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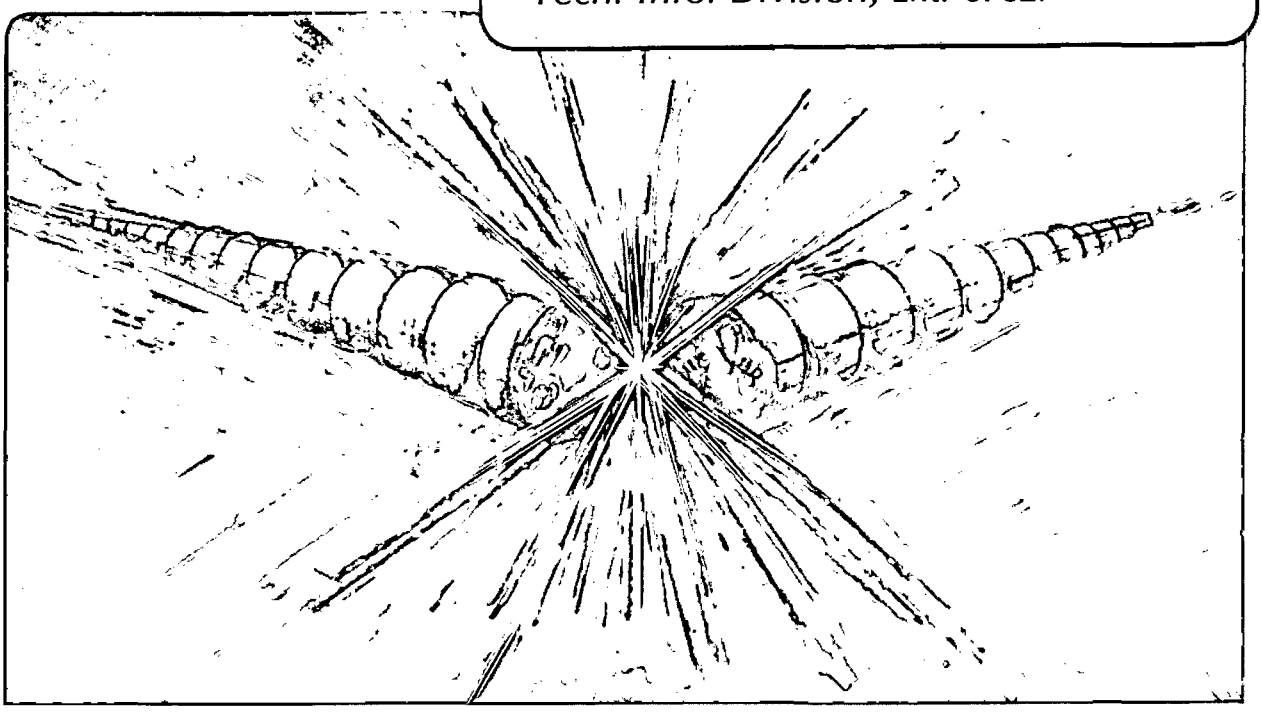

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NATURAL POISSON STRUCTURES OF NONLINEAR PLASMA DYNAMICS*

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Abstract

Hamiltonian field theories, for models of nonlinear plasma dynamics, require a Poisson bracket structure for functionals of the field variables. These are presented, applied, and derived for several sets of field variables: coherent waves, incoherent waves, particle distributions, and multifluid electrodynamics. Parametric coupling of waves and plasma yields concise expressions for ponderomotive effects (in kinetic and fluid models) and for induced scattering.

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I. Introduction

A deeper understanding of plasma processes can be obtained from formulations that exhibit the Hamiltonian structure underlying those processes. Such formulations have only recently been freed[1] from the widespread belief that canonically conjugate fields are required for a Hamiltonian structure. On examining the techniques of Hamiltonian dynamics, one recognizes that its essential ingredients are: (1) a Poisson bracket (PB) rule

$$[A_1, A_2] = A_3, \tag{1}$$

that acts bilinearly, antisymmetrically, and as a first derivative on observables A_j of the system studied; and (2) a Hamiltonian H which governs the evolution of observables by the rule

$$\dot{A} = [A, H]. \tag{2}$$

It is helpful to separate the two parts of the Hamiltonian structure. The PB part, or Poisson structure, appears to be more fundamental, and in some sense unique; we use the term "natural" to characterize those structures which look fundamental.

In this paper we shall present some Poisson structures appropriate to plasma dynamics, showing how they may be used to deal with problems that are much more difficult by non-Hamiltonian methods; we shall also indicate how these structures may be derived.

The dynamical systems we shall deal with are particle distributions $f(\underset{\sim}{z})$ in single-particle phase space; wave-action densities $\mathcal{J}(\underline{x},\underline{k})$ in ray phase space; action $J(\underline{x})$ and phase $\mathcal{H}(\underline{x})$ for eikonal waves in $\underline{x}$-space; fluid models in $\underline{x}$-space.

II. Incoherent Waves

Perhaps the the simplest example of a natural noncanonical Poisson structure is that for action densities. We begin by introducing the canonical phase space $\underset{\sim}{y} = (\underline{x}, \underline{k})$ for rays. For functions $a_i(\underset{\sim}{y})$ on this space, we define the ray PB as

$$\{a_1, a_2\} = \frac{\partial a_1}{\partial \underline{x}} \cdot \frac{\partial a_2}{\partial \underline{k}} - \frac{\partial a_1}{\partial \underline{k}} \cdot \frac{\partial a_2}{\partial \underline{x}} \tag{3}$$

Note that the right side is again a function on phase space; the rule (3) defines a Lie algebra.

Next consider the space of action densities $\mathcal{J}(\underline{x}, \underline{k})$, for a given wave branch; and functionals of action density, $A_i(\mathcal{J})$. As an example, consider the linear wave energy

$$H(\mathcal{J}) = \int d^6y \, \mathcal{J}(\underline{x}, \underline{k}) \omega_0(\underline{x}, \underline{k}), \tag{4}$$

where $\omega_0(\underline{x}, \underline{k})$ is a root of a real (linear) dispersion equation

$$\mathcal{D}(\omega; \underline{k}, \underline{x}) = 0. \tag{5}$$

For weakly nonlinear waves, a suitable model is[2]

$$H(J) = \int d^6y\, J(y)\omega_0(y) + \tfrac{1}{2}\int d^6y \int d^6y'\, J(y)J(y')\omega_2(y,y'). \tag{6}$$

Thus functionals may be linear or nonlinear. Other examples are the (linear) wave momentum

$$\underline{G}(J) = \int d^6y\, \underline{k}\, J(y) \tag{7}$$

and the (nonlinear) wave entropy

$$S(J) = \int d^6y\, \ln J(y). \tag{8}$$

For two functionals A,B, the PB is[2]

$$[A,B] = \int d^6y\, J(y)\{A_J, B_J\}, \tag{9}$$

where $A_J = \delta A/\delta J(\underline{y})$ is the functional derivative, and is a function on ray phase space, for which the PB (3) is defined. The right side of (9) is again a functional. From the general rule (2) for evolution, we now have

$$\begin{aligned}\dot{A} &= \int d^6y\, J\{A_J, H_J\} \\ &= -\int d^6y\, A_J\{J, H_J\},\end{aligned} \tag{10}$$

upon integration by parts. But by implicit differentiation, we also have

$$\dot{A}(\mathcal{J}) = \int d^6y \, A_{\mathcal{J}} \, \partial \mathcal{J}(y)/\partial t . \tag{11}$$

By comparing (10) and (11), we deduce the nonlinear Liouville equation for action density (often called the "wave-kinetic equation"):

$$\partial \mathcal{J}(y)/\partial t + \{\mathcal{J}(y), \omega(y;\mathcal{J})\} = 0 \tag{12}$$

where

$$\omega(y;\mathcal{J}) = \delta H(\mathcal{J})/\delta \mathcal{J}(y) \tag{13}$$

is the local nonlinear wave frequency. For the example (6),

$$\omega(y;\mathcal{J}) = \omega_0(y) + \int d^6y' \, \omega_2(y,y')\mathcal{J}(y'). \tag{14}$$

The Liouville equation states that action density is invariant along rays. The ray equations $\dot{\underline{x}} = \partial\omega/\partial\underline{k}$, $\dot{\underline{k}} = -\partial\omega/\partial\underline{x}$ are the canonical equations for the nonlinear ray Hamiltonian (13), which is the functional derivative of the wave Hamiltonian $H(\mathcal{J})$.

The Hamiltonian functional approach allows us to use Noether's theorem, to relate invariants and symmetries of the Hamiltonian. At a simpler level, we see from (2) that $A(\mathcal{J})$ is invariant under the Hamiltonian iff $[A, H]=0$.

As a first example, suppose that the coefficients ω_0 and ω_2 in (6) exhibit some geometrical symmetry, e.g., axial symmetry. It then follows from (9) and (3) that the wave angular momentum $\int d^6y\, k_\psi \mathcal{J}(\underline{x},\underline{k})$ is invariant.

As another example, we can write

$$[A, H] = \int d^6y\, H_{\mathcal{J}} \{\mathcal{J}, A_{\mathcal{J}}\}, \tag{15}$$

upon integrating (9) by parts. It follows that the set of functionals of the form

$$A(\mathcal{J}) = \int d^6y\, [\mathcal{J}(y)]^n, \tag{16}$$

$n = 1, 2, 3, \ldots$, are invariants under <u>any</u> H. (These are called "Casimir functionals".) This set forms a basis for functionals $A(\mathcal{J}) = \int d^6y\, f(\mathcal{J})$, such as the entropy (8), which are thus invariant.

III. <u>Particle Distributions</u>

As the next example of a natural Poisson structure we consider particle distributions. As in the case of waves, we begin with the six dimensional phase space $\underset{\sim}{z}$ for particle motion. (For some purposes, the eight-dimensional extended phase space may be preferable.) The PB for functions on this space is[3]

$$\{a_1, a_2\} = \frac{\partial a_1}{\partial z^\mu} \frac{\partial a_2}{\partial z^\nu} I^{\mu\nu}(\underset{\sim}{z}), \tag{17}$$

where $I^{\mu\nu}(\underset{\sim}{z})$ is the (antisymmetric) Poisson tensor $\{z^\mu, z^\nu\}$, whose inverse is the symplectic two-form. For particles in a weakly nonuniform magnetic field, we adopt Littlejohn's expression[4] for $I^{\mu\nu}$. However, our present formalism is coordinate-free, so one could use non-physical canonical coordinates $(\underline{r}, \underline{p})$ in (17).

As with waves, we next consider the space of Vlasov distributions $f(\underset{\sim}{z})$, treating one species for simplicity. We are concerned with functionals of f, such as the energy in the Coulomb model (possibly in an ambient magnetic field):

$$H(f) = \int d^6\underset{\sim}{z}\, f(\underset{\sim}{z})\, h_0(\underset{\sim}{z}) + \tfrac{1}{2}\int d^6\underset{\sim}{z}\int d^6\underset{\sim}{z}'\, f(\underset{\sim}{z})\, f(\underset{\sim}{z}')\, h_2(\underset{\sim}{z},\underset{\sim}{z}'), \qquad (18)$$

where h_0 represents kinetic energy, and h_2 represents Coulomb interaction. (Note the analogy to (6).) Other examples of functionals are the spatial density at a point $\underline{x}$:

$$n(\underline{x}; f) = \int d^6\underset{\sim}{z}\, f(\underset{\sim}{z})\, n(\underline{x}; \underset{\sim}{z}), \qquad (19)$$

where $n(\underline{x};\underset{\sim}{z}) = \delta^3(\underline{x} - \underline{r}(\underset{\sim}{z}))$ is the density of one particle; and the entropy:

$$S(f) = -\int d^6\underset{\sim}{z}\, f(\underset{\sim}{z})\, \ln f(\underset{\sim}{z}). \qquad (20)$$

The PB for functionals of f is[5] analogous to (9):

$$[A, B] = \int d^6\underset{\sim}{z}\, f\, \{A_f, B_f\}. \qquad (21)$$

Performing the algebraic steps of (10) and (11), we obtain the nonlinear Vlasov equation:

$$\partial f(\underset{\sim}{z})/\partial t + \{ f(\underset{\sim}{z}), h(\underset{\sim}{z}; f) \} = 0, \tag{22}$$

where the self-consistent particle Hamiltonian is

$$h(\underset{\sim}{z}; f) = \delta H(f) / \delta f(\underset{\sim}{z}). \tag{23}$$

For the model (16), we have the standard result

$$h(\underset{\sim}{z}; f) = h_0(\underset{\sim}{z}) + \int d^6 z' \, h_2(\underset{\sim}{z}, \underset{\sim}{z}') f(\underset{\sim}{z}'). \tag{24}$$

As another illustration of the power of a Hamiltonian formalism, we now introduce a probability functional $\rho(f)$ and the corresponding expectation of $A(f)$:

$$\langle A \rangle = \int df \, A(f) \rho(f). \tag{25}$$

(The integration is functional.) We may now follow the standard methods of statistical mechanics to obtain the Liouville equation for ρ:

$$\partial \rho(f)/\partial t = - [\rho(f), H(f)], \tag{26}$$

and arguments for a coarse-grained approach to a microcanonical ensemble $\rho(f) \sim \delta(H(f) - E)$ or a canonical ensemble $\rho(f) \sim \exp(-\beta H(f))$. In the

latter case, β^{-1} may be intepreted as an effective temperature for correlations. Adding an infinitesimal coupling of H(f) to a time-dependent perturbation yields the Kubo form of the fluctuation-dissipation theorem. The "thermal" fluctuations in f represents waves and clumps; the dissipation is the anti-Hermitian part of the "turbulent" response matrix, related by Kramers-Kronig to the Hermitian part of the response.

IV. Wave-Particle Non-Resonant Coupling

Having introduced Poisson structures for waves and particles separately, we now couple them by going to the oscillation-center description. We use Lie transforms to remove the linear wave oscillation from the particle motion[6], and consider the distribution $F(\underset{\sim}{z})$ of oscillation centers.[7] We adopt the Hamiltonian

$$H(F,J) = H_0(F) + \int d^6y\, J(y)\, \omega_0(y;F), \tag{27}$$

where $H_0(F)$ is the analogue of (18), while ω_0 is a root of the F-dependent dispersion function:

$$\mathcal{D}(\omega;\underset{\sim}{k},\underset{\sim}{x};F) = 0. \tag{28}$$

The natural Poisson structure is now[2]

$$[A,B] = \int d^6y\, J(y)\, \{A_J, B_J\} + \int d^6z\, F(\underset{\sim}{z})\, \{A_F, B_F\}, \tag{29}$$

where the PB's on the right are of course in the two separate phase spaces.

From (2) and (29) we obtain the coupled evolution equations for $\mathcal{J}$ and F:

$$\partial \mathcal{J}(\underset{\sim}{y})/\partial t + \{\mathcal{J}(\underset{\sim}{y}), \omega(\underset{\sim}{y}; F)\} = 0, \tag{30a}$$

$$\partial F(\underset{\sim}{z})/\partial t + \{F(\underset{\sim}{z}), K(\underset{\sim}{z}; \mathcal{J}, F)\} = 0, \tag{30b}$$

where

$$\omega(\underset{\sim}{y}; F) \equiv \delta H(\mathcal{J}, F)/\delta \mathcal{J}(\underset{\sim}{y}), \tag{31a}$$

$$K(\underset{\sim}{z}; \mathcal{J}, F) \equiv \delta H(\mathcal{J}, F)/\delta F(\underset{\sim}{z}). \tag{31b}$$

Thus $\mathcal{J}$ and F each satisfies a Liouville equation in its respective phase space; the ray Hamiltonian $\omega(\underset{\sim}{y})$ depends on F, and the oscillation-center Hamiltonian $K(\underset{\sim}{z})$ depends on $\mathcal{J}$ and F.

Since the ray and oscillation-center Hamiltonians (31) are the two functional derivatives of $H(\mathcal{J}, F)$, a reciprocity relation follows by equating the mixed second functional derivatives:

$$\delta K(\underset{\sim}{z})/\delta \mathcal{J}(\underset{\sim}{y}) = \delta \omega(\underset{\sim}{y})/\delta F(\underset{\sim}{z}). \tag{32}$$

For the model (27), which is linear in $\mathcal{J}$, (32) reduces to

$$\delta K(\underset{\sim}{z})/\delta \mathcal{J}(\underset{\sim}{y}) = -(\partial \mathcal{D}/\partial \omega)^{-1} \delta \mathcal{D}(\omega, \underset{\sim}{k}, \underset{\sim}{x}; F)/\delta F(\underset{\sim}{z}), \tag{33}$$

where (28) has been used. This result, that the ponderomotive contribution to K can be obtained from the linear response D, has previously been derived by more explicit calculations.[6,8]

V. Coherent Waves

So far, our description of waves is appropriate for the incoherent case, where phase information is absent. To include phase information, we introduce the set of phase functions $\psi_i(\underline{x})$, and their canonical conjugates, the wave action densities $J_i(\underline{x})$:

$$\{ J_i(\underline{x}), \psi_j(\underline{x}') \} = \delta_{ij}\, \delta(\underline{x} - \underline{x}'). \tag{34}$$

These represent the amplitude and phase of the eikonal description of the linear wave field:

$$\underline{E}(\underline{x}) = \sum_i \hat{e}_i(\underline{x}) \left[J_i(\underline{x}) (\partial \mathcal{D}/\partial\omega)^{-1} \right]^{1/2} e^{i\psi_i(\underline{x})} + c.c. \tag{35}$$

(Near caustics, an equivalent description is available in $\underline{k}$-space; better yet, the fields can be referred to Lagrangian submanifolds.[9])

For functionals $A(J,\psi)$, the PB follows immediately from (34):

$$[A,B] = \sum_i \int d^3x \left(\frac{\delta A}{\delta J_i(\underline{x})} \frac{\delta B}{\delta \psi_i(\underline{x})} - \frac{\delta A}{\delta \psi_i(\underline{x})} \frac{\delta B}{\delta J_i(\underline{x})} \right). \tag{36}$$

The equations of evolution for $\psi_i(\underline{x})$ and $J_i(\underline{x})$ are canonical:

$$\partial J_i(\underline{x})/\partial t = \delta H/\delta \psi_i(\underline{x}), \tag{37a}$$

$$\partial \psi_i(\underline{x})/\partial t = -\delta H/\delta J_i(\underline{x}). \tag{37b}$$

The simplest application is for the linear Hamiltonian:

$$H(J,\psi) = \sum_i \int d^3x\ J_i(\underline{x})\,\omega_0(\underline{x}, \nabla\psi_i(\underline{x})), \tag{38}$$

where $\omega_0(\underline{x},\underline{k})$ is again a root of (5). The evolution equations (37) are then the standard Hamilton-Jacobi equation of the phase:

$$\partial \psi_i(\underline{x})/\partial t = -\omega_0(\underline{x}, \nabla\psi_i(\underline{x})), \tag{39a}$$

and the standard action "transport" equation:

$$\partial J_i(\underline{x})/\partial t = -\nabla\cdot\left(J_i(\underline{x})\,\frac{\partial \omega_0}{\partial \underline{k}}(\underline{x})\right). \tag{39b}$$

To make use of the phase functions $\psi_i(\underline{x})$, we may now select Hamiltonians that depend on ψ_i, in additon to $\nabla\psi_i$. We begin with a model for three-wave interaction:

$$H = \sum_{i=1}^{3} \int d^3x\ J_i(\underline{x})\,\omega_0(\underline{x}, \nabla\psi_i(\underline{x})) + \int d^3x\ \beta\,(J_1 J_2 J_3)^{1/2} \exp i(\psi_1 - \psi_2 - \psi_3) + c.c., \tag{40}$$

where the coefficient $\beta(\underline{x}, \nabla\psi_1, \nabla\psi_2, \nabla\psi_3)$ can be obtained from the trilinear terms in the oscillation-center Hamiltonian.[10] This wave Hamiltonian yields the local Manley-Rowe relations $\partial J_1/\partial t = -\partial J_2/\partial t = -\partial J_3/\partial t$; the evolution equations are the standard ones[11] for nonuniform media.

VI. Wave-Particle Resonant Coupling

The inclusion of wave phase allows us to treat wave-particle resonances. To illustrate, we consider induced scattering of two waves[12] ("nonlinear Landau damping"); the treatment of linear Landau damping and quasilinear diffusion is similar. We adopt the Hamiltonian

$$H(J,\psi;F) = H_0(F) + \sum_{i=1}^{2}\int d^3x\, J_i(x)\,\omega_0(x,\nabla\psi_i;F) + \int d^3x\,\beta(x,\nabla\psi_1,\nabla\psi_2;F)(J_1J_2)^{1/2}e^{i(\psi_1-\psi_2)} + c.c., \quad (41)$$

where the (complex) coupling coefficient β can be expressed in terms of the linear and bilinear susceptibilities, which in turn are expressible as PB. (For present purposes, the explicit expression for β is not needed; we note, however, that it is nonlinear in F because of shielding effects.)

Letting d/dt denote the contribution of the interaction term (H_β), we see that $dJ_1/dt = -dJ_2/dt$, which is the Manley-Rowe relation. Letting $F = F_0 + F_2$, where F_2 is of order $(J_1J_2)^{1/2}$, we have

$$dF_2/dt = -\{F_0, \delta H_\beta/\delta F\} = -\int d^3x\,\{F_0, \delta\beta/\delta F\}(J_1J_2)^{1/2}e^{i(\psi_1-\psi_2)} + c.c. \quad (42)$$

The action is transferred as

$$dJ_1/dt = \delta H_\beta / \delta \psi_1(x)$$
$$= i\beta(F)(J_1 J_2)^{1/2} e^{i(\psi_1 - \psi_2)} + c.c. \qquad (43)$$

We expand $\beta(F) = \beta(F_0) + \int d^6z\, F_2\, \delta\beta(\underline{x})/\delta F$, and substitute from (42), retaining only the terms which survive phase averaging. We obtain

$$dJ_1/dt = -i J_1 J_2 \int d^4s\, exp[i(\nabla\psi_1 - \nabla\psi_2)\cdot\underline{s} + i\left(\frac{\partial\psi_1}{\partial t} - \frac{\partial\psi_2}{\partial t}\right)\tau] \cdot$$
$$\cdot \int d^6z \{F_0, \delta\beta^*(\underline{x}-\underline{s}, t-\tau)/\delta F\}\, \delta\beta(\underline{x},t)/\delta F(\underline{z})$$
$$= -2 J_1 J_2\, Im \int d^3s \int d\tau\, e^{i(\underline{s}\cdot\Delta\underline{k} - \tau\Delta\omega)} [\beta(\underline{x},t), \beta^*(\underline{x}-\underline{s}, t-\tau)]. \qquad (44)$$

Because this expression is again in terms of PB, it can be immediately applied to plasma in general geometry.

VII. Multifluid Electrodynamics

Natural Poisson structures are known for several fluid models. We select, for discussion here, the model of multifluid electrodynamics;[13] first, because its derivation is elementary,[14] and secondly, because it is easily used to deduce ponderomotive effects. For simplicity, we omit species labels, and ignore thermal effects.

The dynamical variables are mass density $\rho(\underline{x})$, kinetic momentum density $\underline{g}(\underline{x})$, and electromagnetic field $\underline{E}(\underline{x})$, $\underline{B}(\underline{x})$. The PB, for functionals $F(\rho, \underline{g}, \underline{E}, \underline{B})$ is [13]

$$[F, G] = \int d^3x \left(F_{\underline{E}} \cdot \nabla \times G_{\underline{B}} - F_{\underline{B}} \cdot \nabla \times G_{\underline{E}} \right)$$
$$+ \int d^3x \left(G_g \cdot \nabla F_g - F_g \cdot \nabla G_g \right) \cdot g$$
$$+ \int d^3x \left(G_g \cdot \nabla F_\rho - F_g \cdot \nabla G_\rho \right) \rho$$
$$+ \int d^3x \left(F_g \cdot G_{\underline{E}} - G_g \cdot F_{\underline{E}} \right.$$
$$\left. + F_g \times G_g \cdot \underline{B} \right) (e/m) \rho .$$

(45)

To use the PB (45), we need a Hamiltonian functional of the fluid-model variables. We adopt the energy

$$H(\rho, g, \underline{E}, \underline{B}) = \int d^3x \left(\tfrac{1}{2} E^2 + \tfrac{1}{2} B^2 + \tfrac{1}{2} \rho^{-1} g^2 \right).$$

(46)

From (2), (45), and (46), we obtain the evolution equations:

$$\partial \rho / \partial t = - \nabla \cdot (\rho H_g) = - \nabla \cdot g ,$$
$$\partial g / \partial t = - \nabla \cdot (g g / \rho) + (e/m) \rho (\underline{E} + \underline{u} \times \underline{B}),$$
$$\partial \underline{E} / \partial t = \nabla \times \underline{B} - (e/m) g ,$$
$$\partial \underline{B} / \partial t = - \nabla \times \underline{E} .$$

(47)

VIII. Ponderomotive Effects on Fluids

Now, in analogy to our treatment of ponderomotive effects at the kinetic level, we investigate these effects at the fluid level by coupling the fluid Hamiltonian (46) and PB (45) to the wave Hamiltonian (38) and PB (36). For purposes of illustration, we use the coherent wave description (the incoherent one was used for the kinetic problem), and consider a single wave. Thus we adopt the total Hamiltonian

$$H = \int d^3x \left(\tfrac{1}{2} E^2 + \tfrac{1}{2} B^2 + \tfrac{1}{2} \rho^{-1} g^2 + J(x)\, \omega_0 \left(\nabla\psi(x); \rho(x), g(x), B(x) \right) \right), \tag{48}$$

where now (ρ, g, E, B) are interpreted as the slow fluid variables, which appear parametrically in the (high) frequency function $\omega_0(k;\rho,g,B)$, which is a root of the dispersion equation:

$$D(\omega, k; \rho, g, B) = 0. \tag{49}$$

The total PB is now taken as the sum of the fluid PB (45) and the wave PB (36).

It is now straightforward to derive the equations of evolution for the slow field (ρ,u,E,B) and for the wave (J,ψ). We note that the resulting equations automatically conserve energy and momentum, since the Hamiltonian and PB are invariant under space-time translation.

For brevity, we write down only two of the evolution equations. For density, we have

$$\partial\rho(x)/\partial t = -\nabla\cdot\left(g + J\rho\, \partial\omega_0/\partial g \right).$$

Now the Doppler shift implies that $\rho\, \partial\omega_0/\partial \underline{g} = \partial\omega_0/\partial \underline{u} = \underline{k}$, so we obtain

$$\partial\rho/\partial t = -\nabla\cdot(\underline{g} + J\nabla\Psi), \tag{50}$$

indicating that the total mass flux density consists of the quasi-static part $\underline{g}$ and the wave momentum density $J\underline{k}$.

For the other evolution equation, we choose the quasi-static electric field:

$$\partial\underline{E}/\partial t = \nabla\times(\underline{B} + J\,\partial\omega_0/\partial\underline{B}) - (e/m)(\underline{g} + J\nabla\Psi). \tag{51}$$

In the last term, we recognize the wave-momentum contribution to quasi-static current. The other new term is evidently the wave-induced magnetization current $\underline{j}_{mag} = \nabla\times\underline{M}$, with $\underline{M} = -J\,\partial\omega_0/\partial\underline{B}$. By (49), we can write the magnetization as

$$\underline{M} = (\partial\mathcal{D}/\partial\omega)^{-1} J\ \partial\mathcal{D}/\partial\underline{B},$$

and by (35) as

$$\underline{M} = \tilde{E}_i^{*}\tilde{E}_j\,\partial\epsilon_{ij}/\partial\underline{B}, \tag{52}$$

where $\underline{\tilde{E}}$ is the wave amplitude, and $\underset{\sim}{\varepsilon}$ is the cold-fluid dielectric tensor. This is the standard result of Pitaevskii[15], and serves as a check- point for our formalism. In the limit of weak $\underline{B}$, the Hall term in $\underset{\sim}{\varepsilon}$ yields the result

$$\underline{M} = i\,(e/m)\,(\omega_p^2/\omega^3)\,\underline{\tilde{E}}^* \times \underline{\tilde{E}}, \tag{53}$$

which has been derived by many authors and methods.

IX. Derivation of Poisson Structures

Finally, we indicate how these various PB may be derived, touching on the main concepts, but omitting details. We begin with a derivation of the Poisson structure for a single coherent eikonal wave,[16] treating the scalar case for simplicity, and omitting the 2π of fourier transforms.

Let the potential $\phi(x)$ satisfy a self-adjoint linear integral equation:

$$\int d^4x'\ \tilde{\epsilon}(x,x')\,\phi(x') = 0, \tag{54}$$

where $x = (\underline{x},t)$, and $\tilde{\epsilon}(x',x) = \tilde{\epsilon}(x,x')$. This is equivalent to the variational principle $\delta S = 0$, where

$$S(\phi) = \int d^4x \int d^4x'\ \tilde{\epsilon}(x,x')\,\phi(x)\,\phi(x'). \tag{55}$$

Introduce the local spectral density

$$\phi^2(x,k) = \int d^4s\ \phi(x+\tfrac{1}{2}s)\,\phi(x-\tfrac{1}{2}s)\,e^{-ik\cdot s} \tag{56}$$

and the local dielectric function

$$\epsilon(x,k) = \int d^4s\ \tilde{\epsilon}(x+\tfrac{1}{2}s,\ x-\tfrac{1}{2}s)\,e^{-ik\cdot s}, \tag{57}$$

where $k = (\underline{k},\omega_0)$. Then a short calculation yields

$$S(\phi) = \int d^4x \int d^4k\ \epsilon(x,k)\,\phi^2(x,k). \tag{58}$$

Insertion of the eikonal form

$$\phi(x) = \tilde{\phi}(x)\exp i\psi(x) + c.c. \tag{59}$$

into (56), and phase averaging, yields

$$\phi^2(\underline{x},t;\underline{k},\omega) = \tilde{\phi}^2(\underline{x},t)\,\delta^3(\underline{k}-\nabla\psi(\underline{x},t))\,\delta(\omega+\frac{\partial\psi}{\partial t}(\underline{x},t)) \tag{60}$$

Substituting (60) into (58), we have

$$\begin{aligned} S &= \int dt \int d^3x\ \tilde{\phi}^2(\underline{x},t)\,\epsilon(\underline{x},t;\underline{k}=\nabla\psi,\ \omega=-\partial\psi/\partial t) \\ &= \int dt \int d^3x\ L(\underline{x},t,\nabla\psi,\partial\psi/\partial t,\tilde{\phi}). \end{aligned} \tag{61}$$

By the canonical methods of Lagrangian field theory, we deduce that the conjugate to $\Psi(\underline{x})$ is $\delta L/\delta(\partial\Psi/\partial t) = -\tilde{\phi}^2\,\partial\epsilon/\partial\omega \equiv -J(\underline{x})$, so that $\{J(\underline{x}),\Psi(\underline{x}')\} = \delta(\underline{x}-\underline{x}')$, which is (34). Variation of S with respect to $\tilde{\phi}$ yields

$$\epsilon(\underline{x},t;\underline{k},\omega) = 0, \tag{62}$$

the dispersion equation. The canonical Hamiltonian for (61) is, using (62),

$$\begin{aligned} H &= \int d^3x\,[(-J)\,\partial\Psi/\partial t - L\,] \\ &= \int d^3x\; J(\underline{x})\,\omega_0(\underline{k}=\nabla\Psi(\underline{x}),\underline{x},t). \end{aligned} \tag{63}$$

The extension to several waves is trivial.

To find the PB for functionals of $\mathcal{J}$, we first relate $\mathcal{J}$ to J,Ψ:

$$\mathcal{J}(\underline{x},\underline{k}) = \sum_i J_i(\underline{x})\,\delta^3(\underline{k}-\nabla\Psi_i(\underline{x})). \tag{64}$$

Then, using (36), we calculate $[\mathcal{J}(y),\mathcal{J}(y')]$; we omit the details. Finally, we calculate

$$[A(\mathcal{J}),B(\mathcal{J})] = \int d^6y \int d^6y'\, A_{\mathcal{J}(y)}\, B_{\mathcal{J}(y')}\,[\mathcal{J}(y),\mathcal{J}(y')], \tag{65}$$

and substitute the preceding bracket. After some manipulations, the result is (9).

Next we turn to functionals of particle distributions.[17] As in (65), we have

$$[A(f),B(f)] = \int d^6z \int d^6z'\, A_{f(\underline{z})}\, B_{f(\underline{z}')}\,[f(\underline{z}),f(\underline{z}')]. \tag{66}$$

We express $f(\underset{\sim}{z})$ in Klimontovich form (analogous to (64)):

$$f(\underset{\sim}{z}) = \sum_i \delta^6(\underset{\sim}{z} - \underset{\sim}{z}_i), \tag{67}$$

and consider

$$[f(\underset{\sim}{z}), f(\underset{\sim}{z}')] = \sum_{ij} [\delta^6(\underset{\sim}{z} - \underset{\sim}{z}_i), \delta^6(\underset{\sim}{z}' - \underset{\sim}{z}_j)]. \tag{68}$$

Now we interpret the right side of (68) as a PB on the many-particle phase space:

$$[f(\underset{\sim}{z}), f(\underset{\sim}{z}')] = \sum_i I^{\mu\nu}(\underset{\sim}{z}_i)[\partial\delta^6(\underset{\sim}{z}-\underset{\sim}{z}_i)/\partial z_i^\mu] \times [\partial\delta^6(\underset{\sim}{z}'-\underset{\sim}{z}_i)/\partial z_i^\nu]. \tag{69}$$

Substitution of (69) into (66), and manipulation, yield

$$[A,B] = \int d^6z\ f(\underset{\sim}{z})\, I^{\mu\nu}(\underset{\sim}{z})\,(\partial A_f/\partial z^\mu)(\partial B_f/\partial z^\nu), \tag{70}$$

which is (21).

Lastly, we derive one set (as illustrative of the method[11]) of the terms of the PB (43) for fluid electrodynamics. For functionals of momentum density $\underline{g}$, we have

$$[F, G] = \int d^3x \int d^3x'\ \delta F/\delta g_\mu(\underline{x})\ \delta G/\delta g_\nu(\underline{x}') \times [g_\mu(\underline{x}), g_\nu(\underline{x}')]. \tag{71}$$

We represent $\underline{g}(\underline{x})$ analogously to (67):

$$g(\underline{x}) = \sum_i m_i \underline{v}_i\, \delta(\underline{x} - \underline{r}_i), \tag{72}$$

and use the known PB for the noncanonical particle variables $(\underline{r}_i, \underline{v}_i)$. After some algebra, we obtain all the terms of (45) bilinear in $F_{\underline{g}}$ and $G_{\underline{g}}$. For functionals of all the variables, one introduces the usual electromagnetic canonical structure, and obtains (45).

X. Conclusions

In conclusion, the recent discovery of Poisson structures for many systems of interest poses several challenges: How are they related to each other? How can they be derived from first principles? How should dissipative[18] and stochastic perturbations be incorporated? How can they be exploited to simplify old results and derive new results?

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