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The anisotropic Bernstein problem

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The Anisotropic Bernstein Problem

DISSERTATION

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for the degree of

DOCTOR OF PHILOSOPHY

in Mathematics

by

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Dissertation Committee:
Associate Professor Connor Mooney, Chair
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2023

DEDICATION

To my mom

For her endless love, support, and encouragement

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VITA

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ABSTRACT OF THE DISSERTATION

The Anisotropic Bernstein Problem

By

Yang Yang

Doctor of Philosophy in Mathematics

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Associate Professor Connor Mooney, Chair

In this thesis we discuss the Bernstein problem for parametric elliptic functionals. In the first part, we give a proof by foliation that the cones over $\mathbb{S}^k \times \mathbb{S}^l$ minimize parametric elliptic functionals for each $k, l \geq 1$. We also analyze the behavior at infinity of the leaves in the foliations. This analysis motivates conjectures related to the existence and growth rates of nonlinear entire solutions to equations of minimal surface type that arise in the study of such functionals. In the second part we construct nonlinear entire anisotropic minimal graphs over $\mathbb{R}^4$, which completes the solution to the anisotropic Bernstein problem. The examples we construct have a variety of growth rates, and our approach both generalizes to higher dimensions and recovers and elucidates known examples of nonlinear entire minimal graphs over $\mathbb{R}^n, n \geq 8$. The first two parts are joint works with C. Mooney. In the final part, we discuss the work of Ecker-Huisken about a Bernstein theorem for the minimal surface equation with controlled growth condition, and also its possible generalization to the anisotropic case.

Chapter 1

Introduction

A well-known theorem of Bernstein says that entire minimal graphs in $\mathbb{R}^3$ are planes. Building on work of Fleming [13], De Giorgi [7], and Almgren [1], Simons [31] extended this result to minimal graphs in $\mathbb{R}^{n+1}$ for $n \leq 7$. In contrast, there are nonlinear entire solutions to the minimal surface equation in dimension $n \geq 8$ due to Bombieri-De Giorgi-Giusti [2] and Simon [29].

In this thesis, we study the Bernstein problem for a more general class of parametric elliptic functionals. These assign to an oriented hypersurface $\Sigma \in \mathbb{R}^{n+1}$ the value

$$A_\Phi := \int_\Sigma \Phi(\nu) dA, \tag{1.1}$$

where ν is the unit normal to Σ, and Φ is a uniformly elliptic integrand, namely, a one-homogeneous function on $\mathbb{R}^{n+1}$ that is positive and smooth on $\mathbb{S}^n$, and satisfies in addition that $\{\Phi < 1\}$ is uniformly convex. For example, the case $\Phi = 1$ on $\mathbb{S}^n$ corresponds to the area functional. Such functionals have attracted recent attention both for their applied and theoretical interest ([5, 22, 23, 21, 24, 9, 11, 12]). In particular, they arise in models of crystal surfaces and in Finsler geometry, and they present important technical challenges not present for the area functional (especially due to the lack of a monotonicity formula), often leading to more general and illuminating proofs even in the area case.

Almgren-Schoen-Simon proved that the $(n-2)$-dimensional Hausdorff measure of the singular set for a minimizer of (1.1) vanishes [25]. In particular, minimizers are smooth in the case $n = 2$. Morgan later showed that the cone over $\mathbb{S}^k \times \mathbb{S}^k$ in $\mathbb{R}^{2k+2}$ minimizes a parametric elliptic functional for each $k \geq 1$, by constructing a calibration [20]. Thus, there exist singular minimizers of parametric elliptic functionals when $n \geq 3$.

The anisotropic Bernstein problem asks whether critical points of A_Φ which are graphs of functions defined on all of $\mathbb{R}^n$ are necessarily hyperplanes. In the case of the area functional $\Phi(x) = |x|$, the answer is positive if and only if $n \leq 7$. For general uniformly elliptic integrands, it is known that the answer is positive in dimension $n = 2$ by work of Jenkins [15] and in dimension $n = 3$ by work of Simon [28]. It is also known by recent work of Mooney [18] that the answer is negative in dimensions $n \geq 6$. This left open the cases $n = 4, 5$.

The thesis is organized as follows. The second chapter introduces the work of the author and Mooney, proving the cones over $\mathbb{S}^k \times \mathbb{S}^l$ minimize parametric elliptic functionals for each $k, l \geq 1$, by constructing foliations by minimizers in the spirit of [2]. To our knowledge, this is the first application of the foliation approach for integrands other than area (that is, $\Phi|_{\mathbb{S}^n} = 1$).

Chapter 3 is from the paper of the author and Mooney. We discuss the recent solution of the anisotropic Bernstein problem (the answer is yes if and only if $n < 4$). We construct nonlinear entire anisotropic minimal graphs over $\mathbb{R}^4$, and the examples we construct have a variety of growth rates. Our methods both generalize to higher dimensions and shed light on known examples in the case of the area functional.

In Chapter 4, we introduce a theorem of Ecker-Huisken [10], saying that in the area funcational case, Bernstein theorem holds with controlled growth assumption, and we discuss its possible generalization in the parametric elliptic functional case.

Chapter 2

A proof by foliation that Lawson's cones are A_Φ-minimizing

2.1 Introduction

A well-known result in the theory of minimal surfaces is that area-minimizing hypersurfaces in $\mathbb{R}^{n+1}$ are smooth when $n \leq 6$, but can have singularities in higher dimensions. An important tool in the theory is the monotonicity formula, which reduces the regularity problem to establishing the existence or non-existence of singular area-minimizing hypercones. Such cones were ruled out by Simons in the case $n \leq 6$ [31]. On the other hand, Bombieri-De Giorgi-Giusti proved that the cone over $\mathbb{S}^3 \times \mathbb{S}^3$ in $\mathbb{R}^8$ is area-minimizing [2].

In this chapter we consider the regularity problem for minimizers of parametric elliptic functionals, which generalize the area functional. These assign to an oriented hypersurface $\Sigma \subset \mathbb{R}^{n+1}$ the value

$$A_\Phi(\Sigma) := \int_\Sigma \Phi(\nu)\, dA, \tag{2.1}$$

where ν is the unit normal to Σ, and Φ is a one-homogeneous function on $\mathbb{R}^{n+1}$ that is positive and C^2 on $\mathbb{S}^n$, and has uniformly convex sub-level sets. Functionals of the form (2.1) have attracted recent attention for their applied and theoretical interest ([23], [9]). In particular, they arise in models of crystal surfaces and in Finsler geometry, and the lack of a monotonicity formula for critical points of (2.1) presents interesting technical challenges. Almgren-Schoen-Simon proved that the $(n-2)$-dimensional Hausdorff measure of the singular set for a minimizer of (2.1) vanishes [25]. In particular, minimizers are smooth in the case $n = 2$. Morgan later showed that the cone over $\mathbb{S}^k \times \mathbb{S}^k$ in $\mathbb{R}^{2k+2}$ minimizes a parametric elliptic functional for each $k \geq 1$, by constructing a calibration [20]. Thus, there exist singular minimizers of parametric elliptic functionals when $n \geq 3$.

The purpose of this chapter is to prove that the cones over $\mathbb{S}^k \times \mathbb{S}^l$ minimize parametric elliptic functionals for each $k,\, l \geq 1$, by constructing foliations by minimizers in the spirit of [2]. To our knowledge, this is the first application of the foliation approach for integrands other than area (that is, $\Phi|_{\mathbb{S}^n} = 1$).

Remark 2.1.1. The examples here and in [20] show that the best regularity result possible for minimizers of (2.1) is that the singular set has e.g. locally bounded $(n-3)$-dimensional Hausdorff measure. It remains an interesting open problem to determine the maximum possible dimension of the singular set for a minimizer of (2.1). See e.g. [32], pg. 686, for further discussion of this problem.

The approach of constructing a foliation by minimizers has several advantages. The first is that it removes some of the guesswork involved in constructing a calibration. Indeed, the approach involves solving a nonlinear ODE, which we show is possible provided the integrand Φ satisfies analytic conditions that are straightforward to check (see Lemma 2.2.1). The second is that the behavior at infinity of a leaf in the foliation gives quantitative information that is useful in the study of the closely related Bernstein problem for graphical minimizers. When a critical point of (2.1) can be written as the graph of a function u, we say that u

solves an equation of minimal surface type. An interesting question is:

Question 2.1.2. *Are entire solutions to equations of minimal surface type in $\mathbb{R}^n$ necessarily linear?*

For the area functional, the answer to Question 2.1.2 is "yes" if $n \leq 7$ ([13], [7], [1], [31]) and "no" if $n \geq 8$ ([2]). For general parametric elliptic functionals, the answer is "yes" when $n \leq 3$ ([15], [28]) and was recently shown to be "no" when $n \geq 4$ ([19]), and its complete resolution will be further discussed in Chapter 3. Our main theorem in this chapter is an important first step twoards the resolution of the anisotropic Bernstein problem. More precisely, it suggests the existence of nonlinear entire solutions to equations of minimal surface type in dimension $n \geq 4$ that grow sub-quadratically at infinity (see Conjecture 2.3.1 and the discussion after its statement).

For future reference we state our main result of this chapter here. We fix $k,\, l \geq 1$, we let $x \in \mathbb{R}^{k+1}$ and $y \in \mathbb{R}^{l+1}$, and we define the Lawson cones C_{kl} over $\mathbb{S}^k \times \mathbb{S}^l$ by

$$C_{kl} := \{|x| = |y|\} \subset \mathbb{R}^{k+l+2}.$$

Theorem 2.1.3. *For each $k,\, l \geq 1$, there exist parametric elliptic functionals A_Φ such that Φ is analytic away from the origin, and each side of the cone C_{kl} is foliated by analytic minimizers of A_Φ. In particular, C_{kl} minimizes A_Φ.*

Remark 2.1.4. The foliation is generated by dilations of a pair of critical points of A_Φ, each of which lies on one side of C_{kl} and is asymptotic to C_{kl} at infinity. We discuss the precise asymptotic behavior in Section 2.3.

Remark 2.1.5. The A_Φ-minimality of C_{kl} is new in the cases $k \neq l$ and $k+l \leq 5$, or $k+l = 6$ and $\min\{k,\, l\} = 1$. The cases $k = l \leq 2$ were treated in [20], and the remaining examples ($k + l = 6$ and $\min\{k,\, l\} \geq 2$ or $k + l \geq 7$) are known to minimize area, up to making an affine transformation ([16]).

Remark 2.1.6. A proof of minimality by foliation gives rise to a proof by calibration through the following observation: If we denote by $\nu(z)$ the unit normal to the leaf that passes through z, then the vector field $\nabla\Phi(\nu)$ is a calibration on $\mathbb{R}^{k+l+2}$. For a discussion of this connection in the area case see e.g. [6]. Indeed, it is divergence-free by the minimality of the leaves, and satisfies

$$\nabla\Phi(\nu) \cdot w \leq \Phi(w)$$

for all $w \in \mathbb{S}^{k+l+1}$ and at all points x. The inequality follows from the convexity of the hypersurface $\nabla\Phi(\mathbb{S}^{k+l+1})$ and the fact that $\nabla\Phi(w) \cdot w = \Phi(w)$.

The chapter is organized as follows. In Section 2.2 we prove Theorem 2.1.3. We reduce the problem to the careful analysis of a certain nonlinear second-order ODE, using the symmetries of C_{kl}. More precisely, we choose integrands that (like C_{kl}) are invariant under rotations in x and y, and we search for critical points Σ_{kl} of (2.1) that share these symmetries and can be written in the form $\{|y| = \sigma(|x|)\}$ or $\{|x| = \sigma(|y|)\}$ for some function σ of one variable. The condition that Σ_{kl} is a critical point gives rise to a nonlinear second-order ODE for σ. The heart of our construction is Lemma 2.2.1, which gives conditions on the integrand that guarantee the existence of solutions to the ODE with the desired properties. In particular, after a change of variable we can view the ODE as a nonlinear first-order autonomous system. The conditions we impose on the integrand guarantee that the solution trajectory is trapped in a region of the plane that corresponds to a function σ that defines a foliation leaf Σ_{kl} which is asymptotic to C_{kl}, and approaches C_{kl} at a precise rate. We remark that our approach quickly recovers the foliation by area-minimizing hypersurfaces of each side of the Simons cone C_{kk} when $k \geq 3$ (see Remark 2.2.4). Finally, in Section 2.3 we discuss the behavior at infinity of the leaves in the foliation given by Theorem 2.1.3, and the implications for Question 2.1.2. In particular, we state conjectures concerning the existence and growth rates of nonlinear global solutions to equations of minimal surface type in $\mathbb{R}^{k+l+2}$ for each k, $l \geq 1$, and we compare these conjectures with what is known about the minimal surface equation.

2.2 Proof of Theorem 2.1.3

2.2.1 Integrand Notation

We choose integrands Φ that depend only on $|x|$ and $|y|$. We define them by a pair of one-variable functions ϕ and ψ as follows:

$$\Phi(x,\, y) = \begin{cases} |y|\phi\left(\frac{|x|}{|y|}\right), & |y| \geq |x| \\ |x|\psi\left(\frac{|y|}{|x|}\right), & |x| > |y|. \end{cases} \tag{2.2}$$

The functions ϕ and ψ will be chosen to be positive, smooth, even, and locally uniformly convex on $\mathbb{R}$.

2.2.2 Foliation Leaf Notation

Having fixed an appropriate choice of Φ, we will show that there exists a critical point of A_Φ of the form

$$\Sigma_{kl} = \{|y| = \sigma(|x|)\} \subset \{|y| > |x|\}, \tag{2.3}$$

where σ is smooth, even, convex, asymptotic to $|.|$, and $|\sigma'| < 1$. The dilations of Σ_{kl} are then minimizers of A_Φ, and they foliate one side of C_{kl} (namely $\{|y| > |x|\}$), see Figure 2.1. A similar procedure will give a foliation of the other side $\{|x| > |y|\}$ by minimizers of A_Φ.

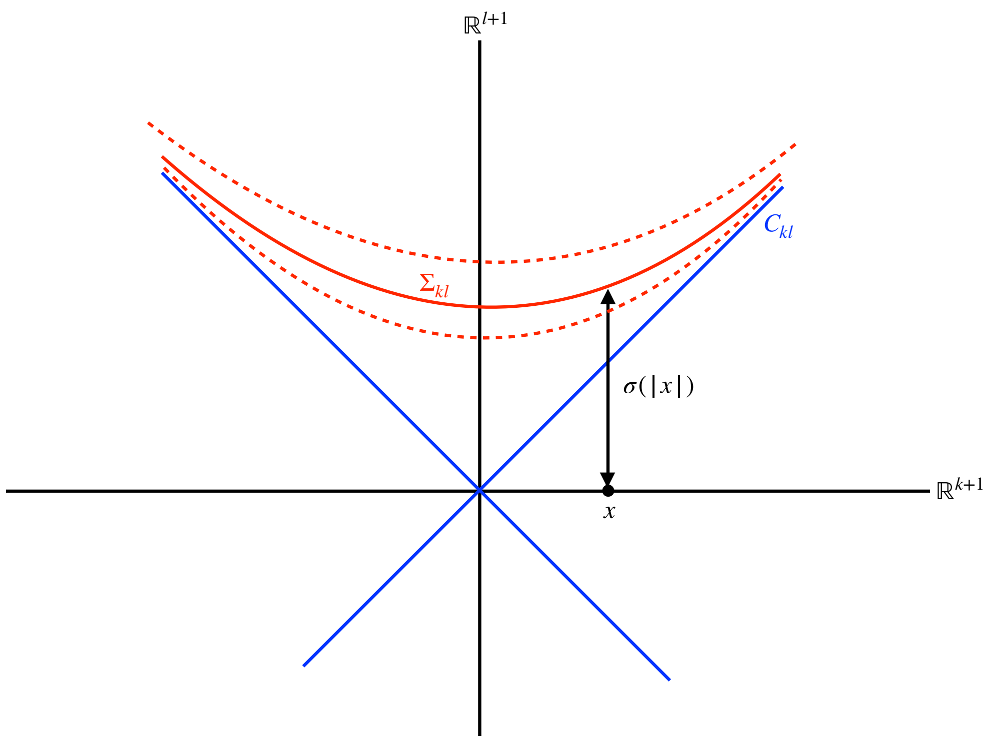

Figure 2.1: The dilations of Σ_{kl} foliate one side of C_{kl}.

2.2.3 Euler-Lagrange ODE

For hypersurfaces of the form (2.3) and integrands of the form (2.2), the condition that Σ_{kl} is a critical point of A_Φ is equivalent to the nonlinear second-order ODE

$$\sigma''(t) + kP(\sigma'(t))\frac{\sigma'(t)}{t} + lQ(\sigma'(t))\frac{1}{\sigma(t)} = 0, \tag{2.4}$$

where

$$P(s) := \frac{\phi'(s)}{s\phi''(s)}, \quad Q(s) := \frac{s\phi'(s) - \phi(s)}{\phi''(s)}. \tag{2.5}$$

This follows from the first variation formula

$$tr(D^2\Phi(\nu(z))II(z)) = 0$$

for critical points of (2.1), where II denotes the second fundamental form of Σ and ν denotes the unit normal. One can also use the symmetries of Σ_{kl} and Φ to reduce the problem to taking the first variation of the one-variable integral

$$A_\Phi(\Sigma_{kl}) = const. \int t^k \sigma^l(t)\phi(\sigma'(t))\, dt.$$

In the following technical lemma we show that there exists a global solution to (2.4) with the desired properties, provided ϕ satisfies certain analytic conditions. We will later give examples of ϕ that satisfy these conditions. To state the lemma we define for a smooth function φ on $\mathbb{R}$ the function $E_{kl}(\varphi)$ by

$$\begin{aligned} E_{kl}(\varphi)(s) := l\frac{k+l-1}{k+l+1}\varphi(s) - \left(k + \left(l - 2\frac{k+l}{k+l+1}\right)s\right)\varphi'(s) \\ - \left(\frac{k+l+1}{2} - s\right)(1-s)\varphi''(s). \end{aligned} \tag{2.6}$$

Lemma 2.2.1. *Assume that $\phi(s)$ is a smooth, even, uniformly convex function on $\mathbb{R}$ that satisfies*

$$\phi(1) = 1, \quad \phi'(1) = \frac{l}{k+l}, \tag{2.7}$$

and in addition that

$$E_{kl}(\phi)(s) \geq \kappa(1-s) \tag{2.8}$$

for some $\kappa > 0$ and all $s \in [0, 1]$. Then there exists a global smooth, even, convex solution σ to the ODE (2.4) that satisfies the initial conditions

$$\sigma(0) = 1, \quad \sigma'(0) = 0 \tag{2.9}$$

and in addition satisfies $\sigma(t) > |t|$, $|\sigma'(t)| < 1$ for all t, and

$$\sigma(t) = |t| + a|t|^{-\mu} + o(|t|^{-\mu}) \tag{2.10}$$

as $|t| \to \infty$ for some $a > 0$, where

$$\mu := \frac{k+l-1}{2} - \sqrt{\left(\frac{k+l-1}{2}\right)^2 - \frac{kl}{\phi''(1)(k+l)}}. \tag{2.11}$$

Remark 2.2.2. It is straightforward to check that any function ϕ satisfying the conditions (2.7) and (2.8) automatically satisfies the inequality

$$\phi''(1) - \frac{4kl}{(k+l)(k+l-1)^2} > 0, \tag{2.12}$$

so μ is well-defined. Conversely, any choice of ϕ that satisfies (2.7) and (2.12) also satisfies (2.8) for $s \in [1-\delta, 1]$, where $\kappa > 0$ and $\delta > 0$ depend only on the left side of (2.12) and

$\|\phi\|_{C^3([-1,\,1])}$.

Proof of Lemma 2.2.1. Standard ODE theory gives the short-time existence of a solution to (2.4) with the desired properties in a neighborhood of 0 (see Remark 2.2.3 below). To proceed we rewrite (2.4) as an autonomous first-order system. In terms of the quantities

$$w(\tau) := e^{-\tau}\sigma(e^{\tau}), \quad z(\tau) := \sigma'(e^{\tau}), \tag{2.13}$$

the second-order ODE (2.4) becomes:

$$\begin{pmatrix} w' \\ z' \end{pmatrix} = \begin{pmatrix} -w+z \\ -l\frac{Q(z)}{w} - kzP(z) \end{pmatrix} := \mathbf{V}(w,\, z). \tag{2.14}$$

We denote the components of the vector field $\mathbf{V}$ by V^i, $i = 1,\, 2$, and the solution curve $(w(\tau),\, z(\tau))$ by $\Gamma(\tau)$. The only zero of $\mathbf{V}$ in the closure of the infinite half-strip

$$\Omega := \{w > 1\} \cap \{0 < z < 1\}$$

occurs at $(1,\, 1)$ (here we used (2.7)). In addition, the linearization of (2.14) around the zero $(1,\, 1)$ has the form $X' = M\, X$, where

$$M = \begin{pmatrix} -1 & 1 \\ -\frac{kl}{\phi''(1)(k+l)} & -k-l \end{pmatrix}. \tag{2.15}$$

The eigenvalues of M are

$$\lambda_{\pm} = -\frac{k+l+1}{2} \pm \sqrt{\left(\frac{k+l-1}{2}\right)^2 - \frac{kl}{\phi''(1)(k+l)}}, \tag{2.16}$$

and these eigenvalues correspond to directions with slopes $1 + \lambda_{\pm}$.

We claim that Γ is contained in the region $R \subset \Omega$ bounded by the curves

$$\Gamma_1 :=\{z=0\}, \quad \Gamma_2 := \{V^2 = 0\} = \left\{ w = \frac{l}{k}\left(\frac{\phi}{\phi'} - z\right)\right\}, \text{ and}$$
$$\Gamma_3 := \left\{ (z-1) = \left(1 + \frac{\lambda_+ + \lambda_-}{2}\right)(w-1)\right\}.$$

We first note that $V^2 > 0$ on $\Gamma_1 \cap \{w > 0\}$ using that $Q(0) < 0$. Next, by the uniform convexity of ϕ, the curve $\Gamma_2 \cap \Omega$ is a graph over the positive z-axis with negative slope, and furthermore $V^1 < 0$ in Ω (in particular, on $\Gamma_2 \cap \Omega$). Finally, after a calculation using the definitions (2.5) of P and Q and inequality (2.8), we have

$$\frac{V^2}{V^1} < 1 + \frac{\lambda_+ + \lambda_-}{2}$$

on $\Gamma_3 \cap \Omega$. Indeed, the preceding inequality can be written

$$-lQ(z) - kwzP(z) > \frac{k+l-1}{2} w(w-z)$$

on $\Gamma_3 \cap \Omega$. Using that

$$\Gamma_3 \cap \Omega = \left\{ w = \frac{k+l+1}{k+l-1} - \frac{2}{k+l-1} z,\ z \in (0,\,1)\right\}$$

and that

$$-Q(z) = \frac{\phi(z) - z\phi'(z)}{\phi''(z)}, \quad -zP(z) = -\frac{\phi'(z)}{\phi''(z)}$$

the previous inequality becomes

$$\begin{aligned} l(\phi - z\phi') - k &\left(\frac{k+l+1}{k+l-1} - \frac{2}{k+l-1} z\right)\phi' \\ &> \frac{k+l-1}{2}\left(\frac{k+l+1}{k+l-1} - \frac{2}{k+l-1} z\right)\frac{k+l+1}{k+l-1}(1-z)\phi'' \\ &= \frac{k+1+1}{k+l-1}\left(\frac{k+l+1}{2} - z\right)(1-z)\phi''. \end{aligned}$$

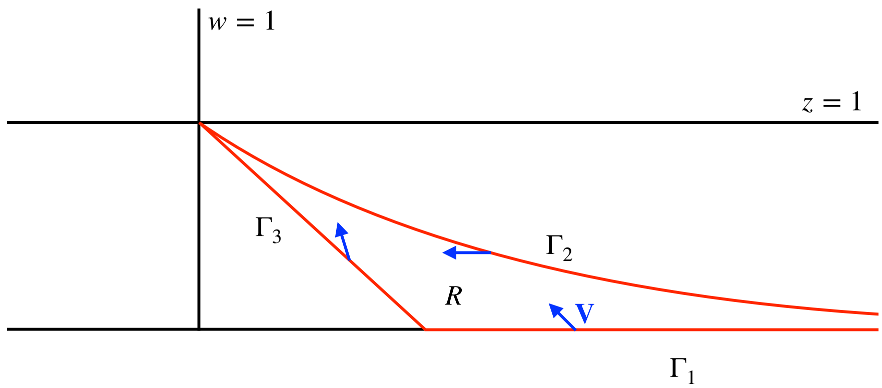

Figure 2.2: The solution curve is contained in the region R bounded by Γ_1, Γ_2 and Γ_3.

After regrouping terms, we see that this inequality holds for $z \in (0, 1)$ using (2.8). We conclude that Γ_2 and Γ_3 meet only at $(1, 1)$, and $\mathbf{V}$ points towards the interior of R on each of the curves $\{\Gamma_i\}_{i=1}^3$ (see Figure 2.2). It only remains to argue that $\Gamma(\tau) \in R$ for all $\tau << 0$, which holds by the initial convexity of σ (note that $V^2 < 0$ "above" Γ_2 in Ω).

Since $R \subset \Omega \cap \{V^2 > 0\}$, we conclude that $\sigma > |.|$, $|\sigma'| < 1$ and that σ is convex. The asymptotic behavior (3.2.2) follows from the linear analysis. Indeed, the region R excludes the line with slope $1 + \lambda_-$ that goes through $(1, 1)$. Thus, in the expansion of Γ for τ large, the coefficient of the principal eigenvector of the linearized operator at $(1, 1)$ (which corresponds to the eigenvalue λ_+ and has slope $-\mu = 1 + \lambda_+$) is nonzero, completing the proof. □

Remark 2.2.3. We could not find a precise reference for short-time existence, so for completeness we sketch the argument. We first rewrite (2.4) in divergence form:

$$[t^k \sigma^l \phi'(\sigma')]' = l t^k \sigma^{l-1} \phi(\sigma').$$

We are thus looking for a continuous function σ' on an interval $[0,\, t_0]$ such that $\sigma'(0) = 0$ and

$$
\begin{aligned}
\sigma'(t) &= (\phi^*)'\left(\frac{l}{t^k\left(1+\int_0^t \sigma'(s)\,ds\right)^l}\int_0^t s^k\left(1+\int_0^s \sigma'(\tau)\,d\tau\right)^{l-1}\phi(\sigma'(s))\,ds\right)\\
&:= G(\sigma')(t),
\end{aligned}
$$

where ϕ^* is the Legendre transform of ϕ. For $t_0 > 0$ small, the operator G is a contraction mapping on the space of continuous functions on $[0,\, t_0]$ that vanish at 0 and are bounded by 1 in the C^0 norm. A fixed point argument then gives the existence of a function $\sigma \in C^1[0,\, t_0] \cap C^\infty(0,\, t_0)$ that solves (2.4) on $(0,\, t_0)$ and satisfies $\sigma(0) = 1$, $\sigma'(0) = 0$. The higher regularity of σ follows from the observation that $\Sigma_{kl} = \{|y| = \sigma(|x|)\}$ can be locally written over its tangent planes as a C^1 graph that solves an equation of minimal surface type (and is thus smooth, see e.g. [14]). Finally, the equation (2.4) itself gives that

$$\sigma''(0) = \frac{l\phi(0)}{(k+1)\phi''(0)} > 0,$$

so σ is convex near 0, concluding the argument.

Remark 2.2.4. In the case of the area functional we have that

$$\phi(s) = \sqrt{1+s^2}. \tag{2.17}$$

Our approach recovers the foliation of each side of the cone C_{kk} by area-minimizing hypersurfaces when $k \geq 3$, as follows. First, a short calculation shows that the function $\sigma_0(t) := (1+t^4)^{1/4}$ is a super-solution of the ODE (2.4) corresponding to the area integrand (2.17) when $k \geq 3$. A similar calculation was performed in [8] to construct a sub-calibration for C_{kk}. If we take Γ_2 as above and we take $\Gamma_3 = \{(e^{-\tau}\sigma_0(e^\tau),\, \sigma_0'(e^\tau)),\, \tau \in \mathbb{R}\}$, then a similar argument to the one above shows that the solution trajectory to the associated autonomous system is trapped between Γ_2 and Γ_3, giving an exact solution σ of (2.4) with the desired properties.

2.2.4 Proof of Main Theorem

In this final subsection we choose the functions ϕ, ψ that define Φ, and we apply Lemma 2.2.1 to prove Theorem 2.1.3.

Proof of Theorem 2.1.3. We first indicate how to choose integrands Φ that are $C^{2,1}$ away from the origin, defined through the notation (2.2). For $p, q > 2$ to be chosen later we take

$$\begin{aligned} \phi(s) &= 1 - \frac{l}{p(k+l)} + \frac{l}{p(k+l)}|s|^p, \\ \psi(s) &= 1 - \frac{k}{q(k+l)} + \frac{k}{q(k+l)}|s|^q, \end{aligned} \tag{2.18}$$

up to making small perturbations near $s = 0$ so that ϕ and ψ are smooth and uniformly convex. It is straightforward to check that if p and q are related by

$$l(p-1) = k(q-1), \tag{2.19}$$

then Φ is $C^{2,1}$ away from the origin. We note that ϕ satisfies (2.7), and we will verify that provided p is sufficiently large, then ϕ also satisfies the desired inequality (2.8). Away from a small neighborhood of $s = 0$, where we perturbed (2.18) and the inequality (2.8) is obvious, the inequality $E_{kl}(\phi)(s) > 0$ becomes

$$\begin{aligned} &\left[(p-1)\left(\frac{k+l+1}{2} - s\right) + ks\right](1-s) \\ &\quad < \frac{(k+l-1)(k+l-l/p)}{k+l+1}(s^{2-p} - s^2). \end{aligned} \tag{2.20}$$

Denote the left side of (2.20) by $L(s)$ and the right side by $R(s)$. Since L is quadratic in s

and R'' is decreasing in s, it suffices to prove the inequalities

$$R'(1) < L'(1), \quad L'' \leq R''(1).$$

The first inequality holds provided

$$p - 1 > \frac{4k}{(k+l-1)^2}, \tag{2.21}$$

in agreement with Remark (2.2.2). The second one holds provided

$$\begin{aligned} (p-1)^2 - \left(\frac{k+2l+2}{k+l} + \frac{4}{(k+l)(k+l-1)}\right)(p-1) \\ + \frac{4k}{(k+l)(k+l-1)} \geq 0. \end{aligned} \tag{2.22}$$

Both (2.21) and (2.22) hold e.g. when $p \geq 6$, regardless of k, $l \geq 1$. When $p \geq 6$ the inequality (2.12) also holds, so by Remark (2.2.2) the desired inequality (2.8) holds for some $\kappa > 0$.

Up to exchanging k and l, the function ψ satisfies (2.7), and a similar analysis shows that $E_{lk}(\psi) \geq \kappa(1-s)$ for some $\kappa > 0$ and all $s \in [0, 1]$ if $q \geq 6$. We conclude using Lemma 2.2.1 that each side of C_{kl} is foliated by smooth critical points of A_Φ when we choose ϕ, ψ as above with p, $q \geq 6$, and furthermore $\Phi \in C^{2,1}(\mathbb{S}^{k+l+1})$ provided p and q are chosen such that (2.19) holds as well.

We now explain how the integrand can be made analytic on $\mathbb{S}^{k+l+1}$, by perturbing the $C^{2,1}$ integrand constructed above. We first improve to smooth. Take ϕ and ψ as above, and let

$$\tilde{\phi}(s) = s\psi(1/s)$$

for $s \leq 1$. We glue ϕ to $\tilde{\phi}$ near $s = 1$ by taking the convex combination

$$\bar{\phi} := \eta_\delta \phi + (1 - \eta_\delta)\tilde{\phi},$$

where η_δ is a smooth function that transitions from 1 to 0 in the interval $[1 - 2\delta,\, 1 - \delta]$ for $\delta > 0$ to be chosen, and satisfies

$$\|\eta_\delta\|_{C^m(\mathbb{R})} \leq C_m \delta^{-m}$$

with C_m independent of δ. Since $\tilde{\phi}$ and ϕ agree to second order at $s = 1$, the inequality (2.8) holds for $\bar{\phi}$ away from $[1 - 2\delta,\, 1 - \delta]$ provided δ is small (see Remark (2.2.2)). Furthermore, we have

$$|\tilde{\phi}^{(m)}(s) - \phi^{(m)}(s)| \leq C_m (1 - s)^{3-m}$$

for each $m \leq 2$ and $s \in [1/2,\, 1]$. It follows that

$$|E_{kl}(\bar{\phi}) - E_{kl}(\phi)| \leq C\delta^2$$

in $[1 - 2\delta,\, 1 - \delta]$. Since $E_{kl}(\phi) \geq \kappa\delta$ in this interval, the inequality (2.8) holds for $\bar{\phi}$ when δ is small, up to reducing κ slightly. After replacing ϕ by $\bar{\phi}$ (and keeping ψ the same), we obtain a new integrand that is smooth on $\mathbb{S}^{k+l+1}$ and by Lemma 2.2.1 satisfies the desired properties.

Finally, we indicate how to improve the regularity from smooth to analytic. We start with a smooth choice of integrand Φ as constructed above. Using the symmetries of Φ we may view it as a smooth function on $\mathbb{S}^1$. We approximate this function by the partial sums S_N of its Fourier series with N terms. We add small correctors of the form $a_N + b_N \cos(2\theta) + c_N \cos(4\theta)$ to S_N to obtain new approximations T_N, with a_N, b_N, c_N chosen such that T_N agrees to second order with Φ at $\theta = \pi/4$. Since S_N converge uniformly in C^m to Φ for any m, the functions T_N do as well. It follows that the one-homogeneous extensions of T_N to $\mathbb{R}^2$ (which we now identify with T_N) have uniformly convex sub-level sets for N large. Since T_N agree to second order with Φ on the diagonals, Remark (2.2.2) implies that the conditions (2.7) and (2.8) hold for the function obtained by restricting T_N to the horizontal lines tangent to $\mathbb{S}^1$ when N is large. The same holds (with k and l exchanged) for the restriction of T_N to the vertical lines tangent to $\mathbb{S}^1$. Hence, after replacing $\Phi(x, y)$ with $T_N(|x|, |y|)$ for N large, we obtain an integrand that is analytic on $\mathbb{S}^{k+l+1}$ and by Lemma 2.2.1 satisfies the desired properties. □

2.3 Discussion

In this section we discuss the implications of the analysis in Section 2.2 for Question 2.1.2. The discussion is motivated by the examples of entire minimal graphs constructed in [2] and [29]. Those examples are asymptotic to area-minimizing cones of the form $K \times \mathbb{R}$, where K is the Simons cone in [2], and any one of a large family of area-minimizing cones with isolated singularities in [29]. In all cases, each side of K is foliated by smooth area-minimizing hypersurfaces. These are closely related to the level sets of the functions u that define the entire minimal graphs. More precisely, each level set of u is a graph over K outside of some ball, with the same leading-order asymptotic behavior at infinity as a leaf in the foliation. Furthermore, if the distance between a leaf in the foliation and K on ∂B_r behaves like $r^{-\mu}$ as $r \to \infty$, then $\sup_{B_r} |\nabla u| \sim r^{\mu}$.

In view of this discussion we conjecture:

Conjecture 2.3.1. *For any integrand Φ as constructed in Theorem 2.1.3, and $k \neq l$, there exists an elliptic extension of Φ to $\mathbb{R}^{k+l+3}$, and a nonlinear global solution to the corresponding equation of minimal surface type in $\mathbb{R}^{k+l+2}$, whose graph is asymptotic to $C_{kl} \times \mathbb{R}$. Moreover, the gradient of this solution grows at the same rate that the leaves in the foliation associated* *to Φ approach C_{kl}.*

In Chapter 3, we will prove the case when $k = l$, but this problem remains open for the case when $k \neq l$. The proof of Theorem 2.1.3 shows that for any $\mu \in (0, \mu_{kl})$, we can choose integrands such that each side of C_{kl} is foliated by minimizers whose distance from C_{kl} on ∂B_r behaves like $r^{-\mu}$, where

$$\mu_{kl} = \frac{k+l-1}{2} - \sqrt{\left(\frac{k+l-1}{2}\right)^2 - \frac{\min\{k,\, l\}}{5}}. \tag{2.23}$$

The formula (2.23) comes from (2.11) and noting that, when choosing ϕ and ψ, we could take any exponents p and q such that (2.19) holds and $p,\, q \geq 6$. Thus, Conjecture (2.3.1) predicts that for any $\mu \in (0, \mu_{kl})$, there exist global solutions to equations of minimal surface type in $\mathbb{R}^{k+l+2}$ whose graphs are are asymptotic to $C_{kl} \times \mathbb{R}$, and have maximum gradient in B_r growing like r^{μ}.

Remark 2.3.2. Mooney showed in [18] that when $k = l = 2$, the graph of $u = |x|^2 - |y|^2$ (which is asymptotic to $C_{22} \times \mathbb{R}$) minimizes a parametric elliptic functional A_Ψ, and each level set of u minimizes A_Φ, where $\Phi = \Psi|_{\{x_7=0\}}$. The perspective in that work is quite different, and the proof is based on solving a linear hyperbolic equation to construct Ψ. However, the discussion at the end of [18] shows that this strategy could be challenging to implement when $2 \leq k + l \leq 3$. In these cases, Question 2.1.2 may instead yield to a combination of the approaches from [2] and [18].

To conclude we discuss the gradient growth rates of the solutions predicted by Conjecture 2.3.1. We first consider the possibility of constructing solutions with fast growth. In the case $k = l = 1$, a closer inspection of inequalities (2.21) and (2.22) shows that we can take any $p = q > 5$ when defining ϕ and ψ. This corresponds to the "optimal" value $\mu_{11} = \frac{1}{2}$. One may hope to show that (2.23) can be improved to $\mu_{kl} = (k + l - 1)/2$ for arbitrary k and l, which corresponds to choices of ϕ such that inequality (2.12) tends to equality. However, we suspect that this is not possible. Indeed, if $k = l$ is large, this corresponds to a small value of $\phi''(1)$. It would follow that the integrand Φ is larger on $\mathbb{S}^k \times \mathbb{S}^k$ than at nearby points on $\sqrt{2}\,\mathbb{S}^{2k+1}$, in which case perturbations of C_{kk} could likely decrease its energy. Since the quantity (2.23) is bounded above independently of k and l, it does not seem likely that the examples from Theorem 2.1.3 can give rise to solutions to equations of minimal surface type with arbitrarily fast gradient growth.

On the other hand, Conjecture 2.3.1 predicts the existence of global solutions to equations of minimal surface type with very slow gradient growth, namely

$$\sup_{B_r} |\nabla u| \sim r^{\mu}$$

with $\mu > 0$ small. However, global solutions to equations of minimal surface type with bounded gradient are linear. This is a consequence of the De Giorgi-Nash-Moser theorem. Indeed, if the gradient of a global solution u to an equation of minimal surface type is bounded, then each derivative u_e is a global bounded solution to a uniformly elliptic equation in divergence form (with ellipticity constants depending on $\|\nabla u\|_{L^\infty}$). Applying e.g. Theorem 8.22 from [14] and sending $R_0 \to \infty$ (keeping in mind the remark at the end of Section 8.9), we see that u_e is constant. We thus expect that the ellipticity of the integrands from Conjecture 2.3.1 will degenerate as μ tends to zero, and we conjecture a "quantitative" version of the rigidity result for solutions with bounded gradient:

Conjecture 2.3.3. *Let u be a global solution to an equation of minimal surface type on $\mathbb{R}^n$, corresponding to a functional A_Ψ. Then for some $\epsilon(n, \Psi) > 0$,*

$$\sup_{B_r} |\nabla u| = O(r^{\epsilon}) \Rightarrow u \text{ is linear.}$$

In [10] the authors give a beautiful proof of Conjecture 2.3.3 for the area functional, for any $\epsilon < 1$ and in arbitrary dimension n. The proof in [10] depends on precise constants in the

Simons inequality for the Laplacian of the second fundamental form on a minimal surface. Although analogues of the Simons inequality exist for critical points of (2.1), the constants degenerate with the ellipticity of Φ, and it is not clear that the same strategy would prove Conjecture 2.3.3.

Chapter 3

The anisotropic Bernstein problem

3.1 Introduction

In this chapter we study graphical critical points of parametric elliptic functionals, which assign to an oriented hypersurface $\Sigma \subset \mathbb{R}^{n+1}$ the value

$$A_\Phi(\Sigma) = \int_\Sigma \Phi(\nu)\, dA. \tag{3.1}$$

Here ν is the unit normal to Σ, and Φ is a uniformly elliptic integrand, namely, a one-homogeneous function on $\mathbb{R}^{n+1}$ that is positive and smooth on $\mathbb{S}^n$, and satisfies in addition that $\{\Phi < 1\}$ is uniformly convex. Such functionals have attracted recent attention both for their applied and theoretical interest ([5, 22, 23, 21, 24, 9, 11, 12]). In particular, they arise in models of crystal surfaces and in Finsler geometry, and they present important technical challenges not present for the area functional (especially due to the lack of a monotonicity formula), often leading to more general and illuminating proofs even in the area case.

The anisotropic Bernstein problem asks whether critical points of A_Φ which are graphs of functions defined on all of $\mathbb{R}^n$ are necessarily hyperplanes. In the case of the area functional $\Phi(x) = |x|$, it is known through spectacular work of Bernstein, Fleming [13], De Giorgi [7], Almgren [1], Simons [31], and Bombieri-De Giorgi-Giusti [2] that the answer is positive if and only if $n \leq 7$. For general uniformly elliptic integrands, it is known that the answer is positive in dimension $n = 2$ by work of Jenkins [15] and in dimension $n = 3$ by work of Simon [28]. It is also known by recent work of Mooney [18] that the answer is negative in dimensions $n \geq 6$. This left open the cases $n = 4, 5$. The purpose of this chapter is to settle the anisotropic Bernstein problem negatively in these remaining cases:

Theorem 3.1.1. *There exists a smooth nonlinear function $u : \mathbb{R}^4 \to \mathbb{R}$ and a uniformly elliptic integrand Φ on $\mathbb{R}^5$ such that the graph of u in $\mathbb{R}^5$ minimizes A_Φ.*

We in fact construct, for any $\mu \in (0, 1/2)$, a pair (u, Φ) proving Theorem 3.1.1 such that

$$\sup_{B_r} u \sim r^{1+\mu}$$

for r large, and our methods both generalize to higher dimensions and shed light on known examples in the case of the area functional.

Here and below we denote for $x \in \mathbb{R}^{k+1}$ and $y \in \mathbb{R}^{l+1}$ the cone $C_{kl} \subset \mathbb{R}^{k+l+2}$ by

$$C_{kl} = \{|x| = |y|\}.$$

The first examples of nonlinear entire minimal graphs were constructed in [2]. The examples in [2] are asymptotic to $C_{kk} \times \mathbb{R} \subset \mathbb{R}^{2k+3}$, for $k \geq 3$. Similarly, for the more general anisotropic case, the example in [18] is asymptotic to $C_{22} \times \mathbb{R}$. However, the approaches in [2] and [18] are completely different. In the former, the method is to carefully construct super- and sub-solutions to the minimal surface equation, with appropriate ordering and symmetries, and then use the maximum principle. In [18], the method is to first fix a choice of solution u, and then construct the integrand Φ by solving a linear *hyperbolic* equation. In the case $k = l = 2$ it turned out that for the simple choice

$$u = |x|^2 - |y|^2,$$

the hyperbolic equation for Φ reduced after a change of variable to the classical two-variable wave equation. This made the construction significantly shorter and more elementary than for the case of the area functional. Unfortunately the analogous choice for u in the case $k = l = 1$ does *not* solve an equation of minimal surface type, as shown in [18]. However, the choice

$$u = |x|^{4/3} - |y|^{4/3}$$

also yields a two-variable wave equation for Φ in the case $k = l = 1$ after a change of variable, so it seemed likely that the methods in [18] could be adapted. The issue is that this choice of u does not solve an equation of minimal surface type near $\{|x||y| = 0\}$, so both u and the integrand Φ obtained by solving the wave equation need to be modified. This seems to be tricky, and we are leaving the pursuit of this approach to its conclusion for future work.

In this chapter we instead proceed by the maximum principle, inspired by [2]. The problem of constructing entire graphical minimizers in $\mathbb{R}^{n+1}$ is closely related to the existence of singular minimizers in $\mathbb{R}^n$. This was certainly recognized in [2], where it was shown that C_{33} is area-minimizing by constructing foliations of each side of the cone by smooth area minimizers. It is clear that the level sets of the entire graphical minimizers in [2] resemble the leaves in this foliation, but no explicit connection is made between the results. Our approach in this chapter is to make this connection explicit. As a consequence we are able to construct, in the general anisotropic case, a variety of examples with many different growth rates in the optimal dimension $n = 4$, and to recover the known examples from [2].

Our starting point is the work in Chapter 2, where we prove that the cones C_{kl} minimize parametric elliptic functionals for all $k, l \geq 1$ by constructing foliations. It was known previously that minimizers of (3.1) are smooth in the case $n = 2$ by deep work of Almgren-Schoen-Simon [25]. Morgan [20] later proved that minimizers are not necessarily smooth in dimension $n = 3$. Indeed, he proved that C_{11} minimizes a parametric elliptic functional, by constructing a calibration. Although the foliation approach in Chapter 2 is more involved, advantages are that it removes some of the guesswork involved in constructing a calibration, and that the leaves in the foliation give a hint as to how to proceed in the anisotropic Bernstein problem. We showed in particular in Chapter 2 that for any $\mu \in (0, 1/2)$, there is an integrand $\overline{\Psi}$ such that the sides of C_{11} are foliated by minimizers of $A_{\overline{\Psi}}$ that closely resemble the level surfaces of locally Lipschitz functions that are homogeneous of degree $1 + \mu$. This involved the careful analysis of a nonlinear second-order ODE.

The first step in this chapter is to deepen the analysis of the nonlinear ODE. By studying its linearization around a solution, we obtain small perturbations of the leaves in our foliation of each side of C_{11} that have the same asymptotic behavior as before, but have strictly positive or negative anisotropic mean curvature. These leaves then define locally Lipschitz functions $\overline{w}$ and $\underline{w}$ that are homogeneous of degree $1 + \mu$, constant on the leaves, vanish on C_{11}, and by virtue of the curvature of their level sets serve as good model candidates for super- and sub-solutions.

The second step in this work is to make a choice of integrand Φ on $\mathbb{R}^5$. Our choice agrees exactly with $\overline{\Psi}$ on $\{x_5 = 0\}$, and can be viewed as way of smoothly extending $\overline{\Psi}$ to $\mathbb{S}^4 \backslash \{x_5 = 0\}$. The case of the area functional suggests taking

$$\Phi|_{\{x_5=1\}} = \left(1 + \overline{\Psi}^2\right)^{1/2},$$

and our integrand indeed resembles this choice.

The final step in this work is to "re-stack" the level sets of the functions $\overline{w}$ and $\underline{w}$ in a way that they become legitimate super- and sub-solutions to the equation of minimal surface type defined by Φ on one side of C_{11}. This is accomplished by composing $\overline{w}$ and $\underline{w}$ with appropriate concave, resp. convex one-variable functions. We can then proceed as in [2], using these super- and sub-solutions to trap the exact solutions to the equation we wish to solve in B_R with appropriate boundary data, and taking R to infinity.

We conclude the introduction with several remarks. The first is that our approach works equally well to construct examples asymptotic to $C_{kk} \times \mathbb{R}$ for all $k \geq 1$. We focus on the case $k = 1$ for simplicity of notation and to emphasize ideas. In a later section we indicate how to generalize to higher dimensions, and we also make explicit how our approach recovers the examples from [2], whose construction seems at first somewhat ad-hoc. The second is that our approach does not work when $k \neq l$, because the argument relies crucially on the odd symmetry over C_{kk} of solutions to the PDE associated to Φ. We intend to pursue this question in future work. In [29] Simon constructs entire minimal graphs that are asymptotic to the cylinders over a variety of area-minimizing cones with isolated singularities (in particular, all of the area-minimizing Lawson cones, which are affine transformations of C_{kl} with $k + l \geq 7$ or $k + l = 6$ and $\min\{k, l\} \geq 2$, see e.g. Chapter 5 in A sufficient criterion for a cone to be area-minimizing by G. Lawlor). The existence of foliations plays an important role in that paper as well, and we believe that a combination of the ideas in that work and ours may bring further clarity to the picture. Finally, we remark that when Φ is close to the area functional on $\mathbb{S}^n$ in a strong topology, e.g. C^4, the results are the same as in the area case. For example, the Bernstein theorem holds up to dimension $n = 7$ [28], regularity of minimizers holds up to dimension $n = 6$ [25], and stable critical points in low dimensions are flat ([4], [5]). We thus know that for our examples, the integrands are necessarily far from area. It would be interesting to weaken the topology required for such results e.g. to closeness in C^2 (which suffices for proving the flatness of stable critical points in dimension $n = 2$ [17]), and our examples may shed light on this question. These remarks are discussed at greater length in a later section.

The chapter is organized as follows. In Section 3.2 we recall our results from Chapter 2 concerning the foliation of each side of C_{11} by minimizers of parametric elliptic functionals, and we refine our analysis of a certain nonlinear ODE to obtain perturbed foliations by hypersurfaces with anisotropic mean curvature of a desired sign. In Section 3.3 we make our choice of integrand Φ, which fixes the equation we wish to solve. In Section 3.4 we construct super- and sub-solutions to this equation using the perturbed foliations. In Section 3.5 we put it all together to prove Theorem 3.1.1. Finally, in Section 3.6 we discuss generalizations of our constructions to higher dimensions, the relation to the case of the area functional, and future work.

3.2 Foliation

In this section we first recall for the reader's convenience the construction of foliations of each side of C_{11} by minimizers of (3.1), accomplished in Chapter 2. We then refine the analysis from Chapter 2, and we use this along with a study of a linearized problem to perturb the leaves of the foliation so that they have positive or negative anisotropic mean curvature. Finally, we use these leaves to define homogeneous functions on $\mathbb{R}^4$ that will serve as models for super- and sub-solutions to the PDE we eventually wish to solve.

3.2.1 Previous Results

In Chapter 2 we constructed uniformly elliptic integrands $\overline{\Psi}$ on $\mathbb{R}^4$ defined by analytic, even, one-variable functions ϕ as follows:

$$\overline{\Psi}(x, y) = \varphi(|x|, |y|) = \begin{cases} |y|\phi(|x|/|y|), \ |y| > 0, \\ |x|\phi(|y|/|x|), \ |x| > 0. \end{cases} \tag{3.2}$$

We showed that for any $\mu \in (0, 1/2)$, the function ϕ could be chosen such that there exists a critical point $\Sigma \subset \{|y| > |x|\}$ of $A_{\overline{\Psi}}$ of the form

$$\Sigma = \{|y| = \sigma(|x|)\},$$

where the function σ is even, analytic, locally uniformly convex, larger than $|.|$, $\sigma(0) = 1$, and

$$\sigma(\tau) = \tau + a\tau^{-\mu} + o(\tau^{-\mu})$$

for some $a > 0$ as $\tau \to \infty$. Furthermore, the curve Γ parameterized for $\tau > 0$ by $(\tau^{-1}\sigma(\tau), \sigma'(\tau))$ tends as $\tau \to \infty$ to $(1, 1)$, and Γ is bounded in a certain region of the (w, z) plane:

$$\Gamma \subset \{0 < z < 1\} \cap \{z > 3/2 - w/2\}. \tag{3.3}$$

We remark that the smoothness of $\overline{\Psi}$ away from 0 ensures that

$$2\phi'(1) = \phi(1), \tag{3.4}$$

and that μ and ϕ are related through the identiy

$$\frac{\phi(1)}{2\phi''(1)} = \mu(1 - \mu). \tag{3.5}$$

The dilations of Σ, along with their reflections over C_{11}, give a foliation of each side of C_{11} by minimizers of $A_{\overline{\Psi}}$. The condition that Σ is a critical point is equivalent to σ solving the ODE

$$G(\sigma)(\tau) := \sigma''(\tau) + \frac{1}{\tau}P(\sigma'(\tau)) + \frac{1}{\sigma}Q(\sigma'(\tau)) = 0, \tag{3.6}$$

where

$$P(s) = \frac{\phi'(s)}{\phi''(s)}, \quad Q(s) = \frac{s\phi'(s) - \phi(s)}{\phi''(s)}, \tag{3.7}$$

and the results were obtained through the analysis of this ODE.

Remark 3.2.1. In Chapter 2 we defined $\overline{\Psi}$ in $\{|x| > 0\}$ and $\{|y| > 0\}$ by two different functions ϕ and ψ. However, the conditions in Chapter 2 that ϕ and ψ are required to satisfy are invariant under taking convex combinations when $k = l$, so we can assume $\phi = \psi$ after replacing $\overline{\Psi}(x, y)$ by $(\overline{\Psi}(x, y) + \overline{\Psi}(y, x))/2$.

3.2.2 Refinement of Foliation Analysis

From hereon out, we fix a choice of $\mu \in (0, 1/2)$ and ϕ as in the previous subsection. For the purposes of this chapter we need to make the asymptotic expansion of σ a little more precise.

We first establish some notation. Here and for the rest of the chapter we will let c denote a small positive constant that depends only on ϕ, and its value may change from line to line. For a smooth function h on $(0, \infty)$, we denote by $\mathcal{F}(h)$ a smooth function on $(0, \infty)$ such that, for all $\tau \geq 1$,

$$|\mathcal{F}(h)(\tau)| \leq c^{-1}|h(\tau)|, \quad |\mathcal{F}(h)'(\tau)| \leq c^{-1}|h'(\tau)|, \quad |\mathcal{F}(h)''(\tau)| \leq c^{-1}|h''(\tau)|.$$

Proposition 3.2.2. *There exists $a > 0$ and $b \in \mathbb{R}$ such that*

$$\sigma = \tau + a\tau^{-\mu} + b\tau^{\mu-1} + \mathcal{F}(\tau^{-1-2\mu}).$$

For our purposes, the following corollary in fact suffices:

$$\sigma = \tau + a\tau^{-\mu} + \mathcal{F}(\tau^{-1/2}) \text{ for some } a > 0. \tag{3.8}$$

Before proving Proposition 3.2.2, we recall a few things from the proof of the results in the previous subsection. First, the ODE (3.6) for σ can be written as a first-order autonomous system for $\mathbf{X}(s) := (e^{-s}\sigma(e^s), \sigma'(e^s))$ of the form

$$\mathbf{X}'(s) = \mathbf{V}(\mathbf{X}(s)),$$

where

$$\mathbf{V}(w, z) = (-w + z, -P(z) - Q(z)/w).$$

Using the identities (3.4) and (3.5) we have that $\mathbf{V}(1, 1) = (0, 0)$ and that

$$D\mathbf{V}(1, 1) = M := \begin{pmatrix} -1 & 1 \\ -\mu(1-\mu) & -2 \end{pmatrix}.$$

The matrix M has eigenvalues $-1-\mu$ and $\mu - 2$, corresponding to eigenvectors in the directions of lines with the slopes $-\mu$ and $\mu - 1$. We proved in Chapter 2 that the solution curve $\mathbf{X}$ tends to $(1, 1)$ as s tends to infinity, and by (3.3) is trapped in a region which excludes the line through $(1, 1)$ with slope $\mu - 1$.

Proof of Proposition 3.2.2: Let

$$\mathbf{Y}(s) = \mathbf{X}(s) - (1, 1), \quad \mathbf{W}(w, z) = \mathbf{V}(w + 1, z + 1),$$

so that

$$\mathbf{Y}' = \mathbf{W}(\mathbf{Y}), \quad \mathbf{W}(0) = 0, \quad D\mathbf{W}(0) = M.$$

The eigenvalues of $M^T + M$ are $-3 \pm \sqrt{1 + (1 - \mu(1-\mu))^2} \in (-5, -3/2)$. Since $|\mathbf{Y}|$ tends to zero, we conclude from the ODE for $|\mathbf{Y}|^2$ that

$$ce^{-5s} \leq |\mathbf{Y}|^2(s) \leq c^{-1} e^{-\frac{3}{2}s}, \quad s \geq 0. \tag{3.9}$$

To obtain (3.9) we use that

$$|\mathbf{Y}' - M\mathbf{Y}| \leq c^{-1} |\mathbf{Y}|^2, \quad s \geq 0. \tag{3.10}$$

Letting

$$\mathbf{p} = (1, -\mu), \quad \mathbf{q} = (1, \mu - 1)$$

denote the eigenvectors of M, decomposing $\mathbf{Y}$ into a linear combination of these vectors, and using (3.9) in (3.10) gives in a similar way that for some $a_0 \in \mathbb{R}$,

$$|\mathbf{Y}(s) - a_0 e^{-(1+\mu)s} \mathbf{p}| \leq c^{-1} e^{-3s/2}, \quad s \geq 0.$$

We conclude from this the following improvement of (3.9):

$$|\mathbf{Y}|^2 \leq c^{-1} e^{-2(1+\mu)s}, \quad s \geq 0.$$

Using this improved estimate in (3.10) we conclude that for some a, $b \in \mathbb{R}$,

$$|\mathbf{Y}(s) - a e^{-(1+\mu)s} \mathbf{p} - b e^{(\mu-2)s} \mathbf{q}| \leq c^{-1} e^{-2(1+\mu)s}, \quad s \geq 0.$$

The conclusion follows from this expression and from the second component of the ODE for $\mathbf{Y}$, provided $a > 0$. To show that $a > 0$, note first that by (3.3), we have $a \geq 0$ and that if $a = 0$ then $b = 0$. In the latter case we would have $|\mathbf{Y}|^2 \leq c^{-1} e^{-4(1+\mu)s}$, $s \geq 0$, and upon examining the ODE (3.10) for $\mathbf{Y}$ again we could improve the expansion of $\mathbf{Y}$ to

$$|\mathbf{Y}(s) - a_1 e^{-(1+\mu)s}\mathbf{p} - b_1 e^{(\mu-2)s}\mathbf{q}| \leq c^{-1} e^{-4(1+\mu)s}, \quad s \geq 0.$$

If $a_1 \neq 0$ then the upper bound $|\mathbf{Y}|^2 \leq c^{-1}e^{-4(1+\mu)s}$, $s \geq 0$ is violated, hence $a_1 = 0$, and again by (3.3) we have $b_1 = 0$. We conclude that $|\mathbf{Y}|^2 \leq c^{-1}e^{-8(1+\mu)s}$, $s \geq 0$, which contradicts the lower bound in (3.9). □

3.2.3 Linear ODE

Our goal now is to perturb the leaves in the foliation determined by the function σ. We let L denote the linearization of the ODE (3.6) at σ, namely,

$$\begin{aligned} Lf &= f'' + \left(\frac{1}{\tau} + \frac{\sigma'}{\sigma} + \frac{\sigma''\phi'''}{\phi''}\right) f' + \frac{\phi - \sigma'\phi'}{\sigma^2\phi''} f \\ &:= f'' + [\log(p)]' f' + qf, \end{aligned}$$

where

$$p(s) = s\sigma(s)\phi''(\sigma'(s)).$$

Using the invariance of the operator G under the Lipschitz rescalings

$$\sigma \to \sigma_\lambda := \lambda^{-1}\sigma(\lambda\tau),$$

we see that the function

$$f_0(\tau) := -\frac{d}{d\lambda}\sigma_\lambda(\tau)|_{\lambda=1} = \sigma(\tau) - \tau\sigma'(\tau)$$

solves the linearized equation

$$Lf_0 = 0.$$

Furthermore, by the properties of σ and the estimate (3.8), f_0 is smooth, positive, even, and satisfies

$$f_0 = a(1+\mu)\tau^{-\mu} + \mathcal{F}(\tau^{-1/2}). \tag{3.11}$$

For g continuous, the solution to the ODE

$$Lf = g, \quad f(0) = f'(0) = 0$$

can be written

$$f(\tau) = f_0(\tau) \int_0^\tau \frac{1}{f_0^2(t)p(t)} \int_0^t g(s)p(s)f_0(s)\, ds\, dt. \tag{3.12}$$

Taking

$$g = \sigma^{-5/2}$$

in (3.12) we obtain a smooth even solution f_1. By the asymptotics (3.11) of f_0 and the formula (3.12) for f_1, we have for some $d \in \mathbb{R}$ that

$$f_1 = d\tau^{-\mu} + \mathcal{F}(\tau^{-1/2}). \tag{3.13}$$

Below we will often bound quantities by powers of σ, which serves as a strictly positive regularization of the function $|\tau|$. Using (3.8) and (3.13) and applying Taylor's theorem with remainder to P and Q, it is straightforward to show for ϵ small that

$$|G(\sigma + \epsilon f_1) - \epsilon L(f_1)| \le c^{-1}\epsilon^2\sigma^{-3-2\mu}. \tag{3.14}$$

Indeed, Taylor expansion gives for $|\epsilon| > 0$ small and any $\tau > 0$ that

$$\begin{aligned}\epsilon^{-2}|G(\sigma+\epsilon f_1)-\epsilon L(f_1)| \le{}& \frac{\|P''\|_{L^\infty([0,\,\sigma'(\tau)+|\epsilon f_1'(\tau)|])}}{\tau} f_1'^2\\ &+\|Q\|_{C^2([0,\,1])}\left(\frac{f_1'^2}{\sigma}+\frac{|f_1 f_1'|}{\sigma^2}+\frac{f_1^2}{\sigma^3}\right)\\ &+|\epsilon|\|Q\|_{C^2([0,\,1])}\left(\frac{|f_1 f_1'^2|}{\sigma^2}+\frac{|f_1^2 f_1'|}{\sigma^3}\right)\\ &+\epsilon^2\|Q\|_{C^2([0,\,1])}\frac{f_1^2 f_1'^2}{\sigma^3}.\end{aligned}$$

Using that P is odd and σ and f_1 are even we have that

$$\frac{\|P''\|_{L^\infty([0,\,\sigma'(\tau)+|\epsilon f_1'(\tau)|])}}{\tau} \le c^{-1}\frac{1}{\sigma},$$

and the remaining terms can be bounded using (3.8) and (3.13).

In summary, we have proven:

Proposition 3.2.3. *There exists $\epsilon_0 > 0$ small such that the functions*

$$\overline{\sigma} := \sigma + \epsilon_0 f_1, \quad \underline{\sigma} := \sigma - \epsilon_0 f_1$$

are even, locally uniformly convex, larger than $|\tau|$, and have the asymptotics

$$\overline{\sigma} = \tau + \overline{a}\tau^{-\mu} + \mathcal{F}(\tau^{-1/2}), \quad \underline{\sigma} = \tau + \underline{a}\tau^{-\mu} + \mathcal{F}(\tau^{-1/2}) \tag{3.15}$$

for some $\overline{a}$, $\underline{a} > 0$. Furthermore, they satisfy for all $\tau \in \mathbb{R}$ that

$$G(\overline{\sigma}) \ge c\sigma^{-5/2}, \quad -G(\underline{\sigma}) \ge c\sigma^{-5/2} \tag{3.16}$$

and

$$\frac{1}{2}\overline{\sigma}(2\tau) \leq \underline{\sigma}(\tau) \leq \overline{\sigma}(\tau). \tag{3.17}$$

The inequality (3.17) follows from the asymptotics (3.8) for σ and (3.13) for f_1, and the smallness of ϵ_0. For (3.16) we use (3.14) and that $5/2 < 3 + 2\mu$.

Geometrically, the surface

$$\overline{\Sigma} := \{|y| = \overline{\sigma}(|x|)\}$$

has anisotropic mean curvature vector pointing "away from" C_{11}, and the opposite is true for the hypersurface

$$\underline{\Sigma} := \{|y| = \underline{\sigma}(|x|)\}.$$

Remark 3.2.4. The choice $g = \sigma^{-5/2}$ is convenient but there is flexibility in the choice of exponent. For all arguments in this chapter, the choice $g = \sigma^{-\beta-2}$ with

$$\mu < \beta < 1 - \mu$$

would suffice. These inequalities guarantee that

$$\sigma = \tau + a\tau^{-\mu} + \mathcal{F}(\tau^{-\beta}),$$

that the solution f to $Lf = g$, $f(0) = f'(0) = 0$ satisfies

$$f = d\tau^{-\mu} + \mathcal{F}(\tau^{-\beta})$$

for some $d \in \mathbb{R}$, and that $G(\sigma + \epsilon f) \geq cg$ for $\epsilon > 0$ small. The arguments below can also be treated in a similar way as presented for such choices of β.

3.2.4 Model Super- and Sub-solutions

To conclude the section we define functions that are homogeneous of degree $1+\mu$ and take the value 1 on $\overline{\Sigma}$ and $\underline{\Sigma}$. These serve as model super- and sub-solutions to the PDE we eventually wish to solve. First, we define functions $\overline{w_0}$, $\underline{w_0}$ of two variables (ξ, ζ) in $\{\zeta > |\xi|\}$ by

$$\overline{w_0}(\lambda^{-1}\tau,\, \lambda^{-1}\overline{\sigma}(\tau)) = \underline{w_0}(\lambda^{-1}\tau,\, \lambda^{-1}\underline{\sigma}(\tau)) = \lambda^{-1-\mu}.$$

Here $\lambda > 0$ and $\tau \in \mathbb{R}$. We extend these functions over the diagonals by odd reflection, and we define them to vanish on the diagonals. The continuity of $\overline{w_0}$ follows from the identity $\overline{w_0}(\tau/\overline{\sigma}(\tau),\, 1) = \overline{\sigma}^{-1-\mu}$ and taking τ to $\pm\infty$, and similarly for $\underline{w_0}$. The calculations below show that $\overline{w_0}$, $\underline{w_0}$ are in fact locally Lipschitz.

Here and below we evaluate the quantities of interest for $\overline{w_0}$ at the point $\lambda^{-1}(\tau,\, \overline{\sigma}(\tau))$, and for $\underline{w_0}$ at $\lambda^{-1}(\tau,\, \underline{\sigma}(\tau))$. We calculate

$$\nabla\overline{w_0} = (1+\mu)\frac{\lambda^{-\mu}}{\overline{\sigma} - \tau\overline{\sigma}'}(-\overline{\sigma}',\, 1), \quad \nabla\underline{w_0} = (1+\mu)\frac{\lambda^{-\mu}}{\underline{\sigma} - \tau\underline{\sigma}'}(-\underline{\sigma}',\, 1). \tag{3.18}$$

Using Proposition 3.2.3 we conclude that

$$c\lambda^{-\mu}\sigma^{\mu} \leq |\nabla\overline{w_0}|,\, |\nabla\underline{w_0}| \leq c^{-1}\lambda^{-\mu}\sigma^{\mu}. \tag{3.19}$$

It follows that $c \leq |\nabla\overline{w_0}(\tau/\overline{\sigma}(\tau),\, 1)|,\, |\nabla\underline{w_0}(\tau/\underline{\sigma}(\tau),\, 1)| \leq c^{-1}$ for all $\tau \in \mathbb{R}$, which establishes the local Lipschitz regularity of these functions.

Next we calculate

$$\begin{aligned} D^2\overline{w_0} =& (1+\mu)\frac{\lambda^{1-\mu}}{(\overline{\sigma} - \tau\overline{\sigma}')^3}\cdot \\ & \left[\mu(\overline{\sigma} - \tau\overline{\sigma}')(-\overline{\sigma}',\, 1)\otimes(-\overline{\sigma}',\, 1) - \overline{\sigma}''(-\overline{\sigma},\, \tau)\otimes(-\overline{\sigma},\, \tau)\right], \end{aligned} \tag{3.20}$$

and the same expression with $\overline{\sigma}$ replaced by $\underline{\sigma}$ for $D^2\underline{w_0}$. By Proposition 3.2.3, each of the four entries in the matrix in brackets is bounded up to multiplying by constants by $\sigma^{-1/2}$, which gives

$$|D^2\overline{w_0}|,\ |D^2\underline{w_0}| \leq c^{-1}\lambda^{1-\mu}\sigma^{3\mu-1/2}. \tag{3.21}$$

For $x,\ y \in \mathbb{R}^2$ we let

$$\overline{w}(x,\ y) = \overline{w_0}(|x|,\ |y|), \quad \underline{w}(x,\ y) = \underline{w_0}(|x|,\ |y|).$$

We evaluate the quantities of interest for $\overline{w}$, resp. $\underline{w}$ on $\{(|x|,\ |y|) = \lambda^{-1}(\tau,\ \overline{\sigma}(\tau)),\ \tau \geq 0,\ \lambda > 0\}$, resp. $\{(|x|,\ |y|) = \lambda^{-1}(\tau,\ \underline{\sigma}(\tau)),\ \tau \geq 0,\ \lambda > 0\}$. By (3.19) we have

$$c\lambda^{-\mu}\sigma^{\mu} \leq |\nabla\overline{w}|,\ |\nabla\underline{w}| \leq c^{-1}\lambda^{-\mu}\sigma^{\mu}. \tag{3.22}$$

Furthermore, for $\tau > 0$ the entries of $D^2\overline{w}$ that are not in $D^2\overline{w_0}$ are $\lambda\tau^{-1}\partial_\xi\overline{w_0}$ and $\lambda\overline{\sigma}^{-1}\partial_\zeta\overline{w_0}$. Using (3.18) we have the bounds

$$\lambda\tau^{-1}|\partial_\xi\overline{w_0}|,\ \lambda\overline{\sigma}^{-1}|\partial_\zeta\overline{w_0}| \leq c^{-1}\lambda^{1-\mu}\sigma^{\mu-1} \leq c^{-1}\lambda^{1-\mu}\sigma^{3\mu-1/2},$$

and similarly for $D^2\underline{w}$. We conclude from (3.21) that

$$|D^2\overline{w}|,\ |D^2\underline{w}| \leq c^{-1}\lambda^{1-\mu}\sigma^{3\mu-1/2}. \tag{3.23}$$

Finally, we relate the curvature of the level sets of $\overline{w}$, $\underline{w}$ to a PDE. Recall that

$$\overline{\Psi}(x,\ y) = \varphi(|x|,\ |y|) = |y|\phi(|x|/|y|) = |x|\phi(|y|/|x|).$$

Using that $\partial_\xi \overline{w_0} < 0$ in $\{\zeta > \xi > 0\}$, we have in $\{|y| > |x|\}$ that

$$\overline{\Psi}_{ij}(\nabla \overline{w})\overline{w}_{ij} = \varphi_{11}\overline{w_0}_{\xi\xi} - 2\varphi_{12}\overline{w_0}_{\xi\zeta} + \varphi_{22}\overline{w_0}_{\zeta\zeta} - \frac{\varphi_1}{|x|} + \frac{\varphi_2}{|y|}, \tag{3.24}$$

where the derivatives of φ are evaluated at $(|\partial_\xi \overline{w_0}|, |\partial_\zeta \overline{w_0}|)$. Using the formulae (3.18) and (3.20) and the relation between φ and ϕ, and evaluating the above expression on $\{(|x|, |y|) = \lambda^{-1}(\tau, \overline{\sigma}(\tau)), \tau > 0, \lambda > 0\}$, we get

$$\overline{\Psi}_{ij}(\nabla \overline{w})\overline{w}_{ij} = -\lambda\phi''(\overline{\sigma}')G(\overline{\sigma}).$$

Indeed, up to the factor $-\lambda\phi''(\sigma')$, the first three terms on the right side of (3.24) contribute the first term in the expression (3.6) for G, and the last two contribute the second two terms in the expression for G. The analogous calculations hold for $\underline{w}$, with $\overline{\sigma}$ replaced by $\underline{\sigma}$. Combining this calculation with Proposition 3.2.3 we obtain:

Proposition 3.2.5. *We have*

$$\overline{\Psi}_{ij}(\nabla \overline{w})\overline{w}_{ij} = -\lambda\phi''(\overline{\sigma}')G(\overline{\sigma}) \leq -c\lambda\sigma^{-5/2}$$

on $\{(|x|, |y|) = \lambda^{-1}(\tau, \overline{\sigma}(\tau)), \tau \geq 0, \lambda > 0\}$, *and*

$$\overline{\Psi}_{ij}(\nabla \underline{w})\underline{w}_{ij} = -\lambda\phi''(\underline{\sigma}')G(\underline{\sigma}) \geq c\lambda\sigma^{-5/2}$$

on $\{(|x|, |y|) = \lambda^{-1}(\tau, \underline{\sigma}(\tau)), \tau \geq 0, \lambda > 0\}$.

To conclude the section we note that inequality (3.17) from Proposition 3.2.3 and homogeneity imply that

$$\underline{w}(\cdot) \leq \overline{w}(2\cdot) = 2^{1+\mu}\overline{w}(\cdot) \text{ in } \{|y| > |x|\}. \tag{3.25}$$

3.3 Choice of Integrand

In this section we fix our choice of integrand. Let $\overline{\Psi}$ be the same as above, and let F be a smooth even convex function on $\mathbb{R}$ such that

$$F(s) = s + \frac{1}{8}s^{-1},\ s > 1/2.$$

We let Ψ be any smooth locally uniformly convex function on $\mathbb{R}^4$, depending only on $|x|$ and $|y|$ and invariant under exchanging x and y, such that

$$\Psi = F(\overline{\Psi}) \text{ in } \{\overline{\Psi} > 1\}.$$

(See Remark 3.3.2). Finally, we define

$$\Phi(x,\, y,\, z) := \begin{cases} |z|\Psi\left(\frac{x}{|z|},\, \frac{y}{|z|}\right), & z \in \mathbb{R}\backslash\{0\}, \\ \overline{\Psi}(x,\, y), & z = 0. \end{cases} \tag{3.26}$$

Proposition 3.3.1. *The function Φ is in $C^\infty(\mathbb{S}^4)$ and is a uniformly elliptic integrand.*

Proof. On $\{|z| > 0\}$ this follows from the local uniform convexity of Ψ, which in $\{\overline{\Psi} > 1\}$ follows from the identity

$$D^2\Psi = F'(\overline{\Psi})D^2\overline{\Psi} + F''(\overline{\Psi})\nabla\overline{\Psi} \otimes \nabla\overline{\Psi},$$

the uniform ellipticity of $\overline{\Psi}$ and the fact that $F''(s) > 0$ for $s > 1$. We now examine the points on $\mathbb{S}^4 \cap \{z = 0\}$. Using the fact that $F(s) = s + s^{-1}/8$ for $s > 1/2$, we have in a neighborhood of $\mathbb{S}^4 \cap \{z = 0\}$ that

$$\Phi(x,\, y,\, z) = \overline{\Psi}(x,\, y) + \frac{z^2}{8\overline{\Psi}(x,\, y)}.$$

Thus, on a hyperplane tangent to $\mathbb{S}^4$ on $\{z = 0\}$, the horizontal second derivatives of Φ at the point of tangency agree with those of $\overline{\Psi}$, the mixed horizontal and vertical second derivatives vanish, and the pure vertical second derivative is strictly positive, completing the proof. □

Remark 3.3.2. This can be accomplished e.g. by taking F to be a positive constant in a small neighborhood of 0 and locally uniformly convex otherwise, and defining Ψ to be the sum of $F(\overline{\Psi})$ and a small multiple of an appropriate radial cutoff of $|x|^2 + |y|^2$.

3.4 Super and Sub Solutions

Let Φ, Ψ be as in the previous section. We note as in [18] that the graph of a function u on a domain $\Omega \subset \mathbb{R}^4$ is a critical point of A_Φ if and only if

$$\Psi_{ij}(\nabla u)u_{ij} = 0 \tag{3.27}$$

in Ω. In this section we “re-stack” the level sets of $\overline{w}$ and $\underline{w}$ to obtain super- and sub-solutions of (3.27) in $\{|y| > |x|\}$, without changing their growth rates.

3.4.1 Supersolution

To begin we fix the quantities

$$\gamma := \frac{2\mu}{1+\mu}, \quad M := \frac{2}{1/2-\mu}, \quad \delta := \frac{\gamma}{M}.$$

We define

$$\overline{u} = \overline{H}(\overline{w}),$$

where $\overline{H}(0) = 0$ and

$$\overline{H}'(s) = A|s|^{-\gamma} + e^{B\int_{|s|}^{\infty} \frac{t^{\delta-1}}{1+t^{2\delta}}\,dt} = A|s|^{-\gamma} + e^{BI(|s|)}.$$

We note that $\overline{H}$ is well-defined because $\gamma < 1$. We claim for $B = A^2$ and A sufficiently large that $\overline{u}$ is a super-solution to (3.27) in $\{|y| > |x|\}$.

To see this, note first that

$$\overline{H}'(s) \geq 1 + As^{-\frac{2\mu}{1+\mu}}.$$

Combining this with the estimate (3.22) we conclude on $\{(|x|,\, |y|) = \lambda^{-1}(\tau, \overline{\sigma}(\tau)),\, \tau \geq 0,\, \lambda > 0\}$ that

$$|\nabla\overline{u}| = \overline{H}'(\lambda^{-1-\mu})|\nabla\overline{w}| \geq c(A\lambda^{\mu} + \lambda^{-\mu}) \geq cA^{1/2}.$$

For $A > 1$ sufficiently large we conclude that

$$\nabla\overline{u}(\{|y| > |x|\}) \subset \{\overline{\Psi} > 1\},$$

thus $\nabla\overline{u}$ always lies in the region where $\Psi = F(\overline{\Psi})$. Henceforth we assume A has been chosen at least this large. We calculate using the one-homogeneity of $\overline{\Psi}$ that

$$\begin{aligned}
\Psi_{ij}(\nabla\overline{u})\overline{u}_{ij} &= F'(\overline{H}'(\overline{w})\overline{\Psi}(\nabla\overline{w}))\overline{\Psi}_{ij}(\nabla\overline{w})\overline{w}_{ij} \\
&\quad + F''(\overline{H}'(\overline{w})\overline{\Psi}(\nabla\overline{w}))\overline{H}''(\overline{w})\overline{\Psi}^2(\nabla\overline{w}) \\
&\quad + F''(\overline{H}'(\overline{w})\overline{\Psi}(\nabla\overline{w}))\overline{H}'(\overline{w})\overline{\Psi}_i(\nabla\overline{w})\overline{\Psi}_j(\nabla\overline{w})\overline{w}_{ij} \\
&:= I + II + III
\end{aligned}$$

in $\{|y| > |x|\}$. We have that $F'(s) \geq 1/2$ and $F''(s) = s^{-3}/4$ when $s > 1$, and that $\overline{H}'' < 0$. Using this, along with Proposition 3.2.5, and the estimates (3.22) and (3.23) on $\nabla\overline{w}$ and

$D^2\overline{w}$, we conclude that

$$I \leq -c\lambda\sigma^{-5/2},\ II \leq -c\left|\overline{H}''\right|\overline{H}'^{-3}\lambda^{\mu}\sigma^{-\mu},\ III \leq c^{-1}\overline{H}'^{-2}\lambda^{1+2\mu}\sigma^{-1/2}$$

on $\{(|x|,\, |y|) = \lambda^{-1}(\tau,\, \overline{\sigma}(\tau)),\, \tau \geq 0,\, \lambda > 0\}$. Here $\overline{H}$ and its derivatives are evaluated at $\lambda^{-1-\mu}$. We conclude in particular that

$$\Psi_{ij}(\nabla\overline{u})\overline{u}_{ij} \leq -c\left(\lambda\sigma^{-5/2} + \left|\overline{H}''\right|\overline{H}'^{-3}\lambda^{\mu}\sigma^{-\mu}\right) + c^{-1}\overline{H}'^{-2}\lambda^{1+2\mu}\sigma^{-1/2}.$$

We can make sure this is nonpositive provided

$$c^{-2} \leq \overline{H}'^{2}(\lambda^{-1-\mu})\lambda^{-2\mu}\sigma^{-2} + \lambda^{-1-\mu}\left|\overline{H}''\right|/\overline{H}'(\lambda^{-1-\mu})\sigma^{1/2-\mu}.$$

After the change of variable $s = \lambda^{-1-\mu}$, $\tilde{\sigma} = \sigma^{1/2-\mu}$ this desired inequality becomes

$$c^{-2} \leq s|\overline{H}''|/\overline{H}'(s)\tilde{\sigma} + \overline{H}'^{2}(s)s^{\gamma}\tilde{\sigma}^{-M} := E \tag{3.28}$$

for all $s > 0$ and $\tilde{\sigma} \geq 1$. We have

$$s|\overline{H}''|/\overline{H}'(s) = \frac{A\gamma s^{-\gamma} + B\frac{s^{\delta}}{1+s^{2\delta}}e^{BI}}{As^{-\gamma} + e^{BI}},$$

and since $\overline{H}' \geq 1 + As^{-\gamma}$,

$$\overline{H}'^{2}(s)s^{\gamma} \geq A^2 s^{-\gamma} + s^{\gamma}.$$

To continue we split the verification of (3.28) into two cases. In the case $s \leq 1$ we have

$$\begin{aligned} E &\geq \frac{A\gamma s^{-\gamma-\delta} + \frac{B}{2}e^{BI}}{As^{-\gamma} + e^{BI}}s^{\delta}\tilde{\sigma} + A^2 s^{-\gamma}\tilde{\sigma}^{-M} \\ &\geq \min\{\gamma,\, 1/2\}X + A^2 X^{-M} \quad \text{(here } X = s^{\delta}\tilde{\sigma}\text{)} \\ &\geq cA^{\frac{2}{M+1}} \end{aligned}$$

for any $B \geq 1$, hence (3.28) is true in the case $s \leq 1$ for A large and any $B \geq 1$. We may take e.g. $B = A^2$. Then in the remaining case $s \geq 1$ we have

$$
\begin{aligned}
E &\geq \frac{B\frac{s^{2\delta}}{1+s^{2\delta}}e^{BI}}{A+e^{BI}}s^{-\delta}\tilde{\sigma} + s^{\gamma}\tilde{\sigma}^{-M} \\
&\geq \frac{1}{2}B\frac{e^{BI}}{A+e^{BI}}X + X^{-M} \qquad (\text{here } X = s^{-\delta}\tilde{\sigma}) \\
&\geq \frac{B}{2(A+1)}X + X^{-M} \\
&\geq \frac{1}{4}AX + X^{-M} \quad (\text{using that } B = A^2 > 1) \\
&\geq cA^{\frac{M}{M+1}},
\end{aligned}
$$

concluding the proof of (3.28) provided $B = A^2$ and A is large.

Remark 3.4.1. Roughly, the term "I" in the PDE, which represents the curvature of the level sets of $\overline{u}$, dominates in a region of $\{|y| > |x|\}$ that lies away from the boundary C_{11}, is tangent to C_{11} near 0, and separates sub-linearly from C_{11} for $|y|$ large. The term "II," which represents the remaining vertical curvature of the graph of $\overline{u}$, dominates near C_{11}.

3.4.2 Subsolution

The subsolution is similar to and in fact easier to construct than the supersolution. Let γ, M, and δ be as above. We first define

$$u_0 = \underline{H}(\underline{w}),$$

with $\underline{H}(0) = 0$ and

$$\underline{H}'(s) = e^{-C\int_s^{\infty} \frac{t^{\delta-1}}{1+t^{2\delta}}\,dt}.$$

We claim that for all $C \geq C_0$ large, there exists $\lambda_C > 0$ small such that the function u_0 is a sub-solution of (3.27) in

$$\Omega_C := \{\underline{w} > \lambda_C^{-1-\mu}\} = \{(|x|,\, |y|) = \lambda^{-1}(\tau,\, \underline{\sigma}(\tau)),\, \tau \geq 0,\, 0 < \lambda < \lambda_C\}.$$

The value λ_C is chosen for any $C \geq 1$ as follows. First, we choose $\lambda_C < 1$ small such that

$$\underline{H}'(\lambda_C^{-1-\mu}) \geq 1/2. \tag{3.29}$$

Using (3.22) we conclude that

$$|\nabla u_0| \geq c\lambda_C^{-\mu} \text{ in } \Omega_C.$$

After possibly taking λ_C smaller we thus have

$$\nabla u_0(\Omega_C) \subset \{\overline{\Psi} > 1\}.$$

This fixes our choice of λ_C for arbitrary $C \geq 1$. Since $\Psi = F(\overline{\Psi})$ in $\{\overline{\Psi} > 1\}$ we conclude in Ω_C that

$$\begin{aligned}
\Psi_{ij}(\nabla u_0)(u_0)_{ij} &= F'(\underline{H}'(\underline{w})\overline{\Psi}(\nabla \underline{w}))\overline{\Psi}_{ij}(\nabla \underline{w})\underline{w}_{ij} \\
&+ F''(\underline{H}'(\underline{w})\overline{\Psi}(\nabla \underline{w}))\underline{H}''(\underline{w})\overline{\Psi}^2(\nabla \underline{w}) \\
&+ F''(\underline{H}'(\underline{w})\overline{\Psi}(\nabla \underline{w}))\underline{H}'(\underline{w})\overline{\Psi}_i(\nabla \underline{w})\overline{\Psi}_j(\nabla \underline{w})\underline{w}_{ij} \\
&= I + II + III.
\end{aligned}$$

Using that $F' \geq 1/2$, $F''(s) = s^{-3}/4$ for $s \geq 1$, Proposition 3.2.5, the estimates (3.22) and (3.23) on $\nabla \underline{w}$ and $D^2\underline{w}$, and (3.29), we conclude on $\{(|x|,\, |y|) = \lambda^{-1}(\tau,\, \underline{\sigma}(\tau)),\, \tau \geq 0,\, 0 < \lambda < \lambda_C\}$ that

$$I \geq c\lambda\sigma^{-5/2},\, II \geq c\underline{H}''(\lambda^{-1-\mu})\lambda^{\mu}\sigma^{-\mu},\, III \geq -c^{-1}\lambda^{1+2\mu}\sigma^{-1/2}.$$

We emphasize here that c does not depend on C. We have in particular that

$$\Psi_{ij}(\nabla u_0)(u_0)_{ij} \geq c\left(\lambda\sigma^{-5/2} + \underline{H}''(\lambda^{-1-\mu})\lambda^{\mu}\sigma^{-\mu}\right) - c^{-1}\lambda^{1+2\mu}\sigma^{-1/2}$$

in Ω_C. After the same change of variable as in the previous subsection, we conclude that u_0 is a sub-solution of (3.27) in Ω_C provided

$$c^{-2} < s\underline{H}''(s)\tilde{\sigma} + s^{\gamma}\tilde{\sigma}^{-M} := E \text{ for all } s \geq \lambda_C^{-1-\mu},\ \tilde{\sigma} \geq 1. \tag{3.30}$$

Since

$$s\underline{H}''(s) = C\frac{s^{\delta}}{1+s^{2\delta}}\underline{H}'(s) \geq \frac{1}{4}Cs^{-\delta}$$

for $s \geq \lambda_C^{-1-\mu}$ (here we use (3.29) again), we have

$$\begin{aligned} E &\geq \frac{1}{4}Cs^{-\delta}\tilde{\sigma} + s^{\gamma}\tilde{\sigma}^{-M} \\ &= \frac{1}{4}CX + X^{-M} \text{ (here } X = s^{-\delta}\tilde{\sigma}) \\ &\geq cC^{\frac{M}{M+1}} \end{aligned}$$

for $s \geq \lambda_C^{-1-\mu}$, $\tilde{\sigma} \geq 1$. Thus, (3.30) holds for any choice $C \geq C_0$ large.

Remark 3.4.2. Again, the term "I" representing the curvature of the level sets of u_0 dominates in a region of Ω_C that lies away from C_{11} and separates sub-linearly from C_{11} as $|y|$ gets large, and the term "II" representing the vertical curvature of the graph of u_0 dominates in the region of Ω_C close to C_{11}.

We have shown that u_0 is a sub-solution of (3.27) in Ω_{C_0}. To get a sub-solution of (3.27) on all of $\{|y| > |x|\}$ we take a truncation of u_0. Namely, the function

$$\underline{u_0} := \max\{0,\, u_0 - \underline{H}(\lambda_{C_0}^{-1-\mu})\}$$

is a sub-solution to (3.27) on $\{|y| > |x|\}$, as is

$$\underline{u_0}_R(\cdot) := R^{-1}\underline{u_0}(R\cdot)$$

for any $R > 0$. To conclude this section we let

$$\underline{u} = \underline{u_0}_R \text{ in } \{|y| > |x|\}, \quad R = 2^{-\frac{1+\mu}{\mu}},$$

and we extend $\underline{u}$ to all of $\mathbb{R}^4$ by odd reflection over C_{11}. For this choice of R, we have in $\{|y| > |x|\}$ that

$$\underline{u} \leq \overline{u}. \tag{3.31}$$

Indeed, in $\{|y| > |x|\}$ we have

$$\begin{aligned}
\underline{u} &= R^{-1}\underline{u_0}(R\cdot) \\
&\leq R^{-1}u_0(R\cdot) \\
&= R^{-1}\underline{H}(\underline{w})(R\cdot) \\
&\leq R^{-1}\underline{w}(R\cdot) \quad (\text{since } \underline{H}' < 1) \\
&= R^{\mu}\underline{w} \quad (\text{homogeneity}) \\
&\leq \overline{w} \quad (\text{choice of } R \text{ and inequality (3.25)}) \\
&\leq \overline{H}(\overline{w}) \quad (\text{since } \overline{H}' > 1) \\
&= \overline{u}.
\end{aligned}$$

Furthermore, since $\underline{H}$ has linear growth, we have that

$$\sup_{B_r} \underline{u} \geq cr^{1+\mu}$$

for all r large.

3.5 Proof of Theorem 3.1.1

Finally, we put everything together to prove the main theorem of this chapter.

Proof of Theorem 3.1.1: The definition and uniform ellipticity of Φ were established in Section 3.3. For $k \geq 1$, solve the Dirichlet problem

$$\Psi_{ij}(\nabla u_k)\partial_i\partial_j u_k = 0, \quad u_k|_{\partial B_k} = \overline{u}.$$

We have the existence of a unique solution $u_k \in C^\infty(B_k) \cap C\left(\overline{B_k}\right)$ by the results in Section 5 of [27]. By the symmetries of the integrand Ψ and the boundary data and uniqueness, the functions u_k depend only on $|x|$ and $|y|$, and are odd over C_{11} (note that $\overline{u}$ is odd over C_{11}). In particular, they vanish on C_{11}. Using (3.31) and the maximum principle we conclude that

$$\underline{u} \leq u_k \leq \overline{u}$$

in $B_k \cap \{|y| > |x|\}$, with the reverse inequality on the other side of C_{11}. Simon's interior gradient estimate (see Section 5 in [27]) implies that for any $R > 0$ and $k > 2R$, the norm $\|u_k\|_{C^1(B_R)}$ is bounded by a constant independent of k. We can in fact replace the space $C^1(B_R)$ in this estimate by $C^m(B_R)$ for any m, using De Giorgi-Nash-Moser theory and Schauder estimates [14]. We may thus extract a subsequence of $\{u_k\}$ that converges locally uniformly along with all its derivatives to a smooth limit u which solves the equation

$$\Psi_{ij}(\nabla u)u_{ij} = 0$$

on $\mathbb{R}^4$ and furthermore satisfies

$$\underline{u} \leq u \leq \overline{u}$$

in $\{|y| > |x|\}$, with the reverse inequality otherwise. Since

$$\sup_{B_r} \underline{u} \geq cr^{1+\mu}$$

for all r large, this completes the proof. □

3.6 Discussion

In this final section we discuss how our methods recover, in a systematic way, known examples of nonlinear entire minimal graphs, and we discuss some open questions related to this work.

3.6.1 Entire minimal graphs asymptotic to $C_{kk} \times \mathbb{R}$

The above approach adapts easily to constructing nonlinear entire solutions to equations of minimal surface type whose graphs are asymptotic to $C_{kk} \times \mathbb{R}$ for any $k \geq 1$. In this subsection we outline how to do this in the case of the minimal surface equation and $k \geq 3$, both recovering the examples in [2] and giving a systematic way of building super- and sub-solutions starting from a foliation.

For x, $y \in \mathbb{R}^{k+1}$, $k \geq 3$, the minimal leaves foliating a side of the cone C_{kk} are dilations of $\{|y| = \sigma(|x|)\}$, where σ solves

$$G(\sigma) := \sigma'' + k(1 + \sigma'^2)\left(\frac{\sigma'}{\tau} - \frac{1}{\sigma}\right) = 0, \quad \sigma(0) = 1, \quad \sigma'(0) = 0.$$

The quantity G is equivalent to the mean curvature of the leaf, up to multiplying by positive constants. An analysis of this ODE similar to that done above in the more general anisotropic setting shows that for

$$\mu = (k - 1/2) - \sqrt{(k-1/2)^2 - 2k},$$

we have

$$\sigma = \tau + a\tau^{-\mu} + \mathcal{F}(\tau^{-\alpha}) \text{ for some } a > 0.$$

Here

$$\begin{aligned}\alpha &= \min\{k - 1/2 + \sqrt{(k-1/2)^2 - 2k},\, 2\mu + 1\} \\ &= \begin{cases} k - 1/2 + \sqrt{(k-1/2)^2 - 2k},\ k = 3 \\ 2\mu + 1,\ k \geq 4. \end{cases}\end{aligned}$$

From here the analysis follows the same lines. Let $\beta \in (\mu, \alpha)$. By analyzing the linearized equation at σ with right hand side $\sigma^{-\beta-2}$, one produces perturbed leaves defined by functions $(\underline{\sigma}, \overline{\sigma})$ with the asymptotic behavior $\tau + (\underline{a}, \overline{a})\tau^{-\mu} + \mathcal{F}(\tau^{-\beta})$ and $\underline{a}, \overline{a} > 0$, and mean curvature $(-G(\underline{\sigma}), G(\overline{\sigma})) \geq c\sigma^{-\beta-2}$. Define $\underline{w}, \overline{w}$ to be functions that are homogeneous of degree $1 + \mu$ with the perturbed leaves defined by $\underline{\sigma}, \overline{\sigma}$ as level sets, and then choose $\underline{H}, \overline{H}$ with linear growth and a constant K such that the minimal surface operator applied to $\overline{H}(\overline{w})$ is nonpositive and to $\max\{0, \underline{H}(\underline{w}) - K\}$ is nonnegative in $\{|y| > |x|\}$ (recall in this case that $F(s) = \sqrt{1+s^2}$). The analysis is nearly identical after changing the parameters γ, M, δ from Section 3.4 to

$$\gamma = \frac{\mu + 1/2}{\mu + 1}, \quad M = \frac{2}{\beta - \mu}, \quad \delta = \frac{1}{M(\mu+1)}.$$

As above, the key terms are a term representing the mean curvature of the level sets (we called it "I" above), which was designed to have a desired sign by perturbing the leaves in the minimal foliation, and terms involving $\underline{H}'', \overline{H}''$ (we called them "II" above) which represent a favorable curvature in the remaining "vertical" direction. These terms dominate in complementary regions of $\{|y| > |x|\}$ (see Remarks 3.4.1 and 3.4.2).

Remark 3.6.1. It is known more generally by work of Hardt-Simon that each side of any area-minimizing hypercone C with an isolated singularity is foliated by smooth area-minimizing hyperfurfaces [30]. In this more general setting, perturbing the leaves in the foliation to have mean curvature of a desired sign amounts to solving the Jacobi field equation with a source term decaying at a certain rate. It is feasible that our approach of perturbing the leaves, then stacking could produce entire minimal graphs asymptotic to $C \times \mathbb{R}$ provided the cone has appropriate symmetries; see next sub-section for issues that can arise.

3.6.2 The case C_{kl}, $k \neq l$

From Chapter 2, we know for any k, $l \geq 1$, each side of C_{kl} is foliated by minimizers of a parametric elliptic functional. Perturbing the leaves in the foliation to have anisotropic mean curvature of a desired sign works in the same way as described above for any k and l (and similarly for the foliations associated to the area-minimizing Lawson cones), as well as the construction of super- and sub-solutions to equations of minimal surface type on each side of the cone that are appropriately ordered and have comparable growth rates. However, to build entire solutions asymptotic to $C_{kk} \times \mathbb{R}$, we rely on the odd symmetry over C_{kk} of solutions to the Dirichlet problem on bounded domains. Thus, another argument is needed to produce examples asymptotic to $C_{kl} \times \mathbb{R}$ when $k \neq l$.

In [29] Simon constructs entire minimal graphs that are asymptotic to $C \times \mathbb{R}$ for a large class of area-minimizing cones C including all of the area-minimizing Lawson cones. The approach in that work is first to solve the Dirichlet problem on B_1 with large values on one side of the cone and small values on the other, and then note that appropriate rescalings and translations of the resulting graphs converge to complete (but not necessarily entire) minimal graphs. It is then delicately argued that these graphs must in fact be entire, using along the way the minimal foliations constructed in [30]. Perhaps a combination of approaches in this chapter and in [29] could elucidate what is happening, and allow the construction of entire anisotropic minimal graphs asymptotic to $C_{kl} \times \mathbb{R}$ in the remaining cases $k \neq l$ and either $k + l < 6$ or $k + l = 6$ and $\min\{k, l\} = 1$.

3.6.3 Controlled Growth Results

It is known that minimal graphs satisfying the controlled growth condition $|\nabla u| = o(|x|)$ are necessarily linear [10]. A key tool in the argument is the Simons identity for the Laplace of the second fundamental form, which has an anisotropic analogue (see [33]). Similar arguments might thus be used to prove controlled growth Bernstein theorems in the anisotropic case, assuming e.g. that $|\nabla u| = O(|x|^{\epsilon})$ for some ϵ small depending on the dimension and the integrand.

3.6.4 Closeness to Area

Finally, the known results for the area functional are robust under small C^4 perturbations of the integrand in (3.1) from area on $\mathbb{S}^n$. For example, the Bernstein theorem still holds up to dimension $n = 7$ ([28]), and the flatness of stable critical points holds in dimension $n = 3$ (this was shown in [4], [5], also for the first time in the case of the area functional). Interestingly, the latter result holds in dimension $n = 2$ under the hypothesis of C^2 closeness to the area functional [17]. It would be interesting to determine whether or not the topology in which closeness is measured could be relaxed e.g. to C^2 in the other cases. By quantifying the closeness to area of the integrands in this chapter or in Chapter 2, one could gain insights into this question.

Chapter 4

Controlled growth anisotropic Bernstein problem

4.1 Introduction

It is well known that the only entire solutions of the minimal surface equation on $\mathbb{R}^n$,

$$\operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = 0,$$

are linear functions, provided that $n \leq 7$, [1], [7], [13], [31].

In 1990, Ecker and Huisken [10] extended this theorem to all dimensions, assuming in addition that the gradient of u grows slightly slower than linearly. Their theorem says the following:

Theorem 4.1.1. *An entire smooth solution u of the minimal surface equation satisfying*

$$|\nabla u(x)| = o(\sqrt{|x|^2 + |u(x)|^2})$$

is a linear function.

If u is an entire solution to the minimal surface equation, and

$$|\nabla u| < C, \tag{4.1}$$

then u is linear. The reason is that the minimal surface equation becomes uniformly elliptic in this case, and then the Harnack inequality of De Giorgi and Nash can be applied [14]. If we assume something weaker, that

$$|u(x)| \leq C(1 + |x|), \tag{4.2}$$

then u is linear. The reason is that linear growth implies the gradient is globally bounded, and this is by the interior gradient estimate of Bombieri-De Giorgi-Miranda [3]. And in Theorem 4.1.1, the assumption is weaker than the previous examples. In fact, it is almost optimal, since in [29], Leon Simon constructed nonlinear global solutions to the minimal surface equation in high dimensions with

$$|\nabla u(x)| \leq c|x|^{1+O(1/n)}.$$

The problem appears to be open if we just assume

$$|\nabla u(x)| = O(|x|).$$

In the anisotropic case, growth conditions (4.1) and (4.2) also imply global solutions are linear. In the case of (4.1), the reason is the same as the minimal surface case; and the case (4.2) is by the interior gradient estimate of Leon Simon [27].

The chapter is organized as follows. In Section 4.2, we give a proof of Theorem 4.1.1. And in Section 4.3, we discuss a possible generalization of Theorem 4.1.1 to the anisotropic case, and also its difficulty.

4.2 Proof of Theorem 4.1.1

Let $\Sigma = \text{graph } u$. Assume without loss of generality that $0 \in \Sigma$. We let $\Sigma \cap B_R = \{(x,\, u(x)) \in \mathbb{R}^{n+1} : |x|^2 + |u(x)|^2 \le R^2\}$, $v = \sqrt{1+|\nabla u|^2}$, and $|II|$ denotes the norm of the second fundamental form of Σ. The Theorem 4.1.1 follows from the following lemma:

Lemma 4.2.1. *We have the curvature estimate:*

$$|II(0)|v(0) \le c(n)R^{-1} \sup_{\Sigma\cap B_R} v \tag{4.3}$$

for all $R > 1$.

Theorem 4.1.1 follows from this estimate and the sub-linear growth hypothesis on $|\nabla u|$:

Proof of Theorem 4.1.1. Suppose the Lemma 4.2.1 is true, then

$$\begin{aligned} |II(0)|v(0) &\le c(n)R^{-1} \sup_{\Sigma\cap B_R} \left(\sqrt{1+o(|x|^2+|u(x)|^2)}\right) \\ &\le c(n)\frac{o(R)}{R}. \end{aligned}$$

After taking $R \to +\infty$, we get $II(0) = 0$. By translation, we have the same holds at any point on Σ. □

Now it suffices to prove the Lemma 4.2.1.

Proof of Lemma 4.2.1. To prove (4.3), we recall two well-known relations for minimal surfaces, the Jacobi equation

$$L_\Sigma \nu^{n+1} = \Delta \nu^{n+1} + |II|^2 \nu^{n+1} = 0, \tag{4.4}$$

and Simons inequality [26]

$$\Delta |II|^2 \geq 2(1 + 2/n)|\nabla |II||^2 - 2|II|^4, \tag{4.5}$$

where ν^{n+1} is the $(n+1)$-th component of ν, L_Σ and Δ denote the Jacobi operator and the Laplace-Beltrami operator on Σ respectively. Notice that $\nu^{n+1} = v^{-1}$, and plug this into the equation (4.4), we arrive at

$$\Delta v = |II|^2 v + 2v^{-1}|\nabla v|^2. \tag{4.6}$$

From (4.5) and (4.6) we compute

$$\begin{aligned} \Delta(|II|^p v^q) \geq &(q-p)|II|^{p+2} v^q + p(p-1+2/n)|II|^{p-2} v^q |\nabla |II||^2 \\ &+ q(q+1) v^{q-2} |II|^p |\nabla v|^2 + 2pq|II|^{p-1} v^{q-1} \nabla |II| \cdot \nabla v. \end{aligned} \tag{4.7}$$

By Young's inequality we have for $\varepsilon > 0$ that

$$2pq|II|v\nabla|II| \cdot \nabla v \geq -pq \left[\varepsilon v^2 |\nabla |II||^2 + \varepsilon^{-1}|II|^2 |\nabla v|^2\right]. \tag{4.8}$$

Using (4.8) inequality in (4.7) we get

$$\begin{aligned} \Delta(|II|^p v^q) \geq &(q-p)|II|^{p+2} v^q + p\left[p - 1 + 2/n - \varepsilon q\right] |II|^{p-2} v^q |\nabla |II||^2 \\ &+ q\left[q + 1 - \varepsilon^{-1} p\right] v^{q-2} |II|^p |\nabla v|^2. \end{aligned}$$

We can choose ε such that each of the last two terms is non-negative provided

$$q(1-2/n) \leq p-1+2/n,$$

in which case

$$\Delta(|II|^p v^q) \geq (q-p)|II|^{p+2}v^q. \tag{4.9}$$

In particular, taking $p = q \geq \frac{n-2}{2}$ we conclude that

$$\Delta(|II|^p v^p) \geq 0.$$

Notice that mean value property also holds on minimal surfaces [14]:

$$\begin{aligned} |II|^p v^p(0) &\leq c(n) \fint_{\Sigma\cap B_R} |II|^p v^p \, d\mathscr{H}^n \\ &\leq c(n)R^{-n/2} \left(\int_{\Sigma\cap B_R} |II|^{2p} v^{2p} \, d\mathscr{H}^n \right)^{1/2} \end{aligned} \tag{4.10}$$

where we used Cauchy-Schwarz inequality and the fact that the n-dimensional Hausdorff measure on minimal graphs can be estimated by $\mathscr{H}^n(\Sigma \cap B_R) \leq c(n)R^n$ [14].

Now, in (4.9), replace p with $p-1$ and q with p, then for $p \geq n-1$ we get

$$\Delta(|II|^{p-1}v^p) \geq |II|^{p+1}v^p. \tag{4.11}$$

In order to estimate the right-hand side of (4.10), we then multiply (4.11) by $|II|^{p-1}v^p\eta^{2p}$ where η is a test function with compact support, then we obtain

$$\int_\Sigma |II|^{2p}v^{2p}\eta^{2p} \leq \int_\Sigma \Delta\left(|II|^{(p-1)}v^p\right) \cdot |II|^{p-1}v^p\eta^{2p}. \tag{4.12}$$

Denote $|II|^{(p-1)}v^p$ by f, then we have

$$\begin{aligned}\int_\Sigma (\Delta f \cdot f)\,\eta^{2p} &= -2p\int_\Sigma f\eta^{2p-1}\nabla f\cdot\nabla\eta - \int_\Sigma |\nabla f|^2\eta^{2p}\\ &\le c(p)\int_\Sigma f^2\eta^{2(p-1)}|\nabla\eta|^2 \\ &= c(p)\int_\Sigma |II|^{2(p-1)}v^{2p}\eta^{2(p-1)}|\nabla\eta|^2.\end{aligned} \tag{4.13}$$

From (4.12) and (4.13), and by using Young's inequality we get

$$\begin{aligned}\int_\Sigma |II|^{2p}v^{2p}\eta^{2p} &\le c(p)\int_\Sigma \left(|II|^{2(p-1)}v^{2(p-1)}\eta^{2(p-1)}\right)\cdot\left(v^2|\nabla\eta|^2\right)\\ &\le c(p)\left(\varepsilon\int_\Sigma |II|^{2p}v^{2p}\eta^{2p} + \varepsilon^{1-p}\int_\Sigma v^{2p}|\nabla\eta|^{2p}\right).\end{aligned}$$

Let ε be small, then we finally arrive at

$$\int_\Sigma |II|^{2p}v^{2p}\eta^{2p} \le c(p)\int_\Sigma v^{2p}|\nabla\eta|^{2p}. \tag{4.14}$$

We now choose η to be the standard cut-off function for $\Sigma\cap B_{2R}$, with $\eta\equiv 1$ in B_R and $\eta\equiv 0$ outside of B_{2R}. Then, since $p = p(n)$, we obtain from (4.14)

$$\left(\int_{\Sigma\cap B_R} |II|^{2p}v^{2p}\,d\mathscr{H}^n\right)^{1/2} \le c(n)R^{n/2}R^{-p}\sup_{\Sigma\cap B_{2R}} v^p \tag{4.15}$$

which in view of (4.10) implies estimate (4.3).

$\square$

4.3 The Anisotropic Case

The Theorem 4.1.1 does not hold for general elliptic functionals, for example in Chapter 3, the nonlinear entire solution u whose graph is a minimizer to some parametric elliptic functional, and its gradient has growth rate between (0, 1/2) at infinity. However, one of the key ingredients in the proof of the Theorem 4.1.1 is Simons identity for the Laplace of the second fundamental form, which has an anisotropic analogue (see [33]). It seems possible that one could use similar arguments to prove a controlled growth Bernstein theorem in the anisotropic case:

Conjecture 4.3.1. *Let u be a global solution to an equation of minimal surface type on $\mathbb{R}^n$, corresponding to a functional A_Φ. Then for some $\varepsilon(n, \Phi) > 0$,*

$$\sup_{B_r} |\nabla u| = O(r^\varepsilon) \Rightarrow u \text{ is linear.}$$

Below we discuss briefly what the main difficulty is in extending the arguments of Ecker-Huisken to the anisotropic case.

We first note that it can be shown that there is an anisotropic analogue for the Jacobi operator for minimal graphs. If a hypersurface Σ is a minimizer of a parametric elliptic functional A_Φ, consider $\tilde{\Sigma}$ a normal variation of Σ, defined as

$$\tilde{\Sigma} = \{\, z + \varepsilon\varphi(z)\nu(z),\ z \in \Sigma \,\},$$

where $\nu(z)$ is the unit normal of Σ at z and φ is a smooth function compactly supported on Σ. After some computation, we see that the anisotropic mean curvature of $\tilde{\Sigma}$ is, to leading order in ε, $\varepsilon L_\Phi \varphi$, where L_Φ is the anisotropic Jacobi operator

$$L_\Phi = \Delta_\Phi + |II_\Phi|^2. \tag{4.16}$$

Here Δ_Φ and $|II_\Phi|^2$ are defined by

$$\Delta_\Phi \varphi = \operatorname{div}\left(D^2\Phi(\nu)\nabla\varphi\right), \quad |II_\Phi|^2 = \operatorname{tr}\left(D^2\Phi(\nu)\cdot II^2\right).$$

Similar to the minimal surfaces case, if Σ is the graph in $\mathbb{R}^{n+1}$ of a function on $\mathbb{R}^n$, we conclude that for the $(n+1)$-th component of the unit normal ν to Σ we have

$$L_\Phi \nu^{n+1} = \Delta_\Phi \nu^{n+1} + |II_\Phi|^2 \nu^{n+1} = 0.$$

Plugging $v^{-1} = \nu^{n+1}$ into the above equation, we arrive at

$$\Delta_\Phi v = |II_\Phi|^2 v + 2v^{-1}\left((D^2\Phi(\nu)\nabla v)\cdot\nabla v\right). \tag{4.17}$$

In [33], there is a generalization of Simons inequality, which says the following:

$$\frac{1}{2}\Delta_\Phi |II_\Phi|^2 \geq \left(\frac{1-\eta}{1+\theta}\right)(1+2/n)(D^2\Phi\nabla|II_\Phi|)\cdot\nabla|II_\Phi| - (\lambda_\Phi^{-1} + C(\eta,\theta)\varepsilon_\Phi)|II_\Phi|^4 \tag{4.18}$$

for all $\eta \in (0,1]$ and $\theta > 0$ with a non-negative constant ε_Φ that tends to 0 as $\|\Phi - A\|_{C^4} \to 0$, where $A(x) = |x|$, and λ_Φ is defined as follows

$$\lambda_\Phi := \inf_{z\in\mathbb{S}^n, V\in z^\perp\setminus\{0\}} \frac{\frac{\partial^2\Phi}{\partial z^\alpha \partial z^\beta}(z)V^\alpha V^\beta}{|V|^2} > 0.$$

For simplicity, in the future we will still use Δ, II instead of Δ_Φ and II_Φ. From (4.17) and

(4.18) we compute

$$\begin{aligned}\Delta(|II|^p v^q) \geq &\left(q - (\lambda_\Phi^{-1} + C(\eta,\theta)\varepsilon_\Phi)p\right)|II|^{p+2}v^q \\ &+ p\left(p - 2 + \left(\frac{1-\eta}{1+\theta}\right)(1+2/n)\right)|II|^{p-2}v^q(D^2\Phi\nabla|II|)\cdot\nabla|II| \\ &+ q(q+1)|II|^p v^{q-2}(D^2\Phi\nabla v)\cdot\nabla v + 2pq|II|^{p-1}v^{q-1}(D^2\Phi\nabla|II|)\cdot\nabla v.\end{aligned}$$

Using Young's inequality we derive

$$\Delta(|II|^p v^q) \geq \left(q - (\lambda_\Phi^{-1} + C(\eta,\theta)\varepsilon_\Phi)p\right)|II|^{p+2}v^q \tag{4.19}$$

provided

$$\left(2 - \left(\frac{1-\eta}{1+\theta}\right)(1+2/n)\right)q \leq p - 2 + \left(\frac{1-\eta}{1+\theta}\right)(1+2/n).$$

It seems that we can continue as in [10], but the issue is that the constant λ_Φ degenerates with the ellipticity of Φ. To be more precise, in order to have

$$\Delta(|II|^p v^q) \geq \left(q - (\lambda_\Phi^{-1} + C(\eta,\theta)\varepsilon_\Phi)p\right)|II|^{p+2}v^q \geq 0,$$

we would need $q > (\lambda_\Phi^{-1} + C(\eta,\theta)\varepsilon_\Phi)p$. Combining this with the constraint from the use of Young's inequality in (4.19), we see that we would need

$$\left(2 - \left(\frac{1-\eta}{1+\theta}\right)(1+2/n)\right)(\lambda_\Phi^{-1} + C(\eta,\theta)\varepsilon_\Phi) < 1,$$

which for λ_Φ small is not possible, so it is not clear that the same strategy would prove Conjecture 4.3.1.

Bibliography

[1] F. J. Almgren. Some interior regularity theorems for minimal surfaces and an extension of bernstein's theorem. *Annals of Mathematics*, 84(2):277–292, 2023/04/20/ 1966. Full publication date: Sep., 1966.

[2] E. Bombieri, E. De Giorgi, and E. Giusti. Minimal cones and the bernstein problem. *Inventiones mathematicae*, 7(3):243–268, Sep 1969.

[3] E. Bombieri, E. De Giorgi, and M. Miranda. Una maggiorazione a priori relativa alle ipersuperfici minimali non parametriche. *Archive for Rational Mechanics and Analysis*, 32(4):255–267, Jan 1969.

[4] O. Chodosh and C. Li. Stable minimal hypersurfaces in $\mathbf{R}^4$. *arXiv e-prints*, page arXiv:2108.11462, Aug. 2021.

[5] O. Chodosh and C. Li. Stable anisotropic minimal hypersurfaces in $\mathbf{R}^4$. *arXiv e-prints*, page arXiv:2206.06394, June 2022.

[6] A. Davini. On calibrations for Lawson's cones. *Rendiconti del Seminario Matematico della Università di Padova*, 111:55–70, 2004.

[7] E. De Giorgi. Una estensione del teorema di bernstein. *Annali della Scuola Normale Superiore di Pisa - Classe di Scienze*, 19(1):79–85, 1965.

[8] G. De Philippis and E. Paolini. A short proof of the minimality of Simons cone. *Rendiconti del Seminario Matematico della Università di Padova*, 121:233–241, 2009.

[9] M. G. Delgadino, F. Maggi, C. Mihaila, and R. Neumayer. Bubbling with l2-almost constant mean curvature and an alexandrov-type theorem for crystals. *Archive for Rational Mechanics and Analysis*, 230(3):1131–1177, jul 2018.

[10] K. Ecker and G. Huisken. A Bernstein result for minimal graphs of controlled growth. *Journal of Differential Geometry*, 31(2):397 – 400, 1990.

[11] A. Figalli and F. Maggi. On the shape of liquid drops and crystals in the small mass regime. *Archive for Rational Mechanics and Analysis*, 201(1):143–207, Jul 2011.

[12] A. Figalli, F. Maggi, and A. Pratelli. A mass transportation approach to quantitative isoperimetric inequalities. *Inventiones mathematicae*, 182(1):167–211, Oct 2010.

[13] W. H. Fleming. On the oriented plateau problem. *Rendiconti del Circolo Matematico di Palermo*, 11(1):69–90, Jan 1962.

[14] D. Gilbarg and N. S. Trudinger. Elliptic partial differential equations of second order. *Springer*, 2001.

[15] H. B. Jenkins. On two-dimensional variational problems in parametric form. *Archive for Rational Mechanics and Analysis*, 8(1):181–206, Jan 1961.

[16] H. B. Lawson. The equivariant plateau problem and interior regularity. *Transactions of the American Mathematical Society*, 173:231–249, 1972.

[17] F. H. Lin. Estimates for surfaces which are stationary for an elliptic parametric integral. *Journal of Partial Differential Equations*, 3(3):78–92, 1990.

[18] C. Mooney. Entire solutions to equations of minimal surface type in six dimensions. *J. Eur. Math. Soc. 24 (2022), no*, 24(12):4353–4361, 2022.

[19] C. Mooney and Y. Yang. The anisotropic Bernstein problem. *arXiv e-prints*, page arXiv:2209.15551, Sept. 2022.

[20] F. Morgan. The cone over the clifford torus in r4 isϕ-minimizing. *Mathematische Annalen*, 289(1):341–354, Mar 1991.

[21] G. D. Philippis and F. Maggi. Regularity of free boundaries in anisotropic capillarity problems and the validity of young's law, 2014.

[22] G. D. Philippis and A. D. Rosa. The anisotropic min-max theory: Existence of anisotropic minimal and cmc surfaces, 2022.

[23] G. D. Philippis, A. D. Rosa, and F. Ghiraldin. Rectifiability of varifolds with locally bounded first variation with respect to anisotropic surface energies, 2016.

[24] A. D. Rosa and R. Tione. Regularity for graphs with bounded anisotropic mean curvature, 2020.

[25] R. Schoen, L. Simon, and F. J. Almgren. Regularity and singularity estimates on hypersurfaces minimizing parametric elliptic variational integrals. *Acta Mathematica*, 139(none):217 – 265, 1977.

[26] R. Schoen, L. Simon, and S. T. Yau. Curvature estimates for minimal hypersurfaces. *Acta Mathematica*, 134(none):275 – 288, 1975.

[27] L. SIMON. Interior gradient bounds for non-uniformly elliptic equations. *Indiana University Mathematics Journal*, 25(9):821–855, 1976.

[28] L. Simon. On some extensions of bernstein's theorem. *Mathematische Zeitschrift*, 154(3):265–273, Oct 1977.

[29] L. Simon. Entire solutions of the minimal surface equation. *Journal of Differential Geometry*, 30(3):643 – 688, 1989.

[30] H. R. Simon Leon. Area minimizing hypersurfaces with isolated singularities. *Journal für die reine und angewandte Mathematik*, 362:102–129, 1985.

[31] J. Simons. Minimal varieties in riemannian manifolds. *Annals of Mathematics*, 88(1):62–105, 1968.

[32] B. White. The mathematics of f. j. almgren, jr. *The Journal of Geometric Analysis*, 8(5):681–702, Sep 1998.

[33] S. Winklmann. Integral curvature estimates for f-stable hypersurfaces. *Calculus of Variations and Partial Differential Equations*, 23(4):391–414, Aug 2005.