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Publication Date

1965-06-11

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Submitted for publication in Physical
Review

UCRL-16193

UNIVERSITY OF CALIFORNIA
Lawrence Radiation Laboratory
Berkeley, California
AEC Contract No. W-7405-eng-48

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ABSTRACT

From the requirements of analyticity and unitarity, we prove that certain normal threshold branch points are absent on unphysical sheets of certain other ones. Also the existence of complex thresholds corresponding to unstable particles is proved, and they are shown to have the discontinuities that one would expect on the basis of "nuclear democracy." To obtain these results it was necessary to derive certain elementary discontinuity formulas, to explicitly construct the inverse of an elastic n -particle-to- n -particle S matrix, and to prove the factorization theorem for the residues of poles in multiparticle amplitudes.

I. INTRODUCTION

We investigate the sheet structure of the two-body scattering amplitude arising from the presence of normal threshold branch points. Some results proved by Olive and Zwanziger for a three-body threshold in a two-body scattering amplitude are generalized.¹⁻⁵

Zwanziger has shown that if a two-body channel (particles a and b) communicates with a three-body channel (particles c, d, and e) and if particles d and e communicate with an unstable resonance r, then the two-body amplitude, continued clockwise about the three-body threshold from the physical sheet, contains a complex branch point at $s = (m_c + m_r)^2$, and the discontinuity across the associated cut is given by the usual type of normal threshold discontinuity equation.

Olive has shown that if d and e communicate with a stable bound state b, then the normal threshold of A_{22} , i.e., $a + b \rightarrow a + b$, corresponding to the 2' branch point [$s = (m_c + m_b)^2$] is not present on the sheet reached by analytic continuation of the amplitude counterclockwise about the three-body threshold.

The phenomenon of a certain threshold singularity's being present on one sheet of a branch point and absent on another is similar to the absence of a bound-state pole in a two-body amplitude on the unphysical sheet reached by continuation around the two-body branch point. In fact, the two are intimately

related. Furthermore, the above properties of real and complex normal thresholds are closely related, since if one could perform an analytic continuation in some coupling constant⁶ so that the resonance gradually became a bound state, the complex cut would move at the same time counterclockwise about the three-body cut onto the real axis. Hence the absence of a resonance cut on the physical sheet corresponds to the absence of a normal threshold branch point on the sheet reached by counterclockwise continuation from the physical sheet around the three-particle branch point.

The above phenomena exhibit certain features of the singularity structure of the S matrix, derived from axiomatic S-matrix theory, that could not be derived from any finite set of Feynman diagrams.⁷ Whereas unitarity guarantees the absence of a bound-state pole on the first unphysical sheet, only an infinite sum of Feynman diagrams could possibly cause the pole to disappear. Hence this cancellation cannot occur in any finite order of perturbation theory that is not unitary.

We will prove the following generalization of these theorems: If a two-body channel (particles 1 and 2) communicates with an n-body channel (particles 1, 2, ..., n) such that a subset of the n particles, (m, m+1, ..., n), form (a) a resonance r or (b) a bound state b, then

(a) A_{22} continued clockwise about the n-body threshold from the physical sheet has a complex m-body threshold branchpoint at $s = (m_1 + \dots + m_{m-1} + m_r)^2$, where m_r is the complex mass of the resonance. The corresponding elementary discontinuity formula will also be derived.

(b) The m-body normal threshold branchpoint of A_{22} at $s = (m_1 + \dots + m_{m-1} + m_b)^2$, where m_b is the mass of the bound state, does not appear on the sheet reached by continuation counterclockwise about the n-body threshold

from the physical sheet.

As an example, consider the $\pi^0 - \pi^0$ scattering amplitude. Since $A_{3\pi \rightarrow 3\pi}^{I=0}$ has a resonance pole at $s = m_\omega^2$, and $A_{3\pi \rightarrow 3\pi}^{I=1}$ has a bound-state pole at $s = m_\pi^2$, analytic continuation of $A_{2\pi^0 \rightarrow 2\pi^0}$ clockwise about the four-body threshold $(\pi\pi\pi\pi)$ leads to a complex normal threshold at $s = (m_\pi + m_\omega)^2$, and continuation of $A_{2\pi^0 \rightarrow 2\pi^0}$ counterclockwise about the same threshold leads to a sheet on which $s = 4m_\pi^2$ is a regular point (see Fig. 1).

Basically, we accept the axiomatic formulation of S-matrix theory due to Olive.³ Thus we take as postulates: (a) physical unitarity, (b) connectedness structure, (c) prescription for normal thresholds, (d) maximal analyticity. We also accept (e) Hermitian analyticity and (f) the normal threshold structure hypothesis, which presumably can be derived from the previous postulates. We make a strenuous effort never to use unphysical unitarity equations, i.e., unitarity-like equations for amplitudes below their physical thresholds. An example of such an equation is

$$\text{---} \bigcirc \text{+} \text{---} - \text{---} \bigcirc \text{-} \text{---} = \text{---} \bigcirc \text{+} \text{---} \bigcirc \text{-} \text{---}$$

If such equations are correct, they should be derivable from the above axioms. Moreover, we do not wish to worry about difficulties that may arise in multi-particle amplitudes due to anomalous threshold singularity structure, which is poorly understood. Physical unitarity together with Hermitian analyticity prescribes the way in which one should treat all singularities, anomalous thresholds included. It is much less clear how to treat anomalous thresholds in unphysical unitarity equations, therefore we shall try to avoid using such equations. We shall consider only spinless particles -- the complications

introduced by spin are quite inessential to our arguments.

In Section II we introduce some notation and conventions. In Section III, elementary discontinuity formulas are derived. The derivations are somewhat more involved than the conventional ones, since unphysical unitarity is not employed. Section IV contains the derivation of an explicit form for the inverse of an elastic n -body S matrix. This section may be read independently of the rest of the paper. In Section V we extend Stapp's proof of the pole-factorization theorem by use of the inversion. Unstable-particle scattering amplitudes are then defined as residue factors. In Sections VI and VII we prove the generalizations of the theorems of Olive and Zwanziger(cited above) respectively. In Section VIII some further generalizations are discussed.

II. PATH NOTATION AND n -BODY PHASE SPACE

We shall use a path notation for labeling the various sheets of multiparticle amplitudes. This notation, introduced by Olive,² labels the various sheets by means of the paths that lead to them from the physical sheet. An n -body threshold of an amplitude in an energy variable s is denoted by n ; n also denotes a path that encircles the threshold n in a counterclockwise direction. Also, n^{-1} denotes the path around n in a clockwise direction; mn is the path that encircles the branch point n first and then the branch point m . (See Fig. 2.)

An $m \rightarrow n$ scattering amplitude has many energy variables. There is the total energy (squared) s , initial subenergies $s_{m-1}, \dots, s_2$, and final subenergies $s'_{n-1}, \dots, s'_2$, where s_k denotes the energy variable of a k -particle subchannel. (An n -body channel has $\binom{n}{k}$ k -body subenergies, of

course.) There are also crossed-channel variables which we do not explicitly display. We use the symbol T to denote the transition amplitude, so that $S = I + iT$, and the symbol A_{mn} to denote the connected part of T_{mn} . We adopt the following ordering of variables: place the initial subenergy variables on the left in "ascending order," the total energy variable in the middle, and the final subenergy variables on the right in "descending order," e.g.,

$$A_{22} = A_{22}(s), \quad \text{---} \quad (2:1a)$$

$$A_{23} = A_{23}(s; s'_2), \quad (2:1b)$$

$$\dots\dots\dots,$$

$$A_{m,n} = A_{mn}(s_2, \dots, s_{m-1}; s; s'_{n-1}, \dots, s'_2). \quad (2:1c)$$

We use the above ordering of variables, but in order to display the sheet structure, we employ path notation, replacing each variable by the path by which it is reached from the physical sheet. Thus 1 denotes a point on the physical sheet, which may for convenience be taken arbitrarily close to the real axis, i.e., the physical region. For the physical amplitudes we would write

$$A_{22} = A_{22}(1), \quad A_{33} = A_{33}(1;1;1), \quad \text{etc.} \quad (2:2)$$

Physical unitarity above the n -body threshold relates $A_{22}(1)$ and

A_{22}^* :

$$A_{22}(1) - A_{22}^* = \sum_{i=2}^n \int d\phi_i A_{2i}(1;1, \dots) A_{2i}^* , \quad (2:3)$$

where $\int d\phi_n$ is n-body phase space, defined by

$$\int d\phi_n(s) = i \int \delta^4(p - \sum_{i=1}^n p_i) \prod_{j=1}^n \delta^+(p_j^2 - \mu^2) \prod_{k=1}^n d^4 p_k . \quad (2:4)$$

For convenience of notation we set all masses equal. Letting s_m denote the invariant energy squared of particles $(1, 2, \dots, m)$, $[s_m = (\sum_{n=1}^m p_i)^2]$, we have

$$\int d\phi_2(s) = \frac{1 q_2}{4s^{1/2}} \int d^2 \hat{q}_2 \quad (2:5a)$$

and

$$\int d\phi_n(s) = \int_{(n-1)^2 \mu^2}^{(s^{1/2} - \mu)^2} ds_{n-1} \frac{q_n}{4s^{1/2}} \int d^2 \hat{q}_n \int d\phi_{n-1}(s_{n-1}), \quad (2:5b)$$

where

$$q_n = \frac{[s - (s_{n-1} + \mu)^2][s - (s_{n-1} - \mu)^2]}{2s^{1/2}}^{1/2} . \quad (2:5c)$$

III. ELEMENTARY DISCONTINUITY EQUATIONS

We now derive for the continuation of A_{22} around the n-body threshold the operation

$$A_{22}(1) - A_{22}(n) = \int d\phi_n A_{2n}(1;1, \dots, 1) A_{n2}^u , \quad (3:1)$$

where A_{n2}^u is defined by the equation

$$A_{n2}(1, \dots, 1; 1) = \int d\varphi_n S_{nn}(1, \dots, 1; 1; 1, \dots; 1) A_{n2}^u. \quad (3:2)$$

In Section IV we prove that S_{nn} is invertible, so that (3:2) defines A_{n2}^u uniquely.

Physical unitarity for A_{22} with the total energy s above the n -body threshold is given by (2:3). We want to find the path notation expression for A_{2k}^* , i.e., to relate A_{2k}^* to an analytic continuation of $A_{k2}(1, \dots, 1; 1)$. This is done, as usual, by invoking Hermitian analyticity,

$$A_{ab}^*(s + i\epsilon) = A_{ba}(s - i\epsilon), \quad (3:3)$$

so that A_{2k}^* corresponds to taking all energy variables with a negative infinitesimal imaginary part in A_{k2} . In other words, it is the boundary value of A_{k2} continued along a path that goes counterclockwise around all the lower-energy branch points (including crossed variables that we do not explicitly exhibit). Thus

$$A_{22}^*(1) = A_{22}(n!) \text{ for } n^2\mu^2 < s < (n+1)^2\mu^2, \quad (3:4)$$

where $n!$ represents the path $n \cdot n-1 \cdot n-2 \dots 3 \cdot 2$ (order is important!).

$A_{22}(n!)$ can then be continued away from the real axis.

When $n^2\mu^2 < s < (n+1)^2\mu^2$, the physical values of the two-body subenergies of A_{32}^* vary from $4\mu^2$ up to $(s^{1/2} - \mu)^2$. Thus the two-body subenergy $s_{2,1}$ is on the sheet $k!$ when $k^2\mu^2 < s_{2,1} < (k+1)^2\mu^2$; k can take the

values 2, 3, ..., n-1. Therefore we must introduce a path notation for the subenergy variables such that the path itself depends on the value of the subenergy. Therefore we write

$$A_{32}^*(1,1) = A_{2,3}(n! ; r_{2,1}^n), \quad (3:5)$$

where

$$r_{2,1}^n = \begin{cases} 2! & \text{for } (2\mu)^2 < s_{2,1} < (3\mu)^2 \\ 3! & \text{for } (3\mu)^2 < s_{2,1} < (4\mu)^2 \\ \vdots & \\ (n-1)! & \text{for } [(n-1)\mu]^2 < s_{2,1} \end{cases}$$

In $A_{k2}^*(1, \dots, 1; 1)$ a k-1 particle subenergy varies from $(k-1)^2\mu^2$ up to $(s^{1/2} - \mu)^2$, a (k-2) particle subenergy from $(k-2)^2\mu^2$ up to $(s^{1/2} - 2\mu)^2, \dots$, a two-body subenergy from $(2\mu)^2$ up to $[s^{1/2} - (k-2)\mu]^2$. Therefore we introduce the path $r_{j,1}^n$ that denotes the sheet of the subenergy $s_{j,1}$ (the i-th j-particle subenergy) in $A_{k,2}^*$ above the n-body threshold:

$$r_{j,1}^n = \begin{cases} j! & j^2\mu^2 < s_{j,1} < (j+1)^2\mu^2 \\ (j+1)! & (j+1)^2\mu^2 < s_{j,1} < (j+2)^2\mu^2 \\ \dots & \dots \\ (n+j-k)! & (n+j-k)^2\mu^2 < s_{j,1} \end{cases} \quad (3:6)$$

where $j = 2, 3, \dots, k-1$ and $n \geq k$. Thus for $n^2\mu^2 < s < (n+1)^2\mu^2$,

$$A_{k,l}^*(1, \dots, 1; 1, \dots, 1) = A_{l,k}(r_{2,1}^n, \dots, r_{l-1,1}^n; n!; r_{k-1,1}^n, \dots, r_{2,1}^n). \quad (3:7)$$

Note that if $n = k$, then $r_{j,1}^n = j!$, so that

$$A_{k2}^*(1, \dots, 1; 1) = A_{2k}(n!; (n-1)!, \dots, 2!). \quad (3:8)$$

We can now analytically continue physical unitarity equations, using (3:7) to eliminate Hermitian adjoints. Consider, for example, physical unitarity for A_{22} when $(n-1)^2 \mu^2 < s < n^2 \mu^2$:

$$\begin{aligned} A_{22}(1) - A_{22}[(n-1)!] &= \int d\varphi_2 A_{22}(1) A_{22}[(n-1)!] + \int d\varphi_3 A_{23}(1; 1) A_{32}[r_{2,1}^{n-1}, (n-1)!] \\ &+ \dots + \int d\varphi_{n-1} A_{2,n-1}(1; 1 \dots 1) A_{n-1}(r_{2,1}^{n-1}, \dots, r_{n-2,1}^{n-1}; (n-1)!). \end{aligned} \quad (3:9)$$

We wish now to continue this equation around the path n in the total energy.

Note that it is not enough merely to continue the integrand in the total energy around n , since in the subenergy integrations parts of the contours will be dragged around thresholds. This will have the effect of changing $A_{k,2}[r_{2,1}^{n-1}, \dots, r_{k-1,1}^{n-1}; (n-1)!]$ to $A_{k,2}(r_{2,1}^n, \dots, r_{k-1,1}^n; n!)$, which is just $A_{2,k}^*$ for $s > n^2 \mu^2$. Also, $A_{2k}(1; 1, \dots, 1)$ will become $A_{2k}(n; p_{k-1,1}^n, \dots, p_{2,1}^n)$, where

$$p_{j,1}^n = \begin{cases} 1 & \text{for } s_{j,1} < (n-1+j-k)^2 \mu^2 \\ n+j-k & \text{for } s_{j,1} > (n-1+j-k)^2 \mu^2, \end{cases} \quad (3:10)$$

where $j = 2, \dots, k-1$.

Notice that

$$p_{j,i}^n r_{j,i}^{n-1} = r_{j,i}^n \quad (3:11)$$

Therefore (3:9) becomes

$$\begin{aligned} A_{22}(n) - A_{22}(n!) &= \int d\varphi_2 A_{22}(n) A_{22}(n!) + \int d\varphi_3 A_{23}(n; p_{2,1}^n) A_{32}(r_{2,1}^n; n!) \\ &+ \dots + \int d\varphi_{n-1} A_{2,n-1}(n; p_{n-2,1}^n, \dots, p_{2,1}^n) A_{n-1,2}(r_{2,1}^n, \dots, \\ &r_{n-2,1}^n; n!) \quad (3:12) \end{aligned}$$

Subtracting (3:12) from the equation for physical unitarity above the n-body threshold gives

$$\begin{aligned} A_{22}(1) - A_{22}(n) &= \int d\varphi_2 [A_{22}(1) - A_{22}(n)] A_{22}(1) + \int d\varphi_3 [A_{23}(1;1) - A_{23}(n; p_{2,1}^n)] A_{32}(r_{2,1}^n, \\ &n!) + \dots + \int d\varphi_{n-1} [A_{2,n-1}(1;1, \dots) - A_{2,n-1}(n; p_{n-1,1}^n, \dots, p_{2,1}^n)] \\ &A_{n-1,2}(r_{2,1}^n, \dots, n!) + \int d\varphi_n A_{2n}(1;1, \dots) A_{n2}(r_{2,1}^n, \dots, r_{n-1,1}^n; n!) \quad (3:13) \end{aligned}$$

Define

$$\begin{aligned} X_{22} &= A_{22}(1) - A_{22}(n) - \int d\varphi_n A_{2n}(1;1, \dots) A_{n2}^u \\ &\dots \dots \dots \\ X_{2,n-1} &= A_{2,n-1}(1;1, \dots) - A_{2,n-1}(n; p_{n-2,1}^n, \dots, p_{2,1}^n) \\ &- \int d\varphi_n A_{2,n-1}(1;1, \dots) T_{n,n-1}^u \quad (3:14) \end{aligned}$$

and derive

$$X_{2,k} = \int d\varphi_2 X_{22} A_{k2}^* + \int d\varphi_3 X_{23} T_{k,3}^* + \dots + \int d\varphi_{n-1} X_{2,n-1} T_{k,n-1}^* . \quad (3:20)$$

In matrix notation (3:18) and (3:20) become

$$(X_{22}, X_{23}, \dots, X_{2,n-1}) \begin{pmatrix} S_{22}^* & \dots & S_{n-1,2}^* \\ S_{23}^* & \dots & \dots \\ \dots & \dots & \dots \\ S_{2,n-1}^* & \dots & S_{n-1,n-1}^* \end{pmatrix} = 0 . \quad (3:21)$$

Even though the above matrix is not unitary [for $S > n^2 \mu^2$ the matrix $S^+ = \begin{pmatrix} S_{22} & \dots & S_{n2} \\ \dots & * & \dots \\ S_{2n} & \dots & S_{nn} \end{pmatrix}$ is unitary], we know that it is invertible. Indeed, its inverse is just

$$\begin{pmatrix} S_{22}(n) & S_{23}(n; p_{2,1}^n) & \dots & S_{2,n-1}(n; p_{n-2,1}^n, \dots) \\ \dots & \dots & \dots & \dots \\ S_{n-1,2}(\dots, p_{n-2,1}^n; n) & \dots & S_{n-1,n-1}(p_{2,1}^n, \dots; n; p_{n-2,1}^n, \dots) \end{pmatrix},$$

as can be seen from examining the continuation of $\sum_{j=2}^{n-1} \int S_{j,i}^* S_{j,k} d\varphi_j = \delta_{ik}$ from below the n -body threshold to above it. We therefore conclude that $X_{2i} = 0$, $i = 1, \dots, n-1$. In particular,

$$A_{22}(1) - A_{22}(n) = \int d\varphi_n A_{2n}(1; 1, \dots, 1) A_{n2}^u . \quad (3:22)$$

If we were to use unphysical unitarity for A_{2n} , i.e., discontinuity equations for A_{2n} around $(n - 1)$ -body thresholds which are not directly derivable from physical unitarity, then we could identify $A_{n,2}^u$ with $A_{n,2}(2,3,\dots, n-1;n)$ and (3:22) would have been easier to prove. One can presumably justify unphysical unitarity by isolating pole terms in crossed energies in physical unitarity equations for multiparticle amplitudes in the same manner as Oliver justifies Hermitian analyticity. To do this for multiparticle amplitudes would be very difficult, if possible at all. But as Olive¹ has emphasized, this is not necessary. It is sufficient for our purposes to define $A_{n,2}^u$ by the integral equation

$$A_{n,2}(1) = \int S_{n,n}(1) A_{n,2}^u d\varphi_n. \quad (3:23)$$

In the next section we invert this equation, thereby showing that $A_{n,2}^u$ is uniquely defined. In Section V we investigate some of its properties.

IV. INVERSION OF S_{nn}

In a sense it is obvious that S_{nn} should be invertible, since if we were dealing with n particles having a certain positive quantum number (e.g., they are all positively charged), then $S_{mn} = \delta_{mn} S_{nn}$ and the S -matrix just

above the n -body threshold is diagonal. Thus in this case S_{nn} is unitary and hence invertible. If S has nondiagonal terms it is not immediately clear what S_{nn}^{-1} is, but one can still reasonably expect it to exist, since it still has the same connectedness structure.

We restrict consideration for the time being to n distinguishable particles, labeled $1, 2, \dots, n$. S_{nn} contains many terms, the general one being denoted by

$$A(i_1 \dots i_{N_1}; i_{N_1+1} \dots i_{N_2}; \dots; i_{N_{m-1}+1} \dots i_{N_m} | i_{N_m+1} \dots i_n), \quad (4:1)$$

where $(i_1 \dots i_n)$ is any permutation of $(1 \dots n)$. This term has the connectedness structure shown in Fig. 3.

Notice that since $S_{nn}(1)$ entered our equations via physical unitarity in the region $n^2 \mu^2 < s < (n+1)^2 \mu^2$, $S_{nn}(1)$ will not contain "inelastic" diagrams such as (This type of diagram appears for the first time in S_{55} .) Such diagrams can be physical only for $s > (n+1)^2 \mu^2$. Therefore they are omitted in the inversion. The sets of lines $H_k = (i_{N_{k-1}+1} \dots i_{N_k})$, $k = 1, 2, \dots, m$, are completely disjoint from one another, but the lines of each set H_k are connected. ($N_0 = 1$ and $N_{m+1} = n$.) The lines $i_{N_{m+1}} \dots i_n$ are disconnected. Each set H_k has at least two lines in it. Now

$$A(i_1 \dots i_{N_1}; \dots i_{N_m} | i_{N_{m+1}} \dots i_n) = A(i_1 \dots i_{N_1}) \dots A(i_{N_{m-1}+1} \dots i_{N_m}) I(i_{N_{m+1}} \dots i_n), \quad (4:2)$$

where $A(1 \dots k)$ is the connected part of the k -body S matrix S_{kk} ; $I(1 \dots k)$ is the identity operator in the space of k particles, i.e., a product of momentum δ functions. Furthermore

(a) A is invariant under any permutation of the lines in any set H_k ,

$k = 1, \dots, m+1.$

(b) A is invariant under any permutation of the sets H_k , $k = 1, \dots, m.$

If the sum is understood to include only distinct terms in the sense of (a) and (b), we then have

$$S_{nn}(1 \dots n) = \sum_{\text{perm.}} A(i_1 \dots i_{N_1}; \dots; i_{N_{m-1}+1} \dots i_{N_m} | i_{N_{m+1}} \dots i_n). \quad (4:3)$$

The inversion of S_{nn} is carried out by factorizing it into a product of terms each of which is invertible by Fredholm techniques, i.e., terms of the form $I + C$ where C is connected.

$$S_{22}(12) = I(12) + iA_{22}(12) \quad (4:4)$$

is already an invertible form. In fact, for $(2\mu)^2 < s < (3\mu)^2$,

$$S_{22}^{-1}(12) = I - iA_{22}^*(12). \quad (4:5)$$

S_{33} is somewhat more complicated:

$$S_{33}(123) = I(123) + iA(12|3) + iA(13|2) + iA(32|1) + iA(123). \quad (4:6)$$

Following Olive,² we notice that the expression

$$P_3(123) = [I(123) + iA(12|3)][I(123) + iA(13|2)][I(123) + iA(32|1)] \quad (4:7)$$

and S_{33} contain the same disconnected parts. Therefore

$$S_{33} = P_3(123)[I + iK_3(123)], \quad (4:8)$$

where

$$K_3(123) = [P_3(123)]^{-1} A'(123), \quad (4:9)$$

$$\begin{aligned} A'(123) = S_{33}(123) - P_3(123) = & A(123) - A(12|3)A(13|2) - A(12|3)A(32|1) \\ & - A(13|2)A(32|1) - A(12|3)A(13|2)A(32|1). \end{aligned} \quad (4:10)$$

But $A'(123)$ is connected and hence $K_3(123)$ is also connected. In bubble notation,

It may appear that A' has terms containing δ functions, since

$$A(12|3)A(13|2) = -2\pi i \delta(u - \mu^2) A(12)A(13), \quad (4:11)$$

where $u = (p_1 + p_2 - p'_2)^2$. But $A(123)$ has a pole at $u = \mu^2$ (this can be shown from unitarity) such that near $u = \mu^2$,

$$A(123) \approx \frac{A(12)A(13)}{u - \mu^2 + i\epsilon}. \quad (4:12)$$

The effect of subtracting expression (4:11) is to change this to

$$A'(123) \approx \frac{A(12)A(13)}{u-m^2-i\epsilon}, \quad (4.13)$$

i.e., the opposite boundary value in the u plane.

We next generalize the factorization (4.8) to S_{nn} by induction.

In order to have a more uniform notation, define

$$K_2(1j) = A_{22}(1j). \quad (4.14)$$

Define

$$\begin{aligned} P_j(1 \cdots j) &= \Pi[1+iK_2(i_1 i_2 | i_3 \cdots i_j)] \times \Pi[1+iK_3(i_1 i_2 i_3 | i_4 \cdots i_j)] \\ &\times \cdots \times \Pi[1+iK_{j-1}(i_1 \cdots i_{j-1} | i_j)] \end{aligned} \quad (4.15)$$

inductively together with the definition

$$\begin{aligned} K_j(1 \cdots j) &= P_j^{-1}(1 \cdots j) \{S_{jj}(1 \cdots j) - P_j(1 \cdots j)\} \\ &= P_j^{-1}(1 \cdots j) A'_j(1 \cdots j). \end{aligned} \quad (4.16)$$

Our induction hypothesis is that for $j = 1, \dots, n$ we have

$$S_{jj}(1 \cdots j) - [P_j(1 \cdots j)]_{\text{disc}} = iA_{jj}(1 \cdots j), \quad (4.17)$$

where "disc" indicates that only the disconnected parts are to be included.

This implies that for $j = 1, 2, \dots, n$, $A'_j(1 \cdots j)$ and $K_j(1 \cdots j)$ are connected.

Our theorem will be proved, therefore, if we can show that (4:17) is valid for $j = n + 1$.

Consider a particular disconnected term in $S_{n+1, n+1}, A(H_1, H_2, \dots, H_k | H_{k+1})$, where the H_i represent sets of lines as before. Clearly this is the only term in $S_{n+1, n+1}$ that has the sets $H_1, H_2, \dots, H_k$ connected and disjoint, and the set H_{k+1} completely disconnected. We can write

$$A(H_1 \dots H_k | H_{k+1}) = A(H_1) \dots A(H_k) I(H_{k+1}), \quad (4:18)$$

where

$$A(H_j) = A(i_{N_{j-1}+1} \dots i_{N_j}) \quad (4:19a)$$

and

$$I(H_{k+1}) = I(i_{N_k+1} \dots i_{N_{k+1}}). \quad (4:19b)$$

All we have done so far is to rewrite (4:2) in a somewhat more convenient notation. If we look for the terms with this connectedness structure in $P_{n+1}(1 \dots n+1)$, as defined by (4:15), and denote them by $P_{n+1}(H_1 \dots H_k, H_{k+1})$, then factors such as $K_\ell(i_1 \dots i_\ell | i_{\ell+1} \dots i_{n+1})$ can contribute only if $(i_1, \dots, i_\ell) \subseteq H_j$ for $j \in (1, 2, \dots, k)$. Since terms whose connected lines belong to disjoint sets commute; i.e., if $(i_1, \dots, i_j)_n (i'_1, \dots, i'_k)_n = 0$, then

$$\begin{aligned} K_j(i_1 \dots i_j | i_{j+1} \dots i_{n+1}) K_k(i'_1 \dots i'_k | i'_{k+1} \dots i'_{n+1}) = \\ K_k(i'_1 \dots i'_k | i'_{k+1} \dots i'_{n+1}) K_j(i_1 \dots i_j | i_{j+1} \dots i_{n+1}). \end{aligned} \quad (4:20)$$

Therefore

$$P_{n+1}(H_1 \cdots H_k | H_{k+1}) = [\prod_{j=1}^k R_j(H_j)], \quad (4:21)$$

where $R_j(H_j)$ denotes the connected part of the product of all those factors $1+1 K_\ell(i_1 \cdots i_\ell | i_{\ell+1} \cdots i_{n+1})$ in (4:15) for which $(i_1 \cdots i_\ell) \subseteq H_j$, taken in the same order as they appear in (4:15). By "connected part" we mean connected with respect to the lines in the set H_j . By use of our induction hypothesis we have

$$\begin{aligned} R_j(H_j) &= \{P_{N_j - N_{j-1}}(H_j) [1 + 1 K_{N_j - N_{j-1}}(H_j | H_1 \cdots H_{j-1} H_{j+1} \cdots H_{k+1})]\}_{\text{conn}} \\ &= 1 P_{N_j - N_{j-1}}(H_j) K_{N_j - N_{j-1}}(H_j | H_1 \cdots H_{j-1} H_{j+1} \cdots H_{k+1}) \\ &\quad + [P_{N_j - N_{j-1}}(H_j)]_{\text{conn}} I(H_1 \cdots H_{k+1}). \end{aligned} \quad (4:22)$$

But for $j = 1, 2, \dots, n$, formulas (4:16) and (4:17) imply

$$P_j(1 \cdots j) K_j(1 \cdots j) = A_{jj}(1 \cdots j) - [P_j(1 \cdots j)]_{\text{conn}} \quad (4:23)$$

and

$$R_j(H_j) = A(H_j | H_1 \cdots H_{j-1} H_{j+1} \cdots H_{k+1}). \quad (4:24)$$

Combining (4:21, 22, 23) we have

$$P_{n+1}(H_1 \cdots H_k | H_{k+1}) = A(H_1) \cdots A(H_k) I(H_{k+1}). \quad (4:25)$$

Hence we have proved (4:17) for $j = n + 1$, and thus $K_{n+1}(1 \dots n + 1)$ is connected. $P_{n+1}(1 \dots n + 1)$ is a product of terms each of which is invertible, and

$$\begin{aligned} S_{n+1,n+1} &= P_{n+1}(1 \dots n+1) [I + i K_{n+1}(1 \dots n+1)] \\ &= \Pi [1 + i K_2(i_1 i_2 | i_3 \dots i_{n+1})] \dots [1 + i K_{n+1}(1 \dots n+1)], \end{aligned} \quad (4:26)$$

Hence we can conclude that $S_{n+1,n+1}$ is invertible.

We have now shown that it is possible to factorize S_{nn} into a product of terms each of which has the form "one plus connected part." Just as in the three-body case, the Fredholm invertibility of S_{nn} follows if each of the connected parts is free from crossed-energy δ functions. It turns out that for $n \geq 4$ the situation is much more complicated. First, there are multiple pole terms, such as the one shown in Fig. 4. Second, the ordering of the factors is crucial. The correct connectedness structure is assured no matter how we order the factors of S_{nn} , provided that the definition of K_m is determined by the relative positions of all the factors $1 + i K_\ell$, $\ell < m$, which act in the same m -body space. However, of all the possible orderings, only some lead to expressions free from troublesome δ functions. In the appendix we display two orderings that work for $n = 4$. The ordering for the general case has not yet been worked out, but we are confident that at least one such ordering always exists. Of course, the inverse is the same however it is obtained.

It is a straightforward task to generalize the preceding discussion to the case in which some of the particles are identical. One can show that the inverse of the symmetrized S-matrix is the inverse we obtained above appropriately symmetrized. In this case the S-matrix is a sum of products, each of which treats subenergies asymmetrically.

When the quantum numbers are such that there is only elastic scattering, S_{nn} is a unitary operator for all n . Then one can show step by step that $1 + iK_2$, $1 + iK_3$, ..., $1 + iK_n$, are all unitary operators. In this case the operator $1 + iK_n$ is unitary, has a simple connectedness structure, and contains all the n -body physics. One might therefore think that K_n is suitable for formulating dynamical equations (e.g., for 3π scattering in the elastic region). Unfortunately, it turns out that K_n is not Hermitian analytic, since initial and final subenergies are treated differently. Therefore the unitarity relation does not provide a discontinuity equation. Attempts to get around this problem seem to create other difficulties.⁸

V. INVESTIGATION OF A_{2n}^u AND DEFINITION OF UNSTABLE PARTICLE SCATTERING AMPLITUDES

We have shown that the continuation of the two-body scattering amplitude around an n -body branch point is given in terms of a function A_{2n}^u , which is defined by

$$A_{2n}^u = A_{2n}(1) S_{nn}^{-1}(1) . \quad (5:1)$$

In order to investigate the properties of $A_{22}(n)$, one must therefore know the properties of A_{2n}^u . Most important is whether A_{2n}^u has subenergy poles, since

they would pinch endpoints of integration contours, thereby causing either complex normal thresholds to appear in $A_{22}(n)$ (Section VI) or stable-particle normal thresholds to disappear (Section VII).

Consider the case in which $n-1$ particles, $2, 3, \dots, n$, say, communicate with a stable particle b so that there is a channel $2' = (1, b)$ that communicates with channel 2. In this case $A_{2n}(1; 1, \dots)$ has a pole in an $(n-1)$ -body subenergy. We demonstrate that this pole is absent in A_{2n}^u .

Recall that⁹

$$S_{nn} = \Pi(1 + iK_2) \dots \prod_{i=2}^n (1 + iK_{n-1}^i)(1 + iK_n) \quad (5:2)$$

and

$$K_n = \prod_{i=1}^n (1 + iK_{n-1}^i)^{-1} \dots \Pi(1 + iK_2)^{-1} \{A_{nn} - [\Pi(1 + iK_2) \dots \prod_{i=2}^n (1 + iK_{n-1}^i)]\}_{\text{conn}}, \quad (5:3)$$

where $K_{n-1}^i = K_{n-1}(1, 2, \dots, i-1, i+1, \dots, n|i)$ in the previous notation. It is clear that we can choose such an order for the factors in S_{nn} .

Suppose that A_{nn} has a pole in its total energy variable s . Since $1 + iK_n$ is the only factor in (5:2) depending upon s , it must contain the pole. Furthermore $1 + iK_n$ has the proper connectedness structure so that its Fredholm determinant exists. If we assume that $\det(1 + iK_n)^{-1}$ has a simple zero at the pole position, then the formula

$$(1 + iK_n)^{-1} \text{adj}(1 + iK_n)^{-1} = I \det(1 + iK_n)^{-1} \quad (5:4)$$

implies that $\text{adj}(1 + iK_n)^{-1}$ is rank one at the pole position, since

$(1 + iK_n)^{-1}$ has nullity one and $\det(1 + iK_n)^{-1}$ vanishes there. This means that we can write

$$\text{adj}(1 + iK_n(s_1; s; s'_1))^{-1} = f_{nl}(s_1, s) f_{ln}(s; s'_1). \quad (5:5)$$

We conclude that the residue of the pole in A_{nn} factorizes, thereby generalizing Stapp's proof¹⁰ to multiparticle amplitudes.

A_{nn} and $A_{n-1, n-1}$ both contain poles in the final subenergy $s_{n-1}^1 = m_b^2$ and their residues factorize; therefore K_{n-1} contains the pole and so does K_n . The residues of K_n and K_{n-1} must contain a common factor $A_{b, n-1}$, in view of (5:3). Define H , the Fredholm resolvent, by

$$I - iH = (1 + iK_n)^{-1}. \quad (5:4)$$

In other words, H is the solution of

$$K_n - H = \int H K_n d\phi_n. \quad (5:5)$$

It is clear from (5:5) that H contains the same final subenergy pole as K_n .

Now

$$A_{2n}^u = A_{2n}(1)[I - iH](1 + iK_{n-1}^1)^{-1} \prod_{i=2}^n (1 + iK_{n-1}^i)^{-1} \dots \prod (1 + iK_2^i)^{-1}. \quad (5:6)$$

Since the pole is in A_{2n} and H , the only terms that could carry it through to the right in (5:6) are those which are disconnected in line 1. Keeping only these terms, we find

$$\text{Res } A_{2n}^u(n) = \left[\int \cdots A_{b,n-1} \times S_{n-1,n-1}^{-1}(1) d\phi_{n-1} \right]_{s_{n-1}^1 = m_b^2} = 0, \quad (5:7)$$

as can be verified from the unitarity equation for $A_{n-1,n-1}$ at the pole position. Hence A_{2n}^u is regular at $s_{n-1}^1 = m_b^2$.

Let us next consider the case of a resonance pole, r . Such a pole is not present on the physical sheet, of course. We now show that it does appear in A_{2n}^u because S_{nn}^{-1} contains it.

In dealing with unstable particles we want to continue clockwise around normal thresholds -- so as to reach the unphysical sheet closest to the physical region. The complex poles on this sheet are the ones with experimental significance. In view of Mermitian analyticity it is obvious that all our previous formulae would apply if paths referred to clockwise continuation and phase-space integration referred to the opposite (bottom) sides of the cuts. In other words we could replace

$$n \leftrightarrow n^{-1}, \quad n \cdot m \leftrightarrow m^{-1} \cdot n^{-1},$$

$$\int d\phi_n \leftrightarrow \int d\phi_n^*. \quad (5:8)$$

Those replacements are understood to be made in the rest of this section and in Section VII.

Suppose that $A_{n-1,n-1}^\dagger(1)$ has a pole in its total energy. ($A^\dagger$ is now the physical amplitude continued clockwise around all the lower branch points.) The residue factorizes in the usual way,¹¹

$$\text{Res } A_{n-1,n-1}^\dagger = A_{n-1,r}^\dagger A_{r,n-1}^\dagger, \quad s = m_r^2. \quad (5:9)$$

As is customary, we take this to imply that the amplitude A_{kn}^+ has a pole in the final $(n-1)$ -body subenergy at $s_{n-1} = m_r^2$ and that its residue factorizes,

$$\text{Res } A_{kn}^+ = A_{k2'}^+ A_{r,n-1}^+, \quad k = 1, 2, \dots, n-1, \quad (5:10)$$

where $2'$ is the channel $(1, r)$.

Now consider (3:15) rewritten in the form

$$\begin{pmatrix} s_{22}^+ & s_{23}^+ & \dots & s_{2, n-1}^+ \\ \dots & \dots & \dots & \dots \\ s_{n-1, 2}^+ & \dots & \dots & s_{n-1, n-1}^+ \end{pmatrix} \begin{pmatrix} T_{2n}^u \\ \vdots \\ T_{n-1, n}^u \end{pmatrix} = \begin{pmatrix} T_{2n}^+ \\ \vdots \\ T_{n-1, n}^+ \end{pmatrix}, \quad s > (m_r)^2 \quad (5:11a)$$

or

$$T_{2n}^u = \sum_{k=2}^{n-1} (s^{+-1})_{2k} T_{kn}^+. \quad (5:11b)$$

By $(S^+)^{-1}$ we mean the inverse of $n-1 \times n-1$ submatrix of S^+ , which is not unity. This point was discussed in Section III. Evaluating the residue of the pole at $s_{n-1} = m_r^2$, we have

$$\text{Res } T_{2n}^u = \sum_{k=2}^{n-1} (s^{+-1})_{2k} A_{k2'}^+ A_{r,n-1}^+. \quad (5:12)$$

Define

$$A_{22'}^u \equiv \sum_{k=2}^{n-1} (s^{+-1})_{2k} A_{k2'}^+ = A_{22', (n)}^+. \quad (5:13)$$

A_{22} is then identified as the unstable particle scattering amplitude ($1 + 2 \rightarrow 1 + r$). We conclude that T_{2r}^u has an unstable pole in the $(n-1)$ -body final subenergy and, in fact,

$$\text{Res } T_{2n}^u = A_{22, (n)} A_{r, n-1}^* , \quad S_{n-1} = m_r^2 . \quad (5:14)$$

In a similar manner one shows that

$$T_{nn}^u(1) = T_{nn}(1) S_{nn}^{-1}(1) \quad (5:15)$$

has the resonance pole in its final subenergy. In fact, the pole appears in two terms of T_{nn}^u :

$$\text{Res } (T_{nn}^u)_{\text{disc}} = A_{n-1, r}^* A_{r, n-1}^* \Delta_1 , \quad (5:16a)$$

$$\text{Res } (T_{nn}^u)_{\text{conn}} = A_{n2}^u A_{r, n-1}^* , \quad (5:16b)$$

and hence

$$\begin{aligned} \text{Res } T_{nn}^u &= A_{n-1, r}^* A_{r, n-1}^* \Delta_1 + A_{n2}^u A_{r, n-1}^* \\ &= T_{n2}^u A_{r, n-1}^* ; \end{aligned} \quad (5:17)$$

Δ_1 is the three-dimensional momentum δ function corresponding to particle 1.

VI. ABSENCE OF A TWO-BODY NORMAL THRESHOLD BRANCH POINT

Having proved that S_{nn} is invertible, we are now in a position to generalize the two theorems due to Zwanziger and Olive quoted in the introduction. The first states that a two-body scattering amplitude containing an n -body branch point n and a second two-body branch point $2'$, corresponding to a bound state of particles $2, 3, \dots, n$, does not have the $2'$ branch point on the second sheet of the n -body cut. Our proof is regarded as an improvement upon that of Olive's for the case $n = 3$, in that it does not use crossed-channel extended unitarity. If we assume that particle b is the same as particle 2 , so that there is only one two-body channel, we can complete the proof without using unphysical unitarity, and therefore we shall consider this case (see Fig. 5).

In path notation the theorem is

$$\text{disc}_2 A_{22}(n) \equiv A_{22}(n) - A_{22}(2n) = 0. \quad (6:1)$$

It is necessary to derive an equation for the amplitude A_{2n}^u first. A_{2n}^u has been defined earlier by the integral equation

$$A_{2n}(1; 1, \dots) = \int d\phi_n A_{2n}^u S_{nn}(1, \dots; 1; 1, \dots). \quad (6:2)$$

Let us take the 2 discontinuity of each side of this equation. Notice that as s approaches branchpoint 2 in the integral on the right-hand side of (6.2), the endpoint of the s_{n-1} integration contour, as defined in (2:56), strikes the pole $s_{n-1} = \mu_b^2 \equiv \mu_2^2$ in parts of $S_{nn}(1, \dots; 1; 1, \dots)$. Therefore, letting "+" refer to the original contour and "-" to the continued contour, we have

$$\begin{aligned}
& \text{disc}_2 \int d\varphi_n A_{2n}^u S_{nn}(1, \dots; 1; 1, \dots) \\
&= \int_+ d\varphi_n A_{2n}^u S_{nn}(1, \dots; 1; 1, \dots) - \int_- A_{2n}^u(2) S_{nn}(1, \dots; 2; 1, \dots) \\
&= \int_+ d\varphi_n A_{2n}^u \text{disc}_2 A_{nn}(1, \dots; 1; 1, \dots) + \int_- d\varphi_n [\text{disc}_2 A_{2n}^u] S_{nn}(1, \dots; 2; 1, \dots) \\
&+ \left[\int_+ - \int_- \right] d\varphi_n A_{2n}^u S_{nn}(1, \dots; 2; 1, \dots) . \tag{6:3}
\end{aligned}$$

$\int_+$ differs from $\int_-$ only by a small loop about the pole position in the s_{n-1} variable, and hence we denote the difference by $\oint$. From the theorem for the factorization of physical poles we know that

$$\text{Res } S_{nn}(1, \dots; 2; 1, \dots) \Big|_{s_{n-1} = m_b^2} = A_{n-1, b}(1, \dots) A_{2n}(2; 1, \dots) . \tag{6:4}$$

Furthermore, by taking residues in the final $(n-1)$ -body subenergy on each side of (6:2) and canceling a common factor, we see that

$$A_{22}(1) = \int A_{2n}^u A_{n2}(1, \dots; 1) d\varphi_n + \int A_{2n}^u [A_{n-1, b}(1; \dots) \Delta_1] d\varphi_n, \tag{6:5}$$

which, if we use the elementary discontinuity formula

$$A_{22}(1) - A_{22}(n) = \int A_{2n}^u A_{n2}(1, \dots; 1) d\varphi_n, \tag{6:6}$$

can be rewritten

$$A_{22}(n) = \int d\varphi_n A_{2n}^u [A_{n-1, b}(1, \dots) \Delta_1] . \tag{6:7}$$

In deriving (6:7) we have taken advantage of the fact proved in the preceding section that A_{2n}^u is regular for $s_{n-1} = m_b^2$, but we have not used unphysical unitarity, or made any further identification of A_{2n}^u . Combining (2:5b), (6:4), and (6:7) results in

$$\begin{aligned} \int d\varphi_n A_{2n}^u S_{nn}(1, \dots; 2; 1, \dots) &= \int d\varphi_n A_{2n}^u \int d\varphi_2 [A_{n-1,b}(1, \dots) \Delta_1] A_{2n}(2; 1, \dots) \\ &= \int d\varphi_2 A_{22}(n) A_{2n}(2; 1, \dots). \end{aligned} \quad (6:8)$$

Combining (6:2, 3, 8) gives

$$\begin{aligned} \text{disc}_2 A_{2n}(1, \dots) &= \int d\varphi_n A_{2n}^u \text{disc}_2 A_{nn}(1, \dots; 1; 1, \dots) \\ &\quad + \int d\varphi_n [\text{disc}_2 A_{2n}^u] S_{nn}(1, \dots; 2; 1, \dots) + \int d\varphi_2 A_{22}(n) A_{2n}(2; 1, \dots). \end{aligned} \quad (6:9)$$

Subtracting $\int d\varphi_2 A_{22}(1) A_{2n}(2; 1, \dots)$ from each side of (6:9) and using (6:6) gives

$$\begin{aligned} \text{disc}_2 A_{2n}(1, \dots) - \int d\varphi_2 A_{22}(1) A_{2n}(2; 1, \dots) &= \int d\varphi_n A_{2n}^u [\text{disc}_2 A_{nn}(1, \dots; 1; 1, \dots) - \int d\varphi_2 A_{n2}(1, \dots; 1) A_{2n}(2; 1, \dots)] \\ &\quad + \int d\varphi_n [\text{disc}_2 A_{2n}^u] S_{nn}(1, \dots; 2; 1, \dots). \end{aligned} \quad (6:10)$$

If we were using unphysical unitarity we could say that the left-hand side of (6:10) and the first term on the right vanish. It would then

follow that $\text{disc}_2 A_{2n}^u = 0$. But this is not necessary for our purposes. Instead, we take the residue of the $(n-1)$ -body final subenergy pole on each side of (6:10), obtaining:

$$\begin{aligned} & \int_+ d\varphi_n A_{2n}^u [\text{disc}_2 A_{n2}(1, \dots; 1) - \int d\varphi_2 A_{n2}(1, \dots; 1) A_{22}(2)] \\ & + \int_- d\varphi_n [\text{disc}_2 A_{2n}^u] A_{n2}(1, \dots; 2) \\ & = \text{disc}_2 A_{22}(1) - \int d\varphi_2 A_{22}(1) A_{22}(2) = 0, \end{aligned} \quad (6:11)$$

where we have set the right-hand side equal to zero because of physical unitarity.

Next consider Eq. (6:6). Taking the discontinuity of each side, we obtain, exactly as in the preceding discussion,

$$\begin{aligned} \text{disc}_2 A_{22}(1) &= \text{disc}_2 A_{22}(n) + \int d\varphi_n [\text{disc}_2 A_{2n}^u] A_{n2}(1, \dots; 2) \\ &+ \int d\varphi_n A_{2n}^u \text{disc}_2 A_{n2}(1) + \int d\varphi_2 A_{22}(n) A_{22}(2), \end{aligned} \quad (6:12)$$

or equivalently,

$$\begin{aligned} & \text{disc}_2 A_{22}(n) + \int d\varphi_n [\text{disc}_2 A_{2n}^u] A_{n2}(1, \dots; 2) \\ & + \int d\varphi_n A_{2n}^u [\text{disc}_2 A_{n2}(1) - \int d\varphi_2 A_{n2}(1, \dots; 1) A_{22}(2)] = 0, \end{aligned} \quad (6:13)$$

where we have used

$$\text{disc}_n A_{22}(1) = \int d\varphi_n A_{2n}^u A_{n2}(1, \dots; 1) \quad (6:14a)$$

and

$$\text{disc}_2 A_{22}(1) = \int d\varphi_2 A_{22}(1) A_{22}(2) , \quad (6:14b)$$

both of which follow from physical unitarity. Finally, combining (6:11) and (6:13), we obtain

$$\text{disc}_2 A_{22}(n) = 0, \quad (6:15)$$

which proves our theorem.

We succeeded in eliminating unphysical unitarity from this proof by assuming that particles 2 and b were identical. Otherwise we would have needed the unphysical discontinuity equation

$$A_{22}(1) - A_{22}(2) = \int d\varphi_2 A_{22}(1) A_{22}(2) . \quad (6:16)$$

VII. COMPLEX NORMAL THRESHOLDS

Our second application of the invertibility of S_{nn} differs from the preceding one in that we consider the case in which r is an unstable particle (resonance) pole. The theorem is illustrated in Fig. 6. In one respect our theorem is considerably simpler than Zwanziger's. He demonstrated the corresponding result for a two-body complex threshold in three-body amplitude A_{33} . That theorem is more complicated because in addition to everything else one has to worry about physical-region poles in crossed-channel energies. As in-

licated in Section V, it is convenient at this point to reverse the sense of our path notation.

Our theorem is

$$A_{22}(n) - A_{22}(2'n) = \int d\varphi_2, A_{22}(n) A_{2,2}(2'n), \quad (7:1)$$

where $A_{22}(n)$ has been defined in Section V and $A_{2,2}(2'n)$ is an analytic continuation of $A_{2,2}(n)$ about the $2'$ branchpoint. Since the complex threshold branch point is absent on the physical sheet,

$$\text{disc}_2, A_{2n}(1) = \text{disc}_2, S_{nn}(1) = 0, \quad (7:2)$$

and we have, in analogy with (6:3),

$$\begin{aligned} \text{disc}_2, \int d\varphi_n A_{2n}^u S_{nn}(1) &= \int d\varphi_n [\text{disc}_2, A_{2n}^u] S_{nn}(1) \\ &+ \oint d\varphi_n A_{2n}^u S_{nn}(1) = 0 \end{aligned} \quad (7:3)$$

and therefore

$$\begin{aligned} &\int d\varphi_n [\text{disc}_2, A_{2n}^u - \int d\varphi_2, A_{22}(n) A_{2,n}^u(2')] S_{nn}(1) \\ &= - \oint d\varphi_n A_{2n}^u S_{nn}(1) - \int d\varphi_2, A_{22}(n) \int d\varphi_n A_{2,n}^u(2') S_{nn}(1), \end{aligned} \quad (7:4)$$

where $A_{2,n}^u(2')$ is an analytic continuation of $A_{2,n}^u$, defined in Section V. We now show that the right-hand side of (7:4) vanishes. Taking residues in the equation

$$T_{nn}(1) = \int T_{nn}^u S_{nn}(1) d\varphi_n \quad (7:5)$$

gives

$$0 = \int_{+} d\varphi_n T_{2,n}^u S_{nn}(1) = \int_{-} d\varphi_n T_{2,n}^u(2') S_{nn}(1), \quad (7:6)$$

and hence, using (5:17), continued around $2'$, we have

$$\int_{-} d\varphi_n (A_{r,n-1}^* \Delta_1) S_{nn}(1) = - \int_{-} d\varphi_n A_{2,n}^u(2') S_{nn}(1). \quad (7:7)$$

Therefore, using (5:14), we have

$$\begin{aligned} \oint A_{2n}^u S_{nn}(1) d\varphi_n &= \int d\varphi_{2'} A_{22'}(n) \int_{-} A_{r,n-1}^* \Delta_1 S_{nn}(1) d\varphi_n \\ &= - \int d\varphi_{2'} A_{22'}(n) \int_{-} A_{2,n}^u S_{nn}(1) d\varphi_n. \end{aligned} \quad (7:8)$$

Therefore the right-hand side of (7:4) vanishes, and the invertibility of S_{nn} allows us to conclude that

$$\text{disc}_{2'} A_{2n}^u = \int d\varphi_{2'} A_{22'}(n) A_{2,n}^u(2'). \quad (7:9)$$

In exactly the same way, one can manipulate

$$A_{22}(n) = \int A_{2n}^u A_{n2}(1) d\varphi_n \quad (7:10)$$

to obtain

$$\begin{aligned}
& \text{disc}_2 A_{22}(n) - \int d\phi_2 A_{22}(n) A_{2,2}(2'n) \\
& = - \int d\phi_n [\text{disc}_2 A_{2n}^u - \int d\phi_2 A_{22}(n) A_{2,n}^u(2')] A_{n2}(1, \dots; 1) = 0, \quad (7:11)
\end{aligned}$$

which proves our theorem.

VIII. SOME FURTHER GENERALIZATIONS

The two preceding theorems can be greatly generalized. We will formulate what we believe is a very general theorem for two-body scattering amplitudes and then sketch some of the methods that would be required in a complete proof.

Consider the two-body scattering amplitude A_{22} . Let $\{\alpha_i, i = 1, 2, \dots\} = E$ denote all the sets of stable particles that communicate with channel 2; α_i will also be used to label the corresponding normal threshold branch point and the path around it. We introduce a partial ordering, which we call inclusion, on the set E as follows: $\alpha_i \subset \alpha_j$ if and only if $\alpha_i \neq \alpha_j$ and there is a partition of the set of particles α_j such that the first particle of α_i communicates with the particles in the first part of the partition of α_j , etc. Another way of expressing this is to say that S_{α_i, α_j} can be obtained as a final subenergy multiple residue of S_{α_i, α_j} [e.g. $(\pi N) \subset (\pi\pi N)$; $(K\bar{K}) \not\subset (\pi N\bar{N})$]. Our conjectured theorem has two parts:

- (a) $\alpha_i \subset \alpha_j \Rightarrow A_{22}(\alpha_j) = A_{22}(\alpha_i \alpha_j)$ or, in other words, $\text{disc}_{\alpha_i} A_{22}(\alpha_j) = 0$;
- (b) $\alpha_i \not\subset \alpha_j$ and $\alpha_j \not\subset \alpha_i \Rightarrow A_{22}(\alpha_i \alpha_j) = A_{22}(\alpha_j \alpha_i)$.

This means that the same sheet is reached regardless of the order in which one goes around the two branch points. In this case we say that the paths α_i and α_j commute. Similarly, the channels α_i and α_j will be called commuting chan-

nels. Thus our conjecture states that given two communicating channels, either one is included in the other or else they commute. As a special case we notice that two n -body channels always commute. This fact is well known for $n = 2$.

The proof of (b) is quite analogous to that of (a) except that the end-point pinch which gave rise to the integral in (6:8), and corresponding integrals in other cases, will not occur. Therefore the theorems for real and complex branch points can be rewritten for this case.

As an initial step toward generalizing the theorem of Section VI to the first part of our conjecture, suppose that particles $(m, m+1, \dots, n)$ communicate with a bound state b , so that $(1, 2, \dots, m-1, b)$ is the channel m' . In this section we shall not go to great pains in defining what we mean by many-particle amplitudes on unphysical sheets, the point already having been made. Once again, we consider

$$A_{2n}(1) = \int d\varphi_n A_{2n}^u S_{nn}(1) . \quad (8:1)$$

Using unphysical unitarity, one can show

$$\begin{aligned} \text{disc}_m A_{2n}(1) &= \int d\varphi_m A_{2m}^u T_{m,n}(2', \dots; m'; (m-1)', \dots, 2', \dots) \\ &+ \int d\varphi_n A_{2n}^u \text{disc}_m A_{nn}(1) \end{aligned} \quad (8:2)$$

and

$$\begin{aligned}
& \text{disc}_m \int d\varphi_n A_{2n}^u S_{nn}(1) \\
&= \int d\varphi_n A_{2n}^u \text{disc}_m A_{nn}(1) + \int [\text{disc}_m A_{2n}^u] S_{nn}(1, \dots, 2'; m'; (m-1)', \dots) d\varphi_n \\
&+ \left[\int_+ - \int_- \right] d\varphi_n A_{2n}^u S_{nn}(1, \dots, 2'; m'; (m-1)'; \dots), \quad (8:3)
\end{aligned}$$

so that

$$\begin{aligned}
& \int [\text{disc}_m A_{2n}^u] S_{nn}(1, \dots, 2', \dots; m'; (m-1)', \dots) d\varphi_n \\
&= \int d\varphi_m A_{2m}^u T_{m,n} [2', \dots; m'; (m-1)', \dots] - \left[\int_+ - \int_- \right] d\varphi_n A_{2n}^u S_{nn}(1, \dots, 2', \dots; m'; \\
& \quad (m-1)', \dots) \quad (8:4)
\end{aligned}$$

We wish to show that the right-hand side of (8:4) vanishes.

By taking the residue of the $(n - m + 1)$ - body final subenergy pole on both sides of (8:1), we find

$$A_{2m}'(1) = \int d\varphi_n A_{2n}^u T_{nm}'(1) + \int d\varphi_n A_{2n}^u [A_{n-m+1,b}(1) \Delta_1 \dots \Delta_{m-1}] \quad (8:5)$$

or

$$A_{2m}' = \int d\varphi_n A_{2n}^u [A_{n-m+1,b}(1) \Delta_1 \dots \Delta_{m-1}], \quad (8:6)$$

which is the generalization of (6:9). By generalizing the discussion of Section V one may introduce an unstable particle amplitude $T_{m,n}$. The relation

we need here is

$$\text{Res } S_{nn}(1, \dots, 2', \dots; m'; (m-1)', \dots) = A_{n-2,p} T_{m',n}(2', \dots; m'; (m-1)', \dots). \quad (8.7)$$

If the only contribution to the $\int_+ - \int_-$ term were due to taking the residue of the $(n - m + 1)$ -body subenergy pole, we would use (8:6) and (8:7) to obtain

$$\begin{aligned} & \left[\int_+ - \int_- \right] d\varphi_n A_{2n}^u S_{nn}(1, \dots; m'; (m-1)', \dots) \\ &= \int d\varphi_n A_{2n}^u \int d\varphi_{m'} [A_{n-2,p} \Delta_{1, \dots} \Delta_{m-1}] T_{m',n}(2', \dots; m'; (m-1)', \dots) \\ &= \int d\varphi_{m'} A_{2m'}^u T_{m',n}(2', \dots; m'; (m-1)', \dots). \end{aligned} \quad (8.8)$$

But this is exactly what we need in order that the expression in (8:4) vanish.

We must therefore understand why the subenergy pole should contribute to

$\int_+ - \int_-$ whereas subenergy branch points should not. To see this we need the formula for phase space,

$$\int d\varphi_n(s) = \int_{(n-m+1)\mu^2}^{(s^{1/2} - (m-1)\mu)^2} d^s \alpha \int_{(m-1)\mu^2}^{(s^{1/2} - s_\alpha^{1/2})^2} d^s \beta \frac{q_{\alpha,\beta}}{4s^{1/2}} \int d^2 \hat{q}_{\alpha,\beta} \int d\varphi_\alpha(s_\alpha) \int d\varphi_\beta(s_\beta), \quad (8.9)$$

where

$$q_{\alpha,\beta} = \frac{[s - (s_\alpha^{1/2} + s_\beta^{1/2})^2][s - (s_\alpha^{1/2} - s_\beta^{1/2})^2]}{2s^{1/2}}; \quad (8.10)$$

β refers to the set $(1, \dots, m-1)$ and α refers to the set $(m, m+1, \dots, n)$. Thus the pole appears for $s_\alpha = M^2$. Also, $\int_- d\varphi_n$ refers to (8:9) with s continued around

$$[M + (m-1)\mu]^2.$$

The shaded area in Fig. 7 represents the region of integration. As $s^{1/2}$ loops $(m-1)\mu + M$ it is clear that the contour loops the pole position $s_{\alpha}^{1/2} = M$, but does not loop the threshold branch points $s_{\beta}^{1/2} = (m-1)\mu$, or any other singularity corresponding to an integration variable. Hence (8:8) is valid and in view of (8:4) we may conclude that

$$\text{disc}_m, A_{2n}^u = 0. \quad (8:11)$$

In the same way one shows

$$\text{disc}_m, A_{22}(n) + \int_{-\infty}^{\infty} d\phi_n [\text{disc}_m, A_{2n}^u] A_{n2}(1, \dots; m') = 0, \quad (8:12)$$

and hence

$$\text{disc}_m, A_{22}(n) = 0. \quad (8:13)$$

The generalization to the case in which there are poles in more than one subenergy is not difficult, either. It is only necessary to use the appropriate phase-space formula for any particular case. One needs to define multiple residues, but this is no real problem when they are taken in disjoint subenergies. We omit the details.

The only complication left to discuss is the case of "overlapping poles" and the related case of identical particles. By "overlapping poles" we mean a situation such as $(n, p) \subset (\pi^0, n, p)$ for which there are two different partitions of (π^0, n, p) that satisfy the conditions of our definition.

These cases should present no problem, however, because in doing the phase-space integrations certain definite subenergies are integrated explicitly to give the appropriate residues, and the singularities of the other subenergies, which are somehow contained in the angular integrations, should not contribute.

There are, of course, corresponding generalizations of our discussion in Section VII of complex normal thresholds. Let β_i denote such a threshold. One expects that if $\beta_i \subset \alpha_j$, where our notation has been extended in an obvious way, then the branch point β_i is present in $A_{22}(\alpha_j)$, and in fact one has the usual elementary discontinuity formula. Correspondingly, we conjecture that if $\beta_i \not\subset \alpha_j$, then $A_{22}(\alpha_j)$ does not contain the branch point β_i .

ACKNOWLEDGMENTS

We wish to express our appreciation to Professor Geoffrey F. Chew for his continuing encouragement and interest in our work. We are also very grateful to Dr. Ian Drummond for his numerous helpful suggestions and criticisms and for his careful reading of the manuscript.

APPENDIX

In proving the inversion theorem in Section IV we showed how to factorize S_{nn} into a product of terms of the form "one plus connected part." However, we pointed out that the definition of the individual factors depends upon the ordering of the factors, and that only a suitable ordering will guarantee the absence of crossed-energy δ functions. Here we shall display two such forms for the case $n = 4$:

$$\begin{aligned}
 P_4 &= [1+K_2(12)][1+K_2(34)][1+K_3(234)][1+K_3(124)] \\
 &\times [1+K_2(13)][1+K_2(24)][1+K_3(134)][1+K_3(123)] \quad (A\circ 1) \\
 &\times [1+K_2(14)][1+K_2(23)] \\
 &= (12)(34) \text{ I III } (13)(24) \text{ II IV } (14)(23)
 \end{aligned}$$

and

$$P_4 = (14)(23)(34) \text{ I II III IV } (12)(13)(24) \quad (A\circ 2)$$

In (A $\circ$ 1) we have introduced a simplified notation. The object is to show that the δ functions of P_4 combine with the poles of A_{44} in such a way that $A'_{44} = A_{44} - [P_4]_{\text{conn}}$ is free from δ functions. It is not clear whether the form (A $\circ$ 1) or (A $\circ$ 2) will be more useful for the generalization to arbitrary n . The techniques used are the same in either case, however. We consider here the latter form.

One can first read off the definition of the three-body factors from (A:2),

$$I \equiv 1 + iK_3(234) = 1 + i(34)^{-1} (23)^{-1} A'(234)(24)^{-1} \quad (A:3a)$$

$$A'(234) = A(234) - \left[(23)(34)(24) \right]_{\text{conn}}. \quad (A:3b)$$

There are similar formulas for II, III, and IV.

Next one chooses one of the twenty-four possible single pole terms, e.g., the one shown in Fig. 8(a). Then one identifies all the terms in (A:2) that can contribute to it. They are

$$(34)(12)(24) + (34)K_3(124)(24) + (34)K_3(123)(24). \quad (A:4)$$

The relevant parts involves only the pole terms of $K_3(124)$ and $K_3(123)$.

We express the result as

$$A(12)_\delta A(34)_\delta A(24) + A(12)_- (34)_\delta A(24) + A(12)_+ (34)_\delta A(23)_\delta (24). \quad (A:5)$$

The +, -, δ subscripts indicate whether the line between blobs has a δ -function or a $\pm i\epsilon$ pole associated with it. The corresponding term in A_{44} is $A(12)_+ A(234)$. When expression (A:5) is subtracted from this, one is left with $A(12)_+ A'(234)$, which is free from δ functions.

Let us also check the pole term in Fig. 8(b), since it is somewhat more complicated. The contributing terms are

$$(23)(34)I(12) + (23)(34)IK_3(124)(12) + (23)(34)IK_3(123)(12) .$$

(A:6)

Throwing away irrelevant parts and simplifying leaves

$$A(234)(24)_\delta^{-1} A(12) + A(234)(24)^{-1} A(24)_+ A(12)$$

$$+ A_{33}(234)(24)_\delta^{-1} (23)^{-1} A(23)_- A(12) \quad (A:7)$$

$$= A_{33}(234)_\delta A(12) + A_{33}(234)(24)^{-1} [A(124) + (23)^{-1} A(23)]_- A(12) .$$

Subtracting this from the pole term $A(234)_+ A(12)$ leaves

$$[A(234)_- A(12)] - A(234)(24)^{-1} [A(24) + (23)^{-1} A(23)]_- A(12)$$

(A:8)

$$= A(234)(24)^{-1} (23)^{-1} A(12) ,$$

which is again free from δ -functions. The remaining twenty-two single poles work out in the same fashion. We remark additionally that once all the single poles have been shown to work out, this completes the proof, because the multiple poles will then automatically be all right.

FOOTNOTES AND REFERENCES

- * Work done under auspices of the U.S. Atomic Energy Commission.
- † National Science Foundation Predoctoral Fellow.
- 1. D. I. Olive, Nuovo Cimento 28, 1318 (1963).
- 2. D. I. Olive, Nuovo Cimento 29, 362 (1963).
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- 4. Daniel Zwanziger, Phys. Rev. 131, 888 (1963).
- 5. J. Gunson (unpublished work)--it is interesting to note that Gunson postulates some of our results in order to prove the existence of stable poles.
- 6. P. V. Landshoff, Nuovo Cimento 28, 123 (1963).
- 7. We are grateful to Professor G. F. Chew for pointing this out to us.
- 8. It does seem to be useful, however, for generalizing Jon Wright's proof of Levinson's theorem for three-body scattering to the n-body case. See Jon Alan Wright, A Generalization of Levinson's Theorem to Three-Body Systems, Lawrence Radiation Laboratory Report UCRL-11896, Jan. 1965.
- 9. This equation and the following ones are only meant to be schematic. The ordering of factors must, of course, conform to the remarks near the end of Sec. IV.
- 10. H. P. Stapp, Nuovo Cimento 32, 103 (1964).
- 11. The proof is the same as in the stable particle case, except that S_{nn} must be replaced by S_{nn}^{-1} .

FIGURE CAPTIONS

- Fig. 1. An example of normal threshold sheet structure.
- Fig. 2. Illustration of the path notation.
- Fig. 3. Connectedness structures of a typical elastic term.
- Fig. 4. Example of a multiple pole term.
- Fig. 5. If x is a point on the physical sheet, continuation along the curve C leads to a regular point at the position 2.
- Fig. 6. The n -body cut is drawn so as to reveal the complex branch point $2'$. It is not present on the physical sheet.
- Fig. 7. The $s_\alpha - s_\beta$ integration contour.
- Fig. 8. Single-pole terms in a four-body amplitude.

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Fig. 1.

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Fig. 2.

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Fig. 3.

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Fig. 4.

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Fig. 5.

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Fig. 6.

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Fig. 7.

(a)

(b)

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Fig. 8.

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