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**Determinacy of Competitive Equilibria in Economies with Many Commodities**

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**Abstract**

This paper provides a framework for establishing the determinacy of equilibria in general equilibrium models with infinitely many commodities and a finite number of consumers and producers. The paper defines a notion of regular economy for such models and gives sufficient conditions on the excess savings equations characterizing equilibria under which regular economies have a finite number of equilibria, each of which is locally stable with respect to perturbations in exogenous parameters, and under which regular economies are generic. For the case of sequence economies in which there are countably many commodities, such as discrete time models or markets with countably many assets, the paper develops sufficient conditions on preferences and technologies for these generic determinacy conclusions to hold. These arguments build on the intuition that these economies can be thought of as the limit of economies with a large finite number of commodities, and conclude that the sharp predictions of generic determinacy in economies with finitely many commodities carry over to economies with countably many commodities under one additional assumption, which prohibits goods from becoming perfect substitutes asymptotically.

# 1 Introduction

The predictions arising from perfect competition in the classical Arrow-Debreu model have been extensively studied over the last forty years. In one sense these predictions are very imprecise, since the set of competitive equilibria is essentially arbitrary as demonstrated by the striking work of Sonnenschein, Debreu, and Mantel. In a more fundamental sense, however, the Arrow-Debreu model of perfect competition yields sharp predictions: almost all smooth economies have only finitely many equilibria, and perhaps more importantly, comparative statics in such regular economies are locally determinate, since equilibria can be expressed locally as smooth functions of underlying exogenous parameters. Furthermore, the restrictions on preferences and production technologies needed to generate a smooth economy are relatively mild, as if preferences are smooth in the sense of Debreu (1972) for example, then the resulting exchange economy is smooth and equilibria are generically determinate.

Part of the predictive power of the classical Arrow-Debreu model derives from the fact that it allows for only a finite number of possible commodities. This restriction means that it must be a static or finite horizon model if goods are interpreted as time-dated, that there can be only finitely many possible states of the world if goods are interpreted as contingent commodities, or that there can be only a finite number of distinct marketed assets if goods are interpreted as portfolio holdings in financial markets. Such restrictions rule out many of the most important features of dynamic economies, models of choice under uncertainty, and financial markets. To model any of these ideas properly requires an infinite number of commodities, as much recent work has emphasized.

When the Arrow-Debreu model is extended to settings involving infinite-dimensional commodity spaces, the issue of determinacy of equilibria loses none of its importance. Indeed, if equilibria are indeterminate, then the slightest changes in exogenous parameters in the economy or the smallest amount of measurement error can lead to large changes in equilibria, resulting in equilibria which may then have vastly different properties from the original prediction. As these models become the standard for addressing dynamic questions, models of choice under uncertainty, or issues in financial markets, resolving questions about determinacy in models with an infinite number of commodities becomes all the more pressing. Moreover in several notable cases, such as overlapping generations economies and incomplete financial markets models (Mas-Colell (1991)), extending the Arrow- Debreu model to accommodate infinitely many commodities introduces robust indeterminacies in the set of competitive equilibria. Both of these examples also involve market distortions which result in a failure of the first welfare theorem, so whether the source of indeterminacy in these economies is the lack of Pareto optimality or the addition of an infinite number of commodities is left unclear by this work.

Furthermore, the existing positive results concerning determinacy in economies with infinitely many commodities do not completely resolve this question. In a discrete time, infinite horizon model with a finite number of infinitely-lived households, Kehoe and Levine (1985) show that equilibria are generically locally unique, as long as each consumer's utility function is additively separable, by relying on Negishi's approach, which uses the welfare theorems to characterize equilibria as the prices and Pareto optimal allocations at which each consumer's budget constraint is satisfied (see Negishi (1960)). Their analysis, however, relies crucially on the additive separability of preferences in their model. When consumers' preferences are additively separable, consumption decisions in one period have no effect on marginal utility in any other period, and hence the infinite-dimensional social planner's problem characterizing Pareto optimal allocations becomes simply a countable sequence of independent finite-dimensional problems, finding the Pareto optimal allocations in a sequence of independent, standard finite-dimensional economies. Familiar arguments and assumptions can be applied in each of these single period, finite-dimensional economies to demonstrate that the Pareto optimal allocations and supporting prices are smooth functions of the welfare weights in each period. Once this has been established, each of the consumers' budget equations can be written as a smooth function of these welfare weights, and standard arguments immediately yield the conclusion that equilibria are generically determinate in this model.

Although the additively separable model yields sharp and straightforward results concerning determinacy of equilibria, the assumption of additive separability is quite restrictive as a model of intertemporal preferences for consumption over an infinite horizon or uncertainty over a countable number of states. For example, the additively separable model implies that consumers' marginal rates of substitution between periods depend only on the amounts of the goods consumed in each period, and are independent of consumption in any other period, which is contradicted by experimental evidence and casual empiricism or introspection (see, e.g., Peleg and Yaari (1973), Strotz (1956), or Thaler (1990)). In models with uncertainty, additive separability implies that the rate of intertemporal substitution and the rate of risk aversion cannot be disentangled (see Epstein and Zin (1989)). As soon as one moves away from this model to one which allows for more interaction across time or between commodities in consumers' utility, such as models incorporating habit formation or recursive preferences, the argument used to establish determinacy in the additively separable case breaks down. No longer can the characterization of Pareto optimal allocations and supporting prices as functions of the welfare weights be decomposed into a sequence of independent finite-dimensional problems; such a characterization is inherently and inextricably an infinite-dimensional problem, and the difficulty of studying determinacy in this more general setting has been well-documented (see, e.g., Mas-Colell (1992) or Mas-Colell and Zame (1991)).

Some of these difficulties are highlighted by the work of Kehoe, Levine, Mas-Colell, and Zame (1989), who give conditions under which equilibria are generically determinate in large square exchange economies by taking as primitives smooth demand functions for all consumers. Although such an approach is standard in economies with a finite number of commodities, and reasonable assumptions on preferences such as differential convexity and strong survival conditions like requiring that indifference curves do not intersect the boundaries of the consumption set ensure the existence of smooth demand in finite economies, this approach is more problematic in models with an infinite number of commodities. For most reasonable preferences over infinite-dimensional commodity spaces, demand functions are not defined for all prices because budget sets are typically unbounded. Moreover, in most of the important economic models with infinitely many commodities, the positive cone, which is the natural domain for consumption bundles and for prices, has an empty interior. In such cases, implicit in the assumption that demand functions are smooth is the assumption that demand functions are well-defined not only for positive prices but for negative prices as well.

To compensate for the emptiness of the interior of the positive cone, Mas-Colell's (1986a) seminal work uses the notion of uniform properness to ensure the existence of prices supporting each Pareto optimal allocation. One explanation for the connection between uniform properness and the existence of supporting prices is the fact that uniform properness of preferences is essentially equivalent to saying preferences can be extended beyond the positive cone to some larger domain which does have a nonempty interior, while still preserving the convexity of preferences, as Richard and Zame (1986) show. This observation is important for determinacy analysis because it means that uniformly proper preferences are incompatible with the survival conditions or Inada conditions necessary to guarantee positive consumption of all commodities. When consumers can shift from consuming a good to not consuming it in response to changes in prices or welfare weights or other parameters, consumption will exhibit kinks. To illustrate this point. consider a simple model of choice under uncertainty over a countable number of states. Considering only consumption claims with finite variances leads to the commodity space $\ell_2$, the space of sequences $x$ such that $\sum_{t=1}^{\infty} |x_t|^2 < \infty$, which is a simple version of the model studied by Duffie and Huang (1985). Consider an exchange economy in this setting composed of two types of consumers, each of which has a von Neumann-Morgenstern utility function of the form $U_i(x) = \sum_{t=0}^{\infty} \beta_i^t u_i(x_t)$, and suppose types have heterogeneous beliefs, so $\beta_1 > \beta_2$. In order for each indifference set to have a well-defined supporting price at each point which is a continuous linear functional, marginal utility for consumption must be bounded, but this assumption rules out standard Inada conditions requiring that the marginal utility for a good go to infinity as the quantity consumed goes to zero. In every Pareto optimal allocation in this model, type 1 consumers eventually consume the entire endowment in each state while type 2 consumers eventually consume nothing. Moreover, the state in which this divergence takes place depends on the weights assigned to each type by the social planner, so as these weights change, consumption of type 2 agents in a given state will go to zero, and the consumption of these consumers exhibits kinks at such weights.[1] Indeed, as is discussed in more detail throughout this paper, these models are ones in which the nonsmoothness of the equilibrium equations is inherent in the problem; no simple assumptions such as differential convexity or that the indifference curves do not intersect the axes are at

[1]Note that the solution to the social planner's problem here is determined by the solution to the social planner's problem in each state, and the relevant first order conditions in each state are $\frac{\lambda_1 u_1'(x_1^t)}{\lambda_2 u_2'(x_2^t)} \geq \left(\frac{\beta_2}{\beta_1}\right)^t$ and $x_2^t(\beta_2^t \lambda_2 u_2'(x_2^t) - \beta_1^t \lambda_1 u_i'(x_1^t)) = 0$. Since $\left(\frac{\beta_2}{\beta_1}\right)^t \to 0$ as $t \to \infty$ and $\frac{u_2'}{u_1'}$ is bounded, $x_2^t = 0$ for $t$ sufficiently large. Similar examples can be easily constructed under the assumption of homogeneous beliefs, in which $\beta_1 = \beta_2$.

once consistent with the economic structure of the model and sufficient to guarantee smoothness.

Studying determinacy in economies with infinitely many commodities while allowing for more interaction across goods or periods, or for a more robust relationship between time and uncertainty, will then require a different mode of analysis, one not predicated on the finite-dimensional structure of the additively separable model, on the notion of excess demand, or on the differential techniques pioneered by Debreu (1970). This paper provides such a framework for establishing the determinacy of equilibria in general equilibrium models with infinitely many commodities and a finite number of consumers and producers in two basic steps. First, the paper defines a notion of regular economy for such models which is independent of smoothness, and gives sufficient conditions on the equilibrium equations under which regular economies have a finite number of equilibria, each of which is locally stable with respect to perturbations in exogenous parameters, and under which regular economies are generic. The primitive notion here, rather than excess demand functions as in Debreu (1970), is the excess savings equations arising from Negishi's characterization of equilibria. Secondly, for the case of sequence economies in which there are countably many commodities, such as discrete time models or markets with countably many assets, the paper develops sufficient conditions on preferences and technologies for these generic determinacy conclusions to hold. These arguments build on the intuition that these economies can be thought of as the limit of economies with a large finite number of commodities, suggested by Bewley's (1972) seminal work and explored recently by Balasko (1995), who studies the structure of the equilibrium set in the context of an additively separable exchange model. This paper concludes that the sharp predictions of generic determinacy in economies with finitely many commodities carry over to economies with countably many commodities under one additional assumption, a stronger notion of concavity prohibiting goods from becoming perfect substitutes asymptotically. In particular, these results are applied to show that when the commodity space is $\ell_2$, as in the previous financial markets model, generic determinacy follows simply from strong differential concavity of consumers' utility functions.

The paper proceeds as follows. Regular economies are defined in section 2, and properties of regular economies are developed for general infinite-dimensional models based on properties of the excess savings equations. Focusing on sequence economies, section 3 introduces the notion of uniform concavity and shows that equilibria are generically determinate under this additional restriction on preferences in an exchange economy with finitely many consumers. In section 4, this analysis is extended to allow preferences to satisfy Inada conditions in the case when the commodity space is $\ell_\infty$, and in section 5, these methods are extended to production economies. Examples are given in all of these sections to illustrate the application of these generic determinacy results.

## 2 Regular Economies

This section develops a framework for establishing the determinacy of equilibria in economies with infinitely many commodities which does not rely on smooth methods. Although the rest of the paper focuses on sequence economies, in which there are a countable number of commodities, the results of this section also apply to more general economies with an infinite-dimensional commodity space. The definitions and results of this section are thus stated for general economies, and these ideas are illustrated using sequence economies. To simplify notation these results are stated for exchange economies, but they carry over in a straightforward manner to production economies. This extension is discussed in section 5. The main conclusions of this section are that even though consumption will typically exhibit kinks in economies with infinitely many commodities, there is a meaningful notion of regular economy which has all of the strong properties of regular economies in a finite-dimensional setting, and as long as the excess savings equations describing equilibria are at least Lipschitz continuous, such regular economies are generic.

The basic assumptions maintained throughout the paper concerning these economies are contained in the following definition.

**Definition 2.1.** *An economy is a* **smooth exchange economy** *if the commodity-price pairing is a symmetric Riesz dual system* $\langle X, X' \rangle$,[2] *and for each consumer* $i = 1, \ldots, m$,

1. $\omega_i \in X_{+-}$;

2. $U_i : X_+ \longrightarrow \mathbb{R}$ *is monotone, strictly concave, and* $\tau$ *continuous for some compatible locally convex topology* $\tau$, *with* $U_i(0) = 0$;

3. $U_i$ *is twice continuously Gateaux differentiable on* $X_+$;[3]

4. $DU_i(x) \in X'_{++}$ *for each* $x \in X_+$;

5. $D^2U_i(x)$ *is negative definite for each* $x \in X_+$.

These assumptions combine the basic structure of infinite-dimensional economies needed to ensure the existence of equilibria with natural analogues of the assumptions maintained in smooth economies with a finite set of commodities. The assumption

[2] A dual system $\langle X, X' \rangle$ is a symmetric Riesz dual system if $X$ is a Riesz space, $X'$ is an ideal of the order dual of $X$, the duality function is the natural one, so that $\langle x, x' \rangle = x'(x)$ for each $x$ and $x'$, and each order interval in $X$ is weakly compact (see Aliprantis, Brown, and Burkinshaw (1989)). Given a dual system $\langle X, X' \rangle$, the ordering conventions used here are as follows: $X_+ = \{x \in X : x \geq 0\}$ and $X_{++} = \{x \in X : \langle x, x' \rangle > 0 \text{ for all } x' \in X'_+ \setminus \{0\}\}$.

[3] Two aspects of this statement may require clarification. First, the assumption that $DU_i$ and $D^2U_i$ are continuous is not explicitly required for this section. The topology in which these derivatives are assumed to be continuous is specified explicitly in all remaining sections. Secondly, a function $f$ is Gateaux differentiable on $X_+$ if there exists a finitely open set $U \supset X_+$ on which $f$ is defined and Gateaux differentiable, where a set is finitely open if and only if its intersection with every finite-dimensional subspace is relatively open. See Hille and Phillips (1957, 1.10 and 3.16) for more on this.

that the relationship between commodities and prices can be described by a symmetric Riesz dual system means that the order intervals in the commodity space $X$ are weakly compact, which ensures that the set of feasible allocations is weakly compact and, together with the continuity of preferences, that Pareto optimal allocations exist. Furthermore, this assumption implies that given any two elements $p, q \in X'$, the functionals $p \vee q$ and $p \wedge q$ are also continuous. As in Mas-Colell and Richard's (1991) striking work, this lattice structure of prices will play a crucial role in characterizing equilibria. This assumption, together with assumptions (1) and (2), are standard assumptions needed to guarantee the existence of equilibria with infinitely many commodities. Note, however, that preferences are not required to be uniformly proper. Instead, assumption (3) imposes a differential version of properness by requiring that at each point in the positive cone, each consumer's utility function has a unique continuous subgradient given by $DU_i(x)$. Mas-Colell (1986b) shows that this condition performs the same function as uniform properness, generating prices supporting each Pareto optimal allocation. The existence of a subgradient at each point is necessary for the existence of prices supporting individual consumption; moreover, this assumption implies that each consumer's utility function can be extended beyond the positive cone, which Richard and Zame (1986) have shown implies, and is essentially equivalent to, uniform properness. Assumptions (3)-(5) strengthen this condition by requiring the subgradient to be strictly positive and to move continuously with its argument. These additional conditions allow us to express supporting prices as a continuous function of the underlying allocation, which will be important for extending the analysis of equilibria in these models beyond questions of existence to encompass questions like determinacy.

As an example, consider sequence economies in which the commodity space is $\ell_p$ for some $p$ such that $1 \leq p < \infty$. The commodity-price pairing $\langle \ell_p, \ell_q \rangle$ is a symmetric Riesz dual system, where $\frac{1}{p} + \frac{1}{q} = 1$. In such a model, if each consumer's utility function is norm continuous and twice norm continuously Gateaux differentiable, differentiably strictly monotone and differentiably strictly concave, then the economy will be a smooth exchange economy.[4] For specific examples of such utility functions, see section 3. Other important symmetric Riesz dual systems include $\langle \mathcal{L}_\infty(\mu), \mathcal{L}_1(\mu) \rangle$ where $\mu$ is $\sigma$ finite, and $\langle \mathcal{L}_p(\mu), \mathcal{L}_q(\mu) \rangle$ for $1 < p < \infty$ (see Aliprantis, Brown and Burkinshaw (1989)).

Not surprisingly, for some commodity spaces it may be possible to weaken some of these assumptions by using specific properties of the space, as is the case when the commodity space is $\ell_\infty$, the natural commodity space for discrete time, infinite horizon economies. In these economies, prices should be representable as sequences, so the most economically meaningful or appealing commodity-price pairing is $\langle \ell_\infty, \ell_1 \rangle$. Furthermore, it is natural to model consumers as being impatient or myopic in this setting, and thus having Mackey continuous preferences (see Brown and Lewis (1981)).[5] In such economies, the analogues of assumptions (1)- (5) are the following conditions:

[4] $U(x)$ is differentiably strictly monotone if $DU(x) \gg 0$ for each $x$ and differentiably strictly concave if $D^2U(x)$ is negative definite for each $x$.

[5] Mackey continuity here means continuity in the Mackey topology for the pairing $\langle \ell_\infty, \ell_1 \rangle$.

$(1)_\infty$ $\omega_i \in \ell_{\infty++}$;

$(2)_\infty$ $U_i : \ell_{\infty+} \to \mathbb{R}$ is monotone, strictly concave, and Mackey continuous, where $U_i(0) = 0$;

$(3)_\infty$ $U_i$ is twice norm continuously Gateaux differentiable on $\ell_{\infty+}$;

$(4)_\infty$ $DU_i(x) \gg 0$ for each $x \in \ell_{\infty+}$;

$(5)_\infty$ $D^2U_i(x)$ is negative definite for each $x \in \ell_{\infty+}$.

An economy satisfying assumptions $(1)_\infty - (5)_\infty$ will be called a **smooth myopic exchange economy**, and in the remainder of the paper, the term "smooth exchange economy" will also include smooth myopic exchange economies. Note that in such an economy, utility functions are required to be Mackey continuous, but only norm continuously differentiable. Furthermore, the assumption that the gradient at each point is strictly positive leaves open the possibility that the gradient, as a norm continuous linear functional on $\ell_\infty$, may not have a sequence representation. Although some elements of the norm dual of $\ell_\infty$, which is $ba$, may fail to have a sequence representation, Mackey continuity and Gateaux differentiability are sufficient to rule out such cases and ensure that all supporting prices lie in $\ell_1$. To say that the derivative lies in $\ell_1$ means that we can think of the derivative as a sequence of marginal utilities or generalized discount factors, and that as time goes to infinity, marginal utility of consumption in period $t$ goes to 0. Then to say that the derivative lies in $\ell_1$ is to say that when calculating the marginal utilities of goods across time, consumers do not put much weight on consumption arbitrarily far in the future, and don't place any weight on consumption "at infinity". As such, this condition is a differential form of impatience or myopia, and so it should not be surprising that this property will always hold if the utility function is Mackey continuous, as the following result shows.

**Theorem 2.1.** *Let $U : \ell_{\infty+} \to \mathbb{R}$ be strictly concave, strictly monotone, and Mackey continuous. If $U$ has a unique subgradient at $x$, then $\partial U(x) \in \ell_1$. In particular, if $U$ is continuously Gateaux differentiable on $\ell_{\infty+}$, then $DU(x) \in \ell_1$ for every $x \in \ell_{\infty+}$.*

*Proof:* Let $x \in \ell_{\infty+}$ be given. Let $p \equiv \partial U(x)$, the set of subgradients of $U$ at $x$. By assumption, $\partial U(x)$ is single-valued, so $p \in ba$. Since $U$ is strictly monotone, $p \geq 0$. By the Hewitt-Yosida theorem, $p = p_c + p_f$, where $p_c \in \ell_1$ and $p_f$ is purely finitely additive. Then it suffices to show that $p_f = 0$. To see this, we will show that $p_c$ is also a subgradient of $U$ at $x$. Suppose not, so that there exists $z \in \ell_{\infty+}$ such that $U(z) > U(x)$ and $p_c \cdot z < p_c \cdot x$. Since $U$ is Mackey continuous, there exists $n$ such that $U(z^n) > U(x)$, where $z^n = (z_1, \ldots, z_n, 0, 0, \ldots)$, and $p_c \cdot z^n \leq p_c \cdot z < p_c \cdot x$. Thus

$$p \cdot z^n = p_c \cdot z^n + p_f \cdot z^n = p_c \cdot z^n < p_c \cdot x + p_f \cdot x = p \cdot x$$

which is a contradiction, since $p$ is a subgradient of $U$ at $x$. Thus $p = p_c \in \ell_1$.

If $U$ is Gateaux differentiable at $x$, then $DU(x) \equiv \partial U(x)$, so the result follows immediately. ∎

As an explicit example of a smooth myopic exchange economy, suppose that the commodity space is $\ell_\infty$ and that each consumer's preferences are represented by a utility function of the form $U_i(x) = \sum_{t=0}^{\infty} \beta_i^t u_i(x_t)$, where $\beta_i \in (0,1)$, $u_i : \mathbb{R}_+ \to \mathbb{R}$ is $C^2$ on $\mathbb{R}_+$ with $u_i'(c) > 0$ and $u_i''(c) < 0$ for each $c \geq 0$. Given any initial endowments $\omega_i \in \ell_{\infty++}$, an economy with such consumers is a smooth exchange economy. as the following result shows.

**Theorem 2.2.** *Let $U : \ell_{\infty+} \to \mathbb{R}$ be given by $U(x) = \sum_{t=0}^{\infty} \beta^t u(x_t)$, where $0 < \beta < 1$, $u : \mathbb{R}_+ \to \mathbb{R}$ is $C^2$, $u'(c) > 0$ and $u''(c) < 0$ for every $c \in \mathbb{R}_+$. Then $U(x)$ satisfies assumptions $(2)_\infty - (5)_\infty$.*

*Proof:* See the appendix. ∎

Now let $\mathcal{E}_\omega$ be a smooth exchange economy. Equilibria of this exchange economy are prices and Pareto optimal allocations at which each consumer's budget constraint is exactly satisfied. To characterize the possible Pareto optimal allocations, for each $\lambda \in \Delta \equiv \{\lambda \in \mathbb{R}_+^m : \sum_{i=1}^{m} \lambda_i = 1\}$ define the social planner's problem:

$$\max \sum_{i=1}^{m} \lambda_i U_i(x_i)$$
$$\text{s.t.} \sum_{i=1}^{m} x_i \leq \omega, \; x_i \in X_+, \; i = 1, \ldots, m.$$

Because each consumer's utility function is strictly monotone, strictly concave and $\tau$ continuous, and the order interval $[0, \omega]$ is weakly compact, the social planner's problem will have a unique solution for each $\lambda \in \Delta$. Thus the Pareto map

$$x(\lambda) = \arg \max \sum_{i=1}^{m} \lambda_i U_i(x_i)$$
$$\text{s.t.} \sum_{i=1}^{m} x_i \leq \omega, \quad x_i \in X_+ \quad i = 1, \ldots, m$$

is well-defined on $\Delta$. The equilibria of the exchange economy $\mathcal{E}_\omega$ are then the solutions $(p, \lambda)$ to the $m-1$ independent budget equations

$$\begin{aligned} p \cdot (x_2(\lambda) - \omega_2) &= 0 \\ &\vdots \\ p \cdot (x_m(\lambda) - \omega_m) &= 0, \end{aligned}$$

where the price $p$ is a price supporting the allocation $x(\lambda)$.

To completely characterize equilibria using the welfare weights, we must also be able to characterize the possible equilibrium prices using these weights. To see how prices can be determined from the welfare weights and the solution to the social planner's problem, note that any equilibrium price $p$ must support the corresponding equilibrium allocation, which is Pareto optimal. Although there may be several prices supporting any given Pareto optimal allocation, especially since these allocations will typically be on the "boundary" in models with an infinite-dimensional commodity space, one such price is always determined by the allocation and consumer preferences. That is, if we define $p(\lambda) \equiv \bigvee_{i=1}^{m} \lambda_i DU_i(x_i(\lambda))$, then $p(\lambda)$ is always a price supporting the allocation $x(\lambda)$.

**Lemma 2.1.** *Let $\mathcal{E}_\omega$ be a smooth exchange economy, and for each $\lambda \in \Delta$, let $p(\lambda) \equiv \bigvee_{i=1}^{m} \lambda_i DU_i(x_i(\lambda))$. Then $p(\lambda)$ supports the Pareto optimal allocation $x(\lambda)$.*

*Proof:* This follows from Mas-Colell (1986b, Lemma p. 327 and Theorem 2). See also Mas-Colell and Richard (1991). ■

This result also suggests that smooth techniques will not necessarily apply in these economies, since $p(\lambda)$ will typically not be smooth unless the maximum operation is trivial. Using this result, equilibria can be unambiguously characterized as the welfare weights $\lambda \in \Delta$ at which the $m-1$ excess savings equations are satisfied, i.e., at which

$$\begin{aligned} p(\lambda) \cdot (x_2(\lambda) - \omega_2) &= 0 \\ &\vdots \\ p(\lambda) \cdot (x_m(\lambda) - \omega_m) &= 0. \end{aligned}$$

To give a simple parameterization of economies, assume that the economy has a fixed structure of revenues, so that each consumer's endowment has the form $\omega_i = \alpha_i \omega$, where $\alpha_i > 0$ and $\sum_{i=1}^{m} \alpha_i = 1$. Prices can be normalized so that the value of the social endowment is always 1 by letting $\tilde{p}(\lambda) = \frac{p(\lambda)}{p(\lambda)\cdot\omega}$. Thus the excess savings equations characterizing equilibria become

$$s(\lambda) \equiv \begin{pmatrix} \tilde{p}(\lambda) \cdot x_2(\lambda) \\ \vdots \\ \tilde{p}(\lambda) \cdot x_m(\lambda) \end{pmatrix} = \begin{pmatrix} \alpha_2 \\ \vdots \\ \alpha_m \end{pmatrix}.$$

Economies are then naturally parameterized by the set $\Omega \equiv \{\alpha \in \mathbb{R}^{m-1}_{++} : \sum_{i=2}^{m} \alpha_i < 1\}$. This set simply represents the possible income distributions among the $m$ consumers that give a positive share to each person. [6]

[6] In sequence economies, a more natural and less restrictive way to parameterize economies can be developed by instead normalizing prices by setting the price of the first good equal to 1. Let $\hat{p}(\lambda) = \frac{1}{p_1(\lambda)} p(\lambda)$ and let $\hat{\omega}_j = (0, \omega_{j_2}, \omega_{j_3}, \ldots)$. Then the excess savings equations characterizing equilibria in a sequence economy become

$$\hat{s}(\lambda) \equiv \begin{pmatrix} \hat{p}(\lambda) \cdot (x_2(\lambda) - \hat{\omega}_2) \\ \vdots \\ \hat{p}(\lambda) \cdot (x_m(\lambda) - \hat{\omega}_m) \end{pmatrix} = \begin{pmatrix} \omega_{2_1} \\ \vdots \\ \omega_{m_1} \end{pmatrix}.$$

With this normalization, equilibria are the solutions to the equation $\hat{s}(\lambda) = \omega^1$, where $\omega^1 = (\omega_{2_1}, \ldots, \omega_{m_1})$, so instead of assuming a fixed structure of revenues, sequence economies can be parameterized using the set $\mathcal{W} \equiv \{\omega^1 \in \mathbb{R}^{m-1}_{++} : \sum_{j=2}^{m} \omega_{j_1} < \omega_1\}$.

Even though the commodity space is infinite-dimensional, the Negishi argument results in a characterization of equilibria which is formally identical to that arising in a standard Arrow-Debreu economy with a finite-dimensional commodity space. Equilibria are solutions to a finite system of equations in a finite number of variables, so a simple counting of equations and unknowns suggests that we might expect the qualitative features of equilibria in these economies to be similar to economies with a finite set of commodities. If so, the methods used to establish such determinacy results will have to be fundamentally different from the smooth techniques prevalent in analyses of finite economies. As the simple example in the introduction highlights, the social planner's allocation problem, and hence the equilibrium equations, will not be smooth when preferences are supportable in general, so it is even unclear how a regular economy should be defined here. A natural definition, suggested by Rader's (1973) work on finite-dimensional economies with nonsmooth demand, is to require that the Jacobian of excess savings be defined and nonsingular at each equilibrium.

**Definition 2.2.** *Given a smooth exchange economy $\mathcal{E}_\omega$, we will say that $\mathcal{E}_\omega$ is a* **regular economy** *if $\alpha$ is a regular value of $s$, that is, for all $\lambda$ such that $s(\lambda) = \alpha$, $Ds(\lambda)$ exists and is nonsingular. Any economy which is not a regular economy will be called a* **critical economy.**[7]

Note that this definition of regular economy makes sense even if $s(\lambda)$ is not smooth, and if $s(\lambda)$ is smooth, then this notion agrees with the standard notion of regular economy. Whether this notion is the right definition of a regular economy obviously depends on the properties of equilibria in regular economies. First, an immediate consequence of this definition is the fact that regular economies have locally unique equilibria.

**Theorem 2.6.** *Let $\mathcal{E}_\omega$ be a smooth exchange economy. If $\mathcal{E}_\omega$ is a regular economy, then each equilibrium is locally unique. If in addition $s(\lambda)$ is continuous, then there are finitely many equilibria.*

*Proof:* By assumption, $\alpha$ is a regular value of $s$ since $\mathcal{E}_\omega$ is a regular economy, and $s : D^o \rightarrow \mathbb{R}^{m-1}$ where $D^o = \{(\lambda_2, \ldots, \lambda_m) \in \mathbb{R}^{m-1}_{++} : 1 - \sum_{i=2}^{m} \lambda_i > 0\}$. Then $D^o$ is open, so each equilibrium is locally unique by Shannon (1994a, Thm. 1). Moreover, if $\lambda \in \partial D^o$, then $\lambda_i = 0$ for some $i = 1, \ldots, m$, which means that $x_i(\lambda) = 0$, and $U_i(x_i(\lambda)) = U_i(0) = 0 < U_i(\omega_i)$. Individual rationality then implies that $s(\lambda) \neq \alpha$ if $\lambda \in \partial D^o$. This argument shows that if $s$ is continuous, then $s^{-1}(\alpha)$ is compact, and so must be a finite set. ■

We would also like to draw conclusions about comparative statics or local stability of equilibria with respect to perturbations in exogenous parameters. In order to examine how equilibria vary as underlying parameters like endowments change, we will need results concerning sensitivity analysis for solutions to nonsmooth equations. In contrast with the smooth case, in general the savings equations will not

[7]Similarly, in sequence economies, we can define a regular economy to be one in which $\omega^1$ is a regular value of $\hat{s}$. All of the results of this section carry over to this alternative definition for sequence economies.

be locally invertible around an equilibrium $\bar{\lambda}$ without stronger assumptions than the assumption that $s$ is continuous and $\alpha$ is a regular value of $s$, so even if $\alpha$ is a regular value of $s(\lambda)$ and we restrict attention to points arbitrarily close to $\bar{\lambda}$ and $\alpha$, the equilibrium set $s^{-1}(\alpha)$ may be multivalued as $\alpha$ varies. In order to answer this sort of sensitivity question relating to the dependence of equilibria on the initial endowments $\alpha$, one must then turn to recent work in nonsmooth analysis, which has concentrated on developing generalized notions of derivatives for correspondences, as well as for Lipschitz continuous and other nonsmooth functions, and building a calculus around such derivatives, in part to answer sensitivity questions for correspondences (see e.g., Clarke (1975), (1983); Rockafellar (1988); Aubin and Frankowska (1990)). By using results from nonsmooth analysis, Shannon (1994a) shows that if we restrict attention to Lipschitz functions, then in a neighborhood of a regular value, the solution set will be locally stable even if it is multi-valued. To make this statement precise requires a notion of stability for correspondences. A correspondence $G : Y \rightrightarrows X$ is said to be *upper Lipschitzian* at $\bar{y} \in Y$ if there exists $k \geq 0$ and a neighborhood $V$ of $\bar{y}$ such that

$$G(y) \subset G(\bar{y}) + k\|y - \bar{y}\|B \quad \forall y \in V,$$

where $B$ is the unit ball in $X$. Note that this says that if $x \in G(y)$, then there exists $\bar{x} \in G(\bar{y})$ such that $\|x - \bar{x}\| \leq k\|y - \bar{y}\|$, so that as $y$ varies, the values of $G(y)$ remain close in a Lipschitzian sense. Results from Shannon (1994a, Thm. 8) imply that if $s$ is locally Lipschitz and $\alpha$ is a regular value of $s$, then the equilibrium set $s^{-1}(\alpha)$ is locally upper Lipschitzian in $\alpha$.

The strongest results concerning determinacy are thus obtained in the case where the savings equations are locally Lipschitz continuous. If the savings equations are locally Lipschitz, the economy $\mathcal{E}_\omega$ will be called a **Lipschitz economy**. As long as the economy is a Lipschitz economy, these strong determinacy results described above will hold: regular economies will have a finite, and in fact odd, number of equilibria, and each equilibrium will be locally upper Lipschitzian in $\alpha$.

**Theorem 2.7.** *Let $\mathcal{E}_\omega$ be a smooth exchange economy which is also a Lipschitz economy. If $\mathcal{E}_\omega$ is a regular economy, then $\mathcal{E}_\omega$ has an odd number of equilibria, each of which is upper Lipschitzian in $\alpha$.*

*Proof:* See the appendix. ∎

As with all such results which "count" the number of equilibria, note that this result automatically implies that equilibria exist in all Lipschitz economies, since this result establishes that equilibria exist in regular Lipschitz economies, and by definition critical economies must have equilibria.

Finally, as in economies with finitely many commodities, these results are more compelling when the set of regular economies is large, when we can assert that critical economies are exceptional cases and that the economy exhibits no robust indeterminacies. To measure the size of the set of regular economies requires a version of Sard's theorem in this nonsmooth setting. One such result has been established by Rader (1973), who shows that Sard's theorem holds for a broad class of nonsmooth functions including Lipschitz continuous functions. As long as the excess savings equations are differentiable almost everywhere and map sets of measure zero into sets of measure zero, then this version of Sard's theorem implies that the set of regular economies will have full measure. In particular, if the excess savings equations are Lipschitz, locally Lipschitz, or pointwise Lipschitz, then they will satisfy these assumptions, and hence almost every economy $\mathcal{E}_\omega$ will be regular.[8]

**Theorem 2.8.** *Let $\mathcal{E}_\omega$ be a smooth exchange economy. If $s : D^o \longrightarrow \mathbb{R}^{m-1}$ is differentiable almost everywhere and has the property that if $B$ has measure 0 then $s(B)$ has measure zero, then the set of critical economies has measure 0. In particular, if $\mathcal{E}_\omega$ is a Lipschitz economy, then the set of critical economies has measure 0.*

*Proof:* This follows immediately from Rader (1973, Lemma 2). ■

For this to be a useful framework for establishing the determinacy of equilibria in general exchange or production economies, without the sort of detailed information about preferences, technologies, and the nature of the Pareto map available in highly parametrically specified models, the obvious problem now is to find conditions on the primitives of the economy, consumers' preferences and firms' production sets, which guarantee that the excess savings equations are Lipschitz continuous. Such results are developed in the next three sections along with a number of examples illustrating these ideas. I have separated the results of this section from those of the following sections to highlight the fact that just as the original results of Debreu (1970) and Dierker (1972) for finite economies relied only on smooth excess demand, the results here rely only on Lipschitzian excess savings. The following sections develop a set of methods which exploit the structure of sequence economies, and build on the intuition that economies with infinitely many commodities are limits of economies with a large finite number of commodities. There may be a variety of methods for establishing that excess savings equations are Lipschitz continuous in different settings, such as using dynamic programming arguments in stationary economies; see Shannon (1994b) for one such example. Regardless of the methods used, however, as long as the excess savings are Lipschitz continuous, the results of this section can be applied to conclude that equilibria will be generically determinate.

# 3 Lipschitz Economies

In order to show that the determinacy results of the previous section apply to a particular economy, we must show that the excess savings equations characterizing equilibria in that model are Lipschitz continuous. Since the prices supporting any Pareto optimal allocation can be expressed as a function of that allocation, the properties of the excess savings equations will be determined by the properties of the

[8] A function is said to be *pointwise Lipschitz* at $x$ if there exists $K > 0$ and a neighborhood $N$ of $x$ such that for all $\bar{x} \in N$, $\|f(x) - f(\bar{x})\| \leq K\|x - \bar{x}\|$. That Lipschitz and locally Lipschitz functions $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ are differentiable almost everywhere follows from Rademacher's theorem (see, e.g., Federer (1969, 3.1.6)), and that such functions map sets of measure 0 into sets of measure 0 is well known (see, e.g., Federer (1969, 2.10.11)); for a proof that pointwise Lipschitz functions $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ satisfy these properties, see Rader (1973, Lemma 3).

Pareto map which solves the social planner's problem. For example, in the sequence economies which will be the focus of the rest of the paper, the economy will be a Lipschitz economy as long as the Pareto map is Lipschitz continuous; Lipschitz continuity of the Pareto map in these models is then sufficient for generic determinacy of equilibria.

To study the planner's problem, we will consider these economies as limits of finite economies, letting the number of commodities grow to infinity, an idea suggested by Bewley's (1972) seminal work.[9] More precisely, let $\mathcal{E}_\omega$ be a smooth myopic economy in which the commodity space is $\ell_\infty$, and consider the exchange economy in which each consumer's endowment bundle is truncated at good $T$, so endowments become $\omega_i^T = (\omega_{i_1}, \ldots, \omega_{i_T}, 0, 0, \ldots)$. Because consumers are impatient, so their preferences are Mackey continuous, the solution to the social planner's problem in this truncated economy should converge to the solution to the social planner's problem in the original economy as $T \to \infty$. Similarly, if the commodity space is $\ell_p$, then the norm continuity of consumers' preferences in a smooth economy has the same effect of ensuring that the distortion caused by truncating the economy becomes arbitrarily small as the number of commodities goes to infinity. This observation suggests that, at least in economies with countably many commodities, we should be able to use properties of the truncated economy, in which the social planner's problem is finite-dimensional, to establish properties of the original economy.[10] This program is carried out in this section, and several examples are given to illustrate these ideas.

To formalize the argument sketched above, for each $T > 0$, define consumer $i$'s utility in the truncated $T$- good economy $U_i^T : \mathbb{R}_+^T \to \mathbb{R}$ by

$$U_i^T(y) = U_i(y_1, \ldots, y_T, 0, 0, \ldots).$$

Since the original utility function $U(x)$ is twice continuously Gateaux differentiable on $\ell_{p+}$, $U_i^T$ is $C^2$ on $\mathbb{R}_+^T$ for each $T$, and its derivatives are just given by the truncations of the corresponding derivatives of $U_i$.[11] Thus

$$DU_i^T(y) = \left(\frac{\partial U_i}{\partial x_1}(y, 0, 0 \ldots), \ldots, \frac{\partial U_i}{\partial x_T}(y, 0, 0 \ldots)\right)$$

and

$$D^2 U_i^T(y) = \left\{\frac{\partial^2 U_i}{\partial x_i \partial x_j}(y, 0, 0 \ldots)\right\}_{i,j=1,\ldots T}.$$

Moreover, $DU_i^T(y) \gg 0$ and $D^2 U_i^T(y)$ is negative definite for each $y \in \mathbb{R}_+^T$. Each $T$-good economy looks very much like an economy in which preferences are smooth

[9] This idea has also been explored recently by Balasko (1995) in the additively separable case. In this case, he shows that every equilibrium in a regular economy is the limit of equilibria in the truncated economies as the number of commodities goes to infinity. See also Mas-Colell (1991).

[10] In particular, note that this argument is fundamentally different from Bewley's argument. While he constructs a convergent net of equilibria in finite-dimensional restrictions of the economy and argues that the limit of this net is an equilibrium in the original infinite-dimensional economy, his argument does not imply that equilibria must be determinate in the limit, even if each equilibrium in the net is determinate.

[11] See the discussion in footnote 3.

in the sense of Debreu (1972), with the exception that indifference curves may intersect the boundaries of the positive cone in these economies. The main result of this section is that in such sequence economies, the powerful properties and predictions of these finite-dimensional truncated economies, in particular the generic determinacy of equilibria, carry over to the original infinite-dimensional economy under one additional restriction on preferences, which strengthens the notion of concavity in these models and prevents goods from becoming perfect substitutes in the limit.

The proof of this result simply involves filling in the steps outlined above. First, for each $\lambda \in \Delta$, define $x(\lambda, \omega)$ to be the solution to the social planner's problem when the social endowment vector is $\omega$ and the welfare weights are $\lambda$. Note that $x(\lambda, \omega^T)$ is just the solution to the social planner's problem in the truncated economy in which the social endowment is $\omega^T$. Moreover, as the first two lemmas below show, as the number of goods grows, these truncated Pareto optimal allocations converge to the Pareto optimal allocation in the original economy.

**Lemma 3.1.** *Let $\mathcal{E}_\omega$ be a smooth exchange economy in which the commodity space is $\ell_p$ for some $p$ such that $1 \leq p < \infty$. Then $x(\lambda, \omega^T) \xrightarrow{n} x(\lambda, \omega)$ for each $\lambda \in \Delta$ and for each $\omega \in \ell_{p+}$. Moreover, $x(\cdot, \omega)$ is norm continuous for each $\omega \in \ell_{p+}$.*

*Proof:* For each $\omega \in \ell_{p+}$, define $C(\omega) \equiv \{x \in \ell^m_{p+} : \sum_{i=1}^m x_i = \omega\}$. Then $C(\omega)$ is convex and norm compact for each $\omega \in \ell_{p+}$, since order intervals in $\ell_p$ are norm compact (see appendix, Lemma A2). Since $U_i(x)$ is norm continuous for each $i$, the result now follows essentially from Berge's theorem. To see this, note that $C(\cdot)$ is clearly upper hemicontinuous, as if $\omega_n \xrightarrow{n} \omega$ and $x_n \in C(\omega_n)$ for each $n$, where $x_n \xrightarrow{n} x$, then $\omega_n = \sum_{i=1}^m x_{i_n} \xrightarrow{n} \sum_{i=1}^m x_i$, so $\sum_{i=1}^m x_i = \omega$. Thus $x \in C(\omega)$. Now let $x \in C(\omega)$ and consider the sequence $\{\omega^T\}$. By definition, $x^T \equiv (x_1^T, \ldots, x_m^T) \xrightarrow{n} x$, and $x^T \in C(\omega^T)$ for all $T$. Thus $C(\omega)$ is lower hemicontinuous along sequences of the form $\{\omega^T\}$. Now I claim that $x(\lambda, \omega^T) \xrightarrow{n} x(\lambda, \omega)$. To show that this is true, suppose not. Then there exists a convergent subsequence $x(\lambda, \omega^S) \xrightarrow{n} x \neq x(\lambda, \omega)$. However, $x \in C(\omega)$ by the argument above, so if $x \neq x(\lambda, \omega)$, then $\sum_{i=1}^m \lambda_i U_i(x_i) < \sum_{i=1}^m \lambda_i U_i(x_i(\lambda, \omega))$. Also, since $x(\lambda, \omega^S) \xrightarrow{n} x$, there exists $T$ large enough such that if $S > T$,

$$\sum_{i=1}^m \lambda_i U_i(x_i(\lambda, \omega^S)) < \sum_{i=1}^m \lambda_i U_i(x_i(\lambda, \omega)).$$

Then $x(\lambda, \omega)^S \in C(\omega^S)$ for all $S$,[12] and $x(\lambda, \omega)^S \xrightarrow{n} x(\lambda, \omega)$, so for $S$ large enough, $x(\lambda, \omega)^S \in C(\omega^S)$ and

$$\sum_{i=1}^m \lambda_i U_i(x_i(\lambda, \omega^S)) < \sum_{i=1}^m \lambda_i U_i(x_i(\lambda, \omega)^S).$$

But this contradicts the definition of $x(\lambda, \omega^S)$, so $x(\lambda, \omega^T) \xrightarrow{n} x(\lambda, \omega)$.

[12] Recall that this notation means that $x(\lambda, \omega)^S = (x_1(\lambda, \omega)^S, \ldots, x_m(\lambda, \omega)^S)$, where $x_i(\lambda, \omega)^S = (x_{i_1}(\lambda, \omega), \ldots, x_{i_S}(\lambda, \omega), 0, 0, \ldots)$.

That $x(\cdot, \omega)$ is norm continuous follows immediately from Berge's theorem, since $C(\omega)$ is independent of $\lambda$. ■

The same result holds in a smooth myopic economy when the commodity space is $\ell_\infty$, using the weak* topology in place of the norm topology.

**Lemma 3.2.** *In a smooth myopic exchange economy,* $x(\lambda, \omega^T) \xrightarrow{w^*} x(\lambda, \omega)$ *for each* $\lambda \in \Delta$ *and for each* $\omega \in \ell_{\infty+}$. *Moreover,* $x(\cdot, \omega)$ *is weak* continuous for each* $\omega \in \ell_{\infty+}$.

*Proof:* Note that $C(\omega)$ is weak* compact for each $\omega \in \ell_{\infty+}$. Moreover, since $U_i(x)$ is Mackey continuous and the Mackey and weak* topologies agree on norm bounded subsets of $\ell_\infty$, $U_i(x)$ is weak* continuous on $[0, \omega]$. The argument is then identical to the proof of Lemma 3.1, with the weak* topology in place of the norm topology. ■

Furthermore, if the social endowment is $\omega^T$, then the social planner faces a finite-dimensional problem, and it is relatively straightforward to show that each such truncated problem has a solution $x(\lambda, \omega^T)$ which is a Lipschitz function of the welfare weights. If in addition we can choose a uniform Lipschitz constant for each of these problems, then the solution to the original social planner's problem will also be Lipschitz continuous.

**Theorem 3.1.** *Let* $\mathcal{E}_\omega$ *be a smooth exchange economy in which the commodity space is* $\ell_p$ *for some* $p$ *such that* $1 \le p \le \infty$. *For each* $T > 0$, $x(\cdot, \omega^T)$ *is a Lipschitz function of* $\lambda$ *with constant* $c_T$. *If* $\{c_T\}$ *is bounded, then* $x(\cdot, \omega)$ *is also Lipschitz continuous.*

*Proof:* To see that $x(\cdot, \omega^T)$ is Lipschitz, note that by definition,

$$\begin{aligned} x(\lambda, \omega^T) = \arg \quad \max \quad & \sum_{i=1}^{m} \lambda_i U_i(x_i) \\ \text{s.t.} \quad & \sum_{i=1}^{m} x_i = \omega^T, \quad x_i \ge 0 \;\; i = 1, \ldots, m. \end{aligned}$$

So $x(\lambda, \omega^T) = (x_1^T(\lambda, \omega^T), 0, 0, \ldots, x_2^T(\lambda, \omega^T), 0, 0, \ldots, \ldots, x_m^T(\lambda, \omega^T), 0, 0, \ldots)$, where

$$\begin{aligned} x^T(\lambda, \omega^T) \equiv \arg \quad \max \quad & \sum_{i=1}^{m} \lambda_i U_i^T(y_i) \\ \text{s.t.} \quad & \sum_{i=1}^{m} y_i = \hat{\omega}^T, \quad y_i \ge 0 \;\; i = 1, \ldots, m \end{aligned}$$

and $\hat{\omega}^T = (\omega_1, \ldots, \omega_T)$. Thus it suffices to show that $x^T(\lambda, \omega^T)$ is Lipschitz in $\lambda$. Since each of these problems is finite-dimensional, arguments similar to those used by Mas-Colell (1985) to establish that demand is Lipschitz in finite-dimensional economies will give the desired result.

For each coordinate subspace $L$ of $\mathbb{R}^{mT}$, define

$$\begin{aligned} x_L^T(\lambda, \omega^T) = \arg \quad \max \quad & \sum_{i=1}^{m} \lambda_i U_i^T(y_i) \\ \text{s.t.} \quad & \sum_{i=1}^{m} y_i = \hat{\omega}^T, \quad y \in L. \end{aligned}$$

Since $D^2U_i^T(y_i)$ is negative definite for all $y_i \in \mathbb{R}_+^T$, $x_L^T(\lambda,\omega^T)$ is $C^1$ in $\lambda$ for each subspace $L$. Moreover,

$$x^T(\lambda,\omega^T) \in \bigcup_{L \in S(\mathbb{R}^{mT})} x_L^T(\lambda,\omega^T)$$

where $S(\mathbb{R}^{mT})$ is the set of all coordinate subspaces of $\mathbb{R}^{mT}$. Since this set has a finite number of elements and $x^T(\lambda,\omega^T)$ is continuous in $\lambda$, $x^T(\lambda,\omega^T)$ is Lipschitz continuous in $\lambda$ with some constant $c_T > 0$ (see MasColell (1985)).

Now suppose $\{c_T\}$ is bounded, and let $c = \sup_T c_T$. We must show that $x(\lambda,\omega)$ is Lipschitz. First suppose $1 \le p < \infty$. Let $x_T(\lambda) \equiv x(\lambda,\omega^T)$. By Lemma 3.1, $x_T(\lambda) \xrightarrow{n} x(\lambda,\omega)$ pointwise. If $\{c_T\}$ is bounded, then each function $x_T$ is Lipschitz with constant $c = \sup_T c_T$. Thus $\{x_T(\lambda)\}$ is an equicontinuous family, so by passing to a subsequence if necessary, $x_T(\lambda) \to x(\lambda,\omega)$ uniformly. This uniform convergence implies that $x(\lambda,\omega)$ is also Lipschitz with constant $c$. To see this, let $\lambda, \lambda' \in \Delta$ be arbitrary, and let $\epsilon > 0$ be given. Then choose $T$ such that $\|x_T(\tilde{\lambda}) - x(\tilde{\lambda},\omega)\| < \epsilon$ for all $\tilde{\lambda} \in \Delta$. Then

$$\begin{aligned} \|x(\lambda,\omega) - x(\lambda',\omega)\| &\le \|x(\lambda,\omega) - x_T(\lambda)\| + \|x_T(\lambda) - x_T(\lambda')\| \\ &\quad + \|x_T(\lambda') - x(\lambda',\omega)\| \\ &\le c\|\lambda - \lambda'\| + 2\epsilon. \end{aligned}$$

Since $\epsilon > 0$ was arbitrary, $x(\cdot,\omega)$ is Lipschitz in $\lambda$ with constant c for each $\omega$.

Now suppose $p = \infty$. By lemma 3.3, $x(\lambda,\omega^T) \xrightarrow{w^*} x(\lambda,\omega)$ pointwise. So for every $n$ and for each $i$, $x_{i_n}(\lambda,\omega^T) \to x_{i_n}(\lambda,\omega)$ pointwise, where $x_{i_n}(\cdot)$ is the $n^{th}$ term of $x_i(\cdot)$. But $\{x(\lambda,\omega^T)\}$ is uniformly Lipschitz with constant $c > 0$, so $\{x_{i_n}(\lambda,\omega^T)\}$ is also uniformly Lipschitz with constant $c$ for each $n$, since $\forall \lambda, \lambda' \in \Delta$,

$$|x_{i_n}(\lambda,\omega^T) - x_{i_n}(\lambda',\omega^T)| \le \|x_i(\lambda,\omega^T) - x_i(\lambda',\omega^T)\|.$$

Thus for each $n$, $\{x_{i_n}(\cdot,\omega^T)\}$ is equicontinuous in $\lambda$. Let $n$ be arbitrary. By passing to a subsequence if necessary, $x_{i_n}(\lambda,\omega^T) \to x_{i_n}(\lambda,\omega)$ uniformly. Then by the same argument used above, this uniform convergence implies that $x_{i_n}(\lambda,\omega)$ is Lipschitz with constant $c$ as well. But then for every $\lambda, \lambda' \in \Delta$,

$$\|x(\lambda,\omega) - x(\lambda',\omega)\| = \sup_{i,n} |x_{i_n}(\lambda,\omega) - x_{i_n}(\lambda',\omega)| \le c\|\lambda - \lambda'\|.$$

So $x(\lambda,\omega)$ is Lipschitz in $\lambda$ with constant $c$. ■

Showing that an economy is a Lipschitz economy, and thus has generically determinate equilibria, then amounts to showing that these truncated planner's problems are uniformly Lipschitz continuous. In some cases this result can be applied directly by calculating the Lipschitz constants in these truncated economies and verifying that they are uniformly bounded. For example, let $U(x)$ be a recursive utility function generated by an aggregator of the form $w(c,y) = u(c) + g(y)$, where $u, g : \mathbb{R}_+ \to \mathbb{R}$

are $C^2$ on $\mathbb{R}_+$, differentiably strictly concave and differentiably strictly monotone, and there exists $\beta \in (0, 1)$ such that $0 < g'(y) \le \beta$ for each $z \in \mathbb{R}_+$. Showing that such preferences satisfy the assumptions of a smooth exchange economy is straightforward, so is left to the reader. To see that such preferences also give rise to Lipschitz economies requires a bit of notation. For each $x$, let ${}_tx \equiv (x_t, x_{t+1}, \ldots)$, and abusing notation slightly, let $g(x) \equiv g(U(x))$. Then note that $\frac{\partial U}{\partial x_1} = u'(x_1)$ and for $t \ge 2$,

$$\frac{\partial U}{\partial x_t}(x) = g'({}_2x)\cdots g'({}_tx)u'(x_t).$$

For each $t$, define $b_t(x) = \frac{\partial U}{\partial x_t}(x)$, so by definition, $DU^T(x) = (b_1(x), \ldots, b_T(x))$. Furthermore, using this notation,

$$D^2U^T(x) = \begin{pmatrix} b_1(x)\frac{u''(x_1)}{u'(x_1)} & 0 & \cdots & 0 \\ 0 & b_2(x)\frac{u''(x_2)}{u'(x_2)} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b_T(x)\frac{u''(x_T)}{u'(x_T)} \end{pmatrix} + \sum_{t=2}^{T} R_t^b(x),$$

where

$$R_t^b(x) = \begin{pmatrix} 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & b_t(x)\frac{g''({}_tx)}{g'({}_tx)}b_1({}_tx) & \cdots & b_t(x)\frac{g''({}_tx)}{g'({}_tx)}b_{T+1-t}({}_tx) \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & b_T(x)\frac{g''({}_tx)}{g'({}_tx)}b_1({}_tx) & \cdots & b_T(x)\frac{g''({}_tx)}{g'({}_tx)}b_{T+1-t}({}_tx) \end{pmatrix}.$$

Although in general $D^2U^T(x)^{-1}$ becomes unbounded as $T$ goes to infinity, this happens essentially because of the consumer's impatience or myopia, as reflected by the sequence of generalized discount factors or beliefs $b(x) = (b_1(x), b_2(x), \ldots)$. This sequence is just equal to the consumer's gradient $DU(x)$, however, which suggests that $\left[D^2U^T(x)\right]^{-1} DU^T(x)$ will be bounded. At least with two consumers, this is essentially sufficient to ensure that the planner's problems are uniformly Lipschitz, and hence that the economy is a Lipschitz economy.

**Theorem 3.2.** *Let $\mathcal{E}_\omega$ be an exchange economy with two consumers in which the commodity space is $\ell_\infty$. For each $i$, let $U_i : \ell_{\infty+} \to \mathbb{R}$ be a recursive utility function generated by a separable aggregator $w_i(c, y) = u_i(c) + g_i(y)$, where $u_i, g_i : \mathbb{R}_+ \to \mathbb{R}$ are $C^2$ on $\mathbb{R}_+$, differentiably strictly concave and differentiably strictly monotone, and there exists $\beta_i \in (0, 1)$ such that $0 < g_i'(y) \le \beta_i$ for each $y \in \mathbb{R}_+$. Then $\mathcal{E}_\omega$ is a Lipschitz economy.*

*Proof:* See the appendix. ∎

Equilibria are generically determinate in this recursive economy, and the crucial condition guaranteeing generic determinacy is that $\frac{u''(x_t)}{u'(x_t)}$ is uniformly bounded away from zero on $[0,\omega]$, which amounts to prohibiting goods from becoming perfect substitutes or prohibiting consumers from becoming risk neutral asymptotically. Furthermore, this argument can be extended to show that equilibria are generically determinate in economies with recursive preferences generated by a nonseparable aggregator $w(c,y)$, provided current consumption and future utility are substitutes, so that $w_{cy}(c,y) \leq 0$ for all $(c,y)$, and provided a suitable upper bound can be put on "risk aversion", so that $\frac{w_{cc}}{w_c}(c,y) < \frac{w_{cy}}{w_y}(c,y)$ and $\frac{w_{yy}}{w_y}(c,y) < \frac{w_{cy}}{w_c}(c,y)$ for all $(c,y)$. In all of these recursive economies, equilibria are generically determinate.

These results will be most useful, however, if we can identify general conditions on preferences which lead to uniformly Lipschitz planner's problems. Each finite-dimensional truncated economy is a Lipschitz economy because consumer's indifference curves have non-zero curvature, since for each $T > 0$ there exists some $\beta_T > 0$ such that

$$z^{\intercal} D^2 U^T(x) z = (z, D^2 U^T(x) z) \leq -\beta_T \|z\|_p^2$$

for each $z \in \mathbb{R}^T$. With a finite number of goods, this condition rules out the robust indeterminacies that arise when goods are perfect substitutes, or when consumers are risk neutral. Intuitively, a similar condition which implies that indifference curves have non-zero curvature, and rules out goods which are perfect substitutes or consumers who are risk neutral in the limit, should be necessary to prohibit robust indeterminacies in infinite-dimensional economies. The difficulty with verifying this intuition comes in defining such a condition which is sufficient to guarantee generic determinacy and also consistent with the other assumptions of smooth exchange economies. A natural condition, analogous to the finite-dimensional one, would be to require that there exist $c > 0$ such that

$$z^{\intercal} D^2 U(x) z = (z, D^2 U(x) z) \leq -c \|z\|_p^2$$

for all $z$, or equivalently, that $\{\beta_T\}$ be bounded away from zero. When the commodity space is $\ell_2$, this condition is simply strong negative definiteness of the second derivative, or strong concavity of the utility function, and the results below show that equilibria are indeed generically determinate in $\ell_2$ when consumers' utility functions are strongly concave. When the commodity space is not $\ell_2$, however, it is typically impossible to rule out preferences violating this restriction due to myopia or impatience. For example, in all smooth myopic exchange economies, $\beta_T \to 0$ as $T \to \infty$, precisely because of the assumption of myopia. When the commodity space is not $\ell_2$ then, a more delicate argument accounting for this myopia or discounting will be required, together with some notion of curvature or negative definiteness valid in these more general settings.

One way to formalize this intuition involves defining a generalized notion of inner product valid outside the setting of $\ell_2$, or outside of a Hilbert space more generally. For a Banach space $X$, we can define the semi-inner product between two elements $x$ and $y$ to be

$$(x, y)_+ \equiv \|y\| \lim_{t \to 0^-} \frac{1}{t} \left[ \|y + tx\| - \|y\| \right].$$

For $\mathbb{R}^n$, $\ell_2$ or for any Hilbert space, this semi-inner product coincides with the standard inner product $(x,y)$. In a general Banach space, this notion essentially measures the Gateaux derivative of the norm of $y$ in the direction of $x$, and shares many of the important properties of an inner product, providing a way to measure curvature or define notions of negative definiteness.[13] In particular, for each $x$, let $F(x) : \ell_p \to \ell_p$ be a linear operator. Then $F(x)$ will be called **uniformly negative definite** if there exists $c > 0$ such that

$$(z, F^T(x)z)_+ \leq -c\|z\|_p^2$$

for each $T > 0$, for each $z \in \mathbb{R}^T$, and for each $x$. This definition suggests a natural way to define stronger notions of concavity in these models.

**Definition 3.1.** *Let $U(x) : \ell_{p+} \to \mathbb{R}$ be twice norm continuously Gateaux differentiable on $\ell_{p+}$. Then $U$ is called* **uniformly concave** *if*

1. *there exists $k > 0$ such that $\left\| \left[D^2U^T(x)\right]^{-1} DU^T(x) \right\|_p \leq k$ for each $x$ and for each $T > 0$, and*

2. *$D^2U(x) = B(x)S(x)$ for each $x \in \ell_{p+}$, where $B(x) = \text{diag}\{\frac{\partial U}{\partial x_t}(x)\}$[14] and $S(x) : \ell_p \to \ell_p$ is uniformly bounded and uniformly negative definite.*

To understand this definition, notice that because of impatience or myopia, the second derivative will typically go to zero in these economies, at least in some directions. To prevent goods from becoming perfect substitutes asymptotically, the first condition specifies that the second derivative cannot go to zero faster than marginal utility does. The second condition says that myopia or impatience is essentially the only reason the second derivative goes to zero. The main result of this section is that these two additional conditions are sufficient to imply that in the resulting economy, equilibria are generically determinate.

**Theorem 3.3.** *Let $\mathcal{E}_\omega$ be a smooth exchange economy in which the commodity space is $\ell_p$ for some $p$ such that $1 \leq p \leq \infty$. If each consumer's utility function is uniformly concave then the economy $\mathcal{E}_\omega$ is a Lipschitz economy.*

*Proof:* See the appendix. ■

To see how to check for uniform concavity, note that by definition

$$(z, S^T(x)z)_+ = \|S^T(x)z\| \lim_{t \to 0^+} \frac{1}{t} \left[\|S^T(x)z + tz\| - \|S^T(x)z\|\right],$$

[13]See the appendix and Deimling (1985), for example, for a discussion of this definition of semi-inner product.

[14]Here $\text{diag}\{c_1, \ldots, c_k\}$ refers to the operator represented by the matrix

$$\begin{pmatrix} c_1 & 0 & \ldots & 0 \\ 0 & c_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & c_k \end{pmatrix}.$$

so uniform negative definiteness can be equivalently characterized by requiring that there exist $c > 0$ such that for each $T$, for each $x$ and for all $z \in \mathbb{R}^T$,

$$\|S^T(x)z + tz\| \leq \|S^T(x)z\| - tc\frac{\|z\|^2}{\|S^T(x)z\|}$$

for all $t \geq 0$ sufficiently small. In many cases, this condition can be simplified even further, as is the case when $S(x)$ is uniformly bounded below, so that there exists $k > 0$ such that $\|S^T(x)z\| \geq k\|z\|$ for all $T > 0$. for each $z \in \mathbb{R}^T$, and for each $x$.

**Lemma 3.3.** *Let $S(x) : \ell_p \to \ell_p$ be uniformly bounded below. If there exists $c > 0$ such that for all $T > 0$, for each $x$ and for all $z \in \mathbb{R}^T$,*

$$\|S^T(x)z + tz\| \leq (1 - tc)\|S^T(x)z\|$$

*for all $t \geq 0$ sufficiently small, then $S^T(x)$ is uniformly negative definite.*

*Proof:* Let $T > 0$ and $x$ be given, and let $z \in \mathbb{R}^T$. By assumption

$$\|S^T(x)z + tz\| \leq (1 - tc)\|S^T(x)z\|$$

for all $t \geq 0$ sufficiently small. Thus there exists $\bar{t} > 0$ such that if $t \leq \bar{t}$, then

$$\|S^T(x)z + tz\| - \|S^T(x)z\| \leq -tc\|S^T(x)z\|,$$

which implies that

$$\|S^T(x)z\|\frac{1}{t}\left[\|S^T(x)z + tz\| - \|S^T(x)z\|\right] \leq -c\|S^T(x)z\|^2 \leq -ck^2\|z\|^2$$

since $\|S^T(x)z\| \geq k\|z\|$. So

$$(z, S^T(x)z)_+ = \lim_{t\to 0^+} \|S^T(x)z\|\frac{1}{t}\left[\|S^T(x)z + tz\| - \|S^T(x)z\|\right] \leq -ck^2\|z\|^2.$$

Thus $S^T(x)$ is uniformly negative definite. ■

A class of examples to which these results immediately apply is the additively separable framework of Kehoe and Levine (1985). In their model, the commodity space is $\ell_\infty$ and preferences are given by utility functions of the form $U_i(x) = \sum_{t=1}^{\infty} \beta_i^t u_i(x_t)$, where $\beta_i \in (0, 1)$. Given such preferences, $D^2U_i(x) = \text{diag}\{\beta_i^t u_i''(x_t)\}$ and $DU_i(x) = \{\beta_i^t u_i'(x_t)\}$ for every $x \in \ell_{\infty+}$, so $(D^2U_i^T(x))^{-1}DU_i^T(x) = \{\frac{u_i'(x_t)}{u_i''(x_t)}\}_{t=1}^T$. Under the assumption that $u_i$ is $C^2$ on $\mathbb{R}_+$, $u_i'(c) > 0$ and $u_i''(c) < 0$ for each $c \geq 0$, this calculation shows that $(D^2U_i^T(x))^{-1}DU_i^T(x)$ is uniformly bounded on any interval of the form $[0, \omega]$. Furthermore, $D^2U(x) = B(x)S(x)$, where $S(x) = \text{diag}\ \{\frac{u''(x_t)}{u'(x_t)}\}$, which is uniformly negative definite on $[0, \omega]$. To see this, first note that $S(x)$ is uniformly bounded below on $[0, \omega]$, since given $T > 0$ and $z \in \mathbb{R}^T$, $S^T(x)z = \{\frac{u''(x_s)}{u'(x_s)}z_s\}$, so

$\|S^T(x)z\| \geq k\|z\|$, where $k = \min_{r\in[0,\|\omega\|]} \left|\frac{u''(r)}{u'(r)}\right| > 0$. Then $S^T(x)z + tz = \{\frac{u''(x_s)}{u'(x_s)} z_s + tz_s\}$, and for $t$ sufficiently small.

$$\begin{aligned}
\left|\frac{u''(x_s)}{u'(x_s)} z_s + tz_s\right| &= \left|\frac{u''(x_s)}{u'(x_s)}\right| \left|z_s + t\frac{u'(x_s)}{u''(x_s)} z_s\right| \\
&= \left|\frac{u''(x_s)}{u'(x_s)}\right| \left(|z_s| - t\left|\frac{u'(x_s)}{u''(x_s)}\right| |z_s|\right) \\
&= \left(1 - t\left|\frac{u'(x_s)}{u''(x_s)}\right|\right) \left|\frac{u''(x_s)}{u'(x_s)} z_s\right| \\
&\leq (1 - tc)\left|\frac{u''(x_s)}{u'(x_s)} z_s\right|,
\end{aligned}$$

where $c = \min_{r\in[0,\|\omega\|]} \left|\frac{u'(r)}{u''(r)}\right| > 0$. So $\|S^T(x)z + tz\| \leq (1 - tc)\|S^T(x)z\|$, which by Lemma 3.3 implies that $S(x)$ is uniformly negative definite. Since all Pareto optimal allocations are contained in the order interval $[0, \omega]$ defined by the social endowment $\omega$, uniform concavity on this interval is sufficient for the generic determinacy results to apply, and again the crucial condition guaranteeing uniform concavity is that $\frac{u''(x_t)}{u'(x_t)}$ is bounded away from zero, ruling out asymptotic risk neutrality or goods becoming perfect substitutes in the limit.

These results can also be extended by explicitly modelling consumers' impatience or myopia. As a motivating example, consider additively separable economies again. For such utility functions, note that we can also write $D^2U_i(x) = B_i(x)S_i(x)$, where $B_i(x) = \text{diag}\{\beta_i^t\}$ and $S_i(x) = \text{diag}\{u_i''(x_t)\}$, and under the standard assumptions on the single good utility functions described above, $S_i(x)$ is uniformly negative definite on any interval of the form $[0, \omega]$. Moreover, the sequence $b_i(x) \equiv \{\beta_i^t\}$ represents the consumer's discount factors or measures his myopia in the sense that his gradient $DU_i(x)$ is bounded with respect to this sequence, so that $[B_i(x)]^{-1}DU_i(x)$ is uniformly bounded on $[0, \omega]$. Modulo this sequence of discount factors or beliefs, such preferences then display the same basic features of uniformly concave preferences. Furthermore, if each consumer has such preferences, then the consumers' beliefs are consistent in the sense that we can substitute one sequence $b_j(x)$ for another $b_i(x)$ and obtain preferences similar to the original ones. These ideas can be formalized by the following two definitions.

**Definition 3.2.** *Let $U(x) : \ell_{p+} \to \mathbb{R}$ be twice norm continuously Gateaux differentiable on $\ell_{p+}$. Then $U(x)$ is called* **myopically concave** *if $D^2U(x) = B(x)S(x)$ for each $x \in \ell_{p+}$, where*

1. *$B(x) : \ell_p \to \ell_q$ has the form $B(x) = \text{diag}\{b(x)\}$, where $b(x) \in \ell_{q++}$ and $[B(x)]^{-1}DU(x)$ is uniformly bounded, and*

2. *$S(x) : \ell_p \to \ell_p$ is uniformly bounded and uniformly negative definite.*

If a consumer's preferences are represented by a myopically concave utility function, then we can essentially decompose the effects on marginal utility into a term reflecting the consumer's discounting or beliefs, represented by the vector $b(x)$, and a second part which is negative definite in a strong sense. As long as these discount factors or beliefs are comparable, this condition suffices to ensure that the social planner's problem is well-behaved in the limit. Here comparability means that if we adjust by another set of beliefs, the result, $B_j(x_j)^{-1}D^2U_i(x_i)$, is at least finitely negative definite, where $F(x) : \ell_p \to \ell_p$ is **finitely negative definite** if

$$(z, F^T(x)z)_+ < 0$$

for each $T$ and for each $z \in \mathbb{R}^T \setminus \{0\}$.

**Definition 3.3.** *Suppose each consumer is myopically concave. Then the consumers are* **consistent** *if for each $i \neq j$, $D(x_j, x_i) \equiv B_j(x_j)^{-1}D^2U_i(x_i)$ is finitely negative definite for all $x_i$ and $x_j$.*

The argument used to prove that uniform concavity is sufficient for an economy to be a Lipschitz economy can then be adapted to show that these two conditions are also sufficient to rule out robust indeterminacies, as the following corollary demonstrates.

**Corollary 3.1.** *Let $\mathcal{E}_\omega$ be a smooth exchange economy in which the commodity space is $\ell_p$ for some $p$ such that $1 \leq p \leq \infty$. If each consumer's utility function is myopically concave and the consumers are consistent, then the economy is a Lipschitz economy.*

*Proof:* See the appendix. ■

Using Corollary 3.1, a class of examples allowing for more interaction between goods in economies with commodity space $\ell_2$ can be developed by considering habit formation preferences, which have the form $U(x) = v(x_0) + \sum_{t=1}^{\infty} \beta^t u^t(x_{t-1}, x_t)$. Suppose here that $0 < \beta < 1$, $u^t, v$ are $C^2$, and that $v'(c) > 0$ and $v''(c) < 0$ for every $c \in \mathbb{R}_+$. Suppose further that $\{Du^t(x_{t-1}, x_t)\} \in \ell_{2++}$ for each $x \in \ell_{2+}$ and that $D^2u^t(c_1, c_2)$ is uniformly negative definite. Here this assumption means that there exists $c > 0$ such that $z^\dagger D^2u^t(c_1, c_2)z \leq -c\|z\|^2$ for each $t$, for each $(c_1, c_2) \in \mathbb{R}^2_+$ and for all $z \in \mathbb{R}^2$. If all consumers have such habit formation preferences, then this economy is also a smooth exchange economy, which can be established by a straightforward adaptation of the proof of Theorem 2.2; the details are left to the reader. Furthermore, under these assumptions, habit formation preferences are myopically concave for all sufficiently large discount factors, as the following result shows.

**Theorem 3.4.** *Suppose $U : \ell_{2+} \to \mathbb{R}$ is given by $U(x) = v(x_0) + \sum_{t=1}^{\infty} \beta^t u^t(x_{t-1}, x_t)$ as described above. There exists $\bar{\beta} \in (0, 1)$ such that if $\beta \geq \bar{\beta}$, then $U(x)$ is myopically concave on $[0, \omega]$ for each $\omega \in \ell_{2+}$.*

*Proof:* See the appendix. ■

When all consumers have the same discount factor, they are also consistent by the same argument. An application of the results of this section then demonstrates that these habit formation economies have generically determinate equilibria. This result can also be extended to a setting in which $c_i \in \mathbb{R}^n_+$ for each $i$, allowing for either $n$ goods each period or a $2n$ period window over which consumption decisions affect current utility.

When the commodity space is $\ell_2$, as in this example and in many financial markets models, these results can be simplified further by making use of the special features of this commodity space. Let $U_i : \ell_{2+} \to \mathbb{R}$ be a consumer's utility function, and suppose that $D^2U_i(x)$ is strongly negative definite, or that there exists $c > 0$ such that for each $x \in \ell_{2+}$,

$$z^{\dagger} D^2U_i(x)z = (z, D^2U_i(x)z) \leq -c\|z\|^2$$

for all $z$, which is equivalent to assuming that $D^2U_i(x)$ is uniformly negative definite.[15] Such utility functions will be called **strongly concave**. Strongly concave utility functions are myopically concave and consistent under the trivial decomposition $D^2U_i(x) = ID^2U_i(x)$, so strong concavity alone suffices to ensure that equilibria are generically determinate when the commodity space is $\ell_2$.

**Theorem 3.5.** *Consider a smooth exchange economy in which the commodity space is $\ell_2$. If each consumer's utility function is strongly concave on $[0, \omega]$, then the economy $\mathcal{E}_\omega$ is a Lipschitz economy.*

*Proof:* This follows from the previous discussion, Lemma A4, and Corollary 3.1. ■

For economies in which the commodity space is $\ell_2$, these results parallel exactly those obtained for economies with finitely many commodities: in a smooth exchange economy, equilibria are generically determinate provided consumers' utility functions are strongly concave, which is the appropriate extension of differential concavity to these economies. When the commodity space is not $\ell_2$, consumers' impatience or myopia will mean that typically there are no smooth economies in which preferences are strongly concave. Uniform concavity and myopic concavity then capture the same idea by explicitly accounting for this discounting.

To develop a further class of examples in which these results apply in commodity spaces other than $\ell_2$, we can again consider additively separable preferences as a motivating example. If preferences are additively separable, then the only thing influencing marginal utility for consumption of good $t$ is consumption of good $t$. So if the utility function $U_i(x)$ is additively separable, then the second derivative $D^2U_i(x)$ is a diagonal matrix with diagonal $\{\beta_i^t u_i''(x_t)\}$, and thus has a dominant diagonal in a very strong sense: all of the off-diagonal elements are 0. If we consider more general preferences, consumption of goods other than $t$ will affect marginal utility for consumption of good $t$, so the second derivative will no longer be a diagonal matrix. As long as the effect of consumption of good $s$ on marginal utility for consumption of good $t$ is small relative to the effect of consumption of the same good $t$ however, the economy will still be a Lipschitz economy.

[15] In a smooth economy, $DU_i(x) \in \ell_{2++}$ for each $x \in \ell_{2+}$, because $\ell_2$ is self-dual. Furthermore, $D^2U_i(x)$ is a linear operator between $\ell_2$ and itself for each $x \in \ell_{2+}$, for the same reason. Note that in this case, strong negative definiteness is also equivalent to requiring that all eigenvalues of $D^2U_i(x)$ be negative and uniformly bounded away from 0.

In order to make sense of that claim, we must first define the notion of dominant diagonal in this setting. If $A(x)$ is a continuous linear operator on $\ell_\infty$ for each $x \in V \subset \ell_\infty$, then $A(x)$ has a **dominant diagonal** $d(x) = \{a_{tt}(x)\}$ if there exists $m \in (0,1)$ such that $\sum_{s \neq t} |\frac{a_{ts}(x)}{a_{tt}(x)}| \leq m < 1$ for every $t$. Moreover, if this bound holds for every $x \in V$, then we say that $A(x)$ has a **uniformly dominant diagonal** over the set $V$, and if in addition $m < \frac{1}{2}$, then $A(x)$ has a **strongly dominant diagonal** over $V$.

Now suppose that for each $i = 1, \ldots, m$, $D^2U_i(x)$ has a strongly dominant diagonal $\{d_{i_{tt}}(x)\}$ which is uniformly bounded with respect to $DU_i(x)$, so that there exists $M > 0$ such that $\frac{\partial U_i}{\partial x_t}(x) \leq M|d_{i_{tt}}(x)|$ for all $x \in V$. If these conditions are satisfied, we will say that $U_i(x)$ has a *dominant diagonal*. If so, then each consumer's utility function is myopically concave.

**Theorem 3.6.** *If $U : \ell_{\infty+} \to \mathbb{R}$ has a dominant diagonal, then $U$ is myopically concave.*

*Proof:* Let $D^2U(x) = \{d_{ts}(x)\}_{t,s=1}^{\infty}$. Then let $B(x) = \text{diag}\{-d_{tt}(x)\}$ and $S(x) = B(x)^{-1}D^2U(x) = \left\{-\frac{d_{ts}(x)}{d_{tt}(x)}\right\}_{t,s=1}^{\infty}$. Then $S(x)$ also has a strongly dominant diagonal given by $(-1, -1, -1, \ldots)$, as does $S^T(x)$ for all $T$. Fix $T > 0$ and $z \in \mathbb{R}^T$ such that $\|z\|_\infty = 1$. Now since $D^2U(x)$ has a strongly dominant diagonal, $S^T(x) = -I + C^T(x)$ for all $x$, where $\|C^T(x)\| \leq m$ for some $m \in (0, \frac{1}{2})$. First, $S^T(x)$ is uniformly bounded below, since

$$\begin{aligned} \|S^T(x)z\| = \|-z + C^T(x)z\| &\geq \|z\| - \|C^T(x)z\| \\ &\geq \|z\| - m\|z\| = (1-m)\|z\| \end{aligned}$$

and $1 - m > 0$. Then

$$\begin{aligned} \|S^T(x)z + tz\| &= \|-z + C^T(x)z + tz\| \\ &= \|-z + C^T(x)z - t(-z + C^T(x)z) + tC^T(x)z\| \\ &\leq (1-t)\|S^T(x)z\| + t\|C^T(x)z\| \\ &\leq (1-t)\|S^T(x)z\| + tm\|z\| \\ &= (1-t)\|S^T(x)z\| + tm\|-z + C^T(x)z - C^T(x)z\| \\ &\leq (1 - t(1-m))\|S^T(x)z\| + tm^2\|z\| \\ &\leq (1 - t(1-m-m^2))\|S^T(x)z\| + tm^3\|z\|. \end{aligned}$$

Repeating this argument shows that

$$\|S^T(x)z + tz\| \leq \left(1 - t\left(1 - \frac{m}{1-m}\right)\right)\|S^T(x)z\| = (1 - tc)\|S^T(x)z\|,$$

and $c = 1 - \frac{m}{1-m} > 0$ since $m < \frac{1}{2}$. Thus by Lemma 3.3, $S(x)$ is uniformly negative definite. ∎

Furthermore, note that in this case, for all $i \neq j$, $(B_j(x_j))^{-1}D^2U_i(x_i)$ also has a strongly dominant diagonal, so $(B_j(x_j))^{-1}D^2U_i(x_i)$ is finitely negative definite for each $x_i$ and $x_j$. Thus consumers with such preferences are consistent. Any economy in which all consumers' utility functions have a dominant diagonal will then generically have determinate equilibria by an immediate application of Corollary 3.1.[16]

**Theorem 3.7.** *Let $\mathcal{E}_{\omega}$ be a smooth myopic exchange economy. If $U_i(x)$ has a dominant diagonal on $[0, \omega]$ for each $i = 1, \ldots, m$, then the economy is a Lipschitz economy.*

One class of economies to which this result can be applied is the habit formation economies discussed above when the commodity space is $\ell_\infty$. For these preferences, if $D^2u(c_1, c_2)$ has a strongly dominant diagonal for each $(c_1, c_2) \in \mathbb{R}^2_+$, then $U(x)$ will have a dominant diagonal on $[0, \omega]$ for each $\omega \in \ell_{\infty+}$, and hence will be myopically concave on this feasible set. Equilibria are thus generically determinate in such economies.

For another example of an economy with dominant diagonal preferences, consider utility functions of the form $U(x) = \sum_{t=1}^{\infty} \beta_t u_t(x_1, \ldots, x_t)$, where $u_t : \mathbb{R}^t_+ \to \mathbb{R}$ and $\{\beta_t\} \in \ell_{1++}$. Suppose that for each $t$, $u_t$ is $C^2$ on $\mathbb{R}^t_+$, differentiably strictly concave and differentiably strictly monotone, and that $u_t$ has a dominant diagonal for each $t$. Moreover, suppose that, given $\omega \in \ell_{\infty+}$, there exist $M_1, M_2 > 0$ such that $\sup_{t,s,x \in [0,\omega]} \frac{\partial u^t}{\partial x_s}(x) \leq M_1$ and $\sup_{t,s,x \in [0,\omega]} \frac{\partial^2 u^t}{\partial x_s^2}(x) \leq -M_2$. Stroyan (1983) has shown that such utility functions are Mackey continuous on $\ell_{\infty+}$, and showing that $U(x)$ is twice continuously Gateaux differentiable is straightforward, as is verifying that $DU(x) \in \ell_{1++}$ and that $D^2U(x)$ is negative definite for each $x \in \ell_{\infty+}$. Moreover, $U(x)$ is myopically concave. To see this, given $T > 0$, for each $t \leq T$, let $\partial^2 u_t(x_1, \ldots, x_t)$ be the $T \times T$ matrix

$$\partial^2 u_t(x_1, \ldots, x_t) = \begin{pmatrix} D^2u_t(x_1, \ldots, x_t) & \mathbf{0}_{t \times (T-t)} \\ \mathbf{0}_{(T-t) \times t} & \mathbf{0}_{(T-t) \times (T-t)} \end{pmatrix}.$$

Then $D^2U^T(x) = \sum_{t=1}^{T} \beta_t \partial^2 u_t(x_1, \ldots, x_t)$, which has a strongly dominant diagonal on any interval of the form $[0, \omega]$. Thus an exchange economy in which consumers have such preferences will have generically determinate equilibria.

All of these results have been in a setting prohibiting Inada conditions commonly found in infinite horizon models. As discussed earlier, such conditions are inconsistent with supportability or properness in almost all economies with infinitely many commodities, since in most of these settings, including all $\ell_p$ spaces except $\ell_\infty$, the positive cone has an empty interior. One notable exception, $\ell_\infty$, is the canonical model for discrete time, infinite horizon economies in which Inada conditions often play a prominent role, so in the next section, I discuss how the results of the paper can be extended to allow for Inada conditions in such economies, and what such conditions might mean in these general infinite horizon models.

[16] This result can also be extended to economies with commodity space $\ell_p$ where $p \neq \infty$ by defining an analogous notion of dominant diagonal, for example requiring that $S^T(x) = \{-\frac{d_{ts}(x)}{d_{tt}(x)}\} = -I + C^T(x)$, where $\|C^T(x)\| \leq m < \frac{1}{2}$.

# 4 Determinacy and Inada Conditions

To allow for Inada conditions when the commodity space is $\ell_\infty$, we must change the notion of a smooth economy. One of the requirements of a smooth economy was the existence of a well-defined continuous linear functional supporting each consumer's indifference set at each point in the positive cone, but natural Inada conditions will give rise to preferences which violate this assumption. For example, consider additively separable preferences of the form $U(x) = \sum_{t=0}^{\infty} \beta^t u(x_t)$; here the natural Inada condition requires that $u'(c) \to \infty$ as $c \to 0$. If the consumption vector $x$ is not uniformly bounded away from 0, then the consumer's indifference set at that bundle typically will not be supported by a continuous linear functional. In contrast, in smooth economies with a finite number of goods, the boundary condition that indifference curves do not intersect the boundary of the positive cone immediately guarantees that every indifference set and every Pareto optimal allocation is supported by a well-defined, unique price. An assumption like that here would mean that consumers cannot be impatient or myopic and must instead be infinitely far-sighted. To see this, note that the assumption of Mackey continuity implies a priori that consumers' indifference curves will intersect the boundary of the positive cone. For example, if $U_i(x) > U_i(y)$ and $U_i(\cdot)$ is Mackey continuous, then there exists $n$ such that $U_i(x^n) = U_i(x_1, \ldots, x_n, 0, 0, \ldots) > U_i(y)$; preferences between bundles do not rely on consumption "at infinity". Any assumption prohibiting the indifference curves from intersecting the boundary of the consumption set will then be incompatible with the assumption of Mackey continuity. With an infinite number of commodities, finding analogous boundary conditions which are consistent with the assumption of Mackey continuity of preferences and still guarantee that the excess savings equations are well-defined becomes more complicated. Thus I first present a series of successively stronger boundary conditions on preferences which imply successively stronger interiority properties for the resulting Pareto optimal allocations, before discussing how to extend the determinacy results of the previous sections to this setting.

The first of these conditions is the weakest, and will ensure that all consumers consume positive quantities of all goods at any Pareto optimal allocation which assigns positive weight to each consumer, and so in particular ensures that all individually rational Pareto optimal allocations are strictly positive.

**Definition 4.1.** *Consumer $i$ satisfies the* **weak survival condition** *if for every $x \in \ell_{\infty+}$ and for every $t$, if $x_t > 0$ then $\frac{\partial U_i}{\partial x_t}(x)$ exists, and*

$$\frac{\frac{\partial U_i}{\partial x_t}}{\frac{\partial U_i}{\partial x_s}}(x_t, x_{-t}) \to \infty \text{ as } x_t \to 0$$

*for all fixed $x_{-t}$ such that $x_s > 0$.*[17]

The weak survival condition says that if consumption of some good goes to zero, then the consumer's marginal rate of substitution for that good with respect to some

[17] Here, $x_{-t}$ denotes the elements of $x$ corresponding to goods $r \neq t$.

other good he consumes in a fixed positive amount goes to infinity as the amount of the diminishing good goes to 0, and this must hold regardless of the (fixed) level of consumption of other goods. This condition, if satisfied by all consumers in the economy, means that consumption bundles are strictly positive in all Pareto optimal allocations giving positive weight to each consumer.

**Theorem 4.1.** *Let $\mathcal{E}_\omega$ be an economy in which each consumer's utility function satisfies the weak survival condition. If $\lambda \in \Delta^o$, then $x(\lambda) \in \ell^m_{\infty++}$.*

*Proof:* See the appendix. ■

In a finite-dimensional model, strictly positive bundles are interior bundles, but since that is not true in infinite-dimensional models, stronger conditions will be necessary to ensure that Pareto optimal allocations lie in the interior of the positive cone.[18] Two such conditions are presented below.

**Definition 4.2.** *In an economy $\mathcal{E}_\omega$, let $x$ be an allocation in which $x_{i_r} \to 0$ for some sequence $r \to \infty$, and in which there exists a consumer $j$ and a fixed constant $c > 0$ such that $x_{j_r} \geq c$ for every $r$. Consumer $i$ satisfies the* **survival condition** *if he satisfies the weak survival condition, and if*

$$\lim\ \sup_r \frac{\frac{\partial U_i}{\partial x_r}(x_i)}{\frac{\partial U_j}{\partial x_r}(x_j)} = \infty.$$

*Let $x^n$ be a sequence of allocations such that $x^n_k \in \text{int } \ell_{\infty+}$ for each $k$ and $n$, and such that for some sequence $\{r(n)\}$ such that $r(n) \to \infty$ as $n \to \infty$, $x^n_{i_{r(n)}} \to 0$ and for some consumer $j$ and some fixed constant $c > 0$, $x^n_{j_{r(n)}} \geq c$ for all $n$. Consumer $i$ satisfies the* **strong survival condition** *if in addition to satisfying the survival condition,*

$$\lim\ \sup_n \frac{\frac{\partial U_i}{\partial x_{r(n)}}(x^n_i)}{\frac{\partial U_j}{\partial x_{r(n)}}(x^n_j)} = \infty.$$

These notions embody several different restrictions. Marginal utility for a good must go to infinity as the quantity of the good consumed goes to zero, and consumers should be discounting at roughly the same rate, which guarantees that the ratio of marginal utilities of consumption of good $t$ of two consumers, calculated at bundles in which one agent's consumption of good $t$ is bounded away from zero uniformly across goods and the other agent's consumption is going to zero as $t$ goes to infinity, should converge to infinity with $t$. The strong survival condition strengthens this notion by requiring this convergence to hold across sequences of allocations. These stronger conditions are sufficient to guarantee that Pareto optimal allocations are interior, and the strong survival condition guarantees in addition that the individually rational Pareto optimal allocations are uniformly bounded away from 0. Alternatively, if we call a consumer an **interior consumer** if $x_j(\lambda) \in \text{int } \ell_{\infty+}$ for all $\lambda \in \Delta^o$, then these survival conditions predict when one or all consumers will be interior consumers.

[18] Recall that the interior of $\ell_{\infty+}$ consists of all bundles uniformly bounded away from 0, so $x \in \text{int } \ell_{\infty+}$ if and only if $\inf_t x_t > 0$.

**Theorem 4.2.** *If each consumer satisfies the weak survival condition and some consumer $j$ satisfies the survival condition, then consumer $j$ is an interior consumer. If all consumers satisfy the survival condition, then all consumers are interior consumers.*

*Proof:* I will prove the second claim; the proof of the first is the same. Let $(x_1, \ldots, x_m)$ be a positive weight Pareto optimal allocation, i.e., an allocation corresponding to weights $\lambda_i > 0$ for each $i$. First note that $x_{i_t} \neq 0$ for all $i$ and $t$ by the previous result. Now suppose there exists $i$ such that $x_i \notin$ int $\ell_{\infty+}$. Then there exists a sequence $r \to \infty$ such that $x_{i_r} \to 0$. Moreover, by feasibility there exists a consumer $j$ such that, by passing to a subsequence and relabeling if necessary, $x_{j_r} \geq \frac{\underline{\omega}}{m}$ for each $r$, where $\underline{\omega} = \inf_t \omega_t > 0$, and $\omega_t$ is the aggregate endowment of good $t$. Define

$$\hat{p}^t = (\frac{\partial U_j}{\partial x_1}(x_j), \ldots, \frac{\partial U_j}{\partial x_t}(x_j))$$

for each $t$. Since $(\hat{x}_1^t, \ldots, \hat{x}_m^t)$ is Pareto optimal in the truncated $t$ good economy for each $t$, the price $\hat{p}^r$ supports $\hat{x}_i^r$ for each $r$. In particular, for each $r$,

$$\frac{\frac{\partial U_i}{\partial x_r}(x_i)}{\frac{\partial U_i}{\partial x_1}(x_i)} = \frac{\frac{\partial U_j}{\partial x_r}(x_j)}{\frac{\partial U_j}{\partial x_1}(x_j)}.$$

Without loss of generality, assume $r_0 = 1$. Thus for each $r$,

$$\frac{\frac{\partial U_j}{\partial x_1}(x_j)}{\frac{\partial U_i}{\partial x_1}(x_i)} = \frac{\frac{\partial U_j}{\partial x_r}(x_j)}{\frac{\partial U_i}{\partial x_r}(x_i)},$$

but this is a contradiction, since the terms on the left become unbounded as $r \to \infty$, and the term on the right is bounded. Thus $x_i \in$ int $\ell_{\infty+}$ for all $i$. ∎

**Theorem 4.3.** *Suppose that each consumer satisfies the strong survival condition. Then there exists $\epsilon > 0$ such that every individually rational Pareto optimal allocation $x$ is bounded below by $\epsilon$, that is, such that $\inf_t |x_{i_t}| \geq \epsilon$ for all $i$.*

*Proof:* See the appendix. ∎

Examples of economies satisfying these various survival conditions are relatively easy to construct. First, the exchange economy in which each consumer's utility function is additively separable provides a range of examples in which, depending on the consumers' discount factors, either the consumers are all interior, or there exists at least one interior consumer. For these examples, suppose that for $i = 1, \ldots, m$, $U_i(x) = \sum_{t=0}^{\infty} \beta_i^t u_i(x_t)$, where $0 < \beta_i < 1$, $u_i : \mathbb{R}_+ \to \mathbb{R}$ is $C^2$ on $\mathbb{R}^2_{++}$, $u_i'(c) > 0$ and $u_i''(c) < 0$ for every $c \in \mathbb{R}_{++}$, and $u_i'(c) \to \infty$ as $c \to 0$. Each consumer in this economy satisfies the weak survival condition, as for all $t$ and $s$,

$$\frac{\frac{\partial U_i}{\partial x_t}(x_i)}{\frac{\partial U_i}{\partial x_s}(x_i)} = \frac{\beta_i^t u_i'(x_{i_t})}{\beta_i^s u_i'(x_{i_s})} \to \infty$$

as $x_{i_t} \to 0$ if $x_{i_s}$ is fixed and positive. Similarly, if all consumers discount at the same rate, so there exists some $\beta \in (0,1)$ such that $\beta = \beta_i$ for all $i$, then all consumers satisfy the other survival conditions as well, as

$$\frac{\frac{\partial U_i}{\partial x_r}(x_i)}{\frac{\partial U_j}{\partial x_r}(x_j)} = \frac{u_i'(x_{i_r})}{u_j'(x_{j_r})}$$

for all $i, j, r$. Furthermore, if consumers discount at different rates, then those consumers with the greatest discount factor are interior consumers. To see this, note that if we choose $k$ such that $\beta_k \geq \beta_i$ for every $i$, $\frac{\beta_k}{\beta_i} \geq 1$ and

$$\frac{\frac{\partial U_k}{\partial x_r}(x_k)}{\frac{\partial U_i}{\partial x_r}(x_i)} = \left(\frac{\beta_k}{\beta_i}\right)^r \frac{u_i'(x_{i_r})}{u_j'(x_{j_r})}.$$

Similarly, suppose consumer's utility functions exhibit habit formation, so that they have the form $U_i(x) = \sum_{t=1}^{\infty} \beta^t u_i(x_{t-1}, x_t)$, where $u_i : \mathbb{R}^2_+ \to \mathbb{R}$ satisfies the following condition: for any sequence $(x_n, y_n)$ in $\mathbb{R}^2_+$ such that $(x_n, y_n) \to (x, y)$, $\frac{\partial u_i}{\partial c_1}(x_n, y_n) \to \infty \iff x_n \to 0$ and similarly $\frac{\partial u_i}{\partial c_2}(x_n, y_n) \to \infty \iff y_n \to 0$.[19] Then all consumers will satisfy the strong survival condition, and thus by Theorem 4.3, Pareto optimal allocations will be uniformly bounded below. If consumers have different discount factors $\beta_i$, then again the most patient, those with the largest discount factors, will be interior consumers.

Similarly, if $m-1$ consumers have additively separable utility functions with constant discount rate $\beta$ as described above, and one consumer has recursive preferences generated by an aggregator of the form $w(c, z) = u(c) + g(z)$, where $u'(c) \to \infty$ as $c \to 0$ and $\alpha \leq g'(z) \leq \gamma$ for some $\alpha, \gamma \in [\beta, 1)$, then the recursive consumer will satisfy the strong survival condition, and hence by Theorem 4.3 she will be an interior consumer.

By these results, if each consumer satisfies the weak survival condition and some consumer satisfies the survival condition, then this consumer always receives an interior bundle in every individually rational Pareto optimal allocation. These survival conditions thus imply that we can characterize equilibria using the excess savings equations much as we did in section 2 while still allowing for Inada type restrictions. Indeed, without loss of generality, we can assume that the first consumer satisfies the survival condition. Given any weights $\lambda$, the price supporting the Pareto optimal allocation $x(\lambda)$ must be proportional to $DU_1(x_1(\lambda))$, so define $p(\lambda) \equiv DU_1(x_1(\lambda))$. Then as in section 2, normalizing prices by setting the price of the first good equal to 1 and letting $\tilde{p}(\lambda)$ be the normalized price vector, we can characterize equilibria as the welfare weights $\lambda$ satisfying the equation $\hat{s}(\lambda) = \omega^1$. The analogue of a smooth economy in the presence of these Inada conditions is then the following.

**Definition 4.3.** *An economy with commodity space $\ell_\infty$ is a* **smooth exchange economy with Inada conditions** *if for each consumer $i = 1, \ldots, m$,*

[19] An example of such a function is $u(x, y) = \sqrt{x} + xy + \sqrt{y}$.

1. $\omega_i \in \text{int } \ell_{\infty+}$;

2. $U_i : \ell_{\infty+} \to \mathbb{R}$ *is monotone, strictly concave, and Mackey continuous, where* $U_i(0) = 0$;

3. $U_i$ *is twice norm continuously Gateaux differentiable on* $\text{int } \ell_{\infty+}$;

4. $DU_i(x) \in \ell_{1++}$ *for each* $x \in \text{int } \ell_{\infty+}$;

5. $D^2U_i(x)$ *is negative definite for each* $x \in \text{int } \ell_{\infty+}$;

6. $U_i$ *satisfies the weak survival condition, and there exists some consumer* $j$ *such that* $U_j$ *satisfies the survival condition.*

In a smooth exchange economy with Inada conditions, all of the strong predictions concerning competitive equilibria in smooth economies established in sections 2 and 3 carry over. For example, by the same arguments used to prove Theorems 2.7 and 2.8, Lipschitz economies in this setting have generically determinate equilibria.

**Theorem 4.4.** *Let* $\mathcal{E}_\omega$ *be a smooth exchange economy with Inada conditions which is also a Lipschitz economy. If* $\mathcal{E}_\omega$ *is a regular economy, then* $\mathcal{E}_\omega$ *has an odd number of equilibria which are upper Lipschitzian in* $\omega^1$. *Moreover, the set of critical economies has measure 0 in* $\mathcal{W}$.

Furthermore, the main result of section 3 remains valid, showing that uniform concavity implies that the economy is a Lipschitz economy, and this result can be established by virtually the same argument. The only difference in the argument comes in defining utility functions in the truncated $T$-good economies. Since each consumer's utility function is Gateaux differentiable on the interior of $\ell_{\infty+}$, define $U_i^T : \mathbb{R}_+^T \to \mathbb{R}$ by

$$U_i^T(y) = U_i(y_1, \ldots, y_T, \omega_{i_{T+1}}, \omega_{i_{T+2}}, \ldots).$$

Then for each $T$, $U_i^T$ is $C^2$ on $\mathbb{R}_{++}^T$, with $DU_i^T(y) \gg 0$ and $D^2U_i^T(y)$ negative definite for each $y \in \mathbb{R}_{++}^T$, since $(y, \omega_{i_{T+1}}, \omega_{i_{T+2}}, \ldots) \in \text{int } \ell_{\infty+}$. With this definition for utility in each $T$-good economy, the analysis of section 3 carries over immediately to these economies.

**Theorem 4.5.** *Let* $\mathcal{E}_\omega$ *be a smooth exchange economy with Inada conditions. If each consumer's utility function is uniformly concave on* $\text{int } \ell_{\infty+}$, *then the economy* $\mathcal{E}_\omega$ *is a Lipschitz economy.*

Finally, myopically concave and dominant diagonal preferences generate classes of economies in which these Inada conditions are consistent with generic determinacy. For example, Theorem 3.6 remains valid in this setting allowing for Inada conditions, and the model of habit formation discussed in section 3 can be modified by adding the simple boundary conditions discussed above to yield an economy satisfying Inada conditions and having generically determinate equilibria. As in section 3, consider habit formation preferences of the form $U(x) = v(x_0) + \sum_{t=1}^{\infty} \beta^t u(x_{t-1}, x_t)$. Suppose here that $0 < \beta < 1$, $u, v$ are $C^2$, $v'(c) > 0, v''(c) < 0$ for every $c \in \mathbb{R}_{++}$, that $Du(c_1, c_2) \gg 0$ and that $D^2u(c_1, c_2)$ is negative definite and has a strongly dominant diagonal for every $(c_1, c_2) \in \mathbb{R}^2_{++}$. Assume in addition that the functions $u_i$ satisfy the boundary condition described above. This assumption means that the individually rational Pareto optimal allocations lie in some interval of the form $[\underline{x}, \bar{x}]$, where $\underline{x} \leq \bar{x}$ and $\underline{x}, \bar{x} \in \text{int } \ell^m_{\infty+}$ by Theorem 4.3. Since these habit formation preferences are myopically concave on any interval of this form, equilibria are generically determinate in this economy.

# 5 Determinacy in Production Economies

Thus far the paper has focused exclusively on determinacy in exchange economies, but many of these results carry over in a straightforward manner to economies with production. This section briefly discusses one such extension. To simplify the discussion, let $Y$ denote the aggregate production set in the economy; the goal of this section is to give restrictions on the aggregate production set and on preferences and endowments under which the corresponding production economy has generically determinate equilibria. As in Mas-Colell (1986b), the production set will be described by a transformation function and a pre-technology set $Z \subset X$ which is a weakly closed, convex sublattice of $X$ containing the origin and satisfying free disposal, so that $Z - X_+ \subset Z$.[20] The following definition collects the basic assumptions regarding the production set $Y$ that will be maintained throughout this section.

**Definition 5.1.** *Let the commodity-price pairing be a symmetric Riesz dual system $\langle X, X' \rangle$. The production set $Y$ is a* **smooth production set** *if*

1. *$Y$ is weakly closed and convex*
2. *$Y \cap X_+ = \{0\}$*
3. *$Y = \{y \in Z : f(y) \leq 0\}$ and $\partial Y = \{y \in Z : f(y) = 0\}$ for a transformation function $f : X \to \mathbb{R}$ satisfying*
   a. *$f$ is strictly convex, strictly monotone, and $\tau$ continuous for some compatible locally convex topology $\tau$, and $f(0) = 0$*
   b. *$f$ is twice continuously Gateaux differentiable on $Y$*
   c. *$Df(y) \in X'_{++}$ for all $y \in \partial Y$*
   d. *$D^2f(y)$ is positive definite for all $y \in \partial Y$.*

If the production set $Y$ is a smooth production set, then at each point on the efficiency frontier $\partial Y$ there is a well-defined vector of marginal rates of transformation

[20] As Mas-Colell (1986b) notes, the use of such a pre- technology set to describe the production set allows us to extend the analysis of production economies beyond the case in which the production set has a nonempty interior, since if $Y = \{y \in X : f(y) \leq 0\}$ for some $\tau$ continuous transformation function $f$ and $f(y) < 0$ for some $y$, then $Y$ will have nonempty $\tau$ interior.

given by $Df(y)$. This assumption plays the same role played by the assumption of Gateaux differentiability of utility functions in exchange economies, substituting for uniform properness of the production set and guaranteeing the existence of prices supporting each Pareto optimal allocation. Thus if consumers' preferences satisfy the assumptions of a smooth exchange economy and the feasible production set is weakly compact, equilibria can be characterized using the excess savings equations as in the case of pure exchange economies. These assumptions are captured by the following definition.

**Definition 5.2.** *An economy is a* **smooth production economy** *if the commodity-price pairing is a symmetric Riesz dual system $\langle X, X' \rangle$ and*

1. *the aggregate production set $Y$ is a smooth production set;*
2. *$\hat{Y} \equiv (Y + \{\omega\}) \cap X_+$ is weakly compact;*
3. *$\omega_i \in X_{++}$ for $i = 1, \ldots, m$, and consumer $i$ receives share $\theta_i$ in profits from aggregate production, where $\theta_i > 0$ and $\sum_{i=1}^{m} \theta_i = 1$;*
4. *for each $i = 1, \ldots, m$,*
   a. *$U_i : X_+ \to \mathbb{R}$ is strictly monotone, strictly concave, and $\tau$ continuous for some compatible locally convex topology $\tau$, with $U_i(0) = 0$;*
   b. *$U_i$ is twice continuously Gateaux differentiable on $X_+$;*
   c. *$DU_i(x) \in X'_{++}$ for each $x \in X_+$;*
   d. *$D^2U_i(x)$ is negative definite for each $x \in X_+$.*

In a smooth production economy, we can define the social planner's problem characterizing Pareto optimal allocations just as in a smooth exchange economy, so given $\lambda \in \Delta$, let

$$\begin{aligned}(y(\lambda), x(\lambda)) &= \arg\max \sum_{i=1}^{m} \lambda_i U_i(x_i) \\ &\quad \text{s.t.} \sum_{i=1}^{m} x_i \in \hat{Y}, \quad x_i \in X_+,\ i = 1, \ldots, m.\end{aligned}$$

Then if we define

$$p(\lambda) = \left( \bigvee_{i=1}^{m} \lambda_i DU_i(x_i(\lambda)) \right) \bigvee \gamma(\lambda) Df(y(\lambda)),$$

where $\gamma(\lambda)$ is the Lagrange multiplier on the feasibility constraint in the social planner's problem, $p(\lambda)$ will be a price supporting the Pareto optimal allocation $(x(\lambda), y(\lambda))$, as shown by Mas-Colell (1986b, Lemma p. 327 and Theorem 2). Thus equilibria in a smooth production economy can be characterized as the welfare weights at which the excess savings equations are satisfied, that is, as the solutions to the equations

$$\begin{aligned} p(\lambda) \cdot (x_2(\lambda) - \theta_2 y(\lambda) - \omega_2) &= 0 \\ &\vdots \\ p(\lambda) \cdot (x_m(\lambda) - \theta_m y(\lambda) - \omega_m) &= 0. \end{aligned}$$

If prices are normalized so that the value of aggregate production is always 1, then these equations can be rewritten

$$s(\lambda) \equiv \begin{pmatrix} \tilde{p}(\lambda) \cdot (x_2(\lambda) - \omega_2) \\ \vdots \\ \tilde{p}(\lambda) \cdot (x_m(\lambda) - \omega_m) \end{pmatrix} = \begin{pmatrix} \theta_2 \\ \vdots \\ \theta_m \end{pmatrix},$$

where $\tilde{p}(\lambda) = \frac{p(\lambda)}{p(\lambda) \cdot y(\lambda)}$. As in the exchange case, this characterization of equilibria suggests a natural parameterization of economies using the shareholdings of consumers, indexed over the set $\Theta = \{\theta \in \mathbb{R}^{m-1}_{++} : \sum_{i=2}^{m} \theta_i < 1\}$.[21]

Using this formulation of the excess savings equations, the notion of a regular economy can be extended to production economies in the natural way. A production economy will be called a **regular economy** if $\theta$ is a regular value of $s(\lambda)$. If the excess savings equations are Lipschitz continuous, then we can draw strong conclusions about determinacy analogous to the results developed in section 2. For example, the following result is a straightforward extension of Theorems 2.6, 2.7, and 2.8; the proof is left to the reader.

**Theorem 5.1.** *Let $\mathcal{E}_\theta$ be a smooth production economy which is also a Lipschitz economy. If $\mathcal{E}_\theta$ is a regular economy, then it has finitely many equilibria, each of which is upper Lipschitzian in $\theta$. Moreover, the set of regular economies has full measure in $\Theta$.*

When the commodity space is a normed space, showing that the social planner's problem has a Lipschitz solution is sufficient to guarantee that the economy is a Lipschitz economy and thus has generically determinate equilibria. In the case of sequence economies, the social planner's problem can be studied by considering truncated versions of the feasible production set, just as in exchange economies the planner's problem can be studied by considering truncated versions of the social endowment vector defining the feasible set. More precisely, if the commodity-price duality is $\langle \ell_p, \ell_q \rangle$ where $\frac{1}{p} + \frac{1}{q} = 1$, then for each $T$ we can define the truncated feasible set

$$\hat{Y}^T \doteq \{y^T \in \ell_{p+} : y \in \hat{Y}\}.$$

[21] As in the exchange case, sequence economies can be alternatively parameterized using initial endowments.

The solution to the social planner's problem given this truncated feasible set is then

$$(y_T(\lambda), x_T(\lambda)) \equiv \arg\max \sum_{i=1}^{m} \lambda_i U_i(x_i)$$
$$\text{s.t.} \sum_{i=1}^{m} x_i \in \hat{Y}^T, \quad x_i \in \ell_{p+}, \; i = 1, \ldots, m.$$

Consumers' myopia in a smooth production economy implies that as $T$ becomes large, the solution to this truncated problem approximates the solution to the original planner's problem, as in the case of pure exchange economies. Lipschitz properties of the planner's problem in these finite-dimensional approximations will then carry over to the infinite-dimensional economy as long as the Lipschitz bounds on the finite-dimensional approximations are uniform. Uniform convexity[22] of the transformation function describing the aggregate production possibility frontier and uniform concavity of consumers' utility functions will suffice to ensure that these Lipschitz constants are bounded, and hence that the resulting production economy is a Lipschitz economy.

**Theorem 5.2.** *Let $\mathcal{E}_\theta$ be a smooth production economy in which the commodity-price duality is $\langle \ell_p, \ell_q \rangle$ where $\frac{1}{p} + \frac{1}{q} = 1$. If each consumer's utility function is uniformly concave and the transformation function $f$ is uniformly convex, then the economy $\mathcal{E}_\theta$ is a Lipschitz economy.*

*Proof:* The proof is similar to the proofs of Lemmas 3.1 and 3.2 and Theorems 3.3 and 3.1. The details are contained in the appendix. ■

As in exchange economies, when the commodity space is $\ell_2$ these results can be simplified significantly along the lines of Theorem 3.5. A smooth production economy with commodity space $\ell_2$ will be a Lipschitz economy as long as the transformation function describing the aggregate production set is strongly convex and each consumer's utility function is strongly concave. Similarly, the results on myopic concavity and dominant diagonal conditions can be extended to characterize classes of uniformly convex or myopically convex transformation functions.

As a last example of such an economy, consider transformation functions of the form $f(y) = \sum_{t=1}^{\infty} \beta_t f_t(y_1, \ldots, y_t)$, where $f_t : \mathbb{R}^t \to \mathbb{R}$ and $\{\beta_t\} \in \ell_{1++}$. Suppose that for each $t$, $f_t$ is $C^2$, differentiably convex and differentiably monotone, and that $f_t$ has a dominant diagonal for each $t$. As in section 3, such transformation functions are Mackey continuous on $\ell_\infty$, and twice continuously Gateaux differentiable, where $Df(y) \in \ell_{1++}$ and $D^2 f(y)$ is positive definite for each $y$. Moreover, $f$ is myopically convex. On the production side, the only assumption which might not be satisfied by such a technology is the assumption that the feasible production set $\hat{Y}$ is weak* compact. Since $\hat{Y}$ is weak* closed, it will be weak* compact if and only if it is norm bounded, thus $\hat{Y}$ is weak* compact if and only if $\{\bar{y}_T\}$ is bounded, where $\bar{y}_T$ solves the equation $f(-\omega_{-T}; \bar{y}_t) = 0$. Even if $\hat{Y}$ is not bounded, however, suitable restrictions on preferences will yield a weak* compact set of Pareto optimal allocations, which

[22] A function $f$ is uniformly convex if $-f$ is uniformly concave.

is sufficient to carry out the analysis of the paper.[23] Finally, although the results in this section are in the setting of an economy with a single producer to simplify the notation and discussion, the same ideas could be used to extend these results to allow for an arbitrary finite number of producers.

# 6 Conclusions

This paper has developed two distinct but complementary ideas. First, the boundary case in which the equilibrium equations are not smooth, rather than being the exception as in the classical Arrow-Debreu model with finitely many commodities, is the rule in economies with infinitely many commodities. The techniques pioneered by Debreu (1970) are thus inapplicable for studying determinacy in most economies with infinitely many commodities. Despite this important difference between the Arrow-Debreu model with finitely many commodities and its extension to a setting with infinitely many commodities, the qualitative features of regular economies with finitely many commodities carry over to infinite-dimensional economies as long as the economy is a Lipschitz economy. There will typically be many different methods for determining whether a particular economy is a Lipschitz economy, which motivates the separation of these ideas from the rest of the results of the paper.

The second theme of the paper, however, is that in many models with infinitely many commodities there is a natural and intuitive method for verifying that an economy is a Lipschitz economy based on the preferences of consumers and the technologies of firms, by thinking of the economy as the limit of economies with a large but finite number of commodities as in Bewley's seminal work. The stronger conclusion sought here, that equilibria are generically determinate rather than simply that equilibria exist, requires stronger assumptions than does Bewley's work, in particular on the structure of the commodity space. Although a number of important economic models have a countable number of commodities, others, such as asset trading in continuous time or markets with differentiated commodities, may require a richer specification of commodities. For these economies, the framework for studying determinacy developed here is suggestive but incomplete. More intricate approximation arguments or some Lipschitz version implicit function theorem, as in Shannon (1994b), will then be required in these economies to establish a general class of preferences and technologies which prohibit robust indeterminacies in the set of competitive equilibria.

[23]See Shannon (1996) for one such set of conditions, and for a further discussion of this point.

# 7 Appendix

The appendix presents the proofs of several results in the text, as well as several additional lemmas needed in the proofs of Theorem 3.3 and Corollary 3.1. These lemmas and the proofs of Theorem 3.3 and Corollary 3.1 are given first.

**Lemma A1.** *Consider a smooth exchange economy in which $X$ is a normed space and $X'$ is a Banach lattice. If $x(\lambda)$ is locally Lipschitz continuous, then $s(\lambda)$ is locally Lipschitz continuous, so the economy is a Lipschitz economy.*

*Proof:* First note that since $DU_i(x)$ is continuously Gateaux differentiable on $X_+$, $DU_i(x)$ is locally Lipschitz on $X_+$. Thus $DU_i(x_i(\lambda))$ is locally Lipschitz on $\Delta^o$ if $x(\lambda)$ is locally Lipschitz. Similarly, $p(\lambda) = \bigvee_{i=1}^m(\lambda_i DU_i(x_i(\lambda)))$ is locally Lipschitz on $\Delta^o$. To see this, let $p^i(\lambda) \equiv \lambda_i DU_i(x_i(\lambda))$. By the above argument, $p^i(\lambda)$ is Lipschitz for each $i$, and $p(\lambda) = p^1(\lambda) \vee \cdots \vee p^m(\lambda)$. Without loss of generality, assume $m = 2$, and let $\lambda \in \Delta^o$. Choose a neighborhood $W$ of $\lambda$ such that $\overline{W} \subset \Delta^o$, and let $\lambda' \in W$. Since $X$ is an Archimedean Riesz space,

$$|p(\lambda) - p(\lambda')| = |p^1(\lambda) \vee p^2(\lambda) - p^1(\lambda') \vee p^2(\lambda')| \leq |p^1(\lambda) - p^1(\lambda')| \vee |p^2(\lambda) - p^2(\lambda')|.$$

Then note that $|p^1(\lambda) - p^1(\lambda')| \geq 0$ and $|p^2(\lambda) - p^2(\lambda')| \geq 0$, so

$$0 \leq |p^1(\lambda) - p^1(\lambda')| \vee |p^2(\lambda) - p^2(\lambda')| \leq |p^1(\lambda) - p^1(\lambda')| + |p^2(\lambda) - p^2(\lambda')|.$$

Putting these together implies that

$$|p(\lambda)-p(\lambda')| \leq |p^1(\lambda)-p^1(\lambda')|+|p^2(\lambda)-p^2(\lambda')| = \Big||p^1(\lambda - p^1(\lambda')| + |p^2(\lambda) - p^2(\lambda')|\Big|.$$

Since $X'$ is a Banach lattice,

$$\begin{aligned} \|p(\lambda) - p(\lambda')\| &\leq \Big\||p^1(\lambda - p^1(\lambda')| + |p^2(\lambda) - p^2(\lambda')|\Big\| \\ &\leq \Big\||p^1(\lambda) - p^1(\lambda')|\Big\| + \Big\||p^2(\lambda) - p^2(\lambda')|\Big\| \\ &= \|p^1(\lambda) - p^1(\lambda')\| + \|p^2(\lambda) - p^2(\lambda')\|, \end{aligned}$$

where the last equality again follows because $X'$ is a Banach lattice. But then since $p^1$ and $p^2$ are locally Lipschitz continuous, there exists some constant $k > 0$ such that $\|p(\lambda) - p(\lambda')\| \leq k\|\lambda - \lambda'\|$. Moreover, $v(\lambda) \equiv p(\lambda) \cdot \omega$ is also locally Lipschitz on $\Delta^o$.

Now let $\lambda \in \Delta^o$, and choose a neighborhood $V$ of $\lambda$ such that $\overline{V} \subset \Delta^o$, and on which $v(\cdot)$ and $p(\cdot)$ are Lipschitz with constants $K_1$ and $K_2$. Then there exists $C > 0$ such that $|v(\lambda)| \leq C$ and $\|p(\lambda)\| \leq C$ for every $\lambda \in \overline{V}$, and hence for every $\lambda \in V$. Moreover, $v(\lambda) > 0$ for every $\lambda \in \Delta$, so there exists $c > 0$ such that $v(\lambda) \geq c$ for every $\lambda \in \overline{V}$, and hence for every $\lambda \in V$. Then if $\lambda^1, \lambda^2 \in V$,

$$\begin{aligned} \left\|\frac{p(\lambda^1)}{v(\lambda^1)} - \frac{p(\lambda^2)}{v(\lambda^2)}\right\| &= \left\|\frac{p(\lambda^1)}{v(\lambda^1)} - \frac{p(\lambda^2)}{v(\lambda^1)} + \frac{p(\lambda^2)}{v(\lambda^1)} - \frac{p(\lambda^2)}{v(\lambda^2)}\right\| \\ &\leq \frac{1}{v(\lambda^1)}\|p(\lambda^1) - p(\lambda^2)\| + \|p(\lambda^2)\|\left\|\frac{1}{v(\lambda^1)} - \frac{1}{v(\lambda^2)}\right\| \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{v(\lambda^1)}\|p(\lambda^1) - p(\lambda^2)\| + \|p(\lambda^2)\|\left\|\frac{v(\lambda^2) - v(\lambda^1)}{v(\lambda^1)v(\lambda^2)}\right\| \\
&\leq \frac{1}{c}K_2\|\lambda^1 - \lambda^2\| + \frac{C}{c^2}\|v(\lambda^2) - v(\lambda^1)\| \\
&\leq K\|\lambda^1 - \lambda^2\|,
\end{aligned}
$$

where $K = \frac{1}{c}K_2 + \frac{C}{c^2}K_1$. So $\tilde{p}(\lambda) = \frac{p(\lambda)}{v(\lambda)}$ is locally Lipschitz on $\Delta$.

Now it suffices to show that since $x_j(\lambda)$ is locally Lipschitz on $\Delta^o$, then $\tilde{p}(\lambda)\cdot x_j(\lambda)$ is locally Lipschitz on $\Delta^o$ as well. Let $\lambda \in \Delta^o$ and choose a neighborhood $U$ of $\lambda$ as before such that $\overline{U} \subset \Delta^o$, and on which $\tilde{p}(\lambda)$ and $x_j(\lambda)$ are Lipschitz with constants $L_p$ and $L_x$. There exists $M > 0$ such that $\|\tilde{p}(\lambda)\| \leq M$ and $\|x_j(\lambda)\| \leq M$ for every $\lambda$ in $\overline{U}$ and hence for every $\lambda$ in $U$. Let $\lambda^1, \lambda^2 \in U$.

$$
\begin{aligned}
\|\tilde{p}(\lambda^1)\cdot x_j(\lambda^1) - \tilde{p}(\lambda^2)\cdot x_j(\lambda^2)\| &\leq \|\tilde{p}(\lambda^1)\cdot x_j(\lambda^1) - \tilde{p}(\lambda^1)\cdot x_j(\lambda^2)\| \\
&\quad + \|\tilde{p}(\lambda^1)\cdot x_j(\lambda^2) - \tilde{p}(\lambda^2)\cdot x_j(\lambda^2)\| \\
&\leq \|\tilde{p}(\lambda^1)\|\|x_j(\lambda^1) - x_j(\lambda^2)\| + \|x_j(\lambda^2)\|\|\tilde{p}(\lambda^1) - \tilde{p}(\lambda^2)\| \\
&\leq ML_x\|\lambda^1 - \lambda^2\| + ML_p\|\lambda^1 - \lambda^2\| \\
&= L\|\lambda^1 - \lambda^2\|.
\end{aligned}
$$

Thus $s(\lambda)$ is locally Lipschitz on $\Delta^o$. ■

**Lemma A2.** *Let $x \in \ell_{p+}$ where $1 \leq p < \infty$. Then the order interval $[0, x]$ is norm compact.*

*Proof:* This follows from the fact that the interval $[0, x]$ is norm totally bounded, and from Theorem 9.1 in Aliprantis and Burkinshaw (1985) and Theorem 1 in DeVito (1990, p. 110). ■

The proofs of Theorem 3.3 and Corollary 3.1 make fundamental use of the notion of uniform negative definiteness introduced in the paper and the properties of the semi-inner product $(\cdot,\cdot)_+$. For these arguments, the most important properties of this semi-inner product are contained in the following result.

**Lemma A3 (Deimling (1985, Proposition 13.1)).** *Let $X$ be a Banach space. Then for all $x, y, z \in X$, $(x + y, z)_+ \leq (x, z)_+ + (y, z)_+$ and $|(x, z)_+| \leq \|x\|\|z\|$. If $\alpha, \beta \geq 0$, then $(\alpha x, \beta z)_+ = \alpha\beta(x, z)_+$.*

The main differences between this semi-inner product and a real inner product are that, with the exception of the case when $X$ is a Hilbert space such as $\ell_2$, the semi-inner product is not necessarily linear or symmetric. The lack of symmetry leads to an alternative definition of "negative definite" to the one given in the paper. Let $F(x) : \ell_p \rightarrow \ell_p$ be a linear operator for each $x$. Then $F(x)$ is **uniformly dissapative** if there exists $c > 0$ such that

$$
(F^T(x)z, z)_+ \leq -c\|z\|_p^2
$$

for all $T > 0$, for all $z \in \mathbb{R}^T$, and for each $x$. and $F(x)$ is **finitely dissapative** if

$$(F^T(x)z, z)_+ < 0$$

for all $T > 0$, for each $x$, and for all $z \in \mathbb{R}^T \setminus \{0\}$. In particular, note that if $F(x)$ is finitely negative definite, then $F^T(x)^{-1}$ exists for each $T$, and $F^T(x)^{-1}$ is finitely dissapative, since if $y \neq 0$, then $(F^T(x)^{-1}y, y)_+ = (z, F^T(x)z)_+ < 0$, where $z = F^T(x)^{-1}y \neq 0$.

**Lemma A4.** *For every $x$, let $S(x) : \ell_p \to \ell_p$ be a linear operator, for some $p$ such that $1 \leq p \leq \infty$. If $S(x)$ is uniformly negative definite (uniformly dissapative), then $[S^T(x)]^{-1}$ is uniformly bounded, i.e., there exists $M > 0$ such that $\|S^T(x)^{-1}\| \leq M$ for all $T$ and for all $x$. Moreover, if in addition $S(x)$ is uniformly bounded, then $S^T(x)^{-1}$ is uniformly dissapative (uniformly negative definite).*

*Proof:* I will prove the lemma for the case in which $S(x)$ is uniformly negative definite; the uniformly dissapative case is analogous. First note that $S^T(x)$ is uniformly bounded below. To see this, let $T$ be given and choose $z \in \mathbb{R}^T$ such that $\|z\|_p = 1$. Let $S^T(x)z = y$. We must show that $\|y\|_p$ is uniformly bounded away from 0. By the assumption that $S$ is uniformly negative definite, there exists $c > 0$, independent of $T$, such that

$$|(z, S^T(x)z)_+| = |(z, y)_+| \geq c.$$

This implies that

$$c \leq |(z, y)_+| \leq \|z\|_p \|y\|_p = \|y\|_p.$$

So $\|y\|_p$ is uniformly bounded away from 0.

Now suppose $w \in \mathbb{R}^T$ and $\|w\|_p = 1$. Since $S^T(x)$ is uniformly negative definite, $S^T(x)^{-1}$ exists, so let $q$ satisfy $S^T(x)^{-1}w = q$. Thus $w = S^T(x)q$, and

$$1 = \|w\|_p = \|S^T(x)q\|_p.$$

But this implies

$$1 = \|S^T(x)q\|_p \geq c\|q\|_p$$

by the previous argument. So $\|q\|_p \leq \frac{1}{c}$, i.e., $\|S^T(x)^{-1}\| \leq \frac{1}{c}$, and this bound is independent of $T$ and $x$.

To see that $S^T(x)^{-1}$ is uniformly dissapative if $S(x)$ is uniformly bounded, let $T$ be arbitrary and let $\|y\|_p = 1$. Then $(S^T(x)^{-1}y, y)_+ = (z, S^T(x)z)_+$, where $z = S^T(x)^{-1}y$. Thus $y = S^T(x)z$, so

$$1 = \|y\|_p = \|S^T(x)z\|_p \leq \|S^T(x)\| \|z\|_p,$$

which implies that $\|z\|_p \geq \frac{1}{\|S^T(x)\|}$. So

$$|(S^T(x)^{-1}y, y)_+| = |(z, S^T(x)z)_+| \geq c\|z\|_p^2 \geq d$$

for some $d > 0$ since $S^T(x)$ is uniformly bounded. Thus $S^T(x)^{-1}$ is uniformly dissapative. ∎

**Lemma A5.** *For every $x$, let $S_1(x), S_2(x) : \ell_p \to \ell_p$ be linear operators. If $S_1(x)$ is uniformly dissapative and $S_2(x)$ is finitely dissapative, then $S_1(x)+S_2(x)$ is uniformly dissapative.*

*Proof:* Let $T > 0$ be given and let $z \in \mathbb{R}^T$. Then by Lemma A3.

$$((S_1^T(x) + S_2^T(x))z, z)_+ \leq (S_1^T(x)z, z)_+ + (S_2^T(x)z, z)_+,$$

and since $S_1(x)$ is uniformly dissapative and $S_2(x)$ is finitely dissapative, there exists $c > 0$ such that

$$((S_1^T(x) + S_2^T(x))z, z)_+ \leq (S_1^T(x)z, z)_+ \leq -c\|z\|_p^2.$$

Thus $S_1(x) + S_2(x)$ is uniformly dissapative. ■.

**Proof of Theorem 3.3:** By Lemma A.1, it suffices to show that $x(\lambda)$ is locally Lipschitz continuous. Then by the previous lemmas, it suffices to show that $\{c_T\}$ is bounded, where $c_T$ is the Lipschitz constant on $x^T(\lambda, \omega^T)$. Using the notation of the previous proof, since $x^T(\lambda, \omega^T) \in \bigcup_{L\in S(\mathbb{R}^{mT})} x_L^T(\lambda, \omega^T)$, $c_T = \max_{L\in S(\mathbb{R}^{mT})} c_{LT}$, where $c_{LT}$ is the Lipschitz constant of $x_L^T(\lambda, \omega^T)$. By the Implicit Function Theorem, each function $x_L^T(\lambda, \omega^T)$ is $C^1$, and

$$\begin{aligned} D_{\lambda_i} x_{Li}^T &= \left[\lambda_i D^2 U_i^T(x_i)\right]^{-1} DU_i^T(x_i) \\ &\quad - \left[\lambda_i D^2 U_i^T(x_i)\right]^{-1} \left[\sum_{j=1}^m \left[\lambda_j D^2 U_j^T(x_j)\right]^{-1}\right]^{-1} \left[\lambda_i D^2 U_i^T(x_i)\right]^{-1} (DU_i^T(x_i) + DU_1^T(x_1)), \end{aligned}$$

and for $j \neq i$,

$$D_{\lambda_j} x_{Li}^T = \left[\lambda_i D^2 U_i^T(x_i)\right]^{-1} \left[\sum_{k=1}^m \left[\lambda_k D^2 U_k^T(x_k)\right]^{-1}\right]^{-1} \left[\lambda_j D^2 U_j^T(x_j)\right]^{-1} (DU_j^T(x_j) + DU_1^T(x_1)).$$

Let $\lambda \in \Delta^o$ and choose a neighborhood $W$ of $\lambda$ such that $\overline{W} \subset \Delta^o$. Since $D_{\lambda_j} x_L^T(\lambda, \omega^T)$ is bounded on $\overline{W}$ for each $j$, $x_L^T(\lambda, \omega^T)$ is Lipschitz in $\lambda$ on $W$ with constant $c_{LT}$ given by $c_{LT} = \sup_{j,\lambda\in\overline{W}} D_{\lambda_j} x_L^T(\lambda, \omega^T)$. Since $x_L(\lambda, \omega^T)$ solves the social planner's problem, $DU_1^T(x_1) = \frac{\lambda_i}{\lambda_1} DU_i^T(x_i)$ for each $i$. To ease the notational burden, for the rest of the argument I will suppress the arguments in these derivatives. Then on $W$, $x^T(\lambda, \omega^T)$ is Lipschitz with constant $c_T$ given by the maximum of

$$\sup_{i,j,x,\lambda\in\overline{W}} \left\| \left[\lambda_i D^2 U_i^T\right]^{-1} DU_i^T - \left[\lambda_i D^2 U_i^T\right]^{-1} \left[\sum_{j=1}^m \left[\lambda_j D^2 U_j^T\right]^{-1}\right]^{-1} \frac{\lambda_1 + \lambda_i}{\lambda_1} \left[\lambda_i D^2 U_i^T\right]^{-1} DU_i^T \right\|.$$

and

$$\sup_{i,j,x,\lambda\in\overline{W}} \left\| \left[\lambda_i D^2 U_i^T\right]^{-1} \left[\sum_{k=1}^m \left[\lambda_k D^2 U_k^T\right]^{-1}\right]^{-1} \frac{\lambda_1 + \lambda_j}{\lambda_1} \left[\lambda_j D^2 U_j^T\right]^{-1} DU_j^T \right\|.$$

On $\overline{W}$, $(\lambda_i)^{-1}$ and $\frac{\lambda_1+\lambda_i}{\lambda_1}$ are bounded for each $i$, so by this calculation, it suffices to show that

$$\sup_{j,T,x}\left\|\left[D^2U_j^T\right]^{-1}DU_j^T\right\|<\infty \text{ and } \sup_{j,T,x}\left\|\left[\lambda_j D^2U_j^T\right]^{-1}\left[\sum_{i=1}^m\left[\lambda_i D^2U_i^T\right]^{-1}\right]^{-1}\right\|<\infty.$$

By uniform concavity there exists $k>0$ such that $\|(D^2U_j^T)^{-1}DU_j^T\|\leq k$ for all $j,T$ and $x$. Now consider $[\lambda_j D^2U_j^T]^{-1}\left[\sum_{i=1}^m[\lambda_i D^2U_i^T]^{-1}\right]^{-1}$. By assumption we can rewrite this expression as

$$\lambda_j^{-1}(S_j^T)^{-1}(B_j^T)^{-1}\left[\sum_{i=1}^m\lambda_i^{-1}(S_i^T)^{-1}(B_i^T)^{-1}\right]^{-1} \qquad (*)$$

where $B_i^T\equiv\text{diag}\{\frac{\partial U_i}{\partial x_t}(x_i)\}$. By definition, $(B_i^T)^{-1}B_j^T=\text{diag}\ \{\frac{\lambda_i}{\lambda_j},\ldots,\frac{\lambda_i}{\lambda_j}\}$ for each $i,j$, and $T$, so $(*)$ can be equivalently written as

$$\lambda_j^{-1}(S_j^T)^{-1}\left[\sum_{i=1}^m\lambda_j^{-1}(S_i^T)^{-1}\right]^{-1}.$$

Since $S_i^T$ is uniformly bounded and uniformly negative definite for each $i$, $(S_i^T)^{-1}$ is uniformly dissapative, thus $\lambda_j^{-1}\sum_{i=1}^m(S_i^T)^{-1}$ is uniformly dissapative on $\overline{W}$. So by Lemma A.4,

$$\left[\lambda_j^{-1}\sum_{i=1}^m(S_i^T)^{-1}\right]^{-1}$$

is uniformly bounded on $\overline{W}$. Since $\lambda_jS_j^T(x_j)$ is uniformly negative definite, $\lambda_j^{-1}(S_j^T)^{-1}$ is also uniformly bounded on $\overline{W}$ by Lemma A4, so $(*)$ is uniformly bounded on $W$. Thus the sequence of Lipschitz constants $\{c_T\}$ is bounded. ■

**Proof of Corollary 3.1:** Let $\lambda\in\Delta^o$ be given, and let $W$ be a neighborhood of $\lambda$ such that $\overline{W}\subset\Delta^o$. To ease the notational burden, I will suppress the dependence on $x$ in the proof. By assumption, $\left[\lambda_jD^2U_j^T\right]^{-1}DU_j^T=\lambda_j^{-1}(S_j^T)^{-1}(B_j^T)^{-1}DU_j^T$, where $(B_j^T)^{-1}DU_j^T$ is uniformly bounded by the assumption of myopic concavity, and $S_j$ is uniformly negative definite, which implies that $(S_j^T)^{-1}$ is uniformly bounded by Lemma A.4. So there exists $k>0$ such that for all $j,x$ and $T$,

$$\left\|\left[\lambda_jD^2U_j^T\right]^{-1}DU_j^T\right\|\leq k.$$

Similarly, consider $[\lambda_jD^2U_j^T]^{-1}\left[\sum_{i=1}^m[\lambda_iD^2U_i^T]^{-1}\right]^{-1}$. By assumption we can rewrite this expression as

$$\lambda_j^{-1}(S_j^T)^{-1}\left[\lambda_j^{-1}(S_j^T)^{-1}+\sum_{i\neq j}\lambda_i^{-1}(S_i^T)^{-1}(B_i^T)^{-1}B_j^T\right]^{-1}.$$

Then since $(B_j^T)^{-1}B_i^T S_i^T$ is finitely negative definite, $(S_i^T)^{-1}(B_i^T)^{-1}B_j^T$ is finitely dissapative, which implies that $\lambda_j^{-1}(S_j^T)^{-1} + \sum_{j\neq i} \lambda_i^{-1}(S_i^T)^{-1}(B_i^T)^{-1}B_j^T$ is uniformly dissapative by Lemma A5. The rest of the argument now follows exactly as the proof of Theorem 3.3. ■

**Proof of Theorem 2.2:** That $U(x)$ is Mackey continuous follows from Bewley (1972, Appendix II), and clearly $U(x)$ is strictly concave and strictly monotone. Then I claim that $U$ is continuously Gateaux differentiable on $\ell_{\infty+}$, and that given $x \in \ell_{\infty+}$, $DU(x) = \{\beta^t u'(x_t)\} \in \ell_{1++}$. To see this, note that by definition it suffices to show that for each $h \in \ell_\infty$,

$$\lim_{r\to 0} \frac{|\sum_{t=0}^{\infty}[\beta^t u(x_t + rh_t) - \beta^t u(x_t) - r\beta^t u'(x_t)h_t]|}{|r|} = 0.$$

But for each $r \neq 0$,

$$\frac{|\sum_{t=0}^{\infty}[\beta^t u(x_t + rh_t) - \beta^t u(x_t) - r\beta^t u'(x_t)h_t]|}{|r|} \leq \sum_{t=0}^{\infty} \beta^t \left|\frac{u(x_t + rh_t) - u(x_t) - ru'(x_t)h_t}{r}\right|.$$

Since $u$ is $C^2$ on $[0, \infty)$, there exists $\eta > 0$ such that on $[-\eta, \|x\| + \eta]$, $u'(z)$ exists and is bounded, so there exists $M > 0$ such that $|u'(z)| \leq M$ for all such $z$. Then given $\epsilon > 0$, choose $T$ such that $\sum_{t=T}^{\infty} \beta^t < \epsilon/2M$. For every $t$,

$$\left|\frac{u(x_t + rh_t) - u(x_t) - ru'(x_t)h_t}{r}\right| \to 0,$$

as $r \to 0$, so for $t = 1, \ldots T$ there exists $\delta_t > 0$ such that for $|r| < \delta_t$,

$$\left|\frac{u(x_t + rh_t) - u(x_t) - ru'(x_t)h_t}{r}\right| < \frac{\epsilon}{1-\beta}.$$

Set $\delta = \min\{\eta, \delta_t : t = 1, \ldots T\} > 0$. For $|r| < \delta$,

$$\begin{aligned}
\frac{\sum_{t=0}^{\infty} |\beta^t u(x_t + rh_t) - \beta^t u(x_t) - r\beta^t u'(x_t)h_t|}{|r|} &\leq \sum_{t=0}^{\infty} \beta^t \left|\frac{u(x_t + rh_t) - u(x_t) - ru'(x_t)h_t}{r}\right| \\
&= \sum_{t=0}^{T-1} \beta^t \left|\frac{u(x_t + rh_t) - u(x_t) - ru'(x_t)h_t}{r}\right| \\
&\quad + \sum_{t=T}^{\infty} \beta^t \left|\frac{u(x_t + rh_t) - u(x_t) - ru'(x_t)h_t}{r}\right| \\
&\leq \sum_{t=0}^{T-1} \beta^t \left|\frac{u(x_t + rh_t) - u(x_t) - ru'(x_t)h_t}{r}\right| \\
&\quad + \sum_{t=T}^{\infty} \beta^t \left|\frac{u'(z_t)|rh_t| - ru'(x_t)h_t}{r}\right|,
\end{aligned}$$

for some $z_t \in (x_t, x_t + rh_t)$,

$$< \quad \epsilon + 2M \cdot \frac{\epsilon}{2M} = 2\epsilon.$$

So $DU(x) = \{\beta^t u'(x_t)\}$ for every $x \in \ell_{\infty+}$. Since $x \in \ell_{\infty+}$ and $u'$ is continuous and strictly positive, $DU(x) \in \ell_{1++}$.

To show $DU(x)$ is norm continuous, suppose $\{x^n\} \in \ell_{\infty+}$ and $x^n \to x$. We must show that $DU(x^n) \to DU(x)$ in $\ell_1$, i.e., that

$$\lim_n \sum_{t=0}^{\infty} |\beta^t u'(x_t^n) - \beta^t u'(x_t)| = 0.$$

Then let $\epsilon > 0$ be given. Since $u'$ is continuous and $\{x, x^n, n = 1, 2, \ldots\}$ is bounded, there exists $M > 0$ such that $|u'(x_t^n)| \leq M$ for all $t$ and all $n$, and $|u'(x_t)| \leq M$ for all $t$. Choose $T$ such that $\sum_{t=T+1}^{\infty} \beta^t < \epsilon/2M$. For $t = 1, \ldots, T$, there exists $\delta_t > 0$ such that $|z - x_t| < \delta_t \Rightarrow |u'(z) - u'(x_t)| < \epsilon/(1-\beta)$, by the continuity of $u'$. Let $\delta = \min\{\delta_1, \ldots, \delta_T\}$ and choose $N$ such that $n \geq N \Rightarrow \|x^n - x\| < \delta$. Then if $n \geq N$,

$$\begin{aligned}
\sum_{t=0}^{\infty} |\beta^t u'(x_t^n) - \beta^t u'(x_t)| &= \sum_{t=0}^{T} \beta^t |u'(x_t^n) - u'(x_t)| + \sum_{t=T+1}^{\infty} \beta^t |u'(x_t^n) - u'(x_t)| \\
&\leq \frac{\epsilon}{1-\beta} \sum_{t=0}^{T} \beta^t + 2M \sum_{t=T+1}^{\infty} \beta^t \\
&< \epsilon + \epsilon = 2\epsilon.
\end{aligned}$$

Thus $\lim_n \sum_{t=0}^{\infty} |\beta^t u'(x_t^n) - \beta^t u'(x_t)| = 0$.

Now I claim that $D^2U(x) : \ell_\infty \to \ell_1$ is the operator such that for $h \in \ell_\infty$, $D^2U(x)h = \{\beta^t u''(x_t)h_t\}$. To establish this, we must show that for each $h \in \ell_\infty$,

$$\lim_{|r| \to 0} \frac{\sum_{t=0}^{\infty} |\beta^t u'(x_t + rh_t) - \beta^t u'(x_t) - r\beta^t u''(x_t)h_t|}{|r|} = 0.$$

A straightforward variation of the argument given above to show that $DU(x) = \{\beta^t u'(x_t)\}$ will show that this holds. Moreover, $D^2U(\cdot)$ is continuous. To see this, let $\{x^n\} \in \ell_{\infty+}$ be such that $x^n \to x$. We must show that $\|D^2U(x^n) - D^2U(x)\| \to 0$. Let $z \in \ell_\infty$ be such that $\|z\| \leq 1$. Then $(D^2U(x^n) - D^2U(x))z = \{\beta^t(u''(x_t^n) - u''(x_t))z_t\}$, so

$$\begin{aligned}
\|(D^2U(x^n) - D^2U(x))z\| &= \sum_{t=0}^{\infty} \beta^t |(u''(x_t^n) - u''(x_t))z_t| \\
&\leq \|z\| \sum_{t=0}^{\infty} \beta^t |u''(x_t^n) - u''(x_t)|.
\end{aligned}$$

Thus $\|D^2U(x^n) - D^2U(x)\| \leq \sum_{t=0}^{\infty} \beta^t |u''(x_t^n) - u''(x_t)| \to 0$ by the same argument given above to show that $DU(x)$ is continuous.

Finally, we must show that $D^2U(x)$ is negative definite. If $z \in \ell_\infty$ and $z \neq 0$, then $z^T D^2U(x)z = \sum_{t=0}^{\infty} \beta^t u''(x_t)(z_t)^2 < 0$, so $D^2U(x)$ is negative definite. ∎

**Proof of Theorem 2.7:** The theorem will be proven by constructing an appropriate homotopy between the equilibrium equations and a set of equations whose degree is known to be 1, and appealing to homotopy invariance. Let $\bar{\lambda} \in \Delta^o$ be fixed, so $\bar{\lambda}_i > 0$ for $i = 1, \ldots, m$. Define $H(\lambda, t) : \Delta \times [0,1] \to \mathbb{R}^{m-1}$ by

$$H(\lambda, t) = (1-t)(\lambda - \bar{\lambda}) + t(s(\lambda) - \alpha).$$

$H(\cdot)$ is locally Lipschitz, and $H(\lambda, 0) \equiv \lambda - \bar{\lambda}$, so $d(H(\lambda,0), \Delta^o, 0) = 1$. Then by homotopy invariance, to show $d(s(\lambda) - \alpha, \Delta^o, 0) = 1$, it suffices to show that $H(\partial\Delta^o, t) \neq 0$ for every $t \in [0,1]$. Clearly if $t = 0$ or if $t = 1$, $H(\partial\Delta^o, t) \neq 0$, so suppose $t \in (0,1)$ and $H(\lambda, t) = 0$. Then

$$t(s(\lambda) - \alpha) = -(1-t)(\lambda - \bar{\lambda}),$$

or

$$t(\tilde{p}(\lambda) \cdot x_j(\lambda) - \alpha_j) = -(1-t)(\lambda_j - \bar{\lambda}_j), \quad j = 2, \ldots, m. \tag{1}$$

If $\lambda \in \partial\Delta^o$, then $\lambda_i = 0$ for some $i = 1, \ldots, m$, where $\lambda_1 = 1 - \sum_{i=2}^{m} \lambda_i$. If $\lambda_i = 0$ for some $i = 2, \ldots, m$, then $x_i(\lambda) = 0$, and (1) implies

$$0 > -t\alpha_i = t(\tilde{p}(\lambda) \cdot x_i(\lambda) - \alpha_i) = (1-t)\bar{\lambda}_i > 0,$$

which is a contradiction. Suppose $\lambda_1 = 0$. By (1),

$$t \sum_{j=2}^{m} (\tilde{p}(\lambda) \cdot x_j(\lambda) - \alpha_j) = -(1-t) \sum_{j=2}^{m} (\lambda_j - \bar{\lambda}_j),$$

and thus

$$-t(\tilde{p}(\lambda) \cdot x_1(\lambda) - \alpha_1) = -(1-t) \sum_{j=2}^{m} (\lambda_j - \bar{\lambda}_j) = (1-t)(\lambda_1 - \bar{\lambda}_1). \tag{2}$$

Since $\lambda_1 = 0$, $x_1(\lambda) = 0$, thus (2) implies that

$$0 < t\alpha_1 = -(1-t)\bar{\lambda}_1 < 0,$$

which is a contradiction. Hence $H(\partial\Delta^o, t) \neq 0$ for every $t \in [0,1]$. By homotopy invariance, $d(s(\lambda) - \alpha, \Delta^o, 0) = 1$. Since $\alpha$ is a regular value of $s$, by Shannon (1994a, Theorem 10),

$$1 = d(s, \Delta^o, \alpha) = \sum_{\lambda \in s^{-1}(\alpha)} \text{sgn } \det Ds(\lambda).$$

Hence $s^{-1}(\alpha)$ has an odd number of elements, which means that the economy $\mathcal{E}_\omega$ has an odd number of equilibria. ∎

**Proof of Theorem 3.2:** Since there are two consumers, it suffices to show that $x_2(\lambda_1)$ is locally Lipschitz. Given $\omega \in \ell_{\infty+}$, by Theorem 3.1 and the calculations in the proof of Theorem 3.3, it suffices to show that $\left[D^2U_i^T(x)\right]^{-1} DU_i^T(x)$ is uniformly bounded on $[0,\omega]$ for $i = 1, 2$, and that

$$\sup_{x_L(\lambda)} \left\| \left[\lambda_2 D^2 U_2^T\right]^{-1} \left[\left[\lambda_1 D^2 U_1^T\right]^{-1} + \left[\lambda_2 D^2 U_2^T\right]^{-1}\right]^{-1} \left[\lambda_1 D^2 U_1^T\right]^{-1} \lambda_1 DU_1^T \right\| < \infty.$$

First, consider $\left[D^2U_i^T(x)\right]^{-1} DU_i^T(x)$, and suppress the subscript $i$. Using the notation defined in section 3, $D^2U^T(x) = B^T(x)S^T(x)$ for each $T > 0$, where $B^T(x) = \operatorname{diag}\{b_t(x)\}_{t=1}^T$ and $S^T(x) = \operatorname{diag}\{\frac{u''(x_t)}{u'(x_t)}\}_{t=1}^T + \sum_{t=2}^T R_t(x)$, with

$$R_t(x) = \begin{pmatrix} 0 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & \dots & 0 & \frac{g''({}_tx)}{g'({}_tx)} b_1({}_tx) & \dots & \frac{g''({}_tx)}{g'({}_tx)} b_{T+1-t}({}_tx) \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & \frac{g''({}_tx)}{g'({}_tx)} b_1({}_tx) & \dots & \frac{g''({}_tx)}{g'({}_tx)} b_{T+1-t}({}_tx) \end{pmatrix}.$$

Since $DU^T(x) = (b_1(x), \dots, b_T(x))$, it suffices to show that $S^T(x)^{-1}\mathbf{1}$ is uniformly bounded, where $\mathbf{1} = (1, 1, \dots, 1)$. Let $T > 0$ be arbitrary, and let $z \in \mathbb{R}^T$ solve the equation $S^T(x)z = \mathbf{1}$. Thus

$$\begin{aligned} \frac{u''(x_1)}{u'(x_1)} z_1 &= 1 \\ \frac{u''(x_2)}{u'(x_2)} z_2 + a_2 &= 1 \\ \frac{u''(x_3)}{u'(x_3)} z_3 + a_2 + a_3 &= 1 \\ &\vdots \\ \frac{u''(x_{T-1})}{u'(x_{T-1})} z_{T-1} + a_2 + \dots + a_{T-1} &= 1 \\ \left(\frac{u''(x_T)}{u'(x_T)} + \frac{g''({}_Tx)}{g'({}_Tx)}\right) z_T + a_2 + \dots + a_{T-1} &= 1 \end{aligned}$$

where $a_t = \frac{g''({}_tx)}{g'({}_tx)} \sum_{s=1}^{T+1-t} b_s({}_tx) z_{t+s-1}$. Equations $(T)$ and $(T-1)$ imply that

$$\frac{u''(x_{T-1})}{u'(x_{T-1})} z_{T-1} = \left(\frac{u''(x_T)}{u'(x_T)} + \frac{g''({}_Tx)}{g'({}_Tx)}\right) z_T,$$

which implies that $\operatorname{sgn} z_{T-1} = \operatorname{sgn} z_T$. Then by definition

$$a_{T-1} = \frac{g''({}_{T-1}x)}{g'({}_{T-1}x)} (b_1({}_{T-1}x) z_{T-1} + b_2({}_{T-1}x) z_T),$$

so sgn $a_{T-1} = -\text{sgn } z_{T-1} = -\text{sgn } z_T$. Similarly, equations $(T-1)$ and $(T-2)$ imply that

$$\frac{u''(x_{T-2})}{u'(x_{T-2})} z_{T-2} = \frac{u''(x_{T-1})}{u'(x_{T-1})} z_{T-1} + a_{T-1},$$

which implies that sgn $z_{T-2} =$ sgn $z_{T-1} =$ sgn $z_T$. and hence sgn $a_{T-2} =$ sgn $a_{T-1} = -$sgn $z_T$. Repeating this argument and recursing backwards shows that sgn $z_t =$ sgn $z_T$ for all $t \geq 2$. and thus that sgn $a_t = -$sgn $z_T$ for all $t$.

Now suppose $z_t > 0$ for all $t \geq 2$. Then the left-hand side in equation $(t)$ is negative while the right-hand side is equal to 1. a contradiction. Thus $z_t \leq 0$ for all $t$, which implies that $a_t \geq 0$ for all $t$. Then for each $t$, equation $(t)$ implies that

$$|z_t| \leq \left| \frac{u'(x_t)}{u''(x_t)} \right|.$$

This argument was independent of $T$, so $\|z\| = \|S^T(x)^{-1}\mathbf{1}\| \leq \sup_t \left|\frac{u'(x_t)}{u''(x_t)}\right|$. Thus $S^T(x)^{-1}\mathbf{1}$ is uniformly bounded on any interval of the form $[0,\omega]$.

Now consider

$$\left[\lambda_2 D^2 U_2^T\right]^{-1} \left[\left[\lambda_1 D^2 U_1^T\right]^{-1} + \left[\lambda_2 D^2 U_2^T\right]^{-1}\right]^{-1} \left[\lambda_1 D^2 U_1^T\right]^{-1} \lambda_1 D U_1^T. \qquad (*)$$

Then $(*)$ can be rewritten as

$$\left[\lambda_1 D^2 U_1^T + \lambda_2 D^2 U_2^T\right]^{-1} \lambda_1 D U_1^T = \left[\lambda_1 B_1^T S_1^T + \lambda_2 B_2^T S_2^T\right]^{-1} \lambda_1 D U_1^T.$$

Since $\lambda_1 B_1 = \lambda_2 B_2 = \lambda_1 \text{diag } \{b_{1_t}(x_1)\}$ and $DU_1(x_1)^T = (b_{1_1}(x_1), \ldots, b_{1_T}(x_1))$, it suffices to show that $\left[S_1^T + S_2^T\right]^{-1} \mathbf{1}$ is uniformly bounded on $[0,\omega]$. But this follows from the same argument given above. ■

**Proof of Theorem 3.4:** Let $x \in [0,\omega]$ and consider an arbitrary row of $D^2U(x) \equiv \{d_{ts}(x)\}$. Letting subscripts on $u$ denote partial derivatives,

$$d_{ts}(x) = \begin{cases} \beta^t u_{21}^t(x_{t-1}, x_t) & \text{if } s = t-1; \\ \beta^t u_{22}^t(x_{t-1}, x_t) + \beta^{t+1} u_{11}^{t+1}(x_t, x_{t+1}) & \text{if } s = t; \\ \beta^{t+1} u_{21}^{t+1}(x_t, x_{t+1}) & \text{if } s = t+1; \\ 0 & \text{otherwise.} \end{cases}$$

We must show that $U^T(x)$ is uniformly concave. By Corollary 3.1 and Lemma A4, it suffices to show that $S^T(x)$ is uniformly negative definite, where $S^T(x)$ is given by

$$\begin{pmatrix} \begin{matrix} v''(x_0) \\ +\beta u_{11}^1(x_0, x_1) \end{matrix} & \beta u_{12}^1(x_0, x_1) & 0 & 0 & \cdots & 0 \\ u_{12}^1(x_0, x_1) & \begin{matrix} u_{22}^1(x_0, x_1) \\ +\beta u_{11}^2(x_1, x_2) \end{matrix} & \beta u_{12}^2(x_1, x_2) & 0 & \cdots & 0 \\ 0 & u_{12}^2(x_1, x_2) & \begin{matrix} u_{22}^2(x_1, x_2) \\ +\beta u_{11}^3(x_2, x_3) \end{matrix} & \beta u_{21}^3(x_2, x_3) & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \begin{matrix} u_{22}^T(x_{T-1}, x_T) \\ +\beta u_{11}^{T+1}(x_T, 0) \end{matrix} \end{pmatrix}.$$

So $z^\dagger S^T(x)z = v''(x_0)z_1^2 + \beta u_{11}^{T+1}(x_T, 0)z_T^2 + \sum_{t=1}^{T}(z_{t-1}, z_t)^\dagger D^t(x_{t-1}, x_t)(z_{t-1}, z_t)$ where

$$D^t(x_{t-1}, x_t) = \begin{pmatrix} \beta u_{11}^t(x_{t-1}, x_t) & \beta u_{12}^t(x_{t-1}, x_t) \\ u_{12}^t(x_{t-1}, x_t) & u_{22}^t(x_{t-1}, x_t) \end{pmatrix}.$$

Note that $D^t(x_{t-1}, x_t)$ is negative definite if and only if $D^t(x_{t-1}, x_t) + D^t(x_{t-1}, x_t)^\dagger$ is negative definite, and

$$D^t(x_{t-1}, x_t) + D^t(x_{t-1}, x_t)^\dagger = \begin{pmatrix} 2\beta u_{11}^t(x_{t-1}, x_t) & (1+\beta)u_{12}^t(x_{t-1}, x_t) \\ (1+\beta)u_{12}^t(x_{t-1}, x_t) & 2u_{22}^t(x_{t-1}, x_t) \end{pmatrix},$$

which is negative definite if and only if $h(\beta) = \frac{4\beta}{(1+\beta)^2} > \frac{(u_{12}^t(x_{t-1}, x_t))^2}{u_{11}^t(x_{t-1}, x_t)u_{22}^t(x_{t-1}, x_t)}$. Then since $h(0) = 0$, $h(1) = 1$, and $h$ is increasing, and given that $\frac{(u_{12}^t(x_{t-1}, x_t))^2}{u_{11}^t(x_{t-1}, x_t)u_{22}^t(x_{t-1}, x_t)} < 1$ for all $x$, let $\bar{\beta} \in (0, 1)$ satisfy the equation

$$\frac{4\bar{\beta}}{(1+\bar{\beta})^2} = \max_{x \in [0, \omega]} \frac{(u_{12}^t(x_{t-1}, x_t))^2}{u_{11}^t(x_{t-1}, x_t)u_{22}^t(x_{t-1}, x_t)}.$$

Then for $\beta > \bar{\beta}$, $D^t(x_{t-1}, x_t)$ is negative definite and continuous, so there exists $d > 0$ such that $y^\dagger D^t(c_1, c_2)y \leq -d\|y\|^2$ for each $y \in \mathbb{R}^2$ and $(c_1, c_2) \in [0, \|\omega\|] \times [0, \|\omega\|]$. Thus

$$z^\dagger S^T(x)z \leq -\hat{d}(z_1^2 + z_T^2 + \sum_{t=1}^{T} \|(z_{t-1}, z_t)\|^2) \leq -\hat{d}\|z\|^2$$

for some $\hat{d} > 0$. Thus $S^T(x)$ is uniformly negative definite on $[0, \omega]$. ■

**Proof of Theorem 4.1:** First I will establish this result when there are only a finite number of commodities. Suppose not, so suppose that there exists a Pareto optimal allocation $(x_1, \ldots, x_m)$ such that $\lambda_i > 0$ for all $i$, so that $U_i(x_i) > 0$ for all $i$, and such that $x_{1_j} = 0$ for some good $j$. Since $U_1(x_1) > 0$, there exists $k$ such that $x_{1_k} > 0$. Moreover, by feasibility there exists $i$ such that $x_{i_j} > 0$. Let $\epsilon, \delta > 0$. Then letting $e_r$ be the $r^{th}$ unit vector,

$$\begin{aligned} U_i(x_i - \epsilon e_j + \delta e_k) - U_i(x_i) &= DU_i(\tilde{x}_i) \cdot (\delta e_k - \epsilon e_j) \\ &= \delta \frac{\partial U_i}{\partial x_k}(\tilde{x}_i) - \epsilon \frac{\partial U_i}{\partial x_j}(\tilde{x}_i) \end{aligned}$$

for some $\tilde{x}_i = \alpha(x_i - \epsilon e_j + \delta e_k) + (1-\alpha)x_i$ where $\alpha \in (0, 1)$. In particular, note that $\tilde{x}_{i_j} > 0$. So

$$U_i(x_i - \epsilon e_j + \delta e_k) - U_i(x_i) \geq 0 \iff \frac{\delta}{\epsilon} \geq \frac{\frac{\partial U_i}{\partial x_j}}{\frac{\partial U_i}{\partial x_k}}(\tilde{x}_i).$$

Moreover, if $c > 0$,

$$\begin{aligned} U_1(x_1 + \epsilon e_j - \delta e_k) - U_1(x_1) &\geq U_1(x_1 + \epsilon e_j - \delta e_k) - U_1(x_1 + \frac{1}{c}\epsilon e_j) \\ &= DU_1(\tilde{x}_1) \cdot (\frac{c-1}{c}\epsilon e_j - \delta e_k) \end{aligned}$$

for some $\tilde{x}_1 = (1-\gamma)(x_1 + \epsilon e_j - \delta e_k) + \gamma(x_1 + \frac{1}{c}\epsilon e_j)$. In particular, note that $\tilde{x}_{1_j} = (1-\gamma)\epsilon + \gamma\frac{1}{c}\epsilon$ and $\tilde{x}_{1_k} = x_{1_k} - (1-\gamma)\delta$. Moreover, note that if $\frac{c}{c-1}\frac{\delta}{\epsilon} < \frac{\frac{\partial U_1}{\partial x_j}}{\frac{\partial U_1}{\partial x_k}}(\tilde{x}_1)$, then $U_1(x_1 + \epsilon e_j - \delta e_k) - U_1(x_1) > 0$. Now we can find constants $\delta, \epsilon, c$ such that

$$\frac{\frac{\partial U_i}{\partial x_j}}{\frac{\partial U_i}{\partial x_k}}(\tilde{x}_i) \leq \frac{\delta}{\epsilon} < \frac{c}{c-1}\frac{\delta}{\epsilon} < \frac{\frac{\partial U_1}{\partial x_j}}{\frac{\partial U_1}{\partial x_k}}(\tilde{x}_1)$$

since the left hand side is bounded and the right hand side goes to infinity as $\epsilon \to 0$. This is a contradiction, since we have thus found a Pareto improving trade.

The argument for the case of commodity space $\ell_\infty$ is similar. Suppose by way of contradiction that there exists $i, t$ such that $x_{i_t} = 0$. Since $U_j(x_j) > 0$ for all $j$, there exists $t'$ such that for each consumer $j$, $x_{j_r} > 0$ for some $r \leq t'$. For each $s$ and for each $z \in \ell_\infty$, define $\hat{z}^s = (z_1, \ldots, z_s)$. Let $T = \max(t, t')$, and consider the truncated $T$ good economy in which consumer $j$'s utility function $\hat{U}_j^T : \mathbb{R}_+^T \to \mathbb{R}$ is given by $\hat{U}_j^T(y) \equiv U_j(y, x_{T+1}, x_{T+2}, \ldots)$ for each $y \in \mathbb{R}_+^T$, and in which consumer $j$'s endowment is $\hat{\omega}_j$. Clearly $(\hat{x}_1^T, \ldots, \hat{x}_m^T)$ is a Pareto optimal allocation in this economy, and each consumer satisfies the weak survival condition in this finite-dimensional economy, so by the previous theorem, $\hat{x}_{j_k}^t = x_{j_k} > 0$ for all $j$ and $k$, which is a contradiction. So $x_i \in \ell_{\infty++}$ for all $i$. ∎

**Proof of Theorem 4.3:** First, by Theorem 4.2, all individually rational Pareto optimal allocations are interior. Now suppose the theorem is false, so that there exists a sequence of individually rational Pareto optimal allocations $x^n$ and there exists $i$ such that $\inf_t |x_{i_t}^n| \equiv b^n \to 0$ as $n \to \infty$. Since each $x^n$ is interior, $\lambda_i^n DU_i(x_i^n) = \lambda_j^n DU_j(x_j^n)$ for each $i, j$ and for every $n$. So for each $i, j, t, n$,

$$\frac{\frac{\partial U_i}{\partial x_t}(x_i^n)}{\frac{\partial U_j}{\partial x_t}(x_j^n)} = \frac{\lambda_j^n}{\lambda_i^n}.$$

For each $i = 1, \ldots, m$, define $F_i = \{\lambda \in \Delta : U_i(x_i(\lambda, \omega)) \geq U_i(\omega_i)\}$. By Lemmas 3.2 and 3.3, $U_i(x_i(\lambda, \omega))$ is a continuous function of $\lambda$, so $F_i$ is a closed set. Thus for each $i$, $F_i$ is compact, since $F_i \subset \Delta$. Moreover, since $U_i(\omega_i) > 0$, $\lambda_i \neq 0$ for all $\lambda \in F_i$. Thus there exists $\underline{\lambda}_i > 0$ such that $\lambda_i \geq \underline{\lambda}_i$ for all $\lambda \in F_i$. Let $F = \bigcap_{i=1}^m F_i$. Then $F$ corresponds to the set of individually rational welfare weights, and by the above argument, $F$ is a compact set on which $\lambda_j \geq \underline{\lambda} > 0$ for some $\underline{\lambda}$ and for all $j$. Thus on $F$, $\frac{\lambda_j^n}{\lambda_i^n}$ is bounded for each $i, j$.

Since $F$ is compact, $x(F)$ is weak* compact, and in particular, for every $t$ there exists $M_t > 0$ such that $|x_{j_t}| \geq M_t$ for all $j$ and all $x \in x(F)$. Thus there exists a sequence $r(n) \to \infty$ as $n \to \infty$ such that $x_{i_{r(n)}}^n \to 0$. Now by feasibility, passing to a subsequence and relabeling if necessary, there exists $j$ such that $x_{j_{r(n)}}^n \geq \frac{\omega}{m}$ for all $n$. Then by the strong survival condition,

$$\limsup_n \frac{\lambda_j^n}{\lambda_i^n} = \limsup_n \frac{\frac{\partial U_i}{\partial x_{r(n)}}(x_i^n)}{\frac{\partial U_j}{\partial x_{r(n)}}(x_j^n)} = \infty.$$

But this contradicts the fact that $\lambda_j^n/\lambda_i^n$ is bounded on $F$. Thus the individually rational Pareto optimal allocations are bounded below. ∎

**Proof of Theorem 5.2:** First note that if $x(\lambda)$ is Lipschitz, so is $y(\lambda)$, since $y(\lambda) = \omega - \sum_{i=1}^{m} x_i(\lambda)$. So it suffices to show that $x(\lambda)$ is Lipschitz. As in the proof of Theorem 3.3 in the exchange case, the proof consists of three steps.

**Claim 1.** *If $1 \leq p < \infty$, then $x_T(\lambda) \xrightarrow{n} x(\lambda)$ for each $\lambda \in \Delta$, and $x(\cdot)$ is norm continuous. If $p = \infty$, then $x_T(\lambda) \xrightarrow{w^*} x(\lambda)$ for each $\lambda \in \Delta$, and $x(\cdot)$ is weak* continuous.*

*Proof of Claim 1:* I will prove the claim for the case $1 \leq p < \infty$; the case $p = \infty$ is analogous. Note that $\hat{Y}$ is norm compact, and for every $T$, $\hat{Y}^T \subset \hat{Y}$ and $\hat{Y}$ is closed. Thus for every $T$, $\hat{Y}^T$ is compact. Suppose by way of contradiction that $x_T(\lambda) \not\to x(\lambda)$ for some $\lambda \in \Delta$. Since $\hat{Y}^T \subset \hat{Y} \subset [0, \bar{y}]$ for some $\bar{y} \in \ell_{p+}$, then for every $T$, $x_T(\lambda) \in [0, \bar{y}]^m$. So there exists some convergent subsequence $x_S(\lambda) \to x \neq x(\lambda)$. Since $\hat{Y}$ is closed, $\sum_{i=1}^{m} x_i \in \hat{Y}$, and since $x \neq x(\lambda)$.

$$\sum_{i=1}^{m} \lambda_i U_i(x_i) < \sum_{i=1}^{m} U_i(x_i(\lambda)).$$

Note that by definition, $\sum_{i=1}^{m} x_i(\lambda)^T \in \hat{Y}^T$ for all $T$. and $x(\lambda)^T \xrightarrow{n} x(\lambda)$, so for some $T$ sufficiently large,

$$\sum_{i=1}^{m} \lambda_i U_i(x_{T_i}(\lambda)) < \sum_{i=1}^{m} U_i(x_i(\lambda)^T).$$

But this contradicts the definition of $x_T(\lambda)$. So $x_T(\lambda) \xrightarrow{n} x(\lambda)$.

**Claim 2.** *For each $T > 0$, $x_T(\lambda)$ is Lipschitz with constant $c_T$, where $c \equiv \sup_T c_T < \infty$.*

*Proof of Claim 2:* To see that $x_T(\lambda)$ is Lipschitz, note that by definition

$$x_T(\lambda) = \arg \max \sum_{i=1}^{m} \lambda_i U_i(x_i)$$
$$\text{s.t.} \quad \sum_{i=1}^{m} x_i \in \hat{Y}^T, \quad x_i \geq 0 \quad i = 1, \ldots, m$$

So $x_{T_i}(\lambda) = (x_i^T(\lambda, \omega^T), 0, 0, \ldots)$, where

$$x^T(\lambda) = \arg \max \sum_{i=1}^{m} \lambda_i U_i^T(z_i)$$
$$\text{s.t.} \quad f^T\Big(\sum_{i=1}^{m} z_i\Big) = 0, \quad z_i \geq 0 \quad i = 1, \ldots, m$$

and $f^T(w) = f(w, 0, 0, \ldots)$ for all $w \in \mathbb{R}^T$. Thus it suffices to show that $x^T(\lambda)$ is Lipschitz in $\lambda$. Since each of these problems is finite-dimensional, arguments similar to those used by Mas-Colell (1985) and in Theorem 3.1 to will give the desired result.

For each coordinate subspace $L$ of $\mathbb{R}^{mT}$, define

$$x_L^T(\lambda, \omega^T) = \arg \quad \max \quad \sum_{i=1}^m \lambda_i U_i^T(z_i)$$
$$\text{s.t.} \quad f^T(\sum_{i=1}^m z_i) = 0, \quad z \in L.$$

Since $D^2 U_i^T(z_i)$ is negative definite for all $z_i \in \mathbb{R}_+^T$ and $D^2 f^T(w)$ is positive definite for all $w$ such that $f^T(w) = 0$, $x_L^T(\lambda)$ is $C^1$ in $\lambda$ for each subspace $L$. Moreover,

$$x^T(\lambda) \in \bigcup_{L \in S(\mathbb{R}^{mT})} x_L^T(\lambda)$$

where $S(\mathbb{R}^{mT})$ is the set of all coordinate subspaces of $\mathbb{R}^{mT}$. Since this set has a finite number of elements and $x^T(\lambda)$ is continuous in $\lambda$, $x^T(\lambda)$ is Lipschitz continuous in $\lambda$ with some constant $c_T > 0$ (see MasColell (1985)) .

Now as in the proof of Theorem 3.3, to show that $\{c_T\}$ is bounded, it suffices to show that $x_L^T(\lambda)$ is uniformly Lipschitz. By the Implicit Function Theorem, each function $x_L^T(\lambda)$ is $C^1$ and $D_{\lambda_j} x_L^T(\lambda)$ is given by

$$\begin{aligned} D_{\lambda_i} x_{L_i}^T &= \left[\lambda_i D^2 U_i^T\right]^{-1} DU_i^T \\ &- \left[\lambda_i D^2 U_i^T\right]^{-1} \left[\sum_{j=1}^m \left[\lambda_j D^2 U_j^T\right]^{-1} + \left[\gamma D^2 f^T\right]^{-1}\right]^{-1} \left[\lambda_i D^2 U_i^T\right]^{-1} (DU_i^T + Df^T), \end{aligned}$$

and for $j \neq i$,

$$D_{\lambda_j} x_{L_i}^T = \left[\lambda_i D^2 U_i^T\right]^{-1} \left[\sum_{k=1}^m \left[\lambda_k D^2 U_k^T\right]^{-1} + \left[\gamma D^2 f^T\right]^{-1}\right]^{-1} \left[\lambda_j D^2 U_j^T\right]^{-1} (DU_j^T + Df^T).$$

where $\gamma$ is the Lagrange multiplier on the feasibility constraint, and again I have supressed the arguments to ease the notational burden. But now an argument identical to that given in the proof of Theorem 3.3 shows that this quantity is bounded for all $j$ and $T$, and thus that $c = \sup_T c_T < \infty$.

Finally, arguments identical to those in the proof of Theorem 3.1 show that because $\{c_T\}$ is bounded, $x_T(\lambda) \longrightarrow x(\lambda)$ uniformly, and thus $x(\lambda)$ is Lipschitz with constant $c$. ■

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