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Publication Date
1986

LBL-20879 c.2
Preprint

Lawrence Berkeley Laboratory
UNIVERSITY OF CALIFORNIA

Physics Division

Submitted for publication

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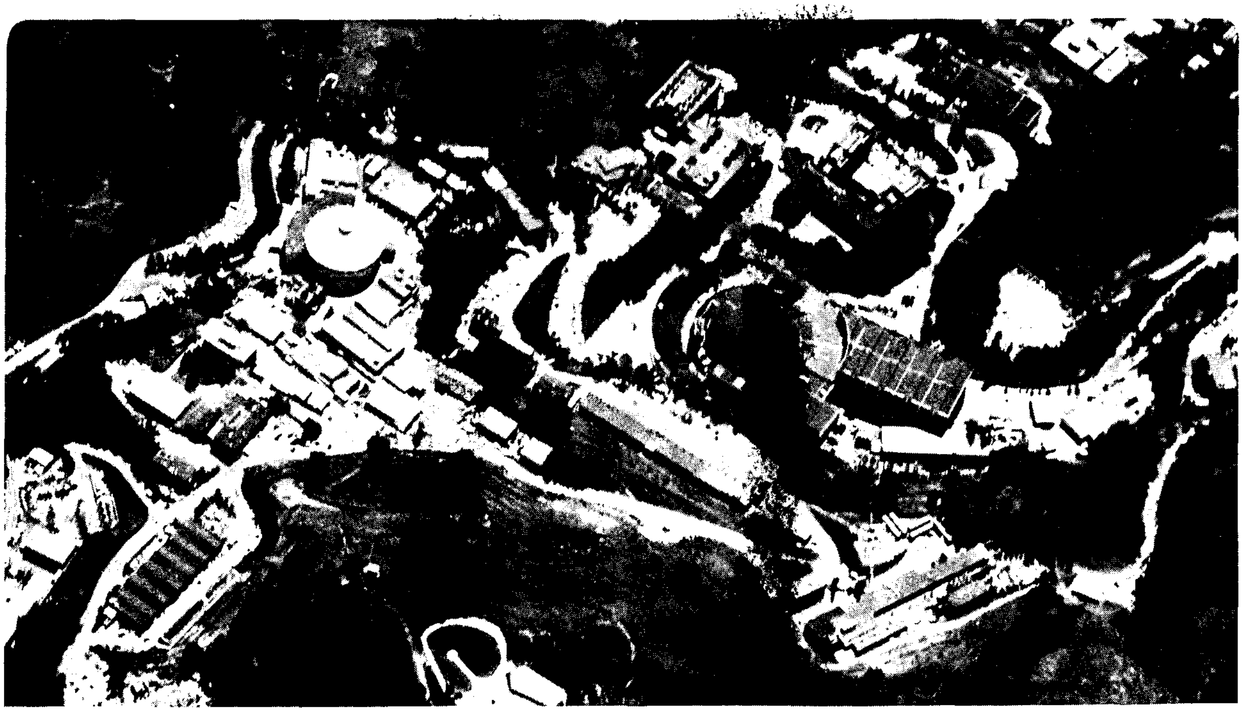
January 1986



Prepared for the U.S. Department of Energy under Contract DE-AC03-76SF00098

January 1986 LBL–20879

Perturbative Solution of the Chiral Schwinger Model *

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Abstract

The anomalous chiral Schwinger model is solved by summing the perturbation series.[1] A gauge invariant spectrum is obtained if a Wess-Zumino term is appended to the original action. The resulting spectrum depends on ambiguities in the specification of the anomaly and is singular at the value corresponding to the minimal anomaly.

*This work was supported by the Director, Office of High Energy and Nuclear Physics, Division of High Energy Physics of the U.S. Department of Energy under contract DE-AC03-76SF00098.

The requirement of anomaly cancellation plays a fundamental role in four dimensional gauge theories and in string theories in higher dimensions. It is important to ask whether it is a truly fundamental requirement or an artifact of our present dependence on perturbative techniques that demand renormalizability in four or more dimensions. Gauge theories in two space-time dimensions are an interesting laboratory in which to examine this question, since they can be solved exactly by a variety of techniques. Jackiw and Rajaraman[1] and Halliday, Rabinovici and Schwimmer *et al.*.[2] have used the method of bosonization to solve versions of the chiral Schwinger model. The former evaluate the fermionic determinant, reducing the theory to a bosonic theory with an arbitrary gauge boson mass parameter. The latter investigated a suggestion by Fadeev[3] that a sensible theory can be constructed by "subtracting" (*i.e.*, adding with a negative sign) the Wess-Zumino action[4] from the action of the original anomalous theory. The authors of ref. (2) found that the action of reference (1) can in fact be realized by the Fadeev construction taken in a "unitary" gauge in which the Wess-Zumino scalar is gauged away.

In this paper I solve the chiral Schwinger model in its aboriginal fermionic form, thereby finessing any subtleties that might arise in bosonization of an anomalous theory. I define the theory as the sum of its perturbation expansion and solve for the spectrum by explicit summation of the perturbation series.[5] I find that the solution depends on ambiguities in the chiral anomaly. The anomaly ambiguities are crucial to the analysis, because it turns out that there is a singularity at the value of the ambiguous parameter corresponding to the minimal anomaly. As in reference 2, the model with an uncancelled anomaly is unitary but violates Lorentz invariance in a temporal gauge, while in covariant gauges the spectrum is Lorentz invariant but depends on the gauge parameter and is unitary or not according to the particular value of the gauge parameter.[6] Upon "subtracting" the Wess-Zumino term the gauge invariance of the spectrum is restored: in temporal gauge the spectrum has a Lorentz invariant dispersion relation, in covariant gauges it is gauge parameter independent, and the two spectra agree. As in refs. 1 and 2 the spectrum depends on a single parameter, seen here to reflect the ambiguity in specifying the anomaly. The spectrum has a massless particle when the parameter assumes the value conventionally associated with the bosonization prescription.

I have also examined the "commutator anomalies" of the generator of time-independent gauge transformations, the operator that in temporal gauge must annihilate physical states to enforce Gauss' law. Though it does not in general commute with the Hamiltonian in the original anomalous theory, it commutes with itself and the Hamiltonian in the Wess-Zumino "subtracted" theory – another sign of the restoration of gauge invariance.

The theory consists of one left and one right chiral massless fermion coupled to a massless "photon",

$$\mathcal{L}_0 = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(i\partial\!\!\!/ + A\!\!\!/ e)\psi \tag{1}$$

where

$$\psi = \begin{pmatrix} \psi_R \\ \psi_L \end{pmatrix}$$

$$e = \begin{pmatrix} e_R & \\ & e_L \end{pmatrix}$$

and the Dirac matrices are $\gamma^0 = \sigma^1$, $\gamma^1 = i\sigma^2$, and $\gamma^5 = -\gamma^0\gamma^1$ so that $(1-\gamma_5)/2$ projects out ψ_L which is left-moving. The Hamiltonian in temporal gauge is

$$\mathcal{H} = \frac{1}{2}\dot{A}^2 - i\bar{\psi}\gamma^1\partial_x\psi + AJ^1 \tag{2}$$

with

$$J^\mu = \bar{\psi}\gamma^\mu e\psi.$$

The vacuum polarization tensor

$$\Pi^{\mu\nu}_{JJ} \equiv \int d^2x e^{ip\cdot x}\langle TJ^\mu(x)J^\nu(0)\rangle \tag{3}$$

may be decomposed by writing J as a sum of vector and axial vector currents,

$$J^\mu = \frac{1}{2}e_+ j^\mu_V + \frac{1}{2}e_- j^\mu_A \tag{4}$$

where $e_\pm = e_R \pm e_L$, $j^\mu_V = \bar{\psi}\gamma^\mu\psi$, and $j^\mu_A = \bar{\psi}\gamma^\mu\gamma_5\psi$. The vacuum polarization tensors of j^μ_V and j^μ_A computed perturbatively are determined up to three real, arbitrary constants, a, b, c:

$$\Pi^{\mu\nu}_{VV} = -\frac{i}{\pi}(S^{\mu\nu} + cg^{\mu\nu}) \tag{5a}$$

$$\Pi^{\mu\nu}_{AA} = -\frac{i}{\pi}(-\mathbf{P}^{\mu\nu} + bg^{\mu\nu}) \tag{5b}$$

$$\Pi^{\mu\nu}_{AV} = \Pi^{\nu\mu}_{VA} = -\frac{i}{\pi}(\epsilon^\mu{}_\alpha S^{\alpha\nu} + a\epsilon^{\mu\nu}) \tag{5c}$$

where $S^{\mu\nu} \equiv g^{\mu\nu} - \frac{p_\mu p^\nu}{p^2}$, $\mathbf{P}^{\mu\nu} \equiv \frac{p^\mu p^\nu}{p^2}$, the metric has $g^{00} = +1$, and the antisymmetric tensor has $\epsilon^{01} = +1$. The theory is completely specified by choosing a, b, c. For

instance, for $e_R = e_L$ our theory is QED and the choice $c = 0$ is necessary to have the usual gauge invariant theory. The formal "duality" relation $j^\mu_5 = \epsilon^\mu{}_\nu j^\nu$ is enforced among the three tensors in eq. (5) if $a = b = c$ (*e.g.*, $\Pi^{\mu\nu}_{AV} = \epsilon^\mu{}_\alpha\Pi^{\alpha\nu}_{VV}$), in which case conservation of VV requires nonconservation of AA. Alternatively we can make VV and AA both conserved (but not VA) by setting $c = b - 1 = 0$, with the sacrifice of the "duality" relationship. These are examples of ambiguities in the anomaly. The essential, unambiguous anomaly resides in the VA tensor (5*b*), since there is no choice of the parameter a that makes $\Pi^{\mu\nu}_{VA}$ conserved on both indices.

Combining eqs. (3-5), the vacuum polarization tensor of the gauge current is

$$\Pi^{\mu\nu} = -i\left\{\frac{e_+^2 + e_-^2}{4\pi}S^{\mu\nu} + c_1 g^{\mu\nu} + c_2(\epsilon^\mu{}_\alpha S^{\alpha\nu} + \epsilon^\nu{}_\alpha S^{\alpha\mu})\right\} \tag{6}$$

where

$$c_1 = \frac{1}{4\pi}(ce_+^2 + (b-1)e_-^2) \tag{7a}$$

$$c_2 = \frac{1}{4\pi}e_+e_- = \frac{e_R^2 - e_L^2}{4\pi} \tag{7b}$$

While c_1 is arbitrary since it depends on b and c, c_2 depends only on the charges $e_{R,L}$. The essential (or minimal) anomaly is the term c_2, which arises in eq. (6) from the VA cross term (5*c*). The complete anomalous Ward identity is just

$$p_\mu\Pi^{\mu\nu} = -ic_1p^\nu - ic_2\epsilon^\nu{}_\alpha p^\alpha. \tag{8}$$

The spectrum is determined by the vacuum polarization tensor, eq. (6). In $A^0 = 0$ gauge the exact photon propagator is obtained[5] by summing the geometric series of bubble graphs

$$\begin{aligned} D &= \frac{i}{p_0^2} + \frac{i}{p_0^2}\left(-\Pi^{11}_{JJ}(p)\right)\frac{i}{p_0^2} + \cdots \\ &= \frac{i}{p_0^2 + i\Pi^{11}_{JJ}}. \end{aligned} \tag{9}$$

Defining

$$i\Pi^{11}_{JJ} = -\frac{p_0^2}{p^2}X_0 \tag{10}$$

the pole is at $p^2 = X_0$ (there is also a pole at $p^0 = 0$, an artifact of the temporal gauge that occurs also in QED) where

$$X_0 = \frac{e_R^2 + e_L^2}{2\pi} - 2c_2\frac{p^1}{p^0} + c_1\frac{p^2}{p_0^2}. \tag{11}$$

To recover QED we set $e_R = e_L = e$ and choose $c = 0$ in eq. (5a), so that $c_1 = c_2 = 0$ and $m^2 = X_0 = e^2/\pi$. For $e_R \neq e_L$, we have a non-Lorentz invariant dispersion relation. For instance, for $c_1 = 0$ the pole is at

$$p^2 \equiv p^{0^2} - p^{1^2} = \frac{e_R^2 + e_L^2}{2\pi} - \frac{p^1}{p^0}\frac{e_R^2 - e_L^2}{2\pi} \tag{12}$$

and in general the position of the pole depends on how we specify c_1.

Eq. (12) is the result obtained in eq. (24) of ref. (2), so apparently the bosonization prescription followed there is equivalent to the choice $c_1 = 0$ and not the choice $a = b = c = 0$ that preserves "duality" and vector current conservation. (In fact, if vector current conservation, $c = 0$, is assumed, then it happens that $c_1 = 0$ follows from the anticommuting γ_5 prescription advocated in dimensional regularization for graphs with an even number of axial vertices.[6])

The conclusion, as in ref. (2), is that in $A^0 = 0$ gauge the breaking of gauge invariance by the anomaly is manifested, as expected, by the breaking of Lorentz invariance. Similarly in a generalized covariant gauge we would find that the spectrum has an explicit dependence on the gauge parameter.[6]

Next we consider the theory in which the Wess-Zumino term is "subtracted" from the initial Lagrangian $\mathcal{L}_0$ of eq. (1). Following the construction of ref. (4) we find

$$\mathcal{L}_{WZ} = \frac{c_1}{2}(\partial_\mu \theta)^2 + \theta(c_1 \partial_\mu A^\mu + c_2 \epsilon_{\mu\nu}\partial^\mu A^\nu) \tag{13}$$

where θ is the Wess-Zumino scalar. We define the theory with the subtraction

$$\mathcal{L}_T = \mathcal{L}_0 - \mathcal{L}_{WZ} \tag{14}$$

For the current of this Lagrangian we now find a <u>conserved</u> vacuum polarization tensor

$$\Pi_T^{\mu\nu} = i\left(\frac{e_+^2 + e_-^2}{4\pi} + \frac{c_2^2}{c_1} + c_1\right) S^{\mu\nu}. \tag{15}$$

Clearly the WZ term has restored the gauge invariance of the theory. It is especially interesting that gauge invariance is restored for any value of the ambiguous parameter c_1.

Since $\Pi_T^{\mu\nu}$ is conserved it comes as no surprise that Lorentz invariance is restored in $A^0 = 0$ gauge. Summing the geometric series for the photon propagator we find that the pole is at

$$p^2 = \frac{e_+^2 + e_-^2}{4\pi} + \frac{c_2^2}{c_1} + c_1 \tag{16}$$

or

$$p^2 = \frac{e_R^2 + e_L^2}{2\pi} + \frac{e_R^2 - e_L^2}{4\pi}\left(\frac{c_2}{c_1} + \frac{c_1}{c_2}\right). \tag{17}$$

Eq. (17) has the same form as the solution obtained in ref. (2) for the WZ modified theory: eq. (17) is identical to their eq. (47) if their parameter α (discussed below) is identified with c_2/c_1. If we set $e_R = 0$ to compare with ref. (1) we again find agreement, identifying their parameter "a" with $c + b = 1 + (4\pi c_1/e_L^2)$ and their e^2 with $e_L^2/4\pi$. In ref. (1) "a" emerges from the fermionic determinant as a mass for the gauge boson.

From the conserved form of the polarization tensor it is clear that the same solution is obtained in generalized covariant gauges.

Notice that the solution (17) is singular at $c_1 = 0$ (though not at $c_2 = 0$). It is already apparent from the form of $\mathcal{L}_{WZ}$ in eq. (13) that $c_1 = 0$ plays a unique role in the analysis. We can still recover the QED limit by respecting the following order of limits: first $e_R \to e_L$, then $c \to 0$.

As noted in reference (2) gauge invariance also allows the addition of a term

$$\mathcal{L}_\alpha = \frac{\alpha}{2}(\partial_\mu \theta - A_\mu)^2 \tag{18}$$

where the gauge transformations are $A^\mu \to A^\mu + \partial^\mu \xi$ and $\theta \to \theta + \xi$. Choosing[2] the unitary gauge $\theta = -\xi$, we have $\mathcal{L}_{WZ} = 0$ and $\mathcal{L}_\alpha$ becomes just a gauge boson mass term, as in the construction of reference (1). Returning to the original non-unitary gauge, if we sum the series for the photon propagator in the theory defined by $\mathcal{L}_0 - \mathcal{L}_{WZ} + \mathcal{L}_\alpha$, the result is eq. (17) with $c_1 \to c_1 + \alpha$,

$$p^2 = \frac{e_R^2 + e_L^2}{2\pi} + c_2\left(\frac{c_2}{c_1 + \alpha} + \frac{c_1 + \alpha}{c_2}\right). \tag{19}$$

$\mathcal{L}_\alpha$ may then "regulate" the theory at $c_1 = 0$. Conversely, while the solution of ref. (2) is singular at $\alpha = 0$, eq. (19) is finite at $\alpha = 0$ if $c_1 \neq 0$. (The singularity of the solution of ref. (2) at $\alpha = 0$ is another indication that their analysis is equivalent to taking $c_1 = 0$.)

Returning to the case $\alpha = 0$ and specifying the anomaly by assuming vector current conservation and "duality", $b = c = 0$, then $c_1 = -e_-^2/4\pi$ and eq. (17) yields a massless pole

$$p^2 = \frac{e_+^2 + e_-^2}{4\pi} - \frac{e_+ e_-}{4\pi}\left(\frac{e_+}{e_-} + \frac{e_-}{e_+}\right) = 0. \tag{20}$$

I have also investigated the equal time commutators $[G, \mathcal{H}]$ and $[G, G]$ in the theory with and without the WZ term. G is the generator of time independent gauge transformations, the residual gauge invariance of a gauge invariant theory in $A^0 = 0$ gauge, given by

$$G(x) = \frac{\partial E}{\partial x} - J^0 \tag{21}$$

with E the electric field. In a gauge invariant theory Gauss' law is imposed in $A^0 = 0$ gauge by requiring that G annihilate the physical states. Since I have defined the theory as the sum of its perturbation expansion, it is appropriate to use the BJL limit[8] to evaluate the equal time commutators. Explicit calculations show[9] that it correctly reproduces ETC's calculated in perturbation theory, where other methods (such as point-splitting at equal times) fail.

For the theory without WZ term the results are

$$[J^0(x,0), J^0(0)] = 2ic_2\delta'(x) \tag{22}$$

$$[J^0(x,0), J^1(0)] = -i\frac{e_R^2 + e_L^2}{2\pi}\delta'(x) \tag{23}$$

$$\left[E(x,0), J^0(0)\right] = ic_2\delta(x) \tag{24}$$

$$\left[\frac{\partial E}{\partial x}(x,0), J^0(0)\right] = ic_2\delta'(x) \tag{24.1}$$

$$\left[\frac{\partial E}{\partial x}(x,0), J^1(0)\right] = -i\left(c_1 + \frac{e_R^2 + e_L^2}{2\pi}\right)\delta'(x) \tag{25}$$

where (24) and (25) are evaluated in $A^0 = 0$ gauge. Using the definition (21) and the Hamiltonian (2) defined in $A^0 = 0$ gauge, we then find

$$[G(x,0), G(0)] = 0 \tag{26}$$

$$[G(x,0), \mathcal{H}(0)] = -i\left(c_1 A(0)\delta'(x) + c_2 E(0)\delta(x)\right) \tag{27}$$

Eq. (26) agrees with the results of Jo[10] and Hwang[11], also computed with the BJL limit. For $e_R = 0$, eqs. (26) and (27) also agree with a BJL calculation[11] and with the Poisson bracket evaluations of Rajaraman[12], based on the bosonized construction of ref. (1); as for the mass, eq. (17), agreement is obtained with eq. (27) if the parameter "a" of refs. (1) and (12) is identified with our $c + b = 1 + (4\pi c_1/e_L^2)$. Eqs. (26) and (27) do not agree with the commutation relations given in ref. (2).[13]

For the theory with WZ term, eq. (14), the commutators are like those of QED.

We have

$$[J^0(x,0), J^0(0)] = \left[E(x,0), J^0(0)\right] = 0 \tag{28}$$

so that trivially

$$[G(x,0), G(0)] = 0. \tag{29}$$

The usual Schwinger and Seagull terms

$$[J^0(x,0), J^1(0)] = -i\left(\frac{e_R^2 + e_L^2}{2\pi} + c_1 + \frac{c_2^2}{c_1}\right)\delta'(x) \tag{30}$$

$$\left[\frac{\partial E}{\partial x}(x,0), J^1(0)\right] = -i\left(\frac{e_R^2 + e_L^2}{2\pi} + c_1 + \frac{c_2^2}{c_1}\right)\delta'(x) \tag{31}$$

cancel in the familiar way

$$[G(x,0), \mathcal{H}(0)] = 0,$$

all as expected in a gauge invariant theory.

To conclude, both the spectrum and the equal time commutators indicate that gauge invariance is restored by "subtracting" a Wess-Zumino term from the original anomalous lagrangian. The resulting theory depends on how the ambiguous anomaly parameter c_1 is specified. To me the most striking feature of this mechanism is that it restores gauge invariance for all nonzero values of c_1. In this respect it seems a "stronger" mechanism than anomaly cancellation by addition of heavy fermions.

Acknowledgments: I am grateful to the authors of ref. (2) for restimulating my interest in this question and for discussions. I also thank R. Jackiw for sharing related correspondence, A. Niemi for an interesting conversation, and D.S. Hwang for finding an error in a previous version of eq. (27). This work was supported by the Director, Office of Energy Research, Office of High Energy and Nuclear Physics, Division of High Energy Physics of the U.S. Department of Energy under contract DE-AC03-76SF00098.

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13. The discrepancy appears to be due to omission of the contribution from the seagull commutator, eq. (24.1); while Schwinger terms are realized canonically in the bosonized theory, seagull commutators are not. In addition there is a sign error in the relative phases of the L and R contributions to $\mathcal{H}$ or G as defined in eqs. (3), (5), (7), and (8) of ref. (2): J^0 appears in G, J^1 appears in $\mathcal{H}$, and J^0 and J^1 should differ by the relative phase of L and R contributions, but in ref. (2) the same relative phase appears in G and $\mathcal{H}$.