Title
The Existence of Maximal Elements and Equilibria in the Absence of Transitivity

Author
Bergstrom, Ted

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THE EXISTENCE OF MAXIMAL ELEMENTS AND EQUILIBRIA IN THE ABSENCE OF TRANSITIVITY

Theodore C. Bergstrom
Department of Economics
Washington University
St. Louis, Mo.
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In the first section of this paper, we generalize results of Sonnenschein [18] and Ky Fan [8] on the existence of maximal elements for non-transitive binary relations. In the second section we prove a natural extension of Nash's theorem on the existence of equilibrium for non-cooperative games [13] to the case of preferences which need not be transitive. In the third section we offer a proof of the existence of competitive equilibrium without assuming either transitivity or free disposability. The theorems of the latter sections show the existence of equilibria as consequences of the existence of maxima for appropriately chosen binary orderings. In fact both Cournot-Nash equilibrium and competitive equilibrium can be treated as maximal elements of binary relations satisfying the conditions of our first section.[1]

Our theorem on the existence of competitive equilibrium is similar to those of Gale and Mas Collel [9], Sonnenschein and Shafer [16], and Shafer [17] and draws heavily on certain technical devices which were first brought to the author's attention in the paper of Gale and Mas Collel. The trick used to eliminate the assumption of free disposal first appeared in Bergstrom [2] and was later applied by Shafer [17] and Bergstrom [3]. In all of the other papers cited, the assumption is made that individual preferences have open graphs. This continuity assumption is in general stronger than the assumption that both upper and lower contour sets are open. (For a discussion of this matter see Bergstrom, Rader, and Parks [4].) Our theorem shows the existence of quasi-equilibrium where preferences exhibit a continuity property weaker than the assumption that lower contour sets are open. Of course to show by usual methods that a quasi-equilibrium is a competitive equilibrium requires the additional assumption that upper contour sets are open.

I. The Existence of Maximal Elements for Non-transitive Binary Relations

Let S be a set and $P \subset S \times S$ a binary relation. If $(y,x) \in P$, we write yPx. Where $x \in S$, let $P(x) = \{y | yPx\}$ and $P^{-1}(x) = \{y | xPy\}$. Where $X \subset S$, $x^* \in X$ is a maximal element for P on X if $P(x^*) \cap X = \Phi$.

Sonnenschein [18] and Ky Fan [8] have proved the following results on the existence of maximal elements.

Sonnenschein's Theorem Let $P \subset E^n \times E^n$ be a binary relation such that:

(i) P is asymmetric,

(ii) For all $x \in E^n$, P(x) is convex or empty,

(iii) For all $x \in E^n$, $P^{-1}(x)$ is open.

Then if $X \subset E^n$ is non-empty, convex and compact, there exists a maximal element x* for P on X.

Ky Fan's Theorem Let $P \subset S \times S$ be a binary relation where S is a linear topological space and where

(i) P is irreflexive,

(ii) For all $x \in X$, P(x) is convex or empty,

(iii) P is an open set in $S \times S$.

Then if $X \subset S$ is non-empty, convex, and compact, there exists a maximal element for P on S.

Although neither result directly implies the other, we can by a method of proof very similar to that used by these authors, generalize both theorems.

Theorem 1 Let $P \subset S \times S$ be a binary relation where S is linear topological space and where:

1.A) For all $x \in S$, $x \notin \text{con}P(x)$.[2]

1.B) For all $x \in S$, $P^{-1}(x)$ is open.

Then if $X \subset S$ is non-empty, convex, and compact, there exists a maximal element for P on X.

To prove Theorem 1, we use a lemma due to Ky Fan which extends the well-known theorem of Knaster, Kuratowski, and Mazurkiewicz to topolocigal linear spaces of arbitrary dimension.

Lemma 1 (Ky Fan) Let S be a linear topological space and $X \subset S$. For each $x \varepsilon X$, let F(x) be a closed set in S such that:

(i) The convex hull of any finite subset $\{x_1,\ldots,x_n\}$ of X is contained in $\bigcup_{i=1}^{m} F(x_i)$.

(ii) F(x) is compact for at least one $x \varepsilon X$.

Then $\bigcap_{x \varepsilon X} F(x) \neq \Phi$.

Proof of Theorem 1

Let $\{x_1,\ldots,x_n\} \subset X$. If $z \varepsilon \bigcap_{i=1}^{n} P^{-1}(x_i)$, then by Assumption 1.A, $z \notin \text{con}\{x_1,\ldots,x_n\}$. Therefore $\bigcap_{i=1}^{n} P^{-1}(x_i) \subset (\text{con}\{x_1,\ldots,x_n\})^c$ and hence $\text{con}\{x_1,\ldots x_n\} \subset \left(\bigcap_{i=1}^{n} P^{-1}(x_i)\right)^c = \bigcup_{i=1}^{n} (P^{-1}(x_i))^c$. For all $x \varepsilon X$, let $F(x) = (P^{-1}(x))^c \cap X$. Then the sets F(x) satisfy condition i of Lemma 1. Also, since $P^{-1}(x)$ is open (by Assumption 1.B) and X is compact, F(x) is compact for all $x \varepsilon X$. Therefore according to Lemma 1, $\bigcap_{x \varepsilon X} F(x) \neq \Phi$. Where $x^* \varepsilon \bigcap_{x \varepsilon X} F(x)$, it must be that x^* is a maximal element for P on X. Q.E.D.

For finite dimensional spaces, Assumption 1.B of Theorem 1 can be weakened in a useful way. The following definitions are required.

Definition 1: Let X and Y be topological spaces and 2^Y the set of all subsets of Y (including the null set). The correspondence $\phi: X \to 2^Y$ is lower semi-continuous (l.s.c.) if for every set V which is open in Y, the set $\{x \varepsilon X \mid \phi(x) \cap V \neq \Phi\}$ is open in X.[3]

Definition 2: Let S be a topological space, $P \subset S \times S$, and $X \subset S$. The relation P is lower semi-continuous on X if the correspondence $\phi: X \to 2^X$

where $\phi(x) = P(x) \cap X$ is l.s.c. (where X is endowed with the relative topology).

Remark 1: Let S be a topological space, $P \subset S \times S$ and $X \subset S$. If for all $x \in S$, $P^{-1}(x)$ is open, then P is l.s.c. on X.

Proof: Let V be an open subset of S. Then $\{x \in X | P(x) \cap V \neq \Phi\} = \bigcup_{y \in V} (P^{-1}(y) \cap X)$ which is the union of sets which are open in X and hence is open in X. Therefore P is l.s.c. on X. Q.E.D.

Theorem 2 Let $P \subset E^n \times E^n$ be a binary relation and let $X \subset E^n$ be non-empty, convex and compact where:

2.A For all $x \in X$, $x \notin \text{con}P(x)$.

2.B P is lower semi-continuous on X.

Then there exists a maximal element for P on X.

To prove theorem 2, we use a lemma which is a special case of a selection theorem proved by Michael [12, Theorem 3.1''').

Lemma 2 (Michael) Let $X \subset E^n$ and $\phi: X \to 2^X$ a l.s.c. correspondence such that for all $x \in X$, $\phi(x)$ is a non-empty convex set. Then there exists a continuous function $f: X \to X$ such that for all $x \in X$, $f(x) \in \phi(x)$.

Proof of Theorem 2

Suppose that for all $x \in X$, $P(x) \cap X \neq \Phi$. By Assumption 2.B, the correspondence $\phi: X \to 2^X$ where $\phi(x) = P(x) \cap X$ is l.s.c. Then the correspondence $\Psi: X \to 2^X$ where $\Psi(x) = \text{con}\phi(x)$ for all $x \in X$ is also l.s.c. (Proposition 3, the Appendix) and $\Psi(x)$ is non-empty and convex for all $x \in X$. By Lemma 2, there exists a continuous function $f: X \to X$ such that for all $x \in X$, $f(x) \in \Psi(x)$. Since X is non-empty, convex, and compact, by Brouwer's fixed point theorem there exists $\bar{x} \in X$ such that $\bar{x} = f(\bar{x}) \in \Psi(\bar{x})$. But then $\bar{x} \in \text{con}P(\bar{x})$ which is contrary to Assumption 2.B. Therefore it must be that $P(x^*) \cap X = \Phi$ for some $x^* \in X$.

Q.E.D.

According to Remark 1, P will be l.s.c. on any set $X \subset S$ if $P^{-1}(x)$ is open for all $x \in S$. Thus for finite dimensional spaces, Theorem 2 generalizes Theorem 1. In the next two sections we show applications of Theorem 2.

II. The Existence of Cournot-Nash Equilibrium

Definition 3: A society $(P_i, X_i)_{i \in I}$ consists of an index set $I = \{1, \ldots, m\}$ and for each $i \in I$, a set X_i and a relation $P_i \subset X \times X$ where $X = \prod_{i \in I} X_i$. For each $i \in I$, define the correspondence $\phi_i : X \to X_i$ so that where $x = (x_1, \ldots, x_m) \in X$, $\phi_i(x) = \{y_i \in X_i \mid (x_1, \ldots, y_i, \ldots, x_m) P_i x\}$.

Definition 4: A Cournot-Nash equilibrium for the society $(P_i, X_i)_{i \in I}$ is a point $\bar{x} \in X$ such that $\phi_i(\bar{x}) = \phi$ for all $i \in I$.

Theorem 3 Let $(P_i, X_i)_{i \in I}$ be a society such that $X_i \subset E^n$ is non-empty, convex and compact for all $i \in I$ and:

3.A. For all $x = (x_1, \ldots, x_m) \in X$ and all $i \in I$, $x_i \notin \text{con}\phi_i(x)$.

3.B. For all $i \in I$, ϕ_i is l.s.c.

Then there exists a Cournot-Nash equilibrium for $(P_i, X_i)_{i \in I}$.

Proof:

For all $i \in I$, define $\Psi_i : X \to 2^X$ so that where $x = (x_1, \ldots, x_m) \in X$, $\Psi_i(x) = \text{con}\{(x_1, \ldots, y_i, \ldots, x_m) \mid y_i \in \phi_i(x)\}$. From Assumption 3.B and Propositions 6 and 4 of the Appendix, it follows that Ψ_i is l.s.c. Let $\Psi : X \to 2^X$ where $\Psi(x) = \bigcup_{i \in I} \Psi_i(x)$. Then Ψ is l.s.c., by Proposition 5. If $y \in \text{con}\Psi(x)$, then $y = \sum_{j \in J} \lambda_j y^j$ where $J \subset I$, where $\sum_{j \in J} \lambda_j = 1$ and for all $j \in J$, $\lambda_j \geq 0$ and $y^j \in \Psi_j(x)$. Then $y - x = \sum_{j \in J} \lambda_j z^j$ where $z^j = y^j - x$. Assumption 3.A requires that $z^j \neq 0$ for $j \in J$. Also, the correspondences Ψ_j are constructed so that the z^j's are linearly independent. Therefore $y - x \neq 0$ and hence for all $x \in X$, $x \notin \text{con}\Psi(x)$. Let $P \subset X \times X$ be the relation such that $P(x) = \Psi(x)$ for all $x \in X$. Then P is l.s.c. on X and for all $x \in X$, $x \notin \text{con}P(x)$. From Theorem 2 it follows that $P(\bar{x}) = \phi$ for some $\bar{x} \in X$.

If $P(\bar{x})=\Phi$, then for all $i\varepsilon I$, $\Psi_i(\bar{x})=\Phi$ and hence $\phi_i(\bar{x})=\Phi$. Q.E.D.

Corollary 1: If $(P_i,X_i)_{i\varepsilon I}$ is a society such that $X_i \subset E^n$ is non-empty, convex, and compact for all $i\varepsilon I$ and:

3.A' For all $x\varepsilon X$ and all $i\varepsilon I$, $x \notin \text{con}P_i(x)$.

3.B' For all $i\varepsilon I$ and all $x_i\varepsilon X_i$, $\phi_i^{-1}(x_i)$ is open in X,

then there exists a Cournot-Nash equilibrium.

Proof: If $x_i' \varepsilon \text{con}\phi_i(x)$ than $(x_1,\ldots,x_i',\ldots,x_m)\varepsilon\text{con}P_i(x)$. Therefore $x\notin\text{con}P_i(x)$ implies $x_i \notin \text{con}\phi_i(x)$. Thus 3.A' implies 3.A of Theorem 3. From Proposition 2 of the A ppendix it is apparent that 3.B' implies 3.B of Theorem 3. Thus the corollary follows. Q.E.D.

III. Existence of Equilibrium in an Exchange Economy

An Exchange economy is described as follows. There are m consumers, indexed by the set $I=\{1,\ldots,m\}$. For each $i\varepsilon I$, there is a consumption set $X_i \subset E^n$ and an initial endowment $w_i \varepsilon E^n$. An allocation is a point $x = (x_1,\ldots,x_m)\varepsilon \prod_I X_i = X$. Each consumer has a preference relation $P_i \subset X \times X$. As in the previous section, the correspondence $\phi_i: X \to X_i$ is defined so that $\phi_i(x) = \{y_i\varepsilon X_i | (x_1,\ldots,y_i,\ldots,x_m)P_i x\}$.

Definition 5: A price vector $\bar{p}$ and an allocation $\bar{x}$ constitute a quasi-equilibrium $(\bar{p},\bar{x})$ if $\bar{p}\neq 0$ and:

(i) For all $i\varepsilon I$, $\bar{p}\bar{x}_i=\bar{p}w_i$.

(ii) For all $i\varepsilon I$, $\phi_i(\bar{x}) \cap \{x_i\varepsilon X_i | \bar{p}x_i < \bar{p}w_i\} = \Phi$.

(iii) $\sum_I \bar{x}_i = \sum_I w_i$.

Definition 6: A price vector $\bar{p}$ and an allocation $\bar{x}$ constitute a competitive equilibrium if conditions (i) and (iii) of Definition 5 are satisfied as well as:

(ii)' For all $i\varepsilon I$, $\phi_i(\bar{x}) \cap \{x_i\varepsilon X_i | \bar{p}x_i \leqq \bar{p}w_i\} = \Phi$.

Theorem 4 There exists a quasi-equilibrium for an exchange economy if:

4.1) For all $i\varepsilon I$, X_i is a convex, compact set and $w_i \varepsilon X_i$.

4.2) For all $i\varepsilon I$, the correspondence ϕ_i is l.s.c. and for all $x\varepsilon X$, $x_i \not\varepsilon \operatorname{con}\phi_i(x)$.

4.3) If x is an allocation such that $\sum_I x_i = \sum_I w_i$, then for all $i\varepsilon I$, $\phi_i(x) \neq \Phi$.

Our strategy for proving Theorem 4 will be to construct an artificial society which satisfies the conditions of Theorem 3. It will be shown that a Cournot-Nash equilibrium for this society is a quasi-equilibrium. To this end we state the following definitions and lemmas.

For all $i\varepsilon I$, let $\phi_i^*:X\to X_i$ be the correspondence such that $\phi_i^*(x) = \{x_i' | x_i' = \lambda y_i + (1-\lambda)x_i$ where $y_i \varepsilon \phi_i(x)$ and $0 < \lambda \leq 1\}$.

Lemma 3 If $x \not\varepsilon \operatorname{con}\phi_i(x)$, then $x \not\varepsilon \operatorname{con}\phi_i^*(x)$. If $\phi_i^*(x) \neq \Phi$, then $x_i \varepsilon$ Boundary $\phi_i^*(x)$. If ϕ_i is l.s.c., then ϕ_i^* is l.s.c.

Proof: If $x\varepsilon \operatorname{con}\phi_i^*(x)$, then $x = \lambda x + (1-\lambda)y$ where $y\varepsilon\operatorname{con}\phi_i(x)$ and $0<\lambda\leq 1$. Therefore x=y and hence $x\varepsilon\operatorname{con}\phi_i(x)$. The first statement in the Lemma then follows. The proof of the second statement is trivial. The third statement is proved as Proposition 7 of the Appendix. Q.E.D.

Let $X_0 = \{p\varepsilon E^n | \ ||p|| \leq 1\}$, let $\hat{I} = \{0\} \cup I = \{0,1,\ldots,m\}$, and let $\hat{X} = X_0 \times X = \prod_{\hat{I}} X_i$. Let $\hat{\phi}_0:\hat{X}\to X_0$ be the correspondence such that for all $(p,x)\varepsilon\hat{X}$, $\phi_0(p,x) = \{p'\varepsilon X_0 | p'\sum_I(x_i-w_i) > p\sum_I(x_i-w_i)\}$. For all $i\varepsilon I$, define $\hat{\phi}_i:X\to X_i$ as follows. If $(p,x)\varepsilon\hat{X}$ and $px_i > pw_i + \frac{1}{m}(1-||p||)$, then $\hat{\phi}_i(p,x) = \{x_i'\varepsilon X_i | px_i' < px_i\}$. If $(p,x)\varepsilon\hat{X}$ and $px_i \leq pw_i + \frac{1}{m}(1-||p||)$, then $\hat{\phi}_i(p,x) = \phi_i^*(x) \cap \{x_i'\varepsilon X_i | px_i' < pw_i + \frac{1}{m}(1-||p||)\}$.[4]

Lemma 4 For all $i\varepsilon\hat{I}$, $\hat{\phi}_i$ is l.s.c.

Proof: Let $(p,x)\varepsilon\hat{X}$ and $p'\varepsilon\phi_0(p,x)$. Then $p'\sum_I(x_i-w_i) > p\sum_I(x_i-w_i)$. Let $(p(n),x(n))\to(p,x)$. Then there exists an interger $N>0$ such that for all $n\geq N$, $p'\sum_I(x_i(n)-w_i) > p(n)\sum_I(x_i(n)-w_i)$. For every interger $k>0$, let $p'(k) = p'$. Then $p'(k)\varepsilon\hat{\phi}_o(p(N+k),x(N+k))$ for all $k>0$ and $p'(k)\to p'$. Therefore, by Proposition 3, $\hat{\phi}_o$ is l.s.c.

Let $(p,x)\varepsilon X$ and $x_i'\varepsilon\hat{\phi}_i(p,x)$ where $i\varepsilon I$. If $px_i > pw_i + \frac{1}{m}(1-||p||)$, then $px_i' < px_i$. Let $(p(n),x(n))\to(p,x)$. Then for some $N>0$, and for all integers $n\geq N$, $p(n)x_i(n) > p(n)w_i + \frac{1}{m}(1-||p(n)||)$, and $p(n)x_i' < p(n)x_i(n)$. For each integer $k>0$, let $x_i'(k) = x_i'$. Then $x_i'(k)\varepsilon\hat{\phi}_i(p(N+k), x(N+k))$ for all $k>0$ and $x_i'(k)\to x_i'$. Therefore $\hat{\phi}_i$ satisfies the convergence property of Proposition 3 at (p,x). Now let $x_i'\varepsilon\hat{\phi}_i(p,x)$ and $px_i \leq pw_i + \frac{1}{m}(1-||p||)$. Then $x_i'\varepsilon\phi_i{}^*(x)$ and $px_i' < pw_i + \frac{1}{m}(1-||p||)$. Let $(p(n),x(n))\to(p,x)$. Since $\phi_i{}^*$ is l.s.c., there is a subsequence $(x(n_k))$ of $(x(n))$ and a sequence $(x_i'(k))$ such that $x_i'(k)\varepsilon\phi_i{}^*(x(n_k))$ for all $k>0$ and $x_i'(k)\to x_i'$. For some integer $K>0$, $p(n_k)x_i'(k) < p(n_k)w_i + \frac{1}{m}(1-||p(n_k)||)$ for all $k\geq K$. Therefore for all $k\geq K$, $x_i'(k)\varepsilon\hat{\phi}_i(p(n_k),x(n_k))$. Since $x_i'(k)\to x_i'$, the convergence property of Proposition 3 is satisfied at (p,x). It follows that $\hat{\phi}_i$ is l.s.c.

Q.E.D.

Define the relation $\hat{P}_o\subset\hat{X}\times\hat{X}$ so that for $(p,x)\varepsilon\hat{X}$, $\hat{P}_o(p,x) = \{(p',x')|p'\varepsilon\hat{\phi}_o(p,x) \text{ and } x'=x\}$. For $i\varepsilon I$, define the relation $\hat{P}_i\subset\hat{X}\times\hat{X}$ so that $\hat{P}_i(p,x) = \{(p',x')|x_i'\varepsilon\hat{\phi}_i(p,x),\ p'=p, \text{ and } x_j'=x_j \text{ for all } j\neq i\}$.

Lemma 5 If for all $i\varepsilon I$, P_i, X_i and ϕ_i satisfy the assumptions of Theorem 4, then there exists a Cournot-Nash equilibrium for the society $(\hat{P}_i,X_i)_{i\varepsilon\hat{I}}$.

Proof: From the definitions of $\hat{P}_i$ for $i\varepsilon\hat{I}$, it follows that $\hat{\phi}_o(p,x) = \{p'\varepsilon X_o|(p',x)\varepsilon\hat{P}_i(p,x)\}$ and that for all $i\varepsilon I$, $\hat{\phi}_i(p,x) = \{x_i'\varepsilon X_i|(p,x_1,\ldots,x_i',\ldots x_m)\varepsilon\hat{P}_i(p,x)\}$. Thus the $\hat{\phi}_i$'s are related to the $\hat{P}_i$'s in the same way as the of conditions are sufficient to guarantee the existence of competitive equilibrium.

The assumption that the consumption sets X_i are compact could be replaced by the weaker assumption that for all $i \varepsilon I$, X_i is closed and bounded from below. The truncation technique devised by Debreu [7], can be adapted in in straightforward fashion for our proof of the existence of quasi-equilibrium. Production can also be incorporated into the model with little difficulty. This can be done either using the model of firms presented in Debreu [6], or the theory of induced preferences expounded by Rader [14] and extended to the case of non-transitive preferences in Rader [15].

Appendix - Properties of Lower Semi-Continuous Correspondences

Here we report several useful properties of l.s.c. correspondences. Propositions 1, 4, and 5 have simple proofs and are reported elsewhere. See Berge [1] and Michael [12]. Proposition 2 is immediate from Proposition 1.

Proposition 1 The correspondence $\phi:X\to 2^Y$ is l.s.c. iff for all $x\in X$, $y\in\phi(x)$ implies that for every open neighborhood V of y in Y there exists an open neighborhood U of x in X such that for all $x'\in U$, $\phi(x')\cap V \neq \phi$.

Proposition 2 Let $\phi:X\to 2^Y$. If for all $y\in Y$, $\{x|y\in\phi(x)\}$ is open in X, then ϕ is l.s.c.

Proposition 3 Where X and Y satisfy the first axiom of countability,[5] the following property is necessary and sufficient that $\phi:X\to 2^Y$ be l.s.c. If $y\in\phi(x)$ and $x(n)\to x$, there exists a subsequence $x(n_k)\to x$ such that for some sequence, $y(k)$ in Y, $y(k)\to y$ and $y(k)\in\phi(x(n_k))$ for every integer k.[6]

Proof of Proposition 3

Suppose that ϕ is l.s.c. Let $y\in\phi(x)$ and let $\{V_k|k=1,2,...\}$ be a countable base at y. For each k, let $\hat{V}_k = \bigcap_{i\leq k} V_i$. Since ϕ is l.s.c., for each k there exists a neighborhood U_k of x such that for all $x'\in U_k$, $\hat{V}_k\cap\phi(x')\neq\phi$. Let $\hat{U}_k = \bigcap_{i\leq k} U_i$. Since $x(n)\to x$, there exists a subsequence $(x(n_k))$ such that for every k, $x(n_k)\in\hat{U}_k$. Then there exists $y(k)\in\hat{V}_k\cap\phi(x(n_k))$. Since $\{\hat{V}_k|k=1,2,...\}$ is a base at y, $y(k)\to y$. Thus $(x(n_k))$ is a subsequence of $x(n)$ such that there exists a sequence $(y(k))$ in Y where $y(k)\to y$ and $y(k)\in\phi(x(n_k))$ for all k. This proves necessity.

Suppose that ϕ is not l.s.c. Then there exist $x\in X$, $y\in Y$ and a neighborhood V of y such that for every neighborhood U of x, there exists $x'\epsilon U$ such that $\phi(x')\cap V=\phi$. Let $\{U_n|n=1,2,\ldots\}$ be a countable base at x and for each k, let $\hat{U}_n=\bigcap_{i\leq n} U_i$. Then there exists a sequence $(x(n))$ such that for all n, $x(n)\epsilon\hat{U}_n$ and $\phi(x(n))\cap V=\phi$. Clearly $x(n)\to x$. But there is no subsequence $(x(n_k))$ of $(x(n))$ for which there exists a sequence $(y(k))$ where $y(k)\to y$ and $y(k)\epsilon\phi(x(n_k))$ for every k. This proves sufficiency. Q.E.D.

<u>Proposition 4</u> If Y is a linear topological space and $\phi:X\to 2^Y$ is l.s.c., then the correspondence, Ψ, $X\to 2^Y$ where $\Psi(x) = \text{con}\phi(x)$ is l.s.c.

<u>Proposition 5</u> Where $\mathcal{A}$ is an index set, let $\phi_\alpha:X\to Y$ be l.s.c. for all $\alpha\epsilon\mathcal{A}$. Then $\Psi:X\to Y$ is l.s.c. where $\Psi(x) = \bigcup_{\alpha\epsilon\mathcal{A}}\phi_\alpha(x)$.

<u>Proposition 6</u> Let $I = \{1,\ldots,m\}$ and $X = \prod_{i\epsilon I} X_i$ where each X_i is a topological space. Let $\phi_i:X\to 2^{X_i}$ be l.s.c. for all $i\epsilon I$. Let $\Psi_i:X\to 2^X$ where $\Psi_i(x) = \{y\epsilon X|y_i\epsilon\phi_i(x)$ and $y_j=x_j$ for all $j\neq i$, $j\epsilon I\}$. Then Ψ_i is l.s.c. for all $i\epsilon I$.

<u>Proof</u>: Let $y\epsilon\Psi_i(x)$ and $x(n)\to x$. Then $y_i\epsilon\phi_i(x)$ and since ϕ_i is l.s.c., there exists a subsequence $x(n_k)\to x$ and a sequence $y_i(k)$ in X_i such that $y_i(k)\to y_i$ and $y_i(k)\epsilon\phi_i(x(n_k))$ for all k. For all k, let $y(k) = (x_1(n_k),\ldots,y_i(k),\ldots,x_m(n_k))$. Then $y(k)\epsilon\Psi_i(x(n_k))$ and $y(k)\to y$. Therefore by Proposition 3, Ψ_i is l.s.c.

Q.E.D.

<u>Proposition 7</u> Let $\phi_i:\Pi X_i\to X_i$ be l.s.c. and define $\phi_i^*:\Pi X_i\to X_i$ so that $\phi_i^*(x) = \{z\epsilon X_i|z = \lambda y_i+(1-\lambda)x_i$ where $y_i\epsilon\phi_i(x)$ and $0<\lambda\leq 1\}$. Then ϕ_i^* is l.s.c.

If $z_i\epsilon\phi_i^*(x)$, then there exists $y_i\epsilon\phi_i(x)$ and a scalar λ such that $0<\lambda\leq 1$ and $z_i = \lambda y_i + (1-\lambda)x_i$. Let V be a neighborhood of z_i in X_i and choose $\varepsilon>0$ so that $V' = \{z_i'\epsilon X_i|\ ||z_i'-z_i|| < \varepsilon\}\subset V$. Since ϕ_i is l.s.c., there exists a neighborhood U of x in X such that for all $x'\epsilon U$, there exists

$y_i'(x')\varepsilon\phi_i(x')\cap V'$. Let $U' = U\cap\{x'\varepsilon X|\ ||x'-x|| < \varepsilon\}$. For all $x'\varepsilon U'$, let $z_i'(x') = \lambda y_i'(x') + (1-\lambda)x_i'$. Then $z_i'(x')\varepsilon\phi_i{}^*(x')$ and $||z_i'(x') - z_i|| \leqq \lambda||y_i'(x') - y_i|| + (1-\lambda)||x_i' - x_i|| < \varepsilon$. Therefore $z_i'(x')\varepsilon\phi_i{}^*(x')\cap V'$. It follows from Proposition 1 that $\phi_i{}^*$ is l.s.c. Q.E.D.

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Footnotes

1. Borglin and Keiting (1), attempt a demonstration of the existence of equilibria by a similar strategy.

2. For any set, S, con S denotes the smallest convex set containing S.

3. Several properties of l.s.c. mappings are studied in the Appendix of t this paper.

4. The mapping ϕ_i for $i \epsilon I$, is motivated as follows. The use of $pw_i + \frac{1}{n}(1-||p||)$ rather than pw_i as a "budget constraint" is a technical device which enables us to deal with economies in which there is neither monotonicity of preferences nor free disposal (see Bergstrom (3)). It will turn out that in "equilibrium", $||p||=1$. Where p and x are such that consumer i can "afford" x_i, we map to consumptions which are both "better than" x_i and can be afforded with less than consumer i's full budget. Where p and x are such that consumer i cannot "afford" x_i, we map to all points "cheaper than" x_i.

5. A topological space satisfies the first axiom of countability if the neighborhood system of every point has a countable base. This is true of all metric spaces. If X and Y lack this property, then lower semi-continuity may be characterized by the convergence of nets rather than sequences. The situation is in close analogy to the characterization of continuous functions by sequences or nets. See Kelley [10].

6. The reader may be familiar with the following characterization of lower semi-continuity. Where $y \epsilon \phi(x)$ and $x(n) \to x$, there exists a sequence $y(n) \to y$ such that $y(n) \epsilon \phi(x(n))$ for all n. If the image sets are always non-empty and X and Y satisfy the first axiom of countability, then this property is necessary and sufficient that ϕ be l.s.c. as defined here. If $\phi(x)$ can be empty this property is not necessary. (In particular it may be that $\phi(x(n))=\Phi$ for some n.)