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Multiple-Time-Scale Approach to Ergodic Adiabatic Systems: Another Look*

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Abstract

We re-examine the multiple-time-scale method as applied to ergodic adiabatic Hamiltonian systems. Solving for the evolution of the phase space density to first order in the slowness parameter, we find a term previously overlooked. The inclusion of this term resolves a standing discrepancy between the multiple-time-scale approach to this problem, and an approach using a Fokker-Planck equation. We apply our solution to the dynamics of a "slow" system coupled to an ensemble of "fast" systems following chaotic trajectories.

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This Letter re-examines the multiple-time-scale approach to the study of ergodic adiabatic systems, i.e. systems evolving chaotically and ergodically under a slowly time-dependent Hamiltonian, H. The relevance of this problem to plasma physics, heavy-ion dynamics (both fission and fusion), cosmic ray acceleration, and microscopic models of dissipation has been discussed in Refs. [1–4]. Ott [1] first used multiple-time-scale analysis in this context to demonstrate the adiabatic invariance of a certain quantity, the *ergodic adiabatic invariant*, and also to study the "goodness" of this quantity as an invariant (the extent to which it is violated when the evolution of H is not perfectly adiabatic). Recently, taking a different approach, we have found disagreement with Ott's results concerning the goodness of this adiabatic invariant [2]. The present Letter is an attempt both to resolve this discrepancy, and to find a solution to the central problem — the evolution in phase space of an ensemble of systems under an ergodic adiabatic Hamiltonian — which we then apply to a dynamical problem recently considered by Berry and Robbins [7].

An *ergodic adiabatic Hamiltonian* $H(\mathbf{z},t)$, where $\mathbf{z}=(\mathbf{q},\mathbf{p})$ denotes a point in phase space and t denotes time, is characterized by two conditions. First, it evolves slowly with time; we express this condition mathematically as $H(\mathbf{z},t)=h(\mathbf{z},\epsilon t)$, where ϵ is a formally small dimensionless parameter. (This defines two time scales: a fast scale over which t changes by order unity, and a slow scale over which ϵt changes by order unity.) Second, if the slow evolution of H is "frozen" at any instant in time, the resulting time-independent Hamiltonian produces trajectories which ergodically and chaotically explore their energy shells (phase space surfaces of constant H). Ott has shown that, for trajectories $\mathbf{z}(t)$ evolving under an ergodic adiabatic Hamiltonian, the volume of phase space enclosed by the instantaneous energy shell on which the trajectory is found, is an adiabatic invariant. That is, let

$$\Omega(E,\epsilon t)\equiv\int d\mathbf{z}\,\theta[E-h(\mathbf{z},\epsilon t)] \tag{1}$$

[where $\theta(x)$ is the unit step function] denote the volume, assumed finite, enclosed by the energy shell E of $h(\mathbf{z},\epsilon t)$; then the quantity $\Omega(H,\epsilon t)$, with H evaluated along $\mathbf{z}(t)$, will remain constant in the limit $\epsilon\rightarrow 0$, over times that scale like ϵ^{-1} ("slow" times of order unity). Ω is the ergodic adiabatic invariant.

Ott, Brown, and Grebogi [1,3] have pursued the question, To what extent is the invariance of Ω violated for slow but finite evolution of the Hamiltonian (small but finite ϵ)? Consider an ensemble of trajectories specified by a uniform distribution of initial conditions over the energy shell E_0 at time $t=0$, then allowed to evolve under $H(\mathbf{z},t)$. After a time t of order ϵ^{-1} (over which H changes by order unity), we expect to find these systems near the energy shell $\mathcal{E}(\epsilon t)$, where $\Omega(\mathcal{E},\epsilon t)=\Omega(E_0,0)$. The "error" in Ω may be measured by the moments

$$M_n(t)\equiv\int d\mathbf{z}\,F(\mathbf{z},t)\left[H(\mathbf{z},t)-\mathcal{E}(\epsilon t)\right]^n, \tag{2}$$

$n=1,2,\cdots$, where $F(\mathbf{z},t)$ is the phase space density representing the ensemble. With multiple-time-scale analysis, Ott [1] has obtained lowest-order expressions for dM_1/dt and dM_2/dt. These rates scale like ϵ^2, and so over times of $O(\epsilon^{-1})$ these two moments will scale like ϵ^1. [Higher moments grow at rates which are $O(\epsilon^3)$ or smaller, and will henceforth be ignored.]

Wilkinson [4] has used Ott's results to write down a Fokker-Planck equation governing the distribution of energies of an ensemble of systems evolving under an ergodic adiabatic

Hamiltonian. Using a different approach, and unaware at the time of Wilkinson's work, we have obtained the same equation [2], and furthermore have shown this equation to be in conflict with Ott's results for dM_1/dt and dM_2/dt. Specifically, the expressions for these rates which follow from the Fokker-Planck equation contain terms not found in Ott's expressions.

We now re-examine the multiple-time-scale method as applied to this problem. Our strategy is as follows. We consider an initial phase space density which is a function of energy shell alone, $F(\mathbf{z},0) = f_{00}(h(\mathbf{z},0))$, with f_{00} an arbitrary function of its argument. Using multiple-time-scale analysis and working to first order in ϵ, we obtain a solution for $F(\mathbf{z},t)$ valid for times of $O(\epsilon^{-1})$. Applying our results to the specific case considered in Refs. [1,3], where the initial density is restricted to a single energy shell, we find expressions for dM_1/dt and dM_2/dt in agreement with those derived from the Fokker-Planck equation. Finally, we discuss the application of our results to a particular dynamical problem.

To apply the multiple-time-scale method [5], we follow Ott's expansion of F:

$$F(\mathbf{z},t) = F_0(\mathbf{z},\tau_2) + \epsilon F_1(\mathbf{z},\tau_1,\tau_2) + \epsilon^2 F_2(\mathbf{z},\tau_1,\tau_2) + \cdots, \tag{3}$$

where $\tau_1 = t$ and $\tau_2 = \epsilon t$ are the "fast" and "slow" times. The initial conditions are: $F_0 = f_{00}(h)$, $F_1 = F_2 = \cdots = 0$, at $\tau_1 = \tau_2 = 0$. Plugging Eq. 3 into the Liouville equation $\partial F/\partial t + \{F, H\} = 0$, and ordering by powers of ϵ, we get, to $O(\epsilon^2)$:

$$\{F_0, h\} = 0 \tag{4a}$$

$$\frac{\partial F_1}{\partial \tau_1} + \{F_1, h\} = -\frac{\partial F_0}{\partial \tau_2} \tag{4b}$$

$$\frac{\partial F_2}{\partial \tau_1} + \{F_2, h\} = -\frac{\partial F_1}{\partial \tau_2}, \tag{4c}$$

where $h = h(\mathbf{z},\tau_2)$.

The solution of Eq. 4a is:

$$F_0(\mathbf{z},\tau_2) = f_0(h,\tau_2), \tag{5}$$

where, aside from the initial condition $f_0(E,0) = f_{00}(E)$, the function f_0 is so far arbitrary. This ambiguity is a feature of the multiple-time-scale method; we remove it by insisting that F_0 remain valid for times of $O(\epsilon^{-1})$, i.e. by removing terms at next order which grow secularly with time.

Following Ott, we multiply Eq. 4b by an arbitrary function $g(h)$ and integrate over phase space, obtaining

$$\frac{\partial}{\partial \tau_1}\int d\mathbf{z}\, g(h) F_1 = -\int d\mathbf{z}\, g(h)\frac{\partial F_0}{\partial \tau_2}. \tag{6}$$

Since the right side of this equation is independent of τ_1, $\int d\mathbf{z}\, g(h) F_1$ will grow secularly unless the term on the right is zero. We therefore set

$$0 = -\int d\mathbf{z}\, g(h)\Big[\frac{\partial f_0}{\partial E}(h,\tau_2)\frac{\partial h}{\partial \tau_2} + \frac{\partial f_0}{\partial \tau_2}(h,\tau_2)\Big], \tag{7}$$

where $\partial f_0/\partial E$ and $\partial f_0/\partial \tau_2$ refer to the derivatives of f_0 with respect to its first and second arguments, respectively. (In what follows, the arguments of h and $\partial h/\partial \tau_2$ are taken to be

$(\mathbf{z},\tau_2)$, if not specified otherwise; those of f_0, and of the functions Σ, u, f_1, and G_2 introduced below, are taken to be (E,τ_2), if not specified.)

Now define $\Sigma(E,\tau_2) \equiv \int d\mathbf{z}\, \delta(E-h) = (\partial/\partial E)\Omega(E,\tau_2)$. Letting $\langle\cdots\rangle_{E,\tau_2}$ denote the phase space average of $(\cdots)$ over the energy shell E of $h(\mathbf{z},\tau_2)$, we have

$$\Big\langle\cdots\Big\rangle_{E,\tau_2} = (1/\Sigma)\int d\mathbf{z}\, \delta(E-h)\cdots, \tag{8}$$

from which

$$\int d\mathbf{z}\cdots = \int dE\, \Sigma\Big\langle\cdots\Big\rangle_{E,\tau_2}. \tag{9}$$

With this result, Eq. 7, which holds for arbitrary g, yields

$$0 = \Sigma\Big(\frac{\partial f_0}{\partial E}u + \frac{\partial f_0}{\partial \tau_2}\Big) = \frac{\partial}{\partial \tau_2}(\Sigma f_0) + \frac{\partial}{\partial E}(u\Sigma f_0), \tag{10}$$

where $u(E,\tau_2) \equiv \langle \partial h/\partial\tau_2\rangle_{E,\tau_2}$, and we have used the identity $\partial\Sigma/\partial\tau_2 + (\partial/\partial E)(\Sigma u) = 0$.

$F_0(\mathbf{z},\tau_2) = f_0(h,\tau_2)$ is now completely specified by the initial conditions, along with Eq. 10. Using Eqs. 5 and 10, we further obtain

$$\frac{\partial F_0}{\partial \tau_2}(\mathbf{z},\tau_2) = \frac{\partial f_0}{\partial E}(h,\tau_2)\Big[\frac{\partial h}{\partial \tau_2} - u(h,\tau_2)\Big], \tag{11}$$

which will be of use below.

We proceed to the evaluation of F_1. The solution of Eq. 4b contains both an inhomogeneous and a homogeneous term:

$$F_1(\mathbf{z},\tau_1,\tau_2) = F_{1i} + F_{1h} = -\int_0^{\tau_1} d\tau_1'\, \frac{\partial F_0}{\partial \tau_2}(\mathbf{Z},\tau_2) + f_1(h,\tau_2), \tag{12}$$

where $\mathbf{Z} = \mathbf{Z}(\mathbf{z},\tau_1,\tau_1',\tau_2)$ is the point in phase space reached by starting at $\mathbf{z}$ at time τ_1, then evolving a trajectory backward in time to τ_1', under the "frozen" (time-independent) Hamiltonian $h(\mathbf{z},\tau_2)$. So far, the homogeneous term f_1 is arbitrary apart from initial conditions ($f_1 = 0$ at $\tau_2 = 0$); to determine it completely, we remove secularities at $O(\epsilon^2)$. We proceed as before, multiplying both sides of Eq. 4c by arbitrary $g(h)$, and integrating:

$$\frac{\partial}{\partial \tau_1}\int d\mathbf{z}\, g(h)F_2 = -\int d\mathbf{z}\, g(h)\frac{\partial F_1}{\partial \tau_2} \tag{13}$$

$$= -\frac{\partial}{\partial \tau_2}\int d\mathbf{z}\, g(h)F_1 + \int d\mathbf{z}\, g'(h)\frac{\partial h}{\partial \tau_2}F_1. \tag{14}$$

(g' denotes the derivative of g with respect to its argument.) With manipulation, the right side becomes

$$-\int dE\, g(E)\Big[\frac{\partial}{\partial \tau_2}\big(\Sigma f_1\big) + \frac{\partial}{\partial E}\big(u\Sigma f_1\big) - \frac{\partial}{\partial E}\Big(\Sigma\frac{\partial f_0}{\partial E}\int_{-\tau_1}^{0} ds\, C(s)\Big)\Big]. \tag{15}$$

The quantity $C(s)$ (whose dependence on E and τ_2 has been suppressed) is an autocorrelation function,

$$C(s) = \left\langle \left(\frac{\partial h}{\partial \tau_2} - u\right) \mathcal{O}_{\tau_2}(s) \left(\frac{\partial h}{\partial \tau_2} - u\right) \right\rangle_{E,\tau_2}, \tag{16}$$

where $\mathcal{O}_{\tau_2}(s)$ is a time-evolution operator which acts to the right, evolving a point $\mathbf{z}$ for a time s under the frozen Hamiltonian h. Note that $C(s) = C(-s)$. For times τ_1 of $O(\epsilon^{-1})$, the integral $\int_{-\tau_1}^{0} ds\, C(s)$ becomes $G_2 = (1/2) \int_{-\infty}^{+\infty} ds\, C(s)$. (We make the assumption that this integral converges; for an example where this assumption fails, see Ref. [3].) The condition for removing secularities at $O(\epsilon^2)$ then becomes

$$\frac{\partial}{\partial \tau_2}\left(\Sigma f_1\right) + \frac{\partial}{\partial E}\left(u \Sigma f_1\right) - \frac{1}{2}\frac{\partial}{\partial E}\left(\Sigma G_2 \frac{\partial f_0}{\partial E}\right) = 0, \tag{17}$$

which, along with the initial conditions, specifies f_1.

We now have our central result, valid to $O(\epsilon^1)$ for times of $O(\epsilon^{-1})$:

$$F(\mathbf{z}, t) = f_0(h, \tau_2) + \epsilon f_1(h, \tau_2) - \epsilon \frac{\partial f_0}{\partial E}(h, \tau_2) \int_0^{\tau_1} d\tau_1' \left[\frac{\partial h}{\partial \tau_2}(\mathbf{Z}, \tau_2) - u(h, \tau_2)\right], \tag{18}$$

where f_0 and f_1 satisfy Eqs. 10 and 17, respectively. Ott's solution for F does not contain a term corresponding to our $F_{1h} = f_1$; we believe this to be the source of the above-mentioned conflict with the Fokker-Planck equation.

Let us now consider the case when the initial conditions are distributed over a single energy shell: $f_0(E, 0) = \delta(E - E_0)/\Sigma(E, 0)$. [The factor $1/\Sigma$ provides normalization: $\int d\mathbf{z}\, F(\mathbf{z}, 0) = 1$.] The solution of Eq. 10 consistent with these initial conditions is

$$f_0(E, \tau_2) = \delta(E - \mathcal{E})/\Sigma(E, \tau_2), \tag{19}$$

where $\mathcal{E} = \mathcal{E}(\tau_2)$ is defined by $\Omega(\mathcal{E}, \tau_2) = \Omega(E_0, 0)$. [To demonstrate by inspection that Eq. 19 is a solution of Eq. 10, one needs the identity

$$d\mathcal{E}/dt = u(\mathcal{E}, \tau_2), \tag{20}$$

which follows from the definitions of $\mathcal{E}$, u, Σ, and Ω.] Eq. 19 shows that, to lowest order, $F(\mathbf{z}, t)$ remains distributed uniformly over the energy shell prescribed by the adiabatic invariance of Ω.

Continuing with these initial conditions, we now consider the moments $M_n(t) = \int d\mathbf{z}\, F(\mathbf{z}, t)(h - \mathcal{E})^n$ which measure the error in the ergodic adiabatic invariant. Of the terms on the right side of Eq. 18, the first and third do not contribute to M_n (the latter because the average of $[\partial h/\partial \tau_2(\mathbf{Z}, \tau_2) - u(h, \tau_2)]$ over any energy shell is zero), leaving

$$M_n(t) = \epsilon \int d\mathbf{z}\, f_1(h, \tau_2)(h - \mathcal{E})^n = \epsilon \int dE\, \Sigma f_1 (E - \mathcal{E})^n. \tag{21}$$

Differentiating with respect to time, one obtains (using Eqs. 17 and 20),

$$\frac{dM_1}{dt} = \epsilon^2 \int dE\, \Sigma f_1 [u - u(\mathcal{E}, \tau_2)] + \epsilon^2 \left[\frac{1}{2\Sigma}\frac{\partial}{\partial E}(\Sigma G_2)\right]_{E=\mathcal{E}} \tag{22a}$$

$$\frac{dM_2}{dt} = 2\epsilon^2 \int dE\, \Sigma f_1 (E - \mathcal{E})[u - u(\mathcal{E}, \tau_2)] + \epsilon^2 G_2(\mathcal{E}, \tau_2). \tag{22b}$$

Expanding $u(E,\tau_2)$ in a Taylor series around $E = \mathcal{E}$, and disregarding moments higher than the second [6], these equations become, to $O(\epsilon^2)$,

$$\frac{dM_1}{dt} = \epsilon\frac{\partial u}{\partial E}(\mathcal{E},\tau_2)M_1 + \frac{1}{2}\epsilon\frac{\partial^2 u}{\partial E^2}(\mathcal{E},\tau_2)M_2 + \epsilon^2\Big[\frac{1}{2\Sigma}\frac{\partial}{\partial E}(\Sigma G_2)\Big]_{E=\mathcal{E}} \tag{23a}$$

$$\frac{dM_2}{dt} = 2\epsilon\frac{\partial u}{\partial E}(\mathcal{E},\tau_2)M_2 + \epsilon^2 G_2(\mathcal{E},\tau_2). \tag{23b}$$

(Since $M_1,\ M_2 \sim \epsilon$ for $t \sim \epsilon^{-1}$, all terms on the right scale like ϵ^2.) These expressions agree with those derived from the Fokker-Planck equation (see Ref. [2], Eqs. 5.13 and 5.14), which resolves the discrepancy mentioned earlier.

Note that Eq. 18 directly leads to an evolution equation for the distribution of energies, $\eta(E,t)$, of an ensemble of systems evolving under an ergodic adiabatic Hamiltonian. We see this by writing $\eta(E,t) = \Sigma(E,\tau_2)\langle F(\mathbf{z},t)\rangle_{E,\tau_2} = \Sigma(f_0 + \epsilon f_1)$. Eqs. 10 and 17 then combine to give, to $O(\epsilon^2)$,

$$\frac{\partial\eta}{\partial t} = -\epsilon\frac{\partial}{\partial E}\big(u\eta\big) + \frac{1}{2}\epsilon^2\frac{\partial}{\partial E}\Big[G_2\Sigma\frac{\partial}{\partial E}\Big(\frac{\eta}{\Sigma}\Big)\Big], \tag{24}$$

which is exactly the Fokker-Planck equation of Refs. [2] and [4].

Finally, Eq. 18 has significance for dynamical problems in which the motion of a "slow" (or "large") system is coupled to an ensemble of noninteracting "fast" systems following chaotic trajectories. Such problems have been considered by Wilkinson [4], Berry and Robbins [7], and, earlier, in the context of nuclear processes, by Błocki, Świątecki, and coworkers [8], and by Koonin and Randrup [9]. In the formulation of Ref. [7], the fast ensemble is described by a density $\rho(\mathbf{z},t)$ evolving under the Hamiltonian $h(\mathbf{z},\mathbf{R}(t))$, where $\mathbf{R}$ denotes the configuration space coordinates of the slow system. (In this formulation, the slowly evolving variable $\mathbf{R}$ becomes a parameter of h, taking the place of the slow time τ_2.) The slow system is subject to a force

$$\mathbf{F}(t) = -\int d\mathbf{z}\,\rho\nabla h \tag{25}$$

(where $\nabla \equiv \partial/\partial\mathbf{R}$) due to its coupling to the fast ensemble [10]. To evaluate this force, we in turn need the response of ρ to the slow evolution of $\mathbf{R}$; this response is precisely the content of Eq. 18.

Taking the fast ensemble to be initially distributed over a single energy shell E_0 of $h(\mathbf{z},\mathbf{R}(0))$, the leading contribution to $\mathbf{F}(t)$ is obtained by averaging $-\nabla h$ over the energy shell $\mathcal{E}(\mathbf{R}(t))$ determined by the ergodic adiabatic invariant. This term is dubbed the "adiabatic" force in Ref. [7]. Following an evaluation of ρ similar to Ott's, Berry and Robbins find, at next order in the rate of change of $\mathbf{R}$, two contributions to $\mathbf{F}$ which they label "deterministic friction" (previously discussed by Wilkinson [4]) and "geometric magnetism"; both follow from a term in ρ corresponding to the term F_{1i} in Eq. 12 above. However, we claim that there also exists at this order a contribution to $\mathbf{F}$ following from a term corresponding to $F_{1h} = f_1$. This contribution can be understood as a correction to the adiabatic force. The latter was determined by assuming that, as the slow system evolves in time, the fast ensemble clings to the energy shell prescribed by the ergodic adiabatic invariant. This is true only at lowest order. At next order, we find ρ distributed over a narrow range of energy shells near $\mathcal{E}(\mathbf{R})$; the true adiabatic force at time t is then a weighted sum of contributions $\langle \nabla h \rangle$ from each of the shells to which the fast ensemble has diffused. This leads to a correction to the adiabatic force given by

$$-\epsilon \int dE \, \Sigma f_1 \left\langle \nabla h \right\rangle_E, \qquad (26)$$

which represents a contribution to $\mathbf{F}$ on the same order as geometric magnetism and deterministic friction.

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[10] This force can be motivated by writing a full (i.e. slow + fast) Hamiltonian $H = (1/2)V^2 + \int d\mathbf{z}\, \rho\, h(\mathbf{z}, \mathbf{R})$, where $\mathbf{V}$ represents the canonical momentum of the slow system. Hamilton's equations then give $d\mathbf{V}/dt = -\int d\mathbf{z}\, \rho \nabla h$, whence Eq. 25.