} \cdots c_{ik} \right|, \tag{1}$$

where c_{ij} is the ith component of column c_j. When D is a regular design, $J_k(s)$ takes on only two values: 0 or N. In general, $0 \le J_k(s) \le N$. If $J_k(s) = N$, these k columns in s form a word of length k.

Suppose that r is the smallest integer such that $\max_{|s|=r} J_r(s) > 0$, where the maximization is over all subsets of r columns of D. The *generalized resolution* (Deng & Tang, 1999) of D is defined as $R(D) = r + [1 - \max_{|s|=r} J_r(s)/N]$. Let

$$A_k(D) = N^{-2} \sum_{|s|=k} [J_k(s)]^2. \quad (2)$$

The vector $(A_1(D), \ldots, A_n(D))$ is the *generalized wordlength pattern*. The *generalized minimum aberration* criterion, called minimum G_2-aberration by Tang & Deng (1999), is to sequentially minimize $A_1(D), A_2(D), \ldots, A_n(D)$. When restricted to regular designs, generalized resolution, generalized wordlength pattern and generalized minimum aberration reduce to the traditional resolution, wordlength pattern and minimum aberration, respectively. In the rest of the paper, we simply use resolution and wordlength pattern for both regular and nonregular designs, but use GMA for generalized minimum aberration and MA for minimum aberration.

A two-level design D of N runs and n factors is an *orthogonal array* (OA) of strength t, denoted by $OA(N, n, 2, t)$, if all possible 2^t level combinations for any t factors appear equally often. Deng & Tang (1999) showed that a design has resolution $r \leq R < r + 1$ if and only if it is an OA of strength $t = r - 1$.

A two-level design is said to have *projectivity* p (Box & Tyssedal, 1996) if any p-factor projection contains a complete 2^p factorial design, possibly with some points replicated. A regular design with resolution $R = r$ is an OA of strength $r - 1$ and hence has projectivity $r - 1$. Deng & Tang (1999) showed that a design with resolution $R > r$ has projectivity $p \geq r$.

A two-level design is also called a binary code in coding theory. For two row vectors a and b, the *Hamming distance* $d_H(a, b)$ is the number of places where they differ. Let

$$B_j(D) = N^{-1} \left| \{(a, b) : a, b \text{ are row vectors of } D, \text{ and } d_H(a, b) = j\} \right|.$$

The vector $(B_0(D), B_1(D), \ldots, B_n(D))$ is called the *distance distribution* of D. The *minimum distance* d is the smallest integer $k \geq 0$ such that $B_k(D) > 0$. A design D of N runs, n factors and minimum distance d is called an (n, N, d) code in coding theory. See Hedayat et al. (1999, Ch. 4) for an introduction to coding theory and applications on OAs.

Xu and Wu (2001) showed that the wordlength pattern is the *MacWilliams transform* of the distance distribution, i.e.,

$$A_j(D) = N^{-1} \sum_{i=0}^{n} P_j(i; n) B_i(D) \text{ for } j = 0, \ldots, n, \quad (3)$$

where $P_j(x; n) = \sum_{i=0}^{j} (-1)^i \binom{x}{i} \binom{n-x}{j-i}$ are the *Krawtchouk polynomials* and $A_0(D) = 1$. By the orthogonality of the Krawtchouk polynomials, it is easy to show that

$$B_j(D) = N\, 2^{-n} \sum_{i=0}^{n} P_j(i;n) A_i(D) \text{ for } j = 0, \ldots, n. \tag{4}$$

The equations (3) and (4) are known as the generalized *MacWilliams identities*.

2.2 Nordstrom and Robinson code

The original NR code (Nordstrom & Robinson, 1967) has 15 columns, labeled as

$$X_0, X_1, \ldots, X_7, Y_0, Y_1, \ldots, Y_6,$$

where $X_0, \ldots, X_7$ are information bits (or independent columns) and $Y_0, \ldots, Y_6$ are redundant bits (or dependent columns). Each Y is a Boolean function of the X's. Y_0 is defined as follows:

$$Y_0 = X_7 \oplus X_6 \oplus X_0 \oplus X_1 \oplus X_3 \oplus (X_0 \oplus X_4)(X_1 \oplus X_2 \oplus X_3 \oplus X_5) \oplus (X_1 \oplus X_2)(X_3 \oplus X_5), \tag{5}$$

where $\oplus$ denotes modulo 2 addition. Note that the X's and Y's take on value 0 or 1 here. The remaining Y's are found by cyclically shifting X_0 through X_6; i.e., for Y_j substitute $X_{i+j \pmod 7}$ for X_i in (5) where $i = 0, 1, \ldots, 6$ for each $j = 0, 1, \ldots, 6$.

The extended NR code has an additional column, labeled as Y_7, where

$$Y_7 = X_0 \oplus X_1 \oplus \cdots \oplus X_7 \oplus Y_0 \oplus Y_1 \oplus \cdots \oplus Y_6.$$

The extended NR code is a design of 256 runs and 16 factors when $X_0, \ldots, X_7$ are evaluated at 2^8 possible level combinations. It is well known in coding theory that the distance distribution coincides with the MacWilliams transform of the distance distribution, and they are:

i :	0	6	8	10	16
$A_i = B_i$:	1	112	30	112	1

The extended NR code is a $(16, 256, 6)$ code and an $OA(256, 16, 2, 5)$.

There are several different constructions for the NR code; see MacWilliams & Sloane (1977, Ch. 2) and Hedayat et al. (1999, Ch. 5 § 10). Nevertheless, it is known that a $(16, 256, 6)$ code is unique up to isomorphism (MacWilliams & Sloane, 1977, p. 74–75). Two designs (or codes) are said to be *isomorphic* if one can be obtained from the other by permuting the rows, the columns and the symbols of each column.

3 Nonregular designs from the Nordstrom and Robinson code

3.1 Designs of 256 runs

First we study the projection property of the extended NR code in term of the $J_k(s)$ values defined in (1). The J_k's are zero except (possibly) for J_6, J_8, J_{10} and J_{16} because the A_k's are zero except for A_6, A_8, A_{10} and A_{16}. With a computer, it is straightforward to verify that there are 448 six-factor projections with $J_6 = 128$, 30 words of length 8 (i.e., $J_8 = 256$), 448 ten-factor projections with $J_{10} = 128$, one word of length 16 (i.e., $J_{16} = 256$), and all other $J_k(s) = 0$. Therefore, the frequencies of the nonzero J_k values are

$J_6 : 128$	$J_8 : 256$	$J_{10} : 128$	$J_{16} : 256$
448	30	448	1

The extended NR code has resolution 6.5 and hence has projectivity at least 6. Indeed, it can be verified that it has projectivity 7. For comparison, a regular MA design of the same size has resolution 5 and projectivity 4.

Note that a regular MA design of 256 runs has resolution 5 for 13–16 factors and resolution 6 for 10–12 factors (Draper & Lin, 1990). An immediate conclusion is that *any projection design with 10–16 columns from the extended NR code has higher resolution and better projectivity than any regular design of the same size.*

Next we study the projection designs (or subdesigns). It is evident that any subdesign of 15 factors is an $OA(256, 15, 2, 5)$ and a $(15, 256, 5)$ code. The wordlength pattern is uniquely determined by the MacWilliams identities because the only nonzero A_i values are $A_0 = 1$, A_6, A_8 and A_{10}, and $B_0 = 1$, $B_i = 0$ for $i = 1, 2, 3, 4$. Indeed, the first three identities of (4) are

$$2^{-7}[1 + A_6 + A_8 + A_{10}] = B_0 = 1,$$
$$2^{-7}[15 + 3A_6 - A_8 - 5A_{10}] = B_1 = 0,$$
$$2^{-7}[105 - 3A_6 - 7A_8 + 5A_{10}] = B_2 = 0.$$

There is a unique solution: $A_6 = 70, A_8 = 15, A_{10} = 42$ and all other $A_i = 0$.

Similarly, any subdesign of 14 factors is an $OA(256, 14, 2, 5)$ and a $(14, 256, 4)$ code. Again, the wordlength pattern is uniquely determined by the MacWilliams identities: $A_6 = 42, A_8 = 7, A_{10} = 14$ and all other $A_i = 0$. Any subdesign of 13 factors is an $OA(256, 13, 2, 5)$ and a $(13, 256, 3)$ code. Again, the wordlength pattern is uniquely determined by the MacWilliams identities: $A_6 =$

Table 1: GMA designs of 256 runs from the NR code

n	Columns	R	$(A_6, \ldots, A_n)$
9	1–8, 16	8	(0,0,1,0)
10	1–9, 16	6.5	(2,0,1,0,0)
11	1–10, 16	6.5	(6,0,1,0,0,0)
12	1–10, 14, 16	6.5	(12,0,3,0,0,0,0)
13	any 13 columns	6.5	(24,0,3,0,4,0,0,0)
14	any 14 columns	6.5	(42,0,7,0,14,0,0,0,0)
15	any 15 columns	6.5	(70,0,15,0,42,0,0,0,0,0)
16	1–16	6.5	(112,0,30,0,112,0,0,0,0,0,1)

$24, A_8 = 3, A_{10} = 4$ and all other $A_i = 0$. The wordlength patterns are not unique for subdesigns of 6–12 factors.

Table 1 shows GMA subdesigns for 9–16 factors from the extended NR code, their resolutions and wordlength patterns ($A_1 = \cdots = A_5 = 0$ are omitted). Corresponding regular MA designs can be found in Chen & Wu (1991) for 9–12 factors, Chen (1992) for 13 factors, and Franklin (1984) for 14–16 factors. Whether Franklin's designs have MA needs to be verified, though. Compared with the regular MA design, the GMA design given in Table 1 has more aberration for 9–10 factors, the same aberration for 11–12 factors and less aberration for 13–16 factors. The GMA design for 9 factors is a regular design.

3.2 Designs of 128 runs

From the extended NR code, one gets a design of 128 runs and 15 factors by taking the runs that begin with $X_0 = 0$ and omitting the column X_0. This technique is known as shortening in coding theory and the resulting design is known as the shortened NR code. It is evident that the shortened NR code is an $OA(128, 15, 2, 4)$ and a $(15, 128, 6)$ code. It is known that a $(15, 128, 6)$ code is unique (MacWilliams & Sloane, 1977, p. 75).

The wordlength pattern of the shortened NR code is again uniquely determined by the MacWilliams identities. The only nonzero B_i values are $B_0 = 1$, B_6, B_8 and B_{10}; and $A_0 = 1$, $A_i = 0$ for $i = 1, 2, 3, 4$. Indeed, the first three identities of (3) are

$$2^{-7}[1 + B_6 + B_8 + B_{10}] = A_0 = 1,$$

Table 2: GMA designs of 128 runs from the NR code

n	Columns	R	$(A_5, \ldots, A_n)$
8	1–3, 5, 7–9, 13	8	(0,0,0,1)
9	1–8, 15	5.5	(1,1,1,0,0)
10	1–9,15	5.5	(3,3,1,0,0,0)
11	1–9, 13, 15	5.5	(6,6,2,1,0,0,0)
12	1–12	5.5	(11,13,2,1,3,1,0,0)
13	any 13 columns	5.5	(18,24,4,3,10,4,0,0,0)
14	any 14 columns	5.5	(28,42,8,7,28,14,0,0,0,0)
15	1–15	5.5	(42,70,15,15,70,42,0,0,0,0,1)

$$2^{-7}[15 + 3B_6 - B_8 - 5B_{10}] = A_1 = 0,$$

$$2^{-7}[105 - 3B_6 - 7B_8 + 5B_{10}] = A_2 = 0.$$

There is a unique solution: $B_6 = 70, B_8 = 15, B_{10} = 42$. By the MacWilliams identities (3), one gets $A_0 = 1, A_5 = 42, A_6 = 70, A_7 = 15, A_8 = 15, A_9 = 70, A_{10} = 42, A_{15} = 1$ and all other $A_i = 0$. It is interesting to note that the wordlength pattern is the MacWilliams transform of the wordlength pattern of the $OA(256, 15, 2, 5)$ described in § 3.1.

The frequencies of the nonzero J_k values are

$J_5 : 64$	$J_6 : 64$	$J_7 : 128$	$J_8 : 128$	$J_9 : 64$	$J_{10} : 64$	$J_{15} : 128$
168	280	15	15	280	168	1

The shortened NR code has resolution 5.5 and projectivity 6 (not 5). For comparison, a regular MA design of the same size has resolution 4 and projectivity 3.

Note that a regular MA design of 128 runs has resolution 4 for 12–15 factors and resolution 5 for 10–11 factors (Draper & Lin, 1990). An immediate conclusion is that *any projection design with 10–15 columns from the shortened NR code has higher resolution and better projectivity than any regular design of the same size.*

Table 2 shows GMA subdesigns for 8–15 factors from the shortened NR code, their resolutions and wordlength patterns. Corresponding regular MA designs can be found in Chen & Wu (1991) for 8–11 factors, Chen (1992) for 12 factors, Chen (1998) for 13–14 factors, and Franklin (1984) for 15 factors. Again, whether the Franklin's design has MA needs to be verified. Compared with the regular MA design, the GMA design given in Table 2 has more aberration for 9 factors, the

Table 3: GMA designs of 64 runs from the NR code

n	Columns	R	$(A_4, \ldots, A_n)$
7	1,2,4,6–8,12	7	(0,0,0,1)
8	1–6,13,14	5.5	(0,2,1,0,0)
9	1–7,11,13	4.5	(1,4,2,0,0,0)
10	1–7,11,13,14	4.5	(2,8,4,0,1,0,0)
11	1–9,11,13	4.5	(4,14,8,0,3,2,0,0)
12	1–9,11,13,14	4.5	(6,24,16,0,9,8,0,0,0)
13	any 13 columns	4.5	(10,36,28,8,21,20,4,0,0,0)
14	1–14	4.5	(14,56,49,16,49,56,14,0,0,0,1)

same aberration for 8, 10–11 factors and less aberration for 12–15 factors. The GMA design for 9 factors is of interest because it has projectivity 6 while a regular MA design of the same size has projectivity 5. The GMA design for 8 factors is a regular MA design.

3.3 Designs of 64 runs

From the shortened NR code, one gets a design of 64 runs and 14 factors by taking the runs that begin with $X_1 = 0$ and omitting the column X_1. It is evident that the resulting design is an $OA(64, 14, 2, 3)$ and a $(14, 64, 6)$ code. It is again known that a $(14, 64, 6)$ code is unique (MacWilliams & Sloane, 1977, p. 75).

The wordlength pattern of the $OA(64, 14, 2, 3)$ is determined by the MacWilliams identities and is the MacWilliams transform of the wordlength pattern of the $OA(256, 14, 2, 5)$ described in § 3.1. The frequencies of the nonzero J_k values are

$J_4 : 32$	$J_5 : 32$	$J_6 : (64, 32)$	$J_7 : 64$	$J_8 : (64, 32)$	$J_9 : 32$	$J_{10} : 32$	$J_{14} : 64$
56	224	(7, 168)	16	(7, 168)	224	56	1

The $OA(64, 14, 2, 3)$ has resolution 4.5 and projectivity 5 (not 4). For comparison, a regular MA design of the same size has resolution 4 and projectivity 3.

Note that a regular MA design of 64 runs and 9–14 factors has resolution 4 (Draper & Lin, 1990). An immediate conclusion is that *any projection design with 9–14 columns from the* $OA(64, 14, 2, 3)$ *has higher resolution and better projectivity than any regular design of the same size.*

Cheng (1998) showed that as long as an OA of strength three has no defining word of length four, its projection onto any five factors allows the estimation of all the main effects and two-factor interactions when the higher-order interactions are negligible. It can be verified that the $OA(64, 14, 2, 3)$ has the stronger property that its projection onto any *seven* factors allows the estimation of all the main effects and two-factor interactions.

Table 3 shows GMA subdesigns for 7–14 factors, their resolutions and wordlength patterns. Compared with the corresponding regular MA design (see Chen, Sun & Wu, 1993), the GMA design given in Table 3 has the same aberration for 7–12 factors and less aberration for 13–14 factors. The GMA design for 8 factors has resolution 5.5 while the regular MA 2^{8-2} design has resolution 5 although they have the same wordlength pattern. The GMA design for 7 factors is a regular MA design.

3.4 Designs of 32 runs

From the $(14, 64, 6)$ code, one gets a design of 32 runs and 13 factors by taking the runs that begin with $X_2 = 0$ and omitting the column X_2. It is evident that the resulting design is an $OA(32, 13, 2, 2)$ and a $(13, 32, 6)$ code. It is again known that a $(13, 32, 6)$ code is unique (MacWilliams & Sloane, 1977, p. 75).

The wordlength pattern of the $OA(32, 13, 2, 2)$ is determined by the MacWilliams identities and is the MacWilliams transform of the wordlength pattern of the $OA(256, 13, 2, 5)$ described in § 3.1. The frequencies of the nonzero J_k values are

$J_3 : 16$	$J_4 : 16$	$J_5 : (32, 16)$	$J_6 : (32, 16)$	$J_7 : (32, 16)$	$J_8 : (32, 16)$	$J_9 : 16$	$J_{10} : 16$	$J_{13} : 32$
16	120	(3, 216)	(12, 96)	(12, 96)	(3, 216)	120	16	1

The $OA(32, 13, 2, 2)$ has resolution 3.5 and projectivity 4 (not 3). Note that a regular MA design of 32 runs and 7–13 factors has resolution 4 and projectivity 3. An immediate conclusion is that *any projection design with 7–13 columns from the* $OA(32, 13, 2, 2)$ *has better projectivity than any regular design of the same size.*

Cheng (1995) showed that as long as an OA of strength two has no defining word of length three or four, its projection onto any four factors allows the estimation of all the main effects and two-factor interactions when the higher-order interactions are negligible. It can be verified that the $OA(32, 13, 2, 2)$ has the stronger property that its projection onto any *five* factors allows the estimation of all the main effects and two-factor interactions.

Table 4 shows GMA subdesigns for 6–13 factors, their resolutions and wordlength patterns. Compared with the corresponding regular MA design (see Chen et al., 1993), the GMA design

Table 4: GMA designs of 32 runs from the NR code

n	Columns	R	$(A_3, \ldots, A_n)$
6	1,3,5–7, 11	6	(0,0,0,1)
7	1–6, 13	4.5	(0,1,2,0,0)
8	1–6, 10, 12	4.5	(0,3,4,0,0,0)
9	1–6, 10, 12, 13	4.5	(0,6,8,0,0,1,0)
10	1–7, 10, 12, 13	3.5	(1,9,14,2,1,4,0,0)
11	1–9, 11, 13	3.5	(2,14,22,8,6,9,2,0,0)
12	1–12	3.5	(3,21,35,19,17,22,9,1,0,0)
13	1–13	3.5	(4,30,57,36,36,57,30,4,0,0,1)

given in Table 4 has the same aberration for 6–9 factors and more aberration for 10–13 factors. The GMA design for 7-9 factors has resolution 4.5 while the corresponding regular MA design has resolution 4 although they have the same wordlength pattern. The GMA design for 6 factors is a regular MA design.

4 Some theoretical results

The MacWilliams identities provide a powerful tool in the study of coding theory and factorial design. Based on the MacWilliams identities and the fact that the wordlength pattern A_i and the distance distribution B_i are always nonnegative, linear programming technique can be used to establish bounds on the maximum size of a code for given length and distance and bounds on the minimum size of an OA for given number of constraints and strength; see MacWilliams and Sloane (1977, Ch. 17 § 4) and Hedayat et al. (1999, Ch. 4 § 5). Here we use MacWilliams identities and linear programming to show that many nonregular designs derived from the NR code indeed have GMA among *all* possible designs.

From the definition (2), it is easy to see that $0 \leq A_k \leq \binom{n}{k} < 2^n$ for all k. Because $N^2 A_k$ is an integer, sequentially minimizing $A_1, A_2, \ldots, A_n$ is equivalent to minimizing $\sum_{j=1}^{n} \lambda^{n-j} A_j$, where λ is any number that is larger than or equal to $N^2 2^n$.

Suppose an $OA(N, n, 2, t)$ exists. Then a GMA design of N runs and n factors must satisfy $A_1 = \ldots = A_t = 0$ and sequentially minimizes $A_{t+1}, A_{t+2}, \ldots, A_n$. So we consider the following linear programming problem:

$$\text{minimize} \sum_{j=t+1}^{n} (N^2 2^n)^{n-j} A_j \tag{6}$$

subject to

$$\sum_{j=t+1}^{n} A_j \geq (N^{-1}2^n) - 1, \tag{7}$$

$$\sum_{j=t+1}^{n} P_i(j;n)A_j \geq -P_i(0;n) \text{ for } i = 1, \ldots, n, \tag{8}$$

$$A_j \geq 0 \text{ for } j = t+1, \ldots, n. \tag{9}$$

where inequality (7) corresponds to $B_0 \geq 1$ and (8) corresponds to $B_i \geq 0$.

An optimal solution to (6) gives a feasible GMA wordlength pattern. If the wordlength pattern of a design coincides with the optimal solution, then the design must have GMA among *all* possible designs.

As an example, consider the case of $N = 32$ and $n = 7$ where an $OA(32, 7, 2, 3)$ exists. The constraints (7)–(9) are

$$A_4 + A_5 + A_6 + A_7 \geq 3, \tag{10}$$

$$-A_4 - 3A_5 - 5A_6 - 7A_7 \geq -7, \tag{11}$$

$$-3A_4 + A_5 + 9A_6 + 21A_7 \geq -21,$$

$$3A_4 + 5A_5 - 5A_6 - 35A_7 \geq -35,$$

$$3A_4 - 5A_5 - 5A_6 + 35A_7 \geq -35,$$

$$-3A_4 - A_5 + 9A_6 - 21A_7 \geq -21,$$

$$-A_4 + 3A_5 - 5A_6 + 7A_7 \geq -7,$$

$$A_4 - A_5 + A_6 - A_7 \geq -1,$$

$$A_4 \geq 0, A_5 \geq 0, A_6 \geq 0, A_7 \geq 0. \tag{12}$$

Adding $3 \times (10)$ to (11) yields $2A_4 - 2A_6 - 4A_7 \geq 2$ or $A_4 \geq A_6 + 2A_7 + 1 \geq 1$ due to (12). When $A_4 = 1$, A_6 and A_7 must be zero, and hence $A_5 \geq 2$ from (10). Therefore, an optimal solution is $A_4 = 1, A_5 = 2, A_6 = 0, A_7 = 0$, which satisfies all constraints. Recall that the GMA design for 7 factors given in Table 4 has this wordlength pattern. Therefore, it has GMA among all possible designs.

Table 5: Optimal solutions to the linear programming problem (6)

N	n	$(A_4, \ldots, A_n)$	Design
32	6	$(0, 0, 1)$	MA
32	7	$(1, 2, 0, 0)$	MA or NR
32	8	$(3, 4, 0, 0, 0)$	MA or NR
32	9	$(6, \frac{15}{2}, 1, 0, 0, \frac{1}{2})$	
32	10	$(10, 16, 0, 0, 5, 0, 0)$	MA
32	11	$(20, 18, 4, 12, 7, 2, 0, 0)$	
32	12	$(\frac{125}{4}, 26, 14, 28, \frac{35}{2}, 10, 0, 0, \frac{1}{4})$	
32	13	$(\frac{93}{2}, \frac{363}{10}, \frac{174}{5}, \frac{286}{5}, \frac{216}{5}, 33, \frac{6}{5}, \frac{6}{5}, \frac{13}{10}, \frac{3}{10})$	
64	7	$(0, 0, 0, 1)$	MA
64	8	$(0, 2, 1, 0, 0)$	MA or NR
64	9	$(1, 4, 2, 0, 0, 0)$	MA or NR
64	10	$(\frac{5}{3}, 8, 5, 0, 0, 0, \frac{1}{3})$	
64	11	$(3, 14, 11, 0, 0, 2, 1, 0)$	
64	12	$(6, 24, 16, 0, 9, 8, 0, 0, 0)$	MA or NR
64	13	$(10, 36, 28, 8, 21, 20, 4, 0, 0, 0)$	NR
64	14	$(14, 56, 49, 16, 49, 56, 14, 0, 0, 0, 1)$	NR
128	8	$(0, 0, 0, 0, 1)$	MA
128	9	$(0, 0, 3, 0, 0, 0)$	MA
128	10	$(0, 2, 5, 0, 0, 0, 0)$	
128	11	$(0, \frac{11}{3}, 11, 0, 0, 0, 0, \frac{1}{3})$	
128	12	$(0, \frac{15}{2}, \frac{79}{4}, 0, 0, \frac{5}{2}, \frac{3}{4}, 0, \frac{1}{2})$	
128	13	$(0, 18, 24, 4, 3, 10, 4, 0, 0, 0)$	NR
128	14	$(0, 28, 42, 8, 7, 28, 14, 0, 0, 0, 0)$	NR
128	15	$(0, 42, 70, 15, 15, 70, 42, 0, 0, 0, 0, 1)$	NR
256	9	$(0, 0, 0, 0, 0, 1)$	MA
256	10	$(0, 0, 1, 2, 0, 0, 0)$	MA
256	11	$(0, 0, 5, 2, 0, 0, 0, 0)$	
256	12	$(0, 0, 10, \frac{24}{5}, 0, 0, 0, 0, \frac{1}{5})$	
256	13	$(0, 0, \frac{94}{5}, 10, 0, 0, 0, \frac{6}{5}, 1, 0)$	
256	14	$(0, 0, 42, 0, 7, 0, 14, 0, 0, 0, 0)$	NR
256	15	$(0, 0, 70, 0, 15, 0, 42, 0, 0, 0, 0, 0)$	NR
256	16	$(0, 0, 112, 0, 30, 0, 112, 0, 0, 0, 0, 0, 1)$	NR

The coefficients of constraints are large and complicated in general, so a computer software is used to solve the linear programming problem. We use Mathematica (a software of Wolfram Research, Inc.) in this task because it gives exact solutions.

Table 5 lists optimal solutions to the linear programming problem (6) for various parameters ($A_1 = A_2 = A_3 = 0$ are omitted). It is obvious that a solution cannot be the wordlength pattern of a design whenever $N^2 A_i$ is not an integer for some i. Nevertheless, there are 20 cases where the wordlength pattern of a design coincides with the optimal solution. The last column of Table 5 indicates such a design where MA refers to a regular MA design and NR refers to a GMA design derived from the NR code. In summary, we have the following result.

Theorem 1. *The nonregular GMA designs given in Tables* 1–4 *have GMA among all possible designs for the following 13 cases: 256 runs and 14–16 factors, 128 runs and 13–15 factors, 64 runs and 8–9, 12–14 factors, and 32 runs and 7–8 factors.*

From Table 5, we observe that a regular MA 2^{n-1} design has GMA for n=6–9. This is true in general because an $OA(2^{n-1}, n, 2, n-1)$ is unique up to isomorphism.

We also observe that a regular MA 2^{n-2} design has GMA for n=7–10 (although there also exist nonregular GMA designs). The following theorem shows that this is also true in general.

Theorem 2. *For n factors and $N = 2^{n-2}$ runs, a regular MA 2^{n-2} design has GMA among all possible designs.*

Proof. Chen & Wu (1991) showed that a regular MA 2^{n-2} design has resolution $R = \lfloor 2n/3 \rfloor$, where $\lfloor x \rfloor$ is the largest integer that is less than or equal to x. They also showed that the MA wordlength pattern is $A_R = 3R - 2n + 3$, $A_{R+1} = 2n - 3R$ and other $A_i = 0$. We show that this is the optimal solution to the linear programming problem (6).

The first two inequalities of (7) and (8) are

$$\sum_{j=R}^{n} A_j \geq (N^{-1}2^n) - 1 = 3, \tag{13}$$

$$\sum_{j=R}^{n} (n-2j) A_j \geq -n. \tag{14}$$

Multiplying (13) by $2R + 2 - n$ and adding it to (14), one gets

$$2A_R + \sum_{j=R+1}^{n} (2R - 2j + 2) A_j \geq 3(2R + 2 - n) - n.$$

Since $2R-2j+2 \leq 0$ for $j \geq R+1$ and $A_j \geq 0$, one gets $A_R \geq 3R-2n+3$. Therefore, the smallest A_R value is $3R-2n+3$. Next multiplying (13) by $2R+4-n$ and adding it to (14), one gets

$$4A_R + 2A_{R+1} + \sum_{j=R+2}^{n} (2R-2j+4)A_j \geq 3(2R+4-n) - n.$$

Since $2R-2j+4 \leq 0$ for $j \geq R+2$ and $A_j \geq 0$, one gets $A_{R+1} \geq 3R-2n+6-2A_R$. When $A_R = 3R-2n+3$, the smallest A_{R+1} value is $2n-3R$. Therefore, the optimal solution is $A_R = 3R-2n+3$, $A_{R+1} = 2n-3R$ and other $A_i = 0$. □

We do not know whether Theorem 2 can be extended to $N = 2^{n-3}$. From Table 5, a regular MA 2^{n-3} design has GMA for $n = 8, 9$. However, the optimal solutions are different from the MA wordlength patterns for $n = 10, 11$. For example, Table 5 shows that the optimal solution is $(0,0,0,0,2,5,0,0,0,0)$ for $n = 10$, whereas a regular MA 2^{10-3} design has wordlength pattern $(0,0,0,0,3,3,1,0,0,0)$. Note that the nonregular GMA design from the NR code has the same wordlength pattern as the regular MA design. It is of interest to investigate whether there exists a nonregular design of 128 runs and 10 factors having wordlength pattern $(0,0,0,0,2,5,0,0,0,0)$.

The linear programming technique can also be used to determine whether an OA has a unique wordlength pattern in some cases. Consider the following linear programming problem:

$$\text{maximize} \sum_{j=t+1}^{n} (N^2 2^n)^{n-j} A_j \tag{15}$$

subject to the constraints (7)–(9). The optimal solution to (15) sequentially maximizes $A_{t+1}, A_{t+2}, \ldots, A_n$ while the optimal solution to (6) sequentially maximizes them. When the two solutions are the same, the wordlength pattern of all $OA(N, n, 2, t)$'s must be unique.

Using Mathematica, we have proved the following result.

Theorem 3. *The following seven OAs have unique wordlength pattern:* $OA(256, n, 2, 5)$ *for* $n = 14, 15, 16$*;* $OA(128, n, 2, 4)$ *for* $n = 13, 14, 15$*; and* $OA(64, 8, 2, 4)$*.*

From Theorem 3, an $OA(256, 16, 2, 5)$ must have the same wordlength pattern and distance distribution as the extended NR code. Therefore, it has minimum distance 6. In other words, an $OA(256, 16, 2, 5)$ is a $(16, 256, 6)$ code. The same argument shows that an $OA(256, 15, 2, 5)$ is a $(15, 256, 5)$ code, an $OA(128, 15, 2, 4)$ is a $(15, 128, 6)$ code and an $OA(128, 14, 2, 4)$ is a $(14, 128, 5)$ code. It is known that the $(16, 256, 6)$, $(15, 256, 5)$, $(15, 128, 6)$ and $(14, 128, 5)$ codes are unique up to isomorphism (MacWilliams & Sloane, 1977, p. 74–75). Therefore, we have the following result.

THEOREM 4. *The* $OA(256, 16, 2, 5)$*,* $OA(256, 15, 2, 5)$*,* $OA(128, 15, 2, 4)$ *and* $OA(128, 14, 2, 4)$ *are unique up to isomorphism.*

Hedayat et al. (1999, p. 109) wrote that the $OA(256, 16, 2, 5)$ is unique up to isomorphism known from coding theory; however, they did not provide the detail. It is interesting to note that the $(14, 64, 6)$ and $(13, 32, 6)$ codes are unique but the $OA(64, 14, 2, 3)$ and $OA(32, 13, 2, 2)$ are not. The $OA(64, 8, 2, 4)$ is not unique although it has a unique wordlength pattern.

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