

# UC San Diego

## UC San Diego Electronic Theses and Dissertations

### Title

Triggering Control Methods for Cyber-Physical Systems : : Security & Smart Grid Applications

### Permalink

<https://escholarship.org/uc/item/6qd9m1mc>

### Author

Foroush, Hamed Shisheh

### Publication Date

2014

Peer reviewed|Thesis/dissertation

UNIVERSITY OF CALIFORNIA, SAN DIEGO

**Triggering Control Methods for Cyber-Physical Systems:  
Security & Smart Grid Applications**

A dissertation submitted in partial satisfaction of the  
requirements for the degree  
Doctor of Philosophy

in

Engineering Sciences (Mechanical Engineering)

by

Hamed Shisheh Foroush

Committee in charge:

Professor Sonia Martínez, Chair  
Professor Jorge Cortés  
Professor Tara Javidi  
Professor Miroslav Krstic  
Professor Jiawang Nie

2014

Copyright  
Hamed Shisheh Foroush, 2014  
All rights reserved.

The dissertation of Hamed Shisheh Foroush is approved, and it is acceptable in quality and form for publication on microfilm and electronically:

---

---

---

---

---

---

Chair

University of California, San Diego

2014

## DEDICATION

This thesis is dedicated to my family and good friends without the help and support of whom the realization of this dissertation would never have been possible.

## EPIGRAPH

Life is a huge complex dynamic programming problem.

## TABLE OF CONTENTS

Signature Page . . . . .	. . . . .	iii
Dedication . . . . .	. . . . .	iv
Epigraph . . . . .	. . . . .	v
Table of Contents . . . . .	. . . . .	vi
List of Figures . . . . .	. . . . .	ix
List of Tables . . . . .	. . . . .	x
Acknowledgements . . . . .	. . . . .	xi
Vita . . . . .	. . . . .	xiv
Abstract of the Dissertation . . . . .	. . . . .	xv
Chapter 1	Introduction . . . . .	1
	1.1 Motivation . . . . .	1
	1.2 Literature Review . . . . .	1
	1.3 Summary of Results & Outline . . . . .	3
Chapter 2	On Event-triggered Control of Linear Systems under Peri- odic Denial-of-Service Jamming Attacks . . . . .	6
	2.1 Summary . . . . .	6
	2.2 Introduction . . . . .	6
	2.3 Problem Formulation . . . . .	9
	2.4 Attack-resilient Event-triggered Strategy . . . . .	11
	2.5 Analysis of the Proposed Triggering Law . . . . .	14
	2.6 Simulation . . . . .	21
	2.7 Conclusions & Future Work . . . . .	24
Chapter 3	On Triggering Control of Single-input Linear Systems under Pulse-Width Modulated DoS Signals . . . . .	26
	3.1 Summary . . . . .	26
	3.2 Introduction . . . . .	27
	3.3 Problem Formulation . . . . .	30
	3.4 Resilient Control & Triggering Strategy . . . . .	34
	3.5 Stability Analysis . . . . .	37
	3.6 Joint Triggering Control & Jammer Identification . . . . .	40

	3.6.1	The JAMCOID FOR PERIODIC SIGNALS Algorithm . . . . .	41
	3.6.2	The JAMCOID Algorithm . . . . .	45
3.7		Simulations . . . . .	49
	3.7.1	Known Jamming Scenario . . . . .	49
	3.7.2	Unknown Jamming Scenario . . . . .	52
3.8		Conclusions & Future Work . . . . .	52
3.9		Omitted Proofs . . . . .	54
	3.9.1	Proof of Lemma 4.5.1 . . . . .	54
	3.9.2	Proof of Proposition 4.5.2 . . . . .	54
	3.9.3	Proof of Theorem 4.6.1 . . . . .	57
	3.9.4	Proof of Corollary 3.5.4 . . . . .	65
	3.9.5	Proof of Inequality (4.13) . . . . .	66
	3.9.6	Proof of Theorem 3.6.1 . . . . .	66
	3.9.7	Proof of Theorem 4.7.1 . . . . .	67
Chapter 4		On Multi-Input Controllable Linear Systems Under Unknown Periodic DoS Jamming Attacks . . . . .	68
	4.1	Sumamry . . . . .	68
	4.2	Introduction . . . . .	68
	4.3	Problem Formulation . . . . .	70
	4.4	Background and Preliminary Results . . . . .	72
	4.5	Jordan Decomposition & Triggering Strategy . . . . .	77
	4.6	Stability Analysis of the Control & Triggering Strategy . . . . .	79
	4.7	Stabilization under unknown jamming signals . . . . .	80
	4.7.1	The JAMCOID Algorithm . . . . .	80
	4.7.2	The Stability of the JAMCOID Algorithm . . . . .	83
	4.8	Simulations . . . . .	84
	4.9	Conclusions & Future Work . . . . .	85
Chapter 5		On the Robustness of Event-Based Synchronization under Switching Interactions . . . . .	86
	5.1	Summary . . . . .	86
	5.2	Introduction . . . . .	87
	5.3	Preliminaries & Problem Formulation . . . . .	88
	5.4	Single Topology: Lyapunov Function & Triggering Strategy Characterization . . . . .	93
	5.5	Multiple Topologies: Arbitrary Switching . . . . .	101
	5.5.1	General Network Dynamics . . . . .	101
	5.5.2	Particular Network Dynamics . . . . .	104
	5.6	Multiple Topologies: Switching Signal Design . . . . .	108
	5.6.1	Regulatory Conditions on $\mathcal{S}_{\text{dwell}[\tau_D]}$ Class . . . . .	108



5.6.2	Regulatory Conditions on $\mathcal{S}_{\text{average}}[\tau_a, N_0]$ Class . . . . .	113
5.7	Particular Network Dynamics: Skew-Symmetric $H$ . . . . .	119
5.8	Simulations . . . . .	127
5.8.1	Simulations Basic . . . . .	127
5.8.2	Simulations for $\mathcal{S}_{\text{dwell}}[\tau_D]$ . . . . .	129
5.8.3	Simulations for $\mathcal{S}_{\text{average}}[\tau_a, N_0]$ . . . . .	132
5.9	Conclusions & Future Work . . . . .	133
Chapter 6	Conclusions & Future Work . . . . .	136
6.1	Conclusions . . . . .	136
6.2	Future Work . . . . .	138
	Bibliography . . . . .	140

## LIST OF FIGURES

Figure 2.1: Problem Architecture . . . . .	10
Figure 2.2: Scheme of Jammer Signal . . . . .	11
Figure 2.3: Temporal evolution of the states . . . . .	23
Figure 2.4: Temporal evolution of the triggering condition, zoomed over the first four periods . . . . .	23
Figure 3.1: Problem Architecture. Two scenarios have been considered: in Scenario 1 measurements are secure or indirectly available to the operator, and in Scenario 2 both the measurement and control channels can be compromised by a DoS signal. . . . .	31
Figure 3.2: Scheme of the Jamming DoS Signal . . . . .	33
Figure 3.3: Flowchart of JAMCOID FOR PERIODIC SIGNALS Algorithm .	43
Figure 3.4: Flowchart of JAMCOID Algorithm . . . . .	46
Figure 3.5: Third-order system, comparing 90% and 50% active jammers	51
Figure 3.6: Temporal results, demonstrating the stability despite DoS sig- nals . . . . .	51
Figure 3.7: Third-order system, evolution of $\tau_\lambda$ . . . . .	52
Figure 3.8: Temporal results, demonstrating the stability despite DoS sig- nals using JAMCOID FOR PERIODIC SIGNALS algorithm . . . . .	53
Figure 4.1: Problem Architecture . . . . .	71
Figure 4.2: Fourth-order multi-input system: evolution of $C(\lambda)$ . . . . .	85
Figure 5.1: Temporal results for uniform switching . . . . .	131
Figure 5.2: Temporal results for average switching . . . . .	134

## LIST OF TABLES

Table 2.1: Guaranteed maximum jammer activity . . . . .	24
Table 2.2: Guaranteed maximum triggering flexibility . . . . .	24

## ACKNOWLEDGEMENTS

At the time that I am writing this acknowledgement to the last dissertation of my academic life, there are many good and bad memories which come and pass in my brain. Indeed, over the past (almost nonstop) twenty-four years of academic life, there have been good and bad events, as well as, good and bad people, which and whom have made me who I am as an individual. While it is difficult (if not impossible) to forget the bad events and forgive the bad people of this story, in what follows I just recall and thus acknowledge, the good portion.

First and foremost, I would like to appreciate very much and proclaim my sincere gratitude to my entire family—from both maternal and paternal sides. In particular, I would like to thank my parents, grandmothers, brother, sister-in-law, and niece, for their (mostly) remote help and support especially over the past four years. Undeniably, without the close, encouraging, and extremely caring guidance of my mom I would never be able to continue until this point. All the knowledge and education that I have gathered thus far is all dedicated to her.

The years of undergraduate in the prestigious Sharif University of Technology, Tehran, Iran, gave me the chance of challenging myself. Indeed, the high-demanding serious education system back in Sharif prepared me to be more disciplined, more organized, stronger and more resilient. In this way, I would like to recall and thank very much distinguished Profs. Bahadori-Nejad, Farrahi, and Vossoughi whose professionalism as well as personality have been always source of inspiration to me. In effect, my main venue in academic life has been chosen thanks to the second-year undergraduate course on linear control taken with Prof. Vossoughi which determined control theory and dynamical systems as my main field of study until today.

After undergraduate, the years of masters at École Polytechnique Fédérale de Lausanne (EPFL), Lausanne, Switzerland, have been perhaps amongst the best years of my life, thus far. Indeed, apart from getting the chance of getting to know many good friends, I had the chance of obtaining unique research

experiences in the world-ranked research centers. In this way, I would like to sincerely thank Profs. Longchamp, Bonvin, and Karimi, who are the head of the Laboratoire d'Automatique (Automatic Control Laboratory) of EPFL who gave me the chance of pursuing two research internship periods at ABB Corporate Research Center and CERN which both have certainly played an important role in building me as who I am as a researcher—I thus highly appreciate the professional trust of these people on me.

The years of doctorate at University of California at San Diego (UCSD) follow the masters years at EPFL. Provided my mostly applied research background back in Switzerland, these years of heavy theoretical studies and research at UCSD have been an excellent complementary part for me. I am, therefore, indebted to the faculty of controls and dynamical systems group of UCSD who made it feasible for me. In particular, I am truly grateful to Prof. Krstic who gave me the chance and opportunity of joining UCSD and who has been always a source of help and support for me during my doctorate studies; I am also grateful to Prof. McEneaney whose mathematical knowledge has been a source of inspiration to me, indeed, it is hard (if not impossible) for me to count the number of hours we discussed math over the past four years; I am also thankful to Prof. de Oliveira whose trust on me as being his teaching assistant for the past six month is highly appreciated. In addition, I would also like to thank the members of my doctoral committee, Profs. Krstic, Cortés, Javidi, and Nie for the time of reviewing this thesis and attending my senate and final defense exams. Last but not the least, special thanks and gratitude go to Ms. Robenko, McKamey, and Carrasco who helped me with miscellaneous administrative issues over the past four years; for sure, beautiful people, e.g., David, Daniele, Pouriya, Ali, and Alireza, are also very much appreciated whose help and support have been a nonstop engine for me to resist these years. I am indebted to all these aforementioned people and I hope a day comes that I can return their favor.

Chapter 2, in part, is a reprint of the material [30] as it appears in 'On event-triggered control of linear systems under periodic Denial of Service at-

tacks' by H. S. Foroush and S. Martínez in proceedings of IEEE Int. Conference on Decision and Control, 2012. The dissertation author was the primary investigator and author of this paper.

Chapter 3, in part, is a reprint of the material [32] as it appears in 'On triggering control of single-input linear systems under pulse-width modulated DoS jamming signals' by H. S. Foroush and S. Martínez in SIAM Journal on Control and Optimization, under review in 2014. The dissertation author was the primary investigator and author of this paper.

Chapter 4, in part, is a reprint of the material [31] as it appears in 'On multi-input controllable linear systems under unknown periodic DoS attacks' by H. S. Foroush and S. Martínez in proceedings of SIAM Conference on Control and its Applications, 2013. The dissertation author was the primary investigator and author of this paper and the submitted material.

Chapter 5, in part, is a reprint of the material [29] as it appears in 'On the Robustness of Event-Based Synchronization under Switching Interactions' by H. S. Foroush and S. Martínez submitted to proceedings of IEEE Int. Conference on Decision and Control, 2014. The dissertation author was the primary investigator and author of this paper and the submitted material.

## VITA

2007	Bachelor of Science in Mechanical Engineering, Sharif University of Technology, Tehran, Iran
2010	Master of Science in Mechanical Engineering, École Polytechnique Fédérale de Lausanne, Lausanne, Switzerland
2010	Master of Science in Biomedical Engineering, École Polytechnique Fédérale de Lausanne, Lausanne, Switzerland
2014	Doctor of Philosophy in Engineering Science (Mechanical Engineering), University of California, San Diego

## PUBLICATIONS

### **Book chapter:**

H. S. Foroush and S. Martínez, "On Triggering Control Techniques for Cyber-Physical Systems", *Principles of Cyber-Physical Systems*, Cambridge University Press, under review.

### **Journal publications:**

H. S. Foroush and S. Martínez, "On event-triggered control of linear systems under periodic Denial-of-Service jamming attacks", *Automatica*, in preparation.

H. S. Foroush and S. Martínez, "On triggering control of single-input linear systems under pulse-width modulated DoS jamming attacks", *SIAM Journal on Control and Optimization*, under review, 2014.

H. S. Foroush and S. Martínez, "On the Robustness of Event-Based Synchronization under Switching Interactions", *IEEE Transactions on Automatic Control*, in preparation.

### **Peer-reviewed conference proceedings:**

H. S. Foroush and S. Martínez, "On event-triggered control of linear systems under periodic Denial of Service attacks," *IEEE Conference on Decision and Control*, Maui, Hawaii, USA, 2012, pp. 2551–2556.

H. S. Foroush and S. Martínez, "On multi-input controllable linear systems under unknown periodic DoS jamming attacks," *SIAM Conference on Control and its Applications*, San Diego, California, USA 2013, pp. 222–229.

H. S. Foroush and S. Martínez, "On the Robustness of Event-Based Synchronization under Switching Interactions", *IEEE Conference on Decision and Control*, Los Angeles, California, USA, 2014, submitted.

ABSTRACT OF THE DISSERTATION

**Triggering Control Methods for Cyber-Physical Systems:  
Security & Smart Grid Applications**

by

Hamed Shisheh Foroush

Doctor of Philosophy in Engineering Sciences (Mechanical Engineering)

University of California, San Diego, 2014

Professor Sonia Martínez, Chair

This thesis contains work on control and monitoring of Cyber-Physical Systems (CPS) using triggering control techniques. Cyber-Physical Systems are remotely controlled and monitored physical systems which pervade our society today in the form of numerous important applications. However, their deployment poses numerous challenges due to their limited computing, communication, and control capabilities and/or environmental constraints. In the controls community, this latter fact has motivated a paradigm shift to a so-called self/event-triggered approach by means of which algorithms employ scarce resources for control only when needed. In this dissertation, we have studied two principal problems considering, respectively, a security and smart grid CPS ap-



plications where we develop novel triggering control techniques to solve these problems. In brief, these problems can be stated as (i) motivated by importance of ensuring security of CPS, to develop failure resilient triggering-based control methods, and (ii) motivated by emergence of smart grid application, to study robustness of event-based synchronization dynamics under switching topologies. In the following lines of this abstract, we shall provide a brief closer look on these problems and our developed solutions to them.

In the first problem, we study the stability of remotely controlled and monitored single-input and multi-input controllable linear class of systems under power-constrained Pulse-Width Modulated (PWM) Denial-of-Service (DoS) signals. The effect of a DoS jamming signal is to corrupt the control and measurement channels, thus preventing the data to be received at its destination. Therefore, a power-constrained DoS signal is modeled as a series of on and off time-intervals, which restricts communications intermittently. In this work, we first assume that the DoS signal is *partially known*, i.e., a uniform lower-bound for the off time-intervals and the on-to-off transiting time-instants are known. Accordingly, we propose our resilient control and triggering strategies which are provably capable of beating partially known jamming signals of this class. Building on this, we then present our joint control and identification algorithms, JAMCOID FOR PERIODIC SIGNALS and JAMCOID, which are provably able to guarantee the system stability under *unknown* jamming signals. More precisely, JAMCOID FOR PERIODIC SIGNALS algorithm is able to partly identify a periodic DoS signal with known uniform lower bound for the off time-intervals, whereas JAMCOID algorithm is capable of dealing with power-constrained, but otherwise unknown, DoS signals whilst ensuring stability. The practicality of the proposed techniques is evaluated on a simulation example under *partially known* and *unknown* jamming scenarios.

In the second problem, we study the robustness of an event-triggered synchronization dynamics for a network of identical nodes under various switching scenarios. We first consider an arbitrary switching scenario where, for a general class of isolated node dynamics we characterize sufficient conditions

in terms of network topologies to maintain synchronization. In particular, we shall also demonstrate that for a specific class of skew-symmetric isolated node dynamics—which play important role in this class of synchronization problems—the asymptotic synchronization is not achievable under arbitrary switching. We then considered two classes of constrained switching signals, namely uniform and average classes, i.e.,  $\mathcal{S}_{\text{dwell}}[\tau_D]$ , and  $\mathcal{S}_{\text{average}}[\tau_a, N_0]$ , respectively, where we characterize sufficient conditions in terms of the associated parameters,  $\tau_D$ ,  $\tau_a$  and  $N_0$  in order to ensure asymptotic synchronization. We shall wrap up our discussion by presenting relevant simulation studies.

# Chapter 1

## Introduction

### 1.1 Motivation

*Cyber-Physical Systems (CPS)* are physical plants which are remotely controlled and monitored via communication networks, and integrate tightly both computational and physical components. Thanks to growing developments in the area of sensing and communication technologies, these systems pervade very diverse areas ranging from aerospace, to energy, to civil infrastructure facilities, manufacturing and transportation networks. Motivated by the use of economic communications and computations in these systems, the adaptation of these processes to external events, as well as by the asynchronous nature of multiple components, the new control paradigm of *triggering control* has emerged. This chapter presents a summary of the main ideas behind the triggering control approach, and then discusses its use on two applications of interest that are the focal points of this dissertation.

### 1.2 Literature Review

As discussed in conventional textbooks [10, 34], one can distinguish between continuous-time and discrete-time control design. In the former, the control is getting updated in a continuous fashion; in the latter, the control is up-

dated at every *prescribed* instants of time which are multiple of a so-called *sampling period*. The discrete-time control methods discussed in these afore cited textbooks do not entail a rigorous analytical study addressing, e.g., how to derive this sampling period, in order to ensure some stability and/or performance properties. This was the initial point of interest in developing rigorous triggering control methods. Some rather old papers, such as [39, 49, 83, 34, 62], also compare and discuss discrete-time periodic control with aperiodic control implementation, these papers along with [9, 8], wherein the advantages of aperiodic implementation are highlighted, can be regarded as the source of *triggering control methods*. One can classify triggering control methods into *time-triggered*, *event-triggered*, and *self-triggered* control approaches.

The time-triggered control, also known as sampled-data control, methods are, in terms of implementation, similar to the aforementioned discrete-time periodic control with the main difference that the sampling period is now derived from a rigorous stability study of the considered class of systems. In this way, representative studies to be mentioned are [21, 64] where, respectively, input-output stability and stability properties of sampled-data systems have been studied, both papers characterize largest sampling period (that they call Maximum Allowable Time-Interval (MATI)) required to ensure some specific notion of input-output stability; the other representative studies include [65, 88] where in both sufficient conditions to ensure stability of a nonlinear sampled-data system have been derived by discrete approximation of the original system.

The event-triggered control methods are based on the continuous measurement of the states and/or outputs of the system along with checking the violation of a triggering condition according to which the next time-instant to update the control is derived, in more specific words, representative studies [78, 55, 72] have studied event-triggered control with more emphasis on state-based notion of stability, e.g., input-to-state stability, papers [40, 79, 46] have also studied event-triggered control, nonetheless, with more emphasis on output based notion of stability, e.g., input-output stability.

The self-triggered control methods were developed to overcome the pos-

sible limitation of event-triggered control in that they do not require continuous measurement of system states/outputs while this comes at the cost of more conservatism at the level of larger sampling time-intervals. On this topic, one may recall [84] as the very first study on the topic of self-triggered control which was then followed by the study [86] in which the stabilizing self-triggered rule is derived based on an input-output stability condition. Besides, papers [6, 58, 60, 5] have also studied self-triggered control based on some state-based stability conditions, e.g., input-to-state stability property.

Besides the principal papers on triggering control methods mentioned in the previous paragraphs, there have been certain studies dedicated to extending those results to other contexts, and dedicated to considering more practical scenarios. From consensus algorithms [24, 75], to deployment [66, 22], to synchronization and distributed optimization [50, 51, 45], the new type of algorithms have proven convergence guarantees and, in most cases, present good non-Zeno behavior. In addition, decentralized event-triggered rules have been developed in [61, 59, 70] based on state-based and small-gain approaches to stability. Besides, in [80] the event-triggered control for tracking scenario for a specific class of nonlinear systems is proposed. Furthermore, [37, 85, 42, 36] have studied event-triggered control with practical considerations such as in presence of data packet dropout, delay, quantization, and potential parameter mismatch.

### 1.3 Summary of Results & Outline

We summarize the results presented throughout this dissertation in very brief words in the following paragraphs. It is then followed by presenting the organization of this dissertation, along with a brief summary of each chapter's contents.

The particular problems chosen in this dissertation to illustrate the use of triggered control are motivated by CPS problems of interest. On the one hand, the secure and resilient control of CPS systems is an area of high concern, see for

instance [20] and references therein. A particular threat to the secure operation of cyber-physical systems arises from vulnerable communication links, which can be disrupted by means of viruses, or external communication-signal interferences. Motivated by this, we summarize an approach to remotely control a linear cyber-physical system subject to a general type of Pulse-Width Modulated (PWM) Denial of Service (DoS) signals by means of adapted triggering control algorithms. A complete version of this work may be found [32], with conference versions in [30, 31].

The synchronization of dynamical systems interacting over a network can model several CPS applications of interest. For example, a smart grid problem entails the coordination of number of power generators to produce and supply electric energy to a network of consumers [43, 7]. The aforementioned synchronization dynamics has been studied for identical nodes or different node (oscillator) dynamics. The major review on synchronization [7] discusses the differences and resemblances between these two problems. In more detailed words, the papers [76, 89, 52] study this problem in the context of switched systems, where switching amongst different potential network topologies has been considered and, thus, some switching rules have been derived in order to achieve network synchronization. In these latter studies, it is nonetheless worth noting that the communication is performed in a continuous fashion. Hence, there is an apparent gap in the literature in terms of studying synchronization dynamics by considering event-triggered communications and in the context of switched systems. This work aims to close this gap by characterizing sufficient conditions on switched networked topologies and robustness conditions on switching signals that can ensure network synchronization.

**Chapter 2:** as the initial step to solve the main problem of failure resilient control of a CPS despite presence of power-constrained DoS jamming signals on the communication channels, this chapter studies resilient event-triggered control strategies in presence of partially known periodic DoS signals. The developed control strategy in this chapter is not parameter-dependent and the main contribution is proposing a sufficient condition

in terms of jamming signal parameters and given stabilizing controller.

**Chapter 3:** in sequel of the previous chapter, this chapter studies triggering control strategies the problem of failure resilient control of a CPS despite presence of power-constrained DoS jamming signals on the communication channels by considering only single-input class of systems in presence of partially known and unknown jamming signals. The proposed control strategy in this chapter is parameter-dependent and thus the main contributions of this chapter are, therefore, (i) discussing and characterizing a parameter-dependent triggering control strategy capable of dealing with partially known class of DoS signals, (ii) discussing and characterizing JAMCOID FOR PERIODIC SIGNALS algorithm capable of dealing with unknown periodic class of DoS signals, and (iii) discussing and characterizing JAMCOID algorithm capable of dealing with unknown class of DoS signals.

**Chapter 4:** as a complementary part to what is introduced and discussed in Chapter 3, this chapter extends those results to the case of multi-input class of systems while preserving the classes of power-constrained DoS signals.

**Chapter 5:** this chapter studies the robustness of event-based synchronization problem under switching interactions, more specifically, the discussion begins with discussing the case of arbitrary switching which is then geared towards the case of constrained switching. In the former, sufficient conditions in terms of network topologies are derived, in the latter, sufficient conditions on the considered class of switching signals are developed, where in both cases the main goal is to ensure the asymptotic synchronization.

**Chapter 6:** this chapter contains the closing remarks on the entire set of results presented in this dissertation which follow with some venues to explore for future work.

## Chapter 2

# On Event-triggered Control of Linear Systems under Periodic Denial-of-Service Jamming Attacks

### 2.1 Summary

In this chapter we study the resilience of a continuous LTI system which is controlled remotely via a wireless channel. An power-constrained periodic (partially known) jammer is corrupting the control communication channel by imposing Denial-of-Service (DoS) attacks. We derive a triggering time-sequence, addressing when to update the control signal under the assumption that the period of the jammer has been detected. Then, we show that, under some sufficient condition, this triggering time-sequence counteracts the effect of the jammer and assures asymptotic stability of the plant. We prove our results theoretically, and demonstrate their validity in a simulation example.

### 2.2 Introduction

Novel advances in communications and sensing technologies are allowing the remote control and monitoring of a variety of physical plants, span-



ning from Unmanned Aerial Vehicles (UAVs) to power reactors. These types of systems integrating computation, communication, and physical processes are called *cyber-physical systems*. While their emergence has come along with many advantages, there are some associated challenges, as well. One of them has to do with *system security*, as vulnerability comes at the price of ease of deployment and hard-to-supervise multiple system components; see [20] and [1].

At the communication level, vulnerabilities can be produced by external communication-signal jammers or attackers. One can distinguish between two types of attacks, namely *Denial-of-Service (DoS)* and *Deceptive* attacks. In the former, the jammer tries to drop the transmitted data, whereas, in the latter, the jammer aims to change the transmitted data, see [87] and [71] for more information. According to [18] and [3], DoS is the most likely type of attack to control systems. Amongst DoS jammers, a simple class is that of *periodic* or *Pulse-Width Modulated (PWM)* jammers. From the point of view of the jammer, periodic signals are motivated by energy constraints, and ease of implementation. It represents a main type of jamming signals studied in the communications literature [23, 11, 82, 33]. Motivated by this, we focus on DoS attacks imposed by PWM jammers whose periodic behavior has already been detected. In particular, we propose an event-triggering control sequence that is compatible with the jammer and study under which conditions the strategy guarantees asymptotic stability.

The topic of security in cyber-physical systems is receiving wide attention from the controls community and has been studied from different viewpoints in the last years. In the framework of multi-agent systems, we refer the reader to [77, 67, 68]. In these papers, the main problem is the identification of the malicious agent, who is part of the network, and the cancellation of its contribution. In [14] and [13], identification is not the main issue, and the specific objective is how to maintain connectivity of the network, despite the presence of the malicious agents. In [91], the authors develop an attack-resilient method subject to deceptive attacks. Our problem formulation is related to the cited previous work in the sense that we assume the jammer has been detected and we

propose an algorithm that aims to counteract its effect.

Other references in the context of secure discrete LTI systems are [26], [3]. In [26], the authors consider deceptive attacks where deception occurs in the observation channel. In [3], the attack is DoS, the problem is formulated in a stochastic setup, and moreover, the attacker obeys an Identically Independent Distributed (IID) assumption, similarly to [74].

The references [2], [38], [81], and [73] model the security problem as a (dynamic) zero-sum non-cooperative game, so they can predict the behavior of the attacker. The authors in [2], [81], and [73] study the vulnerability of the network towards deceptive attacks which differs from our problem. The closest reference to our work is [38], which studies a similar problem in a game-theoretic framework. However, a limitation of [38] is the restriction to scalar dynamics, while the information structure assumed for the jammer is quite rich, which might not be realistic.

Another important topic when it comes to cyber-physical systems, is that of achieving desired control goals with economic communications. This has motivated the topic of *triggering control*, i.e., control actions triggered only when it is necessary. One can distinguish between related *self-triggered control* and *event-triggered control*; see [78], [58] and [86], which study LTI systems. The technique used in [86] is based on Input-Output stability analysis, whereas, the technique used in [78] and [58] is based on Input-to-State Stability (ISS) Lyapunov concept, which also inspires this work. However, a main distinguishing feature is the fact that communications may not be always feasible in our formulation. At last, we would also like to mention [47], which studies a type of system resilience against transient faults using an event-triggering method close to [86].

In this chapter, we address the problem of system resilience in the context of event-triggering control. The types of attacks considered are DoS attacks which we assume have been partially identified. Other than this, we consider a generic class of continuous LTI systems and a generic class of PWM jammers. In particular, we propose a novel triggering time-sequence to counteract jammer effects and derive a sufficient condition under which the asymptotic sta-

bility is ensured, i.e., the system is *safe and secure*. A preliminary version of this work, which omitted all the proofs of the results and some simulations appeared in [30].

The rest of the chapter is organized as follows. Section 2.3 includes the problem formulation and notations. In Section 2.4, we propose a novel attack-resilient event-triggering law consistent with the jammer signal, and in Section 2.5, we analyze and prove the validity of this law. We then demonstrate the functionality of our theoretical results in a specific simulation in Section 2.6. We then conclude in Section 2.7 summarizing the results and future work.

## 2.3 Problem Formulation

In this section, we state, both formally and informally, the main problem analyzed in the chapter.

We consider a remote operator-plant setup, where the operator uses a control channel to send wirelessly a control command to an unstable plant, see Figure 4.1. We assume that the plant has no specific intelligence and is only capable of updating the control based on the data it receives. We also assume that the operator knows the plant dynamics and is able to measure its states continuously.<sup>1</sup>

In this chapter, we assume that the type of jammer and the period of the jamming signal has been identified. Future work will be devoted to enlarge the triggering time-sequence for identification purposes.

Let  $x \in \mathbb{R}^n$  be the state vector and  $u \in \mathbb{R}^m$  be the input vector. We consider the following dynamics:

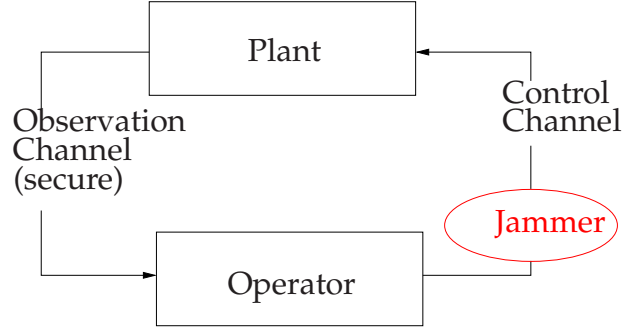
$$\dot{x} = Ax + Bu(t), \quad (2.1a)$$

$$u(t) = Kx(t_k), \quad \forall t \in [t_k, t_{k+1}[, \quad (2.1b)$$

where  $A$ ,  $B$  and  $K$  are matrices of proper dimensions, and  $\{t_k\}_{k \geq 1}$  is the triggering time-sequence to be defined later. We denote  $e(t) = x(t_k) - x(t)$ ,  $\forall t \in$

---

<sup>1</sup>This information can be obtained by using either local “passive” sensors, e.g., a camera network, or positioning systems, e.g., GPS, where cheap and safe communication are possible.



**Figure 2.1:** Problem Architecture

$[t_k, t_{k+1}[$ .

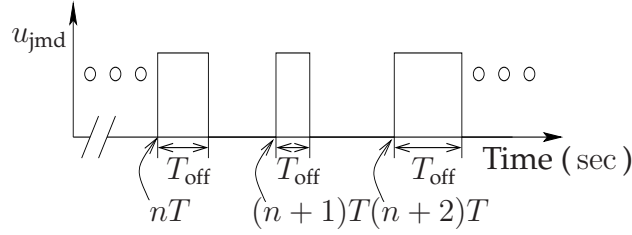
We consider an *power-constrained, periodic* jammer whose signal can be represented as follows:

$$u_{\text{jmd}}(t) = \begin{cases} 1, & (n-1)T \leq t \leq (n-1)T + T_{\text{off}}, \\ 0, & (n-1)T + T_{\text{off}} < t < nT, \end{cases} \quad (2.2)$$

where  $n \in \mathbb{N}$  is the period number,  $T \in \mathbb{R}_{>0}$ , and  $\mathcal{T} = [0, T]$  is the action-period of the jammer. Also,  $T_{\text{off}} \in \mathbb{R}_{>0}$ ,  $T_{\text{off}} < T$ , and  $\mathcal{T}_{\text{off}} = [0, T_{\text{off}}]$  is the time-period where it is sleeping, so communication is possible. We further denote  $T_{\text{on}} \in \mathbb{R}_{>0}$ , and  $\mathcal{T}_{\text{on}} = [T_{\text{on}}, T]$  to be the time-period where the jammer is active, thus no data can be sent. Accordingly, it holds that  $T_{\text{off}} + T_{\text{on}} = T$ . We also note that the parameter  $T_{\text{off}}$  need not be time-invariant which recalls Pulse-Width Modulated (PWM) jamming. Finally, we denote by  $T_{\text{off}}^{\text{cr}}$  a uniform lower-bound for  $T_{\text{off}}$ , i.e.,  $T_{\text{off}}^{\text{cr}} \leq T_{\text{off}}$  which we assume holds for all the periods and we have identified as well. A schematic plot of the jammer signal  $u_{\text{jmd}}(t)$  is shown in Figure 2.2 for some example parameter values.

We can now formulate our main objective:

*[Problem formulation]:* Knowing  $T$  and  $T_{\text{off}}^{\text{cr}} \leq T_{\text{off}}$ , uniformly for all the periods, determine an event-triggering strategy for the operator that is sufficient for system stabilization despite the presence of the jammer.



**Figure 2.2:** Scheme of Jammer Signal

## 2.4 Attack-resilient Event-triggered Strategy

In this section, we introduce an event-triggered strategy which is resilient towards the jamming attack. To do so, we make use of the ISS approach of [78] and [58].

Here, we assume that: (i) the system (4.1a) is open-loop unstable, and (ii) the pair  $(A, B)$  is controllable. The latter guarantees that there exists matrix  $K$  such that  $A + BK$  is Hurwitz. This implies that for every matrix  $Q = Q^T \succ 0$ , there exists a unique matrix  $P = P^T \succ 0$  such that the Lyapunov equation:

$$(A + BK)^T P + P(A + BK) = -Q, \quad (2.3)$$

holds [44]. Given  $Q$ , we consider the Lyapunov function  $V(x) = x^T P x$ . Note that  $V(t) = V(x(t)) = x(t)^T P x(t)$ , so interchangeably, we shall use  $V(x)$  or  $V(t)$ . Since  $Q$  and  $P$  are symmetric, positive-definite matrices, then by applying the Cholesky decomposition, we can express them as  $Q = L^T L$  and  $P = U^T U$ , for some  $L, U \in \mathbb{R}^{n \times n}$ . We also denote by  $\|\cdot\|$  and  $|\cdot|$ , the Euclidean matrix and vector norms, respectively.

We introduce our ISS-Lyapunov function next.

**Proposition 2.4.1.** *Consider the system (4.1), where (2.3) holds. Let  $V(x) = x^T P x$  be the Lyapunov function. If  $\|Q\| > 1$ , then the following holds:*

$$\theta_1 |x|^2 \leq V(x) \leq \theta_2 |x|^2, \quad (2.4)$$

$$\dot{V}(x) \leq -(\|Q\| - 1)|x|^2 + \|PBK\|^2 |e|^2, \quad (2.5)$$

where  $\theta_1, \theta_2 \in \mathbb{R}_{>0}$ . In other words,  $V$  is an ISS-Lyapunov function for (4.1).

*Proof.* We can lower- and upper-bound  $V(x)$  as follows:

$$\lambda_{\min}(P) |x|^2 \leq V(x) \leq \lambda_{\max}(P) |x|^2 ,$$

where  $\lambda_{\min}$  and  $\lambda_{\max}$ , are the minimum and maximum eigenvalues of  $P$ , respectively. Since  $P = P^T \succ 0$ , then (2.4) holds with  $\theta_1 \triangleq \lambda_{\min} > 0$  and  $\theta_2 \triangleq \lambda_{\max} > 0$ .

Let  $\bar{A} = A + BK$  and  $\bar{B} = BK$ . Computing the time derivative of  $V$  and plugging in (4.1a), we obtain:

$$\dot{V}(x) = x^T (\bar{A}^T P + P \bar{A}) x + e^T \bar{B}^T P x + x^T P \bar{B} e .$$

Recalling the following inequality:

$$x^T P \bar{B} e + e^T \bar{B}^T P x \leq x^T x + e^T \bar{B}^T P P \bar{B} e ,$$

we can upper-bound  $\dot{V}$  as:

$$\dot{V}(x) \leq x^T (\bar{A}^T P + P \bar{A}) x + x^T x + e^T \bar{B}^T P P \bar{B} e . \quad (2.6)$$

Using (2.3) and the Cholesky decomposition for  $Q$ , i.e.,  $Q = L^T L$ , we obtain:

$$\dot{V}(x) \leq -(Lx)^T (Lx) + (Ix)^T (Ix) + (P \bar{B} e)^T (P \bar{B} e) .$$

Recalling that  $\|L\|^2 = \|Q\| > 1$ , the latter inequality yields (2.5).  $\square$

Similarly to [78], one can use the ISS-Lyapunov function of Proposition 2.4.1, together with a design parameter  $\sigma \in (0, 1)$ , to determine a stabilizing event-triggering law when the jammer is absent:

**Proposition 2.4.2.** *Consider the system (4.1a), along with the Lyapunov function  $V(x) = x^T P x$  associated with  $\|Q\| > 1$ . If the control (4.1b) is updated at times  $t_k$  governed by the following triggering law:*

$$|e(t_k)|^2 = \sigma \frac{\|Q\| - 1}{\|PBK\|^2} |x(t_k)|^2 , \quad k \geq 1 , \quad (2.7)$$

*then the system is asymptotically stable.*

*Proof.* In order to ensure asymptotic stability, it is sufficient to impose the following constraint on (2.5):

$$\dot{V}(x) \leq -(\|Q\| - 1) |x|^2 + \|PBK\|^2 |e|^2 < 0,$$

which implies:

$$\|PBK\|^2 |e|^2 < (\|Q\| - 1) |x|^2. \quad (2.8)$$

Note that in (2.8), and without loss of generality, we can introduce design parameter  $\sigma \in (0, 1)$ :

$$\|PBK\|^2 |e|^2 \leq \sigma (\|Q\| - 1) |x|^2 < (\|Q\| - 1) |x|^2, \quad (2.9)$$

which still renders the system asymptotically stable. Let  $t_1$  be the first time that (2.9) is violated. Hence, we obtain the following:

$$|e(t_1)|^2 = \sigma \frac{\|Q\| - 1}{\|PBK\|^2} |x(t_1)|^2.$$

By updating the control at  $t_1$ , we get  $e(t_1) = x(t_1) - x(t_1) = 0$  and  $\dot{V}(t_1) < 0$ . Moreover, for  $t > t_1$ , the error  $e(t)$  evolves with time and increasing from  $t_1$ . As long as (2.9) is not violated, i.e., (2.7) does not hold, we have  $\dot{V}(t) < 0$ , by construction. Now, let  $t_2$  be the next time when (2.7) holds. Note that, again,  $e(t_2) = x(t_2) - x(t_2) = 0$  and  $\dot{V}(t_2) < 0$ . Therefore, it follows by induction that by the definition of the triggering sequence according to (2.7),  $\dot{V}(t) < 0, \forall t$ . A standard Lyapunov argument guarantees the result follows.  $\square$

In what follows, we shall study the asymptotic stability of the system despite the jammer presence under a simple modification of the above triggering law. To do this, we assume a “worst-case jamming scenario”, i.e.,  $T_{\text{off}} = T_{\text{off}}^{\text{cr}}$ .

**Definition 1.** We define the triggering time-sequence despite the jammer presence as follows:

$$t_{k,n}^* = \{t_l \text{ satisfying (2.7)} \mid t_l \in [(n-1)T, (n-1)T + T_{\text{off}}^{\text{cr}}] \cup \{nT\}, \quad (2.10)$$

$\forall k \in \mathbb{N}, \forall n \in \mathbb{N}$ . In (2.10),  $k$  denotes the number of triggering times occurring in  $n^{\text{th}}$  jammer action-period.

In order to interpret the triggering law (2.10), let us consider the  $n^{\text{th}}$  action-period, i.e.,  $t \in [(n-1)T, nT]$ . The sequence selects the time-instants given by (2.7) which also lie in the  $[(n-1)T, (n-1)T + T_{\text{off}}^{\text{cr}}]$  time-period along with  $nT$ . In this way, if it ever happens that:

$$t_{k,n}^* = \{t_l \text{ satisfying (2.7) } \mid t_l \in [(n-1)T, (n-1)T + T_{\text{off}}^{\text{cr}}]\} = \emptyset,$$

then the only triggering instant would be  $nT$ .

**Remark 2.4.3.** *In the triggering law (2.10), we have:*

$$\exists \tau > 0, \text{ such that } t_{k+1,n}^* - t_{k,n}^* \geq \tau, \forall k \in \mathbb{N}.$$

*This is based on Theorem III.1 in [78]. In other words, the time-sequence generated by the triggering law (2.10) does not accumulate.*

## 2.5 Analysis of the Proposed Triggering Law

Having introduced the triggering law (2.10), we present our main result in this section which studies the asymptotic stability of the system under attack.

In [53], the author proves the following bound for a matrix  $M \in \mathbb{R}^{n \times n}$ :

$$\|\exp(M)\| \leq \exp(\mu(M)), \quad (2.11)$$

where the  $\mu$ -operator is defined as follows:

$$\mu(M) = \max \left\{ \mu \mid \mu \in \lambda \left( \frac{M + M^T}{2} \right) \right\},$$

with  $\lambda((M + M^T)/2)$  be the spectrum of the matrix  $(M + M^T)/2$ . We shall exploit this bound in the proof of our main result.

**Theorem 2.5.1.** *Consider the system (4.1), along with the triggering law (2.10). The system is asymptotically stable if the following conditions are satisfied:*

$$\frac{(1 - \sigma)T_{\text{off}}^{\text{cr}}(\|Q\| - 1)}{2} > \|P\| \ln(\alpha), \quad (2.12)$$



where,

$$\begin{aligned} \alpha \triangleq & \exp((T - T_{\text{off}}^{\text{cr}})\mu(A + BK)) + \frac{\|BK\|}{\mu(A + BK)} \times \\ & \left( \frac{\|BK\|}{\|A\|} + 1 \right) (1 - \exp((T - T_{\text{off}}^{\text{cr}})\|A\|)) \times \\ & (1 - \exp((T - T_{\text{off}}^{\text{cr}})\mu(A + BK))), \end{aligned} \quad (2.13)$$

and,

$$\mu(A + BK) < 0. \quad (2.14)$$

*Proof.* We shall focus on the first jammer action-period, i.e.,  $0 \leq t \leq T$ . We then show that under the proposed sufficient condition, it holds that  $V(T) < \Upsilon V(0)$ , for some  $\Upsilon \in (0, 1)$ , which can be inductively extended to show  $V((n+1)T) < \Upsilon V(nT)$ ,  $\forall n \in \mathbb{N}$ . From here we then demonstrate that asymptotic stability can be guaranteed. For the sake of brevity, we drop  $n = 1$  in the  $t_{k,n}^*$  annotation. Without loss of generality, let  $\{t_1^* = 0, t_2^*, t_3^*, \dots, t_m^*\}$  be the time-sequence generated by the triggering law (2.10), where it holds that  $t_m^* \leq T_{\text{off}}^{\text{cr}}$  and  $t_{m+1}^* > T_{\text{off}}^{\text{cr}}$ . We note that there must exist such an  $m > 0$ , since according to Remark 2.4.3, this time-sequence does not accumulate.

We consider the evolution of the Lyapunov function in the time-interval  $[t_i^*, t_{i+1}^*]$ , where  $0 \leq t_i^*, t_{i+1}^* \leq t_m^*$ . According to (2.10), in this interval, Equation (2.7) is not yet violated, hence the following holds:

$$|e(t)|^2 < \sigma \frac{\|Q\| - 1}{\|PBK\|^2} |x(t)|^2.$$

Upper-bounding (2.5), by using the latter Equation, yields:

$$\dot{V}(t) < -(1 - \sigma) (\|Q\| - 1) |x(t)|^2. \quad (2.15)$$

Now, we note that:

$$V = x^T P x \Rightarrow V \leq \|U\|^2 |x|^2 \Rightarrow -|x(t)|^2 \leq -\frac{V(t)}{\|U\|^2},$$

with which we can further upper-bound Equation (2.15) as follows:

$$\dot{V}(t) < -\frac{(1 - \sigma) (\|Q\| - 1)}{\|U\|^2} V(t). \quad (2.16)$$

By applying the comparison principle on (2.16), we get:

$$V(t) < V(t_i^*) \exp \left( -\frac{(1-\sigma)(\|Q\| - 1)}{\|U\|^2} (t - t_i^*) \right), \quad (2.17)$$

$\forall t \in [t_i^*, t_{i+1}^*]$ . Using (2.17) in an inductive way, we can express the evolution of Lyapunov function for the time-interval  $[0, t_m^*]$ , as follows:

$$V(t_m^*) < V(0) \prod_{i=1}^{m-1} \exp \left( -\frac{(1-\sigma)(\|Q\| - 1)}{\|U\|^2} (t_{i+1}^* - t_i^*) \right).$$

We note that,  $t_m^* = \sum_{i=1}^{m-1} (t_{i+1}^* - t_i^*)$ , so the latter equation yields:

$$V(t_m^*) < V(0) \exp \left( -\frac{(1-\sigma)(\|Q\| - 1)}{\|U\|^2} t_m^* \right). \quad (2.18)$$

At this stage, note that, according to the triggering law (2.10), the control cannot be updated within the time-interval  $[t_m^*, T]$ . As discussed later in this proof, a sufficient condition for asymptotic stability is given by  $V(T) < V(0)$ . In order for this to hold, we firstly develop some estimate for  $x(T)$ .

We recall the dynamics (4.1), which given the above explanations and notations, can be written under either:

$$\begin{cases} \dot{x}(t) &= Ax(t) + BKx(t_m^*), \\ x(t_m^*) &= x_0, \end{cases} \quad (2.19)$$

or:

$$\begin{cases} \dot{x}(t) &= (A + BK)x(t) + BKe(t), \\ x(t_m^*) &= x_0, \end{cases} \quad (2.20)$$

form. Let us consider (2.20), whose explicit solution evaluated at  $t = T$  is given by:

$$x(T) = \exp((T - t_m^*)(A + BK))x(t_m^*) + \int_{t_m^*}^T \exp((T - s)(A + BK))BKe(s)ds. \quad (2.21)$$

By applying the triangular-inequality on (2.21), we find the following bound:

$$\begin{aligned} |x(T)| \leq & \left\| \exp((T - t_m^*)(A + BK)) \right\| |x(t_m^*)| + \\ & \left| \int_{t_m^*}^T \exp((T - s)(A + BK))BKe(s)ds \right|. \end{aligned} \quad (2.22)$$

Based on (2.11), Equation (2.22) can be further bounded, which gives the following:

$$|x(T)| \leq \exp((T - t_m^*)\mu(A + BK)) |x(t_m^*)| + \int_{t_m^*}^T \exp((T - s)\mu(A + BK)) \|BK\| |e(s)| ds. \quad (2.23)$$

Applying the sup-operator on (2.23), yields:

$$|x(T)| \leq \exp((T - t_m^*)\mu(A + BK)) |x(t_m^*)| + \sup_{s \in [t_m^*, T]} |e(s)| \|BK\| \int_{t_m^*}^T \exp((T - s)\mu(A + BK)) ds. \quad (2.24)$$

We can solve the integral term in (2.24) which gives the following bound:

$$|x(T)| \leq \exp((T - t_m^*)\mu(A + BK)) |x(t_m^*)| - \frac{\sup_{s \in [t_m^*, T]} |e(s)| \|BK\|}{\mu(A + BK)} \times (1 - \exp((T - t_m^*)\mu(A + BK))). \quad (2.25)$$

In order to further progress in our analysis, we need to find an appropriate bound for  $\sup_{s \in [t_m^*, T]} |e(s)|$ . This is done in the following claim.

**Claim 2.5.2.** Consider (2.25),  $\sup_{s \in [t_m^*, T]} |e(s)|$  satisfies the following:

$$\sup_{s \in [t_m^*, T]} |e(s)| \leq |x(t_m^*)| \left(1 + \frac{\|BK\|}{\|A\|}\right) \times (1 - \exp((T - t_m^*)\|A\|)). \quad (2.26)$$

*Proof of Claim 2.5.2:* First, we recall from our notations that  $e(s) = x(t_m^*) - x(s)$ , which yields:

$$|e(s)| = |x(s) - x(t_m^*)|. \quad (2.27)$$

Now, consider the dynamics (2.19), we then find an explicit expression for (2.27):

$$|e(s)| = |x(s) - x(t_m^*)| = |\exp((s - t_m^*)A)x(t_m^*) - x(t_m^*) + \int_{t_m^*}^s \exp((s - s')A)BKx(t_m^*)ds'|.$$

The latter equation can be bounded, by exploiting the triangular inequality, which then results into the following (several algebraic steps are skipped, for

the sake of brevity):

$$|e(s)| \leq \|\exp((s - t_m^*)A) - I\| |x(t_m^*)| + \frac{\|BK\| |x(t_m^*)|}{\|A\|} (\exp((s - t_m^*) \|A\|) - 1). \quad (2.28)$$

In (2.28), we note that the following holds, recalling definition of exponential-matrix and for some  $x \in \mathbb{R}^n$  for which  $|x| = 1$ :

$$\begin{aligned} |(\exp((s - t_m^*)A) - I)x| &= \left| \sum_{k=1}^{\infty} \frac{((s - t_m^*)A)^k}{k!} x \right| \\ &\leq \sum_{k=1}^{\infty} \left| \frac{((s - t_m^*)A)^k}{k!} \right| |x| \leq \sum_{k=1}^{\infty} \frac{|s - t_m^*|^k \|A\|^k}{k!} |x| \\ &= \sum_{k=1}^{\infty} \frac{|s - t_m^*|^k \|A\|^k}{k!} = \exp((s - t_m^*) \|A\|) - 1. \end{aligned} \quad (2.29)$$

Therefore, by definition of the 2-norm of a matrix, we can say

$\|\exp((s - t_m^*)A) - I\| \leq \exp((s - t_m^*) \|A\|) - 1$ . So, by bounding (2.28), with this expression, we get:

$$|e(s)| \leq (\exp((s - t_m^*) \|A\|) - 1) |x(t_m^*)| + \frac{\|BK\| |x(t_m^*)|}{\|A\|} (\exp((s - t_m^*) \|A\|) - 1).$$

By applying the sup operator on the latter inequality, we obtain:

$$\begin{aligned} \sup_{s \in [t_m^*, T]} |e(s)| &\leq \sup_{s \in [t_m^*, T]} (\exp((s - t_m^*) \|A\|) - 1) |x(t_m^*)| + \frac{\|BK\| |x(t_m^*)|}{\|A\|} \times \\ &\quad \sup_{s \in [t_m^*, T]} (\exp((s - t_m^*) \|A\|) - 1). \end{aligned}$$

In this equation, we note that  $\|A\| > 0$ , as  $A \neq 0$ . So, we can compute the sup's, appearing on its RHS, to get:

$$\begin{aligned} \sup_{s \in [t_m^*, T]} |e(s)| &\leq (\exp((T - t_m^*) \|A\|) - 1) |x(t_m^*)| + \\ &\quad \frac{\|BK\| |x(t_m^*)|}{\|A\|} (\exp((T - t_m^*) \|A\|) - 1). \end{aligned} \quad (2.30)$$

Performing some simplifications on (2.30), we obtain (2.26), which then completes the proof of this claim. •

Now, we plug (2.26) into (2.25) which then by some simplifications yields:

$$|x(T)| \leq \alpha' |x(t_m^*)| , \quad (2.31)$$

where the parameter  $\alpha'$  is defined as follows:

$$\begin{aligned} \alpha' \triangleq & \exp((T - t_m^*)\mu(A + BK)) + \frac{\|BK\|}{\mu(A + BK)} \times \\ & \left( \frac{\|BK\|}{\|A\|} + 1 \right) (1 - \exp((T - t_m^*)\|A\|)) \times \\ & (1 - \exp((T - t_m^*)\mu(A + BK))) . \end{aligned} \quad (2.32)$$

It is worth to note that comparing the parameters  $\alpha$  and  $\alpha'$ , introduced in (2.13) and (2.32), respectively, it holds that  $\alpha' \leq \alpha$ , which is because  $t_m^* \leq T_{\text{off}}^{\text{cr}}$  and  $\mu(A + BK) < 0$ , by assumption. According to this observation, Equation (2.31) can be written as:

$$|x(T)| \leq \alpha |x(t_m^*)| . \quad (2.33)$$

The value of the Lyapunov function at  $t = T$ , can be estimated as follows:

$$V(T) = x(T)^T P x(T) \leq \|U\|^2 |x(T)|^2 , \quad (2.34)$$

which then according to (2.33), can be further bounded:

$$V(T) \leq \alpha^2 \|U\|^2 |x(t_m^*)|^2 . \quad (2.35)$$

Besides, let us define the parameter  $\gamma$  to be:

$$\gamma \triangleq - \frac{(1 - \sigma) (\|Q\| - 1)}{\|U\|^2} , \quad (2.36)$$

where we note that  $\gamma < 0$ , based on the assumption of this theorem. Then, we can rewrite (2.18) as in:

$$V(t_m^*) < \exp(\gamma t_m^*) V(0) . \quad (2.37)$$

We note that as  $\gamma < 0$  and  $t_m^* \leq T_{\text{off}}^{\text{cr}}$ , we obtain that

$\exp(\gamma T_{\text{off}}^{\text{cr}}) V(0) \leq \exp(\gamma t_m^*) V(0)$ , based on this inequality, we can impose the following bound on (2.37):

$$V(t_m^*) < \exp(\gamma T_{\text{off}}^{\text{cr}}) V(0) \leq \exp(\gamma t_m^*) V(0) . \quad (2.38)$$

Now, along the same reasoning for (2.34), we note that

$V(t_m^*) \leq \|U\|^2 |x(t_m^*)|^2$ . A conservative bound can be imposed as follows:

$$V(t_m^*) \leq \|U\|^2 |x(t_m^*)|^2 < \exp(\gamma T_{\text{off}}^{\text{cr}}) V(0). \quad (2.39)$$

Applying bound (2.39) on (2.35), we obtain:

$$V(T) < \alpha^2 \exp(\gamma T_{\text{off}}^{\text{cr}}) V(0). \quad (2.40)$$

Let  $\Upsilon \triangleq \alpha^2 \exp(\gamma T_{\text{off}}^{\text{cr}})$ , we note that  $\Upsilon > 0$ , to wrap up this proof, we shall state the next result.

**Claim 2.5.3.** *In (2.40), imposing  $\Upsilon \in (0, 1)$  guarantees the asymptotic stability, i.e.,  $\lim_{n \rightarrow \infty} x(nT) = 0$ .*

*Proof of Claim 2.5.3:* As explained previously, based on  $V(T) < \Upsilon V(0)$ , and in an inductive way, we obtain:  $V(nT) < \Upsilon V((n-1)T), \forall n \in \mathbb{N}$ . Hence, we infer that:  $V(nT) < \Upsilon^n V(0)$ . Based on this latter inequality, along with the notion of Lyapunov function, i.e.,  $V(nT) = x(nT)^T P x(nT)$ , we get:

$$|x(nT)|^2 \lambda_{\min}(P) \leq V(nT) = x(nT)^T P x(nT) < \Upsilon^n V(0) < \Upsilon^n \lambda_{\max}(P) |x_0|^2,$$

which then yields the following:

$$|x(nT)|^2 < \left( \frac{\lambda_{\max}(P)}{\lambda_{\min}(P)} \right) \Upsilon^n |x_0|^2. \quad (2.41)$$

Recalling  $P \succ 0$ , then it holds that  $\lambda_{\max}(P) > 0$ , and  $\lambda_{\min}(P) > 0$ ; so,  $\lambda_{\max}(P)/\lambda_{\min}(P) > 0$ . Then, according to Claim (2.41) and by imposing  $\Upsilon \in (0, 1)$ , it is evident that  $\lim_{n \rightarrow \infty} x(nT) = 0$ ; in other words, the asymptotic stability is guaranteed. •

Therefore, according to 2.5.3, a sufficient condition for maintaining the asymptotic stability is given in the following:

$$0 < \Upsilon < 1. \quad (2.42)$$

Now, we recall the definition of parameter  $\gamma$ , presented in (2.36), and that  $\Upsilon > 0$ , by its construction. Also, we note that  $\|U\|^2 = \|P\|$ , plus given that  $\exp$  is a monotonically increasing function, we can rewrite (2.42) in the form of (2.12). This, hence, completes the proof. □

**Remark 2.5.4.** *The result provided in the Theorem 2.5.1 can be interpreted as a feasibility statement. In other words, for a given system in the form (4.1), one has to find the proper design parameters  $K$ ,  $P$ ,  $Q$ , and  $\sigma \in (0, 1)$  such that the following constraints would be satisfied:*

$$(A + BK): \text{Hurwitz}, \quad (2.43a)$$

$$(A + BK)^T P + P(A + BK) = -Q, \quad (2.43b)$$

$$\frac{(1 - \sigma)T_{\text{off}}^{\text{cr}}(\|Q\| - 1)}{2} > \|P\| \ln(\alpha), \quad (2.43c)$$

$$\mu(A + BK) < 0. \quad (2.43d)$$

We note that, e.g., if  $T_{\text{off}}^{\text{cr}} = T$ , i.e., the jammer is not malicious at all, then  $\alpha = 1$  and so the constraint (2.43c) holds for free. The same will be true for  $T_{\text{off}}^{\text{cr}} \approx T$ . Additionally, note that more relaxed sufficient conditions for stability can be obtained by imposing  $V(knT) \leq \Upsilon V((n - 1)T)$  for some fixed  $k > 1$ , some  $\Upsilon \in (0, 1)$ , and all  $n \in \mathbb{N}$ .

## 2.6 Simulation

In Section 2.4, we have developed a triggering time-sequence and in Section 2.5 proved that under some sufficient conditions, the system under attack is asymptotically stable. In this section, we shall show the validity of these theoretical results on an academic example.

Let us consider the following system:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1.5 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u, \quad (2.44)$$

where  $u \in \mathbb{R}$ . We note that, for this system,  $(A, B)$  is a controllable pair. In addition, it is an open-loop unstable system, provided that eigenvalues of  $A$  have positive real-part. We pick the control gain:  $K = \begin{bmatrix} -2.6 & -1 \end{bmatrix}$ , which renders the matrix  $A + BK$  Hurwitz. Then, we consider the matrix:  $Q = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & \frac{3}{2} \end{bmatrix}$ . We note that  $Q = Q^T \succ 0$ , and that  $\|Q\| > 1$ . Given these matrices, Lyapunov equa-

tion (2.3) gives us:  $P = \begin{bmatrix} 1.1477 & -0.25 \\ -0.25 & 0.5125 \end{bmatrix}$ . We consider the jammer, imposing signal  $u_{\text{jmd}}(t)$ , where  $T = 2$  and  $T_{\text{off}}^{\text{cr}} = 1.8$ .

We then refer to the Theorem 2.5.1. One can compute that:  $\alpha = 1.4413$ ,  $\|P\| = 1.2343$  and  $\|Q\| = 1.559$ . Thus, the condition (2.12) would be translated into:  $0.5036(1 - \sigma) > 0.4512 \implies \sigma < 0.1041$ . It infers that the allowable range for the design parameter is  $\sigma \in (0, 0.1041)$ . To realize, at this point, all the assumptions of this theorem are satisfied, therefore, we expect that the triggering time-sequence (2.10) render the system asymptotically stable.

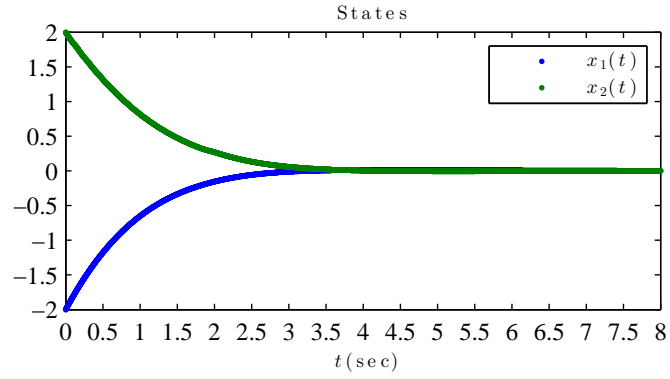
The temporal evolution of the states is shown in Figure 2.3. We can see that the control policy, along with triggering time-sequence has counteracted the effect of jamming attacks.

In order to further demonstrate the triggering time-sequence, we have drawn the temporal evolution of  $|e(t)|^2$  and  $\frac{\sigma(\|Q\|-1)}{\|PBK\|^2} |x(t)|^2$  in Figure 2.4. For the sake of clarity, we have zoomed on the first four periods. According to this figure, we note, e.g., that in the time-interval  $t \in (T_{\text{off}}^{\text{cr}}, T)$ , where the communication is not feasible, the error grows in an unbounded fashion. This effect, however, is accounted for in the next period by triggering more often.

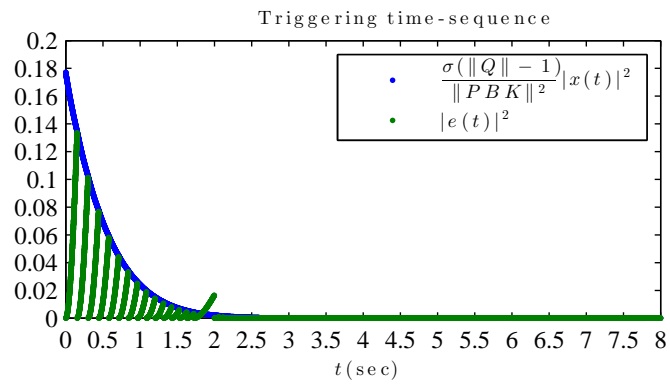
The other interesting observation out of our simulation is explained here. While preserving matrices  $P$  and  $Q$ , for  $T_{\text{off}}^{\text{cr}} \leq 1.7996$ , there is no feasible  $\sigma$ . This is to note that for these *more malicious* attackers, we cannot have a feasible controller. A solution would be to tune matrices  $P$  and  $Q$  which has not been studied in this simulation.

In order to further understand the effect of the length of period,  $T$ , we have conducted a new simulation. In this set of simulation, while preserving the system introduced in (2.44) and the associated matrices  $K, P, Q$ , along with parameter  $\sigma$ , we increase parameter  $T$  and find the following parameter:  $T_{\text{off}, \text{min}}^{\text{cr}} = \min \{T_{\text{off}}^{\text{cr}} | \Upsilon < 1\}$ , which is the shortest  $T_{\text{off}}^{\text{cr}}$  for which the asymptotic stability is guaranteed, according to the Theorem 2.5.1 of this chapter. The result of this simulation is presented in Table 2.1. Referring to this table, we note that by increasing the jammer period  $T$ , the jammer maximum activity decreases,





**Figure 2.3:** Temporal evolution of the states



**Figure 2.4:** Temporal evolution of the triggering condition, zoomed over the first four periods

**Table 2.1:** Guaranteed maximum jammer activity

$T(\text{sec})$	1	2	3	4	10
$T_{\text{off, min}}^{\text{cr}}(\text{sec})$	0.87	1.80	2.75	3.69	9.4
Jammer max. activity $\left(1 - \frac{T_{\text{off, min}}^{\text{cr}}}{T}\right) \times 100$	13%	10%	8.33%	7.75%	6%

**Table 2.2:** Guaranteed maximum triggering flexibility

$T(\text{sec})$	2		5			10		
$T_{\text{off}}^{\text{cr}}(\text{sec})$	1.8	1.9	4.64	4.8	4.9	9.4	9.6	9.8
$\sigma_{\text{max}}$	0.1	0.77	0.11	0.66	0.9	0.17	0.5	0.83

which makes sense because it shows that for a larger jammer period  $T$ , the minimum time jammer sleeps should be larger as well to be able to guarantee the stability.

Moreover, in order to study the effect of the triggering strategy, i.e.,  $\sigma$ , on the stability of the system, we have conducted another set of simulations. We have picked the same system as in the previous parts, and while preserving the matrices  $K$ ,  $P$ , and  $Q$ , for each value of  $T$ , we compute:  $\sigma_{\text{max}} = \max\{\sigma | \Upsilon < 1\}$ , for some  $T_{\text{off, min}}^{\text{cr}} \leq T_{\text{off}}^{\text{cr}} \leq T$ . The result of this simulations for  $T = 2$ ,  $T = 5$ , and  $T = 10$  is presented in the Table 2.2. Referring to this table, we can realize that for each value of  $T$ , by increasing the parameter  $T_{\text{off}}^{\text{cr}}$ ,  $\sigma_{\text{max}}$  also increases which shows that the stability can be guaranteed by lesser number of triggering.

## 2.7 Conclusions & Future Work

We have considered a plant-jammer-operator setup, where the control communication channel (from the operator to the plant) is corrupted by a periodic jammer. For the benefit of maintaining less communication, we have adopted an event-triggering time-sequence to restrict communications when necessary. We have then shown, theoretically and in simulation, that this triggering time-sequence is capable of counteracting the jammer attack and also rendering the system asymptotically stable under some conditions.

As is explained in the manuscript, we assume the jammer has been iden-

tified to the extent that it is periodic and its characteristic parameters are known by the operator. Future work will be devoted to extending our triggering strategy on two fronts: (i) allow for more events so that learning and identification of the jammer is possible, and (ii) exploiting the controllability properties of the linear system to beat a wider class of periodic jammers. In the future, we would like to consider more malicious jammer classes.

## Chapter 3

# On Triggering Control of Single-input Linear Systems under Pulse-Width Modulated DoS Signals

### 3.1 Summary

In this chapter, we study the stability of remotely controlled and observed single-input controllable linear class of systems under power-constrained Pulse-Width Modulated (PWM) Denial-of-Service (DoS) signals. The effect of a DoS jamming signal is to corrupt the control and measurement channels, thus preventing the data to be received at its destination. Therefore, a power-constrained DoS signal is modeled as a series of on and off time-intervals, which restricts communications intermittently. In this work, we first assume that the DoS signal is *partially known*, i.e., a uniform lower-bound for the off time-intervals and the on-to-off transiting time-instants are known. Accordingly, we propose our resilient control and triggering strategies which are provably capable of beating partially known jamming signals of this class. Building on this, we then present our joint control and identification algorithms, JAMCOID FOR PERIODIC SIGNALS and JAMCOID, which are provably able to guarantee the system stability under *unknown* jamming signals. More precisely,

JAMCOID FOR PERIODIC SIGNALS algorithm is able to partly identify a periodic DoS signal with known uniform lower bound for the off time-intervals, whereas JAMCOID algorithm is capable of dealing with power-constrained, but otherwise unknown, DoS signals whilst ensuring stability. The practicality of the proposed techniques is evaluated on a simulation example under *partially known* and *unknown* jamming scenarios.

## 3.2 Introduction

*Cyber-physical systems* comprise a wide range of systems that tightly integrate both computational and physical components. Thanks to growing developments in the area of sensing and communication technologies, these systems are being used in very diverse areas ranging from aerospace, to energy, to civil infrastructure facilities. Whilst the benefits of cyber-physical systems are many, they also come at the price of several challenges. Amongst these, one can highlight a much broader exposure to external actions which threatens their normal operation, i.e., their stability. The latter has brought up and motivated renewed research on the topic of system resilience and security, see for instance [20] and references therein. A particular threat to the secure operation of cyber-physical systems arises from vulnerable communication links, which can be disrupted by means of viruses or external communication-signal jammers. In particular, *Denial-of-Service (DoS)*, resulting in lossy networks, is reported to be the most common type of interference [18]. Motivated by their power-constrained nature, detection avoidance, and ease of implementation, DoS signals can further acquire *Pulse-Width Modulated (PWM)* signal pattern [56, 23]. In this work, we study how to adapt the control of a linear cyber-physical system to power-constrained PWM DoS jamming signals.

The secure operation of cyber-physical systems has been studied in different contexts. The papers [77, 67] characterize topological network conditions that allow a multi-agent system to detect other malicious agents injecting false data; while [13] studies how to maintain group connectivity despite the pres-

ence of malicious external jamming agents. On the other hand, the work [91] proposes a Receding Horizon control methodology to deal with a class of deceptive replay jammers, potentially introducing system delays in formation control missions. However, these previous works can only deal with simple dynamics for each agent (second-order integrators in [91]), and box type of state constraints at best. In [91] resilience comes at the expense of large receding horizons, which can be computationally expensive and difficult to implement.

Some representative studies in the context of Game Theory focus on malicious attacks on linear systems, leading to problem formulations that models the jammer and operator interactions as a dynamic zero-sum non-cooperative game. In this framework, one can single out [63], which consider power-constrained DoS jamming signals on discrete-time systems. The objective of this work is the characterization of equilibrium solutions for fixed-resource agents, which restricts the analytical results to one-dimensional control systems.

The problems of control and estimation over unreliable communication networks have received considerable attention over the last decade [41]. Topics of interest include quantization [17], delays [16], sampling [64], packet dropout [74], DoS jamming signals [3], and clock synchronization [35]. The DoS signals considered in [3] are modeled by means of a stochastic Bernoulli packet drop distribution. The goal is the minimization of a finite-horizon quadratic cost function subject to constraints. This work builds on previous research over lossy networks such as [74]. However, none of the aforementioned papers considers adaptation in the control law in order to exploit an energy limitation of the jamming signals. On estimation, the work [35] provides conditions under which synchronization of a affine-clock network subject to delays is possible. The method assumes information about the clock times is submitted in messages, and does not address how to estimate clocks while maintaining economic communications for an underlying system control. Finally, in the context of discrete-time linear systems, one can also distinguish [26] on deceptive jammers. Using sensor redundancy and compressed sensing techniques, the authors propose an encoding algorithm that can be resilient to this type of attacks. The

algorithm does not account for possible communication interruptions as those imposed by DoS signals.

Motivated by the emerging use of economic communications in modern control systems, we shall address the problem of maintaining system stability in the context of *triggering control* [78, 58, 86]. In better words, we aim to build triggering control actions which rely on limited communications and/or measurements and which are then more robust with respect to a class of DoS PWM jamming signals. In this regard, the works [85, 37] present sufficient conditions on the maximum number of successive data dropouts that guarantee that a distributed system employing an event-triggering algorithm maintains stability. However, communications are not adapted to deal with any type of DoS signal. Finally, the paper [47] considers a resilience problem formulated in the triggering framework. This latter, deals with an alternative type of deceptive signals, which tamper with the control commands. Resilience is based on the switching between a safe and faulty modes to maintain normal system operation at all times. In this setting, the detection of the malfunction above a threshold is always possible, and then the attack has a limited effect on the system performance.

In this chapter, we consider three problem scenarios of increasing difficulty with respect to the assumed knowledge on the DoS signal. First, we consider a partially known PWM DoS jamming signal where the on-to-off time jamming instants are known as well as a guaranteed off period. In this setting, we present a resilient control and triggering strategy that can be tuned arbitrarily to deal with any jammer of this type. Building upon these results, we consider a second setting, where the jamming signal is assumed to be (non-necessarily malicious) periodic but of unknown period. To address this case, we introduce the JAMCOID FOR PERIODIC SIGNALS algorithm that exploits periodicity to both synchronize while sporadically sample the jamming signal, and stabilize the system. Finally, in the third problem scenario we consider an unknown, but power-constrained, PWM DoS jamming signal. For this case, we propose the JAMCOID algorithm, which bestows a joint control

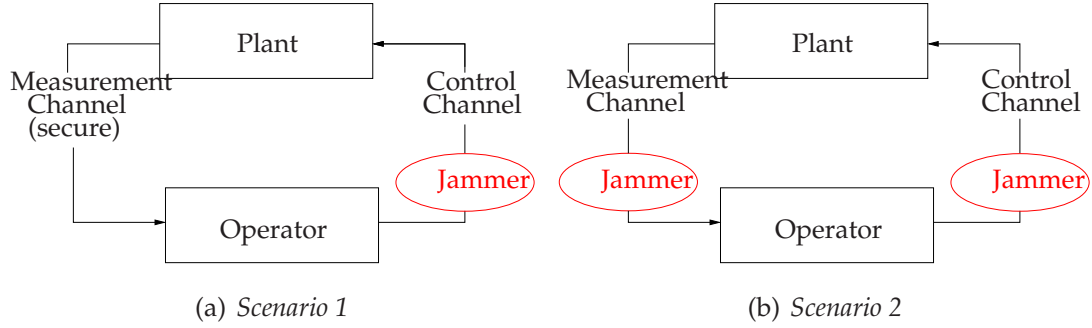
and identification strategies at the expense of a higher number of communications. In these three problems, we prove that our proposed strategies ensure the system's asymptotic stability. In contrast with our earlier work [30], the contributions of this research study may be itemized as follows: (i) proposing a resilient parameter-dependent control and triggering strategies provably capable of dealing with the partially known jamming scenario, (ii) proposing the JAMCOID FOR PERIODIC SIGNALS and JAMCOID algorithms to address the unknown jamming scenario, (iii) simulations on the functionality of both aforementioned contributions. A preliminary version of this work focusing on known jammers and systems of low dimension has appeared in [28]. The other preliminary version, entailing MIMO systems has appeared in [31], where the detailed proofs are omitted. In these studies, comprehensive simulation studies are lacking.

The rest of the chapter is organized as follows. Section 5.3 includes the problem formulation and notations. Then, in Section 3.4, some preliminaries are provided, where then we propose our resilient control and strategy consistent with the jamming signal. In Section 4.6, we analyze and prove the stability of the system equipped with these resilient control and triggering strategies. In what follows in Section 4.7, we shall explain the jammer control and identification algorithms, JAMCOID FOR PERIODIC SIGNALS and JAMCOID, then analyze their asymptotic behavior to prove that they guarantee the system stability. In Section 5.8, we demonstrate, in simulation environment, the functionality of our theoretical results under known and unknown jamming scenarios. At last, in Section 5.9, we summarize the results and state the future work.

### 3.3 Problem Formulation

We consider a remote operator-plant setup, where the operator uses control and measurement channels to respectively send and receive data back from an open-loop unstable plant. The wireless control and measurement channels are prone to be jammed as depicted in Figure 3.1. We assume that the plant has





**Figure 3.1:** Problem Architecture. Two scenarios have been considered: in Scenario 1 measurements are secure or indirectly available to the operator, and in Scenario 2 both the measurement and control channels can be compromised by a DoS signal.

no specific intelligence and is only able to update the control based on the data it receives and to accordingly send back the measurement. We also assume that the operator knows the plant dynamics and is able to compute and send the control and obtain its state measurements at particular times.

More precisely, we consider the following closed-loop dynamics:

$$\dot{x}(t) = Ax(t) + Bu(t), \quad (3.1a)$$

$$u(t) = Kx(t_k), \quad \forall t \in [t_k, t_{k+1}[, \quad (3.1b)$$

where  $x \in \mathbb{R}^d$  is the state vector,  $u \in \mathbb{R}$  is the input,  $A$ ,  $B$  and  $K$  are matrices of proper dimensions, and  $\{t_k\}_{k \in \mathbb{N}}$  is a certain triggering time-sequence. Here, we also assume that: (i) the system (4.1a) is open-loop unstable, and (ii) the pair  $(A, B)$  is controllable.

We now introduce the class of DoS signals studied in this chapter. We consider a *power-constrained* jamming signal or jammer, blocking the control and measurement communication channels as follows, see Figure 3.2 for an illustration:

$$u_{\text{jmd}}(t) = \begin{cases} 0, & T^{n-1} \leq t \leq T^{n-1} + T_{\text{off}}^{n-1}, \\ 1, & T^{n-1} + T_{\text{off}}^{n-1} < t < T^n, \end{cases} \quad (3.2)$$

where we assume that the sequences of real numbers,  $\{T^n\}_{n \in \mathbb{Z}}$ ,  $\{T_{\text{off}}^n\}_{n \in \mathbb{Z}}$ , satisfy  $T^n < T^{n+1}$ ,  $T_{\text{off}}^n \in \mathbb{R}_{>0}$ , and  $T_{\text{off}}^n < T^n - T^{n-1}$ , for  $n \in \mathbb{Z}$ . Using these parameters,

the time-intervals  $[T^n, T^n + T_{\text{off}}^n]$  determine when the signal is off and communication is possible. We further denote the sequence  $\{T_{\text{on}}^n\}_{n \in \mathbb{Z}}$ , with  $T_{\text{on}}^n \in \mathbb{R}$ , and  $]T_{\text{on}}^n, T^{n+1}[$  as the time-interval where the jammer is active, thus no data can be sent or received. Accordingly, it holds that  $T_{\text{off}}^n + T_{\text{on}}^n = T^n - T^{n-1}, \forall n$ . The parameters  $T^n$  and  $T_{\text{off}}^n$  need not be time-invariant which recalls Pulse-Width Modulated (PWM) signals. Finally, we denote by  $T_{\text{off}}^{\text{cr}}$  a uniform lower-bound for  $T_{\text{off}}^n$ , i.e.,  $0 < T_{\text{off}}^{\text{cr}} \leq T_{\text{off}}^n, \forall n$ , where we also denote  $T_{\text{on}}^{\text{cr},n} \triangleq T^n - T^{n-1} - T_{\text{off}}^{\text{cr}}$ . In addition, we assume  $T_{\text{off}}^{\text{cr}} < \infty$  and  $\{T_{\text{on}}^{\text{cr},n}\} < \infty, \forall n \in \mathbb{N}$ , these latter assumptions further justify the power-constrained nature of the considered DoS signal (4.2) because  $\frac{T_{\text{on}}^n}{T^n} \leq \frac{\{T_{\text{on}}^{\text{cr},n}\}}{T_{\text{off}}^{\text{cr}}} < \infty$  holds. The last notation, for the case of  $T^n = nT$ , implies  $T_{\text{on}}^{\text{cr},n} = T - T_{\text{off}}^{\text{cr}}$ , hence, we use  $T_{\text{on}}^{\text{cr}} \equiv T_{\text{on}}^{\text{cr},n}$ .

At this point we shall resort to Figure 3.1 where we have introduced the scenarios considered for the jamming intervention. Then, recalling the system dynamics and jamming signal, respectively introduced in (4.1) and (4.2), we shall be more specific on the jamming intervention in each of these scenarios. The system dynamics (4.1) would be as follows:

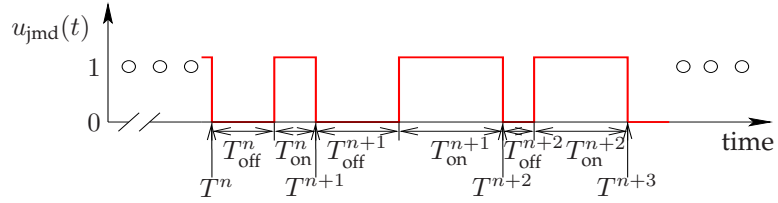
$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \\ u(t) &= Kx(t_k)u_{\text{jmd}}(t_k), \quad \forall t \in [t_k, t_{k+1}[ , \end{aligned}$$

where the operator knowledge about the states of the plant would be  $x(t)$  and  $x(t)u_{\text{jmd}}(t)$  for Scenarios 1 and 2, respectively.

We now consider the following problems:

*[Problem 1]:* Given a power-constrained jamming signal described by (4.2), knowing the sequence  $\{T^n\}$  and the parameter  $T_{\text{off}}^{\text{cr}}$ , determine (i) a time-triggered control strategy under Scenario 2 in Figure 3.1(b), (ii) an event-triggered control strategy under Scenario 1 in Figure 3.1(a), for the system to be resilient to DoS signals.

The solution to this problem will help us address the following more general problems. In order to state them, let  $T^0$  be the time difference between the initial time of the operator's clock and the DoS signal's clock, assumed w.l.o.g. to be  $T^0 \geq 0$ .



**Figure 3.2:** Scheme of the Jamming DoS Signal

*[Problem 2]:* Given a power-constrained jamming signal described by (4.2), assuming that  $T^n = nT$ , and given  $T_{\text{off}}^{\text{cr}}$ , propose a time-triggered control strategy under Scenario 2 to guarantee the asymptotic stability of the system, despite lack of knowledge on  $T$ ,  $T_{\text{on}}^{\text{cr}}$  and the time  $T^0$ .

*[Problem 3]:* Given a power-constrained jammer described by (4.2), propose (i) a time-triggered control strategy under Scenario 2, and (ii) an event-triggered control strategy under Scenario 1, to guarantee the asymptotic stability of the system, despite lack of knowledge on  $\{T^n\}$ ,  $\{T_{\text{on}}^{\text{cr},n}\}$ ,  $T_{\text{off}}^{\text{cr}}$ , and the time,  $T^0$ .

The type of DoS signals considered here constitute a class of resource-constrained jammers, which are not necessarily malicious. Then, it is acceptable to consider a non-malicious periodic type of disturbance as in Problem 2. The special case of Problem 2 is distinguished to show how the periodicity of the DoS signal and the solution to Problem 1 can be exploited to limit communications over the on jamming time-intervals. Problem 3 addresses the case of non-periodic and unknown DoS signals but which are power-constrained. However, in order to deal with any signal of this class, communication over the on periods is necessary as well. It is also nonetheless worth noting that the current structure in presenting the problems has been opted because the results of the main initial Problem 1 serve as the basis for the two afore discussed Problems 2 and 3, which although both deal with the unknown jamming scenarios, they provide different solutions.

### 3.4 Resilient Control & Triggering Strategy

In this section, we first recall some useful properties of the class of systems we study, after which we introduce a proper choice of control matrix,  $K$ . We shall then introduce our class of control strategies which consists of choosing this particular form of  $K$ , along with an associated triggering time-sequence,  $\{t_k\}_{k \in \mathbb{N}}$ , based on an appropriate Jordan decomposition. Unless stated otherwise, we refer the reader to Section 3.9 for the proofs of main results.

Since  $(A, B)$  is controllable, the system (4.1a) can be put into a controllable canonical form by a proper similarity transformation [12]. Thus, we focus on systems of the form:

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_d & -a_{d-1} & -a_{d-2} & \cdots & -a_1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u, \quad (3.4)$$

$$u = [-k_d + a_d, -k_{d-1} + a_{d-1}, \dots, -k_1 + a_1] x.$$

**Lemma 3.4.1.** *Consider  $\lambda \in \mathbb{R}_{>0}$  and system (3.4). By choosing  $K_\lambda = [k_1, \dots, k_d]$  as  $k_i = \binom{d}{i} \lambda^i$ ,  $i \in \{1, \dots, d\}$ , all the closed-loop system poles are placed at  $-\lambda$ . Moreover the eigenvalue  $-\lambda$  has algebraic multiplicity  $d$  and geometric multiplicity 1.*

*Proof.* The proof can be deduced from [12]. □

**Remark 3.4.2.** *Note that matrix  $A + BK_\lambda$  has only one linearly independent eigenvector, therefore it is not diagonalizable. This property holds for all values of  $\lambda \in \mathbb{R}_{>0}$ . Moreover, let  $v$  be an eigenvector of  $A + BK_\lambda$ . Then, since the matrix  $A + BK_\lambda + \lambda I$  depends on  $\lambda$  in a polynomial way, the components of this eigenvector, given by  $(A + BK_\lambda + \lambda I)v = 0$ , become rational functions of  $\lambda$ .*

**Remark 3.4.3.** *For simplicity, we focus here on single-input systems. However, the previous results can be extended to the multi-input case by using the transformation techniques in [4]. The interested reader is referred to our study [31].*

From Lemma 3.4.1, the eigenvalues of the matrix  $A + BK_\lambda$  are at  $-\lambda$ . Thus, the Jordan decomposition of this matrix can be expressed as:

$$A + BK_\lambda = T_\lambda J_\lambda T_\lambda^{-1}, \quad (3.5)$$

where  $J_\lambda = -\lambda I + N$  and  $T_\lambda$  is a matrix built upon the linearly independent and generalized eigenvectors. We note that the matrix  $N$  has a *unique* structure for all values of  $\lambda$ , as the geometric multiplicity of this eigenvalue remains unchanged. Moreover, as discussed in Remark 3.4.2, the only linearly independent eigenvector of  $A + BK_\lambda$  depends in a rational way on  $\lambda$ . Then, by construction of the generalized eigenvectors [25], the matrices  $T_\lambda$  and  $T_\lambda^{-1}$  also depend on  $\lambda$  in a rational way.

Before presenting our control strategy, we introduce a family of coordinate transformations used in this chapter. They are based on the Jordan decomposition technique explained in previous paragraph. Let us consider system (3.4), with the control  $u(t) = K_\lambda x(t_k)$ . Then, the closed-loop dynamics is:

$$\dot{x} = (A + BK_\lambda)x + BK_\lambda e,$$

where,  $e(t) = x(t_k) - x(t)$ . Recalling (4.5), the latter dynamics under the static transformations,  $e = T_\lambda e_\lambda$ , and  $x = T_\lambda x_\lambda$ , yields:

$$\dot{x}_\lambda = J_\lambda x_\lambda + T_\lambda^{-1} BK_\lambda T_\lambda e_\lambda. \quad (3.6)$$

The following result states our first attempt in developing the triggering strategy. Indeed, as it can be verified, the presence of the DoS signal is not accounted for in this lemma; our resilient triggering strategy comes after. The ISS-based triggering approach developed in papers [78] and [58] have inspired the derivation of this lemma.

**Lemma 3.4.4.** *Take  $\lambda > \|N\| + 1/2$ , and  $K_\lambda$  as in Lemma 3.4.1. Then  $V(x_\lambda) = x_\lambda^T x_\lambda$  is a common ISS-Lyapunov function for the system (4.6), and the event-triggered condition:*

$$|e_\lambda(t)|^2 \leq \frac{\sigma(2\lambda - 1 - 2\|N\|)}{\|T_\lambda^{-1} BK_\lambda T_\lambda\|^2} |x_\lambda(t)|^2, \quad (3.7)$$

*guarantees the asymptotic stability of the system, for  $\sigma \in (0, 1)$ .*

**Remark 3.4.5.** Let  $t_k$  and  $t_{k+1}$  be two consecutive time-instants given by the event-triggering strategy (4.7). Then, for each  $\lambda$ , the following holds:

$$\exists \tau_\lambda > 0, \text{ such that } t_{k+1} - t_k \geq \tau_\lambda, \forall k \in \mathbb{N},$$

that is to say, parameter  $\tau_\lambda$  is the uniform lower-bound for the triggering time-sequence,  $\{t_k\}$  given by (4.7). This latter fact is based on Theorem III.1, presented in [78], which also shows how to compute such  $\tau_\lambda$  as recalled in Algorithm 1, later in Section 5.8. This latter observation also implies that the time-sequence,  $\{t_k\}$ , generated by the strategy (4.7) does not accumulate; that is in other words, all two consecutive time-instants,  $t_k$  and  $t_{k+1}$ , are separated by a positive lower-bound,  $\tau_\lambda$ , which then ensures  $\lim_{k \rightarrow \infty} t_{k+1} - t_k \neq 0$ . Since under Scenario 2 we do not assume that the operator can continuously measure the plant states, we adopt this  $\tau_\lambda$  as the basis of our economic time-triggered control strategy.

For parameter  $\tau_\lambda$ , and sequence  $\{t_k\}_{k \in \mathbb{N}}$ , we show the following property.

**Proposition 3.4.6.** Let  $\lambda > \|N\| + 1/2$ , and let  $\{t_k\}_{k \in \mathbb{N}}$  be the associated time-sequence generated by the event-triggering strategy (4.7). Consider the parameter  $\tau_\lambda$  introduced in Remark 3.4.5. Then, the following holds:

$$\lim_{\lambda \rightarrow \infty} \tau_\lambda = 0, \quad \text{and} \quad \lim_{\lambda \rightarrow \infty} t_{k+1} - t_k = 0, \quad \forall k \in \mathbb{N}. \quad (3.8)$$

At this point, we present the class of triggering strategies we consider to solve Problem 1 (both scenarios) starting at  $T^0$ , we shall first introduce these strategies, stability characterization using these strategies is postponed to Section 4.6. To do this, we consider the jammer is constantly maintaining a “worst-case jamming scenario,” i.e.,  $T_{\text{off}}^n = T_{\text{off}}^{\text{cr}}, \forall n \in \mathbb{Z}$ . We would like to clarify that this is a worst case, because in this way the jammer is active the most and is inactive the least, i.e.,  $T_{\text{off}}^n$  takes its least value for each jamming time-interval.

**Definition 2.** A time-triggered control strategy for Problem 1, Scenario 2, consists of  $u_n(t) = K_{\lambda_n} x(t_{k,n}^*)$  during  $t \in [t_{k,n}^*, t_{k+1,n}^*[$ ,  $k, n \in \mathbb{N}$ , where the  $t_{k,n}^*$  are the time-instants:

$$t_{k,n}^* \in \{l\tau_{\lambda_n} \mid l\tau_{\lambda_n} \in [T^{n-1}, T^{n-1} + T_{\text{off}}^{\text{cr}}], l \in \mathbb{N}\} \cup \{T^n\}. \quad (3.9)$$

Here,  $K_{\lambda_n}$  is chosen according to Lemma 3.4.1 and Proposition 4.5.2 to guarantee that the corresponding  $l\tau_{\lambda_n} \in [T^{n-1}, T^{n-1} + T_{\text{off}}^{\text{cr}}]$ , for all  $n \in \mathbb{N}$ .

We note that, based on Proposition 4.5.2, and for a given  $n$ , we are able to find a  $\lambda_c$  so that the multiples of  $\tau_\lambda$  lie in the desired interval, for  $\lambda \geq \lambda_c$ , i.e., the former set introduced in (4.8) is never empty. Note that these strategies limit communications to the off periods of the jamming signal. Similarly, we define our event-triggering strategy to solve Problem 1, Scenario 1, as follows.

**Definition 3.** *An event-triggered control strategy for Problem 1, Scenario 1 consists of  $u_n(t) = K_{\lambda_n}x(t_{k,n}^*)$  during  $t \in [t_{k,n}^*, t_{k+1,n}^*[, k, n \in \mathbb{N}$ , where the  $t_{k,n}^*$  are the time-instants:*

$$t_{k,n}^* \in \{t_l \text{ satisfying (4.7)} \mid t_l \in [T^{n-1}, T^{n-1} + T_{\text{off}}^{\text{cr}}], l \in \mathbb{N}\} \cup \{T^n\}. \quad (3.10)$$

Here,  $K_{\lambda_n}$  is chosen according to Lemma 3.4.1 and Proposition 4.5.2 to guarantee that the corresponding  $t_l, t_{l+1} \in [T^{n-1}, T^{n-1} + T_{\text{off}}^{\text{cr}}]$ , for  $n \in \mathbb{N}$ . It is also worth to mention that according to (3.10),  $t_l$  are the time-instants declared by (4.7) stated in Lemma 4.5.1, which also lie in the desired interval,  $[T^{n-1}, T^{n-1} + T_{\text{off}}^{\text{cr}}]$ , therefore, it does not declare a continuum interval of times.

The choice of  $\lambda_n$ , which influences both the control effort,  $K_{\lambda_n}$ , and the frequency of communications, will be made specific in the following section. We note that both effort  $K_{\lambda_n}$  and frequency of communications will be used to guarantee asymptotic stability of the linear system.

### 3.5 Stability Analysis

In this section, we prove how the class of control and triggering strategies discussed in earlier sections are able to solve Problem 1 (both scenarios) for an appropriate choice of  $\lambda_n$ . The presented analysis provides the foundation to solve Problems 2, and 3, and hence to deal with unknown DoS signals. Unless stated otherwise, we refer the reader to Section 3.9 for the proofs of main results.

Given  $M \in \mathbb{R}^{d \times d}$ , define the  $\mu$  operator:

$$\mu(M) = \max \left\{ \mu \mid \mu \in \text{spec} \left( \frac{M + M^T}{2} \right) \right\}, \quad (3.11)$$

with  $\text{spec}(\cdot)$  be the set of eigenvalues. An upper bound of  $\mu(M)$  is given by:

$$\mu_M \triangleq |\mu(M)| + 1. \quad (3.12)$$

Moreover, we would like to recall the following lemma.

**Lemma 3.5.1** ([57]). *Consider the polynomial:*

$$p(z) = a_0 + a_1 z + \cdots + a_d z^d, \quad (3.13)$$

where,  $z \in \mathbb{R}$ , and  $a_i \in \mathbb{R}$ , for  $i \in \{1, \dots, d\}$ . Then, a lower-bound for all the roots of  $p(z) = 0$  is given as follows:

$$R = \frac{|a_0|}{\max(|a_0|, |a_1| + |a_2| + \cdots + |a_d|)}. \quad (3.14)$$

We can now state the main result of this section.

**Theorem 3.5.2.** *(Stability Characterization of Problem 1, Scenario 2) Consider System (3.4), where  $(A, B)$  is a controllable pair. Given a jamming signal (4.2), where the sequence  $\{T^n\}$  and parameter  $T_{\text{off}}^{\text{cr}}$  are known; consider:*

$$C(n, \lambda) \triangleq \left( \frac{\exp(-(1-\sigma)(2\lambda - 1 - 2\|N\|)T_{\text{off}}^{\text{cr}}/4)}{\|T_\lambda^{-1}\|^{-1}\sqrt{R_\lambda}} \right) \times \quad (3.15)$$

$$\left( \frac{\|BK_\lambda\|}{\mu_A} (\exp(T_{\text{on}}^{\text{cr},n} \mu_A) - 1) + \frac{\exp(-(1-\sigma)(2\lambda - 1 - 2\|N\|)\tau_\lambda)}{\|T_\lambda^{-1}\|^{-1}\sqrt{R_\lambda}} \exp(T_{\text{on}}^{\text{cr},n} \mu_A) \right),$$

wherein  $R_\lambda$  is as defined in (3.14) for the characteristic polynomial of the matrix,  $(T_\lambda^{-1})^T(T_\lambda^{-1})$ . Let  $\lambda_n^* = \inf\{\lambda_n \mid C(n, \lambda_n) < 1 \text{ and } \lambda_n > \|N\| + 1/2\}$ , then, for each  $n \in \mathbb{N}$ , applying control gain  $K_{\lambda_n}$  as chosen in Lemma 3.4.1, along with the time-triggered strategy (4.8), for any  $\lambda_n \geq \lambda_n^*$ , renders the system asymptotically stable.

**Remark 3.5.3.** *We would like to hereby highlight one important feature of the previous result. On the one hand and provided our argument in Lemma 4.5.1, we are imposing the condition,  $\lambda_n \geq \lambda_n^* > \|N\| + 1/2$ , on the other hand, the control strategy resorts*



to time-triggered strategy (4.8), which is based on  $\tau_\lambda$  parameter. Hence, recalling our argument in Proposition 4.5.2, we are inherently imposing  $\tau_{\lambda_n} \leq \tau_{\lambda_n^*} < \tau^*$ , where  $\tau^*$  is associated to  $\lambda^* = \|N\| + 1/2$ . This latter admittedly imposes a constraint in the sense that the frequency of communication cannot go beyond  $\tau^*$  during the off sub-periods of the jammer, nonetheless, this constraint is the price that one has to admit in order to be able to cope with any jammer of this class.

The previous result, i.e., Theorem 4.6.1, is based on the class of time-triggered strategies stated in Definition 3. The following corollary characterizes the alternative class of event-triggered strategy of Definition 3 to solve Problem 1, Scenario 1.

**Corollary 3.5.4.** *(Stability Characterization of Problem 1, Scenario 1) Consider System (3.4), where  $(A, B)$  is a controllable pair. Given a jamming signal (4.2), where the sequence  $\{T^n\}$  and parameter  $T_{\text{off}}^{\text{cr}}$  are known; recall then  $C(n, \lambda)$  as characterized in (3.15) of Theorem 4.6.1 and let  $\lambda_n^* = \inf\{\lambda_n | C(n, \lambda_n) < 1 \text{ and } \lambda_n > \|N\| + 1/2\}$ , then, for each  $n \in \mathbb{N}$ , applying control gain  $K_{\lambda_n}$  as chosen in Lemma 3.4.1, along with the event-triggered strategy (3.10), for any  $\lambda_n \geq \lambda_n^*$ , renders the system asymptotically stable.*

In the following remark, we shall provide some interpretation to what we have stated thus far in Definitions 3, 3, Theorem 4.6.1, and Corollary 3.5.4.

**Remark 3.5.5.** *We would like to emphasize that in our proposed solutions to Problem 1 (both scenarios) and in order to deal with a power-constrained DoS jamming signal, the operator tunes a parameter,  $\lambda$ , and thus employs two resources, i.e., the “frequency of communication,” characterized by  $\tau_\lambda$ , and the “actuation effort,” characterized by  $K_\lambda$ . More specifically, our objective is to determine a least value for  $\lambda$  such that for a given  $T_{\text{off}}^{\text{cr}}$ ,  $\{T^n\}$  and by employing the proposed solutions, we can still guarantee the stability of the system. It is then indeed the coupling between the frequency of communication and actuation effort, determined and tuned by a parameter, which yield the results presented thus far.*

### 3.6 Joint Triggering Control & Jammer Identification

In this section, we propose our solutions to Problems 2 and 3, which are built on the resilient control and triggering strategies introduced in Section 3.4, along with the stability analysis presented in Section 4.6. First we discuss the JAMCOID FOR PERIODIC SIGNALS algorithm to solve Problem 2, i.e., knowing  $T_{\text{off}}^{\text{cr}}$ , and the jamming signal is of form (4.2), with  $T^n = nT$ , for some  $T > 0$ , we show that the JAMCOID FOR PERIODIC SIGNALS algorithm guarantees the asymptotic stability of the system for an unknown  $T$ , and despite presence of an unknown mismatch in the operator's and jammer's clocks initial times. Then, based on the obtained observations we develop the JAMCOID algorithm to solve Problem 3, i.e., to guarantee the system asymptotic stability despite presence of a general jamming signal of the form (4.2), with unknown parameters,  $T^0, \{T^n\}, \{T_{\text{on}}^{\text{cr},n}\}, T_{\text{off}}^{\text{cr}}$ , where only the existence of  $T_{\text{off}}^{\text{cr}}$  is assumed. Unless stated otherwise, we refer the reader to Section 3.9 for the proofs of main results.

First, let us denote by  $u_{\text{id}} : \mathbb{R}_{\geq 0} \rightarrow \{\text{null}\} \cup \{1\}$ , the signal that the operator uses for jammer identification purposes, where  $u_{\text{id}}(t) = 1$  encodes that the operator sends message 1 to the plant at time  $t$ , whereas,  $u_{\text{id}}(t) = \text{null}$  declares that no message is submitted. Let us also denote by  $u_{\text{ste}} : \mathbb{R}_{\geq 0} \rightarrow \{\text{null}\} \cup \mathbb{R}^d$  the signal rebound from the plant, such that  $u_{\text{ste}}(t) \in \mathbb{R}^d$  contains a successfully delivered message containing state-update information at time  $t$ , whereas  $u_{\text{ste}}(t) = \text{null}$  represents no message is delivered at the operator's side. Finally, let  $u_{\text{ctrl}} : \mathbb{R}_{\geq 0} \rightarrow \{\text{null}\} \cup \mathbb{R}$  be the submitted control, where similar to the  $u_{\text{id}}$ -case,  $u_{\text{ctrl}}(t) \neq \text{null}$  induces that a control  $u_{\text{ctrl}}(t)$  is computed and sent, whereas  $u_{\text{ctrl}}(t) = \text{null}$  means that no message is sent.

In fact, we assume that, from the operator viewpoint, the submission of  $u_{\text{id}}$ , receipt of  $u_{\text{ste}}$ , and submission of  $u_{\text{ctrl}}$  happen in a sequential and instantaneous manner. That is to say, first a measurement is requested by sending  $u_{\text{id}} = 1$ , then upon receipt of the measurement via  $u_{\text{ste}}$ , a control is sent to the plant via  $u_{\text{ctrl}}$ . Note that  $u_{\text{ctrl}}(t) = \text{null}$  if  $u_{\text{ste}}(t) = \text{null}$ , i.e., we do *not* send any control if we do *not* receive any measurement, and this happens when the jammer is active at  $t$ .

At last and prior to introducing the afore discussed algorithms, we would like to discuss various symbols and parameters used in these algorithms. The parameter,  $T_j^0$  is the time difference between the jammer's and operator's clocks at the  $j$ -th iteration of the algorithm—with  $T^0 \equiv T_0^0$ . The parameter  $M$  is the sampling time with which the operator communicates with the plant. The parameters,  $\hat{T}_{\text{off}}^{\text{cr}}$  and  $\hat{T}_{\text{on}}^{\text{cr}}$  are, respectively, the estimate of  $T_{\text{off}}^{\text{cr}}$  and  $T_{\text{on}}^{\text{cr}}$ . The parameter,  $\widehat{lT}^{\text{next}}$  is the estimate of the  $l$ -th multiple of the period  $T$  that is used in JAMCOID FOR PERIODIC SIGNALS algorithm. The parameter,  $\sigma \in (1, \infty)$  is also used in order to refine the sampling time,  $M$ , if required.

### 3.6.1 The JAMCOID FOR PERIODIC SIGNALS Algorithm

Unlike in Section 3.4, we assume here that the operator's and jammer's clocks do not have to be synchronous but have similar linear models. Let  $T^0$  be the time difference between the jammer's clock's initial time and the operator's. W.l.o.g. assume  $T^0 \geq 0$ . We realize that, under the “*worst-case jamming scenario*,” there are three unknown parameters,  $T_{\text{on}}^{\text{cr}}$ ,  $T$  and  $T^0$ , which characterize the jammer's signal together with the known parameter,  $T_{\text{off}}^{\text{cr}}$ .

Intuitively, the core idea behind JAMCOID FOR PERIODIC SIGNALS is to intelligently generate the triggering time-sequence  $\{t_k\}$  in order to, (i) bound the asynchronicity,  $T^0$ , (ii) find a valid useful interval to which the parameter  $T$ , or some multiple of this period, belongs. In fact, this latter algorithm fulfills these two latter goals by first employing a periodic time-triggered strategy and second employing a more economic time-triggered strategy closer to Definition 3, along with resetting the operator's clock upon completion of every step of the algorithm. More specifically, using the periodic time-triggered strategy, by verifying the success or failure of the transmitted signal at every sampling time, proper estimates for jamming off sub-period and multiples of the jamming time-interval would be obtained which are helpful in deriving the more economic time-triggered strategy. It is also worth to mention that JAMCOID FOR PERIODIC SIGNALS has an auto-correction module in it in the sense that based on the observation of the economic time-triggered strategy, it

may return to the periodic one in order to refine the estimates of jamming off sub-period and the multiple of its period. These said, by resetting the operator's clock upon retrieving these two estimates, the asynchronicity between jammer's and operator's clocks would be also bounded.

The algorithm is described in the flowchart of Figure 3.3 and summarized in the following lines, where  $u_{\text{ctrl}}(t_k)$  is computed as explained in Section 3.4, that is  $u_{\text{ctrl}}(t_k) = K_\lambda u_{\text{ste}}(t_k)$ , with the gains  $K_\lambda$  given in Lemma 3.4.1. In the flowchart of Figure 3.3, while following the intuitive explanation mentioned earlier: (i)  $law(0)$  refers to a periodic time-triggered strategy defined by the period  $M = \tau_\lambda$ , and with associated  $K_\lambda$ ; (ii)  $law(1)$  refers to a time-triggered strategy of the class in Definition 3 with a  $\lambda$  chosen to guarantee the conditions of Theorem 4.6.1 for an estimated off period of  $\hat{T}_{\text{off}}^{\text{cr}}$ , and assuming that the next on-to-off time instant is given by  $\widehat{lT}^{\text{next}}$ . Briefly, the following steps are performed:

**Step I:** The operator sends messages to the plant with control content following a periodic triggering strategy. During this phase, we can distinguish two cases. **Case (1):** We do *never* hit the jammer's on-subperiod, that is,  $u_{\text{ste}}(t_k) \neq \text{null}, \forall t_k$ . Thus, we can keep updating our control at the prescribed time-instants,  $t_k = kM$ , without interruption. This can happen if, in fact, there is no jammer or, in case there is, the clocks are synchronized and the jamming on-subperiod falls between consecutive triggering time-instants.

**Case (2):** We detect an on-to-off jamming signal transition. That is:

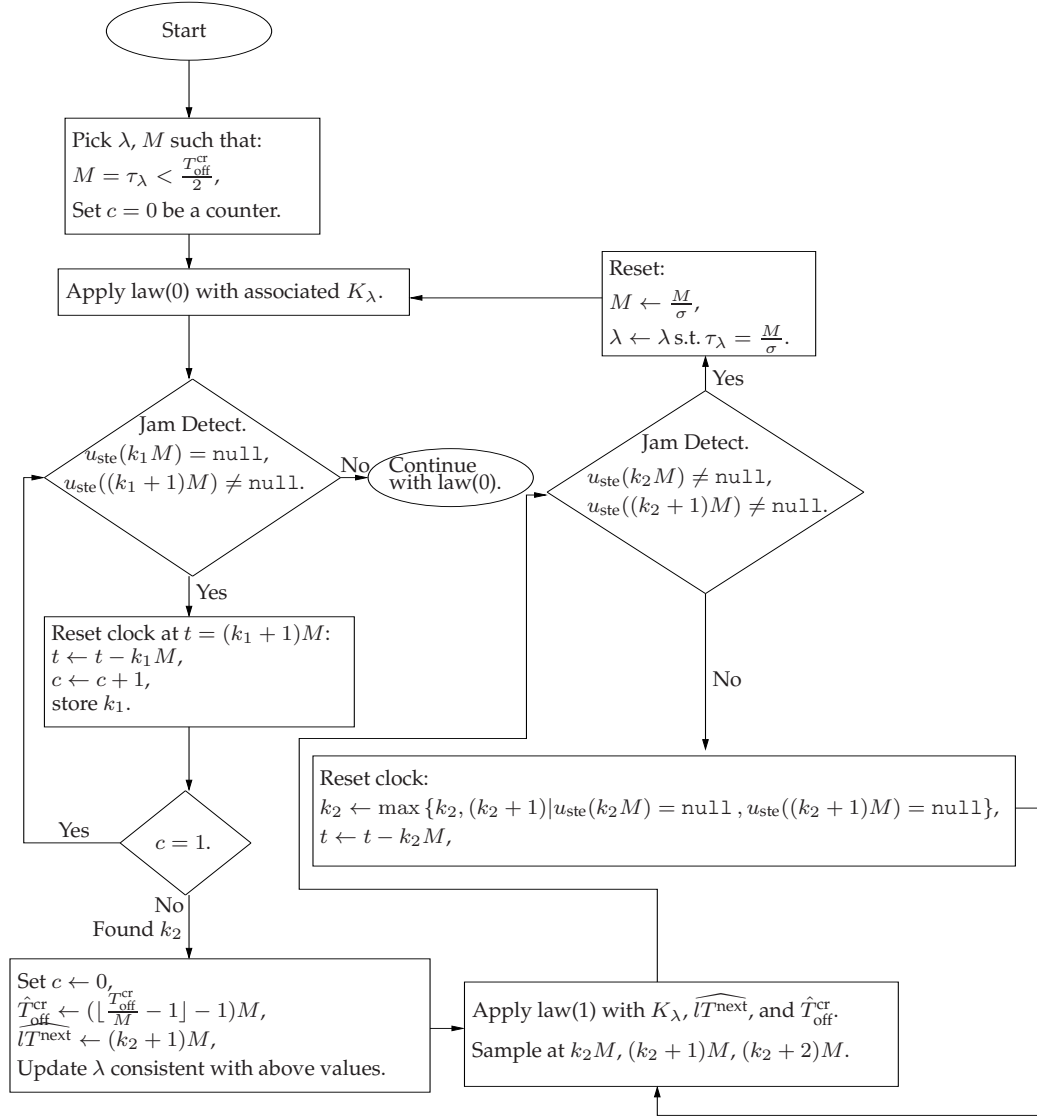
$$\exists k_1 \text{ such that } u_{\text{ste}}(k_1 M) = \text{null} \quad \text{and} \quad u_{\text{ste}}((k_1 + 1)M) \neq \text{null},$$

where recalling the jamming signal shape, the following holds:

$$\exists k_1 \text{ and } l_1 \text{ such that } k_1 M < T_1^0 + l_1 T \leq (k_1 + 1)M. \quad (3.16)$$

**Step II:** After detecting the jammer is on, the operator applies a first clock reset so that the clock time difference is upper bounded by the sampling time,  $M$ . In formal words, at  $t = (k_1 + 1)M$ , we reset  $t \leftarrow t - k_1 M$ . Let us denote  $T_2^0 = T_1^0 + l_1 T - k_1 M$ , then by (4.10) it holds that:

$$0 < T_2^0 \leq M. \quad (3.17)$$



**Figure 3.3:** Flowchart of JAMCOID FOR PERIODIC SIGNALS Algorithm

**Step III:** The operator repeats the strategy described in **Step I** to obtain a first estimate of when the jammer changes activity from on to off. This gives some information on  $T$  which will be used later to limit communications. Again, two cases are possible. In **Case (2)**, we detect an on-to-off signal transition. Let  $k_2 \in \mathbb{N}$  such that:

$$\exists k_2 \text{ such that } u_{\text{ste}}(k_2 M) = \text{null} \quad \text{and} \quad u_{\text{ste}}((k_2 + 1)M) \neq \text{null},$$

where recalling the jamming signal shape, the following holds:

$$\exists k_2 \text{ and } l_2 \text{ such that } k_2 M < T_2^0 + l_2 T \leq (k_2 + 1)M. \quad (3.18)$$

**Step IV:** A new clock reset is applied, to maintain the time offset bounded by the sampling time  $M$ . With additional information,  $T^n = nT$ , the operator will have an estimate of when the jamming signal passes from on- to off-subperiod. This will be helpful to limit the amount of communications used to probe the jammer. In formal words, at time-instant,  $t = (k_2 + 1)M$ , the operator resets the clock as  $t \leftarrow t - k_2 M$ . Further, denote  $T_3^0 = T_2^0 + l_2 T - k_2 M$ , according to Equation (4.12), we get:

$$0 < T_3^0 \leq M, \quad (3.19)$$

where, additionally, it can be proven that:

$$(k_2 - 1)M < T_3^0 + l_2 T \leq (k_2 + 2)M, \quad (3.20)$$

**Step V:** Let  $\tilde{l} = \lfloor \frac{T_{\text{off}}^{\text{cr}}}{M} \rfloor$  and consider the time-interval  $[M, \tilde{l}M]$ . Since  $0 < T_3^0 \leq M$ , from definition of  $\tilde{l}$ ,  $\tilde{l}M \leq T_{\text{off}}^{\text{cr}}$  follows. Also, communication with the plant is feasible at any time in  $[M, \tilde{l}M]$ . Hence,  $[M, \tilde{l}M]$  can play the role of  $[0, T_{\text{off}}^{\text{cr}}]$  in the known jammer scenario. From (4.13), note that  $(k_2 + 2)M$  is a valid upper-bound for the unknown parameter  $T_3^0 + l_2 T$ . Thus, we estimate  $l_2 T$  by  $(k_2 + 2)M$ . In addition, provided these information, we compute  $\hat{T}_{\text{off}}^{\text{cr}} = (\tilde{l} - 1)M$ ,  $\hat{T}_{\text{on}}^{\text{cr}} = (k_2 + 2)M - (\tilde{l} - 1)M$ , and plug these parameters back into Equation (3.15) for  $C(n, \lambda)$ , and retrieve the proper  $\lambda^*$  for which  $C(n, \lambda^*) < 1$ ; accordingly, we update  $K_\lambda \leftarrow K_{\lambda^*}$ . We then keep updating the control at time-instants given by the following triggering strategy:

$$t_k \in \{lM \mid lM \in [M, \tilde{l}M]\} \cup \{(k_2 + 2)M\}, \quad \forall \lambda \in \mathbb{R}_{>0}. \quad (3.21)$$

In addition to communicating with plant at time-instants declared in (4.14), the operator sets  $u_{\text{id}}(k_2 M) = 1$  and  $u_{\text{id}}((k_2 + 1)M) = 1$ , and gathers  $u_{\text{ste}}(k_2 M)$ ,  $u_{\text{ste}}((k_2 + 1)M)$ . To detect the transition from on- to off-subperiod, we consider the following cases: **Case (1):** It holds that  $u_{\text{ste}}(k_2 M) \neq \text{null} \neq u_{\text{ste}}((k_2 + 1)M)$ . Thus, the operator does not detect the jammer's on-to-off transition from  $(l_2 -$

1) $T$  to  $l_2T$ . It also means that the length  $(l_2 - 1)T$  on-subperiod, is shorter than  $M$ . Therefore, in this case, we reset  $M \leftarrow \frac{M}{\sigma'}$ , where  $\sigma' \in (1, \infty)$  is a design parameter. We note that, by construction of  $\tau_\lambda$ ,  $\exists \lambda$ , such that  $\tau_\lambda = \frac{M}{\sigma'}$ . We repeat from **Step I**.

**Case (2):** Either  $u_{\text{ste}}(k_2M) = \text{null}$ , or  $u_{\text{ste}}((k_2 + 1)M) = \text{null}$ , or both. In other words, a jammer's on-to-off transition happens from  $(l_2 - 1)T$  to  $l_2T$ . This is characterized by  $\bar{k}M$ , where:

$$\bar{k} = \max\{k_2, k_2 + 1 \mid u_{\text{ste}}(k_2M) = \text{null}, u_{\text{ste}}((k_2 + 1)M) = \text{null}\}.$$

Reset  $k_2 \leftarrow \bar{k}$ ,  $t \leftarrow t - \bar{k}M$ , and  $T_3^0 \leftarrow T_3^0 + l_2T - \bar{k}M$ , for which (4.13) also holds.

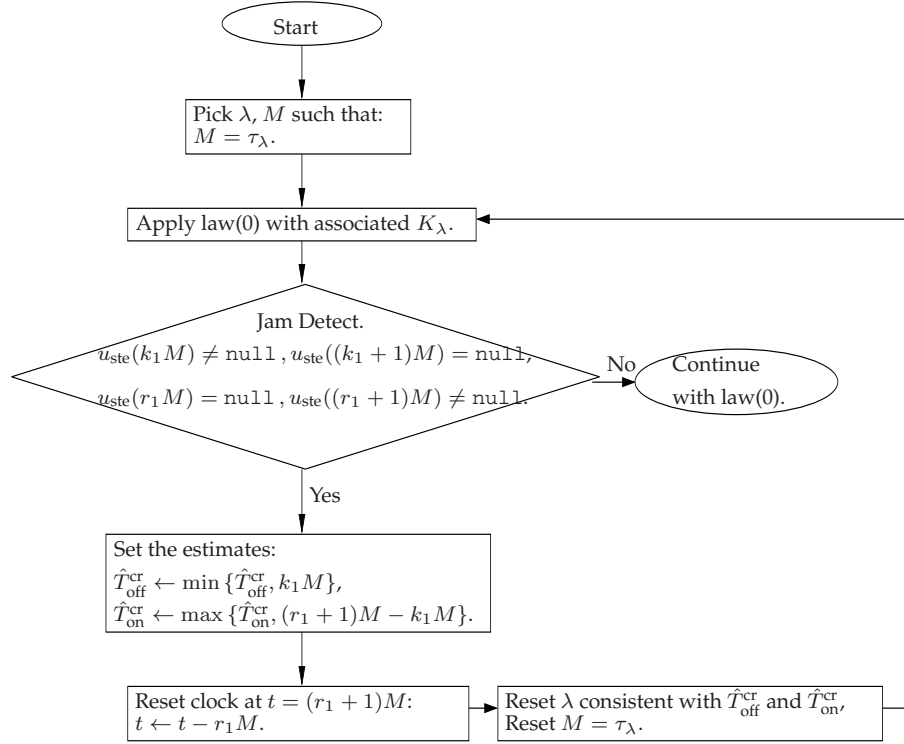
Repeat from **Step V**.

The system asymptotic stability employing JAMCOID FOR PERIODIC SIGNALS, is shown in the next theorem.

**Theorem 3.6.1.** *Consider System (3.4), where  $(A, B)$  is a controllable pair, and a jamming signal (4.2) with constant unknown parameters,  $T$ ,  $T_{\text{off}}^{\text{cr}}$ ,  $T_{\text{on}}^{\text{cr}}$ , and constant known parameter,  $T_{\text{off}}^{\text{cr}}$ . The algorithm JAMCOID FOR PERIODIC SIGNALS renders the system asymptotically stable.*

### 3.6.2 The JAMCOID Algorithm

In order to solve Problem 3, we present here the JAMCOID algorithm. The main idea behind JAMCOID is to generate the triggering time-sequence  $\{t_k\}$  in order to (i) bound the asynchronicity by applying appropriate clock resets, and (ii) find an underestimate of  $T_{\text{off}}^{\text{cr}}$  and an overestimate of all  $\{T_{\text{on}}^{\text{cr},n}\} < \infty$ . In more intuitive words, JAMCOID fulfills these two latter goals by employing a periodic time-triggered strategy which, based on the success or failure of the transmitted signal at every sampling time, computes the proper estimates for the jamming off and on sub-periods and use these estimates to refine its sampling time. The asynchronicity between jammer's and operator's clocks is also bounded by resetting the operator's clock once both estimates have been retrieved. Therefore, this will lead to a conservative algorithm that can handle the power-constrained signal.



**Figure 3.4:** Flowchart of JAMCOID Algorithm

Indeed, as mentioned earlier and as we will be clearer on this point, JAMCOID provides a solution based on a time-triggered control strategy, i.e., under Scenario 2; nonetheless, we shall discuss an extension to an event-triggering control strategy, i.e., under Scenario 2, following it. The algorithm is described in the flowchart of Figure 3.4. In this flowchart, while following the intuitive explanation stated earlier,  $law(0)$  refers to a periodic time-triggered strategy defined by the period  $M = \tau_\lambda$ , and with associated  $K_\lambda$ . Briefly, the following steps are performed:

**Step I:** The operator sends messages to the plant with control content following a periodic triggering strategy. During this phase, we can distinguish between two cases. **Case (1):** We do *never* hit the jammer's on-subperiod, that is,  $u_{ste}(t_k) \neq \text{null}, \forall t_k$ . Thus, we can keep updating our control at the prescribed time-instants,  $t_k = kM$ , without interruption. **Case (2):** We detect an on-to-off



jamming signal transition. That is first we detect:

$$\exists k_1 \text{ such that } u_{\text{ste}}(k_1 M) \neq \text{null} \quad \text{and} \quad u_{\text{ste}}((k_1 + 1)M) = \text{null},$$

and then we detect,

$$\exists r_1 \text{ such that } u_{\text{ste}}(r_1 M) = \text{null} \quad \text{and} \quad u_{\text{ste}}((r_1 + 1)M) \neq \text{null},$$

where recalling the jamming signal shape, the following holds:

$$\exists r_1 \text{ and } s_1 \text{ such that } r_1 M < T_1^0 + T^{s_1} \leq (r_1 + 1)M. \quad (3.22)$$

In addition, the following estimates can be obtained:

$$\hat{T}_{\text{off}}^{\text{cr},1} = k_1 M, \quad \text{and}, \quad \hat{T}_{\text{on}}^{\text{cr},1} = (r_1 + 1)M - k_1 M.$$

**Step II:** After detecting the jammer is on, the operator applies a first clock reset so that the clock time difference is upper bounded by the sampling time,  $M$ . That is, at  $t = (r_1 + 1)M$ , the clock is reset as  $t \leftarrow t - r_1 M$ . Let us denote  $T_2^0 = T_1^0 + T^{l_1} - r_1 M$ , then by (3.22) it holds that  $0 < T_2^0 \leq M$ . In addition, by obtaining the estimates,  $\hat{T}_{\text{off}}^{\text{cr},1}$  and  $\hat{T}_{\text{on}}^{\text{cr},1}$ , we shall find the minimum off-subperiod and maximum on-subperiod that is computed up to this stage of the algorithm. In other words:

$$\hat{T}_{\text{off}}^{\text{cr}} \leftarrow \min \{ \hat{T}_{\text{off}}^{\text{cr},1}, \hat{T}_{\text{off}}^{\text{cr}} \}, \quad \text{and}, \quad \hat{T}_{\text{on}}^{\text{cr}} \leftarrow \max \{ \hat{T}_{\text{on}}^{\text{cr},1}, \hat{T}_{\text{on}}^{\text{cr}} \}.$$

Once we have found the estimates  $\hat{T}_{\text{off}}^{\text{cr}}$  and  $\hat{T}_{\text{on}}^{\text{cr}}$ , we plug the different parameters back into (3.15) for  $C(n, \lambda)$ , and retrieve the proper  $\lambda^*$  for which  $C(n, \lambda^*) < 1$ . We then update  $\tau_\lambda \leftarrow \tau_{\lambda^*}$ ,  $K_\lambda \leftarrow K_{\lambda^*}$ , and go back to **Step I**.

The asymptotic stability of the system, employing JAMCOID, is stated next.

**Theorem 3.6.2.** *Consider System (3.4), where  $(A, B)$  is a controllable pair, and a general jamming signal (4.2) with unknown parameters,  $\{T^n\}$ ,  $\{T_{\text{on}}^{\text{cr},n}\}$ , and  $T_{\text{off}}^{\text{cr}}$ . The algorithm JAMCOID renders the system asymptotically stable.*

The JAMCOID algorithm is based on a time-triggered control strategy, which would be a solution under Scenario 2. In the following remark, however, we discuss an adaptation of this algorithm to deal with Scenario 1, i.e., an event-triggered control strategy.

**Remark 3.6.3.** *The JAMCOID algorithm can be adapted for event-triggered strategies, which then provides a solution under Scenario 1. Let  $\{t_k\}$  be the event-triggered condition as described in Equation (4.7), Lemma 4.5.1. Then, the JAMCOID proposed for time-triggered strategies can be changed as follows:*

1. *In Step I–Case (2), we shall first detect:*

$$\exists k_1 \text{ such that } u_{\text{ste}}(t_{k_1}) \neq \text{null} \text{ and } u_{\text{ste}}(t_{k_1+1}) = \text{null},$$

*and then we detect,*

$$\exists r_1 \text{ such that } u_{\text{ste}}(t_{r_1}) = \text{null} \text{ and } u_{\text{ste}}(t_{r_1+1}) \neq \text{null},$$

*which then implies the following:*

$$\exists r_1 \text{ and } s_1 \text{ such that } t_{r_1} < T_1^0 + T^{s_1} \leq t_{r_1+1},$$

*which is a counterpart to (3.22). This then provides the estimates,  $\hat{T}_{\text{off}}^{\text{cr},1} = t_{k_1}$ , and,  $\hat{T}_{\text{on}}^{\text{cr},1} = t_{r_1+1} - t_{k_1}$ .*

2. *In Step II, the reset is performed at  $t = t_{r_1+1}$  as  $t \leftarrow t - t_{r_1}$ , whereby the estimates are updated as  $\hat{T}_{\text{off}}^{\text{cr}} \leftarrow \min \{\hat{T}_{\text{off}}^{\text{cr},1}, \hat{T}_{\text{off}}^{\text{cr}}\}$ , and,  $\hat{T}_{\text{on}}^{\text{cr}} \leftarrow \max \{\hat{T}_{\text{on}}^{\text{cr},1}, \hat{T}_{\text{on}}^{\text{cr}}\}$ . Then, proper  $\lambda^*$  shall be obtained by resorting to (3.15), by means of which the update on  $K_\lambda \leftarrow K_{\lambda^*}$ , and event-triggered condition (4.7) shall be performed.*

*At last, the proof of this extension can be performed in an exact similar way as in proof of Theorem 4.7.1—this time by resorting to Corollary 3.5.4.*

The following remark shall put in contrast both JAMCOID and JAMCOID FOR PERIODIC SIGNALS algorithms.

**Remark 3.6.4.** *The major difference between JAMCOID and JAMCOID FOR PERIODIC SIGNALS algorithm is that the latter is more economic in terms of communications than the former. In other words, the periodicity property, along with the knowledge of  $T_{\text{off}}^{\text{cr}}$  in JAMCOID FOR PERIODIC SIGNALS, lets us first identify the time-intervals where communications are guaranteed and hence develop a triggering strategy to ensure the stability. Nevertheless, in JAMCOID, because of the non-periodicity of the jamming signal, which prohibits possible predictions on the on-to-off transition time-instants, and lack of knowledge on  $T_{\text{off}}^{\text{cr}}$ , we have to always communicate over the active jamming time-intervals, as well, in order to update our estimates on the minimum of  $T_{\text{off}}^{\text{cr}}$  and  $\max_{n \in \mathbb{N}} \{T_{\text{on}}^{\text{cr},n}\}$  to ensure the system stability. This, hence, prevents economic number of communications.*

## 3.7 Simulations

Having established theoretical results in previous sections, here we demonstrate their functionality on a representative academic example. Hence, we break this section into two parts; first we discuss the known jammer scenario, followed by the unknown jammer scenario.

### 3.7.1 Known Jamming Scenario

We consider the following system:

$$\begin{aligned} \dot{x} &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -3 & -2 & 3 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u, \\ u &= \left[ -\binom{3}{3} \lambda^3 + 3, -\binom{3}{2} \lambda^2 + 2, -3\lambda - 3 \right] x. \end{aligned} \quad (3.23)$$

possesses its only eigenvalue at  $-\lambda$ , with algebraic and geometric multiplicity of 3, and 1, respectively, referring to Lemma 3.4.1. The only linearly independent eigenvector is given by solving the equation  $(A + BK_\lambda + \lambda I)v_1 = 0$  for  $v_1$ . Also, two other generalized eigenvectors, we solve  $(A + BK_\lambda + \lambda I)v_2 = v_1$ , and  $(A +$

---

**Algorithm 1**  $C(\lambda)$ -Seeking
 

---

**Input:** Matrices:  $A$ ,  $B$ , and  $N$ , Sequence:  $\{\lambda_k\}_{k=1}^{N'}$ , Parameters:  $\sigma$ ,  $T_{\text{off}}^{\text{cr}}$  and  $T$ .

- 1: Given controllable pair  $(A, B)$ , compute the proper similarity transformation matrix, and find  $(A_c, B_c)$ —which are in controllable canonical form,
- 2: **for**  $k = 1$  to  $N'$  **do**
- 3: Numerically solve the following ODE, with  $\phi(0) = 0$ :

$$\dot{\phi} = \|A + BK_{\lambda_k}\| + (\|A + BK_{\lambda_k}\| + \|BK_{\lambda_k}\|)\phi + \|BK_{\lambda_k}\|\phi^2,$$

- 4: Find  $\tau_{\lambda_k}$ , such that  $\phi(\tau_{\lambda_k}) = \sigma$ ,
- 5: Compute  $C(\lambda_k)$ , as stated in equation (3.43).
- 6: **end for**

**Output:** Sequences  $\{C(\lambda_k)\}_{k=1}^{N'}$  and  $\{\tau_{\lambda_k}\}_{k=1}^{N'}$ .

---

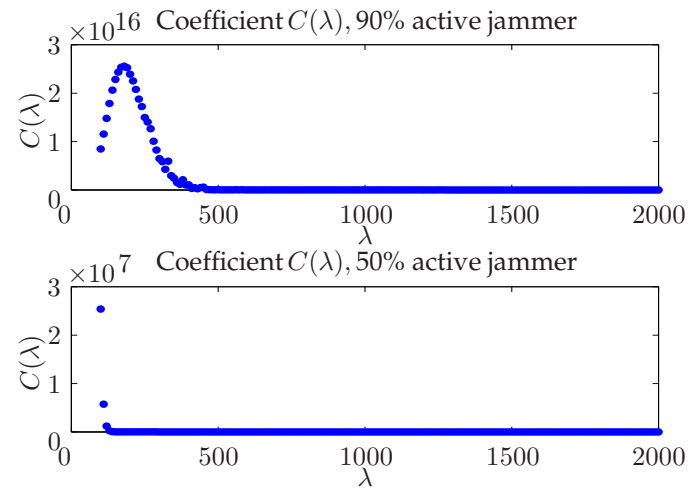
$BK_{\lambda} + \lambda I)v_3 = v_2$  equations. After some algebraic manipulations, we obtain,  $v_1 = \begin{pmatrix} 1 & -\lambda & \lambda^2 \end{pmatrix}^\top$ ,  $v_2 = \begin{pmatrix} \frac{2}{\lambda} & -1 & 0 \end{pmatrix}^\top$ ,  $v_3 = \begin{pmatrix} \frac{3}{\lambda^2} & -\frac{1}{\lambda} & 0 \end{pmatrix}^\top$ . Hence,  $T_{\lambda} = [v_1, v_2, v_3]$ .

In order to perform the simulation, we have chosen  $\sigma = 0.1$ ,  $T = 1$  sec, thus  $T^n = nT$ ,  $T_{\text{on},1}^{\text{cr}} = 0.9T$ ,  $T_{\text{off},1}^{\text{cr}} = 0.1T$ , and  $T_{\text{on},2}^{\text{cr}} = 0.5T$ ,  $T_{\text{off},2}^{\text{cr}} = 0.5T$ . We note  $T_{\text{on},2}^{\text{cr}} < T_{\text{on},1}^{\text{cr}}$ .

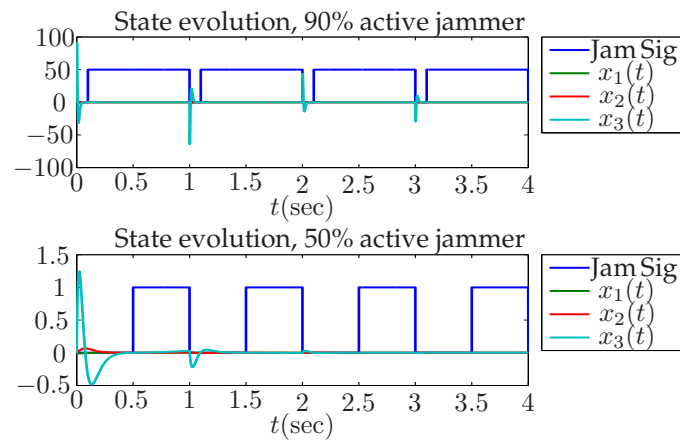
We use the procedure explained in Algorithm 1, to run our simulation. In order to assess the analysis stated in Theorem 4.6.1, we have chosen sequence  $\{\lambda_k = 10k\}_{k=1}^{200}$ . The obtained  $C(\lambda_k)$ -sequences are shown in Figure 3.5, where it confirms  $\lim_{\lambda \rightarrow \infty} C(\lambda) = 0$ . Referring to this figure, we can also list the following remarks.

**Remark 3.7.1.** *Let us define:  $\bar{\lambda} = \min_{1 \leq k \leq N'} \{\lambda_k | \forall \lambda \geq \lambda_k, C(\lambda) < 1\}$ . Then,  $\bar{\lambda}_{90\%} = 1360$  and  $\bar{\lambda}_{50\%} = 210$ . Accordingly, in order to guarantee the asymptotic stability, larger poles are required in the case of 90% active jammer.*

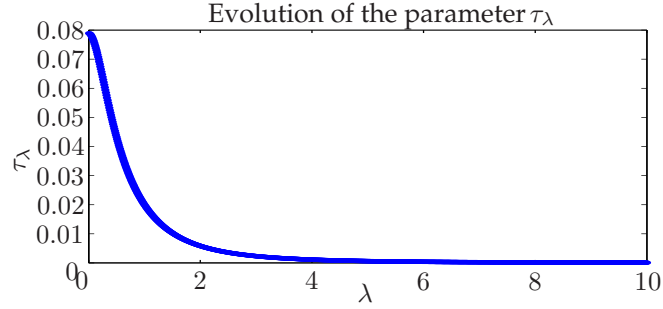
In order to show System (3.23) asymptotic stability, we use  $\bar{\lambda}_{50\%} = 210$ , and  $\bar{\lambda}_{90\%} = 1360$ —introduced in Remark 3.7.1—along with the resilient triggering strategy (3), and the same set of parameters. The temporal evolution of the states is shown in Figure 3.6.



**Figure 3.5:** Third-order system, comparing 90% and 50% active jammers



**Figure 3.6:** Temporal results, demonstrating the stability despite DoS signals



**Figure 3.7:** Third-order system, evolution of  $\tau_\lambda$

Furthermore, on System (3.23), and along the lines of Proposition 4.5.2, we have also conducted a study on parameter  $\tau_\lambda$  evolution. This time,  $\{\lambda_k = 0.01k\}_{k=1}^{1000}$ , and for each  $\lambda_k$ , we ran the procedure explained in Algorithm 1, obtained result is presented in Figure 3.7. This figure confirms our result in Proposition 4.5.2, i.e.,  $\lim_{\lambda \rightarrow \infty} \tau_\lambda = 0$ .

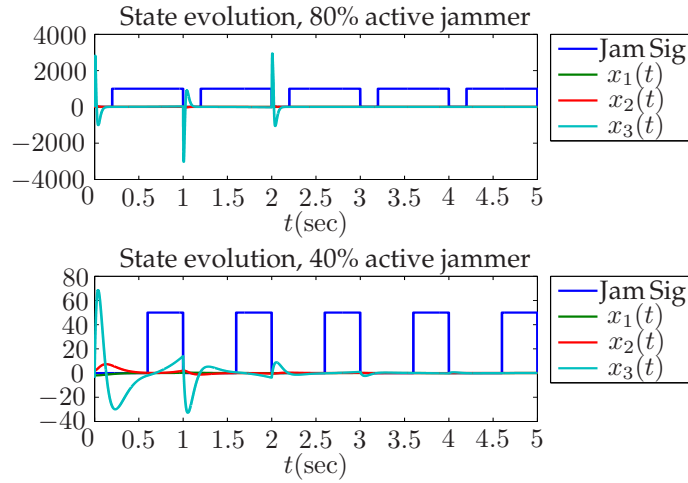
### 3.7.2 Unknown Jamming Scenario

We consider System (3.23) introduced in previous subsection, along with set of parameters  $\sigma = 0.1$ ,  $T = 1$  sec, thus  $T^n = nT$ ,  $T_{\text{on},1}^{\text{cr}} = 0.8T$ ,  $T_{\text{off},1}^{\text{cr}} = 0.2T$ ,  $T_{\text{on},2}^{\text{cr}} = 0.4T$ ,  $T_{\text{off},2}^{\text{cr}} = 0.6T$ , and for initial purposes,  $\lambda_{80\%} = 80$ ,  $\lambda_{40\%} = 14$ . We then run the JAMCOID FOR PERIODIC SIGNALS algorithm,<sup>1</sup> the results as the system state evolution are shown in Figure 3.8. It verifies the asymptotic stability of the system under JAMCOID FOR PERIODIC SIGNALS algorithm.

## 3.8 Conclusions & Future Work

In this chapter, we have considered controllable single-input continuous linear systems subject to power-constrained PWM DoS jamming signals. We have proposed a resilient parameter-dependent control and triggering strategies in three different problem scenarios which guarantee system stability under

<sup>1</sup>Obviously, we note that these aforementioned parameters are required for simulation purposes and not needed in the JAMCOID FOR PERIODIC SIGNALS.



**Figure 3.8:** Temporal results, demonstrating the stability despite DoS signals using JAMCOID FOR PERIODIC SIGNALS algorithm

different assumptions on the knowledge of the jamming signal. The functionality of the theoretical results entailing both partially known and unknown DoS signals has been demonstrated in a simulation environment.

There are several questions that we would like to address in future work. First, the question about how to extend our results to nonlinear systems which are controllable remains. Nonlinearities and the initial system condition will play a role in the definition of the appropriate control laws. In addition, one would have to devise appropriate off-line motion planning algorithms for underactuated systems in order to maintain the system under control during the on periods. Second, although we have also studied a PWM DoS signals characterized by the deterministic sequence  $\{T^n\}$  with variable time-intervals,  $T^n - T^{n-1}$ ; an intriguing question would be what if  $T^n$  are chosen stochastically by the jammer, where the operator is only aware of its probability distribution. In other words, we would like to investigate how to exploit the probability distribution on  $T^n$  to obtain triggering strategies with reduced communications.

## 3.9 Omitted Proofs

### 3.9.1 Proof of Lemma 4.5.1

*Proof.* The proof of this result follows from standard arguments in the event/self-triggering control literature, exploiting the particular structure of  $J_\lambda$ . Let  $\bar{B}_\lambda \triangleq T_\lambda^{-1}BK_\lambda T_\lambda$ . Briefly, the computation of the time-derivative of  $V(x_\lambda)$  leads to the upper bound:

$$\begin{aligned}\dot{V} &= \dot{x}_\lambda^T x_\lambda + x_\lambda^T \dot{x}_\lambda \leq x_\lambda^T (J_\lambda^T + J_\lambda + I)x_\lambda + e_\lambda^T \bar{B}_\lambda^T \bar{B}_\lambda e_\lambda \\ &\leq x_\lambda^T (N^T + N - (2\lambda - 1)I)x_\lambda + e_\lambda^T \bar{B}_\lambda^T \bar{B}_\lambda e_\lambda \\ &\leq -(2\lambda - 1 - 2\|N\|)|x_\lambda|^2 + \|\bar{B}_\lambda\|^2 |e_\lambda|^2.\end{aligned}$$

Hence, for  $\lambda > \|N\| + 1/2$ , and recalling  $V(x_\lambda) = |x_\lambda|^2$ , we conclude that  $V(x_\lambda) = x_\lambda^T x_\lambda$  is an ISS-Lyapunov function for System (4.6). Moreover, let  $\sigma \in (0, 1)$ , and let the time-sequence be given by the times when  $|e_\lambda|^2 \leq \frac{\sigma(2\lambda - 1 - 2\|N\|)}{\|\bar{B}_\lambda\|^2} |x_\lambda|^2$  is violated, it then holds that  $\dot{V} \leq -(1 - \sigma)(2\lambda - 1 - 2\|N\|)|x_\lambda|^2$ . Hence, the event-triggering condition, described by (4.7), guarantees the asymptotic stability of the system.  $\square$

### 3.9.2 Proof of Proposition 4.5.2

*Proof.* Recalling Remark 3.4.5, note that  $\tau_\lambda \leq t_{k+1} - t_k, \forall k \in \mathbb{N}$  holds. Let us, without loss of generality, set  $t_k = 0$ , and denote  $t_\lambda \triangleq t_{k+1}$ . Then,  $0 \leq \tau_\lambda \leq t_\lambda$  holds. In this proof, we shall show  $\lim_{\lambda \rightarrow \infty} t_\lambda = 0$ , which implies both statements in the assertion of this proposition. By construction of (4.7),  $t_\lambda$  is when the following holds:

$$|e_\lambda(t_\lambda)| = \frac{\sqrt{\sigma(2\lambda - 1 - 2\|N\|)}}{\|T_\lambda^{-1}BK_\lambda T_\lambda\|} |x_\lambda(t_\lambda)|. \quad (3.24)$$

On the other hand, according to (4.5),  $BK_\lambda = T_\lambda J_\lambda T_\lambda^{-1} - A$  holds; i.e.,  $T_\lambda^{-1}BK_\lambda T_\lambda = J_\lambda - T_\lambda^{-1}AT_\lambda$ . Hence, (3.24) can be written as follows:

$$|e_\lambda(t_\lambda)| = \frac{\sqrt{\sigma(2\lambda - 1 - 2\|N\|)}}{\|J_\lambda - T_\lambda^{-1}AT_\lambda\|} |x_\lambda(t_\lambda)| \triangleq F(\lambda)|x_\lambda(t_\lambda)|.$$

We continue by presenting the following result.



**Claim 3.9.1.** *It holds that  $\lim_{\lambda \rightarrow \infty} F(\lambda) = 0$ .*

*Proof of Claim 3.9.1:* Recalling  $J_\lambda = -\lambda I + N$ , we rewrite:

$$F(\lambda) = \frac{\sqrt{\sigma(2\lambda - 1 - 2\|N\|)}}{\|(-\lambda I - T_\lambda^{-1}AT_\lambda) - (-N)\|},$$

where by  $\|-\lambda I - T_\lambda^{-1}AT_\lambda\| - \| -N\| \leq \|(-\lambda I - T_\lambda^{-1}AT_\lambda) - (-N)\|$ , we obtain:

$$0 \leq F(\lambda) \leq \frac{\sqrt{\sigma(2\lambda - 1 - 2\|N\|)}}{\|-\lambda I - T_\lambda^{-1}AT_\lambda\| - \| -N\|}.$$

Recalling that the matrix  $N$  does not depend on  $\lambda$ , and  $T_\lambda^{-1}T_\lambda = I$ , we have:

$$0 \leq \lim_{\lambda \rightarrow \infty} F(\lambda) \leq \lim_{\lambda \rightarrow \infty} \frac{\sqrt{2\sigma\lambda}}{\|T_\lambda^{-1}(-\lambda I - A)T_\lambda\|}.$$

On the other hand, for a matrix  $\bar{A} \in \mathbb{R}^{d \times d}$ ,  $\rho(\bar{A}) \leq \|\bar{A}\|$  holds, where  $\rho(\bar{A})$  is the spectral radius of  $\bar{A}$ —for further insight, refer to [12]. Hence, the latter equation can be further bounded as in the following:

$$0 \leq \lim_{\lambda \rightarrow \infty} F(\lambda) \leq \lim_{\lambda \rightarrow \infty} \frac{\sqrt{2\sigma\lambda}}{\rho(T_\lambda^{-1}(\lambda I + A)T_\lambda)}. \quad (3.25)$$

Furthermore, recalling  $T_\lambda^{-1}(\lambda I + A)T_\lambda$  is the similarity transformation of the matrix  $\lambda I + A$ ,  $\rho(T_\lambda^{-1}(\lambda I + A)T_\lambda) = \rho(\lambda I + A)$  holds. Accordingly, we get:

$$0 \leq \lim_{\lambda \rightarrow \infty} F(\lambda) \leq \lim_{\lambda \rightarrow \infty} \frac{\sqrt{2\sigma\lambda}}{\rho(\lambda I + A)}.$$

Now, by means of the Geršgorin disc theorem,

$\rho(\lambda I + A) \in \cup_{i=1}^d D(\lambda + a_{ii}, \sum_{j \neq i} |a_{ij}|)$  holds. This then implies  $\rho(\lambda I + A)$  has a linear growth as  $\lambda$  tends to infinity. Therefore, without loss of generality, we can write

$$0 \leq \lim_{\lambda \rightarrow \infty} F(\lambda) \leq \lim_{\lambda \rightarrow \infty} \frac{\sqrt{2\sigma\lambda}}{\lambda + c} = 0,$$

for some  $c < \infty$  constant. Hence, the result follows. •

**Claim 3.9.2.** *It holds that  $\lim_{\lambda \rightarrow \infty} t_\lambda = 0$ .*

*Proof of Claim 3.9.2:* Recalling (4.7), and by construction of  $t_\lambda$ , we have:

$$|e_\lambda(t_\lambda)| = F(\lambda)|x_\lambda(t_\lambda)|. \quad (3.26)$$

On the other hand, recall that, for  $t \in [0, t_\lambda]$ :  $e_\lambda(t) = T_\lambda^{-1}e(t) = T_\lambda^{-1}(x(t) - x_0)$ , which yields  $e_\lambda(t) = x_\lambda(t) - T_\lambda^{-1}x_0$ . Applying the latter equation on (3.26), bestows:

$$|x_\lambda(t_\lambda) - T_\lambda^{-1}x_0| = F(\lambda)|x_\lambda(t_\lambda)|. \quad (3.27)$$

In order to prove the result, we consider two cases:

**Case (i):**  $|x_\lambda(t_\lambda)| = 0$ . In this case, by (3.27),  $x_0 = 0$  holds. Also, since (4.6) is linear and the control strategy is  $u = Kx_0 = 0$ ,  $x_\lambda(t) = 0$  holds. In other words, there is no need for updating the control strategy,  $t_\lambda = 0$ , which renders  $\tau_\lambda = 0$ .

**Case (ii):**  $|x_\lambda(t_\lambda)| \neq 0$ . In this case, dividing (3.27) by  $|x_\lambda(t_\lambda)|$ , bestows,  $F(\lambda) = \frac{|x_\lambda(t_\lambda) - T_\lambda^{-1}x_0|}{|x_\lambda(t_\lambda)|}$ . Computing the absolute-value of  $F(\lambda)$ , yields:

$$|F(\lambda)| = F(\lambda) = \frac{|x_\lambda(t_\lambda) - T_\lambda^{-1}x_0|}{|x_\lambda(t_\lambda)|} \geq \frac{||x_\lambda(t_\lambda)| - |T_\lambda^{-1}x_0||}{|x_\lambda(t_\lambda)|} \geq 0.$$

Now, according to Claim 3.9.1, we obtain:

$$\lim_{\lambda \rightarrow \infty} \left| 1 - \frac{|T_\lambda^{-1}x_0|}{|x_\lambda(t_\lambda)|} \right| = 0 \Rightarrow \lim_{\lambda \rightarrow \infty} \frac{|x_\lambda(0)|}{|x_\lambda(t_\lambda)|} = 1. \quad (3.28)$$

We will then show  $\lim_{\lambda \rightarrow \infty} t_\lambda = 0$ , using a contradiction argument. Assume  $\lim_{\lambda \rightarrow \infty} t_\lambda$

$\neq 0$ ; it must be that  $\exists t^*$ , such that,  $\forall \lambda > 0, \exists \bar{\lambda} > \lambda$ , such that,  $t_{\bar{\lambda}} > t^*$ . This implies there is a sequence  $\{\lambda_k\}_{k \in \mathbb{N}}$ , where  $\lambda_k \rightarrow \infty$  as  $k \rightarrow \infty$ , and  $t_{\lambda_k} > t^*$ . Because of this, we have  $|x_{\lambda_k}(t_{\lambda_k})| \leq |x_{\lambda_k}(t^*)|$ , which follows from  $\dot{V}(t) < 0$  for  $t < t_{\lambda_k}$ —by the choice of our triggering time-sequence—and that  $V(t) = |x_{\lambda_k}(t)|^2$ . Based on this observation, we derive the following inequality:

$$\lim_{k \rightarrow \infty} \frac{|x_{\lambda_k}(0)|}{|x_{\lambda_k}(t^*)|} \leq \lim_{k \rightarrow \infty} \frac{|x_{\lambda_k}(0)|}{|x_{\lambda_k}(t_{\lambda_k})|}.$$

From Lemma 4.5.1, and the fact that  $t^* < t_{\lambda_k}$ , it holds:

$$|x_{\lambda_k}(t^*)| \leq |x_{\lambda_k}(0)| \exp(-(1 - \sigma)(2\lambda_k - 1 - 2\|N\|)t^*/2),$$

by recalling that  $V$  is an ISS Lyapunov function for System (4.6), and applying the comparison principle, this latter equation further yields:

$$\frac{|x_{\lambda_k}(0)|}{|x_{\lambda_k}(t^*)|} \geq \exp((1 - \sigma)(2\lambda_k - 1 - 2\|N\|)t^*/2).$$

As  $\sigma \in (0, 1)$ , by properly letting  $\lambda_k \geq 1/2 + \|N\|$  tend to infinity, we obtain:

$$\lim_{k \rightarrow \infty} \frac{|x_{\lambda_k}(0)|}{|x_{\lambda_k}(t_{\lambda_k})|} \geq \lim_{k \rightarrow \infty} \frac{|x_{\lambda_k}(0)|}{|x_{\lambda_k}(t^*)|} = \infty,$$

which is in contradiction with (3.28). Therefore, it must be that  $\lim_{\lambda \rightarrow \infty} t_\lambda = 0$ .

Henceforth, from both **Case (i)** and **Case (ii)**, the proof of this claim follows. •

The proof of the proposition follows from the application of both claims. □

### 3.9.3 Proof of Theorem 4.6.1

*Proof.* We shall focus on the first jamming time-interval, i.e.,  $0 \leq t \leq T^1$ . For the sake of brevity, we drop  $n = 1$  in the  $t_{k,n}^*$  annotation. Without loss of generality, let  $t_k^* = k\tau_\lambda$ , for  $k \in \{1, \dots, m\}$ , be the time-sequence generated by (4.8), where  $m$  is such that,  $t_m^* = m\tau_\lambda \leq T_{\text{off}}^{\text{cr}} < t_{m+1}^* = (m+1)\tau_\lambda$ . We note that we can always assume this, since according to Proposition 4.5.2, we can make  $\tau_\lambda$  arbitrarily small by choosing  $\lambda$  large enough. As  $\tau_\lambda > 0$ , we get  $m \leq \frac{T_{\text{off}}^{\text{cr}}}{\tau_\lambda} < m+1$ . Thus,  $\lfloor \frac{T_{\text{off}}^{\text{cr}}}{\tau_\lambda} \rfloor = m$ , where  $\lfloor \cdot \rfloor$  is the floor operator, and

$$t_m^* = \left\lfloor \frac{T_{\text{off}}^{\text{cr}}}{\tau_\lambda} \right\rfloor \tau_\lambda. \quad (3.29)$$

It is easy to see that for all  $a > 0$ , if  $\lfloor a \rfloor \geq 1$ , then  $\lfloor a \rfloor \geq a/2$ . Based on this observation, and as  $\frac{T_{\text{off}}^{\text{cr}}}{\tau_\lambda} \geq 1$ , then  $\lfloor \frac{T_{\text{off}}^{\text{cr}}}{\tau_\lambda} \rfloor \geq \frac{T_{\text{off}}^{\text{cr}}}{2\tau_\lambda}$  holds, which by (3.29), bestows:

$$t_m^* = \left\lfloor \frac{T_{\text{off}}^{\text{cr}}}{\tau_\lambda} \right\rfloor \tau_\lambda \geq \frac{T_{\text{off}}^{\text{cr}}}{2}. \quad (3.30)$$

The rest of the proof goes over the following steps:

1. We break  $[0, T^1]$  into two subintervals,  $[0, t_{m+1}^*]$  and  $[t_{m+1}^*, T^1]$ ; that is, the time-intervals during which the jammer is *inactive* and *active*, respectively,
2. Then, in order to find an estimate for  $|x(t_{m+1}^*)|$ , and  $|x(T^1)|$ , we first transform the original system into new coordinates by the matrix  $T_\lambda$ ; we perform some computations, and transform it back into its original coordi-

nates, by  $T_\lambda^{-1}$ —this is done for each subinterval  $[0, t_{m+1}^*]$ , and  $[t_{m+1}^*, T^1]$ . In this way, the analysis becomes more tractable,

3. Finally, the theorem conclusion will follow by studying the coefficient  $C(n, \lambda)$ , appearing in  $|x(T^n)| < C(n, \lambda)|x(T^{n-1})|$ , and as characterized in (3.15). On this way, we let  $C(1, \lambda) = C(\lambda)$ , we then show that  $\lim_{\lambda \rightarrow \infty} C(\lambda) = 0$ , whereby the existence of a  $\lambda_1^*$  satisfying  $C(\lambda_1^*) < 1$  can be deduced. Then, this latter implication infers that  $\lim_{\lambda \rightarrow \infty} C(n, \lambda) = 0$ , for every  $n \in \mathbb{N}$ ; therefore, it guarantees the existence of  $\lambda_n^*$  satisfying  $C(n, \lambda_n) < 1$ , for  $\lambda_n \geq \lambda_n^*$ , and for each  $n \in \mathbb{N}$ . This fact, accordingly, shows  $\{|x(T^n)|\}$  is a strictly decreasing sequence, which then by using a Lyapunov argument proves the asymptotic stability.

Let us consider the transformed system (4.6) and times  $t \in [0, t_{m+1}^*]$ . We observe that, according to Remark 3.4.5, the event (4.7) introduced in Lemma 4.5.1 holds, and that also  $V(x_\lambda) = x_\lambda^T x_\lambda = |x_\lambda|^2$  is an ISS-Lyapunov function. Hence, resorting to the proof of this result, the following inequality holds,  $\forall t \in [0, t_{m+1}^*]$ :

$$\dot{V}(x_\lambda) \leq -(1 - \sigma)(2\lambda - 1 - 2\|N\|)|x_\lambda|^2 = -(1 - \sigma)(2\lambda - 1 - 2\|N\|)V(x_\lambda).$$

The latter equation, by applying comparison principle, yields  $V(x_\lambda) \leq V(x_\lambda(0)) \times \exp(-(1 - \sigma)(2\lambda - 1 - 2\|N\|)t)$ , which then, recalling  $V(x_\lambda) = x_\lambda^T x_\lambda = |x_\lambda|^2$ , yields:

$$|x_\lambda(t)| \leq |x_\lambda(0)| \exp(-(1 - \sigma)(2\lambda - 1 - 2\|N\|)t/2). \quad (3.31)$$

Now, we have to transform the latter equation into original coordinates. First, by using  $x(t) = T_\lambda x_\lambda(t)$ :

$$\lambda_{\min}((T_\lambda^{-1})^T(T_\lambda^{-1}))|x|^2 \leq |x_\lambda|^2 \leq \|T_\lambda^{-1}\|^2|x|^2. \quad (3.32)$$

The latter equation is obtained noting that (i)  $|x_\lambda|^2 = x^T(T_\lambda^{-1})^T(T_\lambda^{-1})x$ , and (ii) the matrix  $(T_\lambda^{-1})^T(T_\lambda^{-1})$  is a positive-definite symmetric matrix.

According to (3.32), Equation (3.31) implies:

$$|x(t_m^*)| \leq \frac{\|T_\lambda^{-1}\| \exp(-(1-\sigma)(2\lambda-1-2\|N\|)t_m^*/2)}{\sqrt{\lambda_{\min}((T_\lambda^{-1})^T(T_\lambda^{-1}))}} |x_0|, \quad (3.33)$$

which is computed for  $t = t_m^*$ .

In an analogous way, this time considering  $t \in [t_m^*, t_{m+1}^*]$ , we can obtain the following result:

$$|x(t_{m+1}^*)| \leq \frac{\|T_\lambda^{-1}\| \exp(-(1-\sigma)(2\lambda-1-2\|N\|)\tau_\lambda/2)}{\sqrt{\lambda_{\min}((T_\lambda^{-1})^T(T_\lambda^{-1}))}} \times |x(t_m^*)|, \quad (3.34)$$

where we note that  $\tau_\lambda$  appears, as by our resilient triggering strategy,  $t_{m+1}^* - t_m^* = \tau_\lambda$ .

The following derivations will be devoted to obtain a bound for  $|x(T^1)|$ , applying those for  $|x(t_m^*)|$  and  $|x(t_{m+1}^*)|$  found in (3.33) and (3.34). Let us consider the transformed system (4.6), once more. We consider the time-interval  $[t_{m+1}^*, T^1]$ , then  $e_\lambda(t) = x_\lambda(t_m^*) - x_\lambda(t)$  and so an equivalent form of (4.6) can be written as:

$$\dot{x}_\lambda = T_\lambda^{-1}AT_\lambda x_\lambda + T_\lambda^{-1}BK_\lambda T_\lambda x_\lambda(t_m^*), \quad \forall t \in [t_{m+1}^*, T^1].$$

Solving this dynamics for the initial condition  $x_\lambda(t_{m+1}^*)$ , we obtain the following:

$$x_\lambda(t) = \exp((t - t_{m+1}^*)T_\lambda^{-1}AT_\lambda)x_\lambda(t_{m+1}^*) + \int_{t_{m+1}^*}^t \exp((t-s)T_\lambda^{-1}AT_\lambda)T_\lambda^{-1}BK_\lambda T_\lambda x_\lambda(t_m^*) ds, \quad (3.35)$$

which holds for  $t \in [t_{m+1}^*, T^1]$ . In order to further simplify the latter equation, we use the fact that for a given matrix  $A \in \mathbb{R}^{d \times d}$ , and invertible matrix  $T \in \mathbb{R}^{d \times d}$ ,  $\exp(T^{-1}AT) = T^{-1}\exp(A)T$  holds. Hence, Equation (3.35) is simplified as follows:

$$T_\lambda x_\lambda(t) = \exp((t - t_{m+1}^*)A)T_\lambda x_\lambda(t_{m+1}^*) + \int_{t_{m+1}^*}^t \exp((t-s)A)BK_\lambda T_\lambda x_\lambda(t_m^*) ds,$$

and, using  $x = T_\lambda x_\lambda$  to transform it back into the original dynamics, yields:

$$x(t) = \exp((t - t_{m+1}^*)A)x(t_{m+1}^*) + \int_{t_{m+1}^*}^t \exp((t-s)A)BK_\lambda x(t_m^*) ds. \quad (3.36)$$

We upper-bound (3.36), recalling (3.12), whereby  $\|\exp(M)\| \leq \exp(\mu_M)$ , and based on which the following can be derived:

$$|x(t)| \leq |x(t_{m+1}^*)| \exp((t - t_{m+1}^*)\mu_A) + |x(t_m^*)| \|BK_\lambda\| \int_{t_{m+1}^*}^t \exp((t-s)\mu_A) ds.$$

We evaluate the latter equation at  $t = T^1$ , and then solve the integral to obtain:

$$|x(T^1)| \leq |x(t_{m+1}^*)| \exp((T^1 - t_{m+1}^*)\mu_A) + |x(t_m^*)| \frac{\|BK_\lambda\|}{\mu_A} (\exp((T^1 - t_{m+1}^*)\mu_A) - 1). \quad (3.37)$$

Recalling  $T_{\text{on}}^{\text{cr},1} = T^1 - T_{\text{off}}^{\text{cr}}$ , since by construction,  $T_{\text{off}}^{\text{cr}} < t_{m+1}^*$ , we have  $T^1 - t_{m+1}^* < T_{\text{on}}^{\text{cr},1}$ , where in what follows, we use  $T_{\text{on}}^{\text{cr}}$  in lieu of  $T_{\text{on}}^{\text{cr},1}$ . Thus, we can rewrite (3.37):

$$|x(T^1)| \leq |x(t_{m+1}^*)| \exp(T_{\text{on}}^{\text{cr}}\mu_A) + |x(t_m^*)| \frac{\|BK_\lambda\|}{\mu_A} (\exp(T_{\text{on}}^{\text{cr}}\mu_A) - 1). \quad (3.38)$$

Applying now Equation (3.34) on (3.38), we get:

$$\frac{|x(T^1)|}{|x(t_m^*)|} \leq \left( \frac{\|BK_\lambda\|}{\mu_A} (\exp(T_{\text{on}}^{\text{cr}}\mu_A) - 1) + \frac{\exp(-(1-\sigma)(2\lambda - 1 - 2\|N\|)\tau_\lambda/2)}{\|T_\lambda^{-1}\|^{-1} \sqrt{\lambda_{\min}((T_\lambda^{-1})^T(T_\lambda^{-1}))}} \exp(T_{\text{on}}^{\text{cr}}\mu_A) \right). \quad (3.39)$$

Then, combining (3.33) and (3.39), we obtain:

$$\frac{|x(T^1)|}{|x_0|} \leq \left( \frac{\exp(-(1-\sigma)(2\lambda - 1 - 2\|N\|)t_m^*/2)}{\|T_\lambda^{-1}\|^{-1} \sqrt{\lambda_{\min}((T_\lambda^{-1})^T(T_\lambda^{-1}))}} \right) \times \left( \frac{\|BK_\lambda\|}{\mu_A} (\exp(T_{\text{on}}^{\text{cr}}\mu_A) - 1) + \frac{\exp(-(1-\sigma)(2\lambda - 1 - 2\|N\|)\tau_\lambda)}{\|T_\lambda^{-1}\|^{-1} \sqrt{\lambda_{\min}((T_\lambda^{-1})^T(T_\lambda^{-1}))}} \exp(T_{\text{on}}^{\text{cr}}\mu_A) \right). \quad (3.40)$$

We shall use (3.30) to further bound (3.40), which then results in:

$$\begin{aligned} \frac{|x(T^1)|}{|x_0|} &\leq \left( \frac{\exp(-(1-\sigma)(2\lambda-1-2\|N\|)T_{\text{off}}^{\text{cr}}/4)}{\|T_\lambda^{-1}\|^{-1}\sqrt{\lambda_{\min}((T_\lambda^{-1})^T(T_\lambda^{-1}))}} \right) \times \\ &\left( \frac{\|BK_\lambda\|}{\mu_A} (\exp(T_{\text{on}}^{\text{cr}}\mu_A) - 1) + \frac{\exp(-(1-\sigma)(2\lambda-1-2\|N\|)\tau_\lambda)}{\|T_\lambda^{-1}\|^{-1}\sqrt{\lambda_{\min}((T_\lambda^{-1})^T(T_\lambda^{-1}))}} \exp(T_{\text{on}}^{\text{cr}}\mu_A) \right) \\ &\triangleq \hat{C}(\lambda). \end{aligned} \quad (3.41)$$

We hereby note that in (3.41),  $\lambda_{\min}((T_\lambda^{-1})^T(T_\lambda^{-1}))$  is the minimum eigenvalue of matrix  $(T_\lambda^{-1})^T(T_\lambda^{-1})$ , i.e., the smallest root of the characteristic polynomial of this matrix. Now, let  $p(s; \lambda) = 0$  be the characteristic polynomial of  $(T_\lambda^{-1})^T(T_\lambda^{-1})$ , moreover, let  $R_\lambda$  be as defined in (3.14). We note that, (i) because the arrays of  $(T_\lambda^{-1})^T(T_\lambda^{-1})$  depend on  $\lambda$  in a semi-algebraic form, so, by construction, do the coefficients of its characteristic polynomial, and so, regarding  $p(z)$  in Equation (3.13) of Lemma 3.5.1 to be the characteristic polynomial, then resorting to Equation (3.14) stated in Lemma 3.5.1, does the parameter  $R_\lambda$  depend on  $\lambda$  in a semi-algebraic fashion, (ii) according to Lemma 3.5.1, the parameter  $R_\lambda$  lower-bounds all the roots of  $p(s; \lambda) = 0$ , therefore, it lower-bounds  $\lambda_{\min}((T_\lambda^{-1})^T(T_\lambda^{-1}))$  as well, that is:

$$0 < R_\lambda \leq \lambda_{\min}((T_\lambda^{-1})^T(T_\lambda^{-1})). \quad (3.42)$$

Applying Inequality (3.42) on (3.41), bestows:

$$\begin{aligned} \frac{|x(T^1)|}{|x_0|} &\leq \hat{C}(\lambda) \leq \\ &\left( \frac{\exp(-(1-\sigma)(2\lambda-1-2\|N\|)T_{\text{off}}^{\text{cr}}/4)}{\|T_\lambda^{-1}\|^{-1}\sqrt{R_\lambda}} \right) \times \\ &\left( \frac{\|BK_\lambda\|}{\mu_A} (\exp(T_{\text{on}}^{\text{cr}}\mu_A) - 1) + \frac{\exp(-(1-\sigma)(2\lambda-1-2\|N\|)\tau_\lambda)}{\|T_\lambda^{-1}\|^{-1}\sqrt{R_\lambda}} \exp(T_{\text{on}}^{\text{cr}}\mu_A) \right) \\ &\triangleq C(\lambda), \end{aligned} \quad (3.43)$$

in which the upper-bound is indeed  $C(1, \lambda)$  appearing in Equation (3.15) of the theorem statement. In addition, we can note that for the jamming time-interval,  $[T^{n-1}, T^n]$ , and through the same procedure leading in Equation (3.43), we can derive  $C(n, \lambda)$  as appearing in theorem statement.

We shall let  $C(\lambda) = C(1, \lambda)$ , and present now the following result on the coefficient  $C(\lambda)$ .

**Claim 3.9.3.** *In (3.43), it holds:*

$$\lim_{\lambda \rightarrow \infty} C(\lambda) = 0. \quad (3.44)$$

*Proof of Claim 3.9.3:* In order to complete the proof, we shall break  $C(\lambda)$ -expression as  $C(\lambda) = C_1(\lambda)(C_2(\lambda) + C_3(\lambda))$ , where,

$$C_1(\lambda) = \left( \frac{\exp(-(1-\sigma)(2\lambda-1-2\|N\|)T_{\text{off}}^{\text{cr}}/4)}{\|T_\lambda^{-1}\|^{-1}\sqrt{R_\lambda}} \right), C_2(\lambda) = \frac{\|BK_\lambda\|}{\mu_A} (\exp(T_{\text{on}}^{\text{cr}}\mu_A) - 1), \text{ and,}$$

$$C_3(\lambda) = \frac{\exp(-(1-\sigma)(2\lambda-1-2\|N\|)\tau_\lambda)}{\|T_\lambda^{-1}\|^{-1}\sqrt{R_\lambda}} \exp(T_{\text{on}}^{\text{cr}}\mu_A).$$

Then, we shall show  $\lim_{\lambda \rightarrow \infty} C_1(\lambda)C_2(\lambda) = 0$ , and  $\lim_{\lambda \rightarrow \infty} C_1(\lambda)C_3(\lambda) = 0$ . According to (4.5), recalling  $J_\lambda = -\lambda I + N$ , we get  $BK_\lambda = -A + T_\lambda^{-1}(-\lambda I + N)T_\lambda$ , which then results in  $BK_\lambda = -A - \lambda I + T_\lambda^{-1}NT_\lambda$ , and further  $\|BK_\lambda\| = \|-(A + \lambda I) + T_\lambda^{-1}NT_\lambda\|$ . Applying the triangular-inequality, we get  $\|BK_\lambda\| \leq \|-(A + \lambda I)\| + \|T_\lambda^{-1}NT_\lambda\|$ , and further,  $\|BK_\lambda\| \leq \|A\| + |\lambda| + \|T_\lambda^{-1}\| \|N\| \|T_\lambda\|$ .

We shall employ this latter inequality in order to obtain a new upper-bound for  $C_1(\lambda)C_2(\lambda)$ :

$$0 \leq C_1(\lambda)C_2(\lambda) \leq C_1(\lambda) \times \left( \frac{\|A\| + |\lambda| + \|T_\lambda^{-1}\| \|N\| \|T_\lambda\|}{\mu_A} (\exp(T_{\text{on}}^{\text{cr}}\mu_A) - 1) \right). \quad (3.45)$$

In order to show  $\lim_{\lambda \rightarrow \infty} C_1(\lambda)C_2(\lambda) = 0$ , we note that in the upper-bound of (3.45), (i) since  $\lambda > \|N\| + 1/2$ , and  $\sigma \in (0, 1)$ , there exists an exponentially-decaying term in  $C_1(\lambda)$ , (ii) the other terms in the upper-bound of  $C_1(\lambda)C_2(\lambda)$  decay in a semi-algebraic way, that is because matrices  $T_\lambda$ , and  $T_\lambda^{-1}$  depend on  $\lambda$  in a rational way, so the values  $\|T_\lambda\|$ , and  $\|T_\lambda^{-1}\|$  depend on  $\lambda$  in a semi-algebraic form [15], so does  $R_\lambda$ , as discussed earlier prior to developing Equation (3.42), and (iii) by power-constrained nature of the DoS signals and the associated assumption on  $\{T_{\text{on}}^{\text{cr},n}\} < \infty$ , and in particular,  $T_{\text{on}}^{\text{cr}} = T_{\text{on}}^{\text{cr},1} < \infty$ . Therefore, the exponential decay dominates the semi-algebraic one, and so we conclude the upper-bound of  $C_1(\lambda)C_2(\lambda)$  in (3.45), tends to zero, as  $\lambda \rightarrow \infty$ . Henceforth, since the lower-bound of  $C_1(\lambda)C_2(\lambda)$  is zero, then we conclude:

$$\lim_{\lambda \rightarrow \infty} C_1(\lambda)C_2(\lambda) = 0. \quad (3.46)$$



In the following lines, we show  $\lim_{\lambda \rightarrow \infty} C_1(\lambda)C_3(\lambda) = 0$ . First, we note that  $0 \leq \tau_\lambda \leq T^1$ , and  $-(1 - \sigma)(2\lambda - 1 - 2\|N\|) \leq 0$ , therefore, we get:

$$\frac{\|T_\lambda^{-1}\| \exp(-(1 - \sigma)(2\lambda - 1 - 2\|N\|)T^1/2)}{\sqrt{R_\lambda}} \exp(T_{\text{on}}^{\text{cr}}\mu_A) \leq C_3(\lambda) \leq \frac{\|T_\lambda^{-1}\|}{\sqrt{R_\lambda}} \exp(T_{\text{on}}^{\text{cr}}\mu_A).$$

Also,  $C_1(\lambda) > 0, \forall \lambda$ , hence, we multiply the latter inequality by  $C_1(\lambda)$ :

$$\frac{\|T_\lambda^{-1}\| \exp(-(1 - \sigma)(2\lambda - 1 - 2\|N\|)T^1/2)}{\sqrt{R_\lambda} \exp(-T_{\text{on}}^{\text{cr}}\mu_A)} C_1(\lambda) \leq C_1(\lambda)C_3(\lambda) \leq \frac{\|T_\lambda^{-1}\| \exp(T_{\text{on}}^{\text{cr}}\mu_A)}{\sqrt{R_\lambda}} C_1(\lambda). \quad (3.47)$$

Then, we study the limit of upper- and lower-bounds of (3.47). Let us plug  $C_1(\lambda)$ -expression in the lower-bound of (3.47), we obtain:

$$\text{LB}_{C_1C_3}(\lambda) \triangleq \frac{\|T_\lambda^{-1}\|^2 \exp(-(1 - \sigma)(2\lambda - 1 - 2\|N\|)(T^1/2 + T_{\text{off}}^{\text{cr}}/4))}{R_\lambda}.$$

In order to show  $\lim_{\lambda \rightarrow \infty} \text{LB}_{C_1C_3}(\lambda) = 0$ , we recall two facts, (i) since  $\sigma \in (0, 1)$ ,  $\lambda > \|N\| + 1/2$ , then there is an exponentially decaying term in  $\text{LB}_{C_1C_3}(\lambda)$ , (ii) as discussed earlier,  $\|T_\lambda^{-1}\|$  and  $R_\lambda$  depend on  $\lambda$  in a semi-algebraic way dominated by exponential decay.

Having discussed the behavior at infinity of the lower-bound of (3.47), we study the behavior of its upper-bound at infinity. Let us plug the  $C_1(\lambda)$  expression in the upper-bound of (3.47). We then obtain:

$$\text{UB}_{C_1C_3}(\lambda) \triangleq \frac{\|T_\lambda^{-1}\|^2 \exp(-(1 - \sigma)(2\lambda - 1 - 2\|N\|)(T_{\text{off}}^{\text{cr}}/4))}{R_\lambda \exp(-T_{\text{on}}^{\text{cr}}\mu_A)}.$$

Similar to  $\text{LB}_{C_1C_3}(\lambda)$ , it is easy to conclude that  $\lim_{\lambda \rightarrow \infty} \text{UB}_{C_1C_3}(\lambda) = 0$  holds.

In previous paragraphs, we have shown that the limit behavior, as  $\lambda \rightarrow \infty$ , of the lower- and upper-bound of (3.47) is 0. Hence, we infer:

$$\lim_{\lambda \rightarrow \infty} C_1(\lambda)C_3(\lambda) = 0. \quad (3.48)$$

Finally, having demonstrated Equations (3.46) and (3.48), we have proven (3.44). This completes the proof of this claim.  $\bullet$

At this stage, we have proven that in  $|x(T^1)| \leq |x_0|C(\lambda)$ ,  $\lim_{\lambda \rightarrow \infty} C(\lambda) = 0$  holds. Besides, in a similar way, and for every  $n \in \mathbb{N}$ , we can infer  $\lim_{\lambda \rightarrow \infty} C(n, \lambda) = 0$ . The main consequence of this conclusion is:

$$\text{given } \epsilon > 0, \exists \lambda_n^*, \text{ such that } \forall \lambda_n \geq \lambda_n^* \Rightarrow |C(n, \lambda_n)| < \epsilon, \quad (3.49)$$

thereby, in other words, we can arbitrarily tune the *decaying-rate* of the states via  $\lambda$  (and its effect on  $C(n, \lambda)$ ). Therefore, imposing  $\epsilon < 1$ , infers  $C(n, \lambda_n) < 1$  for  $\lambda_n > \lambda_n^*$ . Thus,  $|x(T^n)| < C(n, \lambda_n) < 1|x(T^{n-1})|$ , for  $C(n, \lambda_n) < 1$ , which then in an inductive way infers  $|x(T^n)| < (\prod_{i=1}^n C(i, \lambda_i))|x_0|$  which ensures the system asymptotic stability, i.e.,  $\lim_{n \rightarrow \infty} |x(T^n)| = 0$ , provided that  $\lim_{n \rightarrow \infty} (\prod_{i=1}^n C(i, \lambda_i)) = 0$ , since every  $C(n, \lambda_n) < 1$  and that by construction of  $C(n, \lambda_n)$  in (3.15),  $C(n, \lambda_n) > 0$ .

We would also like to note that a particular case where the Lyapunov function,  $V$  is decreasing at every  $T^n$  time-instant while oscillating with increasing amplitude in between every  $T^{n-1}$  and  $T^n$  is excluded. In what follows, we discuss this latter point resorting to the proof procedure and by considering  $t \in [0, T^1]$  time-interval, where then the conclusion is deduced using inductive argument. We note that (i) the Lyapunov function,  $V(x_\lambda) = x_\lambda^T x_\lambda$ , is a continuous function, provided there is no jump appearing in the state,  $x_\lambda$ , and that moreover,  $V(x_\lambda) > 0$  for every nonzero  $x_\lambda$ , (ii) over the time-interval,  $[0, t_{m+1}^*]$ ,  $\dot{V} < 0$ , and that for  $(t_{m+1}^*, T^1]$  it may hold that  $\dot{V} > 0$ , however, the growth of  $x_\lambda(t)$  characterized by (3.35) is at most exponential and thus no oscillatory behavior may occur, and (iii) we have guaranteed that  $V(T^n) < V(0)$ ; hence it holds that  $\sup_{t \in [0, T^1]} V(x_\lambda) = V(0)$ . Therefore, in an inductive way, it holds that  $\sup_{t \in [T^{n-1}, T^n]} V(x_\lambda(t)) = V(x_\lambda(T^{n-1}))$ . This latter is enough to deduce the fact that  $V$  may not oscillate with increasing amplitude between every  $T^{n-1}$  and  $T^n$ .

□

### 3.9.4 Proof of Corollary 3.5.4

*Proof.* It is sufficient to show that under the resilient triggering strategy (3.10), the analogous equations to (3.30) and (3.34) hold. In order to do so, let us consider the first jamming period. Let us, once more, drop  $n = 1$  in the annotation of  $t_{k,n}^*$ , and let the time-sequence  $\{t_k^*\}_{k=1}^m$  be such that:

$$t_m^* \leq T_{\text{off}}^{\text{cr}} < t_{m+1}^*. \quad (3.50)$$

This is flawless since according to Proposition 4.5.2, we can make  $t_{k+1}^*$  and  $t_k^*$  arbitrarily close, and based on Remark 3.4.5, the sequence  $\{t_k^*\}$  does not accumulate.

On the one hand, according to the definition of resilient triggering strategy (3.10) stated in Definition 3, and Remark 3.4.5,  $t_{k+1}^* - t_k^* \geq \tau_\lambda$ ,  $k \in \{1, \dots, m-1\}$  holds, where, in particular,  $t_1^* - 0 \geq \tau_\lambda$ . Accordingly, we can derive  $t_1^* + \sum_{k=1}^{m-1} t_{k+1}^* - t_k^* \geq m\tau_\lambda$ , which yields  $t_m^* \geq m\tau_\lambda$ . The latter equation, along with (3.50), yields  $m\tau_\lambda \leq t_m^* \leq T_{\text{off}}^{\text{cr}} \leq t_{m+1}^*$ . Given this last inequality, noting  $m\tau_\lambda > 0$ , we can attribute  $\exists L \in \mathbb{N}$ , such that,  $m\tau_\lambda > \frac{T_{\text{off}}^{\text{cr}}}{L}$ , henceforth, we get:

$$t_m^* \geq m\tau_\lambda > \frac{T_{\text{off}}^{\text{cr}}}{L}. \quad (3.51)$$

In fact, (3.51) serves as (3.30), where in the latter,  $L \equiv 2$ .

On the other hand, in an identical way as (3.33) is derived, and for the times  $t \in [t_m^*, t_{m+1}^*]$ , we obtain:

$$|x(t_{m+1}^*)| \leq \frac{\|T_\lambda^{-1}\| \exp(-(1-\sigma)(2\lambda-1-2\|N\|)(t_{m+1}^* - t_m^*)/2)}{\sqrt{\lambda_{\min}((T_\lambda^{-1})^T(T_\lambda^{-1}))}} \times |x(t_m^*)|, \quad (3.52)$$

where recalling  $-(1-\sigma)(2\lambda-1-2\|N\|) < 0$ , and  $t_{m+1}^* - t_m^* \geq \tau_\lambda$ , we further upper-bound (3.52) which yields:

$$|x(t_{m+1}^*)| \leq \frac{\|T_\lambda^{-1}\| \exp(-(1-\sigma)(2\lambda-1-2\|N\|)\tau_\lambda/2)}{\sqrt{\lambda_{\min}((T_\lambda^{-1})^T(T_\lambda^{-1}))}} \times |x(t_m^*)|, \quad (3.53)$$

We note that Equations (3.34) and (3.53) are analogous. Indeed, employing here the same discussion used in Theorem 4.6.1's proof for after (3.34), shall wrap up the proof of this result.  $\square$

### 3.9.5 Proof of Inequality (4.13)

*Proof.* In (4.12), we obtained,  $k_2M < T_2^0 + l_2T \leq (k_2 + 1)M$ , now multiplying this equation by 2 and subtracting  $k_2M + T_2^0$  from it, yields,  $k_2M - T_2^0 < (T_2^0 + l_2T - k_2M) + l_2T \leq (k_2 + 2)M - T_2^0$ . We further upper- and lower-bound this latter equation by (4.11), where we obtain,  $(k_2 - 1)M < (T_2^0 + l_2T - k_2M) + l_2T \leq (k_2 + 2)M$ , where, recalling notation  $T_3^0 = T_2^0 + l_2T - k_2M$ , latter equation yields,  $(k_2 - 1)M < T_3^0 + l_2T \leq (k_2 + 2)M$ , which completes the proof.  $\square$

### 3.9.6 Proof of Theorem 3.6.1

*Proof.* Since we are interested in the system's asymptotic stability, we shall first discuss the asymptotic behavior of the algorithm, whereby we discuss and exclude the other possibilities. The proof is then completed by verifying the stability of each item listed as asymptotic behavior of the algorithm.

We characterize the asymptotic behavior of JAMCOID FOR PERIODIC SIGNALS as one of the following items:

1. **Case (1) in Step I,**
2. **Case (1) in Step III,**
3. **Case (2) in Step V.**

We note that it cannot be otherwise, since **Case (2) in Step I, Step II, Case (2) in Step III,** and **Step IV** are intermediate computations and so cannot be the asymptotic behavior. Moreover, **Case (1) in Step V** is out of sight, as it *cannot* be running indefinitely in the algorithm. This is because repeating this case, with the same parameter  $\sigma'$ , yields the triggering period,  $\frac{M}{\sigma'^n}$ , where given constant  $T_{\text{on}}^{\text{cr}}$ ,  $\sigma' \in (1, \infty)$ , and  $T_{\text{on}}^{\text{cr}} \leq T_{\text{on}}$ . Then, we deduce,  $\exists n^* < \infty \in \mathbb{N}$  such that  $\forall n > n^*$ ,  $\frac{M}{\sigma'^n} < T_{\text{on}}^{\text{cr}}$ . Therefore, in worst case, we shall repeat the **Case (1) in the Step V** only  $n^*$  times.

In order to prove asymptotic stability, we shall study each item. If items 1 and 2 are repeated infinitely often, then the jammer is *not* corrupting the communication channels. Therefore, since the triggering time-sequence is chosen to

be  $k\tau_\lambda$ , with  $k \in \mathbb{N}$ , thus the asymptotic stability is maintained. Moreover, Item 3 leads to the iteration of **Step V** (through **Case (2)**). Stability will follow from the application of Theorem 4.6.1 for each iteration of this item. Specifically, we can approximate  $T \equiv (k_2 + 2)M$ , and choose a  $\lambda^*$  associated with  $(k_2 + 2)M$  which guarantees that the norm of the state decreases in an appropriate manner. This is enough to conclude the JAMCOID FOR PERIODIC SIGNALS algorithm renders the system asymptotically stable.  $\square$

### 3.9.7 Proof of Theorem 4.7.1

*Proof.* Since we are interested in the system's asymptotic stability, we shall first discuss the asymptotic behavior of the algorithm, whereby we discuss and exclude the other possibilities. The proof is then completed by verifying the stability of each item listed as asymptotic behavior of the algorithm.

We characterize the asymptotic behavior of JAMCOID as one of the following items:

1. **Case (1) in Step I,**
2. **Step I (Case (2))-Step II.**

We note that, by construction of the algorithm, it cannot be otherwise.

In order to prove asymptotic stability, we shall study each item. If item 1 is repeated infinitely often, then the jammer is *not* corrupting the communication channels. Therefore, since the triggering time-sequence is chosen to be  $k\tau_\lambda$ , with  $k \in \mathbb{N}$ , thus the asymptotic stability is maintained. Moreover, Item 2 leads to the iteration of **Step I** (through **Case (2)**) and **Step II**. Stability will follow from application of Theorem 4.6.1, provided that at every step and for each iteration of this item, we keep the asynchronicity bounded as in  $0 < T^0 \leq M$ , we find conservative estimates for  $T_{\text{off}}^{\text{cr}}$ , and  $\sup_n \{T_{\text{on}}^{\text{cr},n}\}$ , and that we update  $\tau_\lambda$  with proper  $\lambda^*$  which maintains  $C(n, \lambda^*) < 1$ . This is enough to conclude the JAMCOID algorithm renders the system asymptotically stable.  $\square$

## Chapter 4

# On Multi-Input Controllable Linear Systems Under Unknown Periodic DoS Jamming Attacks

### 4.1 Sumamry

In this chapter, we study remotely controlled and observed multi-input controllable continuous linear systems, subject to periodic Denial-of-Service (DoS) jamming attacks. We first design a control and triggering strategy provenly capable of beating any *partially known* jammer via properly placing the closed-loop poles. Building on it, we then present an algorithm that is able to guarantee the system stability under *unknown* jamming attacks of this class. The functionality of this algorithm is also theoretically proven.

### 4.2 Introduction

Novel developments in the area of sensing and communication technologies have led to the emergence of complex *cyber-physical systems*. As first introduced in [19], cyber-physical systems entail network of physical systems which are remotely controlled and monitored. The advantages of cyber-physical sys-

tems range from ease of implementation to versatile usage in infrastructure facilities [41]. Whilst posing many advantages, they also bear some inherent challenges, including a higher exposure to external attacks. This has resulted in the emergence of an active research on the topic of *system security*, which aims to assess the safety of cyber-physical systems and establish more resilient designs [20, 1].

Indeed, the topic of cyber-physical systems security has been widely appealed within the controls community. To mention a few, in the context of multi-agent systems, [77, 67, 68] aim to identify malicious agents who are part of the network. The main goal of [14, 13] is to maintain group connectivity despite the presence of a malicious agent. Also, within the formation framework, [91] proposes a Receding Horizon Control methodology to deal with a class of deceptive replay attackers inducing system delays. Our problem setup is related to these studies in the way that the jammer has been detected, and the goal is to develop a method to counteract its effect.

The other natural framework to study systems security is Game Theory; to mention a few representative studies, [38, 81, 73]. In these studies, the security problem is formulated as a (dynamic) zero-sum non-cooperative game. In [90], the reinforcement learning technique is employed to beat a deceptive attacker. To the extent of modeling the jammer, the closest work to our studies stated in this chapter are [38, 63], nonetheless, the method exploited to guarantee the stability differs greatly in our chapter since game theoretical framework is not deployed.

In this chapter, we focus on *Denial-of-Service (DoS)* attacks [87, 71], where the attacker aims at dropping the transmitted data. In particular, we narrow our study down to the attacks caused by the so-called *periodic*, or *Pulse-Width Modulated (PWM)* jammers. This type of attack is motivated by the ease of implementation and energy constraints; e.g., see [23, 33].

In particular, we address the problem of system resilience in the context of *triggering control*, i.e., control is updated if required. This is motivated by maintaining the intelligent and economic communications. The recent

works [78, 58, 86] have inspired our research; the distinctive feature in our study is that communication is not always feasible. To cover the globe, [47] addresses the security problem and formulates it in the triggering framework, however, it differs in its attacker model, indeed, [47] considers a class of deceptive attack.

In brief, we first address the problem of partially known DoS attacks caused by PWM jammers on multi-input linear systems to be controlled by sporadic feedback. Then, built on the obtained results, we introduce joint identification and control strategy, JAMCOID, to deal with any unknown DoS jammer of the same class. With respect to our earlier works, [30, 28], the contributions of this note are, (i) the proposal of a parameter-dependent resilient triggering and control strategy for multi-input controllable linear systems, and (ii) the design of JAMCOID algorithm to address unknown periodic DoS PWM jamming attacks.

### 4.3 Problem Formulation

In this section, we state the main problems analyzed in the chapter.

We consider a remote operator-plant setup, where the operator uses a control channel to send wirelessly the control command to an open-loop unstable plant, see Figure 4.1. We assume that the plant has no specific intelligence and is only capable of updating the control based on the data it receives. We also assume that the operator knows the plant dynamics and is able to obtain measurements of its states at particular time-instants.

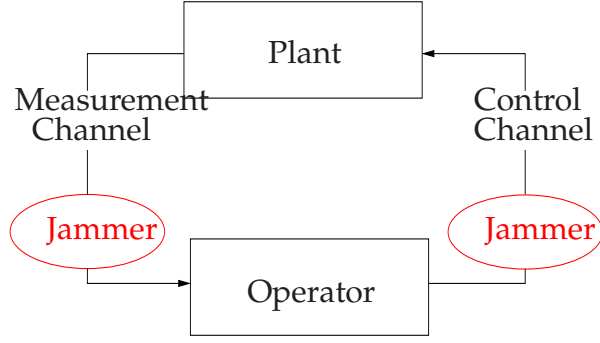
More precisely, consider the following closed-loop dynamics:

$$\dot{x}(t) = Ax(t) + Bu(t), \quad (4.1a)$$

$$u(t) = Kx(t_k), \quad \forall t \in [t_k, t_{k+1}[, \quad (4.1b)$$

where  $x \in \mathbb{R}^d$  is the state vector,  $u \in \mathbb{R}^m$  is the input,  $A$ ,  $B$  and  $K$  are matrices of proper dimensions, and  $\{t_k\}_{k \geq 1}$  is a triggering time-sequence. Here, we also assume that: (i) System (4.1a) is open-loop unstable, and, (ii) the pair  $(A, B)$  is controllable.





**Figure 4.1:** Problem Architecture

We consider an *power-constrained* jammer—causing jamming attack on the control and measurement communication channels—whose signal can be represented as follows:

$$u_{\text{jmd}}(t) = \begin{cases} 0, & (n-1)T \leq t \leq (n-1)T + T_{\text{off}}^{n-1}, \\ 1, & (n-1)T + T_{\text{off}}^{n-1} \leq t \leq nT, \end{cases} \quad (4.2)$$

where  $T \in \mathbb{R}_{>0}$ , and  $n \in \mathbb{N}$ . The sequence  $T_{\text{off}}^n \in \mathbb{R}_{>0}$ ,  $T_{\text{off}}^n < T$ , defines the time-intervals  $[nT, nT + T_{\text{off}}^n]$ , when the jammer is sleeping and communication is possible. We further denote  $T_{\text{on}}^n \in \mathbb{R}_{>0}$ , and,  $[T_{\text{on}}^n, (n+1)T]$  be the time-interval where the jammer is active, thus no data can be sent, and nor the system state can be measured. Accordingly, it holds that  $T_{\text{off}}^n + T_{\text{on}}^n = T$ ,  $n \in \mathbb{N}$ . In this way, the parameter  $T_{\text{off}}^n$  need not be time-invariant which recalls Pulse-Width Modulated (PWM) jamming. Finally, we denote by  $T_{\text{off}}^{\text{cr}}$  a uniform lower-bound for  $T_{\text{off}}^n$ , i.e.,  $T_{\text{off}}^{\text{cr}} \leq T_{\text{off}}^n, \forall n \in \mathbb{N}$ , where also we denote  $T_{\text{on}}^{\text{cr}} \triangleq T - T_{\text{off}}^{\text{cr}}$ .

In this chapter, we shall first assume the type of jammer and the period of jamming signal have been identified, accordingly, we study the system asymptotic stability. Then, we shall address a scenario where the jammer period is not known, we propose a way to tackle this situation. More precisely, we study the following problems:

*[Problem 1]:* Consider any power-constrained jammer described by (4.2) with parameters  $T$  and  $T_{\text{off}}^n$ . Knowing  $T$  and  $T_{\text{off}}^{\text{cr}}$ , design a control and triggering strategy of the form (4.1b) resilient to the action of this jammer.

[Problem 2]: Consider any power-constrained jammer described by (4.2), where also the jammer's and operator's clocks are initially asynchronous by some time,  $t_j$ . Knowing  $T_{\text{off}}^{\text{cr}}$ , propose a method to guarantee the asymptotic stability of the system, despite lack of knowledge on  $T$  and asynchronicity,  $t_j$ .

## 4.4 Background and Preliminary Results

In this section, we briefly discuss the specific canonical form of a multi-input system to be considered in this chapter; it is needed to keep the analysis self-contained and given the fact this form is not unique for this class of systems. The employed technique is inspired from [54, 4], it comes with certain advantages useful in our later analyses. We perform here the explanations to the required extent.

The pair  $(A, B)$  in (4.1) is controllable iff the following matrix is full rank:

$$\Gamma = [B, AB, A^2B, \dots, A^{d-1}B],$$

where,  $\Gamma \in \mathbb{R}^{d \times d.m}$ . Thus, there exist at least  $d$ -linearly independent columns in  $\Gamma$ . The paper [4] describes how to extract these  $d$  columns. Accordingly, [4] derives certain numbers,  $p$ , and  $\{r_i\}_{i=1}^p$ , which define the static similarity transformation matrix,  $T_s$ , to be applied on the system, where it also holds that  $\sum_{i=1}^p r_i = d$ .

Applying this similarity transformation matrix,  $T_s$ , in the following way:

$$\begin{aligned} A &\rightarrow \hat{A} = T_s A T_s^{-1}, & B &\rightarrow \hat{B} = T_s B, \\ K &\rightarrow \hat{K} = T_s^{-1} K, & x &\rightarrow \hat{x} = T_s x, \end{aligned}$$

transforms Dynamics (4.1) into:

$$\dot{\hat{x}}(t) = \hat{A}\hat{x}(t) + \hat{B}u(t), \quad (4.3a)$$

$$u(t) = \hat{K}\hat{x}(t_k), \quad \forall t \in [t_k, t_{k+1}[, \quad (4.3b)$$

where  $\hat{A}$  and  $\hat{B}$ , shall be in a favorite form. In what follows, and with a slight abuse of notation, we denote  $A \equiv \hat{A}$ ,  $B \equiv \hat{B}$ ,  $K \equiv \hat{K}$ , and  $x \equiv \hat{x}$ .



The transformed input-matrix is given by:

$$B = \begin{array}{c} \begin{array}{cc} p & m-p \end{array} \\ \left[ \begin{array}{cccc|cccc} 0 & 0 & \cdot & 0 & \times & \cdot & \cdot & \times \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 1 & \cdot & 0 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & 0 & \cdot & 0 & \times & \cdot & \cdot & \times \end{array} \right]. \end{array}$$

**Lemma 4.4.2** ([4]). *The matrix  $B$  satisfies the following:*

1. *The arrays of  $B$  on its  $j$ -th column, for  $j \in \{1, \dots, p\}$ , are given as follows:*

$$b_{ij} = \begin{cases} 1, & i = d - \sum_{k=1}^{j-1} r_k, \\ 0, & \text{otherwise,} \end{cases}$$

*which recalls the input-matrix of a single-input system in its controllable canonical form,*

2. *The other arrays of  $B$  on its  $j$ -th column, for  $j \in \{p+1, \dots, m\}$ , consist of real values dependent on the original system.*

We now introduce a particular choice of control gains to be applied on the transformed system.

**Proposition 4.4.3.** *Consider System (4.3), let the control matrix  $K_\lambda$  be as follows:*

$$K_\lambda = \begin{array}{c} \begin{array}{cccccccc} 0 & \cdot & 0 & \cdot & 0 & 0 & \cdot & 0 & \kappa_{r_1} & \cdot & \kappa_1 \\ 0 & \cdot & 0 & \cdot & 0 & \lambda_{r_2} & \cdot & \lambda_1 & 0 & \cdot & 0 \\ 0 & \cdot & \mu_{r_3} & \cdot & \mu_1 & 0 & \cdot & 0 & 0 & \cdot & 0 \\ & & & & \cdot & & & & & & \\ & & & & \cdot & & & & & & \\ & & & & \cdot & & & & & & \\ \hline & & & & & & & & \mathbf{0} & & \end{array} \end{array},$$

where the elements on its first three rows are chosen as in:

$$\begin{aligned}\kappa_i &= - \binom{r_1}{i} \lambda^i + \alpha_i, \quad i \in \{1, \dots, r_1\}, \\ \lambda_i &= - \binom{r_2}{i} \lambda^i + \beta_i, \quad i \in \{1, \dots, r_2\}, \\ \mu_i &= - \binom{r_3}{i} \lambda^i + \gamma_i, \quad i \in \{1, \dots, r_3\},\end{aligned}$$

and, similarly, for the other arrays. Then, all the eigenvalues of the closed-loop matrix,  $A + BK_\lambda$ , i.e., the closed-loop system poles, are placed at  $-\lambda$ .

*Proof.* The proof relies on the exploitation of  $A + BK_\lambda$  block diagonal structure, and that every  $i^{\text{th}}$  block is in canonical form. We omit it here.  $\square$

We characterize the algebraic and geometric multiplicities of the eigenvalue  $-\lambda$  in the next result. Beforehand, we also note that the unconventional arrangement of the arrays in matrices  $A$  and  $K_\lambda$  is for the ease and consistency of presentation.

**Proposition 4.4.4.** *Consider System (4.3), along with the control gains stated in Proposition 4.4.3. Also, consider the  $\{1 + \sum_{j=0}^i r_{p-j}\}_{i=-1}^{p-2}$ -th columns of  $A + BK_\lambda$  matrix, and let  $q$  be the number of columns in this sequence with all the zero-elements below its  $\{\sum_{j=0}^i r_{p-j}\}_{i=0}^{p-1}$ -th row. Then, the algebraic multiplicity of  $-\lambda$  is  $d$ , moreover, its geometric multiplicity is  $1 + q$ , where  $1 \leq 1 + q \leq p$ .*

*Proof.* It is easy to see that with this choice of  $K_\lambda$ ,  $\det(sI - (A + BK_\lambda)) = (s + \lambda)^d$  follows. Thus, the algebraic multiplicity of  $-\lambda$  is  $d$ .

Note that the geometric multiplicity of  $-\lambda$  is equal to the kernel of  $A + BK_\lambda + \lambda I$ , given by [12]:

$$\ker(A + BK_\lambda + \lambda I) = d - \text{rank}(A + BK_\lambda + \lambda I). \quad (4.4)$$

For simplicity, let us assume  $p = 2$ . Then, we get:

$$A + BK_\lambda + \lambda I = \begin{array}{c} \begin{array}{cc} & r_2 & & r_1 \\ \left[ \begin{array}{cccc|cccc} \lambda & \cdot & 0 & & 0 & 0 & \cdot & 0 \\ \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & 1 & & 0 & 0 & \cdot & 0 \\ -\lambda^{r_2} & \cdot(-r_2 + 1)\lambda & & & 0 & 0 & \cdot & 0 \\ \hline -m_1 & \cdot & 0 & & \lambda & 1 & \cdot & 0 \\ \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot & \cdot \\ -m_{r_1-1} & \cdot & 0 & & 0 & 0 & \cdot & 1 \\ -m_{r_1} & \cdot & 0 & & -\lambda^{r_1} - r_1 \lambda^{r_1-1} \cdot (-r_1 + 1)\lambda & & & \end{array} \right] & \begin{array}{l} r_2 \\ \\ \\ \\ \\ r_1 \end{array} \end{array} \end{array} .$$

For this matrix, we note, (i) if  $\forall k \in \{1, \dots, r_1\}, m_k = 0$ , then there are  $r_2 - 1$  linearly independent columns in the first  $r_2$ -columns, otherwise, there are  $r_2$ , (ii) there are  $r_1 - 1$  linearly independent columns in the second  $r_1$ -columns of this matrix, and, (iii) the first  $r_1$ -columns cannot influence the linear independence of the second  $r_2$ -columns. Therefore, depending on the values of  $m_k$ , there are either  $r_1 - 1 + r_2 = d - 1$ , or  $r_1 - 1 + r_2 - 1 = d - 2$  linearly independent columns in  $A + BK_\lambda + \lambda I$ . This implies:

$$\text{rank}(A + BK_\lambda + \lambda I) = \begin{cases} d - 2, & \text{if } m_k = 0, \forall k \in \{1, \dots, r_1\}, \\ d - 1, & \text{otherwise.} \end{cases}$$

Let  $q$  be as defined in the proposition statement, then, the last argument attributed for  $p = 2$ , can be also extended, where we conclude:

$$\text{rank}(A + BK_\lambda + \lambda I) = d - 1 - q.$$

Now, plugging the latter equation into (4.4), yields:

$$\ker(A + BK_\lambda + \lambda I) = q + 1,$$

which then implies the geometric multiplicity of  $-\lambda$  is  $q + 1$ . Moreover, by definition of  $q$ , it is at most  $p - 1$  and at least 0, thus  $1 \leq 1 + q \leq p$ . The proof is complete.  $\square$

## 4.5 Jordan Decomposition & Triggering Strategy

In this section, we first present the Jordan decomposition of the closed-loop matrix of System (4.3) where  $K_\lambda$  is chosen as in Proposition 4.4.3. Then, we shall introduce the triggering strategy which solves Problem 1.

According to Proposition 4.4.4, matrix  $A + BK_\lambda$  has at most  $p$  linearly independent eigenvectors, where  $p \leq m < d$ . Thus, this matrix is *not* diagonalizable, this fact motivates us to study its Jordan decomposition. Since the eigenvalues of  $A + BK_\lambda$  are placed at  $-\lambda$ , we have:

$$A + BK_\lambda = T_\lambda J_\lambda T_\lambda^{-1}, \quad (4.5)$$

where,  $J_\lambda = -\lambda I + N$ , and,  $T_\lambda$  is a matrix built upon the linearly independent and generalized eigenvectors of  $A + BK_\lambda$ .

Note that, by Proposition 4.4.4, the geometric multiplicity of  $-\lambda$  is *constant* for all  $\lambda \in \mathbb{R}_{>0}$ . Therefore, matrix  $N$  is *unique* for all values of  $\lambda \in \mathbb{R}_{>0}$ . Moreover, since the arrays of  $A + BK_\lambda$  are polynomial functions of  $\lambda$ , the eigenvectors of  $A + BK_\lambda$  are rational functions of  $\lambda$ . Hence,  $T_\lambda$  and  $T_\lambda^{-1}$  also depend on  $\lambda$  in a rational way. These observations are useful in the stability analysis stated in next section.

Based on this Jordan decomposition technique, we introduce a family of coordinate transformations. Let us consider System (4.3a), with the control,  $u(t) = K_\lambda x(t_k)$ . Then, the closed-loop dynamics is:

$$\dot{x} = (A + BK_\lambda)x + BK_\lambda e,$$

where,  $e(t) = x(t_k) - x(t)$ . Recalling (4.5), the transformations  $e(t) = T_\lambda e_\lambda(t)$ , and,  $x(t) = T_\lambda x_\lambda(t)$  yield:

$$\dot{x}_\lambda = J_\lambda x_\lambda + T_\lambda^{-1} BK_\lambda T_\lambda e_\lambda. \quad (4.6)$$

We state the following result as a first step in developing our triggering strategy.

**Proposition 4.5.1.** *Take  $\lambda > \|N\| + 1/2$  and  $K_\lambda$  as in Proposition 4.4.3. Then,  $V(x_\lambda) = x_\lambda^T x_\lambda$  is an ISS-Lyapunov function for System (4.6) and the event-triggering condition:*

$$|e_\lambda(t)|^2 \leq \frac{\sigma(2\lambda - 1 - 2\|N\|)}{\|T_\lambda^{-1}BK_\lambda T_\lambda\|^2} |x_\lambda(t)|^2, \quad (4.7)$$

*guarantees the asymptotic stability of this system, for  $\sigma \in (0, 1)$ .*

*Proof.* The proof is omitted for space reasons. It follows along the lines of Proposition 4.1 in [28].  $\square$

Let  $t_k$  and  $t_{k+1}$  be two consecutive time-instants given by event-triggering strategy (4.7). Then, for each  $\lambda$ , the following holds:

$$\exists \tau_\lambda > 0, \text{ such that } t_{k+1} - t_k \geq \tau_\lambda, \forall k \in \mathbb{N}.$$

This is based on Theorem III.1, presented in [78]. In particular, [78] shows how to compute such  $\tau_\lambda$ . This implies the time-sequence generated by event-triggering strategy (4.7) does not accumulate. Since in this chapter we do *not* assume the operator can continuously measure the plant states, we adopt this  $\tau_\lambda$  as the basis of our triggering strategy.

**Theorem 4.5.2.** *The parameter,  $\tau_\lambda$ , satisfies the following:*

$$\lim_{\lambda \rightarrow \infty} \tau_\lambda = 0.$$

*Proof.* The proof can be found in Theorem 4.3 in [28]. At a sketch level, the main idea is to use  $\tau_\lambda \leq t_2 - t_1$ , where then letting  $t_1 = 0$ , and denoting  $t_\lambda \triangleq t_2$ , we show  $\lim_{\lambda \rightarrow \infty} t_\lambda = 0$ , which then induces  $\lim_{\lambda \rightarrow \infty} \tau_\lambda = 0$ . In this way, the uniqueness of matrix  $N$ —for all  $\lambda$ —in the Jordan decomposition technique, and, event-triggering condition (4.7) (at which  $t_\lambda$  holds), play important roles.  $\square$

In this chapter, we assume the jammer is causing a “*worst-case jamming scenario*”, i.e.,  $T_{\text{off}}^n = T_{\text{off}}^{\text{cr}}, \forall n \in \mathbb{N}$ . Now, using the parameter  $\tau_\lambda$ , we define our triggering strategy as follows.



**Definition 3.** *The triggering strategy used in this chapter, despite presence of the jammer and to solve Problem 1, is defined as follows:*

$$t_{k,n}^* \in \{l\tau_\lambda \mid l\tau_\lambda \in [(n-1)T, (n-1)T + T_{\text{off}}^{\text{cr}}]\} \cup \{nT\}. \quad (4.8)$$

*In this strategy,  $k \in \mathbb{N}$  denotes the number of triggering time-instants occurring in the  $n^{\text{th}}$  jammer period, and  $l \in \mathbb{N}$  stands for the multiples of  $\tau_\lambda$  starting from  $l = 1$  in the first period and adding up afterwards. We also note that based on Theorem 4.5.2, and for a given  $T$ , we can find a  $\lambda_c$  so that the multiples of  $\tau_\lambda$  lie in the desired interval, i.e., the set introduced in (4.8) is never empty. At last, we also note that for a fixed  $n$ , the largest  $t_{k,n}^*$  is  $nT$  whereas  $t_{1,n+1}^* = nT + \tau_\lambda$ , thus these two time-instants do not coincide.*

## 4.6 Stability Analysis of the Control & Triggering Strategy

Here, we present the main result on the control and triggering strategy which addresses Problem 1.

**Theorem 4.6.1.** *Consider System (4.3), given a jamming signal (4.2) with a known pair  $(T_{\text{off}}^{\text{cr}}, T)$ , then  $\exists \lambda^* > \|N\| + 1/2$ , such that  $\forall \lambda \geq \lambda^*$ , the system with control gain  $K_\lambda$  as chosen in Proposition 4.4.3 and with triggering strategy (4.8) is asymptotically stable.*

*Proof.* The analysis is performed in an analogous way as in the proof of Theorem 5.1 in [28], nonetheless for multi-input systems. At a sketch level, the main idea is to characterize the function  $C(\lambda)$  with the following property:

$$|x(T)| < C(\lambda)|x_0|,$$

and, to further show:

$$\lim_{\lambda \rightarrow \infty} C(\lambda) = 0, \quad (4.9)$$

whereby, the following can be inferred:

$$\exists \lambda^* \text{ such that } \forall \lambda > \lambda^*, C(\lambda^*) < 1.$$

Therefore, by induction argument, we get the sequence  $\{x(nT)\}$  is a strictly decreasing sequence; hence, by a Lyapunov argument, the proof will be completed. On this way, the results explained in Section 4.5, namely, (i) the Jordan decomposition technique, wherein the uniqueness of matrix  $N$  for all values of  $\lambda$  is guaranteed, (ii) the rational dependency of  $T_\lambda$  and  $T_\lambda^{-1}$  matrices on  $\lambda$ , (iii) the ISS Lyapunov function introduced in Proposition 4.5.1, and, (iv) the assertion of Theorem 4.5.2, are extremely helpful. Due to space limits, the details are omitted here.  $\square$

## 4.7 Stabilization under unknown jamming signals

In this section, we propose a solution to Problem 2. It is built on the control and triggering strategy introduced in Section 4.5, along with the stability analysis presented in Section 4.6. First, we shall state our algorithm, and then we analyze the asymptotic stability of the system deploying it.

### 4.7.1 The JAMCOID Algorithm

To begin with, we note that the jammer's and operator's clocks need not be synchronized. Let  $t_j \geq 0$  be this asynchronicity, i.e., the time difference between the jammer clock's initial time and the operator's. We then realize there are three unknown parameters,  $T_{\text{on}}^{\text{cr}}$ ,  $T$ , and  $t_j$ , which characterize the jamming signal, together with the known parameter,  $T_{\text{off}}^{\text{cr}}$ .

Let  $u_{\text{id}} : \mathbb{R}_{\geq 0} \rightarrow \{1\} \cup \{\text{null}\}$  be the signal which operator uses for jammer identification purposes, where  $u_{\text{id}}(t) = 1$  encodes that the operator sends message 1 to the plant, whereas,  $u_{\text{id}}(t) = \text{null}$  declares no message is submitted. Let also  $u_{\text{ste}} : \mathbb{R}_{\geq 0} \rightarrow \{\text{null}\} \cup \mathbb{R}^d$  be the rebound signal from the plant, such that  $u_{\text{ste}}(t) \in \mathbb{R}^d$  is a successfully delivered message containing state information, while  $u_{\text{ste}}(t) = \text{null}$  represents no message is delivered. Finally, let  $u_{\text{ctrl}} : \mathbb{R}^d \rightarrow \{\text{null}\} \cup \mathbb{R}^m$  be the control submitted to the plant, where similar to the  $u_{\text{id}}$ -case,  $u_{\text{ctrl}}(t) \neq \text{null}$  induces that a control  $u_{\text{ctrl}}(t)$  is computed and sent to the plant, whereas  $u_{\text{ctrl}}(t) = \text{null}$  infers that no message is sent.

In fact, we assume that the submission of  $u_{\text{id}}$ , receipt of  $u_{\text{ste}}$ , and submission of  $u_{\text{ctrl}}$  happen in a sequential and instantaneous manner. That is, first a measurement is requested by sending  $u_{\text{id}}$ , then upon its receipt, via  $u_{\text{ste}}$ , a control is sent to the plant, via  $u_{\text{ctrl}}$ . It is nonetheless worth noting that  $u_{\text{ste}}(t) = \text{null}$  if and only if  $u_{\text{ctrl}}(t) = \text{null}$ , i.e., we do *not* send any control if we do *not* receive any measurement, and this happens when the jammer is active at  $t$ .

Intuitively, the core idea behind JAMCOID is to intelligently plan the triggering time-sequence  $\{t_k\}$  in order to (i) bound (not necessarily eliminate) asynchronicity,  $t_j$ , (ii) find a valid useful interval to which  $T$ , or some multiple of this period, belongs. Our JAMCOID algorithm is formally described in the following lines, wherein the control,  $u_{\text{ctrl}}(t_k)$ , is computed as explained in Section 4.4, Proposition 4.4.3.

**Step I:** Set  $u_{\text{id}}(t_k) = 1$ , according to  $t_k = kM$ , where  $k \in \mathbb{N}$ , for  $M = \tau_\lambda < \frac{T_{\text{off}}^{\text{cr}}}{2}$ , and some  $\tau_\lambda$  as introduced in Section 4.5. Because  $T_{\text{on}}^{\text{cr}}$  is unknown, we can distinguish between two cases:

**Case (1):** We do *never* hit the jammer's on-subperiod, that is,  $u_{\text{ste}}(t_k) \neq \text{null}$ ,  $\forall t_k$ . Thus, we keep updating the control at the prescribed times without interruption.

**Case (2):** In this case, we hit the on-subperiod some time on the way. That is:

$$\exists k_1 \text{ such that } u_{\text{ste}}(k_1 M) = \text{null} \text{ and } u_{\text{ste}}((k_1 + 1)M) \neq \text{null},$$

where, recalling the jamming signal, following holds:

$$\exists k_1 \text{ and } l_1 \text{ such that } k_1 M < t_j^1 + l_1 T \leq (k_1 + 1)M. \quad (4.10)$$

If this case occurs, we move on to **Step II**.

**Step II:** At time  $t = (k_1 + 1)M$ , the operator resets his clock as  $t \leftarrow t - k_1 M$ . Let us denote  $t_j^2 = t_j^1 + l_1 T - k_1 M$ , then by (4.10), we obtain:

$$0 < t_j^2 \leq M. \quad (4.11)$$

**Step III:** Similar to **Step I**, we set  $u_{\text{id}}(t_k) = 1$  at  $t_k = kM$ . Again, two cases are possible:

**Case (1):** Same as **Case (1)**, in **Step (1)**.

**Case (2):** In this case, we hit the on-subperiod some time on the way. That is:

$$\exists k_2 \text{ such that } u_{\text{ste}}(k_2M) = \text{null} \text{ and } u_{\text{ste}}((k_2 + 1)M) \neq \text{null} ,$$

where, recalling the jamming signal, following holds:

$$\exists k_2 \text{ and } l_2 \text{ such that } k_2M < t_j^2 + l_2T \leq (k_2 + 1)M . \quad (4.12)$$

If this case occurs, we move on to **Step IV**.

**Step IV:** At time  $t = (k_2 + 1)M$ , the operator resets his clock as  $t \leftarrow t - k_2M$ . Further, let us also denote  $t_j^3 = t_j^2 + l_2T - k_2M$ , by (4.12), we get:

$$0 < t_j^3 \leq M ,$$

where, additionally:

$$(k_2 - 1)M < t_j^3 + l_2T \leq (k_2 + 2)M . \quad (4.13)$$

**Step V:** Let  $\tilde{l} = \lfloor \frac{T_{\text{off}}^{\text{cr}}}{M} \rfloor$  and consider the time-interval  $[M, \tilde{l}M]$ . Since  $0 < t_j^3 \leq M$ , from definition of  $\tilde{l}$ ,  $\tilde{l}M \leq T_{\text{off}}^{\text{cr}}$  follows. Also, communication with the plant is feasible at any time in  $[M, \tilde{l}M]$ . Hence,  $[M, \tilde{l}M]$  plays the role of  $[0, T_{\text{off}}^{\text{cr}}]$  in known jammer scenario; this observation is used in this step.

From (4.13), note that  $(k_2 + 2)M$  is a valid upper-bound for the unknown parameter  $t_j^3 + l_2T$ . Thus we estimate  $l_2T$  by  $(k_2 + 2)M$ . We then keep updating the control at time-instants given by the following triggering strategy:

$$t_k \in \{lM \mid lM \in [M, \tilde{l}M]\} \cup \{(k_2 + 2)M\} , \forall \lambda \in \mathbb{R}_{>0} . \quad (4.14)$$

In addition to communicating with the plant at the time-instants declared in (4.14), the operator sets  $u_{\text{id}}(k_2M) = 1$  and  $u_{\text{id}}((k_2 + 1)M) = 1$ , and obtains  $u_{\text{ste}}(k_2M)$ ,  $u_{\text{ste}}((k_2 + 1)M)$ ; two cases may occur:

**Case (1):**  $u_{\text{ste}}(k_2M) \neq \text{null} \neq u_{\text{ste}}((k_2 + 1)M)$ . Thus, the operator does not detect an on-to-off transition of the jammer's signal from  $(l_2 - 1)T$  to  $l_2T$ . It also means that the length of  $(l_2 - 1)T$  on-subperiod, is shorter than  $M$ . In this case, we reset  $M \leftarrow \frac{M}{\delta}$ , where  $\delta \in (1, \infty)$  is a design parameter. We note that, by construction of  $\tau_\lambda, \exists \lambda$ , such that  $\tau_\lambda = \frac{M}{\delta}$ . Then, repeat from **Step I**.

**Case (2):** Either  $u_{\text{ste}}(k_2M) = \text{null}$ , or  $u_{\text{ste}}((k_2 + 1)M) = \text{null}$ , or both. In other words, an on-to-off transition of the jammer's signal happens from  $(l_2 - 1)T$  to  $l_2T$ . This is characterized by  $\bar{k}M$ , where:

$$\bar{k} = \max\{k_2, k_2 + 1 \mid u_{\text{ste}}(k_2M) = \text{null}, u_{\text{ste}}((k_2 + 1)M) = \text{null}\}.$$

Reset  $k_2 \leftarrow \bar{k}$ ,  $t \leftarrow t - \bar{k}M$ , and  $t_j^3 \leftarrow t_j^3 + l_2T - \bar{k}M$ , for which (4.13) also holds. Then, repeat from **Step V**.

## 4.7.2 The Stability of the JAMCOID Algorithm

Having stated the jammer identification and control algorithm in Subsection 4.7.1, we characterize its convergence properties in this subsection.

**Theorem 4.7.1.** *Consider System (4.3), and a jamming signal described by (4.2), with constant parameters  $T$ ,  $T_{\text{off}}^{\text{cr}}$ , and  $T_{\text{on}}^{\text{cr}}$ , where only  $T_{\text{off}}^{\text{cr}}$  is known. The jammer identification and control algorithm, JAMCOID, renders the system asymptotically stable.*

*Proof.* The asymptotic behavior of JAMCOID is one of the following items:

1. **Case (1) in Step I,**
2. **Case (1) in Step III,**
3. **Case (2) in Step V.**

It cannot be otherwise, since **Case (2) in Step I; Step II; Case (2) in Step III;** and **Step IV** are intermediate computations. Moreover, **Case (1) in Step V** is out of sight, because repeating this case—with the same parameter  $\delta$ —yields

the triggering period,  $\frac{M}{\delta^n}$ , where given  $T_{\text{on}}^{\text{cr}}$  constant,  $\delta \in (1, \infty)$ , and  $T_{\text{on}}^{\text{cr}} \leq T_{\text{on}}$ , then we deduce:

$$\exists n^* < \infty \in \mathbb{N} \text{ such that } \forall n > n^*, \frac{M}{\delta^n} < T_{\text{on}}^{\text{cr}}.$$

Therefore, in worst case, we shall repeat **Case (1)** in **Step V** only  $n^* < \infty$  number of times.

In order to prove asymptotic stability, we assess possible asymptotic behaviors. Under items 1 and 2, the jammer is not corrupting communication channels. Therefore, since the triggering time-sequence is chosen to be  $k\tau_\lambda$ , with  $k \in \mathbb{N}$ , thus the asymptotic stability is maintained.

Item 3 leads to the iteration of **Step V** (through **Case (2)**). Stability will follow from the application of Theorem 4.6.1 for each iteration of this item via approximating  $T \equiv (k_2 + 2)M$ . This completes the proof.  $\square$

## 4.8 Simulations

In this section, we demonstrate the functionality of the aforementioned theoretical results on a representative academic example.

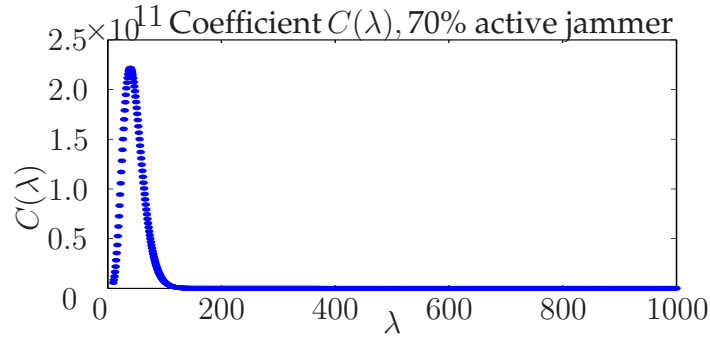
We consider the following system:

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -5 & 7 & 0 & 0 \\ 3 & 0 & 0 & 1 \\ 2 & 0 & 6.5 & 8 \end{bmatrix} x + \begin{bmatrix} 0 & 0 & -6 \\ 0 & 1 & 7.5 \\ 0 & 0 & 8.3 \\ 1 & 0 & 9 \end{bmatrix} u,$$

$$u = \begin{bmatrix} 0 & 0 & -\lambda^2 - 6.5 & -2\lambda - 8 \\ -\lambda^2 + 5 & -2\lambda - 7 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} x,$$

wherein,  $d = 4$ ,  $m = 3$ ,  $p = 2$ ,  $r_1 = 2$ , and  $r_2 = 2$ . For the sake of brevity, we do not introduce the matrices  $A + BK_\lambda$ ,  $T_\lambda$ , and  $N$ , here.

The goal of the simulation is to verify Equation (4.9) stated in the proof sketch of Theorem 4.6.1. In order to do so, we run the  $C(\lambda)$ -Seeking Algorithm



**Figure 4.2:** Fourth-order multi-input system: evolution of  $C(\lambda)$

presented in [28] to obtain the sequence,  $\{C(\lambda_k)\}_{k=1}^{1000}$ , for  $\{\lambda_k = k\}_{k=1}^{1000}$ ; with set of parameters,  $\sigma = 0.001$ ,  $T = 1$  sec,  $T_{\text{on}}^{\text{cr}} = 0.7T$ , and  $T_{\text{off}}^{\text{cr}} = 0.3T$ . The result is presented in Figure 4.2. Referring to this figure, we observe that  $\lim_{\lambda \rightarrow \infty} C(\lambda) = 0$  holds, i.e., (4.9) is verified.

## 4.9 Conclusions & Future Work

In this study, we have considered multi-input controllable continuous linear systems, under periodic PWM DoS jamming attacks. We first recalled a specific canonical form for this class of systems and introduced our control strategy. We then elaborated our triggering strategy, entailing the time-instants to update the control. We then proved this control and triggering strategy is able to beat the considered partially known jamming attacks. Consequently, we proposed JAMCOID algorithm, capable of beating considered unknown jamming attacks.

As future work, we are to extend these results to cope with non-periodic PWM DoS jamming attacks; and to stretch our problem formulation to a multi-agent setup.

## Chapter 5

# On the Robustness of Event-Based Synchronization under Switching Interactions

### 5.1 Summary

In this chapter we study the robustness of an event-triggered synchronization dynamics for a network of identical nodes under various switching scenarios. We first consider an arbitrary switching scenario where, for a general class of isolated node dynamics we characterize sufficient conditions in terms of network topologies to maintain synchronization. In particular, we shall also demonstrate that for a specific class of skew-symmetric isolated node dynamics—which play important role in this class of synchronization problems—the asymptotic synchronization is not achievable under arbitrary switching. We then consider two classes of constrained switching signals, namely uniform and average classes, i.e.,  $\mathcal{S}_{\text{dwell}}[\tau_D]$ , and  $\mathcal{S}_{\text{average}}[\tau_a, N_0]$ , respectively, where we characterize sufficient conditions in terms of the associated parameters,  $\tau_D$ ,  $\tau_a$  and  $N_0$  in order to ensure asymptotic synchronization. This is then followed up by an extensive study on characterization of maintaining a skew-symmetric matrix in the synchronization dynamics and its importance. We shall wrap up our



discussion by presenting relevant simulation studies.

## 5.2 Introduction

Cyber-Physical Systems (CPS) are physical plants which are remotely controlled and monitored via wireless or wired communication channels. Due to the widespread deployment of cps systems in general infrastructure systems, they have gathered significant attention in the past few years. More specifically, amongst various examples of CPS, one may count general example systems such as the *smart power grid*. In nontechnical words, a smart grid entails a number of power generators communicating with each other to produce and supply electric energy to a network of consumers. In more technical words, the dynamics governing the smart grid application is cast under *synchronization* dynamics [43, 7].

The aforementioned synchronization dynamics has been studied mainly under two categories: (i) with identical node (oscillator) dynamics and (ii) with different node (oscillator) dynamics. The major review on synchronization [7] discusses the differences and resemblances between these two classes; in this chapter we shall focus on the the first class of synchronization dynamics, i.e., with identical node dynamics. We note that, as discussed in [43], this class encompasses the dynamics representing a smart grid application—which further motivates the present study.

There exists already a substantial literature within the controls community dedicated to study this specific class of synchronization dynamics. To mention a few, in [69], the authors study the stability of this type of dynamics by introducing a so-called master stability function which characterizes the maximum Lyapunov exponent of the governing dynamics. The papers [76, 89, 52] study this problem in the context of switched systems, where switching amongst different potential network topologies has been considered and, thus, by characterizing the dynamics of the error variable between the network state and the average state, some switching rules have been derived in order to achieve

network synchronization. In these latter studies, it is nonetheless worth noting that the communication is performed in a continuous fashion. In [51, 50] the authors study the synchronization problem in the context of event-triggered dynamics for a fixed network topology, while proposing centralized and distributed event-triggered rules, respectively, to ensure network synchronization. In order to do so, a set-stability technique is exploited.

Hence, there is an apparent gap in the literature in terms of studying synchronization dynamics by considering event-triggered communications and in the context of switched systems. This work aims to close this gap by characterizing sufficient conditions on switched networked topologies and robustness conditions on switching signals that can ensure network synchronization. Indeed, the CPS nature of the smart grid application also motivates this study by economic number of communications governed by considering event-triggered dynamics and the potential unavailability of power generators in a smart grid.

The organization of this chapter is as follows. In Section 5.3, we present the preliminaries, notations and problem formulation. In Section 5.4, we recall the event-triggered rule analyzed in this chapter. In Section 5.5, we present our sufficient conditions in terms of network topologies to ensure synchronization for the case of arbitrary switching. Then, in Section 5.6, we characterize our robustness conditions in terms of switching signals. In Section 5.7, this discussion is then followed up by a study on characterization of the considered event-based synchronization dynamics with a skew-symmetric matrix characterizing its isolated node dynamics. The functionality of these studies is presented in Section 5.8 in a simulation environment. At last, we present in Section 5.9 our conclusions and potential venues for future works.

### **5.3 Preliminaries & Problem Formulation**

In this section, we first present some preliminaries on the synchronization, event-based synchronization and switching concepts; this is then followed by the problems that we have studied in this chapter.

We consider a network of  $N$  identical oscillators, with  $m$  number of possible topologies. Let us consider  $k \in \{1, \dots, m\}$  be the  $k^{\text{th}}$  topology of the network, and  $x_i \in \mathbb{R}^n$ , for  $i \in \{1, \dots, N\}$ , be the states of the  $i^{\text{th}}$  node; then, the dynamics of this node is as follows:

$$\dot{x}_i = Hx_i + c \sum_{j=1}^N a_{ij}^k \Gamma x_j(t_p^k), \quad \forall t \in [t_p^k, t_{p+1}^k[, \quad (5.1)$$

wherein  $H \in \mathbb{R}^{n \times n}$  states the dynamics of each node,  $c > 0$  is the coupling strength, and  $\{t_p^k\}$  is the triggering time-sequence associated to the  $k^{\text{th}}$  topology. Also,  $\Gamma \in \mathbb{R}^{n \times n}$  is the inner-coupling matrix, and  $A^k = [a_{ij}^k] \in \mathbb{R}^{N \times N}$  is the outer-coupling matrix for the  $k^{\text{th}}$  topology. We further assume that each network is undirected, connected, and balanced, which can then be induced that matrices  $A^k$  are symmetric, irreducible, and with zero-sum property, where the last property implies:

$$a_{ii}^k = - \sum_{j=1, j \neq i}^N a_{ij}^k = - \sum_{j=1, j \neq i}^N a_{ji}^k, \quad (5.2)$$

for  $i \in \{1, 2, \dots, N\}$ . Indeed, it can also be observed that in this context, outer-coupling matrix,  $A^k$ , plays the role of the *negative* of Laplacian matrix.

Let also  $\{\lambda_i^k\}_{i \in \{1, \dots, N\}}$  be the set of eigenvalues of the  $k^{\text{th}}$  topology, given the properties of the  $A^k$  matrix, we realize that these eigenvalues are real and that the following holds:

$$\lambda_1^k = 0 > \lambda_2^k \geq \lambda_3^k \cdots \geq \lambda_N^k.$$

In addition, let  $\{\phi_i^k\}_{i \in \{1, \dots, N\}}$ , where  $\phi_i^k \in \mathbb{R}^N$ , be the correspondent set of eigenvectors; then, the following result holds.

**Lemma 5.3.1.** *We recall the set of eigenvectors,  $\{\phi_i^k\}_{i \in \{1, \dots, N\}}$ , of the outer-coupling matrix,  $A^k$ ; then, the following properties hold:*

1.  $\{\phi_i^k\}_{i \in \{1, \dots, N\}}$  can be, without loss of generality, selected to be an orthonormal set of eigenvectors, with  $\phi_1^k = \frac{1}{\sqrt{N}}[1, 1, \dots, 1]^T$  associated to  $\lambda_1^k = 0$  and  $\{\phi_2^k, \dots, \phi_N^k\}$  be such that  $\sum_{j=1}^N \phi_{ij}^k = 0$ , for  $i \in \{2, \dots, N\}$ ,

2. Let  $\Phi^k = [\phi_2^k, \dots, \phi_N^k] \in \mathbb{R}^{N \times N-1}$  then  $\Phi^{k\top} \Phi^k = I_{N-1}$ , and  $\Phi^k \Phi^{k\top} = I_N - \frac{1}{N} 1_N$ , with  $I_N$  and  $1_N$  be, respectively, the identity and matrix of all ones with dimension,  $N$ ,
3. Matrix  $A^k$  is diagonalizable with  $\Phi^{k\top} A^k \Phi^k = \bar{\Lambda}^k = \text{diag} \{\lambda_2^k, \dots, \lambda_N^k\} \in \mathbb{R}^{N-1}$ .

*Proof.* The proof is stated for each item:

1. We recall that matrices  $A^k$  are symmetric, therefore, without loss of generality, the correspondent set of eigenvectors can be selected to be orthonormal. Moreover, we recall the eigenvalue  $\lambda_1^k = 0$ , then the associated eigenvector,  $\phi_1^k$  has to satisfy,  $A^k \phi_1^k = 0$ , which then recalling the zero-sum property—Equation (5.2)— $\phi_1^k = \frac{1}{\sqrt{N}} [1, 1, \dots, 1]^\top$  is a valid eigenvector for  $\lambda_1^k = 0$ . Accordingly, given  $\phi_1^k$  and that  $\{\phi_i^k\}$  is a set of orthonormal eigenvectors,  $\sum_{j=1}^N \phi_{ij}^k = 0$  holds for  $i \in \{2, \dots, N\}$ ,
2. We first check  $\Phi^{k\top} \Phi^k = I_{N-1}$ : we note that  $\Phi^{k\top} \Phi^k = [\langle \phi_i^k, \phi_j^k \rangle] \in \mathbb{R}^{N-1 \times N-1}$  holds for  $i, j \in \{2, 3, \dots, N\}$ , which then by orthonormality of the eigenvalues,  $\langle \phi_i^k, \phi_j^k \rangle = 0$  for  $i \neq j$ , and  $\langle \phi_i^k, \phi_j^k \rangle = 1$  for  $i = j$ , thereby it is easy to verify that  $\Phi^{k\top} \Phi^k = I_{N-1}$ . We then verify  $\Phi^k \Phi^{k\top} = I_N - \frac{1}{N} 1_N$ : let us denote  $U^k \triangleq \Phi^k \Phi^{k\top} \in \mathbb{R}^{N \times N}$ , then the  $i^{\text{th}}$  diagonal element of the matrix  $U^k$  is derived as  $u_{ii}^k = \sum_{j=2}^N \phi_{ji}^k \phi_{ji}^k$  for  $i \in \{2, 3, \dots, N\}$  which, recalling particular form of  $\phi_1^k$  and orthonormality of the eigenvectors, can be further simplified as follows:

$$u_{ii}^k = \sum_{j=1}^N \phi_{ji}^k \phi_{ji}^k - \phi_{1i}^k \phi_{1i}^k = \langle \phi_i^k, \phi_i^k \rangle - \frac{1}{N} = 1 - \frac{1}{N}. \quad (5.3)$$

Moreover, the  $ij^{\text{th}}$  off-diagonal element of  $U^k$  is also derived as  $u_{ij}^k = \sum_{p=2}^N \phi_{pi}^k \phi_{pj}^k$  for  $i, j \in \{2, 3, \dots, N\}$ , which then recalling particular form of  $\phi_1^k$  and orthonormality property, can be further simplified as in:

$$u_{ij}^k = \sum_{p=1}^N \phi_{pi}^k \phi_{pj}^k - \phi_{1i}^k \phi_{1j}^k = \langle \phi_i^k, \phi_j^k \rangle - \frac{1}{N} = -\frac{1}{N}. \quad (5.4)$$

Therefore, by (5.3) and (5.4), we conclude  $\Phi^k \Phi^{k\top} = U^k = I_N - \frac{1}{N} 1_N$ .

3. The proof of this item is straightforward, recalling the algebraic multiplicity of  $\lambda_1^k = 0$  is one and that the matrix  $\Phi^{k\top}$  contains the eigenvectors of  $A^k$  that are correspondent to  $\{\lambda_i^k\}_{i \in \{2,3,\dots,N\}}$ .

The proof is then complete.  $\square$

We then incorporate the notion of Kronecker product,  $\otimes$ , in order to obtain the network dynamics:

$$\dot{x}(t) = (I_N \otimes H)x(t) + c(A^k \otimes \Gamma)x(t_p^k), \quad \forall t \in [t_p^k, t_{p+1}^k[, \quad (5.5)$$

where we recall  $\{t_p^k\}_{p \in \mathbb{N}}$  is the triggering time-sequence associated to the  $k^{\text{th}}$  network topology. In addition, we define the switching signal,  $\sigma : \mathbb{R}_{\geq 0} \rightarrow \{1, 2, \dots, m\}$  in order to declare network topology at the time-instant,  $t$ , therefore, e.g., in (5.5),  $\sigma(t) = k$  declares that the  $k^{\text{th}}$  topology is active where  $k \in \{1, 2, \dots, m\}$ . We assume that the switching signal,  $\sigma(t)$ , is *piecewise constant*, i.e., it has a finite number of discontinuities in any finite time-interval and that it is constant between consecutive discontinuities. We also assume that  $\sigma(t)$  is continuous from above, i.e.,  $\forall t \geq 0, \lim_{s \downarrow t} \sigma(s) = \sigma(t)$ . In addition in this chapter, we shall denote the switching time-instants—which are indeed the discontinuities of  $\sigma(t)$ —to be sequentially  $\tau_j$  for  $j \in \mathbb{N}_0$ .

We shall also define synchronization in the following formal way. Let first  $x(t; x_0) = (x_1(t; x_0)^\top, x_2(t; x_0)^\top, \dots, x_N(t; x_0)^\top)^\top \in \mathbb{R}^{nN}$  be a solution of the network dynamics (5.5) with initial condition  $x_0 = (x_1(t_0)^\top, x_2(t_0)^\top, \dots, x_N(t_0)^\top)^\top$  and some triggering time-sequence  $\{t_p^k\}$  and switching signal  $\sigma(t)$ , the synchronization is then defined as follows that is along the lines of [51].

**Definition 4.** *Let*

$$\mathcal{A}_s = \{x \in \mathbb{R}^{nN} | x_1 = x_2 = \dots = x_N\}, \quad (5.6)$$

*with  $x = (x_1^\top, x_2^\top, \dots, x_N^\top)^\top$ , be the synchronization manifold. If then there exists a  $\delta > 0$  such that the following limit holds*

$$\lim_{t \rightarrow \infty} |x(t; x_0)|_{\mathcal{A}_s} = 0, \quad (5.7)$$

whenever  $|x_0|_{\mathcal{A}_s} < \delta$ , then the network (5.5) is said to achieve local asymptotic synchronization. Moreover, if  $\delta = \infty$ , then global asymptotic synchronization is achieved.

We would like to highlight that in this aforementioned definition,  $|x|_{\mathcal{A}_s}$  states the Euclidean point-to-set distance between the network state vector,  $x$ , and synchronization manifold,  $\mathcal{A}_s$ , defined as follows:

$$|x|_{\mathcal{A}_s} = d(x, \mathcal{A}_s) = \inf_{y \in \mathcal{A}_s} \|x - y\|,$$

with  $\|\cdot\|$  be the Euclidean norm. Therefore, it implies that asymptotic synchronization is achieved if the asymptotic distance to  $\mathcal{A}_s$  vanishes.

We would also like to recall two switching scenarios: (i) arbitrary, and (ii) constrained switching scenarios. More specifically, and for the case of arbitrary switching, the switching signal,  $\sigma(t)$ , does not obey a specific constrained structure, therefore, in other words, the switching time-instants,  $\{\tau_j\}$ , generated by  $\sigma(t)$  do not possess a specific property. On the other hand, for the case of constrained switching, the switching signal,  $\sigma(t)$  has to possess a specific temporal structure. In this chapter, we have considered two classes of constrained switching signals; namely,  $\mathcal{S}_{\text{dwell}}[\tau_D]$ , with  $\tau_D > 0$ , and  $\mathcal{S}_{\text{average}}[\tau_a, N_0]$ , with  $\tau_a, N_0 > 0$ . The class,  $\mathcal{S}_{\text{dwell}}[\tau_D]$ , constitutes of switching signals  $\sigma(t)$  where any two consecutive discontinuities of  $\sigma$  are separated by at least a *dwell time*,  $\tau_D$ , therefore in other words,  $\tau_{j+1} - \tau_j \geq \tau_D, \forall j \in \mathbb{N}_0$ . Moreover, the class,  $\mathcal{S}_{\text{average}}[\tau_a, N_0]$ , contains the switching signals  $\sigma(t)$  for which the following holds:

$$N_\sigma(\tau, t) \leq N_0 + \frac{\tau - t}{\tau_a}, \quad \forall t \geq \tau \geq 0, \quad (5.8)$$

where  $N_\sigma(\tau, t)$  denotes the number of discontinuities of  $\sigma$  in the open interval,  $(t, \tau)$ ; and the constant  $\tau_a$  is called the *average dwell-time* and  $N_0$  is the *chatter bound*.

At last, the problems we have studied in this chapter can be stated as follows:

*[Problem 1]:* Given the network dynamics (5.5), which triggering strategy to be employed in order to generate the triggering time-sequence

$\{t_p^k\}$ ; and in addition, considering arbitrary switching scenario, which sufficient conditions have to be imposed on the network topologies in order to achieve asymptotic synchronization as characterized in Definition 4.

[Problem 2]: Given the network dynamics (5.5), which triggering strategy to be employed in order to generate the triggering time-sequence  $\{t_p^k\}$ ; and in addition, considering constrained switching scenario and under  $\mathcal{S}_{\text{dwell}}$  and  $\mathcal{S}_{\text{average}}$  classes, which sufficient regulatory conditions have to be imposed on the switching signals in order to achieve asymptotic synchronization as characterized in Definition 4.

## 5.4 Single Topology: Lyapunov Function & Triggering Strategy Characterization

In this section, we shall focus on a single topology scenario whereby we first characterize our specific Lyapunov function along with the triggering strategy. We would like to mention that the content of this section is inspired from the results in [51], nonetheless we propose alternative expanded proofs.

We first note that without loss of generality and for the ease of presentation, we shall drop the superscript  $k$  for various variables in this section—this is flawless, provided we focus on a single topology case herein. Let us then recall the matrix of eigenvectors of  $A$ , i.e.,  $\Phi$ , as described and characterized in Lemma 5.3.1. We then introduce  $\bar{\Phi} = \Phi \otimes I_n \in \mathbb{R}^{nN \times n(N-1)}$  whereby we get the following result.

**Lemma 5.4.1.** *Consider network (5.5) and the network state,  $x$ , then the following holds:*

$$\|\bar{\Phi}^\top x\| = |x|_{\mathcal{A}_s}. \quad (5.9)$$

*Proof.* We first recall the notion of Euclidean norm which gives

$$\|\bar{\Phi}^\top x\|^2 = x^\top \bar{\Phi} \bar{\Phi}^\top x = x^\top (\Phi \otimes I_n) (\Phi^\top \otimes I_n) x,$$

which then recalling the properties of Kronecker product and properties of the matrix  $\Phi$  stated in Lemma 5.3.1, yields

$$\|\bar{\Phi}^\top x\|^2 = x^\top (\Phi \Phi^\top) \otimes (I_n) x = x^\top (I_N - \frac{1}{N} \mathbf{1}_N) \otimes I_n x,$$

which can be further expanded as follows

$$\begin{aligned}\|\bar{\Phi}^\top x\|^2 &= x^\top \left( I_{nN} - \frac{1}{N} \mathbf{1}_N \otimes I_n \right) x \\ &= \sum_{i=1}^N \left\| x_i - \frac{1}{N} \sum_{j=1}^N x_j \right\|^2.\end{aligned}\quad (5.10)$$

For notational simplicity, let us also denote  $\bar{x} \triangleq \frac{1}{N} \sum_{j=1}^N x_j$ , then recalling the synchronization manifold,  $\mathcal{A}_s$  as characterized in (5.6), we note that the following holds

$$|x|_{\mathcal{A}_s}^2 = \sum_{i=1}^N \|x_i - \bar{x}\|^2. \quad (5.11)$$

Hence, by (5.10) and (5.11), we note that  $\|\bar{\Phi}^\top x\| = |x|_{\mathcal{A}_s}$  holds, and therefore, the proof is complete.  $\square$

**Remark 5.4.2.** *Indeed, motivated by the previous result, one can define  $y = \bar{\Phi}^\top x$  to be the component of state vector,  $x$ , which evolves transverse to the synchronization manifold. Therefore, resorting to Definition 4, by ensuring  $\lim_{t \rightarrow \infty} \|y(t; y_0)\| = 0$ , we shall ensure (5.7) stated in the latter definition and thus we can ensure the asymptotic synchronization of the original state,  $x$ .*

In the next result and before discussing our triggering strategy, we introduce a proper Lyapunov function to ensure synchronization of the network:

$$\dot{x}_i = Hx_i + c \sum_{j=1}^N a_{ij} \Gamma x_j(t) \quad i \in \{1, 2, \dots, N\}. \quad (5.12)$$

**Proposition 5.4.3.** *If there exist matrices  $P_i = P_i^\top \succ 0 \in \mathbb{R}^{n \times n}$  such that*

$$H_i^\top P_i + P_i H_i \prec 0, \quad i \in \{2, 3, \dots, N\}, \quad (5.13)$$

with  $H_i = H + c\lambda_i\Gamma$ , then

$$V(x) = x^\top \bar{\Phi} P \bar{\Phi}^\top x, \quad (5.14)$$

is a Lyapunov function for the network (5.12), i.e., it ensures its asymptotic synchronization, where  $P = \text{diag}\{P_2, P_3, \dots, P_N\}$  and  $\lambda_i$  are eigenvalues of  $A$  and  $\bar{\Phi} = \Phi \otimes I_n$  obeying the property discussed in Lemma 5.4.1.



*Proof.* Let us first recall dynamics (5.12), where we can derive the network dynamics as in

$$\dot{x} = (I_N \otimes H + c(A \otimes \Gamma))x,$$

where then, along the lines of Remark 5.4.2, we shall transform the dynamics to  $y$ -dynamics with  $y = \bar{\Phi}^\top x$ . This is performed in the following lines.

$$\dot{y} = \bar{\Phi}^\top \dot{x} = (\bar{\Phi}^\top (I_N \otimes H) + c\bar{\Phi}^\top (A \otimes \Gamma))x,$$

where by recalling,  $\bar{\Phi}^\top = \Phi^\top \otimes I_n$ , and properties of Kronecker product, we get

$$\begin{aligned} \dot{y} &= (\Phi^\top \otimes I_n)(I_N \otimes H)x + c(\Phi^\top \otimes I_n)(A \otimes \Gamma)x \\ &= (\Phi^\top I_N \otimes I_n H)x + c(\Phi^\top A \otimes I_n \Gamma)x. \end{aligned} \quad (5.15)$$

In order to further simplify this latter equation, we prove the following claim.

**Claim 5.4.4.** *Consider (5.15), the following holds:*

$$\Phi^\top I_N = I_{N-1} \Phi^\top \quad \text{and} \quad \Phi^\top A = \bar{\Lambda} \Phi^\top, \quad (5.16)$$

with  $\bar{\Lambda} = \text{diag} \{ \lambda_2, \lambda_3, \dots, \lambda_N \}$  be as introduced in Lemma 5.3.1.

*Proof of Claim 5.4.4:* Let us first recall  $\Phi \Phi^\top = I_N - \frac{1}{N} 1_N$  property from Lemma 5.3.1, then multiplying both sides from left by  $\Phi^\top$  yields:

$$\Phi^\top \Phi \Phi^\top = \Phi^\top I_N - \frac{1}{N} \Phi^\top 1_N,$$

where then recalling  $\Phi^\top 1_N = 0_{N-1 \times N}$ —which is induced from the orthonormality property of the set of eigenvectors—and  $\Phi^\top \Phi = I_{N-1}$  as both discussed in Lemma 5.3.1; the latter equation yields  $\Phi^\top I_N = I_{N-1} \Phi^\top$  which proves validity of the first property in (5.16).

Let us then recall  $\Phi^\top A \Phi = \bar{\Lambda}$ —again from Lemma 5.3.1—then multiplying both sides from right by  $\Phi^\top$  yields  $\Phi^\top A \Phi \Phi^\top = \bar{\Lambda} \Phi^\top$ ; where by recalling  $\Phi \Phi^\top = I_N - \frac{1}{N} 1_N$ , we get  $\Phi^\top A (I_N - \frac{1}{N} 1_N) = \bar{\Lambda} \Phi^\top$ . Indeed by recalling the zero-sum property of  $A$ , which governs  $A 1_N = 0$ , this last equation can be further simplified as follows:

$$\Phi^\top A = \bar{\Lambda} \Phi^\top,$$

which proves validity of the second property in (5.16). •

Next, having shown (5.16) in Claim 5.4.4, we can simplify (5.15) in the following way:

$$\begin{aligned}\dot{y} &= (I_{N-1}\Phi^\top \otimes HI_n)x + c(\bar{\Lambda}\Phi^\top \otimes \Gamma I_n)x \\ &= (I_{N-1} \otimes H)(\Phi^\top \otimes I_n)x + c(\bar{\Lambda} \otimes \Gamma)(\Phi^\top \otimes I_n)x,\end{aligned}$$

where recalling  $y = \Phi^\top \otimes I_n x = \bar{\Phi}^\top x$ , the latter dynamics bestows

$$\dot{y} = (I_{N-1} \otimes H)y + c(\bar{\Lambda} \otimes \Gamma)y, \quad (5.17)$$

which given  $\bar{\Lambda} = \text{diag}\{\lambda_2, \lambda_3, \dots, \lambda_N\}$  and by the properties of Kronecker product can be reformulated as follows:

$$\dot{y} = (I_{N-1} \otimes H + c\bar{\Lambda} \otimes \Gamma)y = \text{diag}_{\mathcal{S}_{i \in \{2,3,\dots,N\}}} \{H + c\lambda_i \Gamma\}y, \quad (5.18)$$

where we shall denote  $\mathcal{K} \triangleq \text{diag}_{\mathcal{S}_{i \in \{2,3,\dots,N\}}} \{H + c\lambda_i \Gamma\}$ .

Let us now also recall the Lyapunov function  $V(x) = x^\top \bar{\Phi} P \bar{\Phi}^\top x$ , which under state transformation  $y = \bar{\Phi}^\top x$  can be re-stated as follows:

$$V(y) = y^\top P y. \quad (5.19)$$

Indeed, the rest of this proof is devoted to show that under assumption (5.13), the Lyapunov function (5.19) proves the asymptotic stability of the dynamics (5.18). First, we note that  $P_i = P_i^\top \succ 0$ , therefore  $P = \text{diag}\{P_2, P_2, \dots, P_N\} \succ 0$  which then shows that  $V(y) = y^\top P y$  is a positive-definite function. In addition, given dynamics (5.18), we derive the following:

$$\dot{V}(y) = \dot{y}^\top P y + y^\top P \dot{y} = y^\top (\mathcal{K}^\top P + P \mathcal{K})y,$$

where then recalling the diagonal form of the matrices  $P$  and  $\mathcal{K}$ , we can reformulate the latter equation as follows:

$$\dot{V}(y) = y^\top (\text{diag}_{\mathcal{S}_{i \in \{2,3,\dots,N\}}} \{H_i^\top P_i + P_i H_i\})y, \quad (5.20)$$

with  $H_i = H + c\lambda_i \Gamma$ ; thus, provided (5.20), we note that, under assumption (5.13),  $\dot{V}(y)$  is a negative-definite function. Hence, we conclude that this Lyapunov

function proves asymptotic stability of the transformed dynamics (5.18). Therefore, along the lines of Remark 5.4.2, we can then conclude that Lyapunov function (5.14) proves asymptotic synchronization of the network (5.12) under assumption (5.13). This then completes the proof of this proposition.  $\square$

Having discussed and proven a proper Lyapunov function in the previous proposition, let us recall now the triggered dynamics (5.5) (for the case of single topology):

$$\dot{x}(t) = (I_N \otimes H)x(t) + c(A \otimes \Gamma)x(t_p), \quad \forall t \in [t_p, t_{p+1}[ ,$$

which by introducing the error variable,  $e(t) = x(t_p) - x(t)$ , can be reformulated as follows:

$$\dot{x}(t) = (I_N \otimes H + cA \otimes \Gamma)x + cA \otimes \Gamma e(t), \quad \forall t \in [t_p, t_{p+1}[ . \quad (5.21)$$

The following result characterizes an event-triggering strategy addressing how to generate the time-sequence,  $\{t_p\}_{p \in \mathbb{N}}$ , while maintaining the asymptotic synchronization of the network. We appreciate that the strategy is inspired from [51], nevertheless, we provide a comprehensive proof for this result.

**Proposition 5.4.5.** *Consider the triggered network dynamics (5.21), and assume there exist matrices  $P_i = P_i^\top \succ 0 \in \mathbb{R}^{n \times n}$  such that*

$$H_i^\top P_i + P_i H_i = -2I_n, \quad i \in \{2, 3, \dots, N\}, \quad (5.22)$$

with  $H_i = H + c\lambda_i \Gamma$ , where  $\lambda_i$  be eigenvalues of  $A$  and matrix  $\bar{\Phi} = \Phi \otimes I_n$  be as introduced earlier. Then, the asymptotic synchronization is ensured under the following event-triggered strategy:

$$t_{p+1} = \inf \left\{ t > t_p \mid \|\bar{\Phi}^\top e(t)\| \geq \frac{\delta}{\alpha} \|\bar{\Phi}^\top x(t)\| \right\}, \quad p \in \mathbb{N}, \quad (5.23)$$

with  $e(t) = x(t_p) - x(t)$ ,  $\delta \in (0, 1)$ , and,

$$\alpha = \max_{i \in \{2, 3, \dots, N\}} \{-c\lambda_i \|P_i \Gamma\|\}.$$

In addition, under this triggering strategy, the Lyapunov function  $V(x) = x^\top \bar{\Phi} P \bar{\Phi}^\top x$ , with  $P = \text{diag}_{i \in \{2, 3, \dots, N\}} \{P_i\}$ , satisfies the following dynamics:

$$\dot{V}(x) < -2(1 - \delta) \|\bar{\Phi}^\top x\|^2, \quad \forall t \in [t_p, t_{p+1}[ . \quad (5.24)$$

*Proof.* Let us first define  $e_y(t) \triangleq \bar{\Phi}^\top e(t)$ , therefore, the network triggered dynamics (5.21), can be transformed into the  $y$ -dynamics by multiplying this latter dynamics by  $\bar{\Phi}^\top$  which gives the following equation:

$$\dot{y} = (I_{N-1} \otimes H + c(\bar{\Lambda} \otimes \Gamma))y + c(\bar{\Lambda} \otimes \Gamma)e_y, \quad (5.25)$$

where we have used the similar technique to derive this equation as used to derive (5.17). Then, we further simplify (5.25) as follows:

$$\dot{y} = \text{diag}_{i \in \{2,3,\dots,N\}} \{H + c\lambda_i \Gamma\} y + c \text{diag}_{i \in \{2,3,\dots,N\}} \{\lambda_i \Gamma\} e_y,$$

where then we shall denote  $\mathcal{A} \triangleq \text{diag}_{i \in \{2,3,\dots,N\}} \{\lambda_i \Gamma\}$ , and recall  $\mathcal{K} = \text{diag}_{i \in \{2,3,\dots,N\}} \{H + c\lambda_i \Gamma\}$  from the proof of Proposition 5.4.3 to obtain:

$$\dot{y} = \mathcal{K}y + c\mathcal{A}e_y. \quad (5.26)$$

The next step is to show that the triggering strategy (5.23) guarantees the asymptotic synchronization of the  $x$ -dynamics (or equivalently, the asymptotic stability of the  $y$ -dynamics) based on a Lyapunov argument with given  $V(x) = x^\top \bar{\Phi} P \bar{\Phi}^\top x$ . We then transform this latter  $V(x)$  to obtain  $V(y) = y^\top P y$ , where then we recall the dynamics (5.26) and compute the temporal derivative of  $V(y)$  to obtain the following:

$$\dot{V}(y) = \dot{y}^\top P y + y^\top P \dot{y} = (y^\top \mathcal{K}^\top + e_y^\top \mathcal{A}^\top) P y + y^\top P (\mathcal{K}y + \mathcal{A}e_y),$$

which by some rearrangement of terms results in the following equation:

$$\dot{V}(y) = y^\top (\mathcal{K}^\top P + P \mathcal{K}) y + e_y^\top \mathcal{A}^\top P y + y^\top P \mathcal{A} e_y \quad (5.27)$$

where for the first term in this latter equation, we can derive the following equation:

$$\begin{aligned} \mathcal{K}^\top P + P \mathcal{K} &= \text{diag}_{i \in \{2,3,\dots,N\}} \{H_i^\top P_i + P_i H_i\} = \\ &= \text{diag}_{i \in \{2,3,\dots,N\}} \{-2I_n\} = -2I_{n(N-1)}, \end{aligned} \quad (5.28)$$

wherein Equation (5.22) from the assertion of this proposition have been helpful. Let us then plug (5.28) back in (5.27), where we get

$$\dot{V}(y) = -2y^\top I_{n(N-1)} y + e_y^\top \mathcal{A}^\top P y + y^\top P \mathcal{A} e_y.$$

We then bound  $\dot{V}(y)$  in this latter equation where we get

$$\dot{V}(y) \leq -2\|y\|^2 + \|e_y^\top\| \|\mathcal{A}^\top P\| \|y\| + \|y^\top\| \|P\mathcal{A}\| \|e_y\|, \quad (5.29)$$

wherein recalling the diagonal form of the matrices  $\mathcal{A}$  and  $P$ , we have  $\mathcal{A}^\top P = c \operatorname{diag}_{i \in \{2,3,\dots,N\}} \{\lambda_i \Gamma^\top P_i\}$ ; therefore, for (5.29), we can derive

$$\dot{V}(y) \leq -2\|y\|^2 + 2\|y\| \|c \operatorname{diag}_{i \in \{2,3,\dots,N\}} \{\lambda_i \Gamma^\top P_i\}\| \|e_y\|, \quad (5.30)$$

wherein recalling the norm property of a block diagonal matrix, the following holds:

$$\|c \operatorname{diag}_{i \in \{2,3,\dots,N\}} \{\lambda_i \Gamma^\top P_i\}\| = \max_{i \in \{2,3,\dots,N\}} \{c \lambda_i P_i \Gamma\| \Gamma\| \} = \max_{i \in \{2,3,\dots,N\}} \{-c \lambda_i P_i \Gamma\| \Gamma\| \},$$

where we have used the fact that  $\{\lambda_i(A)\}_{i=2}^N < 0$ . Let us then recall the parameter  $\alpha = \max_{i \in \{2,3,\dots,N\}} \{-c \lambda_i P_i \Gamma\| \Gamma\| \}$  from the assertion of this proposition, thereby (5.30) can be simplified as follows:

$$\dot{V}(y) \leq -2\|y\|^2 + 2\alpha\|y\| \|e_y\|, \quad (5.31)$$

where we note under triggering strategy (5.23), and  $\forall t \in [t_p, t_{p+1}[$ ,  $\|\bar{\Phi}^\top e(t)\| < \frac{\delta}{\alpha} \|\bar{\Phi}^\top x(t)\|$  holds. That is in other words,  $\|e_y(t)\| < \frac{\delta}{\alpha} \|y(t)\|$ ,  $\forall t \in [t_p, t_{p+1}[$ . This latter observation along with Equation (5.31), yields

$$\dot{V}(y) < -2(1 - \delta)\|y\|^2, \quad \forall t \in [t_p, t_{p+1}[. \quad (5.32)$$

At last, derivation of (5.32), under the triggering strategy (5.23), approves:

1. the asymptotic synchronization of  $x$ -dynamics. This is because provided  $\delta \in (0, 1)$ , (5.32) guarantees  $\dot{V}(y) < 0$ ,  $\forall t \in [t_p, t_{p+1}[$ , where also by  $V(y(t_p^-)) = V(y(t_p))$  and that  $V(y) = y^\top P y$  is a positive-definite function, the asymptotic stability of the  $y$ -dynamics is ensured; which then by Remark 5.4.2, the asymptotic synchronization of  $x$ -dynamics is inferred,.
2. the validity of inequality (5.24). This is because recalling the coordinate transformation,  $y = \bar{\Phi}^\top x$ , Equation (5.32) is indeed equivalent to (5.24).

Therefore, the proof of this result is complete.  $\square$

The following corollary characterizes the nonexistence of Zeno behavior for the triggering time-sequence,  $\{t_p\}_{p \in \mathbb{N}}$ , generated by triggering strategy (5.23) stated in the previous proposition.

**Corollary 5.4.6.** *Consider the triggered network dynamics (5.21), and let the assumptions of Proposition 5.4.5 hold, then  $\exists \tau > 0$  such that  $\forall p \in \mathbb{N}$ ,  $t_{p+1} - t_p \geq \tau$ , where  $\{t_p\}_{p \in \mathbb{N}}$  is the triggering time-sequence generated by (5.23).*

*Proof.* We again resort to the coordinate transformation technique, i.e., we recall the following transformed dynamics:

$$\dot{y} = \mathcal{K}y + cAe_y, \quad (5.33)$$

which is stated in the  $y$ -coordinates. In addition, we note that the triggering time-sequence  $\{t_p\}$  given by triggering strategy (5.23):  $\|\bar{\Phi}^\top e(t_{p+1})\| = \frac{\delta}{\alpha} \|\bar{\Phi}^\top x(t_{p+1})\|$  is the same as for the transformed strategy:

$$\|e_y(t_{p+1})\| = \frac{\delta}{\alpha} \|y(t_{p+1})\|, \quad (5.34)$$

this is because the considered transformation under  $\bar{\Phi}^\top$  matrix is static. Hence, it is sufficient to show the existence of the uniform lower-bound,  $\tau$ , based on (5.34) and for (5.33). In order to do so, we shall resort to Theorem III.1 and corollary IV.1 stated in [78]; where we have to show that (i) the dynamics  $\dot{y} = \mathcal{K}y$  is globally asymptotically stable and (ii) the Lyapunov function  $V(y) = y^\top P y$  is an ISS Lyapunov function for (5.33); the rest of this proof is devoted to this aim.

Item (i) can be simply verified, provided that the dynamics  $\dot{y} = \mathcal{K}y$  is globally asymptotically stable given  $\mathcal{K} = \text{diag}_{i \in \{2,3,\dots,N\}} \{H + c\lambda_i \Gamma\}$  and that assumption (5.22) holds. Item (ii) can be also verified, provided  $P = P^\top \succ 0$  and that Equation (5.31) has been proven. Therefore, having discussed the validity of items (i) and (ii), the proof of this corollary is complete.  $\square$

By the end of this subsection, we present the following remark which compares propositions 5.4.3, and 5.4.5.

**Remark 5.4.7.** *In contrast with sufficient condition (5.13), stated in Proposition 5.4.3, we remark that in Proposition 5.4.5, we have specifically used*

$$H_i^\top P_i + P_i H_i = -2I_n, \quad i \in \{2, 3, \dots, N\},$$

*which is indeed helpful in performing the analysis of this latter proposition. The alternative possibilities may be studied which is one of our future works.*

## 5.5 Multiple Topologies: Arbitrary Switching

In this section, we shall address problem 1 stated and discussed in Section 5.3, i.e., we consider the arbitrary switching scenario and shall discuss under which conditions on the network topologies one can guarantee asymptotic synchronization of the network. The analysis of this section is performed under two cases: (i) General Network Dynamics, wherein matrix  $H$  in dynamics (5.5) can be any given matrix, and (ii) Particular Network Dynamics, wherein matrix  $H$  is restricted to be a skew-symmetric matrix. These two latter cases are discussed under two separate subsections which follow next.

### 5.5.1 General Network Dynamics

Let us first recall our discussion in Proposition 5.4.5, where then by resorting to Equation (5.22), and for every  $k^{\text{th}}$  topology, with  $k \in \{1, \dots, m\}$ , we get:

$$H_i^{k\top} P_i^k + P_i^k H_i^k = -2I_n, \quad i \in \{2, \dots, N\}, \quad (5.35)$$

wherein  $H_i^k = H + c\lambda_i^k \Gamma$  with  $\{\lambda_i^k\}_{i=2}^N$  be eigenvalues of the outer-coupling matrix,  $A^k$ —as characterized in Lemma 5.3.1. Let us also recall from Proposition 5.4.5, the quadratic Lyapunov function,  $V^k(x) = x^\top \bar{\Phi}^k P^k \bar{\Phi}^{k\top} x$ , with matrices  $P^k = \text{diag}_{i \in \{2, \dots, N\}} \{P_i^k\}$  and  $\bar{\Phi}^k = \Phi^k \otimes I_n$  where, respectively, matrices  $P_i^k$  and  $\Phi^k$  are as characterized in Equation (5.35) and Lemma 5.3.1.

We then recall from [48] that in order to ensure the uniform asymptotic stability of a switched system,  $\dot{x} = f_p(x)$ , with  $x \in \mathbb{R}^n$ , for the case of arbitrary

switching, and for a family of topologies,  $p \in \mathcal{P}$ , one has to develop a *common Lyapunov function*, i.e., a positive-definite continuously differentiable function,  $V : \mathbb{R}^n \rightarrow \mathbb{R}$ , and a positive-definite continuous function,  $W : \mathbb{R}^n \rightarrow \mathbb{R}$ , such that the following holds:

$$\dot{V}(x) = \frac{\partial V}{\partial x} f_p(x) \leq -W(x), \quad \forall x \in \mathbb{R}^n, \quad \forall p \in \mathcal{P}. \quad (5.36)$$

Hence, in order to adopt this aforementioned argument to the case of switched triggered synchronization dynamics (5.5):

$$\dot{x}(t) = (I_N \otimes H)x(t) + c(A^k \otimes \Gamma)x(t_p^k), \quad \forall t \in [t_p^k, t_{p+1}^k[,$$

one has to develop a Lyapunov function of the earlier-discussed form:  $V^k(x) = x^\top \bar{\Phi}^k P^k \bar{\Phi}^{k\top} x$  which is common for every topology,  $k \in \{1, \dots, m\}$ . The following result studies the existence of such a Lyapunov function for specific network topologies.

**Proposition 5.5.1.** *Consider dynamics (5.5), where  $k \in \{1, \dots, m\}$ , and assume there exist matrices  $P_i^k = P_i^{k\top} \succ 0 \in \mathbb{R}^{n \times n}$ ,  $i \in \{2, 3, \dots, N\}$ , satisfying condition (5.35), for every  $k \in \{1, \dots, m\}$ . Moreover, let  $\delta \in (0, 1)$ ,  $e(t) = x(t_p^k) - x(t)$ ,  $\alpha_k = \max_{i \in \{2, 3, \dots, N\}} \{-c\lambda_i^k \|P_i^k \Gamma\|\}$ ,  $P^k = \text{diag}_{i \in \{2, 3, \dots, N\}} \{P_i^k\}$ , and  $\{t_p^k\}$  be the triggering time-sequence generated by (5.23), that is restated as follows:*

$$\|\bar{\Phi}^{k\top} e(t_p^k)\| = \frac{\delta}{\alpha_k} \|\bar{\Phi}^{k\top} x(t_p^k)\|, \quad p \in \mathbb{N}, \quad k \in \{1, \dots, m\}. \quad (5.37)$$

*Then, the asymptotic synchronization of (5.5) is ensured under arbitrary switching scenario if the following condition holds:*

$$\begin{aligned} & \bar{\Phi}^k (\text{diag}_{i \in \{2, \dots, N\}} (H^\top + H + c\lambda_i^k (\Gamma^\top + \Gamma)))^{-1} \bar{\Phi}^{k\top} = \\ & \bar{\Phi}^{k'} (\text{diag}_{i \in \{2, \dots, N\}} (H^\top + H + c\lambda_i^{k'} (\Gamma^\top + \Gamma)))^{-1} \bar{\Phi}^{k'\top}, \end{aligned} \quad (5.38)$$

*for every  $k, k' \in \{1, \dots, m\}$ .*

*Proof.* The main idea of the proof is to demonstrate that under condition (5.38), the resultant Lyapunov functions of the form  $V^k(x) = x^\top \bar{\Phi}^k P^k \bar{\Phi}^{k\top} x$  are equal for every  $k \in \{1, \dots, m\}$ . We note that this latter argument would be sufficient



provided that under the considered triggering strategy (5.41) and recalling from Proposition 5.4.5, the following inequality, i.e., inequality (5.24), holds:

$$\dot{V}^k(x) < -2(1 - \delta)\|\bar{\Phi}^{k\top}x\|^2, \quad \forall t \in [t_p^k, t_{p+1}^k[ ,$$

therefore:

1. provided under (5.38), the resultant Lyapunov function is the same for every topology, one can consider  $V \equiv V^k$  in inequality (5.36), given it is, by construction, positive-definite and continuously differentiable,
2. function  $W(x)$  in (5.36) can be considered as  $W(x) = 2(1 - \delta)\|\bar{\Phi}^{k\top}x\|^2$ ; this is flawless provided that  $\delta \in (0, 1)$ , therefore,  $2(1 - \delta)$  is a positive coefficient, in addition, function  $\|\bar{\Phi}^{k\top}x\|^2$  is (i) common for all topologies, provided properties of  $\bar{\Phi}^k = \Phi^k \otimes I_n$  matrix—discussed in Lemma 5.3.1—which would ensure the following:

$$\begin{aligned} \|\bar{\Phi}^{k\top}x\| &= \sqrt{x^\top \bar{\Phi}^k \bar{\Phi}^{k\top} x} = \sqrt{x^\top (\Phi^k \otimes I_n) (\Phi^{k\top} \otimes I_n) x} = \\ &= \sqrt{x^\top (\Phi^k \Phi^{k\top} \otimes I_n) x} = \sqrt{x^\top (I_N - \frac{1}{N} 1_N) x}, \end{aligned}$$

(ii) positive-definite by construction, and (iii) continuous, provided that at switching time-instants,  $\tau_j$ , no jump occurs at the state value,  $x(t)$ .

The rest of the proof is devoted to this aim.

Let us consider two topologies,  $k, k' \in \{1, \dots, m\}$ , then ensuring  $V^k(x) = V^{k'}(x)$ , yields:

$$x^\top (\bar{\Phi}^k P^k \bar{\Phi}^{k\top} - \bar{\Phi}^{k'} P^{k'} \bar{\Phi}^{k'\top}) x = 0, \quad \forall x \in \mathbb{R}^n. \quad (5.39)$$

Discussing solutions to this latter equation, we admit, that  $x = 0$  and  $x \in \mathcal{N}(\bar{\Phi}^k P^k \bar{\Phi}^{k\top} - \bar{\Phi}^{k'} P^{k'} \bar{\Phi}^{k'\top})$  satisfy (5.39); nonetheless, this equation has to hold for all  $x \in \mathbb{R}^n$ , therefore, one has to ensure:

$$\bar{\Phi}^k P^k \bar{\Phi}^{k\top} = \bar{\Phi}^{k'} P^{k'} \bar{\Phi}^{k'\top}, \quad (5.40)$$

where we also recall that  $P^k = \text{diag}_{i \in \{2, \dots, N\}} \{P_i^k\}$  with matrices  $P_i^k$  be solutions to Equation (5.35), i.e.:

$$H_i^{k\top} P_i^k + P_i^k H_i^k = -2I_n, \quad i \in \{2, \dots, N\},$$

with  $H_i^k = H + c\lambda_i^k\Gamma$ . We hereby note that indeed this latter equation is a continuous-time Lyapunov equation with matrix  $Q \equiv 2I_n$  and  $P \equiv P_i^k$ , hence, one can solve for this equation, whereby the following shall be obtained:

$$\begin{aligned} P_i^k &= \int_0^\infty (\exp(\tau(H + c\lambda_i^k\Gamma))^\top)(2I_n)(\exp(\tau(H + c\lambda_i^k\Gamma)))d\tau = \\ &= 2 \int_0^\infty \exp[\tau((H + c\lambda_i^k\Gamma)^\top + (H + c\lambda_i^k\Gamma))]d\tau = \\ &= 2 \int_0^\infty \exp[\tau((H^\top + H) + c\lambda_i^k(\Gamma^\top + \Gamma))]d\tau, \end{aligned}$$

which solving for the integration can be further simplified as follows:

$$P_i^k = ((H^\top + H) + c\lambda_i^k(\Gamma^\top + \Gamma))^{-1} \times \exp(\tau[(H^\top + H) + c\lambda_i^k(\Gamma^\top + \Gamma)])|_0^\infty.$$

This latter equation can be further simplified, noting that  $\lim_{\tau \rightarrow \infty} \exp(H + c\lambda_i^k\Gamma)\tau = 0$ , which is flawless provided that matrix  $H_i^k = H + c\lambda_i^k\Gamma$  is Hurwitz given it is a solution to the Lyapunov Equation (5.35). This simplification yields  $P_i^k = -(H^\top + H + c\lambda_i^k(\Gamma^\top + \Gamma))^{-1}$  and similarly,  $P_i^{k'} = -(H^\top + H + c\lambda_i^{k'}(\Gamma^\top + \Gamma))^{-1}$ . Henceforth, plugging these two latter values for matrices,  $P_i^k$  and  $P_i^{k'}$  back in Equation (5.40), bestows:

$$\begin{aligned} \bar{\Phi}^k(\text{diag}_{i \in \{2, \dots, N\}}(H^\top + H + c\lambda_i^k(\Gamma^\top + \Gamma))^{-1})\bar{\Phi}^{k\top} &= \\ \bar{\Phi}^{k'}(\text{diag}_{i \in \{2, \dots, N\}}(H^\top + H + c\lambda_i^{k'}(\Gamma^\top + \Gamma))^{-1})\bar{\Phi}^{k'\top}, & \end{aligned}$$

which is indeed Equation (5.38) stated in the proposition statement. This then completes the proof.  $\square$

## 5.5.2 Particular Network Dynamics

In Subsection 5.5.1, we studied the arbitrary switching scenario for the case of general switched triggered synchronization dynamics (5.5):

$$\dot{x}(t) = (I_N \otimes H)x(t) + c(A^k \otimes \Gamma)x(t_p^k), \quad \forall t \in [t_p^k, t_{p+1}^k[ ,$$

wherein we developed Proposition 5.5.1 to characterize specific class of topologies for which a common Lyapunov function can be derived. Motivated by

these results, in this subsection, we shall narrow our studies down to a particular class of dynamics, for which  $H + H^\top = 0$ , i.e., matrix  $H \in \mathbb{R}^{n \times n}$  which characterizes the dynamics of each node is skew-symmetric. We note that this class of dynamics is interesting provided our discussion in Section 5.7. Having stated these points and along the lines of Proposition 5.5.1, we shall develop the following proposition for this particular class of dynamics.

**Proposition 5.5.2.** *Consider dynamics (5.5), where  $k \in \{1, \dots, m\}$ , and assume first that the matrix  $H$  is skew-symmetric, i.e.,  $H + H^\top = 0$ , and second that there exist matrices  $P_i^k = P_i^{k\top} \succ 0 \in \mathbb{R}^{n \times n}$ ,  $i \in \{2, 3, \dots, N\}$ , satisfying condition (5.35), for every  $k \in \{1, \dots, m\}$ . Moreover, let  $\delta \in (0, 1)$ ,  $e(t) = x(t_p^k) - x(t)$ ,  $\alpha_k = \max_{i \in \{2, 3, \dots, N\}} \{-c\lambda_i^k \|P_i^k \Gamma\|\}$ ,  $P^k = \text{diag}_{i \in \{2, 3, \dots, N\}} \{P_i^k\}$ , and  $\{t_p^k\}$  be the triggering time-sequence generated by (5.23), that is restated as follows:*

$$\|\bar{\Phi}^{k\top} e(t_p^k)\| = \frac{\delta}{\alpha_k} \|\bar{\Phi}^{k\top} x(t_p^k)\|, \quad p \in \mathbb{N}, \quad k \in \{1, \dots, m\}. \quad (5.41)$$

Then, the asymptotic synchronization of (5.5) can never be ensured under arbitrary switching scenario.

*Proof.* The proof of this result shall follow upon the proof of Proposition 5.5.1, in the sense that provided the assumptions of these two results are similar and that the dynamics in Proposition 5.5.1 encompasses the dynamics considered here, one can perform exactly the same procedure in order to develop Equation (5.38), i.e.:

$$\begin{aligned} \bar{\Phi}^k (\text{diag}_{i \in \{2, \dots, N\}} (H^\top + H + c\lambda_i^k (\Gamma^\top + \Gamma))^{-1}) \bar{\Phi}^{k\top} = \\ \bar{\Phi}^{k'} (\text{diag}_{i \in \{2, \dots, N\}} (H^\top + H + c\lambda_i^{k'} (\Gamma^\top + \Gamma))^{-1}) \bar{\Phi}^{k'\top}. \end{aligned}$$

Then, based on this fact and by skew-symmetry property of the matrix  $H$ , which implies  $H^\top + H = 0$ , this latter equation can be further simplified where we get:

$$\bar{\Phi}^k (\text{diag}_{i \in \{2, \dots, N\}} (\lambda_i^k (\Gamma^\top + \Gamma))^{-1}) \bar{\Phi}^{k\top} = \bar{\Phi}^{k'} (\text{diag}_{i \in \{2, \dots, N\}} (\lambda_i^{k'} (\Gamma^\top + \Gamma))^{-1}) \bar{\Phi}^{k'\top}, \quad (5.42)$$

in which the same coefficient  $c$  has been discarded on both sides. Then, in this latter equation, we shall further elaborate on matrix  $\text{diag}_{i \in \{2, \dots, N\}} (\lambda_i^k (\Gamma^\top + \Gamma))^{-1}$ , where we note the following:

$$\text{diag}_{i \in \{2, \dots, N\}} (\lambda_i^k (\Gamma^\top + \Gamma))^{-1} = \text{diag}_{i \in \{2, \dots, N\}} \frac{(\Gamma^\top + \Gamma)^{-1}}{\lambda_i^k} = \quad (5.43)$$

$$\text{diag}_{i \in \{2, \dots, N\}} \frac{1}{\lambda_i^k} \otimes (\Gamma^\top + \Gamma)^{-1}, \quad (5.44)$$

wherein we recall that  $\{\lambda_i^k\}_{i=2}^N$  are nonzero eigenvalues of the outercoupling matrix,  $A^k$ , as discussed in Section 5.3 and further in Lemma 5.3.1. Referring to the same Lemma 5.3.1, we shall also recall  $\Phi^{k\top} A^k \Phi^k = \bar{\Lambda}^k = \text{diag}_{i \in \{2, \dots, N\}} \lambda_i^k$ , based on which and provided  $\bar{\Lambda}^k$  is a diagonal matrix, we note that  $(\bar{\Lambda}^k)^{-1} = \text{diag}_{i \in \{2, \dots, N\}} \frac{1}{\lambda_i^k}$ . Embedding this last observing in derivation (5.43), bestows:

$$\text{diag}_{i \in \{2, \dots, N\}} (\lambda_i^k (\Gamma^\top + \Gamma))^{-1} = (\bar{\Lambda}^k)^{-1} \otimes (\Gamma^\top + \Gamma)^{-1},$$

thanks to which Equation (5.42) can be rephrased as performed in the following:

$$\begin{aligned} & (\Phi^k \otimes I_n) ((\bar{\Lambda}^k)^{-1} \otimes (\Gamma^\top + \Gamma)^{-1}) (\Phi^{k\top} \otimes I_n) = \\ & (\Phi^{k'} \otimes I_n) ((\bar{\Lambda}^{k'})^{-1} \otimes (\Gamma^\top + \Gamma)^{-1}) (\Phi^{k'\top} \otimes I_n), \end{aligned}$$

where we have used  $\bar{\Phi}^k = \Phi^k \otimes I_n$  and  $\bar{\Phi}^{k'} = \Phi^{k'} \otimes I_n$ . This latter equation, by properties of the Kronecker product, yields:

$$(\Phi^k (\bar{\Lambda}^k)^{-1} \Phi^{k\top}) \otimes (\Gamma^\top + \Gamma)^{-1} = (\Phi^{k'} (\bar{\Lambda}^{k'})^{-1} \Phi^{k'\top}) \otimes (\Gamma^\top + \Gamma)^{-1},$$

which, provided the term  $(\Gamma^\top + \Gamma)^{-1}$  is identical on both sides, yields:

$$\Phi^k (\bar{\Lambda}^k)^{-1} \Phi^{k\top} = \Phi^{k'} (\bar{\Lambda}^{k'})^{-1} \Phi^{k'\top}. \quad (5.45)$$

We would now like to recall the property  $\Phi^{k\top} A^k \Phi^k = \bar{\Lambda}^k$ , which indeed provided  $\Phi^k \Phi^{k\top} = I_N - \frac{1}{N} 1_N$  and the zero-sum property of the outercoupling matrix  $A^k$ —as stated in Equation (5.2)—can be re-written as in  $A^k = \Phi^k \bar{\Lambda}^k \Phi^{k\top}$ . Therefore, having obtained this latter equation, and recalling (5.45), we shall assess whether there is a relationship between matrices,  $A^k = \Phi^k \bar{\Lambda}^k \Phi^{k\top}$  and  $\Phi^k (\bar{\Lambda}^k)^{-1} \Phi^{k\top}$ . The next claim is devoted to this aim.

**Claim 5.5.3.** Recall the outercoupling matrix,  $A^k = \Phi^k \bar{\Lambda}^k \Phi^{k\top}$ . Then, the pseudo-inverse of this matrix is given as follows:

$$(A^k)^+ = \Phi^k (\bar{\Lambda}^k)^{-1} \Phi^{k\top}. \quad (5.46)$$

*Proof of Claim 5.5.3:* According to the definition of a pseudo-inverse [12], we shall verify the following properties in order to prove that  $(A^k)^+$  as stated in (5.46) is indeed the pseudo-inverse of  $A^k$ . On this way and for the ease of notation, let first  $X = \Phi^k (\bar{\Lambda}^k)^{-1} \Phi^{k\top}$ , then we shall verify the following, wherein properties  $\Phi^{k\top} \Phi^k = I_{N-1}$  and  $\Phi^k \Phi^{k\top} = I_N - \frac{1}{N} 1_N$  play an important role:

- (i)  $AXA = (\Phi^k \bar{\Lambda}^k \Phi^{k\top})(\Phi^k (\bar{\Lambda}^k)^{-1} \Phi^{k\top})(\Phi^k \bar{\Lambda}^k \Phi^{k\top}) = \Phi^k \bar{\Lambda}^k \Phi^k = A$ ,
- (ii)  $XAX = (\Phi^k (\bar{\Lambda}^k)^{-1} \Phi^{k\top})(\Phi^k \bar{\Lambda}^k \Phi^{k\top})(\Phi^k (\bar{\Lambda}^k)^{-1} \Phi^{k\top}) = (\Phi^k (\bar{\Lambda}^k)^{-1} \Phi^{k\top}) = X$ ,
- (iii) on the one hand,  $AX = (\Phi^k \bar{\Lambda}^k \Phi^{k\top})(\Phi^k (\bar{\Lambda}^k)^{-1} \Phi^{k\top}) = \Phi^k \Phi^{k\top} = I_N - \frac{1}{N} 1_N$ , on the other hand,  $(AX)^\top = (I_N - \frac{1}{N} 1_N)^\top = I_N - \frac{1}{N} 1_N$ ; therefore,  $(AX)^\top = AX$ ,
- (iv) on the one hand,  $XA = (\Phi^k (\bar{\Lambda}^k)^{-1} \Phi^{k\top})(\Phi^k \bar{\Lambda}^k \Phi^{k\top}) = \Phi^k \Phi^{k\top} = I_N - \frac{1}{N} 1_N$ , on the other hand,  $(XA)^\top = (I_N - \frac{1}{N} 1_N)^\top = I_N - \frac{1}{N} 1_N$ ; therefore,  $(XA)^\top = XA$ .

•

At this stage and having developed Claim 5.5.3, we shall resort again to Equation (5.45), where then we get the following:

$$(A^k)^+ = (A^{k'})^+,$$

that is to say that for the case of skew-symmetric matrix  $H$ , and in order to ensure asymptotic synchronization under arbitrary switching, pseudo-inverse of the outercoupling matrices  $A^k$  and  $A^{k'}$  have to be equal. This fact, provided that the pseudo-inverse for a matrix is unique, implies that indeed the two outer coupling matrices have to be the same. This latter observation shall conclude the proof of this proposition.  $\square$

## 5.6 Multiple Topologies: Switching Signal Design

In this section we shall recall two classes of switching signals introduced in Section 5.3, i.e.,  $\mathcal{S}_{\text{dwell}}[\tau_D]$  and  $\mathcal{S}_{\text{average}}[\tau_a, N_0]$ . In order to solve Problem 2, we then characterize sufficient regulatory conditions on the switching signals associated to these classes, such that the asymptotic synchronization of the switched triggered network (5.5), introduced in Section 5.3 be guaranteed. On this way, first  $\mathcal{S}_{\text{dwell}}$  is discussed, and then  $\mathcal{S}_{\text{average}}$ .

### 5.6.1 Regulatory Conditions on $\mathcal{S}_{\text{dwell}}[\tau_D]$ Class

In this subsection we first discuss the switching between two topologies, we then characterize regulatory condition in terms of  $\tau_D$ , and triggering strategy (5.23), discussed in Proposition 5.4.5, such that the asymptotic synchronization of the switched triggered network (5.5) be ensured.

**Theorem 5.6.1.** *Consider dynamics (5.5), where  $k \in \{1, 2\}$ , and assume there exist matrices  $P_i^k = P_i^{k\top} \succ 0 \in \mathbb{R}^{n \times n}$ ,  $i \in \{2, 3, \dots, N\}$ , satisfying condition (5.22), as stated in Proposition 5.4.5, for every  $k \in \{1, 2\}$ . Moreover, let  $\delta \in (0, 1)$ ,  $e(t) = x(t_p^k) - x(t)$ ,  $\alpha_k = \max_{i \in \{2, 3, \dots, N\}} \{-c\lambda_i^k \|P_i^k \Gamma\|\}$ ,  $P^k = \text{diag}_{i \in \{2, 3, \dots, N\}} \{P_i^k\}$ , and  $\{t_p^k\}$  be the triggering time-sequence generated by (5.23), that is restated as follows:*

$$\|\bar{\Phi}^{k\top} e(t_p^k)\| = \frac{\delta}{\alpha_k} \|\bar{\Phi}^{k\top} x(t_p^k)\|, \quad p \in \mathbb{N}, \quad k \in \{1, 2\}. \quad (5.47)$$

Then, the switching signal,  $\sigma : \mathbb{R}_{\geq 0} \rightarrow k$ , where  $\sigma(t) \in \mathcal{S}_{\text{dwell}}[\tau_D]$ , ensures the asymptotic synchronization of (5.5) under the following condition:

$$\tau_D > \frac{1}{2(1-\delta)} \frac{\lambda_{\max}(P^2)\lambda_{\max}(P^1)}{\lambda_{\max}(P^2) + \lambda_{\max}(P^1)} \max\{\Upsilon_1, \Upsilon_2\}, \quad (5.48)$$

with

$$\begin{aligned} \Upsilon_1 &= \log\left(\frac{\lambda_{\max}(P^1)^2 \lambda_{\max}(P^2)}{\lambda_{\min}(P^1)^2 \lambda_{\min}(P^2)}\right), \\ \Upsilon_2 &= \log\left(\frac{\lambda_{\max}(P^2)^2 \lambda_{\max}(P^1)}{\lambda_{\min}(P^2)^2 \lambda_{\min}(P^1)}\right). \end{aligned} \quad (5.49)$$

*Proof.* Let us first denote by  $\{\tau_j\}_{j \in \mathbb{N}}$  the sequence of switching time-instants, which are then the discontinuities of the switching signal,  $\sigma(t)$ . Indeed, provided  $\sigma(t) \in \mathcal{S}_{\text{dwell}}[\tau_D]$ , it holds that  $\tau_{j+1} - \tau_j \geq \tau_D, \forall j \in \mathbb{N}_0$ . The rest of this proof is to characterize  $\tau_D$  such that the synchronization of dynamics (5.5) under triggering rule (5.62) and this class of switching signals be ensured. In order to do so, and without loss of generality, we shall consider the time-intervals,  $[\tau_0, \tau_1[$ ,  $[\tau_1, \tau_2[$ ,  $[\tau_2, \tau_3[$ , and  $[\tau_3, \tau_4[$ ; where, respectively, we assume that the first, and second topologies have been active. The proof shall be complete by characterizing  $\tau_D$  such that  $\|\bar{\Phi}^{2\top} x(\tau_3)\| < \|\bar{\Phi}^{2\top} x(\tau_1)\|$ , and  $\|\bar{\Phi}^{1\top} x(\tau_2)\| < \|\bar{\Phi}^{1\top} x(\tau_0)\|$ —with  $\|\bar{\Phi}^{1\top} x(t)\|$  and  $\|\bar{\Phi}^{2\top} x(t)\|$  be as discussed earlier in Lemma 5.4.1. This latter argument is sufficient to complete this proof because, in an inductive way, one obtains strictly decreasing sequences,  $\{\|\bar{\Phi}^{2\top} x(\tau_{2j+1})\|\}$  and  $\{\|\bar{\Phi}^{1\top} x(\tau_{2j})\|\}$ , that ensure the strictly decreasing distance to the synchronization manifold.

Let us then consider  $[\tau_0, \tau_1[$  where, as assumed earlier, the first topology is active. This latter, together with dynamics (5.5) while incorporating error,  $e(t) = x(t_p^k) - x(t)$ , implies that for the times,  $t \in [\tau_0, \tau_1[$ , the following dynamics is active:

$$\dot{x}(t) = (I_N \otimes H + cA^1 \otimes \Gamma)x(t) + cA^1 \otimes \Gamma e(t).$$

Let us also consider, without loss of generality, that the triggering time-instants,  $\{t_p^1\}_{p=1}^{p_1}$ , occur within  $[\tau_0, \tau_1[$  time-interval, such that  $t_{p_1}^1 < \tau_1$ —this is flawless, recalling corollary 5.4.6 which excludes Zeno behavior from these triggering time-instants. We also recall inequality (5.24), from Proposition 5.4.5, that for every  $t \in [t_{p-1}^1, t_p^1[ \subseteq [\tau_0, \tau_1[$ , the following holds:

$$\dot{V}^1(t) \leq -2(1 - \delta)\|\bar{\Phi}^{1\top} x\|^2, \quad (5.50)$$

where also, recalling Lyapunov function,  $V^1(x) = x^\top \bar{\Phi}^1 P^1 \bar{\Phi}^{1\top} x$ , we obtain  $\lambda_{\min}(P^1)\|\bar{\Phi}^{1\top} x\|^2 \leq V^1(x) \leq \lambda_{\max}(P^1)\|\bar{\Phi}^{1\top} x\|^2$ ; which, in particular, bestows:

$$\|\bar{\Phi}^{1\top} x\|^2 \geq \frac{V^1}{\lambda_{\max}(P^1)}. \quad (5.51)$$

Then, by (5.50) and (5.51), we obtain:

$$\dot{V}^1(t) \leq -2 \frac{(1 - \delta)}{\lambda_{\max}(P^1)} V^1(t), \quad \forall t \in [t_{p-1}^1, t_p^1[ ,$$

which by comparison principle, computing over  $[t_{p-1}^1, t_p^1[$ , yields:

$$V^1(t_p^1) \leq V^1(t_{p-1}^1) \exp\left(-\frac{2(1-\delta)}{\lambda_{\max}(P^1)}(t_p^1 - t_{p-1}^1)\right),$$

therefore, in particular, computing the latter inequality over  $[\tau_0, \tau_1[$  time-interval, yields:

$$V^1(\tau_1) \leq V^1(\tau_0) \exp\left(-\frac{2(1-\delta)}{\lambda_{\max}(P^1)}(\tau_1 - \tau_0)\right). \quad (5.52)$$

Having established the latter equation, we now consider  $[\tau_1, \tau_2[$  time-interval, where the second topology is active. Therefore, in an analogous way as we derived (5.52), this time assuming  $\{t_p^2\}_{p=1}^{P^2}$  to be the triggering time-instants belonging to  $[\tau_1, \tau_2[$  such that  $t_{p_2}^2 < \tau_2$ , we obtain:

$$V^2(\tau_2) \leq V^2(\tau_1) \exp\left(-\frac{2(1-\delta)}{\lambda_{\max}(P^2)}(\tau_2 - \tau_1)\right). \quad (5.53)$$

At this point, and having established (5.53), we assess the relation between  $V^1(\tau_2)$  and  $V^1(\tau_0)$ . However, we have to first study the jumps at switching instants,  $\tau_1$  and  $\tau_2$ . At  $\tau_1$  we switch from the first to second topology, therefore, we have  $V^2(\tau_1) = x(\tau_1)^\top \bar{\Phi}^2 P^2 \bar{\Phi}^{2\top} x(\tau_1)$ , where then we can derive:

$$\lambda_{\min}(P^2) \|\bar{\Phi}^{2\top} x(\tau_1)\|^2 \leq V^2(\tau_1) \leq \lambda_{\max}(P^2) \|\bar{\Phi}^{2\top} x(\tau_1)\|^2. \quad (5.54)$$

Now, in order to connect  $V^2(\tau_1)$  to  $V^1(\tau_1)$ , we have to connect  $\|\bar{\Phi}^{1\top} x(\tau_1)\|$  and  $\|\bar{\Phi}^{2\top} x(\tau_1)\|$ . On this way, provided  $\bar{\Phi} = \Phi \otimes I_n$  and recalling properties of  $\Phi$  matrix stated in Lemma 5.3.1, the following holds:

$$\begin{aligned} \|\bar{\Phi}^{k\top} x\| &= \sqrt{x^\top \bar{\Phi}^k \bar{\Phi}^{k\top} x} = \sqrt{x^\top (\Phi^k \otimes I_n) (\Phi^{k\top} \otimes I_n) x} \\ &= \sqrt{x^\top (\Phi^k \Phi^{k\top} \otimes I_n) x} = \sqrt{x^\top (I_N - \frac{1}{N} 1_N) x}, \end{aligned}$$

which demonstrates that  $\|\bar{\Phi}^{k\top} x\|$  is independent of  $k$ . Therefore, one concludes:

$$\|\bar{\Phi}^{1\top} x\| = \|\bar{\Phi}^{2\top} x\|. \quad (5.55)$$

Accordingly, recalling (5.54) and by (5.55), we develop the following:

$$\begin{aligned} V^2(\tau_1) &\leq \lambda_{\max}(P^2) \|\bar{\Phi}^{2\top} x(\tau_1)\|^2 = \lambda_{\max}(P^2) \|\bar{\Phi}^{1\top} x(\tau_1)\|^2 \\ &= \frac{\lambda_{\max}(P^2)}{\lambda_{\min}(P^1)} \lambda_{\min}(P^1) \|\bar{\Phi}^{1\top} x(\tau_1)\|^2 \leq \frac{\lambda_{\max}(P^2)}{\lambda_{\min}(P^1)} V^1(\tau_1), \end{aligned} \quad (5.56)$$



wherein  $\lambda_{\min}(P^1)\|\bar{\Phi}^{1\top}x(\tau_1)\|^2 \leq V^1(\tau_1)$  is used, which is flawless recalling the aforementioned discussion on  $V^1(x)$  function.

Hence, putting together equations (5.52) and (5.56), we obtain:

$$V^2(\tau_1) \leq \frac{\lambda_{\max}(P^2)}{\lambda_{\min}(P^1)} \exp\left(-\frac{2(1-\delta)}{\lambda_{\max}(P^1)}(\tau_1 - \tau_0)\right) V^1(\tau_0),$$

where then plugging (5.53) into the latter inequality yields:

$$V^2(\tau_2) \leq \frac{\lambda_{\max}(P^2)}{\lambda_{\min}(P^1)} \exp\left(-\frac{2(1-\delta)}{\lambda_{\max}(P^2)}(\tau_2 - \tau_1)\right) \times \\ \exp\left(-\frac{2(1-\delta)}{\lambda_{\max}(P^1)}(\tau_1 - \tau_0)\right) V^1(\tau_0),$$

which then can be simplified by (i) recalling the considered class of switching signals for which  $\tau_2 - \tau_1 \geq \tau_D$  and  $\tau_1 - \tau_0 \geq \tau_D$  hold, and (ii) exploiting inequality  $V^1(\tau_2) \leq \frac{\lambda_{\max}(P^1)}{\lambda_{\min}(P^2)} V^2(\tau_2)$  that can be derived similar to (5.56). The resultant inequality would be as follows:

$$V^1(\tau_2) \leq V^1(\tau_0) \frac{\lambda_{\max}(P^1)\lambda_{\max}(P^2)}{\lambda_{\min}(P^1)\lambda_{\min}(P^2)} \times \\ \exp\left(-\left(\frac{1}{\lambda_{\max}(P^1)} + \frac{1}{\lambda_{\max}(P^2)}\right) 2(1-\delta)\tau_D\right).$$

Then, incorporating inequalities  $\lambda_{\min}(P^1)\|\bar{\Phi}^{1\top}x(\tau_2)\|^2 \leq V^1(\tau_2)$  and  $V^1(\tau_0) \leq \lambda_{\max}(P^1)\|\bar{\Phi}^{1\top}x(\tau_0)\|^2$ , we obtain:

$$\|\bar{\Phi}^{1\top}x(\tau_2)\|^2 \leq \|\bar{\Phi}^{1\top}x(\tau_0)\|^2 \frac{\lambda_{\max}(P^1)^2\lambda_{\max}(P^2)}{\lambda_{\min}(P^1)^2\lambda_{\min}(P^2)} \times \\ \exp\left(-\left(\frac{1}{\lambda_{\max}(P^1)} + \frac{1}{\lambda_{\max}(P^2)}\right) 2(1-\delta)\tau_D\right). \quad (5.57)$$

Henceforth, based on (5.57), in order to impose  $\|\bar{\Phi}^{1\top}x(\tau_2)\| < \|\bar{\Phi}^{1\top}x(\tau_0)\|$ , it is sufficient to guarantee the following:

$$\tau_D > \frac{1}{2(1-\delta)} \frac{\lambda_{\max}(P^1)\lambda_{\max}(P^2)}{\lambda_{\max}(P^1) + \lambda_{\max}(P^2)} \times \log\left(\frac{\lambda_{\min}(P^1)^2\lambda_{\min}(P^2)}{\lambda_{\max}(P^1)^2\lambda_{\max}(P^2)}\right). \quad (5.58)$$

In an analogous way, resulting in (5.58), this time focusing on  $V^2(\tau_1)$  and  $V^2(\tau_3)$ , we obtain:

$$V^2(\tau_3) \leq V^2(\tau_1) \frac{\lambda_{\max}(P^1)\lambda_{\max}(P^2)}{\lambda_{\min}(P^1)\lambda_{\min}(P^2)} \times \\ \exp\left(-\left(\frac{1}{\lambda_{\max}(P^1)} + \frac{1}{\lambda_{\max}(P^2)}\right) 2(1-\delta)\tau_D\right),$$

---

**Algorithm 2**  $\tau_D$ -Seeking
 

---

**Input:**  $m$ : number of topologies; matrices  $P^i$  satisfying condition (5.22) for every  $i \in \{1, \dots, m\}$ ;  $\lambda_{\max}(P^i)$  for every  $i \in \{1, \dots, m\}$ .

- 1: **for**  $i = 1$  to  $m - 1$  **do**
- 2:   **for**  $j = 1$  to  $m - 1$  **do**
- 3:     **for**  $k = 1$  to  $m$  **do**
- 4:       Compute the possible dwell-times:

$$\tau_D(i, j, k) = \frac{1}{2(1 - \delta)} \frac{\prod_{q=i}^j \lambda_{\max}(P^q)}{\sum_{q=i}^j \lambda_{\max}(P^q)} \times \log \left( \frac{\lambda_{\max}(P^k) \prod_{q=i}^j \lambda_{\max}(P^q)}{\lambda_{\min}(P^k) \prod_{q=i}^j \lambda_{\min}(P^q)} \right). \quad (5.61)$$

- 5:     **end for**
- 6:   **end for**
- 7: **end for**

**Output:**  $\tau_D = \max_{i,j,k} \{\tau_D(i, j, k)\}$ .

---

which by incorporating  $\lambda_{\min}(P^2) \|\bar{\Phi}^{2\top} x(\tau_3)\|^2 \leq V^2(\tau_3)$  and  $V^2(\tau_1) \leq \lambda_{\max}(P^2) \|\bar{\Phi}^{1\top} x(\tau_1)\|^2$ , we get:

$$\begin{aligned} \|\bar{\Phi}^{2\top} x(\tau_3)\|^2 &\leq \|\bar{\Phi}^{2\top} x(\tau_1)\|^2 \frac{\lambda_{\max}(P^2)^2 \lambda_{\max}(P^1)}{\lambda_{\min}(P^2)^2 \lambda_{\min}(P^1)} \times \\ &\exp \left( - \left( \frac{1}{\lambda_{\max}(P^1)} + \frac{1}{\lambda_{\max}(P^2)} \right) 2(1 - \delta) \tau_D \right). \end{aligned} \quad (5.59)$$

Hence, in order to ensure  $\|\bar{\Phi}^{2\top} x(\tau_3)\| < \|\bar{\Phi}^{2\top} x(\tau_1)\|$ , it is sufficient to ensure the following condition on  $\tau_D$ :

$$\tau_D > \frac{1}{2(1 - \delta)} \frac{\lambda_{\max}(P^1) \lambda_{\max}(P^2)}{\lambda_{\max}(P^1) + \lambda_{\max}(P^2)} \times \log \left( \frac{\lambda_{\min}(P^2)^2 \lambda_{\min}(P^1)}{\lambda_{\max}(P^2)^2 \lambda_{\max}(P^1)} \right). \quad (5.60)$$

At last, we conclude the proof noting that  $\tau_D$  has to be greater than the lower-bounds introduced in the two inequalities (5.58) and (5.60), which is indeed equivalent to (5.48) stated in theorem statement.  $\square$

The previous result has been established for the case of two topologies;

nonetheless, it can be extended to the case of switching amongst  $m$  topologies. The following corollary elaborates on this point.

**Corollary 5.6.2.** *Consider dynamics (5.5), where  $k \in \{1, \dots, m\}$ , and assume that the assumptions stated in Theorem 5.6.1 hold; moreover, let  $\tau_D^*$  be the dwell time given by Algorithm 2. Then, the asymptotic synchronization of dynamics (5.5) is ensured under triggering strategy (5.62) and switching signal,  $\sigma : \mathbb{R}_{\geq 0} \rightarrow k$ , where  $\sigma(t) \in \mathcal{S}_{\text{dwell}}[\tau_D]$ , for  $\tau_D \geq \tau_D^*$ .*

*Proof.* The proof of this corollary goes along the lines of the proof of Theorem 5.6.1; where it needs to be noted that parameter  $\tau_D^*$  given by Algorithm 2 is indeed the worst-case dwelling time, i.e., it considers switching between all the possible topologies of the network. To be specific, the dwell times,  $\tau_D(i, j, k)$ , as computed in Equation (5.61) are an extension to the uniform lower-bound (5.48) discussed for the case of switching between two topologies.  $\square$

## 5.6.2 Regulatory Conditions on $\mathcal{S}_{\text{average}}[\tau_a, N_0]$ Class

In previous subsection, we discussed regulatory conditions on  $\mathcal{S}_{\text{dwell}}[\tau_D]$  class of switching signals, in this subsection, we shall discuss switching signals,  $\sigma(t) \in \mathcal{S}_{\text{average}}[\tau_a, N_0]$ .

For this class and as briefly discussed in Section 5.3, the number of discontinuities of switching signal,  $\sigma(t)$ , over time-interval  $(t, \tau)$ , i.e.,  $N_\sigma(\tau, t)$ , satisfies  $N_\sigma(\tau, t) \leq N_0 + \frac{\tau-t}{\tau_a}$  with parameters  $N_0$  and  $\tau_a$  be respectively average dwell time and chatter bound. The main goal of this subsection is to characterize sufficient conditions on these two parameters such that the asymptotic synchronization of the triggered network dynamics (5.5) be guaranteed. The next theorem is to serve for this aim.

**Theorem 5.6.3.** *Consider dynamics (5.5), where  $k \in \{1, \dots, m\}$ , and assume there exist matrices  $P_i^k = P_i^{k\top} \succ 0 \in \mathbb{R}^{n \times n}$ ,  $i \in \{2, 3, \dots, N\}$ , satisfying condition (5.22), as stated in Proposition 5.4.5, for every  $k \in \{1, \dots, m\}$ . Moreover, let  $\delta \in (0, 1)$ ,  $e(t) = x(t_p^k) - x(t)$ ,  $\alpha_k = \max_{i \in \{2, 3, \dots, N\}} \{-c\lambda_i^k \|P_i^k \Gamma\|\}$ ,  $P^k = \text{diag}_{i \in \{2, 3, \dots, N\}} \{P_i^k\}$ ,*

and  $\{t_p^k\}$  be the triggering time-sequence generated by (5.23), that is restated as follows:

$$\|\bar{\Phi}^{k\top} e(t_p^k)\| = \frac{\delta}{\alpha_k} \|\bar{\Phi}^{k\top} x(t_p^k)\|, \quad p \in \mathbb{N}, \quad k \in \{1, \dots, m\}. \quad (5.62)$$

Then, the switching signal,  $\sigma : \mathbb{R}_{\geq 0} \rightarrow k$ , where  $\sigma(t) \in \mathcal{S}_{\text{average}}[\tau_a, N_0]$ , ensures the asymptotic synchronization of (5.5) with  $N_0 \in \mathbb{N}$ , and under the following condition:

$$\tau_a > \frac{\bar{\lambda}_{\min}}{2(1-\delta)} \log \left( \frac{\bar{\lambda}_{\max}}{\underline{\lambda}_{\min}} \right), \quad (5.63)$$

where,

$$\begin{aligned} \bar{\lambda}_{\min} &= \max_{k \in \{1, \dots, m\}} \{\lambda_{\min}(P^k)\}, \\ \bar{\lambda}_{\max} &= \max_{k \in \{1, \dots, m\}} \{\lambda_{\max}(P^k)\}, \\ \underline{\lambda}_{\min} &= \min_{k \in \{1, \dots, m\}} \{\lambda_{\min}(P^k)\}. \end{aligned} \quad (5.64)$$

*Proof.* In order to establish the proof of this result, we shall consider, without loss of generality, the time-interval  $(0, T)$ , over which switching between different topologies would occur. Then, at last, we shall express  $\|\bar{\Phi}^{\sigma(T^-)\top} x(T)\|$ , that is the distance to the synchronization manifold at time  $T$  and characterize  $\tau_a$  and  $N_0$  such that  $\lim_{T \rightarrow \infty} \|\bar{\Phi}^{\sigma(T^-)\top} x(T)\| = 0$  holds. This latter argument induces asymptotic synchronization recalling Definition 4 and provided Lemma 5.4.1.

Let us first recall the Lyapunov function,  $V^k(x) = x^\top \bar{\Phi}^k P^k \bar{\Phi}^{k\top} x$ , associated to every topology,  $k \in \{1, \dots, m\}$ , where matrices  $P^k$  are as discussed in theorem statement and matrices  $\bar{\Phi}^k$  are as characterized in Lemma 5.3.1. We note that the inequality  $\lambda_{\min}(P^k) \|\bar{\Phi}^{k\top} x\|^2 \leq V^k \leq \lambda_{\max}(P^k) \|\bar{\Phi}^{k\top} x\|^2$  holds. In addition, we also recall (5.55) from proof of Theorem 5.6.1, by which  $\|\bar{\Phi}^{p\top} x\| = \|\bar{\Phi}^{q\top} x\|$  for every topology,  $p, q \in \{1, \dots, m\}$ . Therefore, one can obtain:

$$\underline{\lambda}_{\min} \|\bar{\Phi}^{k\top} x\|^2 \leq V^k(x) \leq \bar{\lambda}_{\max} \|\bar{\Phi}^{k\top} x\|^2,$$

wherein,  $\underline{\lambda}_{\min}$  and  $\bar{\lambda}_{\max}$  are as introduced in (5.64) of theorem statement.

Moreover, under the triggering rule (5.62) and as discussed in Proposition 5.4.5, Equation (5.24), it holds that  $\dot{V}^k \leq -2(1-\delta) \|\bar{\Phi}^{k\top} x\|^2$ ; nonetheless, we

are interested in obtaining a uniform upper-bound for  $\dot{V}^k$ , therefore we derive the following:

$$\dot{V}^k \leq -2(1 - \delta) \frac{\lambda_{\min}(P^k)}{\lambda_{\min}(P^k)} \|\bar{\Phi}^{k\top} x\|^2 \leq \frac{-2(1 - \delta)}{\lambda_{\min}(P^k)} V^k \leq -2 \left( \frac{1 - \delta}{\bar{\lambda}_{\min}} \right) V^k,$$

where  $\bar{\lambda}_{\min}$  is as introduced in Equation (5.64), of theorem statement. Let us also denote  $\lambda_0 \triangleq \frac{1 - \delta}{\bar{\lambda}_{\min}}$ ; hence, we get:

$$\dot{V}^k \leq -2\lambda_0 V^k. \quad (5.65)$$

Let us also consider some arbitrary  $T > 0$ ,  $t_0 = 0$  and the switching time-instants,  $\tau_1, \tau_2, \dots, \tau_{N_\sigma(T,0)}$ , and the triggering time-instants belonging to each time-interval,  $[\tau_i, \tau_{i+1}[$  to be  $\{t_p^i\}_{p=1}^{p_i}$ . Let us then consider the function  $W(t) \triangleq \exp(2\lambda_0 t) V_{\sigma(t)}(x(t))$ , and time-interval,  $[\tau_i, \tau_{i+1}[$ , then recalling (5.65), we get:

$$\dot{W} = 2\lambda_0 W + \exp(2\lambda_0 t) \dot{V}_{\sigma(\tau_i)} \leq 2\lambda_0 W - 2\lambda_0 \exp(2\lambda_0 t) V_{\sigma(\tau_i)} \leq 0. \quad (5.66)$$

Then, developing  $W(\tau_{i+1})$ , in terms of  $W(\tau_i)$ , yields:

$$\begin{aligned} W(\tau_{i+1}) &= \exp(2\lambda_0 \tau_{i+1}) V_{\sigma(\tau_{i+1})}(x(\tau_{i+1})) \leq \\ &\mu \exp(2\lambda_0 \tau_{i+1}) V_{\sigma(\tau_i)}(x(\tau_{i+1})) = \mu W(\tau_{i+1}^-), \end{aligned} \quad (5.67)$$

wherein  $\mu$  is obtained as follows. Let  $p$  and  $q$  be two topologies belonging to set  $\{1, \dots, m\}$ , then the following holds:

$$\begin{aligned} V_p &\leq \lambda_{\max}(P^p) \|\bar{\Phi}^{p\top} x\|^2 \leq \bar{\lambda}_{\max} \|\bar{\Phi}^{p\top} x\|^2, \\ V_q &\geq \lambda_{\min}(P^q) \|\bar{\Phi}^{q\top} x\|^2 \geq \underline{\lambda}_{\min} \|\bar{\Phi}^{q\top} x\|^2, \end{aligned}$$

which can be rearranged to obtain  $\frac{1}{\lambda_{\max}} V^p \leq \|\bar{\Phi}^{p\top} x\|^2$  and  $\frac{1}{\underline{\lambda}_{\min}} V_q \geq \|\bar{\Phi}^{q\top} x\|^2$ , where then recalling  $\|\bar{\Phi}^{p\top} x\|^2 = \|\bar{\Phi}^{q\top} x\|^2$ , one can obtain:

$$V^p \leq \frac{\bar{\lambda}_{\max}}{\underline{\lambda}_{\min}} V^q = \mu V^q,$$

where  $\mu \triangleq \frac{\bar{\lambda}_{\max}}{\underline{\lambda}_{\min}}$ .

Also, we note that by (5.66), the function  $W(t)$  is a decreasing function over time-horizon  $[\tau_i, \tau_{i+1}[$ . Therefore, by (5.67), we get:

$$W(\tau_{i+1}) \leq \mu W(\tau_{i+1}^-) \leq \mu W(\tau_i),$$

this latter inequality, in an inductive way, and for  $i = 0$  to  $N_\sigma(T, 0) - 1$ , yields:

$$W(T^-) \leq W(\tau_{N_\sigma(T,0)}) \leq \mu^{N_\sigma(T,0)} W(0),$$

which then recalling definition of  $W(t)$  function, bestows:

$$\exp(2\lambda_0 T) V_{\sigma(T^-)}(x(T)) \leq \mu^{N_\sigma(T,0)} W(0) = \mu^{N_\sigma(T,0)} V_{\sigma(0)}(x(0)). \quad (5.68)$$

In addition, as we have considered the switching signal,  $\sigma(t) \in \mathcal{S}_{\text{average}}[\tau_a, N_0]$ , then  $N_\sigma(T, 0) \leq N_0 + \frac{T}{\tau_a}$ . Therefore, Equation (5.68) can further take the following form:

$$\begin{aligned} V_{\sigma(T^-)}(x(T)) &\leq \exp(-2\lambda_0 T) \exp(\log(\mu) N_\sigma(T, 0)) V_{\sigma(0)}(x(0)) \leq \\ &\exp\left(-2\lambda_0 T + \left(N_0 + \frac{T}{\tau_a}\right) \log(\mu)\right) V_{\sigma(0)}(x(0)), \end{aligned}$$

or, equivalently:

$$V_{\sigma(T^-)}(x(T)) \leq \mu^{N_0} \exp\left(\left(\frac{\log(\mu)}{\tau_a} - 2\lambda_0\right) T\right) V_{\sigma(0)}(x(0)). \quad (5.69)$$

At this stage, and recalling our earlier discussion on the proof procedure, we transform inequality (5.69) to obtain proper bound for  $\|\bar{\Phi}^{\sigma(T^-)\top} x(T)\|$  as follows:

$$0 \leq \|\bar{\Phi}^{\sigma(T^-)\top} x(T)\|^2 \leq \frac{\bar{\lambda}_{\max}}{\underline{\lambda}_{\min}} \mu^{N_0} \times \exp\left(\left(\frac{\log(\mu)}{\tau_a} - 2\lambda_0\right) T\right) \|\bar{\Phi}^{\sigma(0)\top} x(0)\|^2, \quad (5.70)$$

hence, based on this latter inequality and in order to ensure

$\lim_{T \rightarrow \infty} \|\bar{\Phi}^{\sigma(T^-)\top} x(T)\| = 0$ , it is sufficient to ensure  $\frac{\log(\mu)}{\tau_a} - 2\lambda_0 < 0$ —that is, in other words, the argument of the second exponential appearing in (5.70) be negative. Let us then recall parameters,  $\mu = \frac{\bar{\lambda}_{\max}}{\underline{\lambda}_{\min}}$  and  $\lambda_0 = \frac{1-\delta}{\lambda_{\min}}$ ; we can reform condition  $\frac{\log(\mu)}{\tau_a} - 2\lambda_0 < 0$  in the following way:

$$\tau_a > \frac{\bar{\lambda}_{\min}}{2(1-\delta)} \log\left(\frac{\bar{\lambda}_{\max}}{\underline{\lambda}_{\min}}\right),$$

which is in accordance with Equation (5.63) stated in theorem statement. This then completes the proof. We would also like to note that referring to (5.70), there is no need to impose specific condition on the chatter bound,  $N_0$ , i.e., it can take any value in  $\mathbb{N}$ ; this is because  $N_0$  appears as in the  $\mu^{N_0}$  term, wherein  $\mu$  is constant for a given set of topologies and which does not depend on  $T$ .  $\square$

In previous result, the asymptotic synchronization property is guaranteed under described conditions, where the decay rate of the synchronization cannot be designed. Indeed, an alteration in the targeted triggering strategy can help us characterize an exponential decay rate for the asymptotic synchronization; the following corollary is devoted to this aim.

**Corollary 5.6.4.** *Consider dynamics (5.5), where  $k \in \{1, \dots, m\}$ , and assume there exist matrices  $P_i^k = P_i^{k\top} \succ 0 \in \mathbb{R}^{n \times n}$ ,  $i \in \{2, 3, \dots, N\}$ , satisfying condition (5.22), as stated in Proposition 5.4.5, for every  $k \in \{1, \dots, m\}$ . Moreover, let  $\delta_1 > 0$ ,  $\delta_2 \in (0, 1)$ , such that  $\delta_1 + \delta_2 \in (0, 1)$ ,  $e(t) = x(t_p^k) - x(t)$ ,  $\alpha_k = \max_{i \in \{2, 3, \dots, N\}} \{-c\lambda_i^k \|P_i^k \Gamma\|\}$ ,  $P^k = \text{diag}_{i \in \{2, 3, \dots, N\}} \{P_i^k\}$ , and  $\{t_p^k\}$  be the triggering time-sequence generated by (5.23), that is restated as follows:*

$$\|\bar{\Phi}^{k\top} e(t_p^k)\| = \frac{\delta_2}{\alpha_k} \|\bar{\Phi}^{k\top} x(t_p^k)\|, \quad p \in \mathbb{N}, \quad k \in \{1, \dots, m\}. \quad (5.71)$$

Then, the switching signal,  $\sigma : \mathbb{R}_{\geq 0} \rightarrow k$ , where  $\sigma(t) \in \mathcal{S}_{\text{average}}[\tau_a, N_0]$ , ensures the asymptotic synchronization of (5.5) with the exponential decay rate of:

$$\lambda^* = \frac{\delta_1}{\bar{\lambda}_{\min}}, \quad (5.72)$$

with  $N_0 \in \mathbb{N}$ , and under the following condition:

$$\tau_a > \frac{\bar{\lambda}_{\min}}{2(1 - (\delta_1 + \delta_2))} \log \left( \frac{\bar{\lambda}_{\max}}{\underline{\lambda}_{\min}} \right), \quad (5.73)$$

where the parameters  $\bar{\lambda}_{\min}$ ,  $\bar{\lambda}_{\max}$ , and  $\underline{\lambda}_{\min}$  are as stated in Equation (5.64) of Theorem 5.6.3.

*Proof.* The proof of this corollary is similar to the proof of Theorem 5.6.3 with the main difference that by triggering strategy (5.71), we shall obtain:

$$\dot{V}^k \leq -2\lambda_0 V^k, \quad (5.74)$$

with  $\lambda_0 \triangleq \frac{1-\delta_2}{\bar{\lambda}_{\min}}$ . It is worth to mention that (5.74) is the counterpart to (5.65) defined earlier, nevertheless,  $\lambda_0$  is defined differently herein.

Then, following the same computations resulting in Equation (5.70), we re-obtain:

$$0 \leq \|\bar{\Phi}^{\sigma(T^-)\top} x(T)\|^2 \leq \frac{\bar{\lambda}_{\max}}{\underline{\lambda}_{\min}} \mu^{N_0} \times \exp \left( \left( \frac{\log(\mu)}{\tau_a} - 2\lambda_0 \right) T \right) \|\bar{\Phi}^{\sigma(0)\top} x(0)\|^2, \quad (5.75)$$

where we recall  $\mu = \frac{\bar{\lambda}_{\max}}{\underline{\lambda}_{\min}}$ . Let us then, this time, impose the following condition:

$$\frac{\log(\mu)}{\tau_a} - 2\lambda_0 < -2\lambda^*, \quad (5.76)$$

with  $\lambda^* \in (0, \lambda_0)$ ; then inequality (5.75) for  $\|\bar{\Phi}^{\sigma(T^-)\top} x(T)\|^2$  can be further upper-bounded as follows:

$$\begin{aligned} 0 \leq \|\bar{\Phi}^{\sigma(T^-)\top} x(T)\|^2 &\leq \frac{\bar{\lambda}_{\max}}{\underline{\lambda}_{\min}} \mu^{N_0} \times \\ \exp\left(\left(\frac{\log(\mu)}{\tau_a} - 2\lambda_0\right)T\right) \|\bar{\Phi}^{\sigma(0)\top} x(0)\|^2 &\leq \\ \frac{\bar{\lambda}_{\max}}{\underline{\lambda}_{\min}} \mu^{N_0} \exp(-2\lambda^*T) \|\bar{\Phi}^{\sigma(0)\top} x(0)\|^2. & \end{aligned} \quad (5.77)$$

Provided (5.77), i.e., the bound

$0 \leq \|\bar{\Phi}^{\sigma(T^-)\top} x(T)\|^2 \leq \frac{\bar{\lambda}_{\max}}{\underline{\lambda}_{\min}} \mu^{N_0} \exp(-2\lambda^*T) \|\bar{\Phi}^{\sigma(0)\top} x(0)\|^2$  it can now be verified that the decay of  $\|\bar{\Phi}^{\sigma(T^-)\top} x(T)\|^2$ —which declares the asymptotic synchronization behavior—is dominated by an exponential term whose decay rate is declared by  $\lambda^*$ ; where also, without loss of generality, we can consider  $\lambda^* = \frac{\delta_1}{\lambda_{\min}}$ , i.e., Equation (5.72) in theorem statement. This way, nevertheless, given  $\lambda^* \in (0, \lambda_0)$ , and that  $\lambda_0 = \frac{1-\delta_2}{\lambda_{\min}}$ , it is easy to verify that  $\delta_1 > 0$  and  $\delta_1 + \delta_2 \in (0, 1)$  which are also attributed on in the theorem statement.

As the last step, let us recall parameters  $\mu, \lambda_0, \lambda^*$ , then by (5.72) and provided  $\lambda^* < \lambda_0$ , we obtain:

$$\tau_a > \log\left(\frac{\bar{\lambda}_{\max}}{\underline{\lambda}_{\min}}\right) \frac{\bar{\lambda}_{\min}}{2(1 - (\delta_1 + \delta_2))},$$

that is in accordance with (5.73). The proof is then complete, where we note that by (5.77), parameter  $N_0$  does not influence the asymptotic behavior of  $\|\bar{\Phi}^{\sigma(T^-)\top} x(T)\|$  and thus it can be chosen to be any value in natural numbers, i.e.,  $N_0 \in \mathbb{N}$ .  $\square$



## 5.7 Particular Network Dynamics: Skew-Symmetric

$H$

In this section, we shall discuss the consequences of the matrix  $H$  be skew-symmetric in our synchronization dynamics (5.5):

$$\dot{x}(t) = (I_N \otimes H)x(t) + c(A^k \otimes \Gamma)x(t_p^k), \quad \forall t \in [t_p^k, t_{p+1}^k[.$$

In particular, we are interested in assessing its effect on dynamics of the average state,  $\bar{x}(t) = \frac{1}{N} \sum_{j=1}^N x_j(t)$ . This latter is motivated by the fact that by Lemma 5.4.1, where we demonstrate by (5.9) that the distance to synchronization manifold,  $|x|_{\mathcal{A}_s}$ , can be characterized by the set of eigenvectors of  $A^k$ ; namely, where we prove that  $\|\bar{\Phi}^\top x\| = |x|_{\mathcal{A}_s}$  holds, we also assert the following:

$$|x|_{\mathcal{A}_s} = \|\bar{\Phi}^\top x\|^2 = \sum_{i=1}^N \|x_i - \frac{1}{N} \sum_{j=1}^N x_j\|^2. \quad (5.78)$$

On the other hand, recalling Definition 4, we note that synchronization is in place as long as  $\lim_{t \rightarrow \infty} |x(t; x_0)|_{\mathcal{A}_s} = 0$  is ensured. This latter, provided the above equation, implies that we would like to ensure:

$$\lim_{t \rightarrow \infty} \|x_i(t) - \frac{1}{N} \sum_{j=1}^N x_j(t)\| = \lim_{t \rightarrow \infty} \|x_i(t) - \bar{x}(t)\| = 0,$$

for every node,  $i \in \{1, \dots, N\}$ —this is provided that (5.78) entails summation of nonnegative terms, thus the summation is zero as long as every individual term is zero. Henceforth, once synchronization is guaranteed, the asymptotic behavior of the states of every node,  $x_i(t)$ , follows indeed the average state,  $\bar{x}(t) = \frac{1}{N} \sum_{j=1}^N x_j(t)$ .

In what follows, we shall first characterize the average dynamics, as performed in the following result.

**Lemma 5.7.1.** *Consider network dynamics (5.5), with outercoupling matrices,  $A^k$ , be symmetric, irreducible, and with zero-sum property. Then, the following holds:*

$$\dot{\bar{x}} = H\bar{x}. \quad (5.79)$$

*Proof.* Given the network dynamics (5.5):

$$\dot{x}(t) = (I_N \otimes H)x(t) + c(A^k \otimes \Gamma)x(t_p^k), \quad \forall t \in [t_p^k, t_{p+1}^k[ ,$$

the dynamics of each node can be then obtained as follows:

$$\dot{x}_i = Hx_i + c \sum_{j=1}^N a_{ij}^k \Gamma x_j(t_p^k), \quad \forall t \in [t_p^k, t_{p+1}^k[ .$$

Thereby recalling the average state,  $\bar{x}(t) = \frac{1}{N} \sum_{j=1}^N x_j(t)$ , we shall perform the following computations:

$$\begin{aligned} \dot{\bar{x}} &= \frac{1}{N} \sum_{i=1}^N \dot{x}_i = \frac{1}{N} \sum_{i=1}^N Hx_i + c \sum_{j=1}^N a_{ij}^k \Gamma x_j(t_p^k) \\ &= \frac{1}{N} \sum_{i=1}^N Hx_i + \frac{1}{N} c \sum_{i=1}^N \sum_{j=1}^N a_{ij}^k \Gamma x_j(t_p^k) \\ &= \frac{1}{N} H \sum_{i=1}^N x_i + \frac{1}{N} c \sum_{j=1}^N \left( \sum_{i=1}^N a_{ij}^k \right) \Gamma x_j(t_p^k) = H\bar{x}, \end{aligned}$$

wherein the zero-sum property of  $A^k$ —as debated in (5.2)—has been helpful. These latter equations indeed confirm that  $\dot{\bar{x}} = H\bar{x}$ , and thus completes this proof.  $\square$

At this stage, and before stepping forward, we shall recall some useful properties of skew-symmetric matrices which would be helpful in our subsequent analysis.

**Lemma 5.7.2** ([12]). *For a given skew-symmetric matrix,  $H \in \mathbb{R}^{n \times n}$ , the following properties hold:*

1. *Arrays of matrix  $H$ :  $h_{ii} = 0, \forall i \in \{1, \dots, n\}$ , and  $h_{ij} = -h_{ji}, \forall i, j \in \{1, \dots, n\}$ .*
2. *Eigenvalues of matrix  $H$ :*
  - (a) *if  $\text{mod}(n, 2) = 0$ :  $\{\pm i\lambda_j\}_{j=1}^{\frac{n}{2}}$ , that is to say all the eigenvalues are purely imaginary eigenvalues,*
  - (b) *if  $\text{mod}(n, 2) = 1$ :  $\lambda_1 = 0$ , with algebraic multiplicity, 1, and  $\{\pm i\lambda_j\}_{j=2}^{\frac{n-1}{2}}$ , that is to say all the remaining eigenvalues are purely imaginary.*

3. *Eigenvectors of matrix  $H$ :*

- (a) if  $\text{mod}(n, 2) = 0$ :  $\{v_k\}_{k=1}^n$  is the set of complex eigenvectors associated to the correspondent set of eigenvalues,  $\{\pm i\lambda_j\}_{j=1}^{\frac{n}{2}}$ , which are also complex conjugate to each other,
- (b) if  $\text{mod}(n, 2) = 1$ : set of eigenvectors consist of a real eigenvector,  $v_1$ , associated to  $\lambda_1 = 0$ , and  $\{v_k\}_{k=2}^n$  the set of complex eigenvectors associated to the correspondent set of eigenvalues,  $\{\pm i\lambda_j\}_{j=2}^{\frac{n-1}{2}}$ , which are also complex conjugate to each other.

Having stated properties of a skew-symmetric matrix in Lemma 5.7.2, in the subsequent result, we shall study the response to the average states dynamics:  $\dot{\bar{x}} = H\bar{x}$  stated in (5.79), Lemma 5.7.1.

**Proposition 5.7.3.** *Consider the average state dynamics (5.79):  $\dot{\bar{x}} = H\bar{x}$ , with some initial condition,  $\bar{x}_0 \in \mathbb{R}^n$ . Let  $H \in \mathbb{R}^{n \times n}$  be skew-symmetric, then the response to this dynamics satisfies the following:*

1. if  $\text{mod}(n, 2) = 0$ :

$$\bar{x}_i(t) = \sum_{j=1}^n \left[ \sum_{k=1}^{\frac{n}{2}} (v_{ki}v_{kj} + \bar{v}_{ki}\bar{v}_{kj}) \cos(\lambda_k t) + \sum_{k=1}^{\frac{n}{2}} (v_{ki}v_{kj} - \bar{v}_{ki}\bar{v}_{kj}) \sin(\lambda_k t) \right] \bar{x}_{0j}, \quad (5.80)$$

for  $i \in \{1, \dots, n\}$ ,

2. if  $\text{mod}(n, 2) = 1$ :

$$\bar{x}_i(t) = \sum_{j=1}^n \left[ (v_{0i})(v_{0j})\bar{x}_{0j} \right] + \sum_{j=1}^n \left[ \sum_{k=1}^{\frac{n-1}{2}} (v_{ki}v_{kj} + \bar{v}_{ki}\bar{v}_{kj}) \cos(\lambda_k t) + \sum_{k=1}^{\frac{n-1}{2}} (v_{ki}v_{kj} - \bar{v}_{ki}\bar{v}_{kj}) \sin(\lambda_k t) \right] \bar{x}_{0j}, \quad (5.81)$$

for  $i \in \{1, \dots, n\}$ ,

where in both (5.80) and (5.81),  $\{\pm i\lambda_j\}_{j=1}^{\frac{n}{2}}$  and  $\{v_k\}_{k=1}^n$  stand, respectively, for the eigenvalues and eigenvectors of  $H$ , as characterized in Lemma 5.7.2. In addition,  $\bar{v}$  stands for the complex conjugate of  $v$ .

*Proof.* The proof of this result is categorized under two parts, in conjunction with items 1 and 2 stated in the assertion. Nonetheless, in order to proceed in both cases we shall use the eigen-decomposition technique for  $H$ . On this way, we denote matrices  $D \triangleq \text{diag}_{i \in \{1, \dots, n\}} \lambda_j$  and  $V \triangleq [v_1, v_2, \dots, v_n]$  with—recalling notations of Lemma 5.7.2— $\lambda_{2j-1} = i\lambda_j$  and  $\lambda_{2j} = -\lambda_j$  for  $j \in \{1, \dots, \frac{n}{2}\}$ , where then  $H = VDV^\top$  holds. This latter technique shall be useful, provided that  $\exp Ht = V \exp(Dt)V^\top$ .

**Case 1:**  $\text{mod}(n, 2) = 0$ . Let  $n' = \frac{n}{2}$ , then diagonal matrix  $D$  can be written as follows:

$$D = \begin{bmatrix} i\lambda_1 & & & & & \\ & -i\lambda_1 & & & & \\ & & \ddots & & & \\ & & & i\lambda_{n'} & & \\ & & & & -i\lambda_{n'} & \\ & & & & & \end{bmatrix},$$

where then, the following holds:

$$\exp(Dt) = \begin{bmatrix} \exp(i\lambda_1 t) & & & & & \\ & \exp(-i\lambda_1 t) & & & & \\ & & \ddots & & & \\ & & & \exp(i\lambda_{n'} t) & & \\ & & & & \exp(-i\lambda_{n'} t) & \\ & & & & & \end{bmatrix}.$$

Having developed the latter equation and recalling eigendecomposition of the matrix,  $H$ , we shall then compute the following expression for  $\exp(Ht)$ :

$$\exp(Ht) = V \exp(Dt)V^\top = \begin{bmatrix} | & | & \cdots & | & | \\ v_1 \bar{v}_1 & \cdots & v_{n'} \bar{v}_{n'} & & \\ | & | & \cdots & | & | \end{bmatrix} \times \begin{bmatrix} \exp(i\lambda_1 t) & & & & \\ & \exp(-i\lambda_1 t) & & & \\ & & \ddots & & \\ & & & \exp(i\lambda_{n'} t) & \\ & & & & \exp(-i\lambda_{n'} t) \end{bmatrix} \begin{bmatrix} \text{---}v_1^\top\text{---} \\ \text{---}\bar{v}_1^\top\text{---} \\ \vdots \\ \text{---}v_{n'}^\top\text{---} \\ \text{---}\bar{v}_{n'}^\top\text{---} \end{bmatrix}.$$

This latter equation can be further expanded as follows, where (i) we note  $\exp(i\lambda t) = \cos(\lambda t) + i \sin(\lambda t)$  and (ii) the eigenvectors are denoted to be  $v_j = [v_{j1}, v_{j2}, \dots, v_{jn}]^\top$  and  $\bar{v}_j = [\bar{v}_{j1}, \bar{v}_{j2}, \dots, \bar{v}_{jn}]^\top$  for  $j \in \{1, \dots, n'\}$ :

$$\exp(Ht) = \begin{bmatrix} v_{11}(\cos(\lambda_1 t) + i \sin(\lambda_1 t)) & \bar{v}_{11}(\cos(\lambda_1 t) - i \sin(\lambda_1 t)) & \vdots & \cdots \\ \vdots & \vdots & \vdots & \vdots \\ v_{1n}(\cos(\lambda_1 t) + i \sin(\lambda_1 t)) & \bar{v}_{1n}(\cos(\lambda_1 t) - i \sin(\lambda_1 t)) & \vdots & \cdots \\ \vdots & \vdots & \vdots & \vdots \\ v_{n'1}(\cos(\lambda_{n'} t) + i \sin(\lambda_{n'} t)) & \bar{v}_{n'1}(\cos(\lambda_{n'} t) - i \sin(\lambda_{n'} t)) & \vdots & \cdots \\ \vdots & \vdots & \vdots & \vdots \\ v_{n'n}(\cos(\lambda_{n'} t) + i \sin(\lambda_{n'} t)) & \bar{v}_{n'n}(\cos(\lambda_{n'} t) - i \sin(\lambda_{n'} t)) & \vdots & \cdots \end{bmatrix} \begin{bmatrix} v_1^\top \\ \bar{v}_1^\top \\ \vdots \\ v_{n'}^\top \\ \bar{v}_{n'}^\top \end{bmatrix}.$$

Let us then denote  $\exp(Ht) \triangleq [\text{EXP}_{ij}]$ , for  $i, j \in \{1, \dots, n\}$ , the following holds:

$$\text{EXP}_{ij} = \sum_{k=1}^{n'} (v_{ki})(v_{kj})(\cos(\lambda_k t) + i \sin(\lambda_k t)) + \sum_{k=1}^{n'} (\bar{v}_{ki})(\bar{v}_{kj})(\cos(\lambda_k t) - i \sin(\lambda_k t)), \quad (5.82)$$

we shall then use this latter equation to derive solution to the average dynamics:  $\dot{\bar{x}} = H\bar{x}$ , with initial condition,  $\bar{x}_0$ . This fact is shown below, where we have considered  $\bar{x}_0 = [\bar{x}_{01}, \bar{x}_{02}, \dots, \bar{x}_{0n}]^\top$ :

$$\bar{x}(t) = \exp(Ht)\bar{x}_0 = \begin{bmatrix} \text{EXP}_{11} & \text{EXP}_{12} & \cdots & \text{EXP}_{1n} \\ \text{EXP}_{21} & \text{EXP}_{22} & \cdots & \text{EXP}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \text{EXP}_{n1} & \text{EXP}_{n2} & \cdots & \text{EXP}_{nn} \end{bmatrix} \begin{bmatrix} \bar{x}_{01} \\ \bar{x}_{02} \\ \vdots \\ \bar{x}_{0n} \end{bmatrix}.$$



Therefore, provided the eigendecomposition of  $H$ , we shall express  $\exp(Ht)$  as follows:

$$\exp(Ht) = V \exp(Dt) V^T = \begin{bmatrix} | & | & | & \cdots & | & | \\ v_0 v_1 \bar{v}_1 \cdots v_{n'} \bar{v}_{n'} \\ | & | & | & \cdots & | & | \end{bmatrix} \times \begin{bmatrix} 1 & & & & & \\ \exp(i\lambda_1 t) & & & & & \\ & \exp(-i\lambda_1 t) & & & & \\ & & \ddots & & & \\ & & & \exp(i\lambda_{n'} t) & & \\ & & & & \exp(-i\lambda_{n'} t) & \end{bmatrix} \begin{bmatrix} \text{---}v_0^T\text{---} \\ \text{---}v_1^T\text{---} \\ \text{---}\bar{v}_1^T\text{---} \\ \vdots \\ \text{---}v_{n'}^T\text{---} \\ \text{---}\bar{v}_{n'}^T\text{---} \end{bmatrix}.$$

This latter equation can be further expanded as follows, where the eigenvectors are denoted to be  $v_j = [v_{j1}, v_{j2}, \dots, v_{jn}]^T$  and  $\bar{v}_j = [\bar{v}_{j1}, \bar{v}_{j2}, \dots, \bar{v}_{jn}]^T$  for  $j \in \{1, \dots, n'\}$ :

$$\exp(Ht) = \begin{bmatrix} v_{01} & v_{11}(\exp(i\lambda_1 t)) & \bar{v}_{11}(\exp(-i\lambda_1 t)) & \cdots & | \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ v_{0n} & v_{1n}(\exp(i\lambda_1 t)) & \bar{v}_{1n}(\exp(-i\lambda_1 t)) & \cdots & | \end{bmatrix} \begin{bmatrix} \text{---}v_0^T\text{---} \\ \text{---}v_1^T\text{---} \\ \text{---}\bar{v}_1^T\text{---} \\ \vdots \\ \text{---}v_{n'}^T\text{---} \\ \text{---}\bar{v}_{n'}^T\text{---} \end{bmatrix}.$$

We then note that  $\exp(i\lambda t) = \cos(\lambda t) + i \sin(\lambda t)$ , let us also denote  $\exp(Ht) \triangleq [\text{EXP}'_{ij}]$ , for  $i, j \in \{1, \dots, n\}$ , then the following holds:

$$\begin{aligned} \text{EXP}'_{ij} &= (v_{0i})(v_{0j}) + \sum_{k=1}^{n'} (v_{ki})(v_{kj})(\cos(\lambda_k t) + i \sin(\lambda_k t)) \\ &\quad + \sum_{k=1}^{n'} (\bar{v}_{ki})(\bar{v}_{kj})(\cos(\lambda_k t) - i \sin(\lambda_k t)), \end{aligned} \quad (5.83)$$

we shall then use this latter equation to derive solution to the average dynamics:  $\dot{\bar{x}} = H\bar{x}$ , with initial condition,  $\bar{x}_0$ . This fact is shown below, where we have

considered  $\bar{x}_0 = [\bar{x}_{01}, \bar{x}_{02}, \dots, \bar{x}_{0n}]^\top$ :

$$\bar{x}(t) = \exp(Ht)\bar{x}_0 = \begin{bmatrix} \text{EXP}'_{11} & \text{EXP}'_{12} & \cdots & \text{EXP}'_{1n} \\ \text{EXP}'_{21} & \text{EXP}'_{22} & \cdots & \text{EXP}'_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \text{EXP}'_{n1} & \text{EXP}'_{n2} & \cdots & \text{EXP}'_{nn} \end{bmatrix} \begin{bmatrix} \bar{x}_{01} \\ \bar{x}_{02} \\ \vdots \\ \bar{x}_{0n} \end{bmatrix}.$$

This latter equation then yields:

$$\bar{x}_i(t) = \sum_{j=1}^n \text{EXP}'_{ij} \bar{x}_{0j}, \quad i \in \{1, \dots, n\}.$$

Therefore, plugging from (5.83) in terms of  $\text{EXP}'_{ij}$  in this latter equation, yields:

$$\begin{aligned} \bar{x}_i(t) = & \sum_{j=1}^n \left[ (v_{0i})(v_{0j})\bar{x}_{0j} \right] + \\ & \sum_{j=1}^n \left[ \sum_{k=1}^{n'} (v_{ki}v_{kj} + \bar{v}_{ki}\bar{v}_{kj}) \cos(\lambda_k t) + \right. \\ & \left. \sum_{k=1}^{n'} (v_{ki}v_{kj} - \bar{v}_{ki}\bar{v}_{kj}) \sin(\lambda_k t) \right] \bar{x}_{0j}, \end{aligned}$$

which, recalling  $n' = \frac{n-1}{2}$ , validates Equation (5.81) stated in the proposition for the case of even  $n$ . Therefore, proof for this case is complete.

Hence, the proof of this proposition is complete.  $\square$

Having discussed the response to the average state dynamics in the previous proposition. We shall discuss the properties of this response in the following remark.

**Remark 5.7.4.** *For the case of even  $n$ : we refer to Equation (5.80), stated in Proposition 5.7.3, where we observe that every  $i^{\text{th}}$  component of the average state,  $\bar{x}_i$  for  $i \in \{1, \dots, n\}$ , is characterized by eigenvectors of the matrix,  $H$ , i.e.,  $\{v_k\}$ , as the amplitudes of the harmonics, along with eigenvalues of  $H$ ,  $\{\pm i\lambda_k\}$ , as the frequencies*



of the harmonics. In addition, it is also observed that the temporal average value of this summation of harmonics is zero, thus there is no biased term in the obtained response.

**For the case of odd  $n$ :** we refer to Equation (5.81), stated in Proposition 5.7.3, where then similar observation as for the case of even  $n$  can be obtained, nonetheless, this time we also observe that the temporal average of this summation of harmonics is not zero. That is to say, there exists a bias term characterized by the purely real eigenvector,  $v_0$  associated to the zero eigenvalue of  $H$ .

## 5.8 Simulations

In this section, we demonstrate the functionality of the regulatory conditions on  $\mathcal{S}_{\text{dwell}}[\tau_D]$  and  $\mathcal{S}_{\text{average}}[\tau_a, N_0]$  classes of switching signals as discussed in Section 5.6. We first introduce the system and different associated parameters necessary to conduct our simulation, it is then followed by discussing simulations for  $\mathcal{S}_{\text{dwell}}[\tau_D]$  and  $\mathcal{S}_{\text{average}}[\tau_a, N_0]$  classes.

### 5.8.1 Simulations Basic

In this set of simulations, we have considered network of 5 agents each of which characterized by 2 states, i.e.,  $N = 5$ , and,  $n = 2$ . Accordingly, the following matrices,  $H$ , and,  $\Gamma$  have been considered:

$$H = \begin{bmatrix} 0 & -0.5 \\ 0.5 & 0 \end{bmatrix}, \quad \Gamma = \begin{bmatrix} 0.25 & 0 \\ -1 & 0.25 \end{bmatrix},$$

where we note that (i) these matrices are indeed common for all the topologies, and (ii) the eigenvalues of matrix  $H$  are  $\lambda_{1,2} = \pm 0.5i$  (with  $i = \sqrt{-1}$ ) which declares that the asymptotic behavior of the average dynamics,  $\dot{\bar{x}} = I_5 \otimes H \bar{x}$ , encompasses two principle time-periodic solutions—in accordance with the even and odd states—with period,  $T = \frac{2\pi}{0.5} = 4\pi$ , to which the synchronized states have to converge, asymptotically. This latter fact is insightful in our discussion of this section.

We have then considered three potential topologies for the network, each characterized by the outer-coupling matrices,  $A^1$ ,  $A^2$ , and  $A^3$ , described as follows:

$$\begin{aligned}
 A^1 &= \begin{bmatrix} -3 & 1 & 1 & 0 & 1 \\ 1 & -3 & 1 & 1 & 0 \\ 1 & 1 & -4 & 1 & 1 \\ 0 & 1 & 1 & -3 & 1 \\ 1 & 0 & 1 & 1 & -3 \end{bmatrix}, \\
 A^2 &= \begin{bmatrix} -2 & 1 & 1 & 0 & 0 \\ 1 & -3 & 1 & 1 & 0 \\ 1 & 1 & -4 & 1 & 1 \\ 0 & 1 & 1 & -3 & 1 \\ 0 & 0 & 1 & 1 & -2 \end{bmatrix}, \\
 A^3 &= \begin{bmatrix} -1 & 1 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix}, \tag{5.84}
 \end{aligned}$$

where we note that the set of eigenvalues of these topologies are, respectively,  $\lambda_1^1 = 0$ ,  $\lambda_{2,3}^1 = -3$ ,  $\lambda_{4,5}^1 = -5$ ;  $\lambda_1^2 = 0$ ,  $\lambda_2^2 = -1.5858$ ,  $\lambda_3^2 = -3$ ,  $\lambda_4^2 = -4.4142$ , and  $\lambda_5^2 = -5$ ;  $\lambda_1^3 = 0$ ,  $\lambda_2^3 = -0.382$ ,  $\lambda_3^3 = -1.382$ ,  $\lambda_4^3 = -2.6188$ ,  $\lambda_5^3 = -3.618$ . This latter, in fact, demonstrates also that the networks associated to topologies 1, 2, and 3 may be ranked from the most to the least connected ones—this is valid recalling the fact that outer-coupling matrix plays the role of the negative of Laplacian matrix and that connectivity can be measured by the second smallest eigenvalue [27], and therefore, it can be observed that  $|\lambda_2^1| > |\lambda_2^2| > |\lambda_2^3|$ . This latter observation will be insightful in our discussion of this section.

In addition to set of eigenvalues, we compute set of eigenvectors for the outer-coupling matrices introduced in (5.84). Recalling then the notations introduced in Lemma 5.3.1, we get  $\phi_1^1 = \phi_1^2 = \phi_1^3 = \frac{1}{\sqrt{5}}[1, 1, 1, 1, 1]^\top$ ; in addition, we

get:

$$\Phi^1 = \begin{bmatrix} -0.7071 & 0 & -0.4082 & 0.3651 \\ 0 & -0.7071 & 0 & -0.5477 \\ 0 & 0 & 0.8165 & 0.3651 \\ 0.7071 & 0 & -0.4082 & 0.3651 \\ 0 & 0.7071 & 0 & -0.5477 \end{bmatrix},$$

$$\Phi^2 = \begin{bmatrix} -0.6533 & -0.5 & 0.2706 & -0.2236 \\ -0.2706 & 0.5 & -0.6533 & -0.2236 \\ 0 & 0 & 0 & 0.8944 \\ 0.2706 & 0.5 & 0.6533 & -0.2236 \\ 0.6533 & -0.5 & -0.2706 & -0.2236 \end{bmatrix},$$

$$\Phi^3 = \begin{bmatrix} -0.6015 & -0.5117 & -0.3717 & -0.1954 \\ -0.3717 & 0.1954 & 0.6015 & 0.5117 \\ 0 & 0.6325 & 0 & -0.6325 \\ 0.3717 & 0.1954 & -0.6015 & 0.5117 \\ 0.6015 & -0.5117 & 0.3717 & -0.1954 \end{bmatrix},$$

where then we get  $\bar{\Phi}^k = \Phi^k \otimes I_2$ , for  $k \in \{1, 2, 3\}$ .

As the next step, recalling our discussion in Proposition 5.4.5, associated matrices  $\{P_i^k\}_{i \in \{2, \dots, 5\}}$  stated in Equation (5.22), can be computed for every topology,  $k \in \{1, 2, 3\}$ —provided that Equation (5.22) is indeed a continuous-time Lyapunov equation.

We have then conducted two sets of simulations, namely in accordance with  $\mathcal{S}_{\text{dwell}}[\tau_D]$  and  $\mathcal{S}_{\text{average}}[\tau_D]$  classes.

### 5.8.2 Simulations for $\mathcal{S}_{\text{dwell}}[\tau_D]$

For this set of simulations, we consider various parameters introduced in the previous subsection. We then consider the “worst-case switching scenario,” where every topology is active for  $\tau_D$  seconds, we note that for  $\mathcal{S}_{\text{dwell}}[\tau_D]$  class of switching signals, this scenario is indeed worst-case, because as discussed in

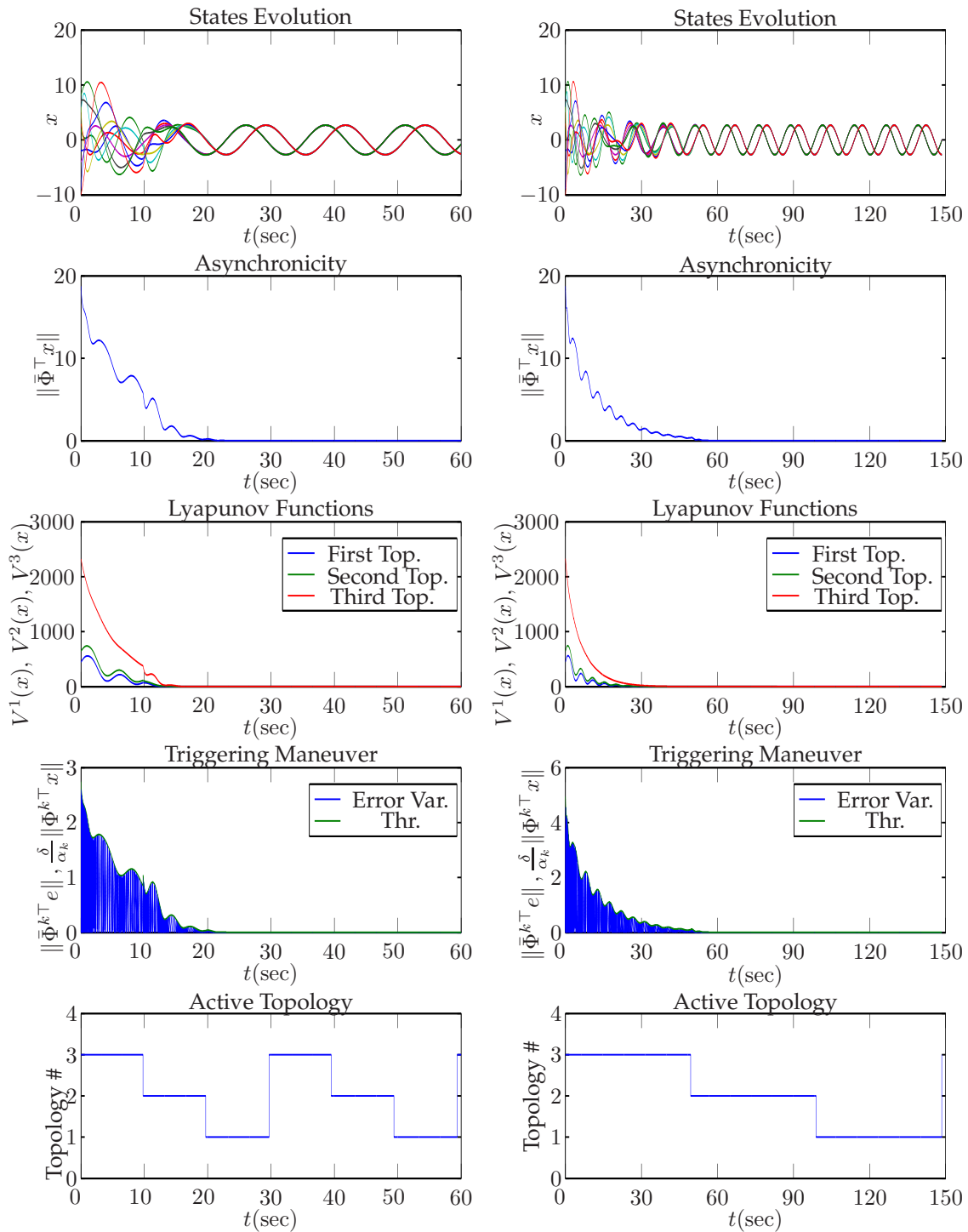
the proof of Theorem 5.6.1,  $\tau_D$  is the minimal amount of time required for every topology to be active in order to ensure the overall stability.

Indeed, provided we have considered 3 topologies in Subsection 5.8.1, the appropriate parameter  $\tau_D$  may be obtained recalling our discussion in Subsection 5.6.1; more specifically, Algorithm 2. In addition, we would like to note the presence of parameter  $\delta \in (0, 1)$  in Equation (5.61) of this aforementioned algorithm as well as its presence in the triggering strategy (5.62). Accordingly, in this set of simulations, we shall study the trade off in terms of choosing higher and lower values for  $\delta$ .

**Case 1:** let  $\delta = 0.5$ , then for the set of parameters stated in Subsection 5.8.1, we obtain  $\tau_D = 9.8993$ . We then conduct the simulations for the afore discussed worst-case switching scenario, where we consider consecutive time-intervals,  $[n\tau_D, (n+1)\tau_D[$ ,  $n \in \mathbb{N}_0$ , where for  $n = 3k$ ,  $n = 3k + 1$ , and  $n = 3k + 2$ , with  $k \in \mathbb{N}_0$ , the first, second, and third topology are active, respectively. The motivation behind choosing this sequence of topologies is that, as discussed previously, this way the topologies are ranked from the least to the most connected ones, rendering the simulations more tractable. It also goes without saying that for every time-interval,  $[n\tau_D, (n+1)\tau_D[$ , the triggering strategy (5.62) is implemented. The results are shown in Figure 5.1(a), where asymptotic synchronization despite presence of switching and under triggering strategy is demonstrated.

**Case 2:** let  $\delta = 0.9$ , then for the set of parameters stated in Subsection 5.8.1, we obtain  $\tau_D = 49.4964$ . Then, we conduct our simulation along the same lines as discussed in **Case 1**. The results are shown in Figure 5.1(b), where asymptotic synchronization despite presence of switching and under triggering strategy is demonstrated.

At this point, comparing the simulation results obtained from Cases 1 and 2, demonstrated in figures 5.1(a) and 5.1(b), we shall validate that increasing  $\delta$  yields less frequent triggering and larger allowable dwell time,  $\tau_D$ . This trade off is indeed in accordance with our theoretical assessment attributed in Subsection 5.6.1.

(a) Case 1:  $\delta = 0.5$ (b) Case 2:  $\delta = 0.9$ **Figure 5.1:** Temporal results for uniform switching

### 5.8.3 Simulations for $\mathcal{S}_{\text{average}}[\tau_a, N_0]$

In this set of simulations, we again consider the set of parameters introduced in Subsection 5.8.1. We then recall the properties of switching signals associated to this class, for which, provided the chatter bound,  $N_0$ , and average dwell time,  $\tau_a$ , the number of discontinuities over every  $(t, \tau)$  time-interval,  $N_\sigma(\tau, t)$ , is characterized by Equation (5.8) that we recall as follows:

$$N_\sigma(\tau, t) \leq N_0 + \frac{\tau - t}{\tau_a} \quad \forall t \geq \tau \geq 0.$$

Also, in our simulations, we consider the time-intervals,  $(t, \tau) = (3n\tau_a, 3(n+1)\tau_a)$ , with  $n \in \mathbb{N}_0$ , where then the number of discontinuities would be characterized as  $N_\sigma(3n\tau_a, 3(n+1)\tau_a) \leq N_0 + 3$ . We then recall that the chatter bound,  $N_0$ , can be chosen as an arbitrary value in natural numbers, provided our discussion in Subsection 5.6.2, more specifically, in Theorem 5.6.3; therefore, in here we consider  $N_0 = 1$ , which then yields  $N_\sigma(3n\tau_a, 3(n+1)\tau_a) \leq 4$ ; hence, we pick  $N_\sigma(3n\tau_a, 3(n+1)\tau_a) = 3$ , which infers that we shall consider the number of discontinuities over every  $(3n\tau_a, 3(n+1)\tau_a)$  time-interval to be 3—this is indeed in accordance with number of topologies we have considered in this set of simulation. In order to complete our simulation setup, we add that within every  $(3n\tau_a, 3(n+1)\tau_a)$ , we shall consider the subintervals,  $(3n\tau_a, (3n+1)\tau_a)$ ,  $((3n+1)\tau_a, (3n+2)\tau_a)$ , and  $((3n+2)\tau_a, 3(n+1)\tau_a)$  wherein the third, second, and first topologies are active, respectively. This latter has been chosen motivated by the fact that this way the topologies have been ranked from the most to the least connected. The simulation results are then discussed in the following under Cases 1 and 2, where again the trade off of choosing  $\delta \in (0, 1)$  on the lower-bound of parameter  $\tau_a$  described in Equation (5.63) and triggering strategy (5.62) both stated and characterized in Theorem 5.6.3 shall be elaborated.

**Case 1:** let  $\delta = 0.5$ , then for the set of parameters stated in Subsection 5.8.1, we obtain  $\tau_a = 1.5350$ . We then conduct our simulation along the lines of the afore discussed switching scenario, where we have considered consecutive time-intervals,  $[n\tau_a, (n+1)\tau_a]$ ,  $n \in \mathbb{N}_0$ , where for  $n = 3k$ ,  $n = 3k+1$ , and  $n = 3k+2$ , with  $k \in \mathbb{N}_0$ , the first, second, and third topology are active, respec-

tively. It also goes without saying that for every time-interval,  $[n\tau_a, (n+1)\tau_a[$ , the triggering strategy (5.62) is implemented. The results are shown in Figure 5.2(a), where asymptotic synchronization despite presence of switching and under triggering strategy is demonstrated.

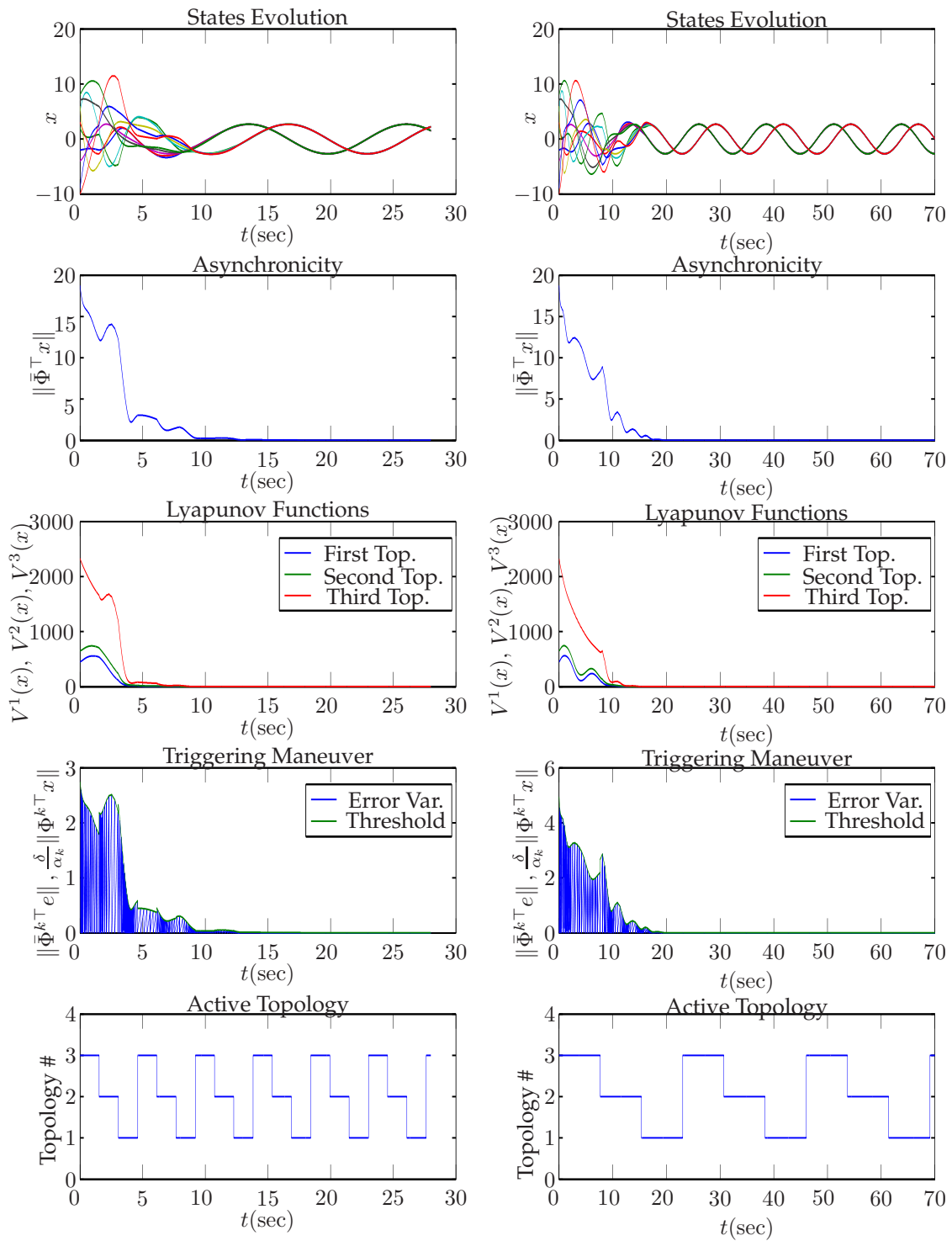
*Case 2:* let  $\delta = 0.9$ , then for the set of parameters stated in Subsection 5.8.1, we obtain  $\tau_a = 7.6751$ . Then, we conduct our simulation along the same lines as discussed in *Case 1*. The results are shown in Figure 5.2(b), where asymptotic synchronization despite presence of switching and under triggering strategy is demonstrated.

At this point, comparing the simulation results obtained from Cases 1 and 2, demonstrated in figures 5.2(a) and 5.2(b), we shall validate that increasing  $\delta$  yields less frequent triggering and larger allowable average dwell time,  $\tau_a$ . This trade off is indeed in accordance with our theoretical assessment attributed in Subsection 5.6.2.

## 5.9 Conclusions & Future Work

In this chapter, we have considered a specific class of synchronization problems with identical nodes. We have then recalled and characterized specific event-triggered rules to ensure the synchronization of this class of dynamics. We have then studied the robustness of this class of event-triggered synchronization dynamics in the face of various switching scenarios. More specifically, under arbitrary switching scenario, we characterized conditions on network topology to achieve asymptotic synchronization. At last, we have also characterized certain robustness conditions for two classes of constrained switching signals, namely uniform and average, under which asymptotic synchronization is ensured.

As future work, there are several venues to be yet explored. This includes, studying the similar type of robustness on switching signals for (i) distributed event-triggered rules—this latter would entail considering a slightly different type of dynamics for the network, for which the analyses presented

(a) Case 1:  $\delta = 0.5$ (b) Case 2:  $\delta = 0.9$ **Figure 5.2:** Temporal results for average switching



in this chapter would serve as an initial point, and (ii) other classes of synchronization problems, i.e., with different isolated node dynamics.

# Chapter 6

## Conclusions & Future Work

This chapter contains a review of the results of this dissertation, as well as potential directions and venues for future work. The presentation of the two subsequent sections of this chapter is then performed in accordance with the two major problems assessed in this dissertation.

### 6.1 Conclusions

In this dissertation, we have studied two major problems, i.e., failure-resilient control using triggering control techniques, and robustness analysis of event-based synchronization dynamics with switching topologies. These problems have been motivated, on the one hand, by the growing interest in CPS and their associated applications, e.g., security and smart grid, and on the other hand, by more economic number of communications that is the result of using triggering control methods. In this section, we shall summarize the representative conclusions deduced studying these problems.

In order to study the failure-resilient problem, we have considered a plant-jammer-operator setup, where the plant is remotely controlled and monitored via unreliable communication channels modeled as some jamming interventions cast under the jammer element. The considered plant is assumed to belong to controllable class of linear systems and the jammer is assumed to maintain power-constrained Denial-of-Service jamming signals, that is to say, it

corrupts the communication channels in the sense that the data can be sent or not. Recalling our motivation on developing and employing triggering control solutions, this major problem has been then studied through analysis of two complementary smaller problems. In the first one, we have developed some sufficient conditions which in conjunction with a resilient event-triggered strategy is capable of ensuring the asymptotic stability of the system. The control law in this latter problem is not in any sort adaptive and thus the main focus of the derived sufficient condition is on jamming parameters. In the second problem, however, while preserving the structure of the problem, the main goal has been to render the triggering control strategy depend on some tunable parameter in order then to be able to deal with any given jamming intervention of the afore discussed class. The second problem has been assessed by first studying single-input class of systems where first principal features of the jamming signal were known, second while adding periodicity assumption, these features were assumed unknown, and third, the periodicity assumption on the jamming signal is also dropped—these results then have been extended to encompass multi-input class of systems, as well—developed algorithms to solve these latter unknown scenarios are, respectively, JAMCOID FOR PERIODIC SIGNALS and JAMCOID algorithms which while tightly depend on the analysis performed on known scenario and are provably functional. In both these aforementioned problems and case scenarios, rigorous analysis on the proposed results have been developed and provided throughout this dissertation.

In order to study the event-based synchronization, we have considered a network of identical oscillators where the topology of the network is prone to switching. We have first considered single topology scenario, thus no switching, where we reviewed and improved a proper event-triggered strategy that is able to ensure asymptotic synchronization of the network. We have then considered the switching case scenario where we solved two complementary problems, i.e., arbitrary and constrained switching scenarios. For the arbitrary switching scenario, we developed some sufficient conditions in terms of network topologies amongst which switching occurs in order to ensure overall asymptotic synchro-

nization. For the constrained switching scenario, we considered two classes of switching signals, i.e., uniform and average switching, where we developed regulatory conditions on switching signal and event-triggered strategy parameters in order to ensure asymptotic synchronization. In both these problems, rigorous analysis on the proposed results have been developed and provided throughout this dissertation.

## 6.2 Future Work

In terms of future work to what have been performed and presented in this dissertation, there are many problems and venues yet to be explored and studied. In what follows in this section, some condensed discussion on these directions shall be provided. The content of this section has been categorized in accordance with the two major problems assessed in this dissertation whose conclusions have been summarized in the previous section along with a summary on leading directions for triggering control methods.

In terms of the failure-resilient problem, the main points to be yet studied are, (i) extending the class of systems to encompass nonlinear class of systems, (ii) extending the triggering strategies to account for the communication issues such as quantization and delay, (iii) extending the class of jamming signals to stochastic class of jamming signals, i.e., where the on and off time-intervals along with the transiting time-instants obey some probability distribution and thus are no longer deterministic, and (iv) extending the problem formulation to account for more distributed/multiagent setup.

In terms of the event-based synchronization problem under switching topologies interaction, the main points to be yet studied are, (i) extending the class of event-triggered strategies to encompass decentralized event-triggered strategies, (ii) extending the class of constrained switching signals to encompass alternative classes as well, (iii) extending the class of synchronization dynamics to encompass alternative dynamics, e.g., with nonidentical nodes.

At last and as it regards the broader area of triggering control, the au-

thors believe that the core concepts of event-triggered, time-triggered, and self-triggered control have been studied within the past decades, the main leading directions and venues for the future would be the application of these concepts to other somewhat “classic” areas of control, this latter includes but is not limited to the fields of robust, adaptive and distributed optimization.

# Bibliography

- [1] N. Adams. Workshop on future directions in cyber-physical systems security. Technical report, Department of Homeland Security (DHS), 2010.
- [2] T. Alpcan and T. Basar. *Network Security: A Decision And Game Theoretic Approach*. Cambridge University Press, 2011.
- [3] S. Amin, A. Cardenas, and S.S. Sastry. Safe and secure networked control systems under denial-of-service attacks. In *Hybrid systems: Computation and Control*, pages 31–45, 2009.
- [4] B.D.O. Anderson and D.G. Luenberger. Design of multivariable feedback systems. *Proceedings of the Institution of Electrical Engineers*, 114:395–399, March 1967.
- [5] A. Anta and P. Tabuada. Exploiting isochrony in self-triggered control. *IEEE Transactions on Automatic Control*, 57(4):950–962, 2010.
- [6] A. Anta and P. Tabuada. To sample or not to sample: Self-triggered control for nonlinear systems. *IEEE Transactions on Automatic Control*, 55(9):2030–2042, 2010.
- [7] A. Arenas, A. Díaz-Guilera, J. Kurths, Y. Moreno, and C. Zhou. Synchronization in complex networks. Preprint available at <http://arxiv.org/pdf/0805.2976.pdf>.
- [8] K.E. Årzén. A simple event-based PID controller. In *IFAC World Congress*, volume 18, pages 423–428, 1999.
- [9] K.J. Åström and B. Bernhardsson. Comparison of periodic and event based sampling for first-order stochastic systems. In *IFAC World Congress*, pages 301–306, 1999.
- [10] K.J. Åström and B. Wittenmark. *Computer-controlled Systems (3rd Ed.)*. Prentice-Hall, Inc., Upper Saddle River, NJ, USA, 1997.

- [11] E. Bayraktaroglu, C. King, X. Liu, G. Noubir, R. Rajaraman, and B. Thapa. On the performance of IEEE 802.11 under jamming. In *IEEE Conf. on Computer Communications*, pages 1265–1274, 2008.
- [12] D.S. Bernstein. *Matrix Mathematics: theory, facts, and formulas with application to linear system theory*. Princeton University Press, 2005.
- [13] S. Bhattacharya and T. Basar. Differential game-theoretic approach to a spatial jamming problem. In *Int. Symposium on Dynamic Games and Applications*, Banff, Canada, June 2010.
- [14] S. Bhattacharya and T. Basar. Graph-theoretic approach for connectivity maintenance in mobile networks in the presence of a jammer. In *IEEE Int. Conf. on Decision and Control*, Atlanta, USA, December 2010.
- [15] J. Bochnak, M. Coste, and M.-F. Roy. *Real Algebraic Geometry*, volume 36 of *Ergeb. Math. Grenzgeb.* Springer-Verlag, New York, 1998.
- [16] M.S. Branicky, S.M. Phillips, and W. Zhang. Stability of networked control systems: explicit analysis of delay. In *American Control Conference*, pages 2352–2357, Chicago, USA, 2000.
- [17] R. W. Brockett and D. Liberzon. Quantized feedback stabilization of linear systems. *IEEE Transactions on Automatic Control*, 45(7):1279–1289, 2000.
- [18] E. Byres and J. Lowe. The myths and facts behind cyber security risks for industrial control systems. In *VDE Congress, VDE Association for Electrical Electronics and Information Technologies*, 2004.
- [19] A. Cardenas, S. Amin, and S.S. Sastry. Secure control: Towards survivable cyber-physical systems. In *Int. Workshop on Cyber-Physical Systems*. IEEE, June 2008.
- [20] A. Cardenas, S. Amin, B. Sinopoli, A. Giani, A. Perrig, and S.S. Sastry. Challenges for securing cyber-physical systems. In *Workshop on Future Directions of Cyber-Physical Systems*. DHS, July 2009.
- [21] T. Chen and B.A. Francis. Input-output stability of sampled-data systems. *IEEE Transactions on Automatic Control*, 36(1):50–58, 1991.
- [22] A. Cortés and S. Martínez. Self-triggered best response dynamics for continuous games. *IEEE Transactions on Automatic Control*, 2013. Under revision.
- [23] B. DeBruhl and P. Tague. Digital filter design for jamming mitigation in 802.15.4 communication. In *Int. Conf. on Computer Communications and Networks*, pages 1–6, 2011.

- [24] D.V. Dimarogonas, E. Frazzoli, and K.H. Johansson. Distributed event-triggered control for multi-agent systems. *IEEE Transactions on Automatic Control*, 57(5):1291–1297, 2012.
- [25] V.N. Faddeeva. *Computational Methods of Linear Algebra*. Dover Publications, 1958.
- [26] H. Fawzi, P. Tabuada, and S. Diggavi. Secure state-estimation for dynamical systems under active adversaries. In *Allerton Conf. on Communications, Control and Computing*, 2011.
- [27] M. Fiedler. Algebraic connectivity of graphs. *Czechoslovak Mathematical Journal*, 23(98):298–305, 1973.
- [28] H. Shisheh Foroush and S. Martínez. On single-input controllable linear systems under periodic DoS jamming attacks. <http://arxiv.org/abs/1209.4101>.
- [29] H. Shisheh Foroush and S. Martínez. On the robustness of event-based synchronization under switching interactions. Preprint available at <http://fausto.dynamic.ucsd.edu/sonia/papers/data/hsf-extended.pdf>.
- [30] H. Shisheh Foroush and S. Martínez. On event-triggered control of linear systems under periodic Denial of Service attacks. In *IEEE Int. Conf. on Decision and Control*, pages 2551–2556, Maui, HI, USA, December 2012.
- [31] H. Shisheh Foroush and S. Martínez. On multi-input controllable linear systems under unknown periodic DoS attacks. In *SIAM Conference on Control and Its Applications (CT)*, January 2013.
- [32] H. Shisheh Foroush and S. Martínez. On triggering control of single-input linear systems under pulse-width modulated DoS jamming attacks. *SIAM Journal on Control and Optimization*, 2013. Submitted. Available at <http://fausto.dynamic.ucsd.edu/sonia/papers/>.
- [33] A.G. Fragkiadakis, V.A. Siris, and N. Petroulakis. Anomaly-based intrusion detection algorithms for wireless networks. In *Int. Conf. on Wired/Wireless Internet Communications*, pages 192–203, 2010.
- [34] G.F. Franklin, J.D. Powell, and A. Emami-Naeini. *Feedback control of dynamic systems*. Prentice Hall, 2006.
- [35] N.M. Freris, S.R. Graham, and P.R. Kumar. Fundamental limits on synchronizing clocks over networks. *IEEE Transactions on Automatic Control*, 56(6):1352–1364, 2011.



- [36] E. Garcia and P.J. Antsaklis. Model-based event-triggered control for systems with quantization and time-varying network delays. *IEEE Transactions on Automatic Control*, 58(2):422–434, 2013.
- [37] M. Guinaldo, D. Lehmann, J. Sánchez, S. Dormido, and K.H. Johansson. Distributed event-triggered control with network delays and packet losses. In *IEEE Int. Conf. on Decision and Control*, pages 1–6, Maui, USA, December 2012.
- [38] A. Gupta, C. Langbort, and T. Basar. Optimal control in the presence of an intelligent jammer with limited actions. In *IEEE Int. Conf. on Decision and Control*, pages 1096–1101, Atlanta, USA, December 2010.
- [39] S. Gupta. Increasing the sampling efficiency for a control system. *IEEE Transactions on Automatic Control*, 8(3):263–264, 1963.
- [40] W.P.M.H. Heemels, J.H. Sandee, and P.P.J. Van Den Bosch. Analysis of event-driven controllers for linear systems. *International Journal of Control*, 81(4):571–590, 2008.
- [41] J. Hespanha, P. Naghshtabrizi, and Y. Xu. A survey of recent results in networked control systems. *Proceedings of IEEE Special Issue on Technology of Networked Control Systems*, 95(1):138–162, 2007.
- [42] L. Hetel, J. Daafouz, and C. Iung. Analysis and control of LTI and switched systems in digital loops via an event-based modeling. *International Journal of Control*, 81(7):1125–1138, 2008.
- [43] D.J. Hill and G. Chen. Power systems as dynamic networks. In *IEEE Int. Symposium on Circuits and Systems*, pages 722–725, May 2006.
- [44] H. K. Khalil. *Nonlinear Systems*. Prentice Hall, 3 edition, 2002.
- [45] S.S. Kia, J. Cortés, and S. Martínez. Distributed convex optimization via continuous time coordination algorithms with discrete-time communication. *Automatica*, 2013. Submitted.
- [46] D. Lehmann and J. Lunze. Event-based output-feedback control. In *Mediterranean Conf. on Control and Automation*, pages 982–987, 2011.
- [47] L. Li, B. Hu, and M.D. Lemmon. Resilient event triggered systems with limited communication. In *IEEE Int. Conf. on Decision and Control*, pages 6577–6582, Hawaii, USA, December 2012.
- [48] D. Liberzon. *Switching in Systems and Control*. Birkhäuser, 2003.

- [49] A.I. Liff and J.K. Wolf. On the optimum sampling rate for discrete-time modeling of continuous-time systems. *IEEE Transactions on Automatic Control*, 11(2):288–290, 1966.
- [50] T. Liu, M. Cao, C. De Persis, and J.M. Hendrickx. Distributed event-triggered control for synchronization of dynamical networks with observers. In *Proc. of 4th IFAC Workshop on Distributed Estimation and Control in Networked Systems*, pages 116–121, September 2013.
- [51] T. Liu, D.J. Hill, and B. Liu. Synchronization of dynamical networks with distributed event-based communication. In *IEEE Int. Conf. on Decision and Control*, pages 7199–7204, Maui, HI, December 2012.
- [52] T. Liu, J. Zhao, and D.J. Hill. Exponential synchronization of complex delayed dynamical networks with switching topology. *IEEE Transactions on Circuits and Systems*, 57(11):2967–2980, 2010.
- [53] C.V. Loan. The sensitivity of the matrix exponential. *SIAM Journal on Numerical Analysis*, 14(6):971–981, 1977.
- [54] D.G. Luenberger. Canonical forms for linear multivariable systems. *IEEE Transactions on Automatic Control*, 12:290–293, 1967.
- [55] J. Lunze and D. Lehmann. A state-feedback approach to event-based control. *Automatica*, 46(1):211–215, 2010.
- [56] X. Luo, E.W.W. Chan, and R.K.C. Chang. Detecting pulsing denial-of-service attacks with nondeterministic attack intervals. *EURASIP Journal on Advances in Signal Processing*, 2009(8):1–13, 2009.
- [57] M. Marden. *Geometry of Polynomials*. Mathematical Surveys. Amer Mathematical Society, 1985.
- [58] M. Mazo, A. Anta, and P. Tabuada. An ISS self-triggered implementation of linear controllers. *Automatica*, 46(8):1310–1314, 2010.
- [59] M. Mazo and M. Cao. Decentralized event-triggered control with asynchronous updates. In *IEEE Int. Conf. on Decision and Control*, pages 2547–2552, 2011.
- [60] M. Mazo and P. Tabuada. Input-to-state stability of self-triggered control systems. In *IEEE Int. Conf. on Decision and Control*, pages 928–933, 2009.
- [61] M. Mazo and P. Tabuada. Decentralized event-triggered control over wireless sensor/actuator networks. *IEEE Transactions on Automatic Control*, 56(10):2456–2461, 2011.

- [62] J. Mitchell and W. McDaniel. Adaptive sampling technique. *IEEE Transactions on Automatic Control*, 14(2):200–201, 1969.
- [63] A. Nayyar, A. Gupta, C. Langbort, and T. Basar. Nash equilibria for stochastic games with asymmetric information-part 1: Finite games. Preprint available at <http://arxiv.org/abs/1209.3549v1>.
- [64] D. Nešić and A. Teel. Input-output stability properties of networked control systems. *IEEE Transactions on Automatic Control*, 49(10):1650–1667, 2004.
- [65] D. Nešić, A.R. Teel, and P.V. Kokotović. Sufficient conditions for stabilization of sampled-data nonlinear systems via discrete-time approximations. *Systems and Control Letters*, 38(4–5):259–270, 1999.
- [66] C. Nowzari and J. Cortés. Self-triggered coordination of robotic networks for optimal deployment. *Automatica*, 48(6):1077–1087, 2012.
- [67] F. Pasqualetti, A. Bicchi, and F. Bullo. Consensus computation in unreliable networks: A system theoretic approach. *IEEE Transactions on Automatic Control*, 57, 2012.
- [68] F. Pasqualetti, R. Carli, and F. Bullo. A distributed method for state estimation and false data detection in power networks. In *IEEE Int. Conf. on Smart Grid Communications*, pages 469–474, October 2011.
- [69] L. Pecora, T. Carroll, G. Johnson, D. Mar, and K.S. Fink. Exponential synchronization of complex delayed dynamical networks with switching topology. *International Journal of Bifurcation and Chaos*, 10(2):273–290, 2000.
- [70] C. De Persis, R. Sailer, and F. Wirth. On a small-gain approach to distributed event-triggered control. In *IFAC World Congress*, volume 18, pages 2401–2406, 2011.
- [71] R.A. Poisel. *Modern Communication Jamming Principles and Techniques*. Artech, 2004.
- [72] R. Postoyan, A. Anta, D. Nešić, and P. Tabuada. A unifying lyapunov-based framework for the event-triggered control of nonlinear systems. In *IEEE Int. Conf. on Decision and Control*, pages 2559–2564, 2011.
- [73] S. Roy, C. Ellis, S. Shiva, D. Dasgupta, V. Shandilya, and Q. Wu. A survey of game theory as applied to network security. In *Int. Conf. on Systems Sciences*, pages 1–10, Hawaii, USA, 2010.
- [74] L. Schenato, B. Sinopoli, M. Franceschetti, K. Poolla, and S.S. Sastry. Foundations of control and estimation over lossy networks. *Proceedings of IEEE Special Issue on Technology of Networked Control Systems*, 95(1):163–187, 2007.

- [75] G.S. Seyboth, D.V. Dimarogonas, and K.H. Johansson. Event-based broadcasting for multi-agent average consensus. *Automatica*, 49(1):245–252, 2013.
- [76] D.J. Stilwell, E.M. Bollt, and D.G. Roberson. Sufficient conditions for fast switching synchronization in time varying network topologies. Preprint available at <http://arxiv.org/pdf/nlin/0502055.pdf>.
- [77] S. Sundaram and C.N. Hadjicostis. Distributed function calculation via linear iterations in the presence of malicious agents - parts I, II. In *American Control Conference*, pages 1350–1362, June 2008.
- [78] P. Tabuada. Event-triggered real-time scheduling of stabilizing control tasks. *IEEE Transactions on Automatic Control*, 52(9):1680–1685, 2007.
- [79] P. Tallapragada and N. Chopra. Output-based event-triggered control with guaranteed  $\mathcal{L}_\infty$ -gain and improved and decentralized event-triggering. *IEEE Transactions on Automatic Control*, 57(6):1362–1376, 2012.
- [80] P. Tallapragada and N. Chopra. On event triggered tracking for nonlinear systems. *IEEE Transactions on Automatic Control*, 58(9):2343–2347, 2013.
- [81] G. Theodorakopoulos and J. S. Baras. Game theoretic modeling of malicious users in collaborative networks. *IEEE Journal of Selected Areas in Communications*, 26(7):1317–1327, 2008.
- [82] D. Thuente and M. Acharya. Intelligent jamming in wireless networks with applications to 802.11b and other networks. In *Military Communications (MILCOM) Conference*, 2006.
- [83] R. Tomovic and G. Bekey. Adaptive sampling based on amplitude sensitivity. *IEEE Transactions on Automatic Control*, 11(2):282–284, 1966.
- [84] M. Velasco, J.M. Fuertes, and P. Marti. The self triggered task model for real-time control systems. In *IEEE Real-Time Systems Symposium*, pages 67–70, 2003.
- [85] X. Wang and M.D. Lemmon. Event-triggering in distributed networked systems with data dropouts and delays. *Hybrid systems: Computation and control*, pages 366–380, 2009.
- [86] X. Wang and M.D. Lemmon. Self-triggered feedback control systems with finite-gain  $\mathcal{L}_2$  stability. *IEEE Transactions on Automatic Control*, 54(3):452–467, 2009.
- [87] W. Xu, W. Trappe, Y. Zhang, and T. Wood. The feasibility of launching and detecting jamming attacks in wireless networks. In *ACM International Symposium on Mobile Ad-Hoc Networking & Computing*, pages 46–57, 2005.

- [88] L. Zaccarian, A.R. Teel, and D. Nešić. On finite gain  $\mathcal{L}_p$  stability of nonlinear sampled-data systems. *Systems and Control Letters*, 49(3):201–212, 2003.
- [89] J. Zhao, D.J. Hill, and T. Liu. Synchronization of complex dynamical networks with switching topology: A switched system point of view. *Automatica*, 45:2502–2511, 2009.
- [90] M. Zhu and S. Martínez. Attack-resilient distributed formation control via online adaptation. In *IEEE Int. Conf. on Decision and Control*, pages 6624–6629, Orlando, FL, USA, December 2011.
- [91] M. Zhu and S. Martínez. On distributed constrained formation control in operator-vehicle adversarial networks. *Automatica*, 2012. Conditionally accepted.