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An Operator Approach To String Equations ¹

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Abstract

In this paper, a new approach to string dynamics is proposed. String coordinates are identified with a non-commuting set of operators familiar from free string quantization, and the dynamics follows from the Virasoro algebra. There is a very large gauge group operating on the non-commuting coordinates. The gauge has to be fixed suitably to make contact with the standard string picture.

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1. Introduction

In the last couple of years, it has become clear that string theory has a very rich structure, which was not apparent in its early formulation [1]. There has been progress in several parallel and overlapping directions: Various string theories which were once considered distinct are now known to be related by duality transformations [2], and new extended objects called branes were shown to play an important role in string dynamics [3,4]. There is also the interesting new idea of duality relating string theory in an AdS background to the conformal field theory on the boundary of the AdS space [5,6]. In view of all of these developments, there is clearly a need for a new approach at a fundamental level which can unify and explain all of these diverse aspects of string theory. Recently, such a unifying approach called M(atrix) theory has been proposed [7,8], and it has already led to a number of interesting results.

A natural idea that has been around for some time is to attribute the rich structure of string theory to an as yet undiscovered gigantic gauge symmetry. From this point of view, the gauge symmetries observed in the zero mass sector, the diffeomorphism invariance associated with the graviton, as well as various dualities are the manifestations of this hidden symmetry. However, the full structure of this conjectured gauge symmetry is yet to be discovered. One reason why this symmetry is so difficult to unmask in the usual perturbative formulation as well as in the string field theory approach [9-12] is that in these formulations, there is always a good deal of gauge fixing to start with. In most versions of string field theory, the gauge fixing is done either through the BRST method or by adopting the light cone gauge [13]. Both the original matrix theory and its compactified version, the matrix string theory [14,15] operate again within the gauge fixed lightcone framework. Still missing is a formulation in terms of a large number of mostly redundant coordinates and a big gauge group. In such a formulation, various different descriptions of string theory will correspond to different choices of gauge.

This paper is an attempt to answer two questions: What is this big gauge gauge group and also what is the space it acts upon? The sigma model [16-20] description of the zero mass modes of the string should shed some light on this problem. Imposing conformal invariance on this model results in equations of motion which are explicitly invariant under the gauge symmetries associated with the massless vector mesons and coordinate transformations associated with gravity. However, the neglect of the heavy modes of the string makes it impossible to go beyond these well known symmetries of the

low energy field theory limit. There is a version [21,22] of the sigma model which incorporates the heavy modes of the string, and conformal invariance in this case is realized through the Wilson renormalization group approach. Unfortunately, the gauge symmetries of the resulting equations are quite obscure, probably because of the particular cutoff used. In a modified version [23] of this approach, invariance under a certain class of coordinate transformations can be made explicit, but this only scratches the surface: A large number of symmetries are still hidden.

The basic idea of this paper is to formulate string dynamics in such a way that coordinates and momenta appear on equal footing. By coordinates and momenta we mean, not just the zero modes, but the full set of quantized modes of, for example, the free string. We will essentially make use of a phase space description, with a doubling of the coordinates. Corresponding to this enlarged coordinate space, there will be an enlarged gauge group, which will now include transformations that mix non-commuting variables. This generalization is motivated by two observations: Various generalized duality transformations [24] interchange coordinates and momenta, and if they are to be part of the proposed gauge group, transformations that mix coordinates and momenta should be included. In addition, the emergence of non-commutative geometry [25-31] as a viable description of some aspects of string theory has been very suggestive. In a sense, what we have here is a generalization of non-commutative geometry to include not just the zero modes, but all of the other modes of the string.

We need to specify a dynamical principle as the basis of our approach. Usually, the starting point is an action, which is required to be conformally invariant. The stress tensor of this action then generates a conformal algebra. We reverse this sequence of steps and start with the conformal algebra as the basic principle. The generators of the conformal algebra are assumed to be expressible in terms of the operators that generate the free string spectrum, and the dynamical equations we are seeking follow the requirement that the algebra close. An important bonus of this approach is that invariance under a big gauge group that mixes non-commuting variables is naturally built into the dynamical framework.

The paper is organized as follows: In section 2, we will first set up the problem in general terms, and then discuss the perturbative expansion around the flat (free string) background. As usual, such a perturbative expansion seems the only way of making progress in solving the dynamics. This enables one to linearize the equations as a first approximation, and the rest

of the section will mostly deal with these linear equations. We will present some special solutions to them based on the vertex construction, and we will then introduce a representation of the operators in terms of states, which will turn out to be useful for a general analysis of the equations. In section 3, we will study the open string dynamics in some detail, with the idea of making contact with the usual free string equations. We will be able to identify a state, which, after some gauge fixing, can be made to depend only on a set of commuting coordinates, which can be identified with position coordinates of the string. Moreover, this state satisfies the standard free string equations of motion. In the next section, these results are generalized to the closed string. In section 5, we will go beyond the linear approximation and turn our attention to the interaction term. A convenient expression for this term will be given in terms of coherent state representation. Finally, in section 6, the basic equations will be reformulated using the BRST formalism.

2. The General Setup

As explained in the introduction, our goal is to derive the dynamics of the bosonic string by constructing representations of the Virasoro algebra

$$[L_m, L_n] = (m - n)L_{m+n} + c(m^3 - m)\delta_{m+n}, \quad (1)$$

where c is the central charge. In the case of the closed string, we have to work with two commuting copies of this algebra. Any conformally invariant two dimensional theory will supply a realization of this algebra; the best known special case is the free field realization in terms of the operators α_m^μ ,

$$L_m^{(0)} = \frac{1}{2} \sum_{-\infty}^{\infty} \alpha_{m-n}^\mu \alpha_n^\mu. \quad (2)$$

The α 's satisfy the commutation relations

$$[\alpha_m^\mu, \alpha_n^\nu] = m \delta_{m+n} \eta^{\mu\nu}.$$

We wish to generalize the free field realization to an interacting one, still using the same set of operators. The most general expression for the L_m 's can somewhat symbolically be written in the form

$$L_m = \sum_i \phi(x)_m^{(i)} (: \prod \alpha :)_i. \quad (3)$$

In this equation, x is the position coordinate and the ϕ 's are the wavefunctions corresponding to various levels of the string. The sum goes over all of the possible products of α 's, normal ordered to avoid possible singular expressions. Imposing the commutation relations of eq.(1) on this ansatz will result in a set of equations for the ϕ 's, which we can identify as string field equations. These equations are easy to write down; but solving them is another matter. The only practical method seems to be a perturbation expansion around a background, which is an explicitly known solution of the same algebra. Therefore, although the equations are background independent, the perturbative solution will depend on the background. In this paper, we will expand the L 's around the free field solution of eq.(2), corresponding to a flat background. Setting

$$L_m = L_m^{(0)} + K_m,$$

gives

$$[L_m^{(0)}, K_n] - [L_n^{(0)}, K_m] + [K_m, K_n] - (m - n)K_{m+n} = 0. \quad (4)$$

Here we have assumed that the central charges of $L^{(0)}$'s and L 's are the same.

In the next section, we will investigate systematically the linear part of this equation, dropping the last term quadratic in the K 's:

$$[L_m^{(0)}, K_n] - [L_n^{(0)}, K_m] - (m - n)K_{m+n} = 0. \quad (5)$$

Eq.(5) is similar to the equation of motion of a non-abelian Chern-Simons gauge theory, with the difference that ordinary commuting coordinate derivatives are replaced by the non-commuting operators $L_m^{(0)}$. An important property it shares with that theory is invariance under the transformations

$$L_m \rightarrow U^{-1} L_m U, \quad (6)$$

where U is an arbitrary unitary operator constructed out of x and the α 's. These transformations can be identified with gauge transformations under which the dynamics is invariant. They will play an important role in what follows.

The linearized equations are invariant under the abelian part of the gauge transformations:

$$K_m \rightarrow K_m + [L_m, H]. \quad (7)$$

As usual, solutions related by gauge transformations will be considered equivalent.

In the case of closed strings, in addition to $L^{(0)}$'s, we have $\tilde{L}^{(0)}$'s, which have an expansion analogous to eq.(2) in terms of the $\tilde{\alpha}$'s, and there is also the corresponding $\tilde{K}$'s. So in addition to eq.(4), there is an identical equation for the variables with tildes, and also an equation that expresses the condition that L 's commute with $\tilde{L}$'s:

$$[L_m^{(0)}, \tilde{K}_n] - [\tilde{L}_n^{(0)}, K_m] + [K_m, \tilde{K}_n] = 0. \quad (8)$$

It is easy to find special solutions to eq.(5) through the standard vertex construction. The vertex operator, given by

$$V(k, \tau) =: \exp(ik \cdot X(\tau)) : \quad (9)$$

where

$$X^\mu(\tau) = x^\mu + \tau \alpha_0^\mu + i \sum_1^\infty \frac{1}{m} \left(e^{-im\tau} \alpha_m^\mu - e^{im\tau} \alpha_{-m}^\mu \right),$$

satisfies the relation[ref]

$$[L_m^{(0)}, V(k, \tau)] = -i e^{im\tau} \frac{d}{d\tau} (V(k, \tau)) + \frac{1}{2} m k^2 e^{im\tau} V(k, \tau). \quad (10)$$

As a result, we can set the K 's equal to the vertex for tachyon emission:

$$K_m = e^{im\tau} V(k, \tau), \quad (11)$$

with the condition

$$k^2 = 2,$$

and eq.(5) is satisfied for any τ and any k that satisfies the tachyon mass shell condition. Furthermore, this solution is abelian; different K 's commute, and as a result, it satisfies the initial exact equation (4). We also note that different values of the parameter τ correspond to gauge equivalent results, since

$$\left[L_m, \int_{\tau_1}^{\tau_2} d\tau V(k, \tau) \right] = -i \left(e^{im\tau_2} V(k, \tau_2) - e^{im\tau_1} V(k, \tau_1) \right). \quad (12)$$

One can easily construct an infinite number of additional solutions for the K 's by replacing the tachyon emission vertex by the emission vertices for the higher excited states.

The solution given above also generalizes to the closed string. In this case, we have two commuting set of operators, α 's and $\tilde{\alpha}$'s, and two commuting

set of Virasoro algebras generated by the L 's and $\tilde{L}$'s. Let $\tilde{V}$ denote the vertex operator of eq.(8), constructed from $\tilde{\alpha}$'s instead of α 's. Then, the corresponding K 's and $\tilde{K}$'s are given by

$$\begin{aligned} K_m &= \int_0^{2\pi} d\sigma e^{im(\tau-\sigma)} \tilde{V}(k, \tau - \sigma) V(k, \tau + \sigma), \\ \tilde{K}_m &= \int_0^{2\pi} d\sigma e^{im(\tau+\sigma)} \tilde{V}(k, \tau - \sigma) V(k, \tau + \sigma), \end{aligned} \quad (13)$$

with

$$k^2 = 2.$$

In this case, the solution only satisfies the linearized equations. Again, vertices for the emission of excited states provide additional solutions.

We now wish to analyze the linear equations for the K 's in general, using the operator expansion of eq.(3). We found it more convenient to carry out this analysis in what amounts to a phase space representation of the operators. In this representation, a one to one correspondence is set up between the normal ordered operator products and a set of states. The states are constructed by the following procedure: First, a new set of operators, β_m^μ , which commute with the α 's, are introduced, as well as a new c-number momentum P , which commutes with x . Next, the old non-commuting set of operators are replaced by a new set of commuting operators according to the recipe

$$K_m \rightarrow |m\rangle, \quad \alpha_{-m} \rightarrow \alpha_{-m}, \quad \alpha_m \rightarrow \beta_{-m}, \quad \alpha_0 = -i\partial \rightarrow P,$$

where $m > 0$. Finally, the application of the resulting operator on the vacuum gives the state which is associated with the original operator. For example, the normal ordered operator product

$$\phi(x) \alpha_0^\mu \alpha_{-m}^\nu \alpha_n^\lambda$$

corresponds to the state

$$\phi(x) P^\mu \alpha_{-m}^\nu \beta_{-n}^\lambda |0\rangle.$$

According to our definition of the normal product, the operators $\alpha_0^\mu = -i\partial^\mu$ are always placed to the right of $\phi(x)$.

We shall need the expressions for $L_m^{(0)}$'s acting on these states. We start with a general normal ordered operator product, and we compute its commutator with $L_m^{(0)}$ given by eq.(2). The original operator product, as well

as the final commutator, can be represented by states, as explained above. We therefore have a mapping from one set of states to another induced by $L_m^{(0)}$'s, and this mapping provides us with a representation different from eq.(2) for the $L^{(0)}$'s. We will call it the state representation for the $L^{(0)}$'s and designate them by the same symbol as before when no confusion can arise. It is straightforward to compute the explicit expressions for $L^{(0)}$'s in this new representation; however, as they stand, these expressions are a bit complicated. They can be simplified considerably by making an additional canonical transformation

$$\alpha_m \rightarrow \alpha_m - \beta_{-m}, \quad \beta_m \rightarrow \beta_m - \alpha_{-m},$$

where $m > 0$. To avoid complicating the notation, we will use the same letters for the new set of operators, it being understood that, from now on, α 's and β 's will stand for the new set of operators. Finally, we have the following simple result:

$$L_m^{(0)} \rightarrow L_m^\alpha - L_{-m}^\beta, \quad (14)$$

where L^α is given by eq.(2) and L^β has an identical expansions in terms of the β 's. The only change is in the definition of the zero modes; they are now given by

$$\alpha_0^\mu = P^\mu - i\partial^\mu, \quad \beta_0^\mu = P^\mu. \quad (15)$$

The lack of symmetry between the zero modes is due to our particular normal ordering prescription for the operators x and $-i\partial$ discussed earlier. We also note that the $L^{(0)}$'s defined by eq.(14) have no central charge. This is to be expected, since the action of $L^{(0)}$'s on the states is defined by their commutators with the corresponding operators, and the central charge drops out of the picture since it commutes with everything.

The linearized equations(eq.(5)), in this new language, are

$$(L_m^\alpha - L_{-m}^\beta) |n\rangle - (L_n^\alpha - L_{-n}^\beta) |m\rangle - (m - n) |m + n\rangle = 0, \quad (16)$$

where the state $|m\rangle$ has replaced the operator K_m . In the language of these new states, hermiticity properties of the original operator products are a bit complicated. Hermitian conjugation of the operator product corresponds to the following interchange of the corresponding states:

$$\alpha_m \leftrightarrow \beta_m, \quad \phi \leftrightarrow \phi^*, \quad |m\rangle \leftrightarrow -|-m\rangle, \quad P \leftrightarrow P - i\partial. \quad (17)$$

3. Solutions to the Linear Equations for the Open String

In this section, we will investigate the linearized equations for the open string (eqn.(5)), and connect the solutions with the well-known states of the free bosonic string. At first sight, it seems that we have both too many states and too many independent variables. Instead of a single string state, there are infinitely many of them labeled by an integer m . Also, the number of coordinates are doubled compared to the standard string picture. The states are now constructed using two new sets of operators: β 's in addition to the α 's, P in addition to x . On the other hand, there are also a large set of gauge transformations given by eq.(7), which translate into

$$|m\rangle \rightarrow |m\rangle + (L_m^\alpha - L_{-m}^\beta) |g\rangle, \quad (18)$$

where $|g\rangle$ represents an arbitrary "gauge" state. As we shall show, equivalence under these gauge transformations will drastically reduce the number of independent states; only the state with $m = -1$ will turn out to be independent. Also, this state will be a function of only the α 's and x , and not of the β 's and P . This reduction from the original set of non-commuting coordinates to a final set of commuting coordinates will be accomplished by a (partial) gauge fixing. To see this, let us organize the states as follows: Consider an expansion of the states in powers of P , and label each term in the expansion by n_p , the number of factors of P it contains. In addition, we label them by the level numbers n_α and n_β ; they are respectively the eigenvalues of the operators

$$N_\alpha = \sum_{m=1}^{\infty} \alpha_{-m} \alpha_m,$$

and

$$N_\beta = \sum_{m=1}^{\infty} \beta_{-m} \beta_m.$$

A given state $|m\rangle$ can therefore be written as a sum:

$$|m\rangle = \sum_{n_p, n_\alpha, n_\beta} |m, n_\alpha, n_\beta, n_p\rangle. \quad (19)$$

We note that all of the n 's are non-negative integers. Also, it is easy to show that the difference $n_\alpha - n_\beta$ is the same for each term of this expansion.

We now focus on the state with $m = -1$. In principle, we could have started with any other value of m , but it turns out that this particular value of m has special properties that makes it easy to work with. We wish to establish the following two results:

- a) In the expansion of this state, all of the terms with either n_p or n_β greater than zero can be gauged away.
- b) After fixing gauge, the complete solution to eq.(16) can be expressed in terms of the single remaining component of $|-1\rangle$, namely $|-1, n_\alpha, 0, 0\rangle$. This state, which depends only x and the α 's, can be identified with usual string state. Furthermore, we will show that it satisfies the correct mass shell equation and the subsidiary conditions.

Since we are contemplating an expansion in powers of P , it is convenient to separate the L 's into two parts; one of them linear in P and the other P independent:

$$L_m^\alpha = L_m^\alpha(0) + P.\alpha_m, \quad L_m^\beta = L_m^\beta(0) + P.\beta_m.$$

The condition for a state $|-1, n_\alpha, n_\beta, n_p\rangle$ to be spurious (pure gauge) follows from eq.(18). If gauge states $|g, n_\alpha, n_\beta, n_p\rangle$, satisfying the equation

$$\begin{aligned} L_1^\beta(0)|g, n_\alpha, n_\beta + 1, n_p\rangle &= |-1, n_\alpha, n_\beta, n_p\rangle + L_{-1}^\alpha(0)|g, n_\alpha - 1, n_\beta, n_p\rangle \\ &+ P.\alpha_{-1}|g, n_\alpha - 1, n_\beta, n_p - 1\rangle + P.\beta_1|g, n_\alpha, n_\beta + 1, n_p - 1\rangle, \end{aligned} \quad (20)$$

can be found, then this state is spurious. We wish to show that this equation has solutions for states with either or both n_β and n_p greater than or equal to 1. The basic idea is to solve it by iteration, starting with the smallest values of n_β and n_p and working up to larger values. The gauge states corresponding to the smallest value of n_β , $n_\beta = 0$, are not determined by the above equation, and we set them equal to zero for simplicity. Starting with $n_\beta = 0$, we note that this is a special case, since the left hand side of eq.(20) vanishes:

$$L_1^\beta(0)|n_\beta = 1\rangle = 0. \quad (21)$$

In this case, we can solve for the gauge state $|g, n_\alpha, 1, n_p\rangle$ by inverting the operator $P.\beta_1$:

$$|g, n_\alpha, 1, n_p\rangle = -\frac{1}{(n_p + 1)!}\beta_{-1}.\frac{\partial}{\partial P}(|-1, n_\alpha, 0, n_p + 1\rangle). \quad (22)$$

This shows that all of the $m = -1$ states with $n_\beta = 0$ and $n_p \geq 1$ can be gauged away. Now let us consider the next set of states at $n_\beta = 2$. In this case, we can write

$$|g, n_\alpha, n_\beta + 1, n_p\rangle = \left(L_1^\beta(0)\right)^{-1} \left(|-1, n_\alpha, n_\beta, n_p\rangle + L_{-1}^\alpha(0)|g, n_\alpha - 1, n_\beta, n_p\rangle + P.\alpha_{-1}|g, n_\alpha - 1, n_\beta, n_p - 1\rangle + P.\beta_1|g, n_\alpha, n_\beta + 1, n_p - 1\rangle\right). \quad (23)$$

We note that the states on the right hand side of this equation all have $n_\beta \geq 1$, and acting on these states, the operator $L_1^\beta(0)$ is invertible. This is in contrast to the previous case when, acting on states with $n_\beta = 0$, $L_1^\beta(0)$ was not invertible, and that is why eq.(20) for $n_\beta = 1$ had to be treated as a special case. Now taking $n_\beta = 1$ and $n_p = 0$ in eq.(23), the right hand side of the equation is completely known. This is because the gauge states with $n_\beta = 1$ are already known (eq.(22)), and the last term on the right hand side vanishes since there is no state with $n_p = -1$. The left hand side of this equation then gives us all of the gauge states with $n_\beta = 2$ and $n_p = 0$. Now take $n_p = 1$ with again $n_\beta = 1$; again the right hand side is completely known, and recycling the equation once more gives us the gauge states with $n_\beta = 2$ and $n_p = 1$. Continuing this iteration, all of the gauge states with $n_\beta = 2$ and arbitrary n_p are determined. Next we set $n_\beta = 2$ in eq.(23) and continue the double iteration process till the gauge states for all of the values of n_β and n_p are determined. This argument establishes the result that all of the states $|-1, n_\alpha, n_\beta, n_p\rangle$ with either of or both n_β, n_p greater than or equal to 1 can be gauged away.

Having shown that the only surviving piece of the state with $m = -1$ has $n_\beta = 0$ and $n_p = 0$, we now derive the equation satisfied by this state. First setting $n = -1$, and $m \geq 0$ in eq.(16), and then projecting out the $n_\beta = 0$, $n_p = 0$ component, we have

$$L_m^\alpha(0)|-1, n_\alpha + m, 0, 0\rangle - L_{-1}^\alpha(0)|m, n_\alpha - 1, 0, 0\rangle - (m + 1)|m - 1, n_\alpha, 0, 0\rangle = 0. \quad (24)$$

Notice that $L_{-m}^\beta(0)$ does not contribute for $m \geq 0$ because of the projection into $n_\beta = 0$ sector, and neither does $L_1^\beta(0)$ because of the eq.(21). Consider the special case $m = 0$ in the above equation:

$$(L_0^\alpha(0) - 1)|-1, n_\alpha, 0, 0\rangle - L_{-1}^\alpha(0)|0, n_\alpha - 1, 0, 0\rangle = 0. \quad (25)$$

We now argue that the state $|0, n_\alpha - 1, 0, 0\rangle$ can be gauged away. Under gauge transformations,

$$|0, n_\alpha, 0, 0\rangle \rightarrow |0, n_\alpha, 0, 0\rangle + L_0^\alpha(0)|g, n_\alpha, 0, 0\rangle.$$

The operator $L_0^\alpha(0)$ is always invertible, since it contains a term of the form $\alpha_0^2 = -\square$, which can always be inverted. Therefore, a suitable gauge state can always be chosen to make the last term in eq.(25) vanish. With this gauge choice, we have the standard mass shell condition:

$$(L_0^\alpha(0) - 1)|-1, n_\alpha, 0, 0\rangle = 0. \quad (26)$$

It is important to notice that the choice $m = -1$ for the state in question was crucial in arriving at the mass shell condition given above. With any other choice of m , there would be extra terms in the corresponding equation; for example, the analogue of eq. (22) for $m = -2$,

$$L_2^\beta(0)|n_\beta = 2\rangle = 0$$

is simply not true. It is possible to go beyond the mass shell condition and establish the standard subsidiary conditions

$$L_m^\alpha(0)|-1, n_\alpha, 0, 0\rangle = 0, \quad m \geq 1, \quad (27)$$

by showing that the states $|m, n_\alpha, 0, 0\rangle$ for $m \geq 1$ can all be gauged away. This can be done, for example, by an adaptation of the standard string theory argument[1].

Having established that the state $|-1, n_\alpha, 0, 0\rangle$ can be made to satisfy the standard string equations (eqs.(26) and (27)), the next step is to show that, with further gauge fixing, all of the other states, up to gauge equivalence, can be determined in terms of this single state. In section (6), this will be shown using the BRST method; here, to illustrate how this process works in a simple example, we will construct an explicit solution for the states with $n_\beta = 1$ and $n_p = 0$ for $m \geq 0$. We take $n = -1$, $m \geq 0$ in eq.(16), and project out the component corresponding to $n_\beta = 0$ and $n_p = 1$:

$$\begin{aligned} & L_m^\alpha(0)|-1, n_\alpha + m, 0, 1\rangle - L_{-1}^\alpha(0)|m, n_\alpha - 1, 0, 1\rangle - (m + 1)|m - 1, n_\alpha, 0, 1\rangle \\ & + P.\alpha_m|-1, n_\alpha + m, 0, 0\rangle + P.\beta_1|m, n_\alpha, 1, 0\rangle = 0. \end{aligned} \quad (28)$$

In this equation, the state $|-1, n_\alpha + m, 0, 1\rangle$ vanishes by the previous gauge choice. The following ansatz provides a solution;

$$\begin{aligned} |m, n_\alpha, 0, 1\rangle &= 0, \quad m \geq 0, \\ |0, n_\alpha, 1, 0\rangle &= i\beta_{-1} \cdot \partial |-1, n_\alpha, 0, 0\rangle, \\ |m, n_\alpha, 1, 0\rangle &= -\beta_{-1} \cdot \alpha_m |-1, n_\alpha + m, 0, 0\rangle \quad m \geq 1. \end{aligned} \quad (29)$$

There is a consistency check: The solution given above should also satisfy the $n_\beta = 1, n_p = 0$ component of eq.(16) for $m, n \geq 0$,

$$\begin{aligned} L_m^\alpha(0)|n, n_\alpha + m, 1, 0\rangle - L_n^\alpha(0)|m, n_\alpha + n, 1, 0\rangle - (m - n)|m + n, n_\alpha, 1, 0\rangle \\ - \delta_{m,0}|n, n_\alpha, 1, 0\rangle + \delta_{n,0}|m, n_\alpha, 1, 0\rangle = 0, \end{aligned} \quad (30)$$

and indeed it does. In principle, solutions for higher values of n_β and n_p , as well as for the negative values of m can be constructed in this fashion, although this brute force method soon becomes too laborious.

We close this section with some further comments on gauge invariance. Starting with a very rich gauge group (eq.(18)), a good deal of gauge fixing had to be done to arrive at the string equations (26) and (27) for $|-1, n_\alpha, 0, 0\rangle$, our candidate for the physical string state. There is, however, still invariance under the residual gauge transformations

$$|-1, n_\alpha, 0, 0\rangle \rightarrow |-1, n_\alpha, 0, 0\rangle + L_{-1}^\alpha(0)|g, n_\alpha - 1, 0, 0\rangle, \quad (31)$$

where the gauge state satisfies

$$L_0^\alpha(0)|g, n_\alpha - 1, 0, 0\rangle = 0.$$

This is the well-known invariance under the shift by zero norm states generated by L_{-1} , and it is valid in any dimension. What is missing is the invariance under shifts by zero norm states generated by L_{-m} 's for $m > 1$, which is only valid in the critical dimension. This is a really troublesome puzzle, and we have only the following tentative suggestion for a solution. It is possible that the choice of the state with $m = -1$, and $n_\beta = n_p = 0$ for the physical string state, although it has the advantage of simplifying various computations, is not suitable for fully understanding the residual gauge invariance. For example, the choice of the state

$$|s, n_\alpha, 0, 0\rangle = \sum_{m=-1}^{n_\alpha} |m, n_\alpha, 0, 0\rangle, \quad (32)$$

for the physical state has the advantage that, under gauge transformations,

$$|s, n_\alpha, 0, 0\rangle \rightarrow |s, n_\alpha, 0, 0\rangle + \sum_{m=1}^{n_\alpha} L_{-m}^\alpha(0) |g, n_\alpha - m, 0, 0\rangle, \quad (33)$$

the gauge states are generated by all of the L_{-m} 's, not just by L_{-1} . However, we have found states of this type difficult to work with and we will not pursue this point any further.

4. Solutions to the Linear Equations for the Closed String

In this section, we will briefly discuss the solutions to the linear closed string equations. Our treatment will be less complete compared to the case of the open string. The operators K_m and $\tilde{K}_m$ are then replaced by the states $|m\rangle$ and $|\tilde{m}\rangle$. There are now two copies of the equation (16), plus an equation expressing the commutativity of the two Virasoro algebras (eq.(8)):

$$L_m^{(0)}|n\rangle - L_n^{(0)}|m\rangle - (m-n)|m+n\rangle = 0, \quad (34)$$

$$\tilde{L}_m^{(0)}|\tilde{n}\rangle - \tilde{L}_n^{(0)}|\tilde{m}\rangle - (m-n)|\tilde{m} + \tilde{n}\rangle = 0, \quad (35)$$

$$\tilde{L}_m^{(0)}|n\rangle - L_n^{(0)}|\tilde{m}\rangle = 0, \quad (36)$$

where,

$$L_m^{(0)} = L_m^\alpha - L_{-m}^\beta, \quad \tilde{L}_m^{(0)} = L_m^{\bar{\alpha}} - L_{-m}^{\bar{\beta}}.$$

The gauge transformations are given by

$$\begin{aligned} |m\rangle &\rightarrow |m\rangle + L_m|g\rangle, \\ |\tilde{m}\rangle &\rightarrow |\tilde{m}\rangle + \tilde{L}_m|g\rangle. \end{aligned} \quad (37)$$

Our discussion of the closed string states will more brief and less complete compared to the open string. As in the last section, we expand the states in powers of P and level numbers n_α, n_β , and now, in addition, in level numbers $n_{\bar{\alpha}}$ and $n_{\bar{\beta}}$. The states are thus labeled as

$$|m, n_\alpha, n_\beta, n_{\bar{\alpha}}, n_{\bar{\beta}}, n_p\rangle, \quad |\tilde{m}, n_\alpha, n_\beta, n_{\bar{\alpha}}, n_{\bar{\beta}}, n_p\rangle.$$

The usual constraint between the right-left level numbers requires that

$$n_\alpha - n_\beta = n_{\bar{\alpha}} - n_{\bar{\beta}}. \quad (38)$$

We have now to make a choice for the physical string state. In parallel with the open string, we will focus on the states with $m = -1$ and set

$$|-1, n_\alpha, 0, n_{\tilde{\alpha}}, 1, 0\rangle = i\tilde{\beta}_{-1} \cdot \partial |s, n_\alpha, 0, n_{\tilde{\alpha}}, 0, 0\rangle, \quad (39)$$

where $n_\alpha = n_{\tilde{\alpha}}$ and the state

$$|s, n_\alpha, 0, n_{\tilde{\alpha}}, 0, 0\rangle$$

is identified with the string state. There is, of course, some arbitrariness in this choice, for example, the reason for having the operator $i\tilde{\beta}_{-1} \cdot \partial$ on the right is not yet clear. Also, this choice is not symmetric between left and right moving (tilde and non-tilde) states. We hope to clarify these issues as we proceed.

We have to show that the state defined above satisfies two sets of equations; one set involving the L 's and the other set $\tilde{L}$'s. The first set of equations are easy to derive. Starting with eqn.(34), and following the same steps as in the last section, one arrives at the analogues of eqns.(26) and (27):

$$\begin{aligned} (L_0^\alpha(0) - 1) |s, n_\alpha, 0, n_{\tilde{\alpha}}, 0, 0\rangle &= 0, \\ L_m^\alpha(0) |s, n_\alpha, 0, n_{\tilde{\alpha}}, 0, 0\rangle &= 0, \quad m \geq 1. \end{aligned} \quad (40)$$

We now turn our attention to eqs.(35) and (36). In particular, eq.(36) imposes stringent conditions on the choice of the string state. Take $m = n = -1$ and set $n_p = 0$ in this equation:

$$\tilde{L}_{-1}^{(0)}(0) |m = -1, n_p = 0\rangle = L_{-1}^{(0)}(0) |\tilde{m} = -1, n_p = 0\rangle. \quad (41)$$

For a non-trivial solution, both sides of this equation must vanish; otherwise, the operators $L_{-1}(0)$ and $\tilde{L}_{-1}(0)$ would be invertible and the resulting solution would be a pure gauge. We therefore set

$$\tilde{L}_{-1}(0) |-1, n_p = 0\rangle = \left(L_{-1}^{\tilde{\alpha}}(0) - L_1^{\tilde{\beta}}(0) \right) |-1, n_p = 0\rangle = 0. \quad (42)$$

To satisfy this condition, we write this state as a sum over $n_{\tilde{\beta}}$,

$$|-1, n_p = 0\rangle = \sum_{n_{\tilde{\beta}}=1}^{\infty} |-1, n_{\tilde{\beta}}, n_p = 0\rangle,$$

and identify the term with $n_{\tilde{\beta}} = 1$ with the state defined by eq.(39). We then have the following solution:

$$|-1, n_p = 0\rangle = \sum_{k=0}^{\infty} \left(L_1^{\tilde{\beta}}(0)\right)^{-k} \left(L_{-1}^{\tilde{\alpha}}(0)\right)^k (i\tilde{\beta}_{-1} \cdot \partial) |s, n_{\alpha}, 0, n_{\tilde{\alpha}}, 0, 0\rangle. \quad (43)$$

We note that the operator $L_1^{\tilde{\beta}}(0)$ is invertible on states with $n_{\tilde{\beta}} \geq 1$, and therefore each term on the right hand side of the above equation is well defined. This is one of the reasons for the presence of the operator $\tilde{\beta}_{-1}$ in the definition of the state given by eq.(39); as a result, the sum in the equation above starts at $n_{\tilde{\beta}} = 1$.

Next, we take $n = -1$ and $m \geq 0$ in eq.(36):

$$\tilde{L}_m(0) |n = -1, n_p = 0\rangle = L_{-1}(0) |\tilde{m} = m, n_p = 0\rangle \quad m \geq 0.$$

Again, unless both sides vanish, the candidate for the string state would be a pure gauge. We must therefore have

$$\left(L_m^{\tilde{\alpha}}(0) - L_{-m}^{\tilde{\beta}}(0)\right) |-1, n_p = 0\rangle = 0, \quad m \geq 0.$$

Now project into the $n_{\tilde{\beta}} = 0$ sector in this equation. Because of this projection, the operator $L_{-m}^{\tilde{\beta}}(0)$ does not contribute for $m > 0$, and at $m = 0$, and acting on the state defined by eq.(39), it gives one. The resulting equations complement eqns.(40):

$$\begin{aligned} \left(L_0^{\tilde{\alpha}}(0) - 1\right) |s, n_{\alpha}, 0, n_{\tilde{\alpha}}, 0, 0\rangle &= 0, \\ L_m^{\tilde{\alpha}}(0) |s, n_{\alpha}, 0, n_{\tilde{\alpha}}, 0, 0\rangle &= 0, \quad m \geq 1. \end{aligned} \quad (44)$$

Another reason for having $\tilde{\beta}_{-1}$ in eq.(39) is that it gives the correct mass shell condition in the above equation.³

As an application of the formalism developed here, we note that T duality, under which the left moving (tilde) components are multiplied by -1, is easily implemented by a transformation of the form given by eq.(6). Similarly, we hope that various other dualities can be implemented by gauge transformations of the same type. An important point is that although these

³Our normalization of the slope parameter for the closed string differs by a factor of two from the conventional one.

symmetries are usually quite transparent in the original non-gauge fixed form of the string equations, they are difficult to see in the gauge fixed form of the equations.

5. The Interaction Term

So far, we have considered only the linearized equations. Our task in this section is to express the non-linear interaction term in eq.(4) in the language of states. For this purpose, it is very convenient to use a coherent state representation for the states:

$$|F\rangle = \int Du Dv F_m(u, v) \exp \left(\sum_1^{\infty} \frac{1}{k} \alpha_{-m} \beta_{-m} \right) \times \exp \left(iu_0 \cdot x + iv_0 \cdot P + \sum_1^{\infty} (u_m \alpha_{-m} + v_m \beta_{-m}) \right) |0\rangle. \quad (45)$$

The states are constructed from the operators α_m , β_m and P described in section (1), but they are now labeled by $F(u, v)$, a functional of variables u_m^μ and v_m^μ . Let the states representing K_m and K_n be labeled by F_m and F_n , and the state representing $[K_m, K_n]$ by $F_{(m,n)}$. The commutator $[K_m, K_n]$ is easily calculated using the coherent state representation. The result is

$$F_{(m,n)}(u, v) = \int Du' Dv' \int Du'' Dv'' F_m(u', v') F_n(u'', v'') \prod_i \delta(u'_i + u''_i - u_i) \prod_j \delta(v'_j + v''_j - v_j) \left(\exp \left(iv'_0 \cdot u''_0 + \sum_1^{\infty} kv'_k \cdot u''_k \right) - \exp \left(v''_0 \cdot u'_0 + \sum_1^{\infty} kv''_k \cdot u'_k \right) \right) \quad (46)$$

It is convenient to define a star product [11] of the states labeled by F_m and F_n to be the state labeled by $F_{(m,n)}$,

$$|F_m\rangle \star |F_n\rangle = |F_{(m,n)}\rangle,$$

where $F_{(m,n)}$ is given by eq.(46). The star product defined in this way is non-commutative but associative. Eq.(4) can now be cast into the form

$$L_m^{(0)}|F_n\rangle - L_n^{(0)}|F_m\rangle - (m - n)|F_m\rangle \star |F_n\rangle = 0. \quad (47)$$

So far, we have considered the open string. However, it is straightforward to generalize everything we have done to the closed string by simply doubling the number of operators.

We have seen that the Virasoro algebra naturally gives rise to an interaction term. The next question is whether it agrees with the standard string interaction. It seems quite plausible that the standard string interaction should provide at least one particular solution to our equations, since it is highly unlikely that standard string theory is eliminated by imposing Virasoro invariance. Of course, the existence of additional solutions cannot a priori be ruled out. One way to attack this problem is to gauge fix to the light cone gauge and to compute the interaction term in that gauge. We hope to return to this problem in the future.

6. The BRST Formulation

An elegant and convenient way of handling a theory invariant under a gauge symmetry is the BRST method, which has been extensively used in the various formulations of the string theory [11]. Here, we will first show how to reformulate the linear equations for the open string (eq.(16)) as the nilpotency condition for the BRST operator, and then we will generalize to include the interaction term. The standard BRST operator for the free Virasoro algebra is given by,

$$Q = \sum_{-\infty}^{\infty} L_m^{(0)} c_{-m} - \frac{1}{2} \sum_{-\infty}^{\infty} (m-n) c_{-m} c_{-n} b_{m+n}, \quad (48)$$

where $L_m^{(0)}$ is given by eq.(2), and b 's and c 's satisfy the usual anticommutation relations:

$$\{c_m, b_n\} = \delta_{m,-n}.$$

We know that Q is nilpotent at the critical dimension:

$$Q^2 = 0.$$

Next, we wish to determine the form of Q in the state representation. This can be done directly by essentially going through the steps that led to eq.(14). In the present case, we start with normal ordered operator products constructed out of b 's and c 's in addition to the α 's, and represent these operators by states as in section (2). To do this, we have to introduce the operators β as before, and also we need two sets of additional anticommuting operators, $\bar{b}_m$ and $\bar{c}_m$, which are "partners" of the b 's and the c 's. Next we compute the anticommutator of Q with the operator product. This anticommutator induces a mapping from the initial set of states to the set of states representing the anticommutator, which then provides the state representation for Q .

This procedure, although straightforward, gives a complicated and unwieldy result. Instead, we will try and guess a simple and economical form for Q , and afterwards check that the equation

$$Q|s\rangle = 0 \quad (49)$$

correctly reproduces eq.(16).

Our guess for the state representation for Q is the following: We will keep eq.(48) as it stands, but the $L_m^{(0)}$'s of eq.(2) will be replaced by those of eq.(14). Also, our choice of the vacuum will be different from the usual one. The standard vacuum is defined so that it is annihilated by all c_m and b_m for $m \geq 1$. Instead, our vacuum will satisfy

$$b_m|0\rangle = 0,$$

for all m , and no other conditions. In particular, none of the c 's annihilate the vacuum. It is important to check whether the nilpotency condition

$$Q^2 = 0$$

is satisfied. Generically, this condition is violated by the central charge of the Virasoro algebra. Only in the critical dimension, the central charge of the Virasoro generators for the matter fields cancel against the central charge of the Virasoro generators for the ghost fields. In the present case, the generators for the matter fields, given by eq.(14), have zero central charge. On the other hand, it is easy to verify that, with our new definition of the vacuum, the Virasoro generators of the ghost sector also have no central charge, and the nilpotency condition is still satisfied.

We will now briefly discuss equation (49). At ghost number zero, there are no solutions to this equation; in particular, the vacuum is no solution. At ghost number 1, the state

$$|s\rangle = \sum_{-\infty}^{\infty} c_k |k\rangle, \quad (50)$$

is a solution if and only if the states $|k\rangle$ satisfy eq.(16). As usual, the gauge transformations (18) are generated by the BRST transformations

$$|s\rangle \rightarrow |s\rangle + Q|g\rangle,$$

where $|g\rangle$ has ghost number zero. It is now easy to incorporate the interaction term; identifying the states $|k\rangle$ in eq.(50) with the coherent states of eq.(45),

$$|s\rangle \rightarrow \sum c_{-k} |F_k\rangle,$$

we define a star product of two states $|s\rangle$ and $|s'\rangle$ by

$$|s\rangle \star |s'\rangle = \sum c_{-k} c_{-l} |F_k\rangle \star |F'_l\rangle.$$

The equations with interaction(eq.(47)) can then be written as

$$Q|s\rangle + |s\rangle \star |s\rangle = 0. \quad (51)$$

It is easy to see that this equation is consistent with the nilpotency of Q . After all, all we have done is to rewrite eq.(4) in a fancier language, and $Q^2 = 0$ is equivalent to the Jacobi identity for the L 's.

The BRST formalism makes it possible to simplify and streamline the derivation results of the earlier sections, although we will not try to do it here. We should also point out that there are some problems with the unconventional definition of the vacuum state we have adopted; for example, the space of states defined in this fashion is not self dual and as a result, the BRST operator is not self adjoint. The alternative construction that we have sketched following eq.(48) avoids this problem, but as we have already mentioned, leads to a complicated expression.

As a final comment, we would like to point out that so far, we have only considered the equations of motion for the string states. However, given the BRST operator and the star product, it is not too difficult to construct an action from which these equations follow. For example, proceeding as in reference [11], one could try the action

$$I = \langle s|Q|s\rangle + \langle s|(|s\rangle \star |s\rangle), \quad (52)$$

where $|s\rangle$ is now a general state of odd ghost number. We hope to return to this problem in the future.

7. Conclusions and Future Directions

The basic idea of this paper was to derive string dynamics from the realization of the conformal algebra on a suitable operator space. The operator space we have chosen is the familiar one that results from quantizing the flat

space free field background. By requiring the closure of the conformal algebra, we have derived a set of dynamical equations for the string. The novelty of this approach lies in the identification of the non-commuting operators with string coordinates. This results in the enlargement of the coordinate space; but in compensation, there is invariance under a large gauge group given by eq.(6). To reach the standard picture of the string, one has to do some suitable gauge fixing.

There are many interesting questions left to answer. For example, can one gain some insight into the non-perturbative symmetries of string theory by exploiting the new gauge invariance? Some other new avenues of exploration are the introduction of fermions and supersymmetry, and non-trivial backgrounds corresponding to various branes.

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