

UC Irvine

Open Access Policy Deposits

Title

Idempotent ideals in integral group rings

Permalink

<https://escholarship.org/uc/item/6gb055h8>

Journal

Journal of Algebra, 23(2)

ISSN

00218693

Author

Akasaki, Takeo

Publication Date

1972-11-01

DOI

10.1016/0021-8693(72)90135-4

Peer reviewed

Idempotent Ideals in Integral Group Rings

TAKEO AKASAKI

Department of Mathematics, University of California, Irvine, Irvine, California 92664

Communicated by I. N. Herstein

Received May 24, 1971

INTRODUCTION

Let π be a finite group and let $\mathbb{Z}\pi$ be its integral group ring. A well-known result of Swan [5, Corollary 8.1] states that $\mathbb{Z}\pi$ has no nontrivial idempotent elements. It follows by a proof in [7, p. 215] that if π is abelian, then $\mathbb{Z}\pi$ has no nontrivial idempotent ideals. On the other hand, if π is perfect (and hence not solvable), then the augmentation ideal of $\mathbb{Z}\pi$ is idempotent [2, p. 190]. The following natural problem is considered here. Determine which finite groups π have the property that $\mathbb{Z}\pi$ be without nontrivial idempotent ideals.

In the present paper we show that if π is nilpotent (e.g., abelian), then $\mathbb{Z}\pi$ has no nontrivial idempotent ideals (Theorem 3.1). Also, if π is not solvable, then $\mathbb{Z}\pi$ has a nontrivial idempotent ideal (Theorem 2.1). It seems likely that $\mathbb{Z}\pi$ can have nontrivial idempotent ideals if and only if π is not solvable. This is false for the case where the idempotent ideal is only one-sided (Section 5). The proof of Theorem 3.1 is different from the abelian case since the argument used in [7, p. 215] does not seem to extend to non-commutative situations. Our proofs in this direction occupy most of the work. The main step is to show that any idempotent ideal is $\mathbb{Z}\pi$ -projective. As an application, we give the analogue (Corollary 4.2) of a result of Jennings [4] for $\mathbb{Z}\pi$.

Given any finite group π , it is not known whether every nonfinitely generated projective $\mathbb{Z}\pi$ -module is free [6, p. 88], [1]. We refer the reader to the final paragraph of this paper for the role that idempotent ideals play in this unsolved problem.

In the sequel, by ideal we shall mean two-sided ideal.

I am grateful to William H. Smoke for his assistance in the problem considered here. Also, I wish to acknowledge my indebtedness to the referee for his suggested modification in the proof of 3.1 and for the examples in §5.

1. Let R be a ring with identity. If M is an R -module, the image of the

natural pairing $\text{Hom}_R(M, R) \otimes M \rightarrow R$ is an ideal $\tau_R(M)$ and called the trace ideal of M . We shall suppress the subscript R when the reference to the ring R is clear.

The following two propositions are known and can be easily verified.

PROPOSITION 1.1. *If A is an idempotent ideal of R , then $\tau(A) = A$.*

PROPOSITION 1.2. *Let M be a projective R -module. Then $\tau(M) = R$ if and only if for any maximal ideal A of R , $\text{Hom}_R(M, R/A) \neq 0$.*

The proof of the next proposition holds in a more general context, but this does not concern us here.

PROPOSITION 1.3. *Let π be any group and M be a projective $\mathbb{Z}\pi$ -module. Then $\tau(M) = \mathbb{Z}\pi$ if and only if for any prime p of $\mathbb{Z}$, $\tau_{\mathbb{Z}_p\pi}(M/pM) = \mathbb{Z}_p\pi$.*

Proof. The necessity follows immediately from the fact that for any R -module M , $\tau(M) = R$ if and only if R is a direct summand of a direct sum of copies of M . For the sufficiency, let A be any maximal ideal of $\mathbb{Z}\pi$. Then $A \cap \mathbb{Z} = (p)$ is a prime ideal of $\mathbb{Z}$. Since $\tau_{\mathbb{Z}_p\pi}(M/pM) = \mathbb{Z}_p\pi$, $\text{Hom}_{\mathbb{Z}_p\pi}(M/pM, \mathbb{Z}_p\pi/A) \neq 0$ by Proposition 1.2. But M is $\mathbb{Z}\pi$ -projective. So it follows that $\text{Hom}_{\mathbb{Z}\pi}(M, \mathbb{Z}\pi/A) \neq 0$. Thus by Proposition 1.2, the proof is complete.

Combining Proposition 1.3 with a major result of Swan [5], we obtain the principal objective of this section.

COROLLARY 1.4. *Let π be any finite group. If P is a finitely generated nonzero projective module over $\mathbb{Z}\pi$, then its trace ideal $\tau(P)$ is equal to $\mathbb{Z}\pi$.*

Proof. Swan [5, Corollary 7.2] has shown that P/mP is free over $\mathbb{Z}_m\pi$ for any integer $m \neq 0$. Since P is $\mathbb{Z}$ -free, P/mP is nonzero. It follows by Proposition 1.3 that $\tau(P)$ coincides with $\mathbb{Z}\pi$ and the proof is complete.

2. THEOREM 2.1. *Let π be a finite group. If π is not solvable, then $\mathbb{Z}\pi$ has a non-trivial idempotent ideal.*

Proof. Since π is not solvable, it contains a normal subgroup $\pi' \neq \{e\}$ such that $\pi' = [\pi', \pi']$. Let I' denote the augmentation ideal of $\mathbb{Z}\pi'$. By [2, p. 190], I' is idempotent. We have an exact sequence

$$0 \rightarrow \mathbb{Z}\pi \cdot I' \rightarrow \mathbb{Z}\pi \rightarrow \mathbb{Z}\pi/\pi' \rightarrow 0.$$

But since π' is normal in π , $\mathbb{Z}\pi \cdot I' = I' \cdot \mathbb{Z}\pi$ and hence the ideal $\mathbb{Z}\pi \cdot I'$ is idempotent. This completes the proof.

3. Recall that a finite group π is nilpotent if and only if π is the direct product of its sylow subgroups.

THEOREM 3.1. *If π is a finite nilpotent group, then $\mathbb{Z}\pi$ has no nontrivial idempotent ideals.*

Proof. Use induction on the order of π . Let p be any prime number dividing $|\pi|$ and let $\pi' \triangleleft \pi$ be a nontrivial p -group. The kernel J of $\mathbb{Z}\pi \rightarrow \mathbb{Z}\pi/\pi'$ is $\mathbb{Z}\pi \cdot I'$ where I' is the augmentation ideal of $\mathbb{Z}\pi'$. Since I'/pI' is the augmentation ideal of $\mathbb{Z}_p\pi'$, it is nilpotent. Therefore, $I'^n \subset pI'$ for some n and so $J^n = \mathbb{Z}\pi \cdot I'^n \subset p\mathbb{Z}\pi$. Now let A be an idempotent ideal of $\mathbb{Z}\pi$. By induction, the image of A in $\mathbb{Z}\pi/\pi'$ is (0) or $\mathbb{Z}\pi/\pi'$. Therefore, $A \subset J$ or $A + J = \mathbb{Z}\pi$. If $A \subset J$ then $A \subset \bigcap J^n \subset \bigcap p^n\mathbb{Z}\pi = (0)$ so if $A \neq (0)$, then $A + J = \mathbb{Z}\pi$. This implies that $A + J^n = \mathbb{Z}\pi$ so $A + p\mathbb{Z}\pi = \mathbb{Z}\pi$. Therefore, $\mathbb{Z}\pi/A$ is divisible by p and so must be finite of order prime to p . This applies to all primes $p \mid |\pi|$ so $[\mathbb{Z}\pi : A]$ is prime to $|\pi|$. According to a result of Swan [5, Proposition 7.1] we conclude that A is $\mathbb{Z}\pi$ -projective. Therefore, by Corollary 1.4 and Proposition 1.1, $A = \mathbb{Z}\pi$. This completes the proof of the theorem.

4. As an easy consequence of Theorem 3.1, we get the analogue of a result of Jennings [4] for $\mathbb{Z}\pi$. (Jennings considered the group ring of a finitely generated torsion free nilpotent group over a field of characteristic zero.) For this purpose we use the following known lemma. I have been unable to find a proof of it in the literature.

LEMMA 4.1. *Let π be any finite group. Then the Jacobson radical of $\mathbb{Z}\pi$ is zero.*

Proof. For a given ring R , let its Jacobson radical be written as $J(R)$. Let p be a prime in $\mathbb{Z}$ and $\eta_p : \mathbb{Z}\pi \rightarrow \mathbb{Z}_p\pi$ be coefficient reduction. Since η_p is a ring epimorphism, $J(\mathbb{Z}\pi)$ is contained in the inverse image $\eta_p^{-1}(J(\mathbb{Z}_p\pi))$. Thus if $p \nmid |\pi|$ then, by Maschke's theorem, $J(\mathbb{Z}\pi) \subset \ker(\eta_p)$. Therefore, if $\lambda \in J(\mathbb{Z}\pi)$, $\lambda \in \ker(\eta_p)$ for infinitely many primes $p \nmid |\pi|$. This means that $\lambda = 0$ and hence $J(\mathbb{Z}\pi) = (0)$.

COROLLARY 4.2. *Let $\pi \neq \{e\}$ be a finite nilpotent group and I be the augmentation ideal in $\mathbb{Z}\pi$. Then I^{n+1} is a proper ideal of I^n for all $n \geq 1$.*

In the context of 4.2, the intersection $\bigcap I^n = (0)$ if and only if π is a p -group by a result of Gruenberg [3].

5. Except for the last step, the argument used for Theorem 3.1 would also apply to one-sided ideals. However, it turns out that the conclusion does not hold for the one-sided case.

EXAMPLE 1. Let $\pi = S_3$, the symmetric group on three letters, $\pi = \langle x, y : x^3 = 1, y^2 = 1, yxy = x^{-1} \rangle$. Let J be the left ideal of $\mathbb{Z}\pi$ generated by $2, y + 1, x^2 + x + 1$. Then $J^2 = J$, but $J \neq \mathbb{Z}\pi$.

EXAMPLE 2. Let $\pi = D_4$, the dihedral group of order 8,

$$\pi = \langle x, y : x^4 = 1, y^2 = 1, yxy = x^{-1} \rangle.$$

Let J be the left ideal of $\mathbb{Z}\pi$ generated by $3, y - 1, x^2 + 1$. Then $J^2 = J$, but $J \neq \mathbb{Z}\pi$.

It is known (and easy to show) that the trace ideal of any projective R -module is idempotent. On the other hand, in $\mathbb{Z}\pi$, π finite, it is not even known whether there is a nontrivial idempotent ideal that can be realized as the trace ideal of a projective $\mathbb{Z}\pi$ -module. Such a module could not be $\mathbb{Z}\pi$ -free and in view of Corollary 1.4, it could not be finitely generated. This cannot occur if π is finite solvable by another theorem of Swan [6, Theorem 7].

REFERENCES

1. H. BASS, Big projective modules are free, *Illinois J. Math.* **7** (1963), 24-31.
2. H. CARTAN AND S. EILENBERG, "Homological Algebra," Princeton University Press, 1956.
3. K. W. GRUENBERG, The residual nilpotence of certain presentations of finite groups, *Arch. Math.* **13** (1962), 408-417.
4. S. A. JENNINGS, The group ring of a class of infinite nilpotent groups, *Can. J. Math.* **7** (1955), 169-187.
5. R. G. SWAN, Induced representations and projective modules, *Ann. Math.* **71** (1960), 552-578.
6. R. G. SWAN, The Grothendieck group of a finite group, *Topology* **2** (1963), 85-110.
7. O. ZARISKI AND P. SAMUEL, "Commutative Algebra," Vol. I, Van Nostrand Company, Princeton, 1958.