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Authors

Thomson, JJ
Benford, Gregory

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THE TURBULENT PARTICLE PROPAGATOR IN A MAGNETIC FIELD*

J. J. THOMSON

Lawrence Livermore Laboratory, University of California, Livermore, CA 94550, U.S.A.

and

GREGORY BENFORD

University of California, Irvine, CA 92717, U.S.A.

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Abstract—We examine qualitatively the requirement that field-particle interactions in turbulence be diffusive. The basic assumption necessary for turbulently dissipative (resonantly broadened) models is that the field-particle correlation time viewed along the particle orbit be much smaller than the field-field correlation time in the laboratory. We assume the orbit variables $\tilde{x}(t)$ and $\tilde{v}(t)$ are normally distributed, but the electric fields need not be. We then use simple arguments from probability theory to derive the particle propagator for a weakly turbulent plasma in a strong magnetic field. We use this propagator to find the effects of cross field diffusion on single wave pulses and plasma echoes.

1. INTRODUCTION

IN A TURBULENT plasma, the trajectory of a charged particle is statistically described by the conditional probability distribution function, $G(\mathbf{R}\mathbf{V}t/\mathbf{R}_0\mathbf{V}_0t_0)$. This quantity, first discussed systematically by ROSTOKER (1961), gives the probability that a particle at $(\mathbf{R}_0, \mathbf{V}_0)$ at time t_0 is at $(\mathbf{R}, \mathbf{V})$ at time t . HINTON and OBERMAN (1968) solved the short-time kinetic equation for G in a weakly turbulent unmagnetized plasma and found the effects of diffusion on single pulses and plasma wave echoes. We later (THOMSON and BENFORD, 1972) derived the same form for G from statistical arguments and showed its relation to the turbulence theory of DUPREE (1966).

Here we employ the statistical approach to derive the probability density function for a plasma in a strong magnetic field. The result, which may be considered an average particle propagator, is used to find the effects of cross-field diffusion on pulses and echoes.

In quasilinear theory, the Vlasov equation is solved in a perturbation series, evaluating each term by the Green's function appropriate for a free particle $\delta(x-x_0-v_0t)\delta(v-v_0)$. This essentially assumes that the zero-order trajectory for a particle is a straight line. Dupree's suggestion was that, since a free particle trajectory is not appropriate for strongly interactive systems, a better perturbation series is obtained by expanding about the zero-order averaged particle trajectory. In many systems, the averaged trajectory is diffusive, with a diffusion coefficient that depends on the turbulent field spectrum.

Now one may calculate the dielectric function $\epsilon_{k\omega}$ which again involves integration over diffusive particle trajectories. This in turn broadens the resonance function $\text{Im}(\epsilon)/\epsilon^2$, giving it a width in frequency equaling approximately the reciprocal of the trajectory diffusion time

$$\Delta\omega \equiv \tau_{ff}^{-1}(L) \simeq \tau_0^{-1},$$

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where $\tau_{ff}(L)$ is the *field-field* correlation time in the laboratory frame. Thus $\Delta\omega$ is the linewidth seen experimentally in the field spectrum.

However, this picture is incomplete. In order that the particle orbit be diffusive, the motion must resemble Brownian dynamics—that is, we require that the *field-particle* correlation time along the trajectory, $\tau_{fp}(T)$ be much less than the diffusion time, i.e.

$$\tau_{fp}(T) \ll \tau_0 = \tau_{ff}(L).$$

Note that there are two correlation times involved in defining our system, $\tau_{fp}(T)$ and $\tau_{ff}(L)$.

Consider a one-dimensional plasma. We require

$$\tau_{fp}(T) \ll \tau_{ff}(L) \equiv 1/\Delta\omega.$$

Now,

$$|\tau_{fp}(T)| = \text{Re} \left| \int_0^\infty e^{i(\omega - kv) - t\Delta\omega} dt \right| = \text{Re} \left| \frac{1}{i(\omega - kv) - \Delta\omega} \right| = \frac{\Delta\omega}{(\omega - kv)^2 + (\Delta\omega)^2}.$$

Our condition is then

$$\frac{\Delta\omega}{(\omega - kv)^2 + (\Delta\omega)^2} \ll \frac{1}{\Delta\omega}$$

or

$$\left(\frac{\omega - kv}{\Delta\omega} \right)^2 \gg 1.$$

Non-resonant particles satisfy this condition by definition. Their nonlinear motion is diffusive. Resonant particles undergo nonlinear oscillations in the wave troughs, however, at roughly the bounce frequency $\omega_B^2 \equiv E_k k e / m$, so $\omega - kv$ is not zero in the nonlinear regime. Then our condition becomes

$$\omega_B^2 \tau_{ff}^2(L) \equiv (\omega_B / \Delta\omega)^2 \gg 1$$

and this defines a sense in which we can describe resonant particles by a diffusive picture. This should not be taken too seriously, however, because the orbit undergoing diffusion is the nonlinear bounce orbit, which itself must be described. If one could reliably write the bounce orbit, a diffusive model for the resonant case could be derived. For our present purposes, note that while $\omega_B \tau_{ff}(L)$ can exceed unity, if we define ω_B^* as a bounce frequency for the *non-resonant* particles,

$$\omega_B^* \tau_{fp}(T) \equiv \frac{\omega_B^* \Delta\omega}{(\omega - kv)^2}$$

which will be small if $\omega_B^* \ll (\omega - kv)$. The above equation is a more specific statement of the usually-invoked $\omega_B^* \tau_{fp}(T) \ll 1$ condition, which justifies cutting off cumulant expansions, etc. (WEINSTOCK, 1970; BENFORD and THOMSON, 1972). Usually turbulence theories do not invoke different conditions for $\tau_{fp}(T)$ and $\tau_{ff}(L)$, but we see here that such specification is necessary.

We follow in this paper a somewhat unusual course in defining the turbulent system. We take the *orbit* variables $\mathbf{v}(t)$, $\mathbf{x}(t)$ to have Gaussian statistics. We do *not* assume the turbulent fields are normally distributed.

This differs from many treatments and, we feel, comes closer to the heart of the matter—the nature of the perturbed orbit.

In general, electric fields which are not normally distributed can still act on particles to make $\tilde{\mathbf{v}}(t)$ and $\tilde{\mathbf{x}}(t)$ Gaussian. (Admittedly, it is relatively easy to measure the probability distribution function of electric fields experimentally, and may be difficult to ascertain whether $\tilde{\mathbf{v}}(x)$ and $\tilde{\mathbf{x}}(t)$ are Gaussian, but this does not mean the field statistics are somehow more fundamental than the orbit statistics.) Our only assumption about the electric fields is that the turbulence be stationary. Other formulations of turbulence theory often assume the *laboratory* variables are normally distributed. In the Appendix we make this assumption. We then estimate the time for *trajectory* variables, which we assume are initially also normally distributed, to build up odd-power correlations such as $\langle (v')^3 \rangle$. We find that

$$\frac{\langle (v')^3 \rangle}{\langle (v')^2 \rangle^{3/2}} \ll 1$$

if

$$1 \ll \frac{t}{\tau_{ff}(L)} \ll [\omega_B \tau_{ff}(L)]^{-4/3}.$$

As t increases eventually the right hand inequality fails and the trajectory variables will no longer be normally distributed. Thus the normality assumption cannot work for both laboratory and trajectory variables for long times. Whether one should assume normality in the laboratory variables or in the trajectory variables should be determined by reference to experimental findings for the distribution in the laboratory frame.

Proceeding from the orbit statistics yields a clearer path to some standard results of weak turbulence theory—the form of $D_{\perp}(v)$, for example—and may provide a more programmatic way to carry forward other calculations, such as single pulse propagation and plasma echoes in turbulent media, and nonlinear frequency shifts (BENFORD, 1976).

2. THE PARTICLE PROPAGATOR

The N -particle distribution function at any time may be written in terms of its value at $t = t_0$ as

$$f_N(\mathbf{R}, \mathbf{V}, t) = \int d\mathbf{R}' d\mathbf{V}' G_N(\mathbf{R}\mathbf{V}t/\mathbf{R}_0'\mathbf{V}_0't_0) f_N(\mathbf{R}', \mathbf{V}', t_0) \quad (1)$$

where

$$f_N(\mathbf{R}, \mathbf{V}, t) = \sum_{R=1}^N \delta[\mathbf{R} - \mathbf{R}_i(t)] \delta[\mathbf{V} - \mathbf{V}_i(t)]$$

and G_N is the particle propagator. A similar equation may be written for the ensemble-averaged 1-particle distribution function

$$f(\mathbf{R}, \mathbf{V}, t) = \int d\mathbf{R}' d\mathbf{V}' g(\mathbf{R}\mathbf{V}t/\mathbf{R}'\mathbf{V}'t_0) f(\mathbf{R}', \mathbf{V}', t_0) \quad (2)$$

in the case where collisions (i.e. strong n -particle correlations) are unimportant. Equation (2) has a simple statistical interpretation; g is the conditional probability that a particle is at $(\mathbf{R}, \mathbf{V})$ at time t_0 .

In the case of weak or broad-band turbulence (BENFORD and THOMSON, 1972), the particles interact with a stochastic field. On a time scale long with respect to the field correlation time along a particle trajectory, the particles undergo diffusion in phase space. On the basis of this simple picture, we will derive the form of $g(RVt/R_0V_0t_0)$.

We assume a strong magnetic field ($\vec{B}$) in the z -direction. For simplicity of notation, we consider a two-dimensional problem, with the perpendicular direction denoted by the subscript $\perp$. The extension to cylindrical symmetry is easily made. The diffusion coefficients are defined as

$$\begin{aligned} D_z &= q^2/m^2 \int_0^\infty \langle E_z'(t)E_z'(t-\tau) \rangle d\tau \\ D_\perp &= c^2/B^2 \int_0^\infty \langle E_\perp'(t)E_\perp'(t-\tau) \rangle d\tau \\ D_{\perp z} &= q/mc/B \int_0^\infty \langle E_\perp'(t)E_z'(t-\tau) \rangle d\tau = D_{z\perp}. \end{aligned} \quad (3)$$

Where $E_{z,\perp}$ is the parallel (perpendicular) electric field, and the primes refer to stochastic quantities, with ensemble average zero. The integration is to be carried out along a particle trajectory. Define the variables

$$\begin{aligned} \xi_\perp &= R_\perp - R_{\perp 0} = c/B \int_{t_0}^t E'(s) ds \\ \eta &= V_z - V_{z0} = q/m \int_{t_0}^t E_z'(s) ds \\ \xi_z &= R_z - R_{z0} - V_{z0}\tau = \int_{t_0}^t \eta(x) ds \\ \tau &= t - t_0 \end{aligned} \quad (4)$$

also integrating along a particle trajectory. In the absence of turbulence, the quantities $\xi_\perp$, η , ξ_z would be zero, and the particle propagator would be given by

$$g_0(\xi_z, \eta, \xi_\perp) = \delta(\xi_z)\delta(\eta)\delta(\xi_\perp) = \delta(R_z - R_{z0} - V_{z0}\tau)\delta(V - V_{z0})\delta(R_\perp - R_{\perp 0}).$$

In the presence of turbulence, these delta-functions are broadened. The calculation of the propagator is complicated by the correlation between the variables. However, making the transformation $(\xi_z, \xi, \xi_\perp) \rightarrow (z_1, z_2, z_3)$, where the z_i 's are independent variables, allows us to write the propagator in the simpler form

$$g(\xi_z, \eta, \xi_\perp, \tau) = g_1(a_1)g_2(z_2)g_3(z_3)J$$

where J is the Jacobian of the transformation. The easiest way to see this is by making the identification of the propagator as the joint probability distribution function, $g(\xi_z, \eta, \xi_\perp, \tau) = f_\tau(\xi_z, \eta, \xi_\perp)$. Then

$$f_\tau(\xi_z, \eta, \xi_\perp) = f_\tau(z_1, z_2, z_3)J = f_\tau(z_1)f_\tau(z_2)f_\tau(z_3)J.$$

The separation is due to independence of the z 's. By our Markovian assumption of short autocorrelation times, η and $\xi_{\perp}$ are normally distributed. Since $\xi_z = \int_0^{\tau} \eta(\tau') d\tau'$, it is also normally distributed. Thus any linear transformation $T(\xi_z, \eta, \xi_{\perp} \rightarrow (z_1, z_2, z_3))$ implies that the z 's are also normal. Independence reduces to the condition $\langle z_i z_j \rangle = 0, i \neq j$.

We may use the familiar Schmidt orthogonalization procedure. An adequate set of independent variables is

$$\begin{aligned} z_1 &= \xi_{\perp} \\ z_2 &= \eta - D_{\perp z} / D_{\perp} \xi_{\perp} \\ z_3 &= \xi_z - \frac{1}{2} \eta \tau. \end{aligned} \quad (5)$$

The Jacobian is unity. Since the z 's are normal, their probability distribution function depends only on the mean and variance.

$$f_z(z_i) = \exp [-(z_i - \langle z_i \rangle)^2 / 2 \langle \Delta z_i^2 \rangle] / (2\pi \langle \Delta z_i^2 \rangle)^{1/2} \quad (6)$$

where $\Delta z_i = z_i - \langle z_i \rangle = 0$. We may calculate the variances with the following easily derived quantities.

$$\langle \xi_{\perp}^2 \rangle = 2D_{\perp} \tau \quad (7a)$$

$$\langle \xi_z^2 \rangle = \frac{2}{3} D_z \tau^3 \quad (7b)$$

$$\langle \eta^2 \rangle = 2D_z \tau \quad (7c)$$

$$\langle \xi_z \eta \rangle = \frac{4}{3} D_z \tau^2 \quad (7d)$$

$$\langle \xi_{\perp} \eta \rangle = 2D_{\perp z} \tau \quad (7e)$$

$$\langle \xi_{\perp} \xi_z \rangle = D_{\perp z} \tau^2. \quad (7f)$$

These expressions follow from the definition of the diffusion coefficients. To show an example, we calculate the last one.

$$\begin{aligned} \langle \xi_{\perp}(\tau) \xi_z(\tau) \rangle &= c/Bq/m \left\langle \int_0^{\tau} E_{\perp}(s) ds \int_0^{\tau} d\tau' \int_0^{\tau'} d\tau'' E_z(\tau'') \right\rangle \\ &= qc/mB \int_0^{\tau} d\tau' \int_0^{\tau'} d\tau'' \left[\int_0^{\tau''} ds + \int_{\tau''}^{\tau} ds \right] \langle E_{\perp}(s) E_z(\tau'') \rangle \\ &= qc/mB \int_0^{\tau} d\tau' \int_0^{\tau'} d\tau'' \left[\int_0^{\tau''} \langle E_{\perp}(\tau'' - s') E_z(\tau'') \rangle ds'' \right. \\ &\quad \left. + \int_0^{\tau - \tau''} \langle E_{\perp}(\tau'' + s') E_z(\tau'') \rangle ds' \right]. \end{aligned}$$

By the assumption of short autocorrelation times, we may extend the limits on the integrals in the brackets to infinity. By the assumption of stationary turbulence,

$$qc/mB \int_0^{\infty} \langle E_{\perp}(\tau'' + s) E_z(\tau'') \rangle ds = qc/mB \int_0^{\infty} \langle E_{\perp}(\tau'' - s) E_z(\tau'') \rangle ds = D_{\perp z}$$

therefore

$$\langle \xi_{\perp} \xi_z \rangle = \int_0^{\tau} d\tau' \int_0^{\tau'} d\tau'' 2D_{\perp z} = D_{\perp z} \tau^2.$$

Using (7a-f) and (6), we may write the particle propagator:

$$g(\xi_z, \eta, \xi_\perp) = \frac{\exp[-\xi_\perp^2/4D_\perp\tau] \exp\left[-\frac{(\eta - (D_{\perp z}/D_\perp)\xi_\perp)^2}{4\alpha D_z\tau}\right]}{(4\pi D_\perp\tau)^{1/2}(4\pi\alpha D_z\tau)^{1/2}} \\ \times \frac{\exp\left[-\frac{(\xi_z - \frac{1}{2}\eta\tau)^2}{\frac{1}{3}D_z\tau^3}\right]}{(\frac{1}{3}\pi D_z\tau^3)^{1/2}} \quad (8)$$

where $\alpha = 1 + D_{\perp z}/D_z(1 - D_{\perp z}/D_\perp)$. From the form of (8), it is easy to see that the delta functions of the propagator in a quiescent medium have spread into Gaussians for the turbulent propagator. When the correlation between the parallel and perpendicular fields disappears ($D_{\perp z} \rightarrow 0$), (8) becomes the product of the parallel propagator $g(\xi_z, \eta, \tau)$, and the perpendicular propagator given by

$$g(\xi_\perp, \tau) = \frac{\exp[-\xi_\perp^2/4D_\perp\tau]}{(4\pi D_\perp\tau)^{1/2}}.$$

Another method for calculating the propagator is by direct solution of the equation that it satisfies, the drift kinetic equation.

$$\left[\frac{\partial}{\partial t} + V_z \frac{\partial}{\partial z} - \frac{\partial}{\partial V_z} D_z \frac{\partial}{\partial V_z} - 2 \frac{\partial}{\partial V_z} D_{\perp z} \frac{\partial}{\partial R_\perp} - \frac{\partial}{\partial R_\perp} D_\perp \frac{\partial}{\partial R_\perp} \right] g(\mathbf{R}, \mathbf{V}, t/R_0, V_0, t_0) = 0 \\ g(\mathbf{R}, \mathbf{V}, t_0/R_0, V_0, t_0) = \delta(\mathbf{R} - \mathbf{R}_0)\delta(\mathbf{V} - \mathbf{V}_0).$$

Following HINTON and OBERMAN (1968), we may transform to variables similar to $\xi_\perp, \eta, \xi_z$ (except that one defines $\xi_z = R - R_0 + V\tau$) and Fourier transform in these variables. The resulting equation may be solved (for constant diffusion coefficients), and then retransformed to obtain g . The result of this lengthy calculation is exactly equation (8).

The advantage of the statistical derivation presented here is two-fold: first, it is simple, and second, we will be able to evaluate integrals by using well-known properties of normal random variables, rather than performing complicated algebra.

3. THE DIFFUSION COEFFICIENT IN A MAGNETIC FIELD

This problem was considered by DUPREE (1967) and WEINSTOCK (1968). Here we rederive a result similar to theirs using the statistical formulation. For electrostatic waves, the diffusion coefficients of equation (3) may be written

$$\begin{bmatrix} D_z \\ D_{\perp z} \\ D_\perp \end{bmatrix} = \int \frac{d\mathbf{k} d\omega}{(2\pi)^4} |\varphi_{k\omega}|^2 \int_0^\infty \exp[i\omega\tau - i\mathbf{k} \cdot \mathbf{R}] \langle \exp[i\mathbf{k} \cdot \mathbf{R}(-\tau)] \rangle \\ \begin{bmatrix} q^2/m^2 k^2 \\ q/mk_z \frac{\mathbf{k} \times \mathbf{B}}{B^2} \\ \frac{c^2}{B^4} (\mathbf{k} \times \mathbf{B})(\mathbf{k} \times \mathbf{B}) \end{bmatrix} \quad (9)$$

where

$$\langle \exp [i\mathbf{k} \cdot \mathbf{R}(-\tau)] \rangle = \int d\mathbf{R}' d\mathbf{v}' g(\mathbf{R}\mathbf{V}t/\mathbf{R}'\mathbf{V}', t-\tau) \exp [i\mathbf{k} \cdot \mathbf{R}'].$$

The brackets $\langle \dots \rangle$ thus refer to an average over statistical trajectories. We may make the $\xi_z, \xi_\perp, \eta, \tau$ transformation of equation (4).

$$\begin{aligned} \langle \exp [i\mathbf{k} \cdot \mathbf{R}(-\tau)] \rangle &= \langle \exp [ik_z \cdot (R - V\tau) - ik_z(\xi_z - \eta\tau) - ik_\perp \xi_\perp] \rangle \\ &= \exp [i(k \cdot R - V\tau)] \langle \exp [-ik_z(\xi_z - \eta\tau) - ik_\perp \xi_\perp] \rangle. \end{aligned}$$

Since the variables $\xi_z, \xi_\perp, \eta$ are normal, we may exactly evaluate the averaged exponential in terms of a cumulant expansion as suggested by WEINSTOCK (1968):

$$\begin{aligned} \langle \exp [-ik_z(\xi_z - \eta\tau) - ik_\perp \xi_\perp] \rangle &= \exp \left[-\frac{1}{2}k_z^2 \langle \xi_z - \eta\tau \rangle^2 \right. \\ &\quad \left. - k_z k_\perp \langle (\xi_z - \eta\tau) \xi_\perp \rangle - \frac{1}{2}k_\perp^2 \langle \xi_\perp^2 \rangle \right] \\ &= \exp \left[-\frac{1}{3}k_z^2 D_z \tau^3 + k_\perp k_z D_{\perp z} \tau^2 - k_\perp^2 D_\perp \tau \right]. \quad (10) \end{aligned}$$

Equation (10) does not agree with the results of DUPREE (1967) in the sign of the $D_{\perp z}$ term, but this is not significant, since we sum over positive and negative k 's.

If one writes the drift kinetic equation in the form

$$\left[\left(\frac{\partial}{\partial t} - k_\perp^2 D_\perp \right) - 2ik_\perp D_{\perp z} \frac{\partial}{\partial V_z} - D_z \frac{\partial^2}{\partial V_z^2} \right] g = 0,$$

it has the appearance of the usual Fokker-Planck equation with collision frequency $k_\perp^2 D_\perp$, and imaginary friction term $A = ik_\perp D_{\perp z}$. One then may use KRUEER's (1971) solution to calculate a diffusion coefficient. (Note that his diffusion coefficients are half ours by choice of definition.) The result agrees with our equation (10).

4. SINGLE PULSE PROPAGATION

We consider the application of a potential pulse at $t=0$. For $t < 0$, the distribution function is $f_0(\mathbf{V})$. At $t=0_+$, it becomes $f_0(\mathbf{V} - \mathbf{V}_0 \cos(\mathbf{k}_0 \cdot \mathbf{R}))$. At any time $t > 0$, it is given by

$$f(\mathbf{R}, \mathbf{V}, t) = \int d\mathbf{R}' d\mathbf{v}' g(\mathbf{R}\mathbf{V}t/\mathbf{R}'\mathbf{V}'0) f_0[\mathbf{V}' - \mathbf{V}_0 \cos(\mathbf{k}_0 \cdot \mathbf{R}')]. \quad (11)$$

We may Taylor expand $f_0(\mathbf{V} - \mathbf{V}_0 \cos(\mathbf{k}_0 \cdot \mathbf{R}))$ and resum the expansion into the displacement operator.

$$\begin{aligned} f_0(\mathbf{V}' - \mathbf{V}_0 \cos(\mathbf{k}_0 \cdot \mathbf{R}')) &= \sum_{n=0}^{\infty} \frac{1}{n!} \left[\cos(\mathbf{k}_0 \cdot \mathbf{R}') \mathbf{V}_0 \cdot \frac{\partial}{\partial \mathbf{V}'} \right]^n f_0(\mathbf{V}') \\ &= \exp \left[\cos(\mathbf{k}_0 \cdot \mathbf{R}') \mathbf{V}_0 \cdot \frac{\partial}{\partial \mathbf{V}'} \right] f_0(\mathbf{V}'). \end{aligned}$$

Now using the Bessel function expansion for $\exp(\cos \theta)$, (12) becomes

$$f(R, V, t) = \int d\mathbf{R}' d\mathbf{V}' g(\mathbf{R}\mathbf{V}t/\mathbf{R}'\mathbf{V}'0) \sum_n (-i)^n e^{in\mathbf{k}_0 \cdot \mathbf{R}'} J_n \left(\mathbf{V}_0 \cdot \frac{\partial}{\partial \mathbf{V}'} \right) f_0(V'). \quad (12)$$

The equivalence of (11) and (12) may be established by expanding in a series.

In their earlier calculations, HINTON and OBERMAN (1968) neglected the statistical fluctuations relative to the thermal velocity. That is, they took $f(V') = f(V)$. We shall temporarily accept this assumption since it simplifies the analysis. Taking V' equal to V in $J_n(V_0 \cdot (\partial/\partial V)f_0(V'))$, (12) may be written

$$f(\mathbf{R}, \mathbf{V}, t) = \sum_n (i)^n \langle \exp(in\mathbf{k}_0 \cdot \mathbf{R}(-t)) \rangle J_n \left(\mathbf{V}_0 \cdot \frac{\partial}{\partial \mathbf{V}} \right) f_0(\mathbf{V}). \quad (13)$$

The averaged exponential is of the same form as required for the diffusion coefficient, equation (10). Using the result of the previous section (13) becomes

$$f(\mathbf{R}, \mathbf{V}, t) = \sum (i)^n \exp[in\mathbf{k}_0 \cdot (\mathbf{R} - \mathbf{V}t) - \frac{1}{2}n^2 k_{0z}^2 D_z t^3 + n^2 k_{0\perp} k_{0z} D_{\perp z} t^2 - n^2 k_{0\perp} D_{\perp} t] J_n \left(\mathbf{V}_0 \cdot \frac{\partial}{\partial \mathbf{V}} \right) f_0(\mathbf{V}). \quad (14)$$

If we follow the analysis of HINTON and OBERMAN (1968), assuming $f_0(V)$ a Maxwellian, only keeping the first term of each Bessel function series and setting $k_{\perp} = 0$, (14) reduces to their equation (27).

Now let us relax the assumption that $f_0(V') = f(V)$. After making the $\xi_{\perp}$, ξ_z , η transformations, (12) may be written

$$f(\mathbf{R}, \mathbf{V}, t) = \int d\xi_{\perp} d\xi_z d\eta g(\xi_z, \eta, \xi_{\perp}, t) \sum_n (i)^n \exp[in\mathbf{k} \cdot (\mathbf{R} - \mathbf{V}t) - i\mathbf{k} \cdot (\xi - \eta t)] J_n \left(\mathbf{V}_0 \cdot \frac{\partial}{\partial \mathbf{V}} \right) f_0(\mathbf{V} - \eta). \quad (15)$$

In general, this would be difficult to evaluate. However, let us integrate (15) over $\mathbf{V}$, to obtain the density fluctuation, $n(\mathbf{R}, t)$.

$$\begin{aligned} n(\mathbf{R}, t) &= \int d\mathbf{V} f(\mathbf{R}, \mathbf{V}, t) \\ &= \int d\mathbf{V} \int d\xi d\eta g(\xi, \eta, t) \sum_n (-i)^n \exp[in\mathbf{k} \cdot (\mathbf{R} - \mathbf{V}t) - i\mathbf{k} \cdot (\xi - \eta t)] J_n \left(\mathbf{V}_0 \cdot \frac{\partial}{\partial \mathbf{V}} \right) f_0(\mathbf{V} - \eta). \end{aligned} \quad (16)$$

Now make the transformation $\mathbf{U} = \mathbf{V} - \eta$, $\eta' = \eta$. Equation (16) becomes

$$\begin{aligned} n(\mathbf{R}, t) &= \int d\mathbf{U} d\xi d\eta' g(\xi, \eta' t) \sum_n (-i)^n \exp[in\mathbf{k} \cdot (\mathbf{R} - \mathbf{U}t) - i\mathbf{k} \cdot \xi] \\ &\quad J_n \left(\mathbf{V}_0 \cdot \frac{\partial}{\partial \mathbf{U}} \right) f_0(\mathbf{U}) \\ &= \int d\mathbf{U} \sum_n (-i)^n \langle \exp(in\mathbf{k}_0 \cdot \xi) \rangle \exp[in\mathbf{k} \cdot (\mathbf{R} - \mathbf{U}t)] \\ &\quad J_n \left(\mathbf{V}_0 \cdot \frac{\partial}{\partial \mathbf{U}} \right) f_0(\mathbf{U}). \end{aligned}$$

The averaged exponential is easy to evaluate. The result is

$$n(\mathbf{R}, t) = \int d\mathbf{U} \sum_n (-i)^n \exp [in\mathbf{k}_0 \cdot (\mathbf{R} - \mathbf{U}t) - \frac{1}{3}n^2 k_{0z}^2 D_z t^3 - n^2 k_\perp k_z D_{\perp z} t^2 - \eta^2 k_\perp^2 D_\perp t] J_n \left[\left(\mathbf{V}_0 \cdot \frac{\partial}{\partial \mathbf{U}} \right) f_0(\mathbf{U}) \right]. \quad (17)$$

By our assumption of strong magnetic field, the particle velocities are directed along z , and the vector velocities in the above equations reduce to the scalar parallel components. If we had wished to include the effects of $V_\perp$, our set of stochastic quantities would have included another term $\eta'_\perp$ and the trajectory integrals would be helices, with considerable complication of the analysis.

5. EFFECTS ON PLASMA ECHOES

We wish to consider the temporal echo caused by applying a second pulse given by

$$-e/m\mathbf{E}_1(\mathbf{x}, t) = \mathbf{V}_1 \delta(t - \tau) \cos \mathbf{k}_1 \cdot \mathbf{x}$$

at a time τ later than the initial pulse. This second pulse can reverse the phase mixing of the first pulse and cause a reconstruction of the initial disturbance.

We use (14) for the distribution function at a time $t = \tau$. At $t = \tau_+$, it is

$$f(\mathbf{R}, V, \tau_+) = \sum_n (-i)^n \exp [n^2 (-\frac{1}{3}k_{0z}^2 D_z \tau^3 + k_{0\perp} k_{0z} D_{\perp z} \tau^2 - k_{0\perp}^2 D_\perp \tau + ink_{0z}(\mathbf{R} - \mathbf{V}\tau)] J_n \left(\mathbf{V}_0 \cdot \frac{\partial}{\partial \mathbf{V}} \right) f_0(\mathbf{V} - \mathbf{V}_1 \cos \mathbf{k}_1 \cdot \mathbf{R}). \quad (18)$$

At time $t > \tau$, it is given by

$$f(\mathbf{R}, \mathbf{V}, t) = \int d\mathbf{R}' d\mathbf{V}' g(\mathbf{R}\mathbf{V}t/\mathbf{R}'\mathbf{V}'\tau) f(\mathbf{R}', \mathbf{V}', \tau_+). \quad (19)$$

We may evaluate (19) in a manner similar to the single-pulse analysis. Using the displacement operator, (19) becomes

$$f(t) = \int d\mathbf{R}' d\mathbf{V}' g(t/\tau) \sum_{n,m} (-i)^{n+m} \exp [-n^2 (\frac{1}{3}k_{0z}^2 D_z \tau^3 - k_{0\perp} k_{0z} D_{\perp z} \tau^2 + k_{0\perp}^2 D_\perp \tau)] \exp [im\mathbf{k}_1 \cdot \mathbf{R}'] J_m \left(\mathbf{V}_1 \cdot \frac{\partial}{\partial \mathbf{V}'} \right) \exp [in\mathbf{k}_0(\mathbf{R}' - \mathbf{V}'\tau)] J_n \left(\mathbf{V}_0 \cdot \frac{\partial}{\partial \mathbf{V}'} \right) f_0(\mathbf{V}').$$

We again make the ξ, η, τ transformation, integrate over velocity and then define $U = V - \eta, \eta' = \eta$. The result is

$$n(\mathbf{R}, t) = \int d\mathbf{U} \sum_{n,m} (-i)^{n+m} \exp [-n^2 (\frac{1}{3}k_{0z}^2 D_z \tau^3 - k_{0\perp} k_{0z} D_{\perp z} \tau^2 + k_{0\perp}^2 D_\perp \tau)] \exp [i(m\mathbf{k}_1 + n\mathbf{k}_0) \cdot \mathbf{R}'(t - \tau)] J_m \left(\mathbf{V}_1 \cdot \frac{\partial}{\partial U} \right) \exp [-in\mathbf{k}_0 \cdot \mathbf{U}\tau] J_n \left(\mathbf{V}_0 \cdot \frac{\partial}{\partial \mathbf{U}} \right) f_0(\mathbf{U}). \quad (20)$$

Evaluating the averaged exponential as above, our result for the density fluctuation is

$$\begin{aligned}
 n(\mathbf{R}, t) = & \int d\mathbf{U} \sum_{n,m} (-i)^{n+m} \exp[i(m\mathbf{k}_1 + n\mathbf{k}_0) \cdot (\mathbf{R} - \mathbf{U}(t - \tau))] \\
 & -\frac{1}{3}(mk_{1z} + nk_{0z})^2 D_z(t - \tau)^3 - (mk_{1z} + nk_{0z})(mk_{1\perp} + nk_{0\perp}) \\
 & \times D_{\perp z}(t - \tau)^2 - (mk_{1\perp} + nk_{0\perp})^2 D_{\perp}(t - \tau) - \frac{1}{3}n^2 k_{0z}^2 D_z \tau^3 \\
 & + n^2 k_{0\perp} k_{0z} D_{\perp z} \tau^2 - n^2 k_{0\perp}^2 D_{\perp} \tau] \\
 & \times J_m\left(\mathbf{V}_1 \cdot \frac{\partial}{\partial \mathbf{U}}\right) \exp[-in\mathbf{k}_0 \cdot \mathbf{U}\tau] J_n\left(\mathbf{V}_0 \cdot \frac{\partial}{\partial \mathbf{U}}\right) f_0(\mathbf{U}). \quad (21)
 \end{aligned}$$

Taking $k = k_z$ and using the small argument expansion for the Bessel functions, we recover a result for $n(R, t)$ with the same physical implications as Hinton and Oberman's equation (29).

As discussed by Hinton and Oberman, the plasma echo occurs at a characteristic time, given by

$$t = mk_1 \tau / (mk_1 + nk_0).$$

In order for the echo to be appreciable, any experimental time scale must be short compared to the characteristic diffusion times. This means that

$$\begin{aligned}
 \tau & \ll (\frac{1}{3}n^2 k_{0z}^2 D_z)^{-1/3}, (n^2 k_{0\perp}^2 D_{\perp})^{-1} \\
 t - \tau & \ll [\frac{1}{3}(mk_{1z} + nk_{0z})^2 D_z]^{-1/3}, [(mk_{1\perp} + nk_{0\perp})^2 D_{\perp}]^{-1}.
 \end{aligned}$$

6. DISCUSSION

We have considered the effect of cross field diffusion on the propagation of pulses in a plasma in the limit of a strong magnetic field. This may be important in the presence of strong perpendicular turbulence; for example, drift modes or ion cyclotron modes in a Q -machine or a tokamak.

We would like to stress the ease of calculation afforded by the statistical approach of considering the fluctuating quantities as normal stochastic variables. As discussed in BENFORD and THOMSON (1972), the normality of these quantities is a consequence of assuming short field-particle correlation times.

The assumption of statistical independence must be justified physically. Even if interactions are strong, if they are normally distributed they lead to diffusion on an appropriately long time scale. This time must exceed the field-particle correlation time viewed along the particle trajectory, $\tau_{fp}(T)$. Our basic expansion parameter is $\omega_B \tau_{fp} \ll 1$, where ω_B is the 'bounce' frequency of non-resonant particles in the wave fields (again, defined in the particle trajectory frame).

What matters here are the statistics, *not* the field strength alone. Strong fields which give normally distributed $x(t)$, $v(t)$ —i.e. have no higher terms in the cumulant expansion of the particle orbit—can still cause diffusive behavior, despite their large amplitude. In this connection, note that RAETHER and YAMATO (1973) found that as the strength of ion acoustic turbulence increased, it became more Gaussian. While normality of the fields does not imply normality of the orbital variables, their result may be indicative.

Since calculations following the formalism of DUPREE (1966) and of WEINSTOCK (1968) are often complicated, especially in a magnetic field with arbitrary $\mathbf{k}$,

the simple statistical model used here may offer an easier starting point for explicit computations. Then further nonlinear effects can be calculated, such as nonlinear frequency shifts and linewidths (BENFORD, 1976). The difficult integrals often encountered in turbulence theory can then be evaluated by invoking the familiar properties of normal random variables. Going beyond the statistical framework seems difficult, however, and new physical aspects will be involved (THOMSON and BENFORD, 1973; MISGUICH and BALESCU, 1975).

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APPENDIX

We have explicitly assumed odd trajectory correlations such as $\langle (v')^3 \rangle$ are small for times of interest. Strictly speaking, this is not always true, as we shall show.

To assess the importance of odd correlations, we calculate $\langle (v')^3 \rangle / \langle (v')^2 \rangle^{-3/2}$ as a measure of the deviation from normality of the trajectory variable v' , where

$$\langle (v')^2 \rangle = \int_0^t \int_0^t dt' dt'' \langle F(t') F(t'') \rangle = 2Dt$$

if the force $F(t)$ is normally distributed. The orbit position variable is

$$r' = \int_0^t F(t') \exp [i(\vec{k} \cdot \vec{v}_0 t' + \vec{k} \cdot \vec{r}')] dt' \quad (\text{A.1})$$

where $\vec{v}_0$ is the unperturbed velocity. We assume now that the $\vec{k} \cdot \vec{r}'$ term is small and take

$$\exp (i\vec{k} \cdot \vec{r}') \approx (1 + i\vec{k} \cdot \vec{r}') \quad (\text{A.2})$$

in the integral. Then

$$\begin{aligned} \langle (v')^3 \rangle &= \left\langle \left\{ \int_0^t dt' F(t') e^{i\vec{k} \cdot \vec{v}_0 t'} [1 + i\vec{k} \cdot \vec{r}'(t')]^3 \right\} \right\rangle \\ &\approx \left\langle \int_0^t \int_0^t \int_0^t dt_1 dt_2 dt_3 F(t_1) F(t_2) F(t_3) \right. \\ &\quad \times [1 + i\vec{k} \cdot \vec{r}(t)] [1 - k^2 r(t_2) r(t_3) + i\vec{k} \cdot \vec{r}(t_3) + i\vec{k} \cdot \vec{r}(t_3)] \Big\rangle \\ &\approx 3ik \int_0^t \int_0^t \int_0^t dt_1 dt_2 dt_3 \langle F(t_1) F(t_2) F(t_3) r(t_3) \rangle. \end{aligned}$$

We assume the odd correlations of F are zero. The first order term in the $\langle Fr \rangle$ correlations is diffusive on a time $\tau_{ff}(L)$, since to first order r is given by equations (A.1) and (A.2) combined. We then find

$$\langle\langle v' \rangle^3 \rangle \approx (3ik)(4D^2)\left(\frac{t^3}{3}\right) = 4ikD^2t^3.$$

Writing

$$D \equiv \left(\frac{q}{m}\right)^2 E_k^2 \tau_{ff}(L)$$

we find

$$\left| \frac{\langle\langle v' \rangle^3 \rangle}{\langle\langle v' \rangle^2 \rangle^{3/2}} \right| \approx \omega_B^2 \tau_{ff}^2(L) \left(\frac{t}{\tau_{ff}(L)}\right)^{3/2}$$

so the odd correlations in the trajectory variables remain small for a time defined by

$$\frac{t}{\tau_{ff}(L)} \ll [\omega_B \tau_{ff}(L)]^{-4/3}.$$

We also require $\tau_{fp}(T) \ll t$, as described in the Introduction.