**Title**

Geometry in the Triangular Spectrum of Perfect Derived Categories



**ISBN**

9798189276507

**Author**

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**Publication Date**

2026-05-01

Peer reviewed|Thesis/dissertation

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by

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A dissertation submitted in partial satisfaction of the

requirements for the degree of

Doctor of Philosophy

in

Mathematics

in the

Graduate Division

of the

University of California, Berkeley

Committee in charge:

Professor David Nadler, Chair
Professor Martin Olsson
Professor Constantin Teleman

Spring 2026

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Abstract

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The derived category $\mathsf{Perf}\, X$ of perfect complexes on a variety $X$ is an algebraic invariant that reflects many geometric properties of $X$. Balmer proved that $\mathsf{Perf}\, X$, together with the monoidal structure $\otimes^{\mathbb{L}}_{\mathcal{O}_X}$ given by the derived tensor product of sheaves, recovers $X$. In the course of the proof, he constructed a locally ringed space $\mathrm{Spec}_{\otimes^{\mathbb{L}}_{\mathcal{O}_X}} \mathsf{Perf}\, X$, called the tensor triangular spectrum, and showed that it is isomorphic to $X$.

In this thesis, in order to study the geometry of varieties with equivalent perfect derived categories, called Fourier–Mukai partners, we consider the triangular spectrum $\mathrm{Spec}_{\vartriangle} \mathsf{Perf}\, X$ of $\mathsf{Perf}\, X$, introduced by Matsui and defined using only the triangulated category structure of $\mathsf{Perf}\, X$. Each Fourier–Mukai partner, realized as a tensor triangular spectrum, defines an open subspace of the common ambient triangular spectrum, and we prove that the interactions among these subspaces reflect the expected birational geometric behavior.

Motivated by the observation that fixed loci of actions of autoequivalences on the triangular spectrum capture geometric information effectively, we introduce the framework of a polarized triangulated category, namely a pair $(\mathcal{T}, \tau)$ consisting of a triangulated category $\mathcal{T}$ and an autoequivalence $\tau$, called a polarization. To such a pair we associate a ringed space called the polarized triangular spectrum. As concrete applications, we generalize several reconstruction results of Bondal–Orlov, Ballard, and Favero, exhibit a universal way to compare birationally equivalent Fourier–Mukai partners with their common canonical model via Iitaka fibrations, and construct mirror partners in homological mirror symmetry of log Calabi–Yau surfaces as polarized triangular spectra. We further consider $\infty$-categorical enhancements of polarized triangulated categories, leading to precise comparisons with the relative triangular spectrum under the monodromy equivalence.

This thesis is based on three papers [Ito23, IM25, Ito25], of which the second is joint work with Hiroki Matsui. We reorganize the statements and proofs of results in [Ito23] using the simplified constructions provided in [IM25] and provide several new results based on [Ito25]. We also draw several results from [IO25, IN25], collaborations with Noah Olander and John Nolan, respectively.

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## Acknowledgments

I would first like to thank my advisor, David Nadler. His insights have always inspired me, and our discussions have shaped the way I think about mathematics. Without his dedicated guidance, constant encouragement, and enthusiasm for my work, I would not have been able to pursue the direction I am following now. I am truly fortunate to have him as my advisor.

I would also like to thank Martin Olsson and Constantin Teleman for their lectures during my first year at Berkeley and for serving on my dissertation committee. I was lucky to gain their perspectives on the subject at an early stage of my career.

I have also benefited greatly from interactions and companionship with many mathematicians. My projects have been deeply inspired by the work of Yuki Hirano, Hiroki Matsui, Hisato Matsukawa, and Genki Ouchi. Discussions with them at workshops and conferences have always been intriguing. I am especially grateful to Hiroki Matsui for kindly agreeing to collaborate when he briefly visited SLMath. I have also learned so much from Yujiro Kawamata through his papers and our conversations. His mathematics has been a foundation of my research. I also gained deeper geometric intuition from János Kollár and Joaquin Moraga, and I am extremely grateful for their continuing encouragement since my undergraduate years at Princeton. I have also learned from and been greatly encouraged by conversations with Jarod Alper, Paul Balmer, David Eisenbud, Martin Gallauer, David Favero, Andrew Hanlon, Tatsuki Kuwagaki, Max Lieblich, Ryo Takahashi, and Xiaolei Zhao.

I cannot be more grateful for Berkeley's colleagues and community. Through my collaboration with Noah Olander, I learned so many techniques in algebraic geometry and derived categories. I have always enjoyed chatting with John Nolan about various ideas, and I am very happy that one of them became a paper. I also benefited a lot from conversations with Ansuman Bardalai, Dennis Chen, Jacob Erlikhman, Swapnil Garg, Peter Haine, Ruoxi Li, Feiyang Lin, Mark Macerato, Claire Mirocha, Yuji Okitani, Jeremy Taylor, and Xiangru Zeng. I would also like to thank the whole Evans Hall community for its warm and welcoming atmosphere. Outside Berkeley, I would also like to thank James Hotchkiss, Kimoi Kemboi, Christopher Kuo, Riku Kurama, Wenyuan Li, Amal Mattoo, Saket Shah, Yu Shen, and Michael Zeng for many interesting discussions and for their companionship.

I cannot overstate how much friendship has made my five years at Berkeley fulfilling. My first-year office mates, Ely Sandine, Garrett Brown, and Jason Zhao, have been great friends. I also appreciate how supportive John Lentfer and Yi Zhao have been. Shingo Tsukamoto has made my life here so stimulating. I would also like to thank my friends Emi Ito, Mitsuhiko Nakamoto, Ayano Tanabe/Yageta, and Kiyotaka Yageta for taking me to many fun activities.

Last, but of course not least, I am deeply grateful to my parents, Kyoko Ito and Toshimitsu Ito, for supporting me throughout my PhD. Without their support, I could not have reached this point. I would also like to thank Yueqing Feng for her unwavering support throughout my time here.

# Chapter 1

# Introduction

## 1.1 Background

When we think of the derived category $\mathsf{Perf}\ X$ of perfect complexes on a variety $X$ as an algebraic invariant of the variety, it is natural to ask how much information about $X$ is encoded in $\mathsf{Perf}\ X$.

In [BO01], Bondal and Orlov showed that if $X$ is a smooth projective variety with ample (anti-) canonical bundle, then the triangulated category $\mathsf{Perf}\ X$ knows everything about $X$, i.e., we can reconstruct $X$ solely from the triangulated category structure of $\mathsf{Perf}\ X$. On the other hand, we cannot expect that this type of complete reconstruction can be generalized too much since there are various non-isomorphic varieties whose derived categories are triangulated equivalent.

In [Bal05], Balmer nevertheless showed that if we in addition remember a monoidal structure on $\mathsf{Perf}\ X$ given by the usual derived tensor product $\otimes^{\mathbb{L}}_{\mathcal{O}_X}$, then the pair $(\mathsf{Perf}\ X, \otimes^{\mathbb{L}}_{\mathcal{O}_X})$ can reconstruct $X$ when $X$ is a noetherian scheme. In the proof, Balmer defined a ringed space $\mathrm{Spec}_{\otimes}\,\mathcal{T}$, called the **tensor-triangular spectrum** (or the **Balmer spectrum**), for a triangulated category $\mathcal{T}$ with a compatible tensor product $\otimes$, called a **tensor triangulated structure**. Points in $\mathrm{Spec}_{\otimes}\,\mathcal{T}$ are certain subcategories of $\mathcal{T}$, called **prime thick $\otimes$-ideals**. Moreover, when $X$ is a noetherian scheme, Balmer showed that we have $X = \mathrm{Spec}_{\otimes^{\mathbb{L}}_{\mathcal{O}_X}}\ \mathsf{Perf}\ X$ as ringed spaces.

We can reinterpret Balmer's work as saying that reconstruction of a variety $X$ from its derived category $\mathsf{Perf}\ X$ can be done by identifying and specifying a "geometric" tensor triangulated structure on $\mathsf{Perf}\ X$. Moreover, if we can classify such tensor triangulated structures, we obtain the classification of **Fourier–Mukai partners** of $\mathcal{T}$, i.e., smooth projective varieties whose derived categories are equivalent to $\mathcal{T}$ as triangulated categories. However, directly working with tensor triangulated structures is hard since we need to keep track of various coherence data and it is in general quite hard to construct examples "by hand."

In this thesis, instead of directly focusing on tensor triangulated structures, we will make use of a ringed space $\mathrm{Spec}_{\vartriangle}\,\mathcal{T}$ recently constructed by Matsui [Mat21, Mat23] only using the triangulated category structure of $\mathcal{T}$, which is called the **triangular spectrum** (or the **Matsui spectrum**) of $\mathcal{T}$. Again, points in $\mathrm{Spec}_{\vartriangle}\,\mathcal{T}$ are certain subcategories of $\mathcal{T}$, called **prime thick subcategories**. A crucial fact about this space due to Matsui [Mat21], Ito–Matsui [IM25] and Matsukawa [Mat25b] is that given a fairly reasonable tensor triangulated structure $\otimes$ on $\mathcal{T}$, we know that there is an open immersion

$$\mathrm{Spec}_{\otimes} \mathcal{T} \subset \mathrm{Spec}_{\vartriangle} \mathcal{T}$$

of ringed spaces. In particular, if a triangulated category $\mathcal{T}$ is triangulated equivalent to $\mathsf{Perf}\ X$ for some variety $X$, then $X$ can be viewed as an open subspace of $\mathrm{Spec}_{\vartriangle} \mathcal{T}$ by the natural inclusion

$$X \cong \mathrm{Spec}_{\otimes^{\mathbb{L}}_{\mathcal{O}_X}} \mathsf{Perf}\ X \subset \mathrm{Spec}_{\vartriangle} \mathcal{T}.$$

Letting $\mathrm{FM}\,\mathcal{T}$ denote the set of isomorphism classes of Fourier–Mukai partners of $\mathcal{T}$, we can rephrase this result as saying that $\mathrm{Spec}_{\vartriangle} \mathcal{T}$ contains any variety $X \in \mathrm{FM}\,\mathcal{T}$ as a tensor triangular spectrum corresponding to $(\mathsf{Perf}\ X, \otimes^{\mathbb{L}}_{\mathcal{O}_X})$ although $\mathrm{Spec}_{\vartriangle} \mathcal{T}$ may contain multiple copies of the same Fourier–Mukai partners and those Fourier–Mukai partners can possibly intersect with each other in $\mathrm{Spec}_{\vartriangle} \mathcal{T}$. Here are some examples, due to Hirano–Ouchi [HO22] and Ito [Ito23], illustrating how Fourier–Mukai partners interact with each other geometrically in the triangular spectrum:

**Example 1.1.1** (Hirano–Ouchi, Ito)**.** Let $\mathcal{T}$ be a triangulated category with $X \in \mathrm{FM}\,\mathcal{T}$.

(i) If $X$ is an elliptic curve, then the triangular spectrum $\mathrm{Spec}_{\vartriangle} \mathcal{T}$ is a disjoint union of infinitely many copies of $X$. See Lemma 4.1.9.

(ii) If $X$ is an abelian variety and $X'$ is a non-isomorphic Fourier–Mukai partner of $X$, then their copies in $\mathrm{Spec}_{\vartriangle} \mathcal{T}$ do not intersect with each other. Indeed, all the copies of Fourier–Mukai partners are disjoint in $\mathrm{Spec}_{\vartriangle} \mathcal{T}$. See Lemma 3.3.10 and Theorem 3.3.12.

(iii) If $X$ is a surface containing a $(-2)$-curve, then $\mathrm{Spec}_{\vartriangle} \mathcal{T}$ contains at least two copies of $X$ which intersect along the complement of the $(-2)$-curve. See Corollary 3.3.21.

(iv) If $X$ is connected with $X' \in \mathrm{FM}\,\mathcal{T}$ via a standard/Mukai flop, then their copies in $\mathrm{Spec}_{\vartriangle} \mathcal{T}$ intersect with each other along the complement of the flopped subvarieties. See Example 3.3.33.

An upshot is that we have an ambient space $\mathrm{Spec}_{\vartriangle} \mathcal{T}$, depending only on the triangulated structure of $\mathcal{T}$, whose subspaces capture many of the expected birational geometric features of Fourier–Mukai partners. This makes it possible to study the information about varieties that is homologically encoded in derived categories through geometric and intuitive methods, without having to keep track of categorical coherence data.

This thesis is based on three papers [Ito23, IM25, Ito25], of which the second is joint work with Hiroki Matsui. We reorganize the statements and proofs of results in [Ito23] using the simplified constructions provided in [IM25] and provide several new results based on [Ito25]. We also draw several results from [IO25, IN25], collaborations with Noah Olander and John Nolan, respectively. See §1.3 for more detailed organization of the thesis.

## 1.2 Overview

### The Fourier–Mukai locus

Motivated by the observation that the triangular spectrum $\operatorname{Spec}_\vartriangle \mathcal{T}$ contains all the Fourier–Mukai partners as open subspaces, we will construct and study the open subspace $\operatorname{Spec}^{\mathsf{FM}} \mathcal{T}$ of $\operatorname{Spec}_\vartriangle \mathcal{T}$, called the **Fourier–Mukai locus**, which is defined to be the union of all the copies of the Fourier–Mukai partners of $\mathcal{T}$ embedded into $\operatorname{Spec}_\vartriangle \mathcal{T}$ as above. In [Ito23], the ringed space structure is constructed by gluing those of the Balmer spectra, but by the results of Ito–Matsui [IM25], it turns out that the ringed space structure constructed in [Ito23] agrees with the restriction of the structure sheaf of the Matsui spectrum.

In this thesis, we will focus on the approach taken in [IM25] and observe that the following characterization of the Fourier–Mukai locus in [Ito23] is enough to identify the two constructions. Moreover, we newly provide self-contained proofs of results on Fourier–Mukai loci in [Ito23], with the approach of [IM25].

**Theorem 1.2.1** (Ito)**.** *Let $\mathcal{T}$ be a triangulated category with $\mathrm{FM}\,\mathcal{T} \neq \emptyset$. Then, $\operatorname{Spec}^{\mathsf{FM}} \mathcal{T}$ is naturally a smooth scheme locally of finite type, into which the tensor triangular spectrum of any $X \in \mathrm{FM}\,\mathcal{T}$ is immersed as an open subscheme. Moreover, the Krull dimension $\dim \operatorname{Spec}^{\mathsf{FM}} \mathcal{T}$ equals the Krull dimension $\dim X$ for any $X \in \mathrm{FM}\,\mathcal{T}$.*

**Notation 1.2.2.** With notations as above, let $\operatorname{Spec}_{\otimes,X} \mathcal{T}$ denote the open subscheme of $\operatorname{Spec}^{\mathsf{FM}} \mathcal{T}$ given by the union of all copies of $X$ in $\operatorname{Spec}_\vartriangle \mathcal{T}$. Equivalently, $\operatorname{Spec}_{\otimes,X} \mathcal{T}$ is the orbit of a single copy of $X$ (as a tensor triangular spectrum) under the action of the group of autoequivalences.

Now, given the scheme structures on $\operatorname{Spec}^{\mathsf{FM}} \mathcal{T}$ and $\operatorname{Spec}_{\otimes,X} \mathcal{T}$, it is interesting to ask their properties as schemes. It turns out that separatedness and irreducibility of $\operatorname{Spec}_{\otimes,X} \mathcal{T}$ are precisely related to standard/birational triangulated equivalences in the following sense.

**Definition 1.2.3.** Let $X$ be a smooth projective variety. Let $\operatorname{Auteq} \mathsf{Perf}\, X$ denote the group of (natural isomorphism classes of) autoequivalences of $\mathsf{Perf}\, X$ and define the subgroup of **standard autoequivalences** to be

$$\operatorname{Auteq}^{\mathrm{st}} \mathsf{Perf}\, X := \operatorname{Aut} X \ltimes \operatorname{Pic} X \times \mathbb{Z}[1] \subset \operatorname{Auteq} \mathsf{Perf}\, X.$$

We say an autoequivalence $\Phi : \mathsf{Perf}\, X \to \mathsf{Perf}\, X$ is a **birational autoequivalence** if there exist closed points $x, y \in X$ such that $\Phi(k(x)) = k(y)$. This implies that there are open neighborhoods $U_x, U_y \subset X$ of $x, y$, respectively, such that skyscrapers on $U_x$ get isomorphically sent to skyscrapers on $U_y$, which explains the name (cf. Lemma 3.1.15). We write the corresponding subgroup by

$$\operatorname{BAuteq} X \subset \operatorname{Auteq} \mathsf{Perf}\, X.$$

It is easy to see in general we have

$$\operatorname{Auteq}^{\mathrm{st}} \mathsf{Perf}\, X \subset \operatorname{BAuteq} X \times \mathbb{Z}[1] \subset \operatorname{Auteq} \mathsf{Perf}\, X.$$

We can organaize

**Theorem 1.2.4** (Ito)**.** *Let $X$ be a smooth projective variety.*

*(i)* $\mathrm{Spec}_{X,\otimes}\,\mathsf{Perf}\,X$ *is separated if and only if* $\mathrm{Auteq}^{\mathrm{st}}\,\mathsf{Perf}\,X = \mathrm{BAuteq}\,X \times \mathbb{Z}[1]$.

*(ii)* $\mathrm{Spec}_{X,\otimes}\,\mathsf{Perf}\,X$ *is irreducible if and only if* $\mathrm{BAuteq}\,X \times \mathbb{Z}[1] = \mathrm{Auteq}\,\mathsf{Perf}\,X$.

*(iii)* $\mathrm{Spec}_{X,\otimes}\,\mathsf{Perf}\,X$ *is irreducible and separated if and only if* $\mathrm{Auteq}^{\mathrm{st}}\,\mathsf{Perf}\,X = \mathrm{Auteq}\,\mathsf{Perf}\,X$.

See Propositions 3.2.3, 3.2.5, 3.2.8 for more equivalent conditions. Let us also note that the first condition implies that $X$ can be reconstructed as any connected component of $\mathrm{Spec}_{X,\otimes}\,\mathsf{Perf}\,X$, and that the last condition implies $\mathrm{FM}\,X = \{X\}$. Later, we will exhibit some specific computations of Fourier–Mukai loci to indicate their relations with geometry. Here are some interesting examples:

**Example 1.2.5** (Ito)**.** Let $\mathcal{T}$ be a triangulated category with a smooth projective variety $X \in \mathrm{FM}\,\mathcal{T}$.

(i) If $X$ is a smooth projective variety with ample (anti-)canonical bundle, then $\mathrm{Spec}^{\mathsf{FM}}\,\mathcal{T} \cong X$ essentially by the result of Bondal–Orlov. In particular, $\mathrm{Spec}_{X,\otimes}\,\mathsf{Perf}\,X$ is irreducible and separated. See Theorem 4.1.5.

(ii) If $X$ is an elliptic curve, then $\mathrm{Spec}^{\mathsf{FM}}\,\mathcal{T}$ is a disjoint union of infinitely many copies of $X$. In particular, $\mathrm{Spec}_{X,\otimes}\,\mathsf{Perf}\,X$ is separated, but not irreducible. See Lemma 4.1.9.

(iii) If $X$ is an abelian variety, then all of its copies in $\mathrm{Spec}^{\mathsf{FM}}\,\mathcal{T}$ are disjoint. In particular, $\mathrm{Spec}_{X,\otimes}\,\mathsf{Perf}\,X$ is separated but not irreducible. See Proposition 3.3.11 and Theorem 3.3.12.

(iv) If $X$ is a toric variety, then any copies of $X, Y \in \mathrm{FM}\,\mathcal{T}$ in $\mathrm{Spec}^{\mathsf{FM}}\,\mathcal{T}$ intersect with each other along open sets containing tori. In particular, $\mathrm{Spec}_{X,\otimes}\,\mathsf{Perf}\,X$ is irreducible, but in general not separated. See Corollary 3.3.27 and Remark 3.3.28.

(v) If $X$ is a surface containing a $(-2)$-curve, then $\mathrm{Spec}_{X,\otimes}\,\mathsf{Perf}\,X$ is not separated, and in general not irreducible either. See Example 3.3.24.

(vi) If $X$ is a Calabi-Yau threefold, then each irreducible component of $\mathrm{Spec}^{\mathsf{FM}}\,\mathcal{T}$ containing a copy of $X$ contains all the copies of smooth projective Calabi-Yau threefolds that are birationally equivalent to $X$. Moreover, $\mathrm{Spec}_{X,\otimes}\,\mathsf{Perf}\,X$ is neither separated nor irreducible in general. See Example 3.3.33 and Remark 3.3.36.

In particular, Fourier–Mukai loci provide novel geometric intuitions to reinterpret behaviors of varieties in birational geometry. Moreover, we will observe relations of Fourier–Mukai loci with the DK hypothesis proposed by Kawamata [Kaw02], which predicts that $K$-equivalent smooth projective varieties are Fourier–Mukai partners. In particular, we will observe that some properties of Fourier–Mukai loci should be a very strong evidence for the DK hypothesis to be true. More precisely, we propose the following version of the DK hypothesis:

**Conjecture 1.2.6** (Conjecture 3.3.46)**.** *Let $X$ be a smooth projective variety. Then, every connected component of* $\mathrm{Spec}^{\mathsf{FM}}\,\mathsf{Perf}\,X$ *containing a copy of $X$ contains all smooth projective varieties that are $K$-equivalent to $X$ as open subschemes.*

## The Serre invariant locus

The study of Fourier–Mukai loci described so far is a priori geometric by definition in the sense that we start with tensor triangulated structures that are equivalent to $(\mathsf{Perf}\ X, \otimes^{\mathbb{L}}_{\mathcal{O}_X})$. In particular, many computations in examples above turn out to be applications of results from birational geometry. Hence, it is natural to try to give categorical characterizations of Fourier–Mukai loci to potentially give backward applications of studies of derived categories to birational geometry.

First, we will discuss comparison of the Fourier–Mukai locus with the **Serre invariant locus** $\mathrm{Spec}^{\mathsf{Ser}}\,\mathcal{T}$ in a triangular spectrum. Here, we recall that points in a triangular spectrum are some special types of subcategories and thus we can talk about Serre functor invariance of the points in the triangular spectrum. By construction it will be straightforward to see

$$\mathrm{Spec}^{\mathsf{FM}}\,\mathcal{T} \subset \mathrm{Spec}^{\mathsf{Ser}}\,\mathcal{T} \subset \mathrm{Spec}_{\vartriangle}\,\mathcal{T}.$$

Note that $\mathrm{Spec}^{\mathsf{Ser}}\,\mathcal{T}$ is defined purely triangulated categorically and in [HO22, HO24], Hirano–Ouchi gave a nice reinterpretation of the Bondal–Orlov reconstruction by showing

$$\mathrm{Spec}^{\mathsf{Ser}}\,\mathsf{Perf}\ X = \mathrm{Spec}_{\otimes_X}\,\mathsf{Perf}\ X \cong X$$

when $X$ is a Gorenstein projective variety with $\otimes$-generating canonical bundle. Here:

**Definition 1.2.7.** Let $X$ be a noetherian scheme. We say a line bundle $\mathscr{L}$ on $X$ is **$\otimes$-generating** if the smallest thick subcategory containing tensor powers $\mathscr{L}^{\otimes n}$ for $n \in \mathbb{Z}$ agrees with $\mathsf{Perf}\ X$.

By Orlov [Orl09], any (anti-)ample line bundle is $\otimes$-generating. In [IO25], Ito–Olander gave a Nakai–Moishezon-style criterion for $\otimes$-generation (cf. Theorem 3.3.3) and provided plenty of examples of varieties with $\otimes$-generating canonical bundle that is neither ample nor anti-ample (cf. Theorem 3.3.5). In this thesis, we will see the following (cf. Theorem 4.1.5, Theorem 4.1.7):

**Theorem 1.2.8** (Hirano–Ouchi, Ito)**.** *Let $\mathcal{T}$ be a triangulated category with $X \in \mathrm{FM}\,\mathcal{T}$. Suppose $X$ is a curve or a variety with $\otimes$-generating canonical bundle. Then*

$$\mathrm{Spec}^{\mathsf{FM}}\,\mathcal{T} = \mathrm{Spec}^{\mathsf{Ser}}\,\mathcal{T}.$$

Since the cases of elliptic curves and non-elliptic curves are two extremes in terms of positivity of (anti-)canonical bundles, we propose the following conjecture (Conjecture 4.1.12):

**Conjecture 1.2.9** (Ito)**.** *Let $\mathcal{T}$ be a triangulated category with $X \in \mathrm{FM}\,\mathcal{T}$. Then we have*

$$\mathrm{Spec}^{\mathsf{FM}}\,\mathcal{T} = \mathrm{Spec}^{\mathsf{Ser}}\,\mathcal{T}.$$

When the conjecture holds, the Fourier–Mukai locus can be obtained purely categorically by considering the Serre invariant locus. The conjecture turns out to be false for $K3$ surfaces of Picard number 1 and abelian varieties that are not isogenous to products of elliptic curves ([HO24, Mat25a]), but other cases (especially when the canonical bundle is non-trivial) are still open.

## Polarizations on a triangulated category

Following the observation of Hirano–Ouchi that the Serre invariant locus recovers the tensor triangular spectrum in the special cases, we consider the following framework of polarized triangulated categories and some associated spaces.

**Definition 1.2.10** (Ito)**.** Let $\mathcal{T}$ be a triangulated category.

(i) A pair $(\mathcal{T}, \tau)$ of a triangulated category $\mathcal{T}$ and its autoequivalence $\tau$ is called a **polarized triangulated category**. In this context, let us refer to $\tau$ as a **polarization** on $\mathcal{T}$.

(ii) For a polarized triangulated category $(\mathcal{T}, \tau)$, define its **$\tau$-fixed locus** to be $\mathrm{Spec}^{\tau}\,\mathcal{T} = \{\mathcal{P} \in \mathrm{Spec}_{\vartriangle}\,\mathcal{T} \mid \tau(\mathcal{P}) = \mathcal{P}\}$, where the structure sheaf is defined in the same way as the Matsui spectrum.

(iii) A **polarized triangulated functor** $(F, \alpha)$ between polarized triangulated categories $(\mathcal{T}_1, \tau_1)$, $(\mathcal{T}_2, \tau_2)$ is defined to be a triangulated functor $F : \mathcal{T}_1 \to \mathcal{T}_2$ together with a natural isomorphism $\alpha : F \circ \tau_1 \cong \tau_2 \circ F$.

A similar notion of a polarized triangulated category is also introduced by Kuznetsov–Perry in [KP21], which is a special case of our definition, so it should not cause confusions. We will also discuss how we can view this framework as local systems of stable $\infty$-categories on $B\mathbb{Z} \simeq S^1$ and clarify relations to relative triangular spectra introduced by Matsukawa [Mat25b] (cf. §5.4). Here are some benefits of having the formalism of polarized triangulated categories in the context of tensor triangular geometry and geometry in the triangular spectrum.

- For a nice enough tensor triangulated category, including the case of $(\mathsf{Perf}\, X, \otimes^{\mathbb{L}}_X)$ for a noetherian scheme $X$, we can view its Balmer spectrum as a $\tau$-fixed locus for a certain (semi-)polarization (cf. Proposition 4.2.15). This realization suggests that the study of polarized triangulated structures could provide new insights into tensor triangulated geometry (cf. Proposition 4.3.2, Theorem 4.3.10 in §4.3).
- The classification of tensor triangulated structures up to symmetric monoidal equivalence is not yet well understood. However, classifying possible polarizations is often more tractable, as the autoequivalence group of $\mathsf{Perf}\, X$ is well understood for many varieties $X$. In particular, we explicitly compute fixed loci for $\mathbb{P}^1$ and an elliptic curve $E$ (Examples 4.2.11, 4.2.12).
- In noncommutative geometry, the perfect derived category $\mathsf{Perf}\, R$ of right modules over a smooth proper dg-algebra $R$ does not generally admit a natural tensor triangulated structure. However, it is known that the Serre functor still exists ([Shk07]), which provides a canonical polarization. In more geometric settings, categories without natural tensor triangulated structures may still have a Serre functor, such as in the case of Kuznetsov components ([KP21]). Moreover, if a triangulated category contains a spherical object, it induces a polarization via its spherical twist ([ST01]).

We will discuss applications of polarizations in Chapter 5.

## Tensor triangular spectra in the triangular spectrum

Now, we go back to the original question of classifying tensor triangulated structures.

First, we aim at understanding classifications of tensor triangulated structures up to **spectral identity**, i.e., up to tensor triangular spectra as subspaces of the triangular spectrum. A key observation is that since a nice enough tensor triangular spectrum agrees with a fixed locus of a polarization as subspaces of the triangular spectrum, the classification of such tensor triangulated structures up to spectral identity is a part of the classification of polarizations up to spectral identity, i.e., up to fixed loci (or more precisely up to polarized triangular spectra).

**Notation 1.2.11.** For a triangulated category $\mathcal{T}$, let $\mathfrak{M}^{\mathsf{pt}}_{\mathcal{T}}$ denote the set of spectrally identical classes of polarizations on $\mathcal{T}$. When $\mathcal{T}$ admits a nice enhancement, we can equip $\mathfrak{M}^{\mathsf{pt}}_{\mathcal{T}}$ with the natural topology coming from the group algebraic space structure on $\operatorname{Auteq}\mathcal{T}$ of Toën–Vaquié [TV07].

For $\mathcal{T} = \mathsf{Perf}\,\mathbb{P}^1$, the topology knows where the tensor triangulated structure $\otimes_{\mathbb{P}^1}$ is in $\mathfrak{M}^{\mathsf{pt}}_{\mathcal{T}}$, providing a variant of Toledo's monoidal Bondal–Orlov reconstruction [Tol23] (Theorem 4.3.10).

**Theorem 1.2.12** (Ito)**.** *We have a homeomorphism*

$$\mathfrak{M}^{\mathsf{pt}}_{\mathsf{Perf}\,\mathbb{P}^1} \cong [\mathbb{A}^3/\mathbb{G}_m](k) \sqcup [\mathbb{A}^3/\mathbb{G}_m](k).$$

*Moreover, a rigid tensor triangulated structure $\otimes$ is spectrally identical to $\otimes^{\mathbb{L}}_{\mathcal{O}_{\mathbb{P}^1}}$ if and only if $\otimes$ corresponds to a closed (stacky) point in $\mathfrak{M}^{\mathsf{pt}}_{\mathsf{Perf}\,\mathbb{P}^1}$.*

It will be interesting to study how general we can gain such characterizations of $\otimes^{\mathbb{L}}_{\mathcal{O}_X}$.

Next, we consider if we can characterize the Fourier–Mukai locus by characterizing "geometric" tensor triangulated structures and considering the union of the corresponding tensor triangular spectra in the triangulated spectrum. In §4.3, we consider multiple necessary properties, but the following corollary of higher Zariski geometry recently developed in [ABC+25] is suggestive:

**Proposition 1.2.13** (Proposition 4.3.22)**.** *Let $(\mathcal{C}, \otimes)$ be an idempotent-complete stably symmetric monoidal $\infty$-category. Then, there exists a variety $X$ and a symmetric monoidal $\infty$-equivalence $(\mathcal{C}, \otimes) \simeq (\mathsf{Perf}\,X, \otimes_X)$ if and only if $\otimes$ is rigid and the 2-ringed space $\operatorname{Spec}_{\otimes}\mathcal{C}$ is equivalent to the 2-ringed space corresponding to a variety.*

**Conjecture 1.2.14.** *Let $(\mathcal{T}, \otimes)$ be a tt-category with $\operatorname{FM}\mathcal{T} \neq \emptyset$. Let $\operatorname{Spec}^{\mathsf{rig,var}}\mathcal{T} \subset \operatorname{Spec}_{\vartriangle}\mathcal{T}$ denote the union of $\operatorname{Spec}_{\otimes}\mathcal{T}$ over rigid tensor triangulated structures whose spectra are varieties. Letting $\operatorname{Spc}^{\mathsf{pFM}}\mathcal{T}$ denote the Fourier–Mukai locus enlarged by allowing smooth proper varieties, we have*

$$\operatorname{Spec}^{\mathsf{pFM}}\mathcal{T} = \operatorname{Spec}^{\mathsf{rig,var}}\mathcal{T} \subset \operatorname{Spec}_{\vartriangle}\mathcal{T}.$$

See Remark 4.3.23 for more discussions on plausibility of the conjecture.

Finally, we explain the classification up to spectral identity is not fine enough to capture interesting variations of tensor triangulated structures on $\mathsf{Perf}\,\mathbb{P}^1$ by recalling the result of Ito–Nolan [IN25].

## Applications of polarizations

Conceptually, the relation to tensor triangular geometry motivates polarizations and their fixed loci, but let us also consider more concrete applications of the framework of polarizations in Chapter 5. The most straightforward application is to reconstruction of varieties, which is stated more generally as Theorem 5.1.3. Note that the latter claim is stronger than Hirano–Ouchi's formulation above.

**Theorem 1.2.15** (Ito)**.** *Let $X$ be a Gorenstein proper variety with $\otimes$-generating canonical bundle. Then, we have $X \cong \operatorname{Spec}^{\mathsf{Ser}} \mathsf{Perf}\, X$. Moreover, if there exists a variety $Y$ with $\mathsf{Perf}\, X \simeq \mathsf{Perf}\, Y$, then $X \cong Y$ and in particular $\omega_Y$ is also $\otimes$-generating.*

It is worth noting that the reconstruction holds for non-projective proper varieties (and there is indeed such an example!), which is unclear from the original approach of Bondal–Orlov using the Proj construction (as below). We also provide generalization of Favero's result on rigidity of polarizations [Fav12], which is again stated more generally as Theorem 5.1.8.

**Theorem 1.2.16** (Ito)**.** *Suppose there is a polarized Fourier–Mukai equivalence*

$$\Phi : (\mathsf{Perf}\, X, \tau_{\mathscr{L}}) \xrightarrow{\sim} (\mathsf{Perf}\, Y, \sigma),$$

*where (i) $X, Y$ are proper varieties, (ii) $\mathscr{L}$ is a $\otimes$-generating line bundle on $X$, and (iii) $\sigma$ is a standard autoequivalence, i.e., $\sigma = f_* \circ \tau_{\mathscr{M}}[n] \in \mathrm{Aut}(X) \ltimes \mathrm{Pic}(X) \times \mathbb{Z}[1]$. Then, $f$ has a fixed point and we have $f^m = id_Y$ and $X \cong Y$.*

In §5.2, we first note that given a projective variety $X$ with an ample line bundle $\mathscr{L}$, there are two ways of recovering $X$ from $(\mathsf{Perf}\, X, \tau_{\mathscr{L}})$:

$$\operatorname{Spec}^{\tau_{\mathscr{L}}} \mathsf{Perf}\, X \cong X \cong \operatorname{Proj} \mathcal{R}_{\mathscr{L}},$$

where $\mathcal{R}_{\mathscr{L}} := \bigoplus_{n\geq 0} \Gamma(X, \mathscr{L}^{\otimes n}) = \bigoplus_{n \geq 0} \mathrm{Hom}_{\mathsf{Perf}\, X}(\mathscr{O}_X, \tau_{\mathscr{L}}^n(\mathscr{O}_X))$ denotes the **section ring** of $\mathscr{L}$.

Based on the isomorphism $X \cong \operatorname{Proj} \mathcal{R}_{\mathscr{L}}$, Artin-Zhang ([AZ94]) introduced the **noncommutative projective scheme** $\operatorname{Proj}^{\mathsf{nc}} A$ associated to a right noetherian $\mathbb{N}$-graded algebra $A$ (cf. Definition 5.2.1). The noncommutative projective scheme naturally provides a polarized triangulated category $(\mathsf{Perf}\, \operatorname{Proj}^{\mathsf{nc}} \mathcal{R}_{\mathscr{L}}, s_{\mathcal{R}_{\mathscr{L}}})$ and we obtain the chain of isomorphisms (cf. Proposition 5.2.4):

$$\operatorname{Spec}^{\tau_{\mathscr{L}}} \mathsf{Perf}\, X \cong \operatorname{Proj} \mathcal{R}_{\mathscr{L}} \cong \operatorname{Spec}^{s_{\mathcal{R}_{\mathscr{L}}}} \mathsf{Perf}\, \operatorname{Proj}^{\mathsf{nc}} \mathcal{R}_{\mathscr{L}}.$$

Now, we can make two key observations from the isomorphisms:

- For a general right noetherian $\mathbb{N}$-graded algebra $A$, $\operatorname{Proj} A$ no longer makes sense, but the ringed space $\operatorname{Spec}^{s_A} \mathsf{Perf}\, \operatorname{Proj}^{\mathsf{nc}} A$ remains well-defined, so we propose that it is a geometric realization of $\operatorname{Proj}^{\mathsf{nc}} A$. A natural direction for future research is to investigate the extent to which the geometry of $\operatorname{Proj}^{\mathsf{nc}} A$ is captured by $\operatorname{Spec}^{s_A} \mathsf{Perf}\, \operatorname{Proj}^{\mathsf{nc}} A$.

- When we set $\mathscr{L} = \omega_X$ for a smooth projective variety $X$, the left hand side becomes the Serre invariant locus $\mathrm{Spec}^{\mathsf{Ser}}\,\mathsf{Perf}\,X$ while the right hand side becomes the so called canonical model $X^{\mathsf{can}} = \mathrm{Proj}\,\mathscr{R}_{\omega_X}$ of $X$, which is a birational invariant of smooth projective varieties. When $\omega_X$ has positive Iitaka dimension, $X$ and $X^{\mathsf{can}}$ are related by a canonical rational map $X \dashrightarrow X^{\mathsf{can}}$, called the **Iitaka fibration**. Namely, we have the following relations:

$$\mathrm{Spec}^{\mathsf{Ser}}\,\mathsf{Perf}\,X \supset \mathrm{Spec}_{\otimes_X}\,\mathsf{Perf}\,X \cong X \dashrightarrow X^{\mathsf{can}} = \mathrm{Proj}\,\mathcal{R}_{\omega_X} \cong \mathrm{Spec}^{S_{\mathcal{R}_{\omega_X}}}\,\mathsf{Perf}\,\mathrm{Proj}^{\mathsf{nc}}\,\mathcal{R}_{\omega_X}.$$

  Since both the left-hand side and the right-hand side are birational invariants of Fourier–Mukai partners, further investigation can lead to a universal understanding and categorification of Iitaka fibrations across all such partners. This is an ongoing project.

In §5.3, we explore whether algebraic mirror partners in homological mirror symmetry can be realized as fixed loci. To fix notations, suppose we have an equivalence

$$\mathsf{Perf}\,X \simeq \mathsf{Fuk}(M,\omega),$$

where $X$ is a smooth variety and $\mathsf{Fuk}(M,\omega)$ denotes a Fukaya-like category associated to a symplectic manifold $(M,\omega)$, possibly with additional data. Given that there can exist non-isomorphic Fourier–Mukai partners, the following natural questions arise from our perspective.

**Question 1.2.17.** Is there a symplecto-geometric way to construct a polarization $\tau$ on $\mathsf{Fuk}(M,\omega)$ such that $\mathrm{Spec}^{\tau}\,\mathsf{Fuk}(M,\omega)$ defines an algebraic mirror partner of a symplectic manifold $(M,\omega)$ possibly together with additional data? Furthermore, how canonical is such a mirror partner?

In certain cases, the answer is affirmative. In this paper, we focus on homological mirror symmetry of log Calabi–Yau surfaces, as studied by Hacking–Keating in [HK21] and [HK23]. In particular, we reinterpret their results to show that algebraic mirrors can indeed be constructed as fixed loci of the Matsui spectrum of Fukaya categories, using only symplecto-geometric data (Proposition 5.3.15). Our approach makes the relation of choices of Lagrangian fibrations and Fourier–Mukai partners under homological mirror symmetry more explicit.

In §5.4, we discuss that the natural $\infty$-categorical enhancement of polorizations corresponds to considering local systems of perfect $\infty$-categories on $B\mathbb{Z} = S^1$, and under the monodromy equivalence, it is equivalent to giving $\mathsf{Perf}(B\mathbb{G}_{m,k})$-module structures to perfect $\infty$-categories (Proposition 5.4.10). This perspective first lets us realize the fixed locus/polarized triangular spectrum as the relative triangular spectrum introduced by Matsukawa [Mat25b]. Moreover, we can consider polarizations corresponding to any monoid $M$ in spaces by considering the local systems on the classifying category $\mathsf{B}M$, and if $M$ is group-like, there is the corresponding spectrum given as the relative triangular spectrum. This story has a natural higher categorical enhancement, and it will be interesting to see the relation to 3-dimensional mirror symmetry (as proposed by Teleman [Tel14], see also [GHMG22, PPS25]) in the future research.

In §5.5, we briefly discuss the functoriality issue and how to get around.

## 1.3 Organization of the thesis

This thesis is primarily based on [Ito23, IM25, Ito25], where the second work is a joint work with Hiroki Matsui. In this thesis, we modified some notations/statements for clarity and we newly rewrite some proofs of results in [Ito23] using the results of [IM25]. Some more examples are added for clarity. Furthermore, there are several new results particularly in §4.3 and §5.4.

Chapter 1 is based on the introductions of [Ito23, Ito25].

Chapter 2 is based on the preliminary section of [Ito25] and some materials from [Ito23].

Chapter 3 is based on [Ito23, IM25, Ito25]. Most of the examples are coming from [Ito23] except for the case of general abelian varieties from [IM25]. Throughout, the construction of the Fourier–Mukai locus is based on the simplified version of [IM25].

Chapter 4 is based on [Ito23, Ito25]. §4.3 contains some new results including Theorem 4.3.10 and Proposition 4.3.22. Moreover, we explain the motivation/background for [IN25], which is a joint work with John S. Nolan.

Chapter 5 is based on [Ito25]. We also simplify some assumptions and give more examples by citing the results of [IO25], which is a joint work with Noah Olander. §5.4 consists of new results/observations.

# Chapter 2

# Preliminaries

## 2.1 The tensor triangular spectrum and the triangular spectrum

Throughout this thesis, we work over a field $k$. In particular, a scheme and an algebra are assumed to be over $k$ and a variety is assumed to be an integral and separated scheme of finite type over $k$, unless otherwise specified. Given a ringed space $X$, $|X|$ denotes its underlying topological space.

Now, let us quickly fix notations for Balmer's theory of tensor triangular geometry and Matsui's theory of triangular spectra. For details of constructions and results, see [Bal05, Mat23].

**Notation 2.1.1.** A triangulated category is assumed to be $k$-linear and **essentially small** (i.e., having a set of isomorphism classes of objects) and functors/structures are assumed to be $k$-linear.

(i) Let $\mathcal{T}$ be a triangulated category and let $\mathrm{Th}(\mathcal{T})$ denote the set of **thick subcategories** (i.e., triangulated full subcategories closed under direct summand). Note that $\mathrm{Th}(\mathcal{T})$ is a poset (and indeed a complete and distributive lattice) with respect to inclusions and that we can equip $\mathrm{Th}(\mathcal{T})$ with topology, which we call the **Balmer topology**, whose closed subsets are of the form
$$V(\mathcal{E}) = \{\mathcal{I} \in \mathrm{Th}(\mathcal{T}) \mid \mathcal{I} \cap \mathcal{E} = \emptyset\},$$
where $\mathcal{E} \subset \mathrm{Ob}\,\mathcal{T}$ is a collection of objects. Also, for a collection $\mathcal{E}$ of objects in $\mathcal{T}$, let $\langle \mathcal{E} \rangle$ denote the smallest thick subcategory of $\mathcal{T}$ containing $\mathcal{E}$. We say an object $E$ is a **split-generator** if $\langle E \rangle = \mathcal{T}$.

(ii) We say a thick subcategory $\mathcal{P}$ of a triangulated category $\mathcal{T}$ is a **prime thick subcategory** if the subposet $\{\mathcal{I} \mid \mathcal{I} \supsetneq \mathcal{P}\} \subset \mathrm{Th}(\mathcal{T})$ has the smallest element, or equivalently, $\mathcal{P} \subsetneq \cap_{\mathcal{I} \supsetneq \mathcal{P}} \mathcal{I}$. Let $\mathrm{Spc}_{\vartriangle} \mathcal{T}$ denote the subspace of prime thick subcategories in $\mathrm{Th}\,\mathcal{T}$, which we call the **Matsui spectrum** or the **triangular spectrum** of $\mathcal{T}$.

(iii) Let $(\mathcal{T}, \otimes)$ be a **tensor triangulated category** (in short, a **tt-category**), i.e., a triangulated category $\mathcal{T}$ with a symmetric monoidal structure $\otimes$ that is triangulated in each variable, called a **tt-structure**. A **tt-functor** is a monoidal functor between tt-categories whose underlying functor is triangulated.

(iv) A tt-category $(\mathcal{T}, \otimes)$ is said to be **rigid** if the functor $- \otimes M$ admits a right adjoint $[M, -]$ for any object $M \in \mathcal{T}$ and if the canonical map $[M, \mathbb{1}] \otimes N \to [M, N]$ is an isomorphism for any object $M, N \in \mathcal{T}$. In particular, every object is dualizable.

(v) Let $(\mathcal{T}, \otimes)$ be a tt-category. A thick subcategory $\mathcal{I}$ is said to be a **$\otimes$-ideal** if for any $M \in \mathcal{T}$ and $N \in \mathcal{I}$, we have $M \otimes N \in \mathcal{I}$. Similarly, a $\otimes$-ideal $\mathcal{P}$ is said to be a **prime $\otimes$-ideal** if $\mathcal{P} \neq \mathcal{T}$ and $M \otimes N \in \mathcal{P}$ implies $M \in \mathcal{P}$ or $N \in \mathcal{P}$. Similarly, the radical $\sqrt{\mathcal{I}}$ of a $\otimes$-ideal $\mathcal{I}$ is

$$\sqrt{\mathcal{I}} := \{M \in \mathcal{T} \mid M^{\otimes n} \in \mathcal{I} \text{ for some } n > 0\}$$

and we say a $\otimes$-ideal $\mathcal{I}$ is a **radical $\otimes$-ideal** if $\mathcal{I} = \sqrt{\mathcal{I}}$. Recall that if $(\mathcal{T}, \otimes)$ is a rigid tt-category, every $\otimes$-ideal is radical ([Bal05, Remark 4.3 and Proposition 4.4]).

Let $\operatorname{Spc}_{\otimes} \mathcal{T} \subset \operatorname{Th} \mathcal{T}$ denote the subspace of prime $\otimes$-ideals in $\operatorname{Th} \mathcal{T}$, which we call the **Balmer spectrum** or the **tt-spectrum** of $(\mathcal{T}, \otimes)$.

(vi) Let $(\mathcal{T}, \otimes)$ be a tt-category. We say an object $M \in \mathcal{T}$ is **$\otimes$-invertible** if there exists $N \in \mathcal{T}$ such that $M \otimes N \cong \mathbb{1}$ and let $\operatorname{Pic}(\mathcal{T}, \otimes)$ denote the group of isomorphism classes of $\otimes$-invertible objects. For an invertible object $M \in \operatorname{Pic}(\mathcal{T}, \otimes)$, let

$$\tau_M := - \otimes M$$

denote the corresponding autoequivalence of $\mathcal{T}$ if doing so will not cause any confusion about which tt-structure we are using.

(vii) The group $\operatorname{Auteq} \mathcal{T}$ of the natural isomorphism classes of autoequivalences of $\mathcal{T}$ acts on $\operatorname{Th}(\mathcal{T})$ both as lattice isomorphisms and as homeomorphisms by

$$\operatorname{Auteq} \mathcal{T} \times \operatorname{Th} \mathcal{T} \ni (\tau, \mathcal{I}) \mapsto \tau^{-1}(\mathcal{I}) \in \operatorname{Th} \mathcal{T}.$$

The action restricts to an action on $\operatorname{Spc}_{\vartriangle} \mathcal{T}$. On the other hand, the action does not restrict to $\operatorname{Spc}_{\otimes} \mathcal{T}$ in general for a tt-structure $\otimes$. Letting $\operatorname{End} \mathcal{T}$ denote the monoid of natural isomorphism classes of triangulated endofunctors on $\operatorname{Th} \mathcal{T}$, this action naturally extends to the monoid action

$$\operatorname{End} \mathcal{T} \times \operatorname{Th} \mathcal{T} \ni (\tau, \mathcal{I}) \mapsto \tau^{-1}(\mathcal{I}) \in \operatorname{Th} \mathcal{T}.$$

(viii) For $\tau \in \operatorname{End} \mathcal{T}$, let

$$\operatorname{Spc}^{\tau} \mathcal{T} = \{\mathcal{P} \in \operatorname{Spc}_{\vartriangle} \mathcal{T} \mid \mathcal{P} = \tau^{-1}(\mathcal{P})\} \subset \operatorname{Spc}_{\vartriangle} \mathcal{T}$$

denote the subspace of fixed points by the action of $\tau$. Similarly, for $\tau \in \operatorname{End} \mathcal{T}$, write

$$\operatorname{Th}^{\tau} \mathcal{T} = \{\mathcal{I} \in \operatorname{Th} \mathcal{T} \mid \mathcal{I} = \tau^{-1}(\mathcal{I})\}.$$

Although tt-categories are ubiquitous in many fields (cf. e.g. [Bal20]), we will primarily focus on ones coming from algebraic geometry and quiver representations. Before introducing them, let us fix some notations for the latter. For example, see [Ela22].

**Construction 2.1.2.** Let $\mathsf{Q}$ be a quiver and let $\mathsf{Q}_0$ (resp. $\mathsf{Q}_1$) denote the set of vertices (resp. arrows) of $\mathsf{Q}$ together with the source and the target maps denoted by $s, t : \mathsf{Q}_1 \to \mathsf{Q}_0$. Now let $R = k\mathsf{Q}$ denote the path algebra of $\mathsf{Q}$ and let $M$ be a right $R$-module. For a vertex $v \in \mathsf{Q}_0$, let $M_v$ denote the subspace $Me_v \subset M$, where $e_v \in R$ denotes a path of length 0 at $v$. Note we have a decomposition

$$M = \oplus_{v \in \mathsf{Q}_0} M_v$$

by $k$-vector spaces and a quiver arrow $a$ from $u$ to $v$ induces a $k$-linear map $M_v \to M_u$. Therefore, giving a right $R$-module $M$ is equivalent to giving a collection $\{M_v \mid v \in \mathsf{Q}_0\}$ of $k$-vector spaces and a collection $\{M_a : M_{s(a)} \to M_{t(a)} \mid a \in \mathsf{Q}_1\}$ of $k$-linear maps and this data is called a **representation** of a quiver $\mathsf{Q}$. Note we can similarly define maps of representations so that they are equivalent to homomorphisms of right $R$-modules. In particular, the category of right $k\mathsf{Q}$-modules and the category of representations of a quiver $\mathsf{Q}$ are equivalent.

Now, for a vertex $v \in \mathsf{Q}_0$, let $P_v$ denote the right submodule $e_v R \subset R$. Then we clearly have

$$k\mathsf{Q} = \oplus_{v \in \mathsf{Q}_0} P_v.$$

Therefore, $P_v$ are projective and indeed it can be shown that any indecomposable projective module is isomorphic to $P_v$ for a unique vertex $v \in \mathsf{Q}_0$. Moreover, the Nakayama functor $\mathbb{S}$ (cf. Example 2.2.6) sends each $P_v$ to an indecomposable injective module $I_v := \mathbb{S}(P_v)$ and it can be shown that any indecomposable injective module is isomorphic to $I_v$ for a unique vertex $v \in \mathsf{Q}_0$. Also, for $v \in \mathsf{Q}_0$, let $S_v$ denote the corresponding simple right $R$-module $k$, on which the idempotent $e_v$ acts by the identity and any other path acts by zero.

Furthermore, if $\mathsf{Q}$ is finite and has no oriented cycles, then we can order $\mathsf{Q}_0 = \{1, \dots, n\}$ compatibly with arrows and then it is a standard result that $\langle P_1, \dots, P_n \rangle$ is a full and strong exceptional collection of $\mathsf{Perf}\ R$ (cf. Example 2.1.3 (ii)) and $R \cong \mathrm{End}_R(\oplus_i P_i)$. From now on, we will assume a finite quiver with no oriented cycles is ordered.

Finally, note that given two representations $M = (\{M_v \mid v \in \mathsf{Q}_0\}, \{M_a : M_{s(a)} \to M_{t(a)} \mid a \in \mathsf{Q}_1\})$ and $M' = (\{M'_v \mid v \in \mathsf{Q}_0\}, \{M'_a : M'_{s(a)} \to M'_{t(a)} \mid a \in \mathsf{Q}_1\})$ of a quiver $\mathsf{Q}$ (or equivalently given two right $k\mathsf{Q}$-modules $M$ and $M'$), we can define a **quiver tensor product** $M \otimes_{\mathsf{rep}} M'$ to be the vertex-wise and arrow-wise tensor product

$$(\{M_v \otimes_k M'_v \mid v \in \mathsf{Q}_0\}, \{M_a \otimes_k M'_a : M_{s(a)} \otimes_k M_{s(a)} \to M_{t(a)} \otimes_k M_{t(a)} \mid a \in \mathsf{Q}_1\}).$$

Note the quiver tensor product defines a symmetric monoidal structure on the category of right $k\mathsf{Q}$-modules.

Now, the following are main examples we will deal with:

**Example 2.1.3.**

(i) Let $X$ be a noetherian scheme and let $\mathsf{D}(X)$ denote the derived category of $\mathcal{O}_X$-modules. A complex $\mathcal{F}$ of $\mathcal{O}_X$-modules is said to be **perfect** if it is locally quasi-isomorphic to a bounded complex of locally free $\mathcal{O}_X$-modules of finite rank. Define $\mathsf{Perf}\ X \subset \mathsf{D}(X)$ to be the full subcategory on perfect complexes. Then, the derived tensor product $\otimes_X := \otimes^{\mathbb{L}}_{\mathcal{O}_X}$ gives a rigid tt-structure on $\mathsf{Perf}\ X$. Recall when $X$ is a smooth variety, $\mathsf{Perf}\ X$ is equivalent to the bounded derived category $\mathsf{D}^b\mathsf{coh}X$ of coherent $\mathcal{O}_X$-modules.

(ii) Let $\mathsf{Q}$ be a finite quiver with no oriented cycles and let $\mathsf{D}(k\mathsf{Q})$ denote the derived category of right $k\mathsf{Q}$-modules. A complex of right $k\mathsf{Q}$-modules are said to be **perfect** if it is quasi-isomorphic to a bounded complex of finitely generated projective right $k\mathsf{Q}$-module. Let $\mathsf{Perf}\ k\mathsf{Q} \subset \mathsf{D}(k\mathsf{Q})$ denote the full subcategory on perfect complexes. Then, the derived tensor product $\otimes_{\mathsf{rep}} := \otimes^{\mathbb{L}}_{\mathsf{rep}}$ gives a tt-structure on $\mathsf{Perf}\ k\mathsf{Q}$. Note that since $k\mathsf{Q}$ is finite-dimensional (and hence noetherian) and has finite global dimension, we see that $\mathsf{Perf}\ k\mathsf{Q}$ is equivalent to the bounded derived category $\mathsf{D}^b\mathsf{Mod}^{\mathsf{fg}}k\mathsf{Q}$ of finitely generated right $k\mathsf{Q}$-modules.

The corresponding tt-spectra are known:

**Theorem 2.1.4** ([Bal10b, Theorem 54])**.** *Let $X$ be a quasi-compact and quasi-separated scheme. Then, there is a homeomorphism*

$$S_X : |X| \xrightarrow{\sim} \mathrm{Spec}_{\otimes^{\mathbb{L}}_{\mathcal{O}_X}} \mathsf{Perf}\ X$$

*of ringed spaces, where the continuous map on the underlying topological spaces is given by*

$$x \mapsto S_X(x) := \{\mathcal{F} \in \mathsf{Perf}\ X \mid x \notin \mathrm{Supp}\,\mathcal{F}\}.$$

*Here, for $\mathcal{F} \in \mathsf{Perf}\ X$ we define the **support** of $\mathcal{F}$ to be*

$$\mathrm{Supp}\,\mathcal{F} := \{x \in X \mid \mathcal{F}_x \not\cong 0 \textit{ in } \mathsf{Perf}\ \mathcal{O}_{X,x}\}.$$

**Theorem 2.1.5** ([SL13, (2.1.5.1), (2.2.4.1)])**.** *Let $Q$ be a finite quiver without oriented cycles. Then,*

$$\mathrm{Spc}_{\otimes^{\mathbb{L}}_{\mathsf{rep}}} \mathsf{Perf}\ kQ = \bigsqcup_{v \in Q_0} \mathrm{Ker}(P_v \otimes_{\mathsf{rep}} -).$$

Now, let us recall the construction of the structure sheaf on the tt-spectrum.

**Notation 2.1.6.** In the rest of the paper, we fix the following models for localization and idempotent completion, which are the same as ones fixed in [Bal02, 2.11, 2.12]. See loc. cit. for more details.

(i) Let $\mathcal{K}$ be a triangulated category and let $\mathcal{I}$ be a thick subcategory of $\mathcal{K}$. Then, the localization $\mathcal{K}/\mathcal{I}$ can be modeled by setting objects to be the same as ones of $\mathcal{K}$ and morphisms to be equivalence classes of fractions

$$\xleftarrow{s} \rightarrow$$

with $\mathrm{cone}(s) \in \mathcal{I}$ (cf. e.g., [Ver96, Chapter II]). Note that given a chain of thick subcategories $\mathcal{I} \subset \mathcal{J} \subset \mathcal{K}$, there is a canonical strictly commutative diagram

$$\begin{tikzcd} & \mathcal{K} \arrow[dl] \arrow[dr] & \\ \mathcal{K}/\mathcal{I} \arrow[rr] & & \mathcal{K}/\mathcal{J} \end{tikzcd}$$

where each functor is the identity on the object and the canonical quotient map on the hom set.

(ii) Let $\mathcal{K}$ be a triangulated category (or more generally an additive category). We fix the model of idempotent completion $\mathcal{K}^\natural$ to be the **Karoubi envelope**, i.e., we set objects to be the pair $(M, e)$, where $e : M \to M$ is an idempotent in an object $M$, and morphisms $(M, e) \to (N, f)$ are morphisms $\phi : N \to N$ in $\mathcal{K}$ such that $f \circ \phi = \phi = \phi \circ e$. Note any triangulated functor $F : \mathcal{K} \to \mathcal{L}$ induces a unique canonical functor

$$F^\natural : \mathcal{K}^\natural \to \mathcal{L}^\natural,$$

which sends an object $(M, e)$ to $(F(M), F(e))$ and a morphism $\phi$ to $F(\phi)$.

For a triangulated category $\mathcal{K}$ and a thick subcategory $\mathcal{I} \subset \mathcal{K}$, we write the compositions of the canonical functors by

$$q_{\mathcal{I}} : \mathcal{K} \to \mathcal{K}/\mathcal{I} \to (\mathcal{K}/\mathcal{I})^\natural.$$

One reason for fixing models is to make sense of the strict equalities in the following construction:

**Construction 2.1.7.** Let $\mathcal{K}$ be a triangulated category and let $\mathcal{I} \subset \mathcal{J} \subset \mathcal{K}$ be a chain of thick subcategories. Then, we get a canonical functor

$$\rho_{\mathcal{J}\mathcal{I}} : (\mathcal{K}/\mathcal{I})^\natural \to (\mathcal{K}/\mathcal{J})^\natural$$

induced by the canonical functor $\mathcal{K}/\mathcal{I} \to \mathcal{K}/\mathcal{J}$. In particular, the following diagram strictly commutes:

$$\begin{tikzcd} & \mathcal{K} \arrow[dl] \arrow[dr] & \\ \mathcal{K}/\mathcal{I} \arrow[rr] \arrow[d] & & \mathcal{K}/\mathcal{J} \arrow[d] \\ (\mathcal{K}/\mathcal{I})^\natural \arrow[rr, "\rho_{\mathcal{J}\mathcal{I}}"'] & & (\mathcal{K}/\mathcal{J})^\natural \end{tikzcd}$$

and hence we have the strict equality

$$q_{\mathcal{J}} = \rho_{\mathcal{J}\mathcal{I}} \circ q_{\mathcal{I}}.$$

Now, by construction, for any chain $\mathcal{I} \subset \mathcal{J} \subset \mathcal{L} \subset \mathcal{K}$ of thick subcategories, we have the strict equality:

$$\rho_{\mathcal{L}\mathcal{I}} = \rho_{\mathcal{L}\mathcal{J}} \circ \rho_{\mathcal{J}\mathcal{I}}.$$

Now, we construct a structure sheaf on the Balmer spectrum.

**Construction 2.1.8.** Let $(\mathcal{T}, \otimes)$ be a tt-category with unit $\mathbb{1}$. We define a structure sheaf $\mathcal{O}_{\mathcal{T},\otimes}$ on $\mathrm{Spc}_\otimes \mathcal{T}$ as follows. First, for an open subset $U \subset \mathrm{Spc}_\otimes \mathcal{T}$, set $\mathcal{T}^U = \cap_{\mathcal{P}\in U}\mathcal{P}$, which is clearly a $\otimes$-ideal (where we set $\mathcal{T}^\emptyset = \mathcal{T}$). Then, the triangulated category

$$\mathcal{T}(U) := (\mathcal{T}/\mathcal{T}^U)^\natural$$

has a canonical tt-structure $\otimes_U$ induced by $\otimes$ whose unit $\mathbb{1}_U$ is the image of $\mathbb{1}$ under the canonical functor $q_U := q_{\mathcal{T}^U} : \mathcal{T} \to \mathcal{T}(U)$ (cf. Construction 2.1.7). Now, we set

$$\mathcal{O}^{\mathsf{pre}}_{\mathcal{T},\otimes}(U) := \mathrm{End}_{\mathcal{T}(U)}(\mathbb{1}_U),$$

which is commutative since $\mathbb{1}_U$ is the unit of the (symmetric) monoidal structure $\otimes_U$. Now, recall the following claim, which follows by Construction 2.1.7.

**Lemma 2.1.9** ([Bal02, Proposition 5.3])**.** *Let* $(\mathcal{T}, \otimes)$ *be a tt-category. Let* $V_1 \subset V_2$ *be open subsets of* $\mathrm{Spc}_\otimes \mathcal{T}$*. Then, for the canonical tt-functor*

$$\rho_{V_1,V_2} := \rho_{\mathcal{T}^{V_1},\mathcal{T}^{V_2}} : \mathcal{T}(V_2) \to \mathcal{T}(V_1),$$

*we have the strict equality* $q_{V_1} = \rho_{V_1V_2} \circ q_{V_2}$*. Moreover, for any triple of open subsets* $V_1 \subset V_2 \subset V_3$ *in* $\mathrm{Spc}_\otimes \mathcal{T}$*, we have the strict equality:*

$$\rho_{V_1V_3} = \rho_{V_1V_2} \circ \rho_{V_2V_3}.$$

Therefore, the association

$$\mathcal{O}^{\mathsf{pre}}_{\mathcal{T},\otimes} : (U \subset V) \mapsto \left(\mathrm{End}_{\mathcal{T}(V)}(\mathbb{1}_V) \xrightarrow{\rho_{UV}} \mathrm{End}_{\mathcal{T}(U)}(\mathbb{1}_U)\right)$$

is a presheaf of commutative rings and we define the structure sheaf $\mathcal{O}_{\mathcal{T},\otimes}$ on $\mathrm{Spc}_\otimes \mathcal{T}$ to be its sheafification. The corresponding ringed space is denoted by $\mathrm{Spec}_\otimes \mathcal{T}$.

Note that we can generalize the construction to any subspace $X \subset \mathrm{Th}\,\mathcal{T}$ and an object $M \in \mathcal{T}$ instead of $\mathrm{Spc}_\otimes \mathcal{T} \subset \mathrm{Spc}_\vartriangle \mathcal{T}$ and $\mathbb{1} \in \mathcal{T}$. More precisely, the association

$$\mathcal{O}^{\mathsf{pre}}_{X,M} : (U \subset V \subset X) \mapsto \left(\mathrm{End}_{\mathcal{T}(V)}(q_V(M)) \xrightarrow{\rho_{UV}} \mathrm{End}_{\mathcal{T}(U)}(q_U(M))\right)$$

is a presheaf of (not necessarily commutative) rings, where $\rho_{UV}$'s and $q_U$'s are suitably generalized, e.g., for any open subset $U \subset X$, we set

$$\mathcal{T}(U) := (\mathcal{T}/\cap_{\mathcal{P}\in U}\mathcal{P})^\natural.$$

Now, we can define the sheaf $\mathcal{O}_{X,M}$ of rings on $\mathrm{Spc}_\otimes \mathcal{T}$ to be its sheafification.

We can also equip the Matsui spectrum with a ringed space structure $\mathrm{Spec}_\vartriangle \mathcal{T} = (\mathrm{Spc}_\vartriangle \mathcal{T}, \mathcal{O}_{\mathcal{T},\vartriangle})$ as follows:

**Construction 2.1.10.** Let $\mathcal{T}$ be a triangulated category. The **center** $Z(\mathcal{T})$ of a triangulated category $\mathcal{T}$ is defined to be the commutative ring of natural transformations $\alpha : \mathrm{id}_{\mathcal{T}} \to \mathrm{id}_{\mathcal{T}}$ with $\alpha[1] = [1]\alpha$, where the multiplication is defined by composition. We say an element $\alpha \in Z(\mathcal{T})$ is **locally nilpotent** if $\alpha_M$ is a nilpotent element in the endomorphism ring $\mathrm{End}_{\mathcal{T}}(M)$ for each $M \in \mathcal{T}$. Let $Z(\mathcal{T})_{\mathsf{lnil}}$ denote the ideal of locally nilpotent elements and set $Z(\mathcal{T})_{\mathsf{lrad}} := Z(\mathcal{T})/Z(\mathcal{T})_{\mathsf{lnil}}$.

As in Construction 2.1.8, we can make the association

$$U \mapsto Z(\mathcal{T}(U))_{\mathsf{lred}}.$$

into a presheaf $\mathcal{O}^{\mathsf{pre}}_{\mathcal{T},\vartriangle}$ on $\mathrm{Spc}_{\vartriangle}\mathcal{T}$ and define the structure sheaf $\mathcal{O}_{\mathcal{T},\vartriangle}$ on $\mathrm{Spc}_{\vartriangle}\mathcal{T}$ to be its sheafification. The corresponding ringed space is denoted by $\mathrm{Spec}_{\vartriangle}\mathcal{T}$. See [Mat23] for more details.

Furthermore, as remarked in [HO24, Section 1.3] (see also [Bal02, Definition 6.6]), note that the same construction indeed defines a sheaf on any subspace $X \subset \mathrm{Th}\,\mathcal{T}$, which will be denoted by $\mathcal{O}_{X,\vartriangle}$ or simply $\mathcal{O}_X$ if doing so may not cause any confusion. To be more precise, $\mathcal{O}_{X,\vartriangle}$ is the sheafification of the presheaf

$$X \underset{\text{open}}{\supset} U \mapsto Z(\mathcal{T}(U))_{\mathsf{lred}},$$

where as before we set

$$\mathcal{T}(U) := (\mathcal{T}/\cap_{\mathcal{P}\in U}\mathcal{P})^{\natural}.$$

Note that Balmer's definition ([Bal02, Definition 6.6]) corresponds to the case when we set $X = \mathrm{Spc}_{\otimes}\mathcal{T}$ for a tt-structure $\otimes$ on $\mathcal{T}$. Namely, using their notation, we have the relation

$$\mathrm{Space}_{\mathrm{pt.red}}(\mathcal{T}, \otimes) = (\mathrm{Spc}_{\otimes}\mathcal{T}, \mathcal{O}_{\mathrm{Spc}_{\otimes}\mathcal{T},\vartriangle}).$$

The tensor triangular spectrum and the triangular spectrum are closely related.

**Theorem 2.1.11** ([Mat21, Proposition 4.8], [Mat23, Corollary 3.2, Theorem 4.6])**.** *Let* $(\mathcal{T}, \otimes)$ *be a rigid tt-category so that every* $\otimes$*-ideal is radical.*

*(i) Any prime thick subcategory that is a (radical)* $\otimes$*-ideal is a prime* $\otimes$*-ideal.*

*(ii) If* $(\mathcal{T}, \otimes)$ *is moreover idempotent complete and locally monogenic (cf.* [Mat23, p.9] *for definition) with noetherian Balmer spectrum, then any prime* $\otimes$*-ideal is a prime thick subcategory, i.e., we have* $\mathrm{Spc}_{\otimes}\mathcal{T} \subset \mathrm{Spc}_{\vartriangle}\mathcal{T}$.

*(iii) For a noetherian scheme* $X$*, the tt-category* $(\mathsf{Perf}\,X, \otimes_X)$ *is rigid, idempotent complete, and locally monogenic. In particular, a thick subcategory of* $\mathsf{Perf}\,X$ *is a prime* $\otimes$*-ideal if and only if it is a prime thick subcategory that is a* $\otimes$*-ideal (which is also directly shown in* [Mat21, Corollary 4.9]*).*

For convenience, let us use the following notion in this thesis.

**Definition 2.1.12.** A tt-category $(\mathcal{T}, \otimes)$ (or a tt-structure $\otimes$) is said to be ***M*-compatible** ($M$ for Matsui) if we have that a thick subcategory of $\mathcal{T}$ is a prime $\otimes$-ideal if and only if it is a prime thick subcategory that is a $\otimes$-ideal.

By Theorem 2.1.11, a rigid, idempotent complete and locally monogenic tt-category with noetherian Balmer spectrum is $M$-compatible. On the other hand, this is not a necessary condition.

**Example 2.1.13.** For a finite acyclic quiver $\mathsf{Q}$, the tt-category $(\mathsf{Perf}\ k\mathsf{Q}, \otimes_{\mathsf{rep}})$ is in general not rigid, but since $\mathrm{Spec}_{\otimes_{\mathsf{rep}}} \mathsf{Perf}\ k\mathsf{Q}$ consists of maximal thick subcategories (as $\mathrm{Spec}\ k$'s) (cf. Theorem 2.1.5), we have

$$\mathrm{Spec}_{\otimes_{\mathsf{rep}}} \mathsf{Perf}\ k\mathsf{Q} \subset \mathrm{Spec}_{\vartriangle} \mathsf{Perf}\ k\mathsf{Q}.$$

Moreover, it is easy to see that a thick subcategory is a prime $\otimes$-ideal if and only if it is a prime thick subcategory that is a $\otimes$-ideal.

On the other hand, Example 4.2.9 shows that $(\mathsf{Perf}(B\mu_2), \otimes_{B\mu_2})$ is not $M$-compatible, while being rigid and idempotent complete.

## 2.2 Triangular spectra in algebraic geometry

Now, the following are important aspects of the theories of Balmer and Matsui spectra in terms of algebraic geometry, where the order of references aligns with the order of the statements.

**Theorem 2.2.1** ([Bal05, Theorem 6.3], [Mat21, Corollary 3.8], [Bal02, Theorem 8.5], [Mat25b, Theorem 7.7])**.** *Let $\mathcal{T}$ be a triangulated category with $\mathcal{T} \simeq \mathsf{Perf}\ X$ for a noetherian scheme $X$. Fix a tt-structure $\otimes$ so that $(\mathcal{T}, \otimes)$ is tt-equivalent to $(\mathsf{Perf}\ X, \otimes_X)$. The following assertions hold:*

*(i) We have $\mathrm{Spec}_{\otimes} \mathcal{T} \cong X$ as ringed spaces.*

*(ii) We have the inclusion $\mathrm{Spc}_{\otimes} \mathcal{T} \subset \mathrm{Spc}_{\vartriangle} \mathcal{T}$ as topological spaces.*

*(iii) We have $(\mathrm{Spc}_{\otimes} \mathcal{T}, \mathcal{O}_{\mathrm{Spc}_{\otimes} \mathcal{T}, \vartriangle}) \cong X_{\mathsf{red}}$ as ringed spaces (cf. Construction 2.1.10 for notations).*

*(iv) We have an open immersion*

$$X_{\mathsf{red}} \cong (\mathrm{Spc}_{\otimes} \mathcal{T}, \mathcal{O}_{\mathrm{Spc}_{\otimes} \mathcal{T}, \vartriangle}) \hookrightarrow \mathrm{Spec}_{\vartriangle} \mathcal{T}$$

*of ringed spaces whose underlying continuous map is the inclusion in (ii).*

*Proof.* Only for readers' convenience, we reproduce a proof of part (iv) although it is a mere specialization of the proof of Matsukawa [Mat25b, Theorem 7.7] (see also [Mat25b, Remark 7.8]) to our setting. Let $X = \bigcup_{i \in I} U_i$ be an affine open cover and set $Z_i = X \setminus U_i$. Also, fix a split-generator $\mathscr{E}$ of $\mathsf{Perf}\ X$. (Recall if $X$ is a quasi-compact and quasi-separated scheme, then $\mathsf{Perf}\ X$ has a split generator ([BB02, Corollary 3.1.2]).) By [Tho97, Lemma 3.4] there is $\mathscr{F}_i \in \mathsf{Perf}\ X$ such that $\mathrm{Supp}\, \mathscr{F}_i = Z_i$. Since $\langle \mathscr{F}_i \otimes_X \mathscr{E} \rangle = \langle \mathscr{F}_i \otimes_X \mathsf{Perf}(X) \rangle \subset \mathsf{Perf}\ X$ is a $\otimes$-ideal and

$$\mathscr{F}_i \in \langle \mathscr{F}_i \otimes_X \mathscr{E} \rangle \subset \mathsf{Perf}_{Z_i} X := \{ \mathscr{G} \in \mathsf{Perf}\ X \mid \mathrm{Supp}\, \mathscr{G} \subset Z_i \}$$

we have $\langle \mathscr{F}_i \otimes_X \mathscr{E} \rangle = \mathsf{Perf}_{Z_i} X$ by [Tho97, Theorem 3.15]. Now, by [Mat23, Corollary 4.4], the natural inclusion

$$\mathrm{Spc}_{\vartriangle} \mathsf{Perf}\ X / \mathsf{Perf}_{Z_i} X \hookrightarrow \mathrm{Spc}_{\vartriangle} \mathsf{Perf}\ X$$

is an open embedding, where we identify $\mathrm{Spc}_\vartriangle \mathsf{Perf}\, X/\mathsf{Perf}_{Z_i} X$ with the subspace on prime thick subcategories of $\mathsf{Perf}\, X$ containing $\mathsf{Perf}_{Z_i} X$. Now, recall from [Bal02] that

$$\mathrm{Spc}_{\otimes_X} \mathsf{Perf}\, X = \{S_X(x) \mid x \in X\},$$

where $S_X(x) = \{\mathscr{F} \in \mathsf{Perf}\, X \mid F_x \cong 0 \text{ in } \mathsf{Perf}\, \mathscr{O}_{X,x}\}$ (and $x \in X$ is not necessarily a closed point). We claim

$$\mathrm{Spc}_\vartriangle \mathsf{Perf}\, X/\mathsf{Perf}_{Z_i} X = \{S_X(x) \mid x \in U_i\}.$$

Indeed, it is clear $\mathsf{Perf}_{Z_i} X \subset S_X(x)$ for $x \in U_i$, so we have $\{S_X(x) \mid x \in U_i\} \subset \mathrm{Spc}_\vartriangle \mathsf{Perf}\, X/\mathsf{Perf}_{Z_i} X$. On the other hand, recall that an open immersion $j_i : U_i \hookrightarrow X$ induces a fully faithful dense functor

$$j_i^* : \mathsf{Perf}\, X/\mathsf{Perf}_{Z_i} X \hookrightarrow \mathsf{Perf}\, U_i$$

(cf. [TT90] and [Bal02, Theorem 2.13]) and since $j_i^* S_X(x) = S_{U_i}(x)$ for $x \in U_i$, we have a commutative diagram:

$$\begin{tikzcd}
\{S_X(x) \mid x \in U_i\} \arrow[r, hook] \arrow[d, "j_i^*"'] & \mathrm{Spc}_\vartriangle \mathsf{Perf}\, X/\mathsf{Perf}_{Z_i} X \arrow[d, "j_i^*"] \\
\mathrm{Spc}_{\otimes_{U_i}} \mathsf{Perf}\, U_i \arrow[r, hook] & \mathrm{Spc}_\vartriangle \mathsf{Perf}\, U_i
\end{tikzcd}$$

where vertical arrows are bijections by [Mat21, Proposition 2.11]. Moreover, the bottom inclusion is the identity by [Mat21, Corollary 4.9] (as any thick subcategory of $\mathsf{Perf}\, U_i$ is a $\otimes$-ideal) and therefore we see that the top inclusion is the identity as desired. Now, we have that

$$\mathrm{Spc}_{\otimes_X} \mathsf{Perf}\, X = \bigcup_{i \in I} \mathrm{Spc}_\vartriangle \mathsf{Perf}\, X/\mathsf{Perf}_{Z_i} X \subset \mathrm{Spc}_\vartriangle \mathsf{Perf}\, X$$

is an open subspace and hence we are done by [Mat23, Theorem 4.6]. $\square$

Now, using the same notations as in Theorem 2.2.1, we can interpret those results by the following slogans:

- Part (i) claims that a noetherian scheme can be viewed as a choice of a tt-structure on its perfect derived category.
- Part (ii) claims that topologically such a choice of a tt-structure may be viewed as a choice of a subspace in the Matsui spectrum.
- Part (iii) claims that the reduced structure sheaf on $|X|$ can be recovered as the structure sheaf $\mathscr{O}_{\mathrm{Spc}_\otimes \mathcal{T},\vartriangle}$ of $\mathrm{Spc}_\otimes \mathcal{T}$ in the sense of Construction 2.1.10, which only uses the triangulated category structure once we identify the subspace $\mathrm{Spc}_\otimes \mathcal{T} \subset \mathrm{Spc}_\vartriangle \mathcal{T}$.
- Part (iv) claims that the structure sheaf of $X_{\mathsf{red}}$ can be indeed recovered as the restriction of the structure sheaf $\mathscr{O}_{\mathcal{T},\vartriangle}$ of the entire Matsui spectrum $\mathrm{Spc}_\vartriangle \mathcal{T}$.

An upshot is that understanding subspaces of the Matsui spectra provides a novel way to understand relations of derived categories and algebraic geometry. Now, let us see some examples to get some sense of those definitions. We will come back to those examples later.

**Example 2.2.2.**

(i) [Mat23] Let $X = \mathbb{P}^1_k$. Then we have

$$\mathrm{Spec}_{\vartriangle} \mathsf{Perf}\, \mathbb{P}^1_k \cong \mathbb{P}^1_k \sqcup \bigsqcup_{i\in\mathbb{Z}} \mathrm{Spec}\, k$$

as ringed spaces, where the underlying topological space of each $\mathrm{Spec}\, k$ corresponds to a prime thick subcategory of the form $\langle \mathcal{O}_{\mathbb{P}^1_k}(i)\rangle$ for $i \in \mathbb{Z}$.

(ii) [HO22, Mat23] Let $E$ be an elliptic curve. Then we have

$$\mathrm{Spec}_{\vartriangle} \mathsf{Perf}\, E \cong E \sqcup \bigsqcup_{(r,d)\in I} E_{r,d}$$

as ringed spaces, where $I := \{(r, d) \in \mathbb{Z} \mid r > 0, \gcd(r, d) = 1\}$ and $E_{r,d}$ is a copy of $E$ for each $(r, d) \in I$.

(iii) [GS23, Example 5.1.3] Let $R = k\mathsf{A}_2$ be the path algebra of the $\mathsf{A}_2$-quiver. Then we have

$$\mathrm{Spec}_{\vartriangle} \mathsf{Perf}\, R \cong \mathrm{Spec}\, k \sqcup \mathrm{Spec}\, k \sqcup \mathrm{Spec}\, k$$

as ringed spaces, where the underlying topological space of each $\mathrm{Spec}\, k$ corresponds to points $\langle P_1\rangle, \langle P_2\rangle, \langle S_2\rangle$, respectively. Here, the structure sheaf $\mathcal{O}_{R,\vartriangle}$ of $\mathrm{Spec}_{\vartriangle} \mathsf{Perf}\, R$ can be computed as follows: First of all, since $\mathrm{Spec}_{\vartriangle} \mathsf{Perf}\, R$ is discrete, it suffice to find $\mathcal{O}_{R,\vartriangle}(\langle P_1\rangle)$, $\mathcal{O}_{R,\vartriangle}(\langle P_2\rangle)$, and $\mathcal{O}_{R,\vartriangle}(\langle S_2\rangle)$, which are isomorphic to the stalks of $\mathcal{O}_{R,\vartriangle}$ at the corresponding points. On the other hand, we know

$$\mathcal{O}^{\mathsf{pre}}_{R,\vartriangle}(\langle P_1\rangle) = \mathcal{O}^{\mathsf{pre}}_{R,\vartriangle}(\langle P_2\rangle) = \mathcal{O}^{\mathsf{pre}}_{R,\vartriangle}(\langle S_2\rangle) = Z(\mathsf{Perf}\, k)_{\mathsf{lrad}} = k.$$

Since stalks are preserved under sheafification, we obtain the desired structure sheaf $\mathcal{O}_{R,\vartriangle}$.

We also note that the triangular spectrum remembers semi-orthogonal decompositions.

**Proposition 2.2.3** (Matsui, Matsukawa)**.** *Let $X$ be a noetherian scheme and suppose we have a semi-orthogonal decomposition*

$$\mathsf{Perf}\, X = \langle S_1, \ldots, S_n\rangle.$$

*Then, there is a natural open immersion*

$$\mathrm{Spec}_{\vartriangle} S_1 \sqcup \cdots \sqcup \mathrm{Spec}_{\vartriangle} S_n \hookrightarrow \mathrm{Spec}_{\vartriangle} \mathsf{Perf}\, X.$$

*Proof.* This follows by iteratively applying [Mat25a, Proposition 3.1] since $\mathsf{Perf}\, X$ has a split-generator and hence is classically finitely generated (see also [Mat25a, Example 2.5]). Here, for example, the embedding $\mathrm{Spec}_\vartriangle S_1 \hookrightarrow \mathrm{Spec}_\vartriangle \mathsf{Perf}\, X$ is given as the composition

$$\mathrm{Spec}_\vartriangle S_1 \cong \mathrm{Spec}_\vartriangle \mathcal{T}/\langle S_2, \dots, S_n\rangle \subset \mathrm{Spec}_\vartriangle \mathcal{T}. \qquad \square$$

This allows us to observe more geometric information contained in the Matsui spectrum.

**Example 2.2.4** (Hirano–Ouchi)**.** Let $f : Y \to X$ be a blow-up of a smooth variety $X$ along a smooth center $Z$ of codimension $c$. By Orlov's blow-up formula [Orl93], we have

$$\mathsf{Perf}\, Y = \langle \mathbb{L}f^* \mathsf{Perf}\, X, \Psi_0(\mathsf{Perf}\, Z), \dots, \Psi_{c-2}(\mathsf{Perf}\, Z)\rangle,$$

where $f^* : \mathsf{Perf}\, X \hookrightarrow \mathsf{Perf}\, Y$ and $\Psi_i : \mathsf{Perf}\, Z \hookrightarrow \mathsf{Perf}\, Y$ are fully faithful. Hence, we have an open immersion

$$X \sqcup Z \sqcup \cdots \sqcup Z \subset \mathrm{Spec}_\vartriangle \mathsf{Perf}\, X \sqcup \mathrm{Spec}_\vartriangle \mathsf{Perf}\, Z \sqcup \cdots \sqcup \mathrm{Spec}_\vartriangle \mathsf{Perf}\, Z \hookrightarrow \mathrm{Spec}_\vartriangle \mathsf{Perf}\, Y.$$

Moreover, the computations in [HO22, Example 3.8] show that the image of $X$ intersects with $\mathrm{Spec}_{\otimes_Y} \mathsf{Perf}\, Y \cong Y$ along the complement of $Z$ and the exceptional divisor. On the other hand, the image of $c-1$ copies of $Z$'s do not intersect with $Y$ since $\mathbb{L}f^* \mathsf{Perf}\, X$ and hence any prime thick subcategory in the image of $\mathrm{Spec}_\vartriangle \mathsf{Perf}\, Z$ contains $\mathcal{O}_Y$, but $S_Y(y)$ does not.

Given other semi-orthogonal decompositions coming from projective bundles [Orl93], Severi–Brauer varieties [Ber09, Mat25b], standard flips [BO95, HO22], quadratic fibrations [Kuz08], etc., we obtain similar examples.

Finally, let us recall that perfect derived categories on smooth projective varieties have a particularly nice property.

**Definition 2.2.5.** Let $\mathcal{T}$ be a triangulated category with finite dimensional Hom-spaces. A **Serre functor** $\mathbb{S} : \mathcal{T} \simeq \mathcal{T}$ is a triangulated autoequivalence such that for any $F, G \in \mathcal{T}$, there is a functorial isomorphism

$$\mathrm{Hom}_\mathcal{T}(F, G) \cong \mathrm{Hom}_\mathcal{T}(G, \mathbb{S}(F))^*,$$

where $(-)^*$ denotes the dual of a vector space over $k$. It is well-known that a Serre functor, if exists, is unique up to natural isomorphism and commutes with any $k$-linear autoequivalence of $\mathcal{T}$ (e.g. [Huy06, Lemma 1.30]). When $\mathcal{T}$ admits the Serre functor $\mathbb{S}$, we define the **Serre invariant locus** to be

$$\mathrm{Spc}^{\mathsf{Ser}}\, \mathcal{T} := \mathrm{Spc}^{\mathbb{S}}\, \mathcal{T}.$$

**Example 2.2.6.**

(i) If $\mathcal{T} = \mathsf{Perf}\, X$ for a Gorenstein proper variety $X$, then we have $\mathbb{S} = - \otimes_{\mathcal{O}_X} \omega_X[\dim X]$, where $\omega_X$ denotes the canonical bundle of $X$. In particular, if $X$ has the trivial canonical bundle, then $\mathbb{S}$ is given by the shift functor $[\dim X]$.

(ii) If $\mathcal{T} = \mathsf{Perf}\, k\mathsf{Q}$ for a finite quiver $\mathsf{Q}$ with no oriented cycle, then a Serre functor $\mathbb{S}$ is given by the Nakayama functor, which is the composition $\mathbb{R}\operatorname{Hom}_{k\mathsf{Q}}(-, k\mathsf{Q}) \circ \mathbb{R}\operatorname{Hom}_k(-, k)$.

Let us recall some facts about the following simplest case:

**Example 2.2.7.** Let $R = k\mathsf{A}_2$ be the path algebra of the $\mathsf{A}_2$-quiver. Then it is easy to see there are only three indecomposable representations:

- $(k \leftarrow 0) \cong e_1 R = P_1 = S_1$;
- $(0 \leftarrow k) \cong S_2 = I_2$;
- $(k \xleftarrow{\sim} k) \cong e_2 R = P_2 = I_1$;

and thus the Nakayama functor $\mathbb{S}$ acts by $\mathbb{S}(P_1) = I_1 = P_2$, $\mathbb{S}(P_2) = I_2 = S_2$, and $\mathbb{S}(S_2) = P_1[1]$, where the last equality follows since $\mathbb{S}$ is triangulated and we have a distinguished triangle $P_1 \to P_2 \to I_2 \to P_1[1]$.

# Chapter 3

# The Fourier–Mukai locus in the triangular spectrum

In [Ito23], we constructed the Fourier–Mukai locus by gluing the Balmer spectra in the triangular spectrum. Later in [IM25], we proved that the Fourier–Mukai locus constructed in [Ito23] is naturally isomorphic to an open ringed subspace of the triangular spectrum. In this thesis, we take the latter approach to construct the Fourier–Mukai locus and observe that they indeed coincide. In particular, this chapter also serves as a self-contained exposition of [Ito23] with the approach of [IM25].

## 3.1 Construction of the Fourier–Mukai locus and orbits of Fourier–Mukai partners

**Definition 3.1.1.** Let $\mathcal{T}$ be a triangulated category.

(i) We say a smooth projective variety $X$ is a **Fourier-Mukai partner** of $\mathcal{T}$ if there exists a triangulated equivalence $\mathcal{T} \simeq \mathsf{Perf}\, X$. Let $\mathrm{FM}\,\mathcal{T}$ denote the set of isomorphism classes of Fourier-Mukai partners of $\mathcal{T}$.

(ii) We say a tt-structure $\otimes$ on $\mathcal{T}$ is **geometric in** $X \in \mathrm{FM}\,\mathcal{T}$ if there exists an equivalence

$$(\mathcal{T}, \otimes) \simeq (\mathsf{Perf}\, X, \otimes_X)$$

of tt-categories. We say a tt-structure $\otimes$ on $\mathcal{T}$ is **geometric** if there exists $X \in \mathrm{FM}\,\mathcal{T}$ such that $\otimes$ is geometric in $X$.

(iii) Define the **Fourier-Mukai locus** of $\mathcal{T}$ to be the subspace

$$\mathrm{Spc}^{\mathsf{FM}}\,\mathcal{T} := \bigcup_{\text{geom. tt-str. } \otimes \text{ on } \mathcal{T}} \mathrm{Spc}_{\otimes}\,\mathcal{T} \subset \mathrm{Spc}_{\vartriangle}\,\mathcal{T}.$$

Now, we have the following consequences of Theorem 2.2.1.

**Proposition 3.1.2.** *Let $\mathcal{T}$ be a triangulated category with* $\mathrm{FM}\,\mathcal{T} \neq \emptyset$. *Then, the following hold:*

*(i) For a geometric tt-structure $\otimes$ on $\mathcal{T}$, the inclusion* $\mathrm{Spc}_{\otimes}\,\mathcal{T} \subset \mathrm{Spc}_{\vartriangle}\,\mathcal{T}$ *is open.*

*(ii) The Fourier-Mukai locus of $\mathcal{T}$ is open in the triangular spectrum of $\mathcal{T}$.*

*(iii) In* [Ito23]*, the topology on the Fourier-Mukai locus is defined to be the one generated by open subsets of the Balmer spectrum for each geometric tt-structure. This topology on* $\mathrm{Spc}^{\mathsf{FM}}\,\mathcal{T}$ *agrees with the subspace topology on* $\mathrm{Spc}^{\mathsf{FM}}\,\mathcal{T}$ *in* $\mathrm{Spc}_{\vartriangle}\,\mathcal{T}$.

*Proof.* For a fixed $X \in \mathrm{FM}\,\mathcal{T}$ and a tt-equivalence $\Phi : \mathcal{T} \xrightarrow{\sim} \mathsf{Perf}\,X$, we have a commutative diagram

$$\begin{tikzcd}
\mathrm{Spc}_{\otimes}\,\mathcal{T} \arrow[r, "\subseteq"] \arrow[d, "\Phi"', "\cong"] & \mathrm{Spc}^{\mathsf{FM}}\,\mathcal{T} \arrow[r, "\subseteq"] \arrow[d, "\Phi"', "\cong"] & \mathrm{Spc}_{\vartriangle}\,\mathcal{T} \arrow[d, "\Phi"', "\cong"] \\
\mathrm{Spc}_{\otimes}\,\mathsf{Perf}\,X \arrow[r, "\subseteq"] & \mathrm{Spc}^{\mathsf{FM}}\,\mathsf{Perf}\,X \arrow[r, "\subseteq"] & \mathrm{Spc}_{\vartriangle}\,\mathsf{Perf}\,X
\end{tikzcd}$$

where the vertical maps induced from $\Phi$ are homeomorphisms. By Theorem 2.2.1, we have that the inclusion $\mathrm{Spc}_{\otimes_X}\,\mathsf{Perf}\,X \subset \mathrm{Spc}_{\vartriangle}\,\mathsf{Perf}\,X$ is an open embedding. Therefore, we have part (i). Part (ii) and (iii) immediately follow from part (i). □

Note by [Ito23, Theorem 4.7], we can glue the Balmer spectra corresponding to geometric tt-structures to equip $\mathrm{Spc}^{\mathsf{FM}}\,\mathcal{T}$ with a scheme structure, where the corresponding scheme is denoted by $\mathrm{Spec}^{\mathsf{FM}}\,\mathcal{T}$. Recall by construction the scheme structure on the Fourier-Mukai locus satisfies the following properties.

**Theorem 3.1.3** ([Ito23, Theorem 4.7])**.** *Let $\mathcal{T}$ be a triangulated category with* $\mathrm{FM}\,\mathcal{T} \neq \emptyset$. *Then, the scheme* $\mathrm{Spec}^{\mathsf{FM}}\,\mathcal{T}$ *is a smooth scheme locally of finite type and for any geometric tt-structure $\otimes$, we have a canonical open immersion*

$$\mathrm{Spec}_{\otimes}\,\mathcal{T} \hookrightarrow \mathrm{Spec}^{\mathsf{FM}}\,\mathcal{T}$$

*of schemes whose underlying continuous map is the inclusion.*

Now, we show that we can obtain the same scheme structure on the Fourier-Mukai locus by simply restricting the structure sheaf on the triangular spectrum to the Fourier-Mukai locus. Logically, the following steps are unnecessary for the later parts and readers may safely skip to Notation 3.1.7, but we present them for readers' convenience. To see this, let us recall the following classical result (e.g. [Gro64, Proposition 10.9.6] and [Har77, Proposition I.3.5]):

**Lemma 3.1.4.** *Let $X$ and $Y$ be reduced schemes locally of finite type over an algebraically closed field $k$ and let $f, g : X \to Y$ be morphisms of schemes over $k$. If $f$ and $g$ agree on the set of closed points, then they agree as a morphism of schemes over $k$.*

Now, we are ready to show the following.

**Theorem 3.1.5.** *Let $\mathcal{T}$ be a triangulated category with* $\operatorname{FM}\mathcal{T} \neq \emptyset$. *Then, there is an isomorphism*

$$\operatorname{Spec}^{\mathsf{FM}}\mathcal{T} \xrightarrow{\sim} (\operatorname{Spc}^{\mathsf{FM}}\mathcal{T}, \mathcal{O}_{\mathcal{T},\Delta}|_{\operatorname{Spc}^{\mathsf{FM}}\mathcal{T}})$$

*of ringed spaces whose underlying continuous map is the identity.*

*Proof.* First, note that by Theorem 2.2.1, $(\operatorname{Spc}^{\mathsf{FM}}\mathcal{T}, \mathcal{O}_{\mathcal{T},\Delta}|_{\operatorname{Spc}^{\mathsf{FM}}\mathcal{T}})$ is a smooth scheme locally of finite type. Take an open covering $\{\operatorname{Spc}_{\otimes_\alpha}\mathcal{T}\}_{\alpha\in A}$ of $\operatorname{Spc}^{\mathsf{FM}}\mathcal{T}$, where each $\otimes_\alpha$ is a geometric tt-structure on $\mathcal{T}$. Then, for each $\alpha \in A$, there exists a canonical open immersion

$$\iota_\alpha : \operatorname{Spec}_{\otimes_\alpha}\mathcal{T} \hookrightarrow \operatorname{Spec}^{\mathsf{FM}}\mathcal{T}$$

whose underlying continuous map is the inclusion by Theorem 3.1.3 and therefore there is an open immersion

$$i_\alpha : \iota_\alpha(\operatorname{Spec}_{\otimes_\alpha}\mathcal{T}) \hookrightarrow (\operatorname{Spc}^{\mathsf{FM}}\mathcal{T}, \mathcal{O}_{\mathcal{T},\Delta}|_{\operatorname{Spc}^{\mathsf{FM}}\mathcal{T}})$$

whose underlying continuous map is the inclusion by Theorem 2.2.1. Now, by Lemma 3.1.4, we see that $\{i_\alpha\}_{\alpha\in A}$ glues to a morphism

$$\phi : \operatorname{Spec}^{\mathsf{FM}}\mathcal{T} \to (\operatorname{Spc}^{\mathsf{FM}}\mathcal{T}, \mathcal{O}_{\mathcal{T},\Delta}|_{\operatorname{Spc}^{\mathsf{FM}}\mathcal{T}})$$

whose underlying continuous map is the identity. Since $\phi$ is a homeomorphism and locally an isomorphism, it is an isomorphism. □

*Remark* 3.1.6. Note that Theorem 3.1.5 makes the construction of the structure sheaf on the Fourier-Mukai locus purely triangulated categorical. Therefore, if we can determine the underlying topological space of the Fourier-Mukai locus categorically, then we can reconstruct information coming from the gluings of structure sheaves of Fourier-Mukai partners performed in [Ito23]. In particular, we have more hope to have backward applications of the Fourier-Mukai locus to birational geometry.

**Notation 3.1.7.** In the sequel, the **Fourier–Mukai locus** refers to the ringed space

$$(\operatorname{Spc}^{\mathsf{FM}}\mathcal{T}, \mathcal{O}_{\mathcal{T},\Delta}|_{\operatorname{Spc}^{\mathsf{FM}}\mathcal{T}}).$$

Now, let us recall some geometric results on the Fourier-Mukai locus from [Ito23], to which we provide self-contained proofs. First, we recall basic notations and terminologies.

**Definition 3.1.8.** Let $\mathcal{T}$ be a triangulated category with $X \in \operatorname{FM}\mathcal{T}$. Define the **full orbit** of $X$ in the Fourier-Mukai locus to be the open subscheme

$$\operatorname{Spec}_{\otimes,X}\mathcal{T} := \bigcup_{\text{geom. tt-str. } \otimes \text{ in } X} \operatorname{Spec}_{\otimes}\mathcal{T} \subset \operatorname{Spec}^{\mathsf{FM}}\mathcal{T}.$$

In particular, the underlying topological space $\operatorname{Spc}_{\otimes,X}\mathcal{T}$ is the union of all Balmer spectra corresponding to geometric tt-structures in $X$.

The name "full orbit" is justified by the following more general construction.

**Definition 3.1.9.** Let $\mathcal{T}$ be a triangulated category with $X \in \mathrm{FM}\,\mathcal{T}$ and fix a geometric tt-structure $\otimes_0$ in $X$. For a subgroup $G \subset \mathrm{Auteq}\,\mathcal{T}$, define the ***G*-orbit** at $\otimes_0$ to be the open subscheme

$$G \cdot \mathrm{Spec}_{\otimes_0} \mathcal{T} := \bigcup_{\tau \in G} \tau(\mathrm{Spc}_{\otimes_0} \mathcal{T}) \subset \mathrm{Spec}^{\mathsf{FM}} \mathcal{T}.$$

When $G = \mathrm{Auteq}\,\mathcal{T}$, the $G$-orbit at any geometric tt-structure in $X$ coincides with the full orbit of $X$.

*Remark* 3.1.10. In [Ito23], the scheme $G \cdot \mathrm{Spec}_{\otimes_0} \mathcal{T}$ is denoted by $\mathrm{Spec}^{G\cdot\eta}_{\otimes,X} \mathcal{T}$, where $\otimes_0 = \otimes_{X,\eta}$.

**Example 3.1.11.** Let $X$ be a smooth projective variety. We set

$$\mathrm{Auteq}^{\mathsf{st}} \mathsf{Perf}\, X := \mathrm{Aut}\, X \ltimes \mathrm{Pic}\, X \times \mathbb{Z}[1] \subset \mathrm{Auteq}\, \mathsf{Perf}\, X$$

to be the group of **standard autoequivalences**. Clearly, we have

$$\mathrm{Auteq}^{\mathsf{st}} \mathsf{Perf}\, X \cdot \mathrm{Spec}_{\otimes_X} \mathsf{Perf}\, X = \mathrm{Spec}_{\otimes_X} \mathsf{Perf}\, X.$$

It is natural to ask how those copies of Fourier-Mukai partners interact with each other in the triangular spectrum. Indeed, the following results and examples in [Ito23] show that the topology of the Fourier-Mukai locus is closely related to types of possible equivalences between Fourier-Mukai partners, which are then related to (birational) geometric properties of varieties.

**Definition 3.1.12.** Let $X$ and $Y$ be smooth projective varieties. We say a triangulated equivalence

$$\Phi : \mathsf{Perf}\, X \to \mathsf{Perf}\, Y$$

is **birational** if there exists a closed point $x \in X$ such that $\Phi(k(x))$ is isomorphic to $k(y)$ for some closed point $y \in Y$. If a birational equivalence exists, $X$ and $Y$ are said to be **birational Fourier–Mukai partners**. Let

$$\mathrm{BAuteq}\, X \subset \mathrm{Auteq}\, \mathsf{Perf}\, X$$

denote the subgroup of birational autoequivalences.

**Example 3.1.13.** For a smooth projective variety $X$, since any standard autoequivalence sends a skyscraper sheaf to a skyscraper sheaf up to shift, we have

$$\mathrm{Auteq}^{\mathsf{st}} \mathsf{Perf}\, X \subset \mathrm{BAuteq}\, X \times \mathbb{Z}[1].$$

Recall that for a noetherian scheme $X$, the isomorphism $X \cong \mathrm{Spec}_{\otimes_X} \mathsf{Perf}\, X$ is given by

$$X \ni x \mapsto S_X(x) := \{\mathscr{F} \in \mathsf{Perf}\, X \mid \mathscr{F}_x \cong 0\} \in \mathrm{Spec}_{\otimes_X} \mathsf{Perf}\, X.$$

Let us also recall the following useful observation by Hirano–Ouchi.

**Lemma 3.1.14.** [HO22, Lemma 3.2] *Let $X$ and $Y$ be smooth varieties and suppose that we have a fully faithful functor $\Phi : \mathsf{Perf}\, X \hookrightarrow \mathsf{Perf}\, Y$ that admits a right adjoint functor. Then, for $x \in X$ and $y \in Y$, we have $\Phi(S_X(x)) = S_Y(y)$ if and only if $\Phi(k(x)) \cong k(y)[l]$ for some $l \in \mathbb{Z}$.*

As a consequence, we obtain a useful criterion for birational triangulated equivalences.

**Lemma 3.1.15.** *A triangulated equivalence $\Phi : \mathsf{Perf}\, X \to \mathsf{Perf}\, Y$ is birational up to shift (i.e., $\Phi \in \mathrm{BAuteq}\, X \times \mathbb{Z}[1]$) if and only if*

$$\Phi(\mathrm{Spec}_{\otimes_X} \mathsf{Perf}\, X) \cap \mathrm{Spec}_{\otimes_Y} \mathsf{Perf}\, Y \neq \emptyset.$$

*Moreover, birational Fourier–Mukai partners $X$ and $Y$ are birationally equivalent (and indeed $K$-equivalent (cf. Definition 3.3.38)).*

*Proof.* By Lemma 3.1.14, $\Phi(\mathrm{Spec}_{\otimes_X} \mathsf{Perf}\, X) \cap \mathrm{Spec}_{\otimes_Y} \mathsf{Perf}\, Y \neq \emptyset$ if and only if $\Phi(k(x)) \cong k(y)[l]$ for some $x \in X$, $y \in Y$, and $l \in \mathbb{Z}$, i.e., $\Phi$ is birational up to shift by definition. Now, since $X \cong \Phi(\mathrm{Spec}_{\otimes_X} \mathsf{Perf}\, X)$ and $Y \cong \mathrm{Spec}_{\otimes_Y} \mathsf{Perf}\, Y$ are both open in $\mathrm{Spec}_{\vartriangle} \mathsf{Perf}\, Y$, the intersection $\Phi(\mathrm{Spec}_{\otimes_X} \mathsf{Perf}\, X) \cap \mathrm{Spec}_{\otimes_Y} \mathsf{Perf}\, Y$ defines isomorphic open subschemes in $X$ and $Y$, i.e., $X$ and $Y$ are birationally equivalent. For $K$-equivalences, see Theorem 3.3.42. □

**Definition 3.1.16.** Given a birational triangulated equivalence $\Phi : \mathsf{Perf}\, X \to \mathsf{Perf}\, Y$ up to shift, we define the **associated birational map** to be the birational map given by

$$X \supset \mathrm{Spec}_{\otimes_X} \mathsf{Perf}\, X \cap \Phi^{-1}(\mathrm{Spec}_{\otimes_X} \mathsf{Perf}\, X) \overset{\Phi}{\cong} \Phi(\mathrm{Spec}_{\otimes_X} \mathsf{Perf}\, X) \cap \mathrm{Spec}_{\otimes_Y} \mathsf{Perf}\, Y \subset Y.$$

*Remark* 3.1.17. In [Ito23], we made use of Inaba's moduli of simple objects of $\mathsf{Perf}\, X$ ([Ina02]) and Calabrese's results in [Cal17] to show the openness of the domain of definition, but because of the openness of the tt-spectrum shown in [IM25], we no longer need to use them.

Let us also record when a triangulated equivalence is not birational.

**Lemma 3.1.18.** *Suppose the kernel $\mathscr{P} \in \mathsf{Perf}(X \times Y)$ of a Fourier-Mukai equivalence $\Phi_{\mathscr{P}} : \mathsf{Perf}\, X \simeq \mathsf{Perf}\, Y$ is a locally free sheaf on $X \times Y$. Then, $\Phi$ is not birational, i.e.,*

$$\Phi(\mathrm{Spec}_{\otimes_X} \mathsf{Perf}\, X) \cap \mathrm{Spec}_{\otimes_Y} \mathsf{Perf}\, Y = \emptyset.$$

*Proof.* Since $\mathscr{P}$ is flat over $X$, we see that $\Phi_{\mathscr{P}}(k(x)) = \mathscr{P}|_{\{x\}\times Y}$ for any closed point $x \in X$ by [Huy06, Example 5.4], which is locally free and hence $\mathrm{Supp}\, \Phi_{\mathscr{P}}(k(x)) = Y$. □

Using the action of (birational) autoequivalences, let us give the decomposition theorem of the full orbit $\mathrm{Spec}_{\otimes,X} \mathsf{Perf}\, X$ of $X$.

**Lemma 3.1.19.** *Let $X$ be a scheme admitting an open cover $\{U_i\}_{i\in I}$ by irreducible open subschemes $U_i$. Then a connected component and an irreducible component coincide. In particular, such a component is given as a union of mutually intersecting irreducible open subschemes of the form $U_i$.*

*Proof.* First, we can write

$$X = \bigsqcup_{j \in J} V_j,$$

where each $V_j$ is a union of mutually intersection irreducible open subschemes of the form $U_i$. Note that each $V_j$ is irreducible and hence connected, which suffices for a proof. □

In general, a smooth projective variety is not tt-irreducible, but we have an important symmetry for irreducible components of the full orbit $\mathrm{Spec}_{\otimes,X} \mathsf{Perf}\, X$:

**Lemma 3.1.20.** *Let $X$ be a smooth projective variety. Then, the irreducible component of the full orbit $\mathrm{Spec}_{\otimes,X} \mathsf{Perf}\, X$ containing $\mathrm{Spec}_{\otimes} \mathsf{Perf}\, X$ agrees with*

$$\mathrm{BAuteq}\, X \cdot \mathrm{Spec}_{\otimes_X} \mathsf{Perf}\, X \subset \mathrm{Spec}^{\mathsf{FM}} \mathsf{Perf}\, X,$$

*which we call the **birational orbit** of $X$ in the Fourier-Mukai locus.*

*Proof.* Since for $\tau \in \mathrm{Auteq}\, \mathsf{Perf}\, X$, $\tau(\mathrm{Spec}_{\otimes_X} \mathsf{Perf}\, X) \cap \mathrm{Spec}_{\otimes_X} \mathsf{Perf}\, X \neq \emptyset$ if and only if $\tau \in \mathrm{BAuteq}\, X \times \mathbb{Z}[1]$, the claim follows from Lemma 3.1.19 noting shifts act trivially. □

**Theorem 3.1.21.** *Let $X$ be a smooth projective variety and consider the set of left cosets*

$$I_X := \mathrm{Auteq}\, \mathsf{Perf}\, X / \mathrm{BAuteq}\, X \times \mathbb{Z}[1].$$

*Then, we have the decomposition of the full orbit:*

$$\mathrm{Spec}_{\otimes,X} \mathsf{Perf}\, X = \bigsqcup_{[\tau] \in I_X} \tau(\mathrm{BAuteq}\, X \cdot \mathrm{Spec}_{\otimes_X} \mathsf{Perf}\, X) \cong \bigsqcup_{I_X} \mathrm{BAuteq}\, X \cdot \mathrm{Spec}_{\otimes_X} \mathsf{Perf}\, X.$$

*In particular, each irreducible component of the full orbit is isomorphic to the birational orbit of $X$. See Corollary 3.3.13 for an example of this expression in the case of an abelian variety.*

*Proof.* First, note by definition that the scheme $\tau(\mathrm{BAuteq}\, X \cdot \mathrm{Spec}_{\otimes_X} \mathsf{Perf}\, X)$ does not depend on the choice of the representative of $[\tau]$. Moreover, by definition we have

$$\mathrm{Spec}_{\otimes,X} \mathsf{Perf}\, X = \bigcup_{[\tau] \in I_X} \tau(\mathrm{BAuteq}\, X \cdot \mathrm{Spec}_{\otimes_X} \mathsf{Perf}\, X).$$

Therefore, it is enough to see the right hand side is disjoint, which indeed follows since we have $\tau(\mathrm{Spec}_{\otimes_X} \mathsf{Perf}\, X) \cap \mathrm{Spec}_{\otimes_X} \mathsf{Perf}\, X \neq \emptyset$ if and only if $\tau \in \mathrm{BAuteq}\, X \times \mathbb{Z}[1]$. □

Since we by definition have

$$\mathrm{Spec}^{\mathsf{FM}}\, \mathcal{T} = \bigcup_{X \in \mathrm{FM}\, \mathcal{T}} \mathrm{Spec}_{\otimes,X}\, \mathcal{T},$$

the study of the Fourier–Mukai locus reduces to understanding the birational orbit for each Fourier–Mukai partner and how birational orbits interact for different Fourier–Mukai partners. Finally, let us introduce the following notion for convenience.

**Proposition 3.1.22.** *Let $\mathcal{T}$ be a triangulated category with $X, Y \in \mathrm{FM}\,\mathcal{T}$. Then, $X$ and $Y$ are birational Fourier-Mukai partners if and only if*

$$\mathrm{Spec}_{\otimes,X}\,\mathcal{T} \cap \mathrm{Spec}_{\otimes,Y}\,\mathcal{T} \neq \emptyset.$$

*Therefore, any birational Fourier-Mukai partner is isomorphic if and only if we have*

$$\mathrm{Spec}^{\mathsf{FM}}\,\mathcal{T} = \bigsqcup_{X \in \mathrm{FM}\,\mathcal{T}} \mathrm{Spec}_{\otimes,X}\,\mathcal{T}$$

*in which case $\mathcal{T}$ is said to be of* ***disjoint Fourier-Mukai type****.*

*Proof.* The first claim is a direct consequence of Lemma 3.1.15 and the second claim is the direct consequence of the first. □

Finally, let us pose the following natural question.

**Question 3.1.23.** Let $\mathcal{T}$ be a triangulated category with $X, Y \in \mathrm{FM}\,\mathcal{T}$. Does

$$\mathrm{Spec}_{\otimes,X}\,\mathcal{T} = \mathrm{Spec}_{\otimes,Y}\,\mathcal{T}$$

imply $X \cong Y$? Note that the equality implies that the birational orbits of $X$ and $Y$ are isomorphic.

## 3.2 Properties of the Fourier–Mukai locus and orbits as schemes

Now, let us observe some basic properties of Fourier-Mukai loci and full orbits, and their relations with (birational) geometry.

**Definition 3.2.1.** Let $X$ be a smooth projective variety. For a property $\mathcal{P}$ of a scheme, we say $X$ satisfies **tt-**$\mathcal{P}$ (tt for tensor triangular) if $\mathrm{Spec}_{\otimes,X}\,\mathsf{Perf}\,X$ satisfies $\mathcal{P}$.

**Example 3.2.2.** Any smooth projective variety is tt-smooth and tt-locally of finite type, so we are mainly interested in understanding if a smooth projective variety is tt-separated, tt-irreducible, or tt-quasi-compact. Since a universally closed morphism is quasi-compact and in this thesis only examples of tt-quasi-compact varieties will be smooth projective varieties with $\otimes$-generating canonical bundles (e.g. with ample (anti)canonical bundles), we will not elaborate on tt-universally closedness, let alone tt-properness/projectivity.

### Separatedness

Separatedness of Fourier–Mukai loci and full orbits implies that varieties appearing are birationally rigid.

**Proposition 3.2.3.** *Let $X$ be a smooth projective variety. The following are equivalent:*

*(i) $X$ is tt-separated;*

*(ii) The birational orbit coincides with* $\mathrm{Spec}_{\otimes_X} \mathsf{Perf}\, X$*, i.e.,*

$$\mathrm{BAuteq}\, X \cdot \mathrm{Spec}_{\otimes_X} \mathsf{Perf}\, X = \mathrm{Spec}_{\otimes_X} \mathsf{Perf}\, X.$$

*(iii) We have the decomposition*

$$\mathrm{Spec}_{\otimes,X} \mathsf{Perf}\, X = \bigsqcup_{[\tau]\in I_X} \tau(\mathrm{Spec}_{\otimes_X} \mathcal{T}) \cong \bigsqcup_{I_X} X;$$

*(iv) If $\Phi : \mathsf{Perf}\, X \to \mathsf{Perf}\, X$ is a birational triangulated equivalence, then for any closed point $x \in X$, there exists a closed point $x' \in X$ such that $\Phi(k(x)) \cong k(x')$;*

*(v) We have*

$$\mathrm{BAuteq}\, X \times \mathbb{Z}[1] = \mathrm{Auteq}^{\mathrm{st}} \mathsf{Perf}\, X.$$

*Equivalently, any birational autoequivalence is standard.*

*In light of Remark 3.1.6, a tt-separated smooth projective variety $X$ can be reconstructed as a connected component of* $\mathrm{Spec}_{\otimes,X} \mathsf{Perf}\, X$ *if we can categorically determine* $\mathrm{Spc}_{\otimes,X} \mathsf{Perf}\, X$.

*Proof.* First of all, (ii) $\Rightarrow$ (iii) follows by Theorem 3.1.21 and (iii) $\Rightarrow$ (i) is clear. Also, by [Huy06, Corollary 5.23] and Lemma 3.1.14, (iv) $\iff$ (v) $\iff$ (ii) follows. Therefore, it remains to show (i) $\Rightarrow$ (ii). We show its contrapositive. Suppose there exists a birational autoequivalence $\Phi \in \mathrm{BAuteq}\, X$ such that

$$U := \Phi(\mathrm{Spec}_{\otimes_X} \mathsf{Perf}\, X) \cup \mathrm{Spec}_{\otimes_X} \mathsf{Perf}\, X \supsetneq \mathrm{Spec}_{\otimes_X} \mathsf{Perf}\, X.$$

Then, $U$ is not a separated scheme since otherwise $X \cong \mathrm{Spec}_{\otimes_X} \mathsf{Perf}\, X$ is closed in $U$ (as it is an image of a proper scheme in a separated scheme) and hence $\mathrm{Spec}_{\otimes_X} \mathsf{Perf}\, X$ is a closed and open subscheme of $U$, i.e.,

$$U = \Phi(\mathrm{Spec}_{\otimes_X} \mathsf{Perf}\, X) \sqcup \mathrm{Spec}_{\otimes_X} \mathsf{Perf}\, X$$

which is absurd as $\Phi$ is birational (cf. Lemma 3.1.15). Therefore, $\mathrm{Spec}_{\otimes,X} \mathsf{Perf}\, X$ has a non-separated open subscheme $U$ and hence is not separated either. □

**Proposition 3.2.4.** *Let $\mathcal{T}$ be a triangulated category with $\mathrm{FM}\,\mathcal{T} \neq \emptyset$. The following are equivalent:*

*(i) $\mathrm{Spec}^{\mathsf{FM}}\, \mathcal{T}$ is a separated scheme;*

*(ii) For any $X_1, X_2 \in \mathrm{FM}\,\mathcal{T}$ and $\Phi : \mathsf{Perf}\, X_1 \simeq \mathsf{Perf}\, X_2$, if $\Phi$ is a birational triangulated equivalence up to shifts, i.e., if*

$$\Phi(\mathrm{Spec}_{\otimes_{X_1}} \mathsf{Perf}\, X_1) \cap \mathrm{Spec}_{\otimes_{X_2}} \mathsf{Perf}\, X_2 \neq \emptyset,$$

*then we have*

$$\Phi(\mathrm{Spec}_{\otimes_{X_1}} \mathsf{Perf}\, X_1) = \mathrm{Spec}_{\otimes_{X_2}} \mathsf{Perf}\, X_2.$$

*In particular, we have $X_1 \cong X_2$.*

*(iii)* $\mathcal{T}$ *is of disjoint Fourier–Mukai type and each* $X \in \mathrm{FM}\,\mathcal{T}$ *is tt-separated. Equivalently,*

$$\mathrm{Spec}^{\mathsf{FM}}\,\mathcal{T} = \bigsqcup_{X\in \mathrm{FM}\,\mathcal{T}} \bigsqcup_{[\tau]\in I_X} \tau(\mathrm{Spec}_{\otimes_X} \mathsf{Perf}\, X) \cong \bigsqcup_{X\in \mathrm{FM}\,\mathcal{T}} \bigsqcup_{I_X} X.$$

*In particular, the set of isomorphism classes of connected components of* $\mathrm{Spec}^{\mathsf{FM}}\,\mathcal{T}$ *coincides with* $\mathrm{FM}\,\mathcal{T}$.

*Proof.* (ii) $\Rightarrow$ (iii) and (iii) $\Rightarrow$ (i) are clear, so it suffices to show (i) $\Rightarrow$ (ii). We show its contrapositive. Suppose there exist $X_1, X_2 \in \mathrm{FM}\,\mathcal{T}$ and a triangulated equivalence $\Phi : \mathsf{Perf}\, X_1 \to \mathsf{Perf}\, X_2$ such that

$$U := \Phi(\mathrm{Spec}_{\otimes_{X_1}} \mathsf{Perf}\, X_1) \cap \mathrm{Spec}_{\otimes_{X_2}} \mathsf{Perf}\, X_2 \neq \emptyset$$

but $\Phi(\mathrm{Spec}_{\otimes_{X_1}} \mathsf{Perf}\, X_1) \neq \mathrm{Spec}_{\otimes_{X_2}} \mathsf{Perf}\, X_2$. Then,

$$X := \Phi(\mathrm{Spec}_{\otimes_{X_1}} \mathsf{Perf}\, X_1) \cup \mathrm{Spec}_{\otimes_{X_2}} \mathsf{Perf}\, X_2 \subset \mathrm{Spec}^{\mathsf{FM}}\,\mathsf{Perf}\, X_2$$

is not a separated scheme since otherwise $X_1 \cong \Phi(\mathrm{Spec}_{\otimes_{X_1}} \mathsf{Perf}\, X_1)$ is closed in $X$ (as it is an image of a proper scheme in a separated scheme) and hence $U$ is a closed and open subscheme of $\mathrm{Spec}_{\otimes_{X_2}} \mathsf{Perf}\, X_2 \cong X_2$, which is absurd as $X_2$ is connected. Therefore, $\mathrm{Spec}^{\mathsf{FM}}\,\mathsf{Perf}\, X_2 \cong \mathrm{Spec}^{\mathsf{FM}}\,\mathcal{T}$ has a non-separated open subscheme and hence is not separated either. □

## Irreducibility

Irreducibility of Fourier–Mukai loci and full orbits implies that all autoequivalences are birational in nature, and we expect variates appearing have rich birational geometry.

**Proposition 3.2.5.** *Let $X$ be a smooth projective variety. The following are equivalent:*

*(i)* $X$ *is tt-irreducible;*

*(ii)* $X$ *is tt-connected;*

*(iii) For any triangulated autoequivalence* $\Phi : \mathsf{Perf}\, X \simeq \mathsf{Perf}\, X$, *we have*

$$\Phi(\mathrm{Spec}_{\otimes_X} \mathsf{Perf}\, X) \cap \mathrm{Spec}_{\otimes_X} \mathsf{Perf}\, X \neq \emptyset.$$

*In particular, any tt-spectrum of geometric tt-structure in* $X$ *intersects with each other in* $\mathrm{Spec}_{\otimes,X} \mathsf{Perf}\, X$.

*(iv) We have*

$$\mathrm{BAuteq}(X) \times \mathbb{Z}[1] = \mathrm{Auteq}(\mathsf{Perf}\, X).$$

*Equivalently, any autoequivalence is birational up to shift.*

*(v) We have*

$$\mathrm{Spec}_{\otimes,X} \mathsf{Perf}\, X = \mathrm{BAuteq}\, X \cdot \mathrm{Spec}_{\otimes_X} \mathsf{Perf}\, X.$$

*Proof.* First of all, (iii) $\iff$ (iv) is Lemma 3.1.15. Now, (i) $\iff$ (ii) $\iff$ (iv) $\iff$ (v) follow by Theorem 3.1.21. □

*Remark* 3.2.6. In [Ueh17], a smooth projective variety $X$ is said to be **of $K$-equivalent type** if

$$\mathrm{BAuteq}\, X \ltimes \mathbb{Z}[1] = \mathrm{Auteq}\,\mathsf{Perf}\ X$$

(cf. Remark 3.3.43). Therefore, $X$ is tt-irreducible if and only if $X$ is of $K$-equivalent type.

**Proposition 3.2.7.** *Let $\mathcal{T}$ be a triangulated category with $\mathrm{FM}\,\mathcal{T} \neq \emptyset$. The following are equivalent:*

*(i)* $\mathrm{Spec}^{\mathsf{FM}}\,\mathcal{T}$ *is an irreducible scheme;*

*(ii) For any $X_1, X_2 \in \mathrm{FM}\,\mathcal{T}$, any triangulated equivalence $\Phi : \mathsf{Perf}\ X_1 \simeq \mathsf{Perf}\ X_2$ is birational up to shifts, i.e.,*

$$\Phi(\mathrm{Spec}_{\otimes_{X_1}}\mathsf{Perf}\ X_1) \cap \mathrm{Spec}_{\otimes_{X_2}}\mathsf{Perf}\ X_2 \neq \emptyset.$$

*(iii) Any $X \in \mathrm{FM}\,\mathcal{T}$ is tt-irreducible and for any $X, Y \in \mathrm{FM}\,\mathcal{T}$,*

$$\mathrm{Spec}_{\otimes,X}\,\mathcal{T} \cap \mathrm{Spec}_{\otimes,Y}\,\mathcal{T} \neq \emptyset.$$

*Proof.* By Proposition 3.1.22 and Proposition 3.2.5, (ii) $\Rightarrow$ (iii) follows. By Lemma 3.1.19, (iii) $\Rightarrow$ (i) and (i) $\Rightarrow$ (ii) follow. □

Combining tt-separatedness and tt-irreducibility, we obtain the following.

**Proposition 3.2.8.** *Let $X$ be a smooth projective variety. The following are equivalent.*

*(i) $X$ is tt-separated and tt-irreducible;*

*(ii)* $\mathrm{Auteq}\,\mathsf{Perf}\ X = \mathrm{BAuteq}\, X \times \mathbb{Z}[1] = \mathrm{Auteq}^{\mathsf{st}}\,\mathsf{Perf}\ X$*;*

*(iii)* $\mathrm{Spec}^{\mathsf{FM}}\,\mathsf{Perf}\ X = \mathrm{Spec}_{\otimes_X}\mathsf{Perf}\ X$.

*In these cases,* $\mathrm{FM}\, X = \{X\}$.

*Proof.* The implications (i) $\iff$ (ii) $\iff$ (iii) follow by Proposition 3.2.3 and Proposition 3.2.5. The last implications follow since for any $Y \in \mathrm{FM}\,\mathcal{T}$, $Y$ is a closed and open subscheme of $X$. □

*Remark* 3.2.9. Note that $\mathrm{FM}\, X = \{X\}$ does not imply either tt-separatedness nor tt-irreducibly. For example, the Hirzebruch surface $\mathbb{F}_2$ has no non-trivial Fourier–Mukai partner, but by [BP14] and Corollary 3.3.27,

$$\mathrm{Auteq}\,\mathsf{Perf}\ X = \mathrm{BAuteq}\, X \times \mathbb{Z}[1] \supsetneq \mathrm{Auteq}^{\mathsf{st}}\,\mathsf{Perf}\ X,$$

i.e., $\mathbb{F}_2$ is tt-irreducible but not tt-separated. On the other hand, an elliptic curve $E$ has no non-trivial Fourier–Mukai partner, but by Theorem 4.1.5 and Theorem 3.3.12,

$$\mathrm{Auteq}\,\mathsf{Perf}\ X \supsetneq \mathrm{BAuteq}\, X \times \mathbb{Z}[1] = \mathrm{Auteq}^{\mathsf{st}}\,\mathsf{Perf}\ X,$$

i.e., $E$ is tt-separated but not tt-irreducible.

## Quasi-compactness

**Definition 3.2.10.** Let $\mathcal{T}$ be a triangulated category with $\mathrm{FM}\,\mathcal{T} \neq \emptyset$. We say $\mathcal{T}$ has a **finite Fourier-Mukai cover** if there is a finite subset $S \subset \mathrm{FM}\,\mathcal{T}$ such that $\mathrm{Spc}^{\mathsf{FM}}\,\mathcal{T} = \bigcup_{X \in S} \mathrm{Spc}_{\otimes,X}\,\mathcal{T}$.

If $\mathrm{FM}\,\mathcal{T}$ is a finite set, then $\mathcal{T}$ has a finite Fourier-Mukai cover. Moreover, the assumption that $\mathrm{FM}\,\mathcal{T}$ is finite is not too special. Indeed, Kawamata conjectured that for any smooth projective variety $X$, $\mathrm{FM}\,X$ is finite ([Kaw02, Conjecture 1.5]) and the conjecture holds for curves and surfaces ([Kaw02], [Huy06]), for abelian varieties and varieties with ample (anti-)canonical bundle ([Fav12]), and for toric varieties ([Kaw13]). On the other hand, a counter-example is also given by Lesieutre using an infinite family of blow-ups of $\mathbb{P}^3$ at certain 8 points, which are all connected by Cremona transformations ([Les14]).

**Question 3.2.11.** Does Lesieutre's example have a finite Fourier–Mukai cover? More generally, does every smooth projective variety have a finite Fourier–Mukai cover?

As before, we have the following:

**Proposition 3.2.12.** *Let $X$ be a smooth projective variety. Then, the following are equivalent:*

*(i) $X$ is tt-quasi-compact;*

*(ii) $I_X = \mathrm{Auteq}\,\mathsf{Perf}\,X / \mathrm{BAuteq}\,X \times \mathbb{Z}[1]$ is finite and the birational orbit of $X$ is quasi-compact.*

*Proof.* By Theorem 3.1.21, $I_X$ is finite if and only if the full orbit $\mathrm{Spec}_{\otimes,X}\,\mathsf{Perf}\,X$ has only finitely many connected components. Since every connected component of the full orbit is isomorphic to the birational orbit, (ii) $\Rightarrow$ (i). Similarly, if $I_X$ is infinite or the birational orbit is not quasi-compact, then $X$ is not tt-quasi-compact. □

**Proposition 3.2.13.** *Let $\mathcal{T}$ be a triangulated category with $\mathrm{FM}\,\mathcal{T} \neq \emptyset$. The following are equivalent:*

*(i) $\mathrm{Spec}^{\mathsf{FM}}\,\mathcal{T}$ is quasi-compact;*

*(ii) $\mathcal{T}$ has a finite Fourier-Mukai cover and every $X \in \mathrm{FM}\,\mathcal{T}$ is tt-quasi-compact.*

*Proof.* First, suppose $\mathrm{Spec}^{\mathsf{FM}}\,\mathcal{T}$ is quasi-compact. Then, $\mathcal{T}$ clearly has a finite Fourier-Mukai cover. Since $\mathrm{Spec}^{\mathsf{FM}}\,\mathcal{T}$ is of finite type, it is in particular noetherian and hence each $\mathrm{Spc}_{\otimes,X}\,\mathcal{T}$ is quasi-compact for $X \in \mathrm{FM}\,\mathcal{T}$. The other direction is clear. □

**Observation 3.2.14.** Note that tt-quasi-compactness of a smooth projective variety $X$ may fail in various ways:

(i) For abelian varieties (Corollary 3.3.13), the birational orbit is quasi-compact but $I_X$ is infinite.

(ii) For toric varieties (Corollary 3.3.27 and Remark 3.3.28), $I_X$ is finite but the birational orbit is in general not quasi-compact.

(iii) For some $K3$ surfaces (Example 3.3.24), $I_X$ is infinite and the birational orbit is not quasi-compact.

It is indeed hard to come up with tt-quasi-compact smooth projective varieties except for ones with $\otimes$-generating canonical bundle. From observations so far, it is reasonable to expect the following.

**Conjecture 3.2.15.** *A smooth projective variety $X$ is tt-separated and tt-quasi-compact if and only if* $\mathrm{Spec}_{\otimes,X} \mathsf{Perf}\ X = \mathrm{Spec}_{\otimes_X} \mathsf{Perf}\ X$. *This follows from a stronger conjecture that there is no smooth projective variety $X$ such that* $1 < |I_X| < \infty$.

To author's knowledge, we do not have a counter-example to the stronger conjecture either.

## 3.3 Examples

Now let us apply results so far to compute Fourier-Mukai loci and full orbits for some specific cases. In this section, $k$ is an algebraically closed field of characteristic 0.

### Smooth projective varieties with $\otimes$-generating canonical bundle

The easiest case is when a smooth projective variety has a $\otimes$-generating canonical bundle. This can be understood as generalization of the Bondal–Orlov reconstruction.

**Definition 3.3.1.** For a noetherian scheme $X$, a line bundle $\mathscr{L}$ on $X$ is said to be **$\otimes$-generating** if $\langle \mathscr{L}^{\otimes n} \mid n \in \mathbb{Z} \rangle = \mathsf{Perf}\ X$.

Since $\mathsf{Perf}\ X$ has a split-generator and $\tau_{\mathscr{L}} = - \otimes_X \mathscr{L}$ is an autoequivalence, the condition is equivalent to saying there exists $N > 0$ such that $\mathscr{O}_X \oplus \cdots \oplus \mathscr{L}^{\otimes N}$ is a split-generator of $\mathsf{Perf}\ X$ ([IO25, Lemma 2.2]).

**Example 3.3.2.** By Orlov [Orl09], any (anti-)ample line bundle is $\otimes$-generating.

In [IO25], we provided a Nakai–Moishezon-style criterion for $\otimes$-generation, which in particular provides a lot of examples of $\otimes$-generating line bundles that are neither ample nor anti-ample.

**Theorem 3.3.3** (Ito–Olander)**.** *Let $X$ be a proper variety and $\mathscr{L}$ a line bundle on $X$. The following are equivalent:*

*(i) $\mathscr{L}$ is $\otimes$-generating;*

*(ii) For any closed subvariety $Z \subset X$, $\mathscr{L}|_Z$ or $\mathscr{L}^{-1}|_Z$ is big.*

Indeed, if we disallow the condition "$\mathscr{L}^{-1}|_Z$ is big" from condition (ii) above, then it is exactly the Nakai–Moishezon criterion for ampleness. As a result, we obtain the following examples.

**Theorem 3.3.4** (Ito–Olander)**.** *The following schemes admit $\otimes$-generating line bundles which are neither ample nor anti-ample:*

*(i) A reducible union of two copies of the projective line meeting at a node.*

*(ii) Any smooth projective surface containing an integral curve of negative self-intersection.*

*(iii) A blowup of a quasi-projective variety in a closed point.*

*(iv) Affine space of any dimension with doubled origin. In fact, the structure sheaf is $\otimes$-generating but not ample. In particular, a scheme with a $\otimes$-generating line bundle need not be separated and need not satisfy the resolution property.*

*(v) The blowup of the affine plane at the origin minus a closed point of the exceptional divisor. In fact, the structure sheaf is $\otimes$-generating but not ample in this case.*

*(vi) Hironaka's smooth proper, non-projective threefold.*

**Theorem 3.3.5** (Ito–Olander)**.** *The following provide $\otimes$-generating canonical bundles which are neither ample nor anti-ample*

*(i) Let $X$ be a smooth projective surface over an algebraically closed field such that $\omega_X$ or $\omega_X^{-1}$ is big (e.g. toric surfaces or surfaces of general type). Then, $\omega_X$ is $\otimes$-generating if and only if $X$ contains a $(-2)$-curve. In particular, most smooth projective toric surfaces have $\otimes$-generating canonical bundle.*

*(ii) Let $X$ be the blow-up of a smooth variety with ample canonical bundle at finitely many points. Then, $\omega_X$ is $\otimes$-generating.*

*(iii) Let $k$ be an algebraically closed field and let $X$ be the blowup of $\mathbb{P}^2_k$ in $n$ closed points. Assume either: (a) all the points lie on a line and $n \neq 3$, or (b) all the points lie on a conic and $n \neq 6$. Then, $\omega_X$ is $\otimes$-generating.*

*(iv) Let $E$ be an elliptic curve over the complex numbers. Let $\mathscr{F}$ be a rank two vector bundle on $E$ and let $X = \mathbb{P}_E(\mathscr{F})$. If $\mathscr{F}$ is unstable, then $\omega_X$ is $\otimes$-generating.*

*(v) Let $Y \subset \mathbb{P}^4_{\mathbb{C}}$ be a smooth projective hypersurface of general type over the complex numbers. Assume $Y$ contains a line $\ell \subset \mathbb{P}^4_{\mathbb{C}}$ which moves on $Y$, i.e., $H^0(\ell, \mathscr{N}_{\ell/Y}) \neq 0$. If $X$ is the blowup of $Y$ in $\ell$, then $\omega_X$ is $\otimes$-generating.*

Now, in these cases, the Fourier–Mukai locus is as simple as possible, and as a result so is the group of autoequivalences.

**Proposition 3.3.6.** *Let $X$ be a smooth projective variety with $\otimes$-generating canonical bundle. Then, $X$ is tt-separated and tt-irreducible, i.e,*

$$\mathrm{Spec}^{\mathsf{FM}}\,\mathsf{Perf}\,X = \mathrm{Spec}_{\otimes_X}\,\mathsf{Perf}\,X.$$

*In particular, by Proposition 3.2.8,*

$$\operatorname{Auteq} \mathsf{Perf}\, X = \operatorname{BAuteq} X \times \mathbb{Z}[1] = \operatorname{Auteq}^{\mathrm{st}} \mathsf{Perf}\, X.$$

Since the proof gets much simpler if we consider the Serre invariant locus, let us delay it to Theorem 4.1.5. It is worth mentioning that we deduce the equality $\operatorname{Auteq} \mathsf{Perf}\, X = \operatorname{Auteq}^{\mathrm{st}} \mathsf{Perf}\, X$ from the computation of the Fourier–Mukai locus, not the other way around (cf. Proposition 3.2.8).

## Abelian varieties

First, let us recall some basics of the derived category of coherent sheaves on an abelian variety.

**Notation 3.3.7.** Let $X$ be an abelian variety and let $\hat{X}$ denote its dual. For a closed point $x \in X$, let $t_x : X \xrightarrow{\sim} X; y \mapsto y + x$ denote the translation. Moreover, for a closed point $\alpha \in \hat{X}$, let $\mathscr{L}_\alpha \in \operatorname{Pic}^0(X)$ denote the corresponding line bundle of degree 0.

In [Orl02], Orlov gave several important results on the derived category of coherent sheaves on an abelian variety.

**Definition 3.3.8.** Let $X$ be an abelian variety. Then, define the group of **symplectic automorphisms** of $X \times \hat{X}$ (with respect to the natural symplectic form) to be

$$\operatorname{Sp}(X \times \hat{X}) := \left\{ \begin{pmatrix} f_1 & f_2 \\ f_3 & f_4 \end{pmatrix} \in \operatorname{Aut}(X \times \hat{X}) \,\middle|\, \begin{pmatrix} f_1 & f_2 \\ f_3 & f_4 \end{pmatrix} \begin{pmatrix} \hat{f}_4 & -\hat{f}_2 \\ -\hat{f}_3 & \hat{f}_1 \end{pmatrix} = \mathrm{id}_{X \times \hat{X}} \right\},$$

where $\operatorname{Aut}(X \times \hat{X})$ denotes the group of automorphisms of abelian varieties and $\hat{f}_i$ denotes the transpose of $f_i$. Here, we are writing an automorphism $f : X \times \hat{X} \to X \times \hat{X}$ with matrix form, where $f_1 : X \to X$, $f_2 : \hat{X} \to X$, etc. We say a symplectic automorphism $f$ is **elementary** if $f_2$ is an isogeny.

**Theorem 3.3.9** ([Orl02, Theorem 2.10, Corollary 2.13, Proposition 3.2, Construction 4.10, Proposition 4.12])**.** *Let X be an abelian variety. Then, there is a surjective group homomorphism*

$$\gamma : \operatorname{Auteq} \mathsf{Perf}\, X \to \operatorname{Sp}(X \times \hat{X})$$

*such that for any $\Phi_{\mathscr{E}} \in \operatorname{Auteq} \mathsf{Perf}\, X$ with Fourier-Mukai kernel $\mathscr{E} \in \mathsf{Perf}(X \times X)$ (which is necessarily isomorphic to a sheaf on $X \times X$ up to shift) and for any $(a, \alpha), (b, \beta) \in X \times \hat{X}$, we have that $\gamma(\Phi_{\mathscr{E}})(a, \alpha) = (b, \beta)$ if and only if*

$$t_{(0,b)_*}\mathscr{E} \otimes_{\mathscr{O}_{X \times X}} \pi_2^* \mathscr{L}_\beta \cong t^*_{(a,0)} \mathscr{E} \otimes_{\mathscr{O}_{X \times X}} \pi_1^* \mathscr{L}_\alpha$$

*where $\pi_i$ denotes projections $\pi_i : X \times X \to X$ so that $\Phi_{\mathscr{E}}(-) = \mathbb{R}\pi_{2*}(\pi_1^*(-) \otimes_{X \times X} \mathscr{E})$. Moreover, we have*

$$\operatorname{Ker} \gamma = (X \times \hat{X})_k \times \mathbb{Z}[1] \subset \operatorname{Auteq} \mathsf{Perf}\, X,$$

*where each component corresponds to translations, tensor products with lines bundles of degree* 0*, and shifts, respectively. Furthermore, for any elementary symplectic automorphism $f \in \operatorname{Sp}(X \times \hat{X})$, there exists a (semihomogeneous) vector bundle $\mathscr{E}$ on $X \times X$ such that $\gamma(\Phi_{\mathscr{E}}) = f$.*

We have the following "global" understanding of the Fourier-Mukai locus of an abelian variety:

**Lemma 3.3.10.** *Let $\mathcal{T}$ be a triangulated category with an abelian variety $X \in \mathrm{FM}\,\mathcal{T}$. Then, any $Y \in \mathrm{FM}\,\mathcal{T}$ is also an abelian variety and we have*

$$\mathrm{Spec}^{\mathsf{FM}}\,\mathcal{T} = \bigsqcup_{Y \in \mathrm{FM}\,\mathcal{T}} \mathrm{Spec}_{\otimes,Y}\,\mathcal{T}.$$

*as a scheme. In particular, $\mathcal{T}$ is of disjoint Fourier–Mukai type.*

*Proof.* By [HNW11, Theorem 0.4], any Fourier-Mukai partner of $X$ is also an abelian variety. Moreover, it is well-known that any birationally equivalent abelian varieties are isomorphic. □

In particular, in order to understand the Fourier-Mukai locus, we can focus on the locus $\mathrm{Spec}_{\otimes,X}\,\mathcal{T}$ for a single abelian variety $X \in \mathrm{FM}\,\mathcal{T}$. Now, we show any variety is tt-separated, but not tt-irreducible.

**Proposition 3.3.11.** *An abelian variety $X$ is not tt-irreducible.*

*Proof.* By Proposition 3.2.5, it suffices to show there is a triangulated equivalence $\Phi : \mathsf{Perf}\,X \to \mathsf{Perf}\,X$ such that

$$\Phi(\mathrm{Spc}_{\otimes_X}\,\mathsf{Perf}\,X) \cap \mathrm{Spc}_{\otimes_X}\,\mathsf{Perf}\,X = \emptyset.$$

Take an ample line bundle $\mathscr{L}$ on $\hat{X}$ with isogeny $\phi_{\mathscr{L}} : \hat{X} \to \hat{\hat{X}} \cong X, x \mapsto t_x^*\mathscr{L} \otimes \mathscr{L}^{-1}$. First, note that by [Plo05, Example 4.5], we have

$$\gamma(- \otimes_{\mathcal{O}_{\hat{X}}} \mathscr{L}) = \begin{pmatrix} \mathrm{id}_{\hat{X}} & 0 \\ \phi_{\mathscr{L}} & \mathrm{id}_X \end{pmatrix} \in \mathrm{Sp}(\hat{X} \times X)$$

and in particular $\phi_{\mathscr{L}} = \hat{\phi_{\mathscr{L}}}$. Thus, we have

$$f := \begin{pmatrix} \mathrm{id}_X & \phi_{\mathscr{L}} \\ 0 & \mathrm{id}_{\hat{X}} \end{pmatrix} \in \mathrm{Sp}(X \times \hat{X}).$$

Now, since $f$ is, in particular, an elementary symplectic isomorphism, we have a vector bundle $\mathscr{E}$ on $X \times X$ such that $\Phi_{\mathscr{E}} \in \mathrm{Auteq}\,\mathsf{Perf}\,X$ (with $\gamma(\Phi_{\mathscr{E}}) = f$) by Theorem 3.3.9. Therefore, we see that

$$\Phi_{\mathscr{E}}(\mathrm{Spec}_{\otimes_X}\,\mathsf{Perf}\,X) \cap \mathrm{Spec}_{\otimes_X}\,\mathsf{Perf}\,X = \emptyset.$$

by Lemma 3.1.18 as desired. □

**Theorem 3.3.12** (Ito–Matsui)**.** *An abelian variety $X$ is tt-separated.*

*Proof.* Take a birational autoequivalence $\Phi \in \mathrm{Auteq}\,\mathsf{Perf}\,X$, i.e., suppose there exist $x_0, y_0 \in X$ such that $\Phi(k(x_0)) \cong k(y_0)$. By Proposition 3.2.3, it suffices to show that for any $x \in X$, there exists $y \in X$ such that $\Phi(k(x)) \cong k(y)$. Now, by [MGP13, Proposition 3.2], we have $\Phi = \Phi_{\mathscr{K}}$ for a sheaf $\mathscr{K}$ on $X \times X$ that is flat along each projection and therefore by [Huy06, Example 5.4] we have $\mathscr{K}|_{\{x_0\}\times X} \cong \Phi(k(x_0)) \cong k(y_0)$ under the canonical identification $\{x_0\} \times X \cong X$. Moreover, by Theorem 3.3.9, there is a corresponding isomorphism $f_{\mathscr{K}} : X \times \hat{X} \to X \times \hat{X}$ satisfying $f_{\mathscr{K}}(a, \alpha) = (b, \beta)$ if and only if

$$t^*_{(a,0)}\mathscr{K} \otimes_{\mathscr{O}_{X\times X}} \pi_1^*\mathscr{L}_\alpha \cong t_{(0,b)_*}\mathscr{K} \otimes_{\mathscr{O}_{X\times X}} \pi_2^*\mathscr{L}_\beta,$$

which is equivalent to

$$t^*_{(a,0)}\mathscr{K} \cong t^*_{(0,-b)}\mathscr{K} \otimes_{\mathscr{O}_{X\times X}} \pi_2^*\mathscr{L}_\beta \otimes_{\mathscr{O}_{X\times X}} (\pi_1^*\mathscr{L}_\alpha)^{-1}.$$

Under canonical identifications $\{x\} \times X \cong X \cong X \times \{y\}$ for closed points $x, y \in X$, we therefore see that for any $a \in X$, we can take $\alpha, b, \beta$ such that

$$\begin{aligned}
\Phi(k(x_0 + a)) &\cong \mathscr{K}|_{\{x_0+a\}\times X} \cong (t^*_{(a,0)}\mathscr{K})|_{\{x_0\}\times X} \\
&\cong \left(t^*_{(0,-b)}\mathscr{K} \otimes_{\mathscr{O}_{X\times X}} \pi_2^*\mathscr{L}_\beta \otimes_{\mathscr{O}_{X\times X}} (\pi_1^*\mathscr{L}_\alpha)^{-1}\right)|_{\{x_0\}\times X} \\
&\cong t^*_{-b}(\mathscr{K}|_{\{x_0\}\times X}) \otimes_X \left(\pi_2^*\mathscr{L}_\beta \otimes_{\mathscr{O}_{X\times X}} (\pi_1^*\mathscr{L}_\alpha)^{-1}\right)|_{\{x_0\}\times X} \\
&\cong k(y_0 + b) \otimes_{\mathscr{O}_X} \left(\pi_2^*\mathscr{L}_\beta \otimes_{\mathscr{O}_{X\times X}} (\pi_1^*\mathscr{L}_\alpha)^{-1}\right)|_{\{x_0\}\times X} \cong k(y_0 + b)
\end{aligned}$$

as desired. □

By Theorem 3.2.4, Lemma 3.3.10, and Theorem 3.3.12, we obtain some kind of a generalization of [Mat23, Corollary 4.10] (see also [HO22, Theorem 4.11] and [HO24, Theorem 5.3]). Here we note that $\mathrm{Spec}^{\mathsf{FM}}\,\mathsf{Perf}\,X = \mathrm{Spec}_\vartriangle\,\mathsf{Perf}\,X$ holds for an elliptic curve $X$ by [HO22, Proposition 4.10] and the subsequent argument.

**Corollary 3.3.13.** *Let $X$ be an abelian variety. Then there is an isomorphism*

$$\mathrm{Spec}^{\mathsf{FM}}\,\mathsf{Perf}\,X \cong \bigsqcup_{Y\in \mathrm{FM}\,\mathsf{Perf}\,X}\ \bigsqcup_{I_Y} Y$$

*of schemes. Moreover, $I_Y$ is infinite for each abelian variety $Y \in \mathrm{FM}\,X$.*

*Proof.* The first claim follows from Theorem 3.2.4, Lemma 3.3.10, and Theorem 3.3.12. For the latter claim, note that the image of standard autoequivalences (and hence birational autoequivalences up to shift) in $\mathrm{Sp}(Y \times \hat{Y})$ is in the subgroup of lower triangular matrices by [Plo05, Example 4.5]. Hence, the powers of $\Phi_{\mathscr{E}}$ constructed in the proof of Proposition 3.3.11 define infinitely many distinct elements in the left coset $I_Y$. □

By Theorem 3.1.5 and Corollary 3.3.13, we can reconstruct all the Fourier-Mukai partners of an abelian variety $X$ if we can identify the Fourier-Mukai locus

$$\mathrm{Spec}^{\mathsf{FM}}\,\mathsf{Perf}\,X \subset \mathrm{Spec}_\vartriangle\,\mathsf{Perf}\,X$$

purely categorically.

## Spherical twists and surfaces

Next, let us make some observations on spherical twists and the case of surfaces. First, let us recall the following notions.

**Definition 3.3.14.** Let $X$ be a smooth projective variety. An object $\mathscr{F} \in \mathsf{Perf}\ X$ is said to be **spherical** if

(i) $\mathscr{F} \cong \mathscr{F} \otimes \omega_X$;

(ii) $\mathrm{Hom}(\mathscr{F}, \mathscr{F}[i]) = \begin{cases} k & \text{if } i = 0, \dim X \\ 0 & \text{otherwise.} \end{cases}$

Note if $\mathscr{F}$ is spherical, then $\mathscr{F}^\vee$, $\mathscr{F}[i]$ for any $i \in \mathbb{Z}$ and $\mathscr{F} \otimes \mathscr{L}$ for any line bundle $\mathscr{L}$ are spherical.

**Construction 3.3.15.** Let $X$ be a smooth projective variety and take an object $\mathscr{E} \in \mathsf{Perf}\ X$. Define $\mathscr{P}_{\mathscr{E}} \in \mathsf{Perf}\ X \times X$ to be

$$\mathscr{P}_{\mathscr{E}} := \mathrm{cone}(p_1^* \mathscr{E}^\vee \otimes p_2^* \mathscr{E} \to \mathscr{O}_\Delta),$$

where $p_i : X \times X \to X$ denotes the $i$-th projection, $\mathscr{O}_\Delta$ denotes the structure sheaf on the (image of) diagonal $\Delta \subset X \times X$, and the morphism is given by the composition

$$p_1^* \mathscr{E}^\vee \otimes p_2^* \mathscr{E} \to \Delta_* \Delta^* (p_1^* \mathscr{E}^\vee \otimes p_2^* \mathscr{E}) = \Delta_* (\mathscr{E}^\vee \otimes \mathscr{E}) \to \Delta_* \mathscr{O}_X = \mathscr{O}_\Delta.$$

When $\mathscr{E}$ is spherical, the associated Fourier-Mukai transform

$$T_{\mathscr{E}} := \Phi_{\mathscr{P}_{\mathscr{E}}} : \mathsf{Perf}\ X \to \mathsf{Perf}\ X$$

is called the **spherical twist** associated to $\mathscr{E}$. It is a standard result (e.g. [Huy06, Proposition 8.6]) that any spherical twist is an autoequivalence.

Recall the following result:

**Lemma 3.3.16.** [Ueh17, Example 4.2] *Let $X$ be a smooth projective variety and let $\mathscr{E} \in \mathsf{Perf}\ X$ be a spherical object. If $x \in \mathrm{Supp}\,\mathscr{E}$, then $\mathrm{Supp}\, T_{\mathscr{E}}(k(x)) = \mathrm{Supp}\,\mathscr{E}$. If $x \in X \setminus \mathrm{Supp}\,\mathscr{E}$, then $T_{\mathscr{E}}(k(x)) = k(x)$.*

Now, we get the following useful corollary:

**Proposition 3.3.17.** *Let $X$ be a smooth projective variety of dimension $\geq 1$ and let $\mathscr{E} \in \mathsf{Perf}\ X$ be a spherical object. Then, $T_{\mathscr{E}}$ is birational if and only if $\mathrm{Supp}\,\mathscr{E} \neq X$. If $\mathrm{Supp}\,\mathscr{E}$ is moreover not a singleton, then the associated birational map $X \dashrightarrow X$ is given by the identity on $X \setminus \mathrm{Supp}\,\mathscr{E}$, i.e.,*

$$\mathrm{Spec}_{\otimes_X} \mathsf{Perf}\ X \cap T_{\mathscr{E}}(\mathrm{Spec}_{\otimes_X} \mathsf{Perf}\ X) \cong X \setminus \mathrm{Supp}\,\mathscr{E}$$

*Proof.* The first part follows from Lemma 3.3.16. In particular, in such a case, the domain of definition of $T_{\mathscr{E}}$ contains $X \setminus \operatorname{Supp} \mathscr{E}$. If $\operatorname{Supp} \mathscr{E}$ is moreover not a singleton, then for any $x \in \operatorname{Supp} \mathscr{E}$, $\operatorname{Supp} T_{\mathscr{E}}(k(x))$ is not a singleton (i.e., $x$ is not in the domain of definition of $T_{\mathscr{E}}$ by Lemma 3.1.14), which shows the second part. □

In particular, we can see the following by Proposition 3.2.3:

**Corollary 3.3.18.** *Let $X$ be a smooth projective variety (of dimension $\geq 1$). If there exists a spherical object $\mathscr{E} \in \mathsf{Perf}\, X$ with $\operatorname{Supp} \mathscr{E}$ neither being $X$ nor being a singleton, then $X$ is not tt-separated.*

**Example 3.3.19.** One of the easiest examples of spherical twists arises for elliptic curves. Let $X$ be an elliptic curve. Recall as a variant of [Orl02], we have the following short exact sequence:

$$1 \to \operatorname{Aut} X \ltimes \operatorname{Pic}^0 X \times \mathbb{Z}[2] \to \operatorname{Auteq} \mathsf{Perf}\, X \xrightarrow{\theta} \mathrm{SL}(2, \mathbb{Z}) \to 1,$$

where $\theta$ is given by the action on the image of the charge map

$$Z : K(X) \twoheadrightarrow \mathbb{Z}^2, \mathscr{F} \mapsto (\operatorname{rank} \mathscr{F}, \chi(\mathscr{F})).$$

Since the author could not find a proof of this fact, let us include it for completeness. The surjectivity of $\theta$ follows by noting that the images of the spherical twists $T_{\mathscr{O}_X}$ and $T_{k(x)}$ for a closed point $x \in X$ generate $\mathrm{SL}(2, \mathbb{Z})$ (cf. [ST01]). On the other hand, to show $\operatorname{Aut} X \ltimes \operatorname{Pic}^0 X \times \mathbb{Z}[2] = \operatorname{Ker} \theta$, it suffices to show any $\Phi \in \operatorname{Ker} \theta$ is a standard autoequivalence since then by the Riemann-Roch theorem, the line bundle part needs to have degree 0 and the shift part needs to have even degree. Indeed, since any object in $\mathsf{Perf}\, X$ is a direct sum of shifts of coherent sheaves, any $\Phi \in \operatorname{Ker} \theta$ sends any indecomposable skyscraper sheaf to an indecomposable skyscraper sheaf (up to shift) as desired.

Now, noting that the images of $\mathscr{O}_X$ and $k(x)$ under $\theta$ generate $\mathrm{SL}(2, \mathbb{Z})$, we see that any autoequivalence can be written as compositions of spherical twists and standard equivalences. Moreover, it is easy to see that $T_{k(x)} \simeq \mathscr{O}_X(x) \otimes -$ (e.g. [Huy06, Example 8.10 (i)]) and hence is indeed a standard equivalence. This illustrates that for the latter claim in Proposition 3.3.17 we really need to exclude the case when the support of a spherical object is a singleton since in this example the birational map associated to $T_{k(x)}$ is the identity on the whole $X$, not just on $X \setminus \{x\}$. Note, on the other hand, that $T_{\mathscr{O}_X}$ is not birational by Proposition 3.3.17.

Now, let us make some observations on $\operatorname{Spec}_{\otimes, X} \mathsf{Perf}\, X$. Recall we have the following relation (called a **Braid group relation**) ([ST01]):

$$T_{\mathscr{O}_X} T_{k(x)} T_{\mathscr{O}_X} = T_{k(x)} T_{\mathscr{O}_X} T_{k(x)}.$$

Noting that $T_{k(x)}$ is a standard autoequivalence, this relation says

$$T_{\mathscr{O}_X} T_{k(x)} T_{\mathscr{O}_X} (\operatorname{Spec}_{\otimes_X} \mathsf{Perf}\, X) = T_{k(x)} T_{\mathscr{O}_X} (\operatorname{Spec}_{\otimes_X} \mathsf{Perf}\, X) \subset \operatorname{Spec}_{\otimes, X} \mathsf{Perf}\, X.$$

In other words, $T_{\mathscr{O}_X}$ acts on $T_{k(x)} T_{\mathscr{O}_X} (\operatorname{Spec}_{\otimes_X} \mathsf{Perf}\, X)$, but does not act on $\operatorname{Spec}_{\otimes_X} \mathsf{Perf}\, X$. Also, see Observation 4.1.11 for a thorough description of $\operatorname{Spec}_{\otimes, X} \mathsf{Perf}\, X$.

Spherical twists give the following key example for Fourier-Mukai loci corresponding to surfaces.

**Example 3.3.20.** ([Huy06, Example 8.10(iii)]) Let $X$ be a smooth projective surface and $C \subset X$ a $(-2)$**-curve**, i.e., a smooth rational curve with $C^2 = -2$. Then $\mathcal{O}_C$ is a spherical object. More generally, the pull-back $\mathcal{O}_C(k)$ of $\mathcal{O}_{\mathbb{P}^1}(k)$ under an isomorphism $C \cong \mathbb{P}^1$ is spherical. Note that any smooth (irreducible) rational curve in a $K3$ surface is a $(-2)$-curve.

By Proposition 3.3.17, we get the following:

**Corollary 3.3.21.** *Let $X$ be a smooth projective surface. Suppose $X$ admits a $(-2)$-curve $C$. Then,*

$$\mathrm{Spec}_{\otimes_X} \mathsf{Perf}\, X \cap T_{\mathcal{O}_C}(\mathrm{Spec}_{\otimes_X} \mathsf{Perf}\, X) \cong X \setminus C.$$

*In particular, $X$ is not tt-separated.*

*Remark* 3.3.22. Note that by blowing up a surface twice and looking at the strict transform of the exceptional curve of the first blow-up, we can always get a surface with a $(-2)$-curve. Therefore, tt-separatedness is not birational invariance.

In the rest of this subsection, let us consider specific cases of $K3$ surfaces although we will indeed see the following claim holds for any surface in the next subsection (Corollary 3.3.48).

**Lemma 3.3.23.** *Let $\mathcal{T}$ be a triangulated category with a $K3$ surface $X \in \mathrm{FM}\,\mathcal{T}$. Then, $\mathcal{T}$ is of disjoint Fourier-Mukai type, i.e,*

$$\mathrm{Spec}^{\mathsf{FM}}\, \mathcal{T} = \bigsqcup_{X \in \mathrm{FM}\,\mathcal{T}} \mathrm{Spec}_{\otimes, X}\, \mathcal{T}.$$

*Proof.* It is well-known that a Fourier-Mukai partner is also a $K3$ surface (e.g. [Huy06, Corollary 10.2]). Moreover, note that any birationally equivalent $K3$ surfaces are isomorphic since they are minimal models. □

In particular, this gives a different proof and generalization of [HO22, Example 3.5] when a $K3$ surface $X$ is not isomorphic to the moduli space $M_H(v)$ of $H$-semistable sheaves for an ample divisor $H$ with a Mukai vector $v = (r, c, d) \in \mathrm{H}^*(X, \mathbb{Z})$ such that $c \in \mathrm{NS}(X)$, $r \geq 0$, and $\gcd(r, cH, d) = 1$ (cf. [Huy06, Proposition 10.20]).

**Example 3.3.24.** In contrast with the case of abelian varieties, a $K3$ surface $X$ is not tt-separated in general. For example, if we have the Picard number $\rho(X) \geq 12$, then $X$ admits a $(-2)$-curve ([Huy16, Corollary 14.3.8]). Moreover, if $X$ contains a $(-2)$-curve and $\mathrm{Aut}\, X$ is infinite, then there are infinitely many $(-2)$-curves in $X$ ([Huy16, Corollary 8.4.7]) and hence $X$ is far from being separated. On the other hand, there are $K3$ surfaces with no $(-2)$-curves (while having the infinite automorphism group) (cf. [Huy16, §15.2.5]).

Note that when $\rho(X) \geq 12$, we have $\mathrm{FM}\,\mathsf{Perf}\, X = \{X\}$ ([Huy16, Corollary 16.3.8]), so although such a $K3$ surface has a $(-2)$-curve, the whole Fourier-Mukai locus can be simpler than other $K3$ surfaces with multiple Fourier-Mukai partners.

In [HO24], Hirano–Ouchi provided the full description when $\rho(X) = 1$.

**Theorem 3.3.25** (Hirano–Ouchi)**.** *Let $X$ be a complex algebraic $K3$ surface of Picard number 1. Then, for every $Y \in \mathrm{FM}\, X$,*

$$\mathrm{BAuteq}\, Y \times \mathbb{Z}[1] = \mathrm{Auteq}^{\mathrm{st}}\, \mathsf{Perf}\, Y \subset \mathrm{Auteq}\, \mathsf{Perf}\, Y.$$

*In particular, every $Y \in \mathrm{FM}\, X$ is tt-separated and*

$$\mathrm{Spec}^{\mathsf{FM}}\, \mathsf{Perf}\, X \cong \coprod_{Y \in \mathrm{FM}\, X} \coprod_{I_Y} \mathrm{Spec}_{\otimes_Y}\, \mathsf{Perf}\, Y.$$

## Toric varieties and varieties of general type

Let us briefly look at cases of toric varieties and varieties of general type. First, the following result by Kawamata [Kaw13, Theorem 5] shows any toric varieties are tt-irreducible:

**Theorem 3.3.26** (Kawamata)**.** *Let $X$ be a smooth projective toric variety and $Y$ a smooth projective variety. Suppose there is a triangulated equivalence $\Phi : \mathsf{Perf}\, X \simeq \mathsf{Perf}\, Y$. Then, $Y$ is also a toric variety and $\Phi$ is birational up to shift with the associated birational map being a toric map. Moreover, there exist only finitely many such birational maps when $X$ is fixed and $Y$ is varied.*

**Corollary 3.3.27.** *Let $\mathcal{T}$ be a triangulated category with a toric variety $X \in \mathrm{FM}\, \mathcal{T}$. Then, any $Y \in \mathrm{FM}\, \mathcal{T}$ is toric and tt-irreducible, and $\mathrm{Spec}^{\mathsf{FM}}\, \mathsf{Perf}\, X$ is irreducible.*

*Proof.* Follows from Proposition 3.2.5 and Proposition 3.2.7. □

*Remark* 3.3.28. Note that Theorem 3.3.26 is not enough to show that $X$ is tt-quasi-compact since in general there may be birational autoequivalences such that the associated birational maps agree, but the action on the triangular spectrum is different over the complement of the domain of definition. Also, note that $X$ is not tt-separated in general by Corollary 3.3.21 since a toric surface may contain a $(-2)$-curve.

Since a toric variety has a big anti-canonical bundle (for example [BP14, Lemma 11]), they are special cases of the following result by Kawamata (in the proof of [Kaw02, Theorem 2.3 (ii)]):

**Theorem 3.3.29** (Kawamata)**.** *Let $X$ be a smooth projective variety with big (anti-)canonical bundle and suppose there is a triangulated equivalence $\Phi : \mathsf{Perf}\, X \simeq \mathsf{Perf}\, Y$ with $Y$ a smooth projective variety. Then, $Y$ has the big (anti-)canonical bundle and $\Phi$ is birational.*

**Corollary 3.3.30.** *Let $\mathcal{T}$ be a triangulated category with $X \in \mathrm{FM}\, \mathcal{T}$ for a smooth projective variety $X$ with big (anti-)canonical bundle. Then, any $Y$ is tt-irreducible and $\mathrm{Spec}^{\mathsf{FM}}\, \mathcal{T}$ is irreducible.*

*Proof.* Follows from Proposition 3.2.5 and Proposition 3.2.7. □

**Question 3.3.31.** If $X$ is tt-irreducible (equivalently, $\mathrm{BAuteq}\, X \times \mathbb{Z}[1] = \mathrm{Auteq}\, \mathsf{Perf}\, X$), what can we say about positivity of its (anti-)canonical bundle?

## Flops and K-equivalences

Now, let us move on to a more general approach to study Fourier-Mukai loci using birational geometry. For this, let us introduce the following standard notions in birational geometry (e.g. [Kaw02], [KM98], and [Tod06]).

**Definition 3.3.32.** Let $X$ and $Y$ be projective varieties with only canonical singularities.

(i) A birational map $\alpha : X \dashrightarrow Y$ is said to be **crepant** if there exist a smooth projective variety $Z$ and birational morphisms $f : Z \to X$ and $g : Z \to Y$ such that $\alpha \circ f = g$ and $f^*K_X \sim_{\mathbb{Q}} g^*K_Y$.

(ii) A birational map $\alpha : X \dashrightarrow Y$ is said to be a **flop** if there exist a normal projective variety $W$ and crepant birational morphisms $\phi : X \to W$ and $\psi : Y \to W$ such that

- $\phi = \psi \circ \alpha$;
- $\phi$ and $\psi$ are isomorphisms in codimension 1;
- the relative Picard numbers of $\phi$ and $\psi$ are 1;
- for any $\phi$-ample divisor $H$ on $X$, $-H'$ is $\psi$-ample, where $H'$ is the strict transform of $H$ on $Y$.

Here are some examples of flops and applications to our situations.

**Example 3.3.33.**

(i) (See [Huy06, Section 11.3] for details.) Let $X_1$ be a smooth projective variety of dimension $2k+1$ and suppose $X_1$ contains a closed subvariety $Y_1$ isomorphic to $\mathbb{P}^k$ whose normal bundle is isomorphic to $\mathcal{O}_{\mathbb{P}^k}(-1)^{\oplus k+1}$. Now, writing the blow-up along $Y_1$ by $p_1 : \tilde{X} \to X_1$, we see that its exceptional divisor $E$ is isomorphic to $\mathbb{P}^k \times \mathbb{P}^k$ and moreover that we can blow-down to the other projection, which yields a birational morphism $p_2 : \tilde{X} \to X_2$. Then, the birational map $p_2 \circ p_1^{-1} : X_1 \dashrightarrow X_2$ is called the **standard flop**. In short, the standard flop is constructed so that it fits into the following diagram:

$$\begin{tikzcd}
 & & & E \cong \mathbb{P}^k \times \mathbb{P}^k \arrow[d, hook] \arrow[dlll] \arrow[drrr] & & & \\
 & & & \tilde{X} \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & & & \\
\mathbb{P}^k \cong Y_1 \arrow[r, hook] & X_1 & & & & X_2 & Y_2 \cong \mathbb{P}^k \arrow[l, hook']
\end{tikzcd}$$

where the normal bundle of $Y_i \cong \mathbb{P}^k$ in $X_i$ is isomorphic to $\mathcal{O}_{\mathbb{P}^k}(-1)^{\oplus k+1}$ (so that $\omega_{X_i}|_{\mathbb{P}^k} \cong \mathcal{O}_{\mathbb{P}^k}$) and $p_i$ is the blow-up along $Y_i$ with exceptional divisor $E$. In particular, it can be shown that the birational map $p_2 \circ p_1^{-1} : X_1 \dashrightarrow X_2$ is indeed a flop by considering contractions $X_i \to W$

of $\mathbb{P}^k$ for each $i$. Here, note that $X_2$ is not necessarily projective in general (cf. Remark 3.3.36), so let us suppose $X_2$ is projective in this paper. By [BO95, Theorem 3.6], the functor

$$\Phi = p_{2*} \circ p_1{}^* = \Phi_{\mathcal{O}_Z} : \mathsf{Perf}\ X_1 \to \mathsf{Perf}\ X_2$$

is an equivalence, where $Z = X_1 \times_W X_2 \subset X_1 \times X_2$. Since $p_1$ and $p_2$ are isomorphisms away from exceptional loci, $\Phi$ is moreover birational and indeed the associated birational map $X_1 \dashrightarrow X_2$ is the isomorphism $X_1 \setminus Y_1 \cong X_2 \setminus Y_2$, i.e., the standard flop itself (cf. [HO22, Lemma 3.7]). Note that in general the birational Fourier–Mukai partners $X_1$ and $X_2$ are not isomorphic and therefore

$$\emptyset \neq \Phi(\mathrm{Spec}_{\otimes_{X_1}} \mathsf{Perf}\ X_1) \cap \mathrm{Spec}_{\otimes_{X_2}} \mathsf{Perf}\ X_2 \subsetneq \mathrm{Spec}_{\otimes_{X_2}} \mathsf{Perf}\ X_2.$$

Indeed, if $\Phi(\mathrm{Spec}_{\otimes_{X_1}} \mathsf{Perf}\ X_1) \cap \mathrm{Spec}_{\otimes_{X_2}} \mathsf{Perf}\ X_2 = \mathrm{Spec}_{\otimes_{X_2}} \mathsf{Perf}\ X_2$, then $X_2$ is a closed and open subscheme of $X_1$, which is absurd. These arguments reproduce [HO22, Example 3.9].

(ii) (See [Huy06, Section 11.4] for details.) Let $X_1$ be a smooth projective variety of dimension $2k$ and suppose $X_1$ contains a closed subvariety $Y_1$ isomorphic to $\mathbb{P}^k$ whose normal bundle is isomorphic to the cotangent bundle $\Omega_{Y_1}$. As in the construction of the standard flop, we can construct a birational map $X_1 \dashrightarrow X_2$, called the **Mukai flop**, that fits into the following diagram:

$$\begin{tikzcd}
 & & E \cong \mathbb{P}(\Omega_{Y_1}) \arrow[r, hook] \arrow[d, hook] \arrow[ddll] \arrow[ddrr] & Y_1 \times Y_2 & \\
 & & \tilde{X} \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & & \\
\mathbb{P}^k \cong Y_1 \arrow[r, hook] & X_1 & & X_2 & Y_2 \cong (\mathbb{P}^k)^* \arrow[l, hook']
\end{tikzcd}$$

where the normal bundle of $Y_i$ in $X_i$ is isomorphic to $\Omega_{Y_i}$, $(\mathbb{P}^k)^*$ denotes the dual projective space of $\mathbb{P}^k$, $p_1$ is the blow-up of $X_1$ along $Y_1$, and $p_2$ is a birational morphism contracting $E$ to the other direction, noting $\mathcal{O}_E(E) \cong \mathcal{O}_E(-1,-1)$ and the Fujiki-Nakano criterion (cf. [Huy06, Remark 11.10]). Again, $X_2$ is not necessarily projective, so let us suppose it is projective in this paper. However, in contrast to the standard flop, the functor

$$\Phi = p_{2*} \circ p_1{}^* : \mathsf{Perf}\ X_1 \to \mathsf{Perf}\ X_2$$

is **not** even fully faithful for $n > 1$ by [Nam03a]. In the paper, Namikawa nevertheless proves the following: First, consider the contractions $X_i \to W$ of $Y_i$ and write $Z' = X_1 \times_W X_2$ with projections $q_i : \hat{X} \to X_i$. Then, the functor

$$\Psi = q_{2*} \circ q_1{}^* = \Phi_{\mathcal{O}_{Z'}} : \mathsf{Perf}\ X_1 \to \mathsf{Perf}\ X_2$$

is an equivalence. In particular, since $q_1$ and $q_2$ are isomorphisms away from exceptional loci, $\Psi$ is birational and indeed the associated birational map $X_1 \dashrightarrow X_2$ is the isomorphism $X_1 \setminus Y_1 \cong X_2 \setminus Y_2$, i.e., the Mukai flop itself.

(iii) Let us focus on the case of threefolds. First, recall the following important results:

**Theorem 3.3.34** (Kollár)**.** [Kol89, Theorem 4.9] *Let $X_1$ and $X_2$ be projective threefolds with $\mathbb{Q}$-factorial terminal singularities. Suppose $K_{X_1}$ and $K_{X_2}$ are both nef. Then any birational map $X_1 \dashrightarrow X_2$ can be written as a finite composite of flops. Moreover, each flop does not change analytic singularity and only involves at worst terminal singularities for flopping contractions (cf.* [Kol89, Theorem 2.4]*).*

**Theorem 3.3.35** (Bridgeland)**.** [Bri02, Theorem 1] *If $X$ is a projective threefold with terminal singularities and*

$$\begin{tikzcd} Y_1 \arrow[dr, "f_1"'] & & Y_2 \arrow[dl, "f_2"] \\ & X & \end{tikzcd}$$

*are crepant birational morphisms with smooth projective varieties $Y_1, Y_2$, then there is an equivalence $\mathsf{Perf}\ Y_1 \simeq \mathsf{Perf}\ Y_2$. Moreover, such an equivalence can be taken to be birational.*

In particular, if $X_1$ and $X_2$ are birationally equivalent smooth projective threefolds with nef canonical bundles (e.g. smooth projective Calabi-Yau threefolds), then there is a birational triangulated equivalence $\mathsf{Perf}\ X_1 \simeq \mathsf{Perf}\ X_2$. In other words, each irreducible component of $\mathrm{Spec}^{\mathsf{FM}}\ \mathsf{Perf}\ X_1$ containing a copy of $X_1$ contains all the copies of smooth projective threefolds with nef canonical bundles that are birationally equivalent to $X_1$. In particular, a smooth projective threefold with nef canonical bundle is not tt-separated in general.

*Remark* 3.3.36.

(i) Given the first two examples, Orlov-Bondal conjectured that if smooth varieties $X$ and $Y$ are related by flops, then they are Fourier-Mukai partners ([BO95]). Note this is a special case of Conjecture 3.3.40. On the other hand, a choice of a Fourier-Mukai kernel is a delicate problem. In the first two examples, given a flop $X_1 \to W \leftarrow X_2$, we can always take a kernel to be $\mathcal{O}_Z$ for $Z = X_1 \times_W X_2 \subset X_1 \times X_2$, but Namikawa showed that this is not the case for a stratified Mukai flop in [Nam03b] although it is shown that there is a (birational) equivalence of derived categories corresponding to such a flop in [CKL13].

(ii) Theorem 3.3.34 is generalized to minimal models of general dimension by Kawamata in [Kaw08] and Theorem 3.3.35 is generalized to more general singularities in [Che02] and [VdB04]. In particular, Van den Bergh uses a different method in his proof, which contributed to the development of noncommutative crepant resolutions.

(iii) In the last example, we have seen that any birationally equivalent Calabi-Yau threefolds are Fourier-Mukai partners, but the converse is not true. In [OR18, Theorem 4.1], it is shown that there are Calabi-Yau threefolds that are Fourier-Mukai partners, but not birationally equivalent. In particular, $X$ is not tt-irreducible in general. More generally, Uehara also produces threefolds of Kodaira dimension 1 that are Fourier-Mukai partners, but not birationally equivalent ([Ueh12]).

Now, by the symmetry of flops, note that Example 3.3.33 (and conjecturally any flop not changing class of singularity) produces autoequivalences by applying triangulated equivalences corresponding to flops back and forth. Those autoequivalences, called **flop-flop autoequivalences**, can indeed be non-standard autoequivalences and thus their actions on tt-spectra would be interesting.

**Example 3.3.37.** We will use the same notations as in Example 3.3.33 in each respective part.

(i) Consider the following diagram of the standard flop:

$$\begin{tikzcd}
 & & E \cong \mathbb{P}^k \times \mathbb{P}^k \arrow[d, hook] \arrow[dll] \arrow[drr] & & \\
 & & \tilde{X} \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & & \\
\mathbb{P}^k \cong Y_1 \arrow[r, hook] & X_1 & & X_2 & Y_2 \cong \mathbb{P}^k \arrow[l, hook']
\end{tikzcd}$$

Then, we get an autoequivalence

$$\Phi = p_{1*}p_2{}^* \circ p_{2*}p_1{}^* : \mathsf{Perf}\ X_1 \xrightarrow{\sim} \mathsf{Perf}\ X_2 \xrightarrow{\sim} \mathsf{Perf}\ X_1.$$

Note that $\mathcal{O}_{\mathbb{P}^k}(i)$ is a spherical object in $\mathsf{Perf}\ X_1$ for any $k \in \mathbb{Z}$ (e.g. [Huy06, Example 8.10 (v)]). By [ADM19, Theorem A], we have

$$\Phi = T^{-1}_{\mathcal{O}_{\mathbb{P}^k}(-1)} \circ T^{-1}_{\mathcal{O}_{\mathbb{P}^k}(-2)} \circ \cdots \circ T^{-1}_{\mathcal{O}_{\mathbb{P}^k}(-k)}.$$

More generally, any flop-flop autoequivalence

$$p_{1*}(\mathcal{O}_{\tilde{X}}(mE) \otimes p_2{}^*(-)) \circ p_{2*}(\mathcal{O}_{\tilde{X}}(nE) \otimes p_1{}^*(-))$$

can be written as compositions of (inverses of) spherical twists around $\mathcal{O}_{\mathbb{P}^k}(l)$ for some $l$. In particular, any flop-flop autoequivalence associated to a standard flop glue copies of $X_1$ along an open neighborhood of $X_1 \setminus Y_1$. Here, note that a priori the open neighborhood can be strictly larger than $X_1 \setminus Y_1$ since Proposition 3.3.17 can only tell behaviors on $X_1 \setminus Y_1$ and it is a priori possible that a skyscraper sheaf $k(x)$ at some $x \in Y_1$ gets mapped to a skyscraper after several compositions of spherical twists. In particular, if we suppose $X$ is a threefold so that $k = 1$ and $Y_i$ are $(-1, -1)$-curves (in which case the standard flop is called the **Atiyah flop**), then we have $\Phi = T^{-1}_{\mathcal{O}_{\mathbb{P}^1}(-1)}$, i.e., $\Phi$ is the inverse of the spherical twist and we can apply Proposition 3.3.17.

(ii) In the same paper, the authors show similar results for Mukai flops by using $\mathbb{P}$-twists instead of spherical twists ([ADM19, Theorem B]). In particular, let us consider the case when $k = 1$, i.e., when $X_1$ is a surface. Then, the Mukai flop indeed does not do anything, but the corresponding flop-flop autoequivalence $q_{1*}q_2{}^* \circ q_{2*}q_1{}^*$ is the inverse of the spherical twist $T_{\mathscr{O}_{\mathbb{P}^1}(-1)}$ so we can apply Proposition 3.3.17. In higher dimensions, it will be interesting to investigate $\mathbb{P}$-twists more closely.

(iii) In addition to the Atiyah flops, which flop $(-1,-1)$-curves in threefolds, we can also think of flopping other types of rational curves in threefolds, namely, $(-2,0)$-curves and $(-3,1)$-curves, which are the only possibilities (cf. [Pin83, Proposition 2]). Note, in those cases, the corresponding sheaves are not spherical, so the straightforward generalization cannot be expected. On the other hand, Toda showed that the corresponding flop-flop autoequivalences can be described as generalized spherical twists [Tod07]. Moreover, the case of $(-3,1)$-curves is considered by Donovan-Wemyss [DW16], where they define noncommutative generalization of twist functors. It turns out that cases of $(-1,-1)$-curves and $(-2,0)$-curves end up staying in commutative world while we strictly need noncommutative methods for the case of $(-3,1)$-curves. It will also be interesting to investigate their actions on tt-spectra in the future.

Now, we will see that birational Fourier-Mukai partners are closely related to a well-studied topic in birational geometry, namely $K$-equivalences. First, let us recall the definition.

**Definition 3.3.38.** Let $X$ and $Y$ be smooth projective varieties. We say $X$ and $Y$ are **$K$-equivalent** if there is a crepant birational map $X \dashrightarrow Y$.

*Remark* 3.3.39. Any birationally equivalent minimal models are $K$-equivalent by [Wan98, Corollary 1.10].

One of the main conjectures about $K$-equivalences is proposed by Kawamata [Kaw02]:

**Conjecture 3.3.40** (DK hypothesis)**.** *Let $X$ and $Y$ be smooth projective varieties. If $X$ and $Y$ are $K$-equivalent, then they are Fourier-Mukai partners. Conversely, if $X$ and $Y$ are birationally equivalent Fourier-Mukai partners and have non-negative Kodaira dimension, then they are $K$-equivalent.*

*Remark* 3.3.41. The first implication of this conjecture can be thought of the combination of the following two conjectures.

(i) Kawamata ([Kaw17, Conjecture 3.6]) conjectured that any $K$-equivalence factors into compositions of flops, which is shown for $K$-equivalences between threefolds ([Kaw02]), minimal models ([Kaw08]) and toric varieties ([Kaw13]), respectively.

(ii) Bondal–Orlov ([BO95]) conjectured that varieties related by a flop are Fourier-Mukai partners. This has been also shown in various cases as we have seen in Example 3.3.33. Moreover, Example 3.3.33 suggests that such varieties are birational Fourier-Mukai partners.

Now $K$-equivalence comes into our contexts due to the following result:

**Theorem 3.3.42.** [Tod06, Lemma 7.3.] *Let $X$ and $Y$ be smooth projective varieties. If $X$ and $Y$ are birational Fourier-Mukai partners, then they are $K$-equivalent under the birational map associated to a birational triangulated equivalence.*

*Remark* 3.3.43. For this reason, a birational triangulated equivalence up to shift is called a triangulated equivalence of $K$-equivalent type by Uehara in [Ueh17].

We can rephrase this result as follows:

**Corollary 3.3.44.** *Let $\mathcal{T}$ be a triangulated category and suppose $X, Y \in \mathrm{FM}\,\mathcal{T}$. If there is a prime thick subcategory of $\mathcal{T}$ that is an ideal with respect to both tt-structures transported from $(\mathsf{Perf}\, X, \otimes_X)$ and $(\mathsf{Perf}\, Y, \otimes_Y)$, then $X$ and $Y$ are $K$-equivalent.*

Now, we have the following natural question arising from Theorem 3.3.42:

**Question 3.3.45.** If $X$ and $Y$ are $K$-equivalent Fourier-Mukai partners, then are they birational Fourier-Mukai partners?

Note that conjectures in Remark 3.3.41 suggest the question has a positive answer. Indeed, if Question 3.3.45 has a positive answer, one direction of the DK hypothesis is equivalent to the non-emptiness of intersections of tt-spectra of birationally equivalent Fourier-Mukai partners by Lemma 3.1.15 and Theorem 3.3.42. Therefore, we propose the following version of the DK hypothesis:

**Conjecture 3.3.46** (DK hypothesis)**.** *Let $X$ be a smooth projective variety. Then, every connected component of $\mathrm{Spec}^{\mathsf{FM}}\, \mathsf{Perf}\, X$ containing a copy of $X$ contains all smooth projective varieties that are $K$-equivalent to $X$ as open subschemes.*

Note in particular that this version of the DK hypothesis has a topological necessary condition to be true. Now, let us note that for surfaces, $K$-equivalence is indeed a quite strong condition.

**Lemma 3.3.47.** [Huy06, Corollary 12.8.] *If $X$ and $Y$ are $K$-equivalent smooth projective surfaces, then they are isomorphic. In particular, any birational Fourier-Mukai partners are isomorphic.*

As a corollary, we obtain generalization of Lemma 3.3.23:

**Corollary 3.3.48.** *Let $\mathcal{T}$ be a triangulated category with a surface $X \in \mathrm{FM}\,\mathcal{T}$. Then $\mathcal{T}$ is of disjoint Fourier-Mukai type and hence we have*

$$\mathrm{Spec}^{\mathsf{FM}}\, \mathcal{T} = \bigsqcup_{X \in \mathrm{FM}\,\mathcal{T}} \mathrm{Spec}_{\otimes, X}\, \mathcal{T}.$$

Finally, let us summarize notions and implications we have seen so far.

**Observation 3.3.49.** Let $X$ and $Y$ be smooth projective varieties. Write relations of $X$ and $Y$ as follows:

ISO: $X$ and $Y$ are isomorphic;

FOE: There exists a triangulated category $\mathcal{T}$ such that $X, Y \in \mathrm{FM}\,\mathcal{T}$ and $\mathrm{Spec}_{\otimes,X}\,\mathcal{T} = \mathrm{Spec}_{\otimes,Y}\,\mathcal{T} \subset \mathrm{Spec}_{\vartriangle}\,\mathcal{T}$;

BFM: $X$ and $Y$ are birational Fourier-Mukai partners, or equivalently there exists a triangulated category $\mathcal{T}$ such that $X, Y \in \mathrm{FM}\,\mathcal{T}$ and $\mathrm{Spec}_{\otimes,X}\,\mathcal{T} \cap \mathrm{Spec}_{\otimes,Y}\,\mathcal{T} \neq \emptyset$;

KEFM: $X$ and $Y$ are $K$-equivalent Fourier-Mukai partners;

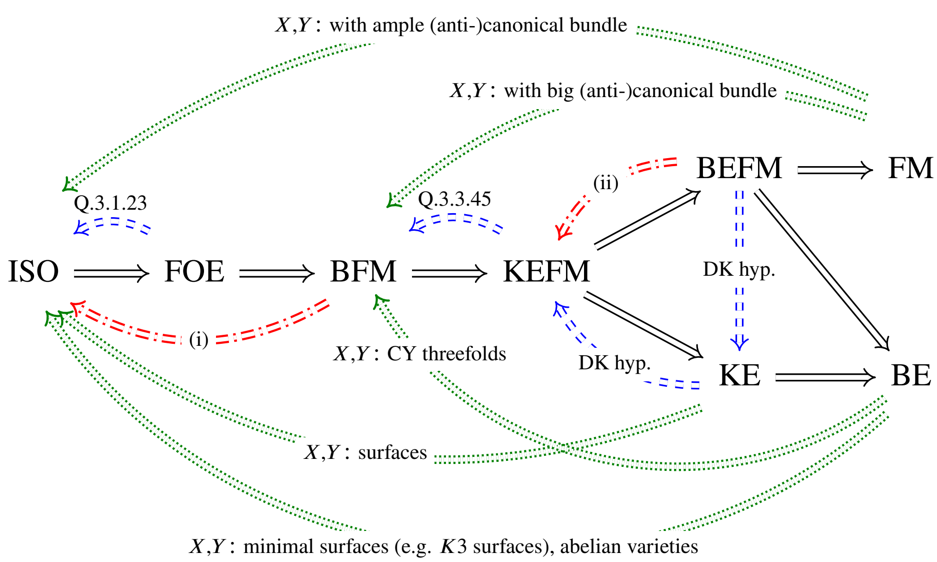
BEFM: $X$ and $Y$ are birationally equivalent Fourier-Mukai partners;

FM: $X$ and $Y$ are Fourier-Mukai partners;

KE: $X$ and $Y$ are $K$-equivalent;

BE: $X$ and $Y$ are birationally equivalent.

Now, we clearly have the following implications:




where solid black arrows are proved implications, dashed blue arrows are implications under a question or conjecture, dotted green arrows are implications known under the specified additional hypotheses, and dash–dot red arrows indicate implications for which counterexamples are listed below:

(i) This is equivalent to saying that there is a triangulated category $\mathcal{T}$ that is not of disjoint Fourier-Mukai type, which we have already seen (e.g. standard/Mukai flops between non-isomorphic varieties or any birationally equivalent non-isomorphic Calabi-Yau threefolds).

This shows at least one of BFM $\Rightarrow$ FOE and FOE $\Rightarrow$ ISO is false, where the former is more likely to be false since it is sufficient to find $K$-equivalent Fourier-Mukai partners that are not birationally equivalent around some points.

(ii) In [Ueh04], Uehara constructed a family of rational elliptic surfaces that are Fourier-Mukai partners (and birationally equivalent), which are not $K$-equivalent. Note they have Kodaira dimension equal to $-\infty$ and hence they do not contradict with the DK hypothesis.

So far, we have been assuming projectivity of varieties, but it is also natural to only assume properness, since properness is a categorical property while projectivity is not, in the sense that if we have varieties $X, Y$ with $\mathsf{Perf}\, X \simeq \mathsf{Perf}\, Y$, then $X$ being proper implies $Y$ being proper, but $X$ being projective does not imply $Y$ being projective. Hence, we consider the following analogue of the Fourier-Mukai locus.

**Definition 3.3.50.** Let $\mathcal{T}$ be a triangulated category. Let $\mathrm{pFM}\,\mathcal{T}$ denote the set of isomorphism classes of smooth **proper** varieties that are Fourier-Mukai partners of $\mathcal{T}$, i.e., whose perfect derived categories are equivalent to $\mathcal{T}$. As before, noting Theorem 2.2.1 (iv), we define the **proper Fourier–Mukai locus** to be the open subscheme

$$\mathrm{Spec}^{\mathsf{pFM}}\,\mathcal{T} := \bigcup_{X \in \mathrm{pFM}\,\mathcal{T}} \mathrm{Spec}_{\otimes, X}\,\mathcal{T} \subset \mathrm{Spec}_{\vartriangle}\,\mathcal{T}.$$

**Observation 3.3.51.** Since a smooth proper variety of dimension $\leq 2$ is projective, we see that if we have a curve or surface $X \in \mathrm{FM}\,\mathcal{T}$, then $\mathrm{Spec}^{\mathsf{FM}}\,\mathcal{T} = \mathrm{Spec}^{\mathsf{pFM}}\,\mathcal{T}$. On the other hand, $\mathrm{Spec}^{\mathsf{pFM}}\,\mathcal{T}$ can be a priori strictly larger than $\mathrm{Spec}^{\mathsf{FM}}\,\mathcal{T}$ since flops of 3-dimensional varieties may not preserve projectivity while inducing equivalences of derived categories.

In general, by the result of Matsukawa (Theorem 2.2.1 (iv)), we may consider the union of all (reduced) noetherian schemes whose derived category is equivalent to a prescribed triangulated category, but in this thesis, we will only consider up to the generality of the proper Fourier–Mukai locus.

# Chapter 4

# Cutting out the triangular spectrum

In the preceding chapters, we have observed that the triangular spectrum contains subspaces that capture (birational) geometric data of varieties. Nevertheless, what we have seen so far has been geometric in nature since we start with tt-structures that actually come from the tt-categories of the form $(\mathsf{Perf}\ X, \otimes_X)$. Indeed, many results/examples in the last chapter were obtained by applications of birational geometry.

In this chapter, we aim at giving more categorical descriptions of Fourier-Mukai loci, which could potentially lead to reverse applications to birational geometry. Furthermore, we set up the framework of polarizations on a triangulated category as a way to cut out loci in the triangular spectrum.

## 4.1 The Serre invariant locus and the Fourier–Mukai locus

In this section, we will focus on the Serre invariant locus of the triangular spectrum introduced in [HO22]. This will naturally lead to the notion of polarizations in the next section.

**Definition 4.1.1** (Hirano–Ouchi)**.** Let $\mathcal{T}$ be a triangulated category with a Serre functor $\mathbb{S}$. A prime thick subcategory $\mathcal{P}$ of $\mathcal{T}$ is said to be **Serre invariant** if $\mathbb{S}(\mathcal{P}) = \mathcal{P}$. Let

$$\mathrm{Spc}^{\mathsf{Ser}}\, \mathcal{T} \subset \mathrm{Spc}_{\vartriangle}\, \mathcal{T}$$

denote the subspace of Serre invariant prime thick subcategories. Since thick subcategories are replete, the Serre invariant locus does not depend on a choice of Serre functors. Recalling Construction 2.1.10, set

$$\mathrm{Spec}^{\mathsf{Ser}}\, \mathcal{T} = (\mathrm{Spc}^{\mathsf{Ser}}\, \mathcal{T}, \mathcal{O}_{\mathrm{Spc}^{\mathsf{Ser}}\, \mathcal{T}, \vartriangle}).$$

We have the following quick observation.

**Proposition 4.1.2.** *For any triangulated category $\mathcal{T}$, we have*

$$\mathrm{Spc}^{\mathsf{FM}}\, \mathcal{T} \subset \mathrm{Spc}^{\mathsf{Ser}}\, \mathcal{T} \subset \mathrm{Spc}_{\vartriangle}\, \mathcal{T}.$$

*Moreover, any triangulated equivalence $\Phi : \mathcal{T} \simeq \mathcal{T}'$ induces compatible homeomorphisms:*

$$\begin{tikzcd} \mathrm{Spc}^{\mathsf{FM}}\,\mathcal{T} \arrow[r, "\subseteq"] \arrow[d, "\Phi"', "\cong"] & \mathrm{Spc}^{\mathsf{Ser}}\,\mathcal{T} \arrow[r, "\subseteq"] \arrow[d, "\Phi"', "\cong"] & \mathrm{Spc}_{\vartriangle}\,\mathcal{T} \arrow[d, "\Phi"', "\cong"] \\ \mathrm{Spc}^{\mathsf{FM}}\,\mathcal{T}' \arrow[r, "\subseteq"] & \mathrm{Spc}^{\mathsf{Ser}}\,\mathcal{T}' \arrow[r, "\subseteq"] & \mathrm{Spc}_{\vartriangle}\,\mathcal{T}' \end{tikzcd}$$

*Proof.* For the first claim, let $\mathbb{S}$ denote a Serre functor of $\mathcal{T}$ and take $X \in \mathrm{FM}\,\mathcal{T}$ and a triangulated equivalence $\Phi : \mathsf{Perf}\,X \simeq \mathcal{T}$. Note that any point in $\Phi(\mathrm{Spc}_{\otimes_X}\,\mathcal{T})$ is of the form $\Phi(S_X(x))$ for some (not necessarily closed) point $x \in X$. Now, by the uniqueness of Serre functors and the commutativity with equivalences, we have

$$\mathbb{S}(\Phi(S_X(x))) = \Phi(\mathbb{S}(S_X(x))) = \Phi(\omega_X[\dim X] \otimes_X S_X(x)) = \Phi(S_X(x)).$$

Thus,

$$\Phi(\mathrm{Spc}_{\otimes_X}\,\mathsf{Perf}\,X) \subset \mathrm{Spc}^{\mathsf{Ser}}\,\mathcal{T}$$

for any $X \in \mathrm{FM}\,\mathcal{T}$ and an equivalence $\Phi : \mathsf{Perf}\,X \simeq \mathcal{T}$. Since $\mathrm{Spc}^{\mathsf{FM}}\,\mathcal{T}$ is the union of these images, we see

$$\mathrm{Spc}^{\mathsf{FM}}\,\mathcal{T} \subset \mathrm{Spc}^{\mathsf{Ser}}\,\mathcal{T} \subset \mathrm{Spc}_{\vartriangle}\,\mathcal{T}.$$

The latter claims follow from commutativity of a Serre functor with triangulated equivalences and the construction of Fourier-Mukai loci. □

The following lemma is a useful criterion for checking if a thick subcategory is Serre invariant. This should be known to experts, but let us include a proof for completeness.

**Lemma 4.1.3.** *Let $\mathcal{T}$ be a triangulated category with a Serre functor $\mathbb{S}$. Moreover, suppose $\mathcal{T}$ is* ***connected****, i.e., cannot be written as a direct sum of nontrivial triangulated subcategories. Then, any nontrivial admissible subcategory $\mathcal{I}$ of $\mathcal{T}$ is not Serre invariant.*

*Proof.* By the definition of a Serre functor, if an admissible subcategory $\mathcal{I}$ is Serre invariant, then the right orthogonal and the left orthogonal agree, which implies $\mathcal{T} = \mathcal{I} \oplus \mathcal{I}^{\perp}$ since for any $X \in \mathcal{T}$, there is a split triangle $Y \to X \to Z \xrightarrow{0} Y[1]$ with $Y \in \mathcal{I}$ and $Z \in \mathcal{I}^{\perp} = {}^{\perp}\mathcal{I}$. Thus, since $\mathcal{T}$ is connected, such an $\mathcal{I}$ should be trivial. □

Combined with Corollary 4.1.2, this provides another proof for [HO22, Lemma 5.7].

**Example 4.1.4.**

(i) [HO22, Remark 5.2] If $\mathcal{T} = \mathsf{Perf}\,X$ for a smooth projective variety $X$ with trivial canonical bundle, then $\mathbb{S} = (-)[\dim X]$ and hence

$$\mathrm{Spec}^{\mathsf{Ser}}\,\mathcal{T} = \mathrm{Spec}_{\vartriangle}\,\mathcal{T}.$$

(ii) If $\mathcal{T} = \mathsf{Perf}\ kQ$ for a quiver $Q$ of type $A_2$, then by Example 2.2.7, we can see

$$\mathrm{Spec}^{\mathsf{Ser}}\,\mathcal{T} = \emptyset.$$

As pointed out right after [GS23, Example 7.1.7] by Gratz-Stevenson, the same results hold for any quiver of type $\mathsf{A}_n$ and $\mathsf{D}_n$. More generally, since any thick subcategory of $\mathsf{Perf}\ k\mathsf{Q}$ for any Dynkin quiver $\mathsf{Q}$ is admissible (for example, by [IT09]), the same results hold by Lemma 4.1.3.

One crucial observation by Hirano–Ouchi ([HO22, Proposition 5.3]) is the following. Note that in [HO22], they only show the equality $\mathrm{Spc}_{\otimes_X}\,\mathsf{Perf}\ X = \mathrm{Spc}^{\mathsf{Ser}}\,\mathsf{Perf}\ X$ of the underlying spaces and the equality as ringed spaces is proven in [IM25, HO24], independently.

**Theorem 4.1.5** (Hirano–Ouchi)**.** *Let $X$ be a Gorenstein projective variety $X$ with $\otimes$-generating canonical bundle. Then, we have*

$$X \cong \mathrm{Spec}_{\otimes_X}\,\mathsf{Perf}\ X = \mathrm{Spec}^{\mathsf{FM}}\,\mathsf{Perf}\ X = \mathrm{Spec}^{\mathsf{Ser}}\,\mathsf{Perf}\ X.$$

In particular, we can reconstruct a Gorenstein projective variety $X$ with $\otimes$-generating canonical bundle from $\mathsf{Perf}\ X$ as a triangulated category. This is a strict generalization of the Bondal–Orlov reconstruction by Theorem 3.3.5. See Theorem 5.1.3 for further discussions and generalizations.

We provide a proof for readers' convenience. For future use, let us record the following simple observation as a lemma.

**Lemma 4.1.6.** *Let $(\mathcal{T}, \otimes)$ be a tt-category with a split generator $E$. Then, a thick subcategory $\mathcal{I} \subset \mathcal{T}$ is a $\otimes$-ideal if and only if $\mathcal{I} \otimes E \subset \mathcal{I}$.*

*Proof.* One direction is clear. Conversely, suppose $\mathcal{I} \otimes E \subset \mathcal{I}$ for $\mathcal{I} \in \mathrm{Th}\,\mathcal{T}$. Let

$$\mathcal{J} = \{M \in \mathcal{T} \mid \mathcal{I} \otimes M \subset \mathcal{I}\}.$$

Then $\mathcal{J}$ is clearly a thick subcategory of $\mathcal{T}$. Now, since $E \in \mathcal{J}$, we have $\mathcal{J} = \mathcal{T}$, i.e., $\mathcal{I}$ is a $\otimes$-ideal. □

*Proof of Theorem 4.1.5.* The inclusion

$$\mathrm{Spc}_{\otimes_X}\,\mathsf{Perf}\ X \subset \mathrm{Spc}^{\mathsf{Ser}}\,\mathsf{Perf}\ X$$

is clear by the definition of $\otimes$-ideals since the Serre functor can be written as

$$\mathbb{S} = - \otimes_X \omega_X[\dim X].$$

Conversely, by the definition of $\otimes$-generation (cf. Definition 3.3.1), there is $N > 0$ such that $\mathscr{E} = \mathscr{O}_X \oplus \cdots \oplus \omega_X^{\otimes n}$ is a split-generator of $\mathsf{Perf}\ X$. Since for any $\mathcal{P} \in \mathrm{Spc}^{\mathsf{Ser}}\,\mathsf{Perf}\ X$, we have

$$\mathcal{P} \otimes \mathscr{E} = \mathcal{P},$$

$\mathcal{P}$ is a $\otimes$-ideal by Lemma 4.1.6 and hence by Theorem 2.1.11 (iii), $\mathcal{P}$ is a prime $\otimes$-ideal. Therefore, $\mathrm{Spc}_{\otimes_X}\,\mathsf{Perf}\ X = \mathrm{Spc}^{\mathsf{Ser}}\,\mathsf{Perf}\ X$ and we are done for the underlying spaces by Proposition 4.1.2. The structure sheaves also agree by Theorem 2.2.1 (iv) and Notation 3.1.7. □

For curves over an algebraically closed field of characteristic 0, we have the following.

**Theorem 4.1.7.** *Let $\mathcal{T}$ be a triangulated category with a smooth projective curve $X \in \mathrm{FM}\,\mathcal{T}$. Then,*

$$\mathrm{Spec}^{\mathsf{FM}}\,\mathcal{T} = \mathrm{Spec}^{\mathsf{Ser}}\,\mathcal{T}.$$

For a proof let us introduce some notions and results discussed in [HO22].

**Construction 4.1.8.** Let $X$ be an elliptic curve. Let $I := \{(r,d) \mid r > 0, \gcd(r,d) = 1\}$ and $M(r,d)$ denote the fine moduli space of $\mu$-semistable sheaves having Chern character $(r,d)$ with universal family $\mathscr{U}_{r,d} \in \mathsf{coh}(X \times M(r,d))$. Let

$$\Phi_{r,d} : \mathsf{Perf}\, M(r,d) \simeq \mathsf{Perf}\, X$$

denote the Fourier-Mukai equivalence given by $\mathscr{U}_{r,d}$. Then, we have the corresponding open subscheme

$$M_{r,d} := \Phi_{r,d}(\mathrm{Spec}_{\otimes_{M(r,d)}} \mathsf{Perf}\, X) \subset \mathrm{Spec}^{\mathsf{FM}}\,\mathsf{Perf}\, X.$$

Note that if $\mathscr{U}'_{r,d}$ is another universal family, we have an invertible sheaf $\mathscr{L}$ on $M(r,d)$ such that $\mathscr{U}'_{r,d} \cong \mathscr{U}_{r,d} \otimes p^*\mathscr{L}$, where $p : X \times M(r,d) \to M(r,d)$ is a canonical projection. Thus, writing the corresponding Fourier-Mukai equivalence by $\Phi'_{r,d}$, we see that $\Phi'_{r,d} = \Phi_{r,d} \circ (- \otimes_{M(r,d)} \mathscr{L})$ and thus $\Phi_{r,d}$ and $\Phi'_{r,d}$ define the same subspace of $\mathrm{Spc}_{\vartriangle}\,\mathsf{Perf}\, X$, so in particular $M_{r,d}$ does not depend on a choice of universal families.

**Lemma 4.1.9.** [HO22, Lemma 4.13] *Let $X$ be an elliptic curve. Then,* $\mathrm{FM}(X)$ *is a singleton and*

$$\mathrm{Spec}_{\vartriangle}\,\mathsf{Perf}\, X = \mathrm{Spec}^{\mathsf{Ser}}\,\mathsf{Perf}\, X = \mathrm{Spec}_{\otimes_X}\,\mathsf{Perf}\, X \sqcup \bigsqcup_{(r,d)\in I} M_{r,d}.$$

*In particular, we have:*

*(i)* $M_{r,d} \cap \mathrm{Spc}_{\otimes_X}\,\mathsf{Perf}\, X = \emptyset$ *for any* $(r,d) \in I$*;*

*(ii)* $M_{r,d} \cap M_{r',d'} = \emptyset$ *for any* $(r,d) \neq (r',d') \in I$.

Now, we can put the ingredients together to show Theorem 4.1.7:

*Proof of Theorem 4.1.7.* By Corollary 4.1.5 we only need to consider the case when $X$ is an elliptic curve. By Proposition 4.1.2 we may assume $\mathcal{T} = \mathsf{Perf}\, X$ and by Construction 4.1.8 we have

$$\mathrm{Spc}_{\otimes_X}\,\mathsf{Perf}\, X \cup \bigcup_{(r,d)\in I} M_{r,d} \subset \mathrm{Spc}^{\mathsf{FM}}\,\mathsf{Perf}\, X.$$

Thus we conclude by Lemma 4.1.9. □

*Remark* 4.1.10. Since an elliptic curve is tt-separated by Theorem 3.3.12, the categorical construction of the Fourier–Mukai locus as the Serre invariant locus lets us reconstruct the elliptic curve as a connected component of the Serre invariant locus. This is observed in [Mat23, Corollary 4.10].

**Observation 4.1.11.** Let $\mathcal{T}$ be a triangulated category with an elliptic curve $X \in \mathrm{FM}\,\mathcal{T}$. Let us reinterpret Hirano–Ouchi's presentation

$$\mathrm{Spec}_{\vartriangle} \mathsf{Perf}\, X = \mathrm{Spec}^{\mathsf{Ser}} \mathsf{Perf}\, X = \mathrm{Spec}_{\otimes, X} \mathsf{Perf}\, X = \mathrm{Spec}_{\otimes_X} \mathsf{Perf}\, X \sqcup \bigsqcup_{(r,d)\in I} M_{r,d}$$

in terms of the presentation in Theorem 3.1.21. First, recall that the determinant map gives an isomorphism $\det : M(r,d) \to \mathrm{Pic}^d(X)$. Hence, fixing a line bundle on $X$ of degree 1 (or equivalently fixing a base point of $X$), we obtain a canonical isomorphism $z_{r,d} : X \cong M(r,d)$ for any $(r,d) \in I$ and let $\zeta_{r,d} : \mathsf{Perf}\, X \to \mathsf{Perf}\, M(r,d)$ denote the corresponding equivalence. Therefore, writing $\tau_{r,d} := \Phi_{r,d} \circ \zeta_{r,d} \in \mathrm{Auteq}\, \mathsf{Perf}\, X$ with $\tau_{(0,1)} = \mathrm{id}_{\mathsf{Perf}\, X}$, we have

$$M_{r,d} = \Phi_{r,d}(\mathrm{Spec}_{\otimes_{M(r,d)}} \mathsf{Perf}\, X) = \tau_{r,d}(\mathrm{Spec}_{\otimes_X} \mathsf{Perf}\, X)$$

and thus we can write

$$\mathrm{Spec}_{\otimes, X} \mathsf{Perf}\, X = \bigsqcup_{\tau_{r,d} \in I \cup \{(0,1)\}} \tau_{r,d}(\mathrm{Spec}_{\otimes_X} \mathsf{Perf}\, X).$$

In particular, this shows $I_X = I \cup \{(0,1)\}$ and hence Hirano–Ouchi's presentation corresponds to a specific choice of the representatives in the left coset

$$I_X = \mathrm{Auteq}\, \mathsf{Perf}\, X / \mathrm{Auteq}^{\mathsf{st}}\, \mathsf{Perf}\, X.$$

Since the cases of elliptic curves and non-elliptic curves are two extremes in terms of positivity of (anti-)canonical bundles, we proposed the following conjecture in [Ito23]:

**Conjecture 4.1.12** (Ito)**.** *Let $\mathcal{T}$ be a triangulated category with $X \in \mathrm{FM}\,\mathcal{T}$. Then we have*

$$\mathrm{Spc}^{\mathsf{Ser}}\, \mathcal{T} = \mathrm{Spc}^{\mathsf{FM}}\, \mathcal{T}.$$

*In particular,* $\dim \mathrm{Spc}^{\mathsf{Ser}}\, \mathcal{T} = \dim \mathrm{Spc}^{\mathsf{FM}}\, \mathcal{T} = \dim X$.

*Remark* 4.1.13. Note the conjecture also holds for $\mathcal{T} = \mathsf{Perf}\, R$ when $R$ is the path algebra of a Dynkin quiver by Example 4.1.4 (ii).

The conjecture tends to be false when a canonical bundle $\omega_X$ is trivial so that $\mathrm{Spec}^{\mathsf{Ser}} \mathsf{Perf}\, X = \mathrm{Spec}_{\vartriangle} \mathsf{Perf}\, X$. The following are given in [HO24, Mat25a] (over a field of characteristic 0).

**Theorem 4.1.14** (Hirano–Ouchi, Matsukawa)**.** *Let $X$ be a $K3$ surface with Picard number* 1*. Then,*

$$\mathrm{Spec}^{\mathsf{FM}} \mathsf{Perf}\, X \subsetneq \mathrm{Spec}^{\mathsf{Ser}} \mathsf{Perf}\, X = \mathrm{Spec}_{\vartriangle} \mathsf{Perf}\, X.$$

**Theorem 4.1.15** (Matsukawa)**.** *Let $X$ be an abelian variety that is not isogenous to any product of elliptic curves. Then,*

$$\mathrm{Spec}^{\mathsf{FM}} \mathsf{Perf}\, X \subsetneq \mathrm{Spec}^{\mathsf{Ser}} \mathsf{Perf}\, X = \mathrm{Spec}_{\vartriangle} \mathsf{Perf}\, X.$$

On the other hand, the conjecture is widely open for varieties with "positive" (anti-)canonical bundle, such as toric varieties and varieties of general type.

## 4.2 Fixed points of autoequivalences and polarizations

In this section, we introduce the framework of polarizations on triangulated categories and their corresponding loci/spectra following [Ito25, IO25]. In short, a polarization on a triangulated category is a fixed autoequivalence, and we can consider its fixed points in the triangular spectrum. This is a direct generalization of the construction of the Serre invariant locus.

It turns out that the framework of polarizations is closely related to tensor triangular geometry, although studies of polarizations are rich in its own right. To mimic the construction of tt-spectra, we will first introduce our guiding example from algebraic geometry, which was essentially observed in [HO22, Mat23] and made explicit in [IM25]. For notations and terminology, recall that Notation 2.1.1 (vi), (viii) and Definition 3.3.1.

**Proposition 4.2.1** (Guiding example)**.** *Let $X$ be a noetherian scheme and let $\mathscr{L}$ be a $\otimes$-generating line bundle. Then, we have*

$$\mathrm{Spec}_{\otimes} \mathsf{Perf}\, X = \mathrm{Spec}^{\tau_{\mathscr{L}}} \mathsf{Perf}\, X \subset \mathrm{Spec}_{\vartriangle} \mathsf{Perf}\, X.$$

*Proof.* The proof is exactly the same as Theorem 4.1.5. □

Let us list some takeaways from this observation.

- To get the space of fixed points, we only need a single autoequivalence $\tau_{\mathscr{L}} \in \mathrm{Auteq}\, \mathsf{Perf}\, X$, which does not require any other extra structures on the triangulated category $\mathsf{Perf}\, X$ such as monoidal structures.
- When $X$ is reduced, we can reconstruct the structure sheaf of the tt-spectrum on
$$\mathrm{Spc}_{\otimes} \mathsf{Perf}\, X = \mathrm{Spc}^{\tau_{\mathscr{L}}} \mathsf{Perf}\, X$$
as the restriction of the structure sheaf of the Matsui spectrum by Theorem 2.2.1 (iv). Namely, we do not need the data corresponding to the tensor unit $\mathscr{O}_X$ to define the structure sheaf (cf. Construction 2.1.8).
- It is also worth emphasizing that the classification of tt-structures is not yet well-formulated while classifications of autoequivalences just happen inside of $\mathrm{Auteq}\, \mathsf{Perf}\, X$, which is well-understood in various cases. In other words, we can hope for classifying geometry of
$$\mathrm{Spc}^{\tau} \mathsf{Perf}\, X \subset \mathrm{Spc}_{\vartriangle} \mathsf{Perf}\, X$$
when we vary $\tau \in \mathrm{Auteq}\, \mathsf{Perf}\, X$.

## Generalization with tt-structures and $\otimes$-generating objects

Let us first generalize the guiding example in the presence of tt-structures to clarify which classes of tt-categories are in our scope and below we will recast this phenomenon in the absence of tt-structures so that it is applicable to more general classes of triangulated categories not admitting natural (symmetric) monoidal structures.

The following property extracts the essence of the proof of Proposition 4.2.1.

**Definition 4.2.2.** Let $(\mathcal{T}, \otimes)$ be a tt-category. An object $\mathscr{L} \in \mathcal{T}$ is said to be **$\otimes$-generating** if $\mathscr{L}$ is $\otimes$-invertible and if $\langle \mathscr{L}^{\otimes n} \mid n \in \mathbb{Z} \rangle = \mathcal{T}$. Note that for a noetherian scheme $X$, a line bundle $\mathscr{L}$ on $X$ is $\otimes$-generating if $\mathscr{L}$ is $\otimes$-generating in $(\mathsf{Perf}\ X, \otimes_X)$.

Indeed, by the identical arguments as in the proof of Proposition 4.2.1 (and hence Theorem 4.1.5), we immediately obtain the following generalization. Note however that we are not claiming the identification as ringed spaces below, since a priori a tt-spectrum may not be open in the triangular spectrum.

**Proposition 4.2.3.** *Let $(\mathcal{T}, \otimes)$ be an $M$-compatible rigid tt-category with a $\otimes$-generating object $\mathscr{L}$. Then, we have*

$$\operatorname{Spc}_{\otimes} \mathcal{T} = \operatorname{Spc}^{\tau_{\mathscr{L}}} \mathcal{T} \subset \operatorname{Spc}_{\vartriangle} \mathcal{T}.$$

## Generalization in the absence of tt-structures and polarizations

Above, the data of an autoequivalence coming from tensoring with $\otimes$-generating objects recovered tt-spectra. Based on this observation, let us consider an analogue of tt-spectra in the absence of tt-structures as follows.

**Definition 4.2.4.** Let $\mathcal{T}$ be a triangulated category.

(i) For an autoequivalence $\tau \in \operatorname{Auteq} \mathcal{T}$, the pair $(\mathcal{T}, \tau)$ is said to be a **polarized triangulated category** (in short, a **pt-category**). In this context, a chosen autoequivalence $\tau$ of $\mathcal{T}$ may be interchangeably referred to as a **polarization** on $\mathcal{T}$.

(ii) For a pt-category $(\mathcal{T}, \tau)$, define its **$\tau$-fixed locus** (of the triangular spectrum) to be the ringed space

$$\operatorname{Spec}^{\tau} \mathcal{T} = (\operatorname{Spc}^{\tau} \mathcal{T}, \mathscr{O}_{\operatorname{Spc}^{\tau} \mathcal{T}, \vartriangle}).$$

See Construction 2.1.10 for the definition of the structure sheaf $\mathscr{O}_{\operatorname{Spc}^{\tau} \mathcal{T}, \vartriangle}$. If there is no confusion, let us write $\mathscr{O}_{\tau} := \mathscr{O}_{\operatorname{Spc}^{\tau} \mathcal{T}, \vartriangle}$.

(iii) For a pt-category $(\mathcal{T}, \tau)$, define its **pt-spectrum** (**polarized Matsui spectrum**) to be the ringed space

$$\operatorname{Spec}_{\vartriangle}(\mathcal{T}, \tau) = (\operatorname{Spc}_{\vartriangle}(\mathcal{T}, \tau), \mathscr{O}_{\operatorname{Spc}_{\vartriangle}(\mathcal{T}, \tau), \vartriangle}),$$

where

$$\operatorname{Spc}_{\vartriangle}(\mathcal{T}, \tau) = \{\mathcal{P} \in \operatorname{Th}^{\tau} \mathcal{T} \mid \text{the poset } \{\mathcal{Q} \in \operatorname{Th}^{\tau} \mathcal{T} \mid \mathcal{P} \subsetneq \mathcal{Q}\} \text{ has a unique smallest element}\}.$$

Similarly, if there is no confusion, let us write $\mathcal{O}_\tau := \mathcal{O}_{\mathrm{Spc}_\triangle(\mathcal{T},\tau),\triangle}$.

(iv) Let $(\mathcal{T}_1, \tau_1)$ and $(\mathcal{T}_2, \tau_2)$ be pt-categories. A **pt-functor** (resp. **pt-equivalence**) $(F, \alpha)$ is defined to be a triangulated functor (resp. equivalence) $F : \mathcal{T}_1 \to \mathcal{T}_2$ together with a natural isomorphism $\alpha : \tau_2 \circ F \cong F \circ \tau_1$. In this paper, we abuse notations by saying a triangulated functor $F : \mathcal{T}_1 \to \mathcal{T}_2$ is a **pt-functor** if there exists $\alpha$ as above, as the data of $\alpha$ does not play a role in studying fixed loci/pt-spectra.

As in the case of triangular spectra (cf. [Mat23, Proposition 4.2]), a pt-equivalence $F : (\mathcal{T}_1, \tau_1) \xrightarrow{\sim} (\mathcal{T}_2, \tau_2)$ induces an isomorphism of fixed loci/pt-spectra as ringed spaces (which does not depend on the choice of $\alpha$).

In this thesis, we focus on fixed loci since they are more computable for our purposes and we want to study geometry of subspaces in the triangular spectrum, noting that the pt-spectrum is not necessarily contained in the triangular spectrum (cf. Example 4.2.9). On the other hand, the pt-spectrum aligns better with the theory of relative Matsui spectra developed by Matsukawa [Mat25b]. To maintain our focus, let us delay the following topics to later sections.

*Remark* 4.2.5. Indeed, when pt-categories can be enhanced to the dg/$\infty$-categorical setup, we may identify a pt-category as a $\mathsf{Perf}(B\mathbb{G}_{m,k})$-module (or equivalently a local system of stable $\infty$-categories on $S^1$) and we can realize its pt-spectrum as a relative Matsui spectrum with respect to this action. See §5.4 for more discussions.

*Remark* 4.2.6. Another immediate question is whether or not the association of pt-categories with pt-functors to fixed loci/pt-spectra as above is functorial. Unfortunately, it is not even well-defined on morphisms (as in the case of the triangular spectrum) for the similar reason that the maximal ideals of a ring are not necessarily pulled back to maximal ideals by ring homomorphisms. Nevertheless, the association $(\mathcal{T}, \tau) \mapsto \mathrm{Th}^\tau(\mathcal{T})$ can be made into a functor (into the category of topological complete lattices), so the theory of pt-spectra can be extracted from a functorial theory. See §5.5 for more discussions.

First, let us quickly note the following features of fixed loci/pt-spectra with respect to localizations, which are analogous to what happens to tt-spectra and relative Matsui spectra.

**Lemma 4.2.7.** *Let $(\mathcal{T}, \tau)$ be a pt-category. For any open subset $U$ of $\mathrm{Spc}^\tau \mathcal{T}$ or $\mathrm{Spc}_\triangle(\mathcal{T}, \tau)$, the polarization $\tau$ descends to a polarization $\tau_U$ on $\mathcal{T}(U) := (\mathcal{T}/\bigcap_{\mathcal{P}\in U} \mathcal{P})^\natural$. Moreover, fixing data for a quasi-inverse, $\tau$ induces automorphisms $\tau_*$ of $\mathrm{Spec}^\tau \mathcal{T}$ and $\mathrm{Spec}_\triangle(\mathcal{T}, \tau)$ as ringed spaces, whose underlying map is the identity.*

*Proof.* Fix an open subset $U$ of $\mathrm{Spc}^\tau \mathcal{T}$ or $\mathrm{Spc}_\triangle(\mathcal{T}, \tau)$. Since $\tau(\mathcal{P}) = \mathcal{P}$ for any $\mathcal{P} \in U$, we have $\tau(\bigcap_{\mathcal{P}\in U} \mathcal{P}) = \bigcap_{\mathcal{P}\in U} \mathcal{P}$. Therefore, $\tau$ induces an autoequivalence on $\mathcal{T}/\bigcap_{\mathcal{P}\in U} \mathcal{P}$ and hence on the Karoubi envelope $\mathcal{T}(U)$, which is denoted by $\tau_U$. Now, since $\tau_U$ induces a functorial automorphism $\mathcal{O}^{\mathsf{pre}}_\tau(U) \cong \mathcal{O}^{\mathsf{pre}}_\tau(U)$ by conjugation (which depends on the choice of the data for a quasi-inverse of $\tau$), $\tau$ induces an automorphism $\mathcal{O}_\tau \cong \mathcal{O}_\tau$ of sheaves. Since $\tau$ acts on the underlying topological space as the identity, $\tau$ induces the automorphism of $\mathrm{Spec}^\tau \mathcal{T}$ and $\mathrm{Spec}_\triangle(\mathcal{T}, \tau)$ as ringed spaces. $\square$

Next, let us clarify the relationship among fixed loci, pt-spectra, and triangular spectra.

**Proposition 4.2.8.** *Let $(\mathcal{T}, \tau)$ be a pt-category. Then,*

$$\operatorname{Spc}^\tau \mathcal{T} = \operatorname{Spc}_\vartriangle(\mathcal{T}, \tau) \cap \operatorname{Spc}_\vartriangle \mathcal{T}.$$

*In particular,* $\operatorname{Spc}^\tau \mathcal{T} = \operatorname{Spc}_\vartriangle(\mathcal{T}, \tau)$ *if and only if* $\operatorname{Spc}_\vartriangle(\mathcal{T}, \tau) \subset \operatorname{Spc}_\vartriangle \mathcal{T}$*, in which case we say $\tau$ or $(\mathcal{T}, \tau)$ is **$M$-compatible**.*

*Proof.* By definition, the containment $\operatorname{Spc}_\vartriangle(\mathcal{T}, \tau) \cap \operatorname{Spc}_\vartriangle \mathcal{T} \subset \operatorname{Spc}^\tau \mathcal{T}$ and $\operatorname{Spc}^\tau \mathcal{T} \subset \operatorname{Spc}_\vartriangle \mathcal{T}$ are clear, so it remains to show

$$\operatorname{Spc}^\tau \mathcal{T} \subset \operatorname{Spc}_\vartriangle(\mathcal{T}, \tau)$$

Take $\mathcal{P} \in \operatorname{Spc}^\tau \mathcal{T}$. It suffices to show that the unique smallest element $\overline{\mathcal{P}}$ of $\{\mathcal{Q} \in \operatorname{Th} \mathcal{T} \mid \mathcal{P} \subsetneq \mathcal{Q}\}$ is fixed by $\tau$. Indeed, since $\tau$ acts lattice-isomorphically on the lattice $\{\mathcal{Q} \in \operatorname{Th} \mathcal{T} \mid \mathcal{P} \subsetneq \mathcal{Q}\}$ as $\mathcal{P}$ is fixed by $\tau$, it preserves the unique smallest element $\overline{\mathcal{P}}$. □

Note in general that the pt-spectrum is not necessarily contained in the triangular spectrum. The following example is well-known to experts, but let us record it for completeness.

**Example 4.2.9.** Let $X = B\mu_2$ (and assume char $k \neq 2$) and let $\tau = \tau_{\mathcal{O}_X(\chi)}$, where $\chi$ is the non-trivial character of $\mu_2$. Now, since $\mathsf{Perf}\ X = \langle \mathcal{O}_X, \mathcal{O}_X(\chi) \rangle \simeq \mathsf{Perf}\ k \oplus \mathsf{Perf}\ k$, we have

$$\operatorname{Spc}_\vartriangle \mathsf{Perf}\ X = \langle \mathcal{O}_X \rangle \sqcup \langle \mathcal{O}_X(\chi) \rangle$$

and $\operatorname{Spc}^\tau \mathsf{Perf}\ X = \emptyset$. On the other hand, since $\operatorname{Th}^\tau \mathsf{Perf}\ X = \{\langle 0 \rangle, \mathsf{Perf}\ X\}$, we have

$$\operatorname{Spc}_\vartriangle(\mathsf{Perf}\ X, \tau) = \operatorname{Spc}_{\otimes_X} \mathsf{Perf}\ X = \langle 0 \rangle$$

and hence

$$\operatorname{Spc}_\vartriangle(\mathsf{Perf}\ X, \tau) \cap \operatorname{Spc}_\vartriangle \mathcal{T} = \emptyset = \operatorname{Spc}^\tau \mathcal{T}.$$

Indeed, fixed loci, pt-spectra, and tt-spectra relate as follows in general.

**Proposition 4.2.10.** *Let $(\mathcal{T}, \otimes)$ be a rigid tt-category (so that any $\otimes$-ideal is radical) with noetherian tt-spectrum and let $\mathscr{L}$ be a $\otimes$-generating object. Then, we have*

$$\operatorname{Spc}_\otimes \mathcal{T} = \operatorname{Spc}_\vartriangle(\mathcal{T}, \tau_\mathscr{L}) \subset \operatorname{Th} \mathcal{T}.$$

*In particular, $\otimes$ is $M$-compatible if and only if $\tau_\mathscr{L}$ is $M$-compatible, in which case we have*

$$\operatorname{Spc}^{\tau_\mathscr{L}} \mathcal{T} = \operatorname{Spc}_\otimes \mathcal{T} = \operatorname{Spc}_\vartriangle(\mathcal{T}, \tau_\mathscr{L}).$$

*Proof.* By the same arguments as Proposition 4.2.1, $\operatorname{Th}^{\tau_\mathscr{L}} \mathcal{T}$ agrees with the lattice of radical thick $\otimes$-ideals of $(\mathsf{Perf}\ X, \otimes_X)$. Hence, by [Mat21, Proposition 4.7], $\operatorname{Spc}_\otimes \mathcal{T} = \operatorname{Spc}_\vartriangle(\mathcal{T}, \tau_\mathscr{L})$. The latter claim follows from Proposition 4.2.8 (see also Proposition 4.2.3). □

We have observed (e.g. in Theorem 3.3.5) that there is subtle difference between the notion of (anti-)ample line bundles and the notion ⊗-generating line bundles, but further generalizations to pt-categories reveal that there are even more subtleties for such properties of line bundles.

We begin with a simpler example of $\mathbb{P}^1$ and then move on to the case of elliptic curves where we can already observe such complexity, namely, there is a line bundle that is not ⊗-generating, but whose $\tau$-fixed locus still agrees with tt-spectrum at all $k$-points.

**Example 4.2.11.** Let $X$ be a smooth projective variety with ample (anti-)canonical bundle. Then, by [BO01], polarizations on $\mathsf{Perf}\, X$ are classified by

$$\mathrm{Auteq}\,\mathsf{Perf}\, X = \mathrm{Aut}\, X \ltimes \mathrm{Pic}\, X \times \mathbb{Z}[1].$$

In particular, if $X = \mathbb{P}^1$, then $\mathrm{Auteq}\,\mathsf{Perf}\,\mathbb{P}^1$ is a group generated by $\mathrm{Aut}\,\mathbb{P}^1 \cong \mathrm{PGL}(2)$, $\mathrm{Pic}(X) = \mathbb{Z}$ and shifts $\mathbb{Z}[1]$. Also, note that by [Mat21, Example 4.10], we have

$$\mathrm{Spc}_{\vartriangle}\,\mathsf{Perf}\,\mathbb{P}^1 = \mathbb{P}^1 \sqcup \mathbb{Z},$$

where $\mathbb{Z}$ is equipped with discrete topology and each point corresponds to the thick subcategory generated by $\mathcal{O}(i)$ for $i \in \mathbb{Z}$. It is straightforward to observe the following:

(i) For $f_* \in \mathrm{Aut}\,\mathbb{P}^1$, we have

$$\mathrm{Spc}^{f_*}\,\mathsf{Perf}\,\mathbb{P}^1 = \{\text{fixed points of } f : \mathbb{P}^1 \to \mathbb{P}^1\} \sqcup \mathbb{Z} \subset \mathbb{P}^1 \sqcup \mathbb{Z}.$$

(ii) For $\tau_{\mathcal{O}_{\mathbb{P}^1}(i)} \in \mathrm{Pic}\,\mathbb{P}^1$ with $i \neq 0$, we have

$$\mathrm{Spc}^{\tau_{\mathcal{O}(i)}}\,\mathsf{Perf}\,\mathbb{P}^1 = \mathbb{P}^1 \subset \mathbb{P}^1 \sqcup \mathbb{Z}.$$

(iii) For $[n] \in \mathbb{Z}$, we have

$$\mathrm{Spc}^{[n]}\,\mathsf{Perf}\,\mathbb{P}^1 = \mathbb{P}^1 \sqcup \mathbb{Z}.$$

Therefore, for a general autoequivalence $\tau = \mathcal{O}_{\mathbb{P}^1}(i) \otimes_{\mathcal{O}^1_{\mathbb{P}}} f_*(-)[n]$, we have

$$\mathrm{Spc}^{\tau}\,\mathsf{Perf}\,\mathbb{P}^1 = \begin{cases} (\mathbb{P}^1)^f \sqcup \mathbb{Z} & \text{if } i = 0; \\ (\mathbb{P}^1)^f & \text{if } i \neq 0. \end{cases}$$

To get a clearer presentation of the classification, let us compute the following set. Define the equivalence relation $\sim$ on $\mathrm{Aut}\,\mathbb{P}^1$ by setting $f \sim g$ if $(\mathbb{P}^1)^f = (\mathbb{P}^1)^g$. Note that under $\mathrm{Aut}\,\mathbb{P}^1 = \mathrm{PGL}_2(k)$, fixed points correspond to eigenlines. Now, we claim that there is a natural bijection

$$(\mathrm{Aut}(\mathbb{P}^1) \setminus \{\mathrm{id}_{\mathbb{P}^1}\}/\sim) \cong \mathrm{Sym}^2\,\mathbb{P}^1 = \mathbb{P}^2.$$

Indeed, it is clear that any $f \in \mathrm{Aut}\,\mathbb{P}^1$ except for $\mathrm{id}_{\mathbb{P}^1}$ has two fixed points (counted with multiplicity) and any pair of two points (counted with multiplicity) in $\mathbb{P}^1$ can be realized as fixed points of certain $f \in \operatorname{Aut} \mathbb{P}^1$. In other words, the set of configurations of fixed points is in bijection with the set of degree 2 divisors in $\mathbb{P}^1$, which is naturally in bijection with $\operatorname{Sym}^2 \mathbb{P}^1$.

Thus, we have a complete classification of polarizations on $\mathsf{Perf}\, \mathbb{P}^1$ up to the geometry of its fixed locus as follows:

$$\{\text{polarizations on } \mathsf{Perf}\, \mathbb{P}^1 \text{ up to fixed loci in } \operatorname{Spc}_\vartriangle \mathsf{Perf}\, \mathbb{P}^1\} = (\{\tau_{\mathcal{O}_{\mathbb{P}^1}}\} \sqcup \{\tau_{\mathcal{O}_{\mathbb{P}^1}(1)}\}) \times (\mathbb{P}^2 \sqcup \{\mathrm{id}_{\mathbb{P}^1}\})$$

See Theorem 4.3.10 for a natural algebraic stack structure on this set.

On the other hand, to the author's knowledge, it is still not known if we have a complete classification of tt-structures on $\mathsf{Perf}\, \mathbb{P}^1$ up to tt-geometry/tt-equivalences.

Also, note that not all tt-spectra are realized as fixed points of autoequivalences since the tt-spectrum for the quiver tensor product $\otimes_{\mathsf{rep}}$ on the Kronecker quiver consists of two points in $\mathbb{Z}$ (Example 4.3.16). We do not know if the converse is also false. That is, we do not know if given a polarization on $\mathsf{Perf}\, X$, there exists a tt-structure on $\mathsf{Perf}\, \mathbb{P}^1$ whose tt-spectrum agrees with the $\tau$-fixed locus.

**Example 4.2.12.** Let us observe geometry of fixed loci in the case of an elliptic curve $E$ over an algebraically closed field $k$ of characteristic zero. We will classify fixed loci for generators of $\operatorname{Auteq} \mathsf{Perf}\, E$ and observe that a line bundle is $\otimes$-generating if and only if the corresponding $\tau$-fixed locus agrees with the tt-spectrum (see the third bullet). On the other hand, there is a non-$\otimes$-generating line bundle whose $\tau$-fixed locus agrees with the tt-spectrum on all $k$-points.

First, recall from Example 3.3.19 that there is a short exact sequence

$$1 \to \operatorname{Aut} E \ltimes \operatorname{Pic}^0 E \times \mathbb{Z}[2] \to \operatorname{Auteq} \mathsf{Perf}\, E \xrightarrow{\theta} \mathrm{SL}(2, \mathbb{Z}) \to 1$$

where $\theta$ is given by the action on the image of the charge map

$$Z : K(E) \twoheadrightarrow \mathbb{Z}^2, \mathscr{F} \mapsto (\operatorname{rank} \mathscr{F}, \deg \mathscr{F}).$$

Also recall from [HO22, Theorem 4.11] (cf. Construction 4.1.8) that we have

$$\operatorname{Spec}_\vartriangle \mathsf{Perf}\, E = \operatorname{Spec}_{\otimes_E} \mathsf{Perf}\, E \sqcup \bigsqcup_{(r,d)\in I} M_{r,d},$$

where $I = \{(r, d) \in \mathbb{Z}_{>0} \times \mathbb{Z} \mid \gcd(r, d) = 1\}$ and

$$M_{r,d} := \Phi_{\mathscr{U}_{r,d}} \left(\operatorname{Spec}_{\otimes_{M(r,d)}} \mathsf{Perf}\, M(r, d)\right)$$

for the moduli space $M(r, d)$ of stable sheaves of rank $r$ and degree $d$ and the Fourier–Mukai transform $\Phi_{\mathscr{U}_{r,d}}$ with respect to a universal family $\mathscr{U}_{r,d}$. For convenience, let us write the subspace

$$M_{0,1} := \operatorname{Spec}_{\otimes_{\mathcal{O}_E}} \mathsf{Perf}\, E \subset \operatorname{Spec}_\vartriangle \mathsf{Perf}\, E.$$

It is easy to check

$$M_{r,d}(k) = \{S_{r,d}(\mathscr{F}) \mid [\mathscr{F}] \in M(r, d)(k)\},$$

where we define

$$S_{r,d}(\mathscr{F}) := \langle \mathscr{G} \mid [\mathscr{G}] \in M(r,d)(k), \mathscr{G} \ncong \mathscr{F} \rangle \subset \mathsf{Perf}\, E.$$

Note the generic point of $M_{r,d}$ is given by

$$\eta_{r,d} := \langle \mathscr{G} \mid [\mathscr{G}] \in M(r,d)(k) \rangle \subset \mathsf{Perf}\, E.$$

Also, recall that any object in $\mathsf{Perf}\, E$ can be written as a direct sum of shifts of indecomposable coherent sheaves on $E$ and hence that any autoequivalence of $\mathsf{Perf}\, E$ takes an indecomposable coherent sheaf to a shift of an indecomposable coherent sheaf. Hence, as a very rough estimate, for an autoequivalence $\tau \in \operatorname{Auteq} \mathsf{Perf}\, E$, we have

$$\operatorname{Spc}^{\tau} \mathsf{Perf}\, E \subset \bigsqcup_{(r,d) \in I^{\tau}} M_{r,d},$$

where $I^{\tau} := \{(r,d) \in I \cup \{(0,1)\} \mid \theta(\tau)(r,d) = \pm(r,d)\}$, noting that for a coherent sheaf $\mathscr{F}$ on $E$, we have $(\operatorname{rank} \mathscr{F}[n], \deg \mathscr{F}[n]) = (-1)^n (\operatorname{rank} \mathscr{F}, \deg \mathscr{F})$.

Now, let us consider the $\tau$-fixed locus corresponding to generators of $\operatorname{Auteq} \mathsf{Perf}\, E$.

- To get more explicit descriptions of generators of $\mathrm{SL}(2,\mathbb{Z})$, note that $\theta$ maps $T_{k(x)} (= \tau_{\mathscr{O}_E(x)})$ and $T_{\mathscr{O}_E}$ to $\alpha := \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ and $\beta := \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$, respectively, where $x$ is a closed point of $E$ and $T_{\mathscr{E}}$ denotes the spherical twist with respect to an spherical object $\mathscr{E}$. To visualize the corresponding fixed loci, note that $T_{k(x)}$ maps the component $M_{r,d}$ to $M_{r,r+d}$ while $T_{\mathscr{O}_E}$ maps the component $M_{r,d}$ to $M_{r-d,d}$ if $r - d \geq 0$ and to $M_{-r+d,-d}$ otherwise. Thus, we have

  $$\operatorname{Spc}^{T_{k(x)}} \mathsf{Perf}\, E = M_{0,1}$$

  and

  $$\operatorname{Spc}^{T_{\mathscr{O}_E}} \mathsf{Perf}\, E = \bigsqcup_{r \in \mathbb{Z}_{>0}} M_{r,0}.$$

  For the latter equality, we need to show that for any stable sheaf $\mathscr{F}$ of rank $r$ and degree 0, we have

  $$T_{\mathscr{O}_E} S_{r,0}(\mathscr{F}) = S_{r,0}(\mathscr{F}).$$

  To see this, note that we have

  $$T_{\mathscr{O}_E} \mathscr{F} \cong \operatorname{cone}((\mathrm{H}^0(E,\mathscr{F}) \oplus \mathrm{H}^1(E,\mathscr{F})[-1]) \otimes_{\mathbb{C}} \mathscr{O}_E \xrightarrow{\mathsf{ev}} \mathscr{F})$$

  (e.g., cf. [Huy06]) and there exists a unique stable sheaf $\mathscr{F}_{r,0}$ of rank $r$ and degree 0 with global sections (up to isomorphism) by [Ati57, Theorem 5]. Therefore, for any stable sheaf $\mathscr{F} \ncong \mathscr{F}_{r,0}$ of rank $r$ and degree 0, we have $\mathrm{H}^0(E,\mathscr{F}) = \mathrm{H}^1(E,\mathscr{F}) = 0$ and hence $T_{\mathscr{O}_E} \mathscr{F} = \mathscr{F}$, i.e., $T_{\mathscr{O}_E} S_{r,0}(\mathscr{F}) = S_{r,0}(\mathscr{F})$. Since $T_{\mathscr{O}_E}$ induces an automorphism on the component $M_{r,0}$, we see that $T_{\mathscr{O}_E} S_{r,0}(\mathscr{F}_{r,0}) = S_{r,0}(\mathscr{F}_{r,0})$ as well.

- Consider $\tau = f_*$ for $f \in \operatorname{Aut} E$. By definition we have

$$\operatorname{Spc}^\tau \mathsf{Perf}\, E = \{\text{fixed points of } f\} \sqcup \bigsqcup_{(r,d)\in I} \left\{ S_{r,d}(\mathscr{F}) \mid \mathscr{F} \cong f_*\mathscr{F} \right\} \cup \{\eta_{r,d}\}.$$

  Note a special case is when $f$ is a translation, in which case

$$\operatorname{Spc}^\tau \mathsf{Perf}\, E = \bigsqcup_{r\in\mathbb{Z}_{>0}} M_{r,0} \sqcup \bigsqcup_{(r,d)\in I, d\neq 0} \{\eta_{r,d}\}.$$

  since the only homogeneous stable bundles are ones of degree 0 by Atiyah's classification. Here, it is worth noting that the fixed loci for $T_{\mathscr{O}_E}$ and translations are the same while their images under $\theta$ are different.

- Consider $\tau = \tau_{\mathscr{L}}(:= - \otimes_{\mathscr{O}_E} \mathscr{L})$ for $\mathscr{L} \in \operatorname{Pic}^0 E$. By definition we have

$$\operatorname{Spc}^\tau \mathsf{Perf}\, E = M_{0,1} \sqcup \bigsqcup_{(r,d)\in I} \{ S_{r,d}(\mathscr{F}) \mid \mathscr{F} \cong \mathscr{F} \otimes \mathscr{L} \} \cup \{\eta_{r,d}\}.$$

  By [Ati57, Corollary of Theorem 7], we see that:

  - if $\mathscr{L}^{\otimes r} \cong \mathscr{O}_E$, then $\bigsqcup_{d\in\mathbb{Z}} M_{r,d} \subset \operatorname{Spc}^\tau \mathsf{Perf}\, E$ and
  - if $\mathscr{L}^{\otimes r} \ncong \mathscr{O}_E$ for all $r > 0$, then

$$\operatorname{Spc}^\tau \mathsf{Perf}\, E = M_{1,0} \sqcup \bigsqcup_{(r,d)\in I} \{\eta_{r,d}\}$$

  In particular, a line bundle is $\otimes$-generating if and only if the corresponding $\tau$-fixed locus agrees with the tt-spectrum. On the other hand, the second example is quite close to reconstruct the tt-spectrum in the sense that the $k$-points of its $\tau$-fixed locus agree with those of the tt-spectrum. In particular, the tt-spectrum is realized as a unique maximal dimensional connected component of $\operatorname{Spc}^\tau \mathsf{Perf}\, E$.

So far, we have observed the fixed loci for the generators of $\operatorname{Auteq} \mathsf{Perf}\, E$, so for fun, let us consider some explicit examples:

- Set

$$\tau = \left(T_{\mathscr{O}_E} T_{k(x)}\right)^3 = i^*[1],$$

  where $i$ denotes the inverse map $E \to E$ with respect to the identity element $e$ (cf. e.g., [ST01]). Clearly, we have

$$M_{0,1} \cap \operatorname{Spc}^\tau \mathsf{Perf}\, E = E_2,$$

  where $E_2$ denotes the set of 2-torsion points of $E$ (with respect to the fixed point $e$). To understand the action of $i^*$ on the moduli spaces, let us recall we have an isomorphism

$$\det : M_{r,d} \xrightarrow{\sim} M_{1,d} = \operatorname{Pic}^d E$$

so that $i^* \det \mathscr{F} \cong \det i^*\mathscr{F}$ for any $\mathscr{F} \in M_{r,d}(k)$. Now, for a line bundle $\mathscr{L} = \mathscr{O}_E((d-1)e+p)$ of degree $d$, note that $i^*\mathscr{L} \cong \mathscr{L}$ if and only if $p \in E_2$. Therefore, setting

$$\mathscr{F}_{r,d}(p) := \det{}^{-1}(\mathscr{O}_E((d-1)e+p)),$$

we see that

$$\mathrm{Spc}^\tau \,\mathsf{Perf}\, E = E_2 \sqcup \bigsqcup_{(r,d)\in I} \bigsqcup_{p \in E_2} \{\mathscr{F}_{r,d}(p)\},$$

where the generic points of all the connected components are included.

- Set

$$\tau = \Phi_{\mathscr{U}_{1,0}} \circ f_{1,0_*},$$

where $f_{1,0} : E \to M_{1,0}(= \mathrm{Pic}^0(E))$ is an isomorphism given by the fixed point $e$. It is easy to check $\theta(\tau) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and hence we see

$$\mathrm{Spc}^\tau \,\mathsf{Perf}\, E = \emptyset.$$

## Variants of polarizations

Now, let us introduce the following natural generalizations of pt-categories, which will be used to deal with reconstructions of not necessarily quasi-projective schemes.

**Definition 4.2.13.** Let $\mathcal{T}$ be a triangulated category.

(i) For an endofunctor $\tau \in \mathrm{End}\,\mathcal{T}$, a pair $(\mathcal{T}, \tau)$ is said to be a **semi-pt-category**. In this context, an endofunctor $\tau$ may be interchangeably referred to as a **semi-polarization**. We define the corresponding fixed locus to be

$$\mathrm{Spec}^\tau \,\mathcal{T} := (\mathrm{Spc}^\tau \,\mathcal{T}, \mathscr{O}_{\mathrm{Spc}^\tau \mathcal{T}, \Delta}).$$

(ii) For a collection $\tau = \{\tau_i\}_{i\in I}$ of polarizations on $\mathcal{T}$, a pair $(\mathcal{T}, \tau)$ is said to be a **multi-pt-category** and the collection $\tau$ is said to be a **multi-polarization**. We define the corresponding fixed locus to be

$$\mathrm{Spec}^\tau \,\mathcal{T} := (\mathrm{Spc}^\tau \,\mathcal{T}, \mathscr{O}_{\mathrm{Spc}^\tau \mathcal{T}, \Delta})$$

where we set

$$\mathrm{Spc}^\tau \,\mathcal{T} := \bigcap_{i \in I} \mathrm{Spc}^{\tau_i} \,\mathcal{T}.$$

We define the corresponding notions of the pt-spectrum for each. See §5.4 for more discussions. In particular, the notion of semi- (resp. multi-) polarizations correspond to “local systems” on $B\mathbb{Z}_{\geq 0}$ (resp. $B\mathbb{Z}^{*n}$ for the free group $\mathbb{Z}^{*n}$ in $n$ elements), whereas usual polarizations correspond to “local systems” on $B\mathbb{Z} \simeq S^1$.

To begin with, we have the following generalization of our guiding example (Proposition 4.2.1) for weakly pt-categories.

**Lemma 4.2.14.** *Let $X$ be a noetherian scheme and $\mathscr{E}$ a split generator of* $\mathsf{Perf}\ X$. *Write* $\otimes := \otimes_X$ *and set* $\tau_{\mathscr{E}} := - \otimes \mathscr{E} \in \operatorname{End} \mathsf{Perf}\ X$. *Then, we have*

$$|X| \cong \operatorname{Spc}_{\otimes} \mathsf{Perf}\ X = \operatorname{Spc}^{\tau_{\mathscr{E}}} \mathsf{Perf}\ X \subset \operatorname{Spc}_{\vartriangle} \mathsf{Perf}\ X.$$

*In particular, we have* $X_{\mathsf{red}} \cong \operatorname{Spec}^{\tau_{\mathscr{E}}} \mathsf{Perf}\ X$.

*Proof.* First, take $\mathcal{P} \in \operatorname{Spc}^{\tau_{\mathscr{E}}} \mathsf{Perf}\ X \subset \operatorname{Spc}_{\vartriangle} \mathsf{Perf}\ X$. Then, since we have

$$(\tau_{\mathscr{E}})^{-1}(\mathcal{P}) = \{\mathscr{F} \in \mathsf{Perf}\ X \mid \mathscr{F} \otimes \mathscr{E} \in \mathcal{P}\},$$

we have

$$\mathcal{P} = \{\mathscr{F} \in \mathsf{Perf}\ X \mid \mathscr{F} \otimes \mathscr{E} \in \mathcal{P}\}$$

and hence for any $\mathscr{F} \in \mathcal{P}$, we have $\mathscr{F} \otimes \mathscr{E} \in \mathcal{P}$, i.e., $\mathcal{P}$ is a $\otimes$-ideal by Lemma 4.1.6 as $\mathscr{E}$ is a split generator. Since $\mathcal{P}$ is moreover a prime thick subcategory, we have $\mathcal{P} \in \operatorname{Spc}_{\otimes} \mathsf{Perf}\ X$ by Theorem 2.1.11.

Conversely, take $\mathcal{P} \in \operatorname{Spc}_{\otimes} \mathsf{Perf}\ X$. Since it is in particular a $\otimes$-ideal, we have

$$\mathcal{P} \subset \{\mathscr{F} \in \mathsf{Perf}\ X \mid \mathscr{F} \otimes \mathscr{E} \in \mathcal{P}\}.$$

On the other hand, since $\mathcal{P}$ is a prime $\otimes$-ideal and it cannot contain a split generator (so $\mathscr{F} \otimes \mathscr{E} \in \mathcal{P}$ implies $\mathscr{F} \in \mathcal{P}$), we have

$$\mathcal{P} \supset \{\mathscr{F} \in \mathsf{Perf}\ X \mid \mathscr{F} \otimes \mathscr{E} \in \mathcal{P}\}.$$

Therefore,

$$\mathcal{P} = \{\mathscr{F} \in \mathsf{Perf}\ X \mid \mathscr{F} \otimes \mathscr{E} \in \mathcal{P}\} = (\tau_{\mathscr{E}})^{-1}(\mathcal{P}),$$

i.e., $\mathcal{P} \in \operatorname{Spc}^{\tau_{\mathscr{E}}} \mathsf{Perf}\ X$. The latter claim follows from Theorem 2.2.1 (iii). □

By the same proof as above (and Proposition 4.2.10), we can generalize Proposition 4.2.10 to the following claim:

**Proposition 4.2.15.** *Let* $(\mathcal{T}, \otimes)$ *be a rigid tt-category with a split generator* $\mathscr{E}$ *of* $\mathcal{T}$. *Then, we have*

$$\operatorname{Spc}_{\otimes} \mathcal{T} = \operatorname{Spc}_{\vartriangle}(\mathcal{T}, \tau_{\mathscr{E}}) \subset \operatorname{Th} \mathcal{T}.$$

*If* $\otimes$ *is moreover* $M$*-compatible, then*

$$\operatorname{Spc}_{\otimes} \mathcal{T} = \operatorname{Spc}_{\vartriangle}(\mathcal{T}, \tau_{\mathscr{E}}) = \operatorname{Spc}^{\tau_{\mathscr{E}}} \mathcal{T} \subset \operatorname{Spc}_{\vartriangle} \mathcal{T}.$$

Now, let us use multi-polarizations to give another generalization of Proposition 4.2.1. First, let us recall the following notion:

**Definition 4.2.16.** Let $X$ be a quasi-compact quasi-separated scheme and $\{\mathscr{L}_i\}_{i\in I}$ a collection of line bundles on $X$. Write $\otimes := \otimes_X$. We say $\{\mathscr{L}_i\}_{i\in I}$ is an **ample family** if the following equivalent conditions hold (cf. [BGI71, 6.II.2.2.3]):

(i) For any quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$ of finite type, there exist integers $n_i, k_i > 0$ such that $\mathscr{F}$ is a quotient of $\oplus_{i\in I}(\mathscr{L}_i^{\otimes -n_i})^{\oplus k_i}$;

(ii) There is a family of sections $f \in \Gamma(X, \mathscr{L}_i^{\otimes n})$ such that $X_f$ forms an affine open cover of $X$, where $X_f$ denotes the complement of the vanishing locus of $f$.

In condition (i), we can always take such a direct sum to be finite. If a quasi-compact and quasi-separated scheme $X$ admits an ample family, then $X$ is said to be **divisorial**.

**Proposition 4.2.17.** *Let $X$ be a divisorial scheme and let $\{\mathscr{L}_i\}_{i\in I}$ be an ample family. Then, we have*

$$\langle \mathscr{L}^{\otimes d} \mid d \in \mathbb{Z}^{\oplus I}\rangle = \mathsf{Perf}\, X.$$

*where we are using multi-grading notations $d = (d_i)_{i\in I} \in \mathbb{Z}^{\oplus I}$ and $\mathscr{L}^{\otimes d} = \bigotimes_{i\in I} \mathscr{L}^{\otimes d_i}$.*

*Proof.* The proof is essentially the same as [Orl09, Theorem 4], but let us provide a proof for readers' convenience. First, we take a locally free sheaf $\mathscr{F}$ of finite rank on $X$ and we want to show $\mathscr{F} \in \langle \mathscr{L}^{\otimes d} \mid d \in I\rangle$. By the definition of an ample family, there exists a bounded above complex $\mathscr{P}$ each of whose terms are finite direct sums of line bundles $\mathscr{L}_i^{\otimes k}$ together with a quasi-isomorphism $\mathscr{P} \to \mathscr{F}$. Consider the brutal truncation $\sigma^{\geq -(m+1)}\mathscr{P}$ for $m = \dim X$. Since the cone of the map $\sigma^{\geq -(m+1)}\mathscr{P} \to \mathscr{F}$ is isomorphic to $\mathscr{G}[m+1]$ for some locally free sheaf $\mathscr{G}$ and we have $\mathrm{Hom}(\mathscr{F}, \mathscr{G}[m+1]) = \mathrm{H}^{m+1}(X, \mathscr{F}^\vee \otimes_{\mathscr{O}_X} \mathscr{G}) = 0$, we see that $\mathscr{F}$ is a direct summand of $\sigma^{\geq -(m+1)}\mathscr{P}$, i.e., $\mathscr{F} \in \langle \mathscr{L}^{\otimes d} \mid d \in I\rangle$. Now, since any perfect complex over a divisorial scheme is quasi-isomorphic to a bounded complex of vector bundles (e.g., cf. [TT90, 2.3.1.d]), we see $\langle \mathscr{L}^{\otimes d} \mid d \in I\rangle = \mathsf{Perf}\, X$. □

As a direct corollary, we obtain the following:

**Theorem 4.2.18.** *Let $X$ be a noetherian divisorial scheme and take an ample family $\{\mathscr{L}_i\}_{i\in I}$. Set $\otimes := \otimes_X$ and take the corresponding multi-polarization $\tau := \{\tau_{\mathscr{L}_i}\}_{i\in I}$. Then, we have*

$$\mathrm{Spc}_\otimes \mathsf{Perf}\, X = \mathrm{Spc}^\tau \mathsf{Perf}\, X \subset \mathrm{Spc}_\vartriangle \mathsf{Perf}\, X.$$

*In particular, we have $X_{\mathsf{red}} \cong \mathrm{Spec}^\tau \mathcal{T}$.*

*Proof.* The containment $\mathrm{Spc}_\otimes \mathsf{Perf}\, X \subset \mathrm{Spc}^\tau \mathsf{Perf}\, X$ is clear. Conversely, if $\mathcal{P} \in \mathrm{Spc}^\tau \mathsf{Perf}\, X$, then for any $d \in \mathbb{Z}^{\oplus I}$, we have $\mathscr{L}^{\otimes d} \otimes \mathcal{P} = \mathcal{P}$. Since $\langle \mathscr{L}^{\otimes d} \mid d \in \mathbb{Z}^{\oplus I}\rangle = \mathsf{Perf}\, X$ by Proposition 4.2.17, we see that $\mathcal{P}$ is a $\otimes$-ideal and hence $\mathcal{P}$ is a prime $\otimes$-ideal by Theorem 2.1.11 (ii). The latter claim follows from Theorem 2.2.1 (iii). □

*Remark* 4.2.19. Similarly, we can define and consider the notion of **$\otimes$-generating family** of line bundles (see [IO25, §6]). Clearly, Theorem 4.2.18 extends to such family.

Finally, let us mention that we may talk about an open subvariety of the original variety as a spectrum of a multi-pt-category.

**Example 4.2.20.** Let $S$ be a smooth projective surface over an algebraically closed field with a $(-2)$-curve $C \subset S$. Then recall that $\mathcal{O}_C$ is a spherical object in $\mathsf{Perf}\, S$ and there is the associated spherical twist $T_{\mathcal{O}_S}$. Note in [Ito23, Corollary 5.14], it is essentially observed that

$$\mathrm{Spec}_{\otimes_S} \mathsf{Perf}\, S \cap \mathrm{Spec}^{T_{\mathcal{O}_C}} \mathsf{Perf}\, S \cong S \setminus C,$$

so if we take a $\otimes$-generating line bundle $\mathscr{L}$ on $S$ and set $\tau = \{T_{\mathcal{O}_C}, \tau_{\mathscr{L}}\}$, we have

$$\mathrm{Spec}^{\tau} \mathsf{Perf}\, S \cong S \setminus C.$$

In particular, classifications of multi-polarizations up to their spectra must involve classifications of $(-2)$-curves.

## 4.3 Tensor triangular spectra in the triangular spectrum

In this section, we study ($M$-compatible) tt-structures up to geometry in the triangular spectrum following [Ito23, Ito25], and we explain the relation to the work of Ito–Nolan [IN25].

**Definition 4.3.1.** Let $\mathcal{T}$ be a triangulated category.

(i) We say tt-structures $\otimes$ and $\otimes'$ are **spectrally identical** if

$$\mathrm{Spc}_{\otimes} \mathcal{T} = \mathrm{Spc}_{\otimes'} \mathcal{T} \subset \mathrm{Th}\, \mathcal{T}.$$

Recall that we say a tt-structure $\otimes$ on $\mathcal{T}$ is said to be **$M$-compatible** if

$$\mathrm{Spc}_{\otimes} \mathcal{T} \subset \mathrm{Spc}_{\vartriangle} \mathcal{T}.$$

(ii) We say (semi-, multi-)polarizations $\tau, \tau'$ are **spectrally identical** if

$$\mathrm{Spc}_{\vartriangle}(\mathcal{T}, \tau) = \mathrm{Spc}_{\vartriangle}(\mathcal{T}, \tau') \subset \mathrm{Th}\, \mathcal{T}.$$

Recall that we say a polarization $\tau$ on $\mathcal{T}$ is said to be **$M$-compatible** if

$$\mathrm{Spc}_{\vartriangle}(\mathcal{T}, \tau) \subset \mathrm{Spc}_{\vartriangle} \mathcal{T},$$

which implies $\mathrm{Spc}^{\tau} \mathcal{T} = \mathrm{Spc}_{\vartriangle}(\mathcal{T}, \tau)$ by Proposition 4.2.8.

The classifications of ($M$-compatible) rigid tt-structures and polarizations up to spectral identity are parts of the classification of ($M$-compatible) semi-polarizations up to spectral identity in the following sense. Let

$$\mathfrak{M}_{\mathcal{T}}^{\mathsf{tt,rig}}, \mathfrak{M}_{\mathcal{T}}^{\mathsf{tt,rig,M}}, \mathfrak{M}_{\mathcal{T}}^{\mathsf{pt}}, \mathfrak{M}_{\mathcal{T}}^{\mathsf{pt,M}}, \mathfrak{M}_{\mathcal{T}}^{\mathsf{spt}}, \mathfrak{M}_{\mathcal{T}}^{\mathsf{spt,M}}$$

denote the sets of the corresponding spectrally identical classes. We clearly have

$$\mathfrak{M}_{\mathcal{T}}^{\mathsf{pt}} \subset \mathfrak{M}_{\mathcal{T}}^{\mathsf{spt}}.$$

**Proposition 4.3.2.** *Let $\mathcal{T}$ be a triangulated category with a split-generator $\mathscr{E}$. Then, the map*

$$\mathfrak{M}_{\mathcal{T}}^{\mathsf{tt,rig}} \to \mathfrak{M}_{\mathcal{T}}^{\mathsf{spt}}, \quad \otimes \mapsto - \otimes \mathscr{E}$$

*is well-defined and injective, and does not depend on the choice of $\mathscr{E}$. The same claims follow in the $M$-compatible case. In particular, we can canonically view*

$$\mathfrak{M}_{\mathcal{T}}^{\mathsf{tt,rig}} \subset \mathfrak{M}_{\mathcal{T}}^{\mathsf{spt}}, \quad \mathfrak{M}_{\mathcal{T}}^{\mathsf{tt,rig,M}} \subset \mathfrak{M}_{\mathcal{T}}^{\mathsf{spt,M}}$$

*Proof.* All of the first claims are direct consequences of Proposition 4.2.15. The $M$-compatible case follows by Proposition 4.2.8. □

**Example 4.3.3.** Let $X$ be a noetherian scheme with a $\otimes$-generating line bundle (e.g., a quasi-projective scheme). Then, by definition,

$$\otimes_X \in \mathfrak{M}_{\mathsf{Perf}\,X}^{\mathsf{tt,rig,M}} \cap \mathfrak{M}_{\mathsf{Perf}\,X}^{\mathsf{pt,M}} \subset \mathfrak{M}_{\mathsf{Perf}\,X}^{\mathsf{spt,M}}$$

Therefore, we can categorically reconstruct $\otimes_X$ if we can specify $\otimes_X$ in the locus $\mathfrak{M}_{\mathsf{Perf}\,X}^{\mathsf{tt,rig,M}} \cap \mathfrak{M}_{\mathsf{Perf}\,X}^{\mathsf{pt,M}}$.

To specify $\otimes_X$ in the locus $\mathfrak{M}_{\mathsf{Perf}\,X}^{\mathsf{tt,rig,M}} \cap \mathfrak{M}_{\mathsf{Perf}\,X}^{\mathsf{pt,M}}$, let us note that we may equip $\mathfrak{M}_{\mathcal{T}}^{\mathsf{pt}}$ with natural topology as follows when $\mathcal{T}$ has a suitable enhancement with nice properties.

**Notation 4.3.4.** In the rest of discussions on $\mathfrak{M}_{\mathsf{Perf}\,X}$'s, we assume our ground field $k$ is **algebraically closed of characteristic zero**.

**Construction 4.3.5** (Topology on $\mathfrak{M}_{\mathcal{T}}^{\mathsf{pt}}$)**.** In this construction, we follow [HP24] for the terminology on $\infty$-categories. See also §5.4 for more expositions. Suppose that a triangulated category $\mathcal{T}$ is the homotopy category of a $k$-linear smooth proper idempotent-complete stable $\infty$-category $\mathcal{C}$ that is connected (in the sense of [HP24, Definition 5.18]). Let $\mathrm{Aut}_T(\mathcal{C}_T)$ denote the $\infty$-groupoid of $T$-linear $\infty$-autoequivalences of $\mathcal{C}_T$ for a $k$-scheme $T \to \operatorname{Spec} k$ (cf. [HP24, §4.1]).

By [HP24, Proposition 8.2] (see also [TV07, Corollary 3.24]), $\mathrm{Aut}_T(\mathcal{C}_T)$ is an ordinary groupoid under any base change to a $k$-scheme $T \to \operatorname{Spec} k$ and moreover the sheafification of the presheaf

$$(\mathsf{Sch}_k)^{\mathsf{op}} \to \mathsf{Set}, \quad T \mapsto \pi_0(\mathrm{Aut}_T(\mathcal{C}_T))$$

is a group algebraic space $\mathrm{Aut}(\mathcal{C}/k)$ such that $\mathrm{Aut}(\mathcal{C}/k)(k) = \pi_0(\mathrm{Aut}_k(\mathcal{C}))$ (cf. [Rom05, Remark 5.2 (i)] with $H = \mathbb{G}_m$). Therefore, $\pi_0(\mathrm{Aut}_k(\mathcal{C}))$ has the subspace topology from $|\mathrm{Aut}(\mathcal{C}/k)|$.

Now, suppose that any autoequivalence of $\mathcal{T} = \mathsf{Ho}(\mathcal{C})$ has an enhancement to an autoequivalence of $\mathcal{C}$ uniquely up to natural isomorphism, i.e., that

$$\operatorname{Auteq} \mathcal{T} = \pi_0(\mathrm{Aut}_k(\mathcal{C})).$$

Then, since $\mathfrak{M}_{\mathcal{T}}^{\mathsf{pt}}$ is a quotient set of $\operatorname{Auteq} \mathcal{T}$, $\mathfrak{M}_{\mathcal{T}}^{\mathsf{pt}}$ has the natural quotient topology. We also put the subspace topology on $\mathfrak{M}_{\mathcal{T}}^{\mathsf{pt,M}} \subset \mathfrak{M}_{\mathcal{T}}^{\mathsf{pt}}$.

*Remark* 4.3.6. By considering the higher locally algebraic moduli stack $\mathcal{M}(\mathrm{Fun}_k(\mathcal{C},\mathcal{C})/k)$ of endofunctors (see [HP24, §7.3] for notations), we may also expect natural topology on $\mathfrak{M}^{\mathsf{spt}}_{\mathcal{T}}$. Since the natural morphism $\mathcal{A}ut(\mathcal{C}/k) \to \mathcal{M}(\mathrm{Fun}_k(\mathcal{C},\mathcal{C})/k)$ is a Zariski open immersion ([HP24, Proposition 8.2]), we expect that $\mathfrak{M}^{\mathsf{pt}}_{\mathcal{T}} \subset \mathfrak{M}^{\mathsf{spt}}_{\mathcal{T}}$ is an open embedding with respect the topology.

**Example 4.3.7.** Let $X$ be a smooth proper variety. Then, $\mathsf{Perf}\ X$ is the homotopy category of the $k$-linear smooth proper connected idempotent-complete stable $\infty$-category $\mathsf{Perf}^{\infty}\,\mathcal{T}$ in a natural way (cf. [BZFN10, HP24]). Moreover, since any autoequivalence of $\mathsf{Perf}\ X$ is a Fourier–Mukai transform by [Ola24, Theorem 1], we see that

$$\mathrm{Auteq}\,\mathsf{Perf}\ X = \pi_0(\mathrm{Aut}_k(\mathsf{Perf}^{\infty}\ X))$$

by [BZFN10, Theorem 1.2]. Therefore, $\mathfrak{M}^{\mathsf{pt}}_{\mathsf{Perf}\ X}$ has natural topology.

*Remark* 4.3.8. In the enhanced set-up of polarizations (§5.4), it may be more natural to construct $\mathfrak{M}^{\mathsf{pt}}_{\mathcal{C}}$ as a certain geometric quotient/stratification of the (derived) stack $\mathrm{Aut}_k(\mathcal{C})$ (or its $\mathbb{G}_m$-gerbe $\mathcal{A}ut_k(\mathcal{C})$). We postpone such considerations to future works, but for $\mathcal{C} = \mathsf{Perf}\ \mathbb{P}^1$, we expect it to be the stack $[\mathbb{A}^3/\mathbb{G}_m] \sqcup [\mathbb{A}^3/\mathbb{G}_m]$ (cf. the proof of Theorem 4.3.10).

**Proposition 4.3.9.** *Let $X$ be a smooth proper variety. Then, the natural homomorphism*

$$\rho : \mathrm{Aut}_{X/k} \ltimes \mathrm{Pic}_{X/k} \times \underline{\mathbb{Z}} \to \mathrm{Aut}(\mathsf{Perf}\ X/k)$$

*of group algebraic spaces is an open immersion, where* $\mathrm{Aut}_{X/k}$ *denotes the automorphism group scheme of $X$,* $\mathrm{Pic}_{X/k}$ *the Picard scheme of $X$, and* $\underline{\mathbb{Z}}$ *the constant group scheme (over $k$). If we moreover have*

$$\mathrm{Auteq}\,\mathsf{Perf}\ X = \mathrm{Auteq}^{\mathrm{st}}\,\mathsf{Perf}\ X,$$

*then the open immersion $\rho$ is an isomorphism.*

*Proof.* This should be well-known to experts and essentially follows from [HP24, Lemma 8.14] (see also [Ros09, Proposition 2.10] and [Ols25, Example 1.2, Theorem 1.7, Proposition 5.5]), but let us include arguments for readers' convenience. First of all, consider the natural homomorphism

$$\rho : \mathrm{Aut}_{X/k} \ltimes \mathrm{Pic}_{X/k} \times \underline{\mathbb{Z}} \to \mathrm{Aut}(\mathsf{Perf}\ X/k)$$

of group algebraic spaces given by the actions of $\mathrm{Aut}_{X/k}$ and $\mathrm{Pic}_{X/k}$ on $\mathsf{Perf}(X)_{(-)} \simeq \mathsf{Perf}(X_{(-)})$ by $\rho(f,\mathscr{L}) = f_*(-) \otimes \mathscr{L}$. More precisely, we first define the natural morphism $\mathrm{Aut}_{X/k} \ltimes \mathcal{P}ic_{X/k} \times \underline{\mathbb{Z}} \to \mathcal{A}ut(\mathsf{Perf}\ X/k)$ on $\mathbb{G}_m$-gerbes defined by the formula above and then rigidify it to get $\rho$ (cf. [Sta21, Tag 06QD]). In particular, $\rho$ is a monomorphism, as it is clear on the presheaf level by the uniqueness of Fourier–Mukai kernels and the sheafification preserves monomorphisms. Since we can directly see the identity component of $\mathrm{Aut}_{X/k} \ltimes \mathrm{Pic}_{X/k} \times \underline{\mathbb{Z}}$ is $\mathrm{Aut}^0_{X/k} \ltimes \mathrm{Pic}^0_{X/k}$, which is smooth as it is a group scheme of finite type over a field of characteristic zero (cf. [Sta21, Tag 047N]), and the identity component of $\mathrm{Aut}^0(\mathsf{Perf}\ X/k)$ of $\mathrm{Aut}(\mathsf{Perf}\ X/k)$ exists and is smooth by [HP24, Lemma 8.10], the homomorphism $\rho$ restricts to the homomorphism

$$\rho^0 : \mathrm{Aut}^0_{X/k} \ltimes \mathrm{Pic}^0_{X/k} \to \mathrm{Aut}^0(\mathsf{Perf}\ X/k).$$

Now, we show $\rho^0$ is an isomorphism. Let $e \in \mathrm{Aut}^0_{X/k}(k) \ltimes \mathrm{Pic}^0_{X/k}(k)$ and $e' \in \mathrm{Aut}^0(\mathsf{Perf}\ X/k)(k)$ be the identity elements. By the proof of [HP24, Lemma 8.10], we have

$$\mathrm{T}_{e'}\,\mathrm{Aut}(\mathsf{Perf}\ X/k) = \mathrm{HH}^1(\mathsf{Perf}\ X/k) = \mathrm{Ext}^1_{X\times X}(\mathcal{O}_\Delta, \mathcal{O}_\Delta)$$

and as in the proof of [HP24, Lemma 8.14], the Lie algebra map

$$d\rho^0_e : \mathrm{T}_e(\mathrm{Aut}^0_{X/k} \ltimes \mathrm{Pic}^0_{X/k}) = \mathrm{H}^0(X, \mathrm{T}_{X,k}) \oplus \mathrm{H}^1(X, \mathcal{O}_X) \to \mathrm{T}_{e'}\,\mathrm{Aut}(\mathsf{Perf}\ X/k)$$

agrees with the Hochschild–Kostant–Rosenberg isomorphism. Therefore, the homomorphism $\rho^0$ is étale (at $e$ and hence on $\mathrm{Aut}^0_{X/k} \ltimes \mathrm{Pic}^0_{X/k}$) and since the image of $\rho^0$ is an open subgroup of the connected group algebraic space $\mathrm{Aut}^0(\mathsf{Perf}\ X/k)$, it needs to agree with $\mathrm{Aut}^0(\mathsf{Perf}\ X/k)$ and hence $\rho^0$ is surjective. Since $\rho$ and hence $\rho^0$ is a monomorphism, $\rho^0$ is an open immersion ([Sta21, Tag 05W5]) and hence an isomorphism. Similarly, since $\rho^0$ is étale, $\rho$ is also an étale monomorphism and hence an open immersion.

Now, if we moreover have

$$\mathrm{Aut}(\mathsf{Perf}\ X/k)(k) = \mathrm{Auteq}\,\mathsf{Perf}\ X = \mathrm{Auteq}^{\mathrm{st}}\,\mathsf{Perf}\ X = (\mathrm{Aut}_{X/k} \ltimes \mathrm{Pic}_{X/k} \times \underline{\mathbb{Z}})(k),$$

then the open immersion $\rho$ is surjective on $k$-points. Since any algebraic space of finite type over an algebraically closed field $k$ has a $k$-point, $\rho$ is surjective by considering the complement of the (set-theoretic open) image of $\rho$ and hence $\rho$ is an isomorphism. $\square$

To be more concrete, let us consider the case $X = \mathbb{P}^1$. The following results can be understood as a variant of the monoidal Bondal–Orlov reconstruction by Toledo [Tol24, Corollary 3.18]. Note that our approach does not need to specify the $\otimes$-unit.

**Theorem 4.3.10** (Monoidal Bondal–Orlov reconstruction for $\mathbb{P}^1$)**.** *A rigid tt-structure $\otimes$ on $\mathsf{Perf}\ \mathbb{P}^1$ is spectrally identical to $\otimes_{\mathbb{P}^1}$ if and only if $\otimes$ defines a closed point in $\mathfrak{M}^{\mathrm{pt}}_{\mathsf{Perf}\ \mathbb{P}^1}$.*

*Proof.* By the computations in Example 4.2.11, we see that

$$\mathfrak{M}^{\mathrm{pt}}_{\mathsf{Perf}\ \mathbb{P}^1} \cong [\mathbb{A}^3/\mathbb{G}_m](k) \sqcup [\mathbb{A}^3/\mathbb{G}_m](k)$$

as sets (noting $[\mathbb{A}^3/\mathbb{G}_m](k) = \mathbb{P}^2(k) \sqcup B\mathbb{G}_m(k)$ as sets), where $\mathbb{G}_m$ acts on $\mathbb{A}^3$ by the standard scaler action and the origins (i.e., $B\mathbb{G}_m(k)$'s) correspond to (the classes of) $\mathrm{id}_{\mathsf{Perf}\ \mathbb{P}^1}$ and $\tau_{\mathcal{O}_{\mathbb{P}^1}(1)}$. Now, we claim that the bijection is indeed a homeomorphism, where we put the subspace topology $[\mathbb{A}^3/\mathbb{G}_m](k) \subset |[\mathbb{A}^3/\mathbb{G}_m]|$. By Proposition 4.3.9, we have

$$\mathrm{Aut}(\mathsf{Perf}\ \mathbb{P}^1/k) \cong \mathrm{PGL}_2 \times \underline{\mathbb{Z}} \times \underline{\mathbb{Z}}.$$

First, consider the morphism

$$f : \mathrm{GL}_2 \to \mathbb{A}^3, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto (c, d-a, -b).$$

Then, $f$ is clearly $\mathbb{G}_m$-equivariant for the standard scaler actions of $\mathbb{G}_m$ on $\mathrm{GL}_2$ and $\mathbb{A}^3$. Also, note that $f$ is surjective and flat (as every fiber is 1-dimensional and $\mathrm{GL}_2, \mathbb{A}^3$ are both regular). Since the quotient $\mathrm{GL}_2 \to \mathrm{PGL}_2$ is the canonical $\mathbb{G}_m$-torsor, the surjective flat morphism $f$ induces the surjective flat morphism

$$\tilde{f} : \mathrm{PGL}_2 \to [\mathbb{A}^3/\mathbb{G}_m],$$

as flatness can be checked after the surjective flat base change along $\mathbb{A}^3 \to [\mathbb{A}^3/\mathbb{G}_m]$ [Sta21, Tag 06PZ]) and surjectivity is clear. Since $\tilde{f}$ is also of finite type, we see that

$$|\tilde{f}| : |\mathrm{PGL}_2| \to |[\mathbb{A}^3/\mathbb{G}_m]|$$

is a surjective open continuous map by [Sta21, Tag 06R7], which then induces a surjective open continuous map on $k$-points (equipped with the subspace topology):

$$q : \mathrm{PGL}_2(k) \to [\mathbb{A}^3/\mathbb{G}_m](k); \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto \text{the orbit } k^\times \cdot (c, d-a, -b).$$

By construction, it is direct to see that $A, B \in \mathrm{PGL}_2(k)$ has the same configuration of fixed points in $\mathbb{P}^1$ if and only if $q(A) = q(B)$, noting that $[x : y] \in \mathbb{P}^1$ is fixed by $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{PGL}_2(k)$ if and only if $cx^2 + (d-a)xy - by^2 = 0$. Therefore, the quotient of $\mathrm{PGL}_2(k)$ by spectral identity is homeomorphic to $[\mathbb{A}^3/\mathbb{G}_m](k)$ by the universal property of quotient topology. Also note that $[\mathbb{A}^3/\mathbb{G}_m](k)$ has a unique closed point at the origin. Therefore, since $(\underline{\mathbb{Z}} \times \underline{\mathbb{Z}})(k)$ has discrete topology, the computations in Example 4.2.11 give the homeomorphism

$$\mathfrak{M}^{\mathsf{pt,M}}_{\mathsf{Perf}\,\mathbb{P}^1} \cong [\mathbb{A}^3/\mathbb{G}_m](k) \sqcup [\mathbb{A}^3/\mathbb{G}_m](k).$$

In particular, $\mathfrak{M}^{\mathsf{pt,M}}_{\mathsf{Perf}\,\mathbb{P}^1}$ has only two closed points, which correspond to $\mathrm{id}_{\mathsf{Perf}\,\mathbb{P}^1}$ and $\tau_{\mathcal{O}_{\mathbb{P}^1}(1)}$.

Now, we claim $\mathrm{id}_{\mathsf{Perf}\,\mathbb{P}^1} \notin \mathfrak{M}^{\mathsf{tt,rig,M}}_{\mathsf{Perf}\,X}$. Indeed, assume that there is $\otimes \in \mathfrak{M}^{\mathsf{tt,rig}}_{\mathsf{Perf}\,\mathbb{P}^1}$ such that $\otimes \simeq \mathrm{id}_{\mathsf{Perf}\,\mathbb{P}^1}$ in $\mathfrak{M}^{\mathsf{tt,rig}}_{\mathsf{Perf}\,\mathbb{P}^1} \cap \mathfrak{M}^{\mathsf{pt}}_{\mathsf{Perf}\,\mathbb{P}^1}$, then we have $\tau_{l,m} := - \otimes (\mathcal{O}_{\mathbb{P}^1}(l) \oplus \mathcal{O}_{\mathbb{P}^1}(m)) \simeq \mathrm{id}_{\mathsf{Perf}\,\mathbb{P}^1} \in \mathfrak{M}^{\mathsf{pt}}_{\mathsf{Perf}\,\mathbb{P}^1}$ for all $l, m \in \mathbb{Z}$ with $l \neq m$, as $\mathcal{O}_{\mathbb{P}^1}(l) \oplus \mathcal{O}_{\mathbb{P}^1}(m)$ is a split-generator when $l \neq m$. Now, since $\tau_{l,m}^{-1}(\langle \mathcal{O}_{\mathbb{P}^1}(n) \rangle) = \langle \mathcal{O}_{\mathbb{P}^1}(n) \rangle$ for all $n \in \mathbb{Z}$ by the assumption, we have

$$\langle \mathcal{O}_{\mathbb{P}^1}(n) \otimes \mathcal{O}_{\mathbb{P}^1}(l) \oplus \mathcal{O}_{\mathbb{P}^1}(n) \otimes \mathcal{O}_{\mathbb{P}^1}(m) \rangle = \langle \mathcal{O}_{\mathbb{P}^1}(n) \rangle$$

for all $l, m, n \in \mathbb{Z}$ with $l \neq m$. In particular, for any $n, m \in \mathbb{Z}$, we have

$$\mathcal{O}_{\mathbb{P}^1}(n) \otimes \mathcal{O}_{\mathbb{P}^1}(m) \in \langle \mathcal{O}_{\mathbb{P}^1}(n) \rangle \cap \langle \mathcal{O}_{\mathbb{P}^1}(m) \rangle.$$

Thus, if $n \neq m$, then we have

$$\mathcal{O}_{\mathbb{P}^1}(n) \otimes \mathcal{O}_{\mathbb{P}^1}(m) = 0.$$

This is absurd since then for 3 distinct integers $l, m, n \in \mathbb{Z}$, we have

$$\langle \mathcal{O}_{\mathbb{P}^1}(n) \otimes \mathcal{O}_{\mathbb{P}^1}(l) \oplus \mathcal{O}_{\mathbb{P}^1}(n) \otimes \mathcal{O}_{\mathbb{P}^1}(m) \rangle = 0 \subsetneq \langle \mathcal{O}_{\mathbb{P}^1}(n) \rangle.$$

Therefore, $\otimes_X = \tau_{\mathcal{O}_{\mathbb{P}^1}(1)}$ is a unique closed point in $\mathfrak{M}^{\mathsf{pt}}_{\mathsf{Perf}\,\mathbb{P}^1}$ that is also in the image of $\mathfrak{M}^{\mathsf{tt,rig}}_{\mathsf{Perf}\,\mathbb{P}^1}$. $\square$

**Question 4.3.11.** Here are natural questions arising from this observation.

(i) For a smooth proper connected idempotent-complete stable $\infty$-category $\mathcal{C}$, can we naturally make $\mathfrak{M}^{\mathsf{pt}}_{\mathcal{C}}$ into a stack so that $\mathfrak{M}^{\mathsf{pt}}_{\mathsf{Perf}\,\mathbb{P}^1} = [\mathbb{A}^3/\mathbb{G}_m] \sqcup [\mathbb{A}^3/\mathbb{G}_m]$? How about other sets such as $\mathfrak{M}^{\mathsf{spt}}_{\mathcal{C}}$, $\mathfrak{M}^{\mathsf{tt,rigid}}_{\mathcal{C}}$?

(ii) Can we prove the monoidal Bondal–Orlov reconstruction in the above sense more generally? If not, can we characterize tt-structures corresponding to closed points/stacky points (or more generally living in closed subvarieties/substacks)?

(iii) Take Fourier–Mukai transforms $\Phi_{\mathscr{P}}, \Phi_{\mathscr{P}'} : \mathsf{Perf}\, X \to \mathsf{Perf}\, X$. Can we give geometric criterion on $\mathscr{P}, \mathscr{P}' \in \mathsf{D}_{\mathsf{qc}}(X \times X)$ for $\Phi_{\mathscr{P}}, \Phi_{\mathscr{P}'}$ to define spectrally identical semi-polarizations?

(iv) If $X$ is a noetherian scheme with a $\otimes$-generating line bundle, does every noetherian scheme $Y$ with $\mathsf{Perf}\, Y \simeq \mathsf{Perf}\, X$ admits a $\otimes$-generating line bundle? In this case, the locus $\mathfrak{M}^{\mathsf{tt,rig,M}}_{\mathsf{Perf}\,X} \cap \mathfrak{M}^{\mathsf{pt,M}}_{\mathsf{Perf}\,X}$ contains $\otimes_Y$ for every $Y$ with $\mathsf{Perf}\, Y \simeq \mathsf{Perf}\, X$.

Next, let us consider several properties of tt-structures and the corresponding subsets of the set $\mathfrak{M}^{\mathsf{tt,M}}_{\mathcal{T}}$ of spectral identity classes of $M$-compatible tt-structures on $\mathcal{T}$.

**Definition 4.3.12.** Let $\mathcal{T}$ be a triangulated category. An $M$-compatible tt-structure $\otimes$ on $\mathcal{T}$ is:

(i) **geometric** if $(\mathcal{T}, \otimes) \simeq (\mathsf{Perf}\, X, \otimes_X)$ for a smooth projective variety $X$;

(ii) **schematic** if $\operatorname{Spec}_{\otimes} \mathcal{T}$ is a scheme;

(iii) **Serre-compatible** if a Serre functor $\mathbb{S}$ exists and can be written as $\mathbb{S} \cong - \otimes \mathbb{S}(\mathbb{1})$.

Write the corresponding subsets in $\mathfrak{M}^{\mathsf{tt,M}}_{\mathcal{T}}$ by

$$\mathfrak{M}^{\mathsf{tt,geom,M}}_{\mathcal{T}}, \mathfrak{M}^{\mathsf{tt,sch,M}}_{\mathcal{T}}, \mathfrak{M}^{\mathsf{tt,Sc,M}}_{\mathcal{T}} \subset \mathfrak{M}^{\mathsf{tt,M}}_{\mathcal{T}}.$$

Recall that we have

$$\operatorname{Spc}^{\mathsf{FM}} \mathcal{T} = \bigcup_{\otimes \in \mathfrak{M}^{\mathsf{tt,geom,M}}_{\mathcal{T}}} \operatorname{Spc}_{\otimes} \mathcal{T}.$$

Write the loci corresponding to $\mathfrak{M}^{\mathsf{tt,rig,M}}_{\mathcal{T}}$, $\mathfrak{M}^{\mathsf{tt,sch,M}}_{\mathcal{T}}$, $\mathfrak{M}^{\mathsf{tt,Sc,M}}_{\mathcal{T}}$ by

$$\operatorname{Spc}^{\mathsf{rig}} \mathcal{T},\ \operatorname{Spc}^{\mathsf{sch}} \mathcal{T},\ \operatorname{Spc}^{\mathsf{Sc}} \mathcal{T} \subset \operatorname{Spc}_{\vartriangle} \mathcal{T}.$$

**Example 4.3.13.** Let $X$ be a smooth projective variety.

(i) If $\omega_X$ is $\otimes$-generating, then $\mathfrak{M}^{\mathsf{tt,M,geom}}_{\mathsf{Perf}\,X} = \{\otimes_X\}$ by the Bondal–Orlov reconstruction (cf. Theorem 5.1.3).

(ii) If $X$ is an elliptic curve, then $\mathfrak{M}^{\mathsf{tt,M,geom}}_{\mathsf{Perf}\,X} = \{\otimes_X\} \sqcup \coprod_{(r,d)\in I} \{\otimes_{X,\tau_{r,d}}\}$, where $\otimes_{X,\tau_{r,d}}$ denotes a tt-structure given by transporting $\otimes_X$ along the autoequivalence $\tau_{r,d}$ (cf. Construction 4.1.11).

*Remark* 4.3.14. Note if $\otimes$ is Serre-geometric, then $\mathbb{S}(\mathbb{1})$ is necessarily $\otimes$-invertible with inverse $\mathbb{S}^{-1}(\mathbb{1})$.

By definition, we have the following inclusions.

**Corollary 4.3.15.** *Let $\mathcal{T}$ be a triangulated category. Then, we have*

$$\mathrm{Spc}^{\mathsf{FM}}\,\mathcal{T} \subset \mathrm{Spc}^{\mathsf{pFM}}\,\mathcal{T} \subset \mathrm{Spc}^{\mathsf{sch}}\,\mathcal{T} \subset \mathrm{Spc}_{\vartriangle}\,\mathcal{T}.$$

*In particular, if $\mathrm{Spc}^{\mathsf{FM}}\,\mathcal{T} = \mathrm{Spc}_{\vartriangle}\,\mathcal{T}$ (e.g. if an elliptic curve $X \in \mathrm{FM}\,\mathcal{T}$), then all of them agree.*

Let us provide several examples of schematic tt-structures to observe how these loci change.

**Example 4.3.16.** Let $\mathcal{T}$ be a triangulated category.

(i) We see the inclusion $\mathrm{Spc}^{\mathsf{FM}}\,\mathcal{T} \subset \mathrm{Spc}^{\mathsf{sch}}\,\mathcal{T}$ can be strict. Define a tt-structure on the (underived) category $\mathsf{Vect}_k$ of **finite dimensional** $k$-vector spaces as follows. First, we define a triangulated category structure on $\mathsf{Vect}_k$ by setting the shift functor to be the identity functor and by setting a triangle $X \to Y \to Z \to X$ to be distinguished if it is an exact sequence (or equivalently if it is a finite direct sum of (shifts of) the triangle $k \xrightarrow{\mathrm{id}} k \to 0 \to k$). Then, $(\mathsf{Vect}_k, \otimes_k)$ is a tt-category with the usual tensor product $\otimes_k$. Since the only additive proper subcategory of $\mathsf{Vect}_k$ is 0, the underlying topological space of $\mathrm{Spec}_{\otimes_k}\mathsf{Vect}_k$ is a singleton and since $\mathrm{End}_{\mathsf{Vect}_k}(k) = Z(\mathsf{Vect}_k)_{\mathsf{lrad}} = k$, it follows that

$$\mathrm{Spec}_{\vartriangle}\,\mathsf{Vect}_k = \mathrm{Spec}_{\otimes_k}\mathsf{Vect}_k = \mathrm{Spec}\,k$$

as ringed spaces. Therefore, the tt-structure $\otimes_k$ is schematic, while there is no geometric tt-structure on $\mathsf{Vect}_k$.

(ii) We can also cook up schematic tt-structures that are not geometric as follows. We follow the proof of [Sos08, Proposition 4.0.9], which is suggested here [AT023]. Let $X$ be a smooth projective variety (or more generally a connected noetherian scheme). Define a tt-structure $\boxtimes$ on $\mathsf{Perf}(X \sqcup X) = \mathsf{Perf}\,X \oplus \mathsf{Perf}\,X$ by setting

$$(A, B) \boxtimes (C, D) := (A \otimes_X C, A \otimes_X D \oplus C \otimes_X B),$$

where the unit is given by $(\mathcal{O}_X, 0)$. Also note that the projection

$$\Phi : \mathsf{Perf}\,X \oplus \mathsf{Perf}\,X \to \mathsf{Perf}\,X; \quad (A, B) \mapsto A$$

is a tt-functor and hence we have a morphism

$$\phi : \mathrm{Spec}_{\otimes_X}\mathsf{Perf}\,X \to \mathrm{Spec}_{\boxtimes}(\mathsf{Perf}\,X \oplus \mathsf{Perf}\,X); \quad \mathcal{P} \mapsto \mathcal{P} \oplus \mathsf{Perf}\,X.$$

of ringed spaces, which was shown to be a homeomorphism with inverse $\psi$ induced by the inclusion tt-functor

$$\Psi : \mathsf{Perf}\,X \to \mathsf{Perf}\,X \oplus \mathsf{Perf}\,X; \quad A \mapsto (A, 0).$$

Now, by construction (cf. Construction 2.1.8), for any open subset $U \subset \mathrm{Spec}_{\otimes_X} \mathsf{Perf}\, X$, $\Phi$ induces an equivalence

$$(\mathsf{Perf}\, X \oplus \mathsf{Perf}\, X)(U \oplus \mathsf{Perf}\, X) \simeq \mathsf{Perf}\, X(U)$$

sending $\mathbb{1}_{U \oplus \mathsf{Perf}\, X} = (\mathbb{1}_U, 0)$ to $\mathbb{1}_U$ and hence by the definition of structure sheaves on tt-spectra, $\Phi$ induces an isomorphism

$$X \cong \mathrm{Spec}_{\otimes_X} \mathsf{Perf}\, X \cong \mathrm{Spec}_{\boxtimes} \mathsf{Perf}(X \sqcup X)$$

as ringed spaces and hence $\boxtimes$ is a schematic tt-structure on $\mathsf{Perf}\, X \oplus \mathsf{Perf}\, X$. On the other hand, it is not geometric since $\mathsf{Perf}\, X \not\simeq \mathsf{Perf}(X \sqcup X)$ and indeed $\mathrm{FM}\, X \sqcup X = \emptyset$ as $\mathsf{Perf}\, X \sqcup X$ is not connected. Now, by construction we see that

$$\mathrm{Spec}_{\boxtimes} \mathsf{Perf}(X \sqcup X) \subset \mathrm{Spec}_{\otimes_{X \sqcup X}} \mathsf{Perf}(X \sqcup X)$$

is the inclusion of $X$ into the first component of $X \sqcup X$.

(iii) We see that the inclusion $\mathrm{Spc}^{\mathsf{FM}}\, \mathcal{T} \subset \mathrm{Spc}^{\mathsf{sch}}\, \mathcal{T}$ can be strict even when $\mathrm{FM}\, \mathcal{T} \neq \emptyset$. Let $X = \mathbb{P}^1$. First, it is well-known that $\mathsf{Perf}\, X = \mathsf{Perf}\, k\mathsf{Kr}$, where $k\mathsf{Kr}$ is the path algebra of the Kronecker quiver $\mathsf{Kr}$. Thus, we can consider the tt-category $(\mathsf{Perf}\, X, \otimes_{\mathsf{rep}})$. By Theorem 2.1.5 we have $\mathrm{Spec}_{\otimes_{\mathsf{rep}}} \mathsf{Perf}\, X = \mathrm{Spec}\, k \sqcup \mathrm{Spec}\, k$, where those two points correspond to prime thick subcategories $\langle \mathcal{O}_X \rangle$ and $\langle \mathcal{O}_X(1) \rangle$ and hence $\otimes_{\mathsf{rep}}$ is a schematic tt-structure and $\mathrm{Spc}^{\mathsf{Ser}}\, \mathsf{Perf}\, X \cap \mathrm{Spc}_{\otimes_{\mathsf{rep}}} \mathsf{Perf}\, X = \emptyset$ by Example 2.2.2. Thus, we have

$$\mathrm{Spc}^{\mathsf{Ser}}\, \mathsf{Perf}\, X = \mathrm{Spc}^{\mathsf{FM}}\, \mathsf{Perf}\, X \subsetneq \mathrm{Spc}^{\mathsf{sch}}\, \mathsf{Perf}\, X.$$

By transporting the tt-structure $\otimes_{\mathsf{rep}}$ under autoequivalences coming from $\mathrm{Pic}(X) = \mathbb{Z}$, we see that

$$\mathrm{Spc}^{\mathsf{sch}}\, \mathsf{Perf}\, X = \mathrm{Spc}_{\triangle}\, \mathsf{Perf}\, X.$$

In particular, $\mathrm{Spc}^{\mathsf{sch}}\, \mathsf{Perf}\, X \backslash \mathrm{Spc}^{\mathsf{FM}}\, \mathsf{Perf}\, X = \mathbb{Z}$. By similar procedures, we expect any variety $X$ with full exceptional collections has

$$\mathrm{Spc}^{\mathsf{FM}}\, \mathsf{Perf}\, X \subsetneq \mathrm{Spc}^{\mathsf{sch}}\, \mathsf{Perf}\, X$$

(cf. [SL13, (2.1.6)]). However, we do not know if $\mathrm{Spc}^{\mathsf{sch}}\, \mathsf{Perf}\, X = \mathrm{Spc}_{\triangle}\, \mathsf{Perf}\, X$ in general.

(iv) Let $X$ be a smooth projective variety with ample (anti-)canonical bundle and suppose $X \in \mathrm{FM}\, \mathcal{T}$. In this case, we can see schematic tt-structures behave similarly as geometric tt-structures in the following sense. Suppose that $\otimes$ is a schematic tt-structure on $\mathcal{T}$ with $\mathrm{Spec}_{\otimes}\, \mathcal{T} \cong X$ with unit $\Phi(\mathcal{O}_X)$ for some $\Phi : \mathsf{Perf}\, X \simeq \mathcal{T}$. Then, by [Tol24, Corollary 3.18], we have that $\Phi$ is monoidal on objects, i.e., for any $\mathcal{F}, \mathcal{G} \in \mathsf{Perf}\, X$, we have

$$\Phi(\mathcal{F} \otimes_X \mathcal{G}) = \Phi(\mathcal{F}) \otimes \Phi(\mathcal{G}).$$

In particular, $\Phi$ induces a bijection $\mathrm{Spc}_{\otimes} \mathcal{T} \cong \mathrm{Spc}_{\otimes_X} \mathsf{Perf}\, X$ of the underlying topological spaces. It is worth mentioning that in [Tol23], Toledo extensively considers deformation of tt-structures (up to tt-equivalences) using Davydov-Yetter cohomology, but note that in general spectral identity does not imply tt-equivalence and vice versa.

*Remark* 4.3.17. The author would like to thank Paul Balmer for sharing some thoughts on this topic, in particular, Example 4.3.16 (i).

Next, let us provide motivations to consider a Serre-compatible tt-structure:

**Proposition 4.3.18.** *Let $\mathcal{T}$ be a triangulated category with a Serre functor and take a tt-structure $(\otimes, \mathbb{1})$. If $(\otimes, \mathbb{1})$ is rigid, then we have a natural isomorphism $\mathbb{S}(-) \cong - \otimes \mathbb{S}(\mathbb{1})$. In particular, any $M$-compatible rigid tt-structure is Serre-compatible.*

*Proof.* For the first claim, note that we have the following natural isomorphisms for any $M, N \in \mathcal{T}$:

$$\begin{aligned}\mathrm{Hom}_{\mathcal{T}}(M, \mathbb{S}(N)) &\cong \mathrm{Hom}_{\mathcal{T}}(N, M)^* \\ &\cong \mathrm{Hom}_{\mathcal{T}}(\mathbb{1}, N^\vee \otimes M)^* \\ &\cong \mathrm{Hom}_{\mathcal{T}}(N^\vee \otimes M, \mathbb{S}(\mathbb{1})) \\ &\cong \mathrm{Hom}_{\mathcal{T}}(M, N \otimes \mathbb{S}(\mathbb{1})).\end{aligned}$$

Hence, we have $\mathbb{S}(-) \cong - \otimes \mathbb{S}(\mathbb{1})$. □

**Proposition 4.3.19.** *Let $\mathcal{T}$ be a triangulated category with $\mathrm{FM}\,\mathcal{T} \neq \emptyset$. We have the following inclusions:*

$$\mathrm{Spc}^{\mathsf{FM}}\,\mathcal{T} \subset \mathrm{Spc}^{\mathsf{pFM}}\,\mathcal{T} \subset \mathrm{Spc}^{\mathsf{rig}}\,\mathcal{T} \subset \mathrm{Spc}^{\mathsf{Sc}}\,\mathcal{T} \subset \mathrm{Spc}^{\mathsf{Ser}}\,\mathcal{T}.$$

*Proof.* The first two inclusions are trivial. The third one follows from Proposition 4.3.18. The last one follows from definition. □

*Remark* 4.3.20. Let $\mathcal{T}$ be a triangulated category with a Serre functor. Conjecture 4.1.12 implies that all of the inclusions $\mathrm{Spc}^{\mathsf{FM}}\,\mathcal{T} \subset \mathrm{Spc}^{\mathsf{pFM}}\,\mathcal{T} \subset \mathrm{Spc}^{\mathsf{rig}}\,\mathcal{T} \subset \mathrm{Spc}^{\mathsf{Sc}}\,\mathcal{T} \subset \mathrm{Spc}^{\mathsf{Ser}}\,\mathcal{T}$ are the equalities if $\mathrm{FM}\,\mathcal{T}$ is not empty. Indeed, each equality is a priori non-trivial. Although Conjecture 4.1.12 is false in general, it will be interesting to see if there is any equality that still holds.

Given the observations, let us propose the following conjecture that is weaker than Conjecture 4.1.12, but still provide a categorical characterization of the proper Fourier–Mukai locus. Here, recall that if $\mathrm{FM}\,\mathcal{T} \neq \emptyset$ and $\mathsf{Perf}\, X \simeq \mathcal{T}$ for a variety $X$, then $X$ is necessarily smooth and proper.

**Conjecture 4.3.21.** *Let $(\mathcal{T}, \otimes)$ be a tt-category with $\mathrm{FM}\,\mathcal{T} \neq \emptyset$. Let $\mathrm{Spc}^{\mathsf{rig,var}}\,\mathcal{T} \subset \mathrm{Spc}_{\vartriangle}\,\mathcal{T}$ denote the union of $\mathrm{Spc}_{\otimes}\,\mathcal{T}$ over rigid tensor triangulated structures whose spectra are varieties. Then, we have*

$$\mathrm{Spc}^{\mathsf{pFM}}\,\mathcal{T} = \mathrm{Spc}^{\mathsf{rig,var}}\,\mathcal{T} \subset \mathrm{Spc}_{\vartriangle}\,\mathcal{T}.$$

Note that the conjecture indeed holds as a formal consequence of higher Zariski geometry developed in [ABC+25] if we upgrade the meaning of "whose spectra are varieties" in the following sense.

**Proposition 4.3.22.** *Let $(\mathcal{C}, \otimes)$ be a* ***2-ring****, i.e., an idempotent-complete stably symmetric monoidal $\infty$-category. Then, there exists a variety $X$ and a symmetric monoidal $\infty$-equivalence $(\mathcal{C}, \otimes) \simeq (\mathsf{Perf}\ X, \otimes_X)$ if and only if $\otimes$ is rigid and the 2-ringed space $\mathrm{Spec}_{\otimes} \mathcal{C}$ is equivalent to the 2-ringed space corresponding to a variety. In particular, we have the equality of the corresponding loci:*

$$\mathrm{Spc}^{\mathsf{pFM}} \mathsf{Ho}(\mathcal{C}) = \mathrm{Spc}^{\mathsf{rig,2var}} \mathsf{Ho}(\mathcal{C}) \subset \mathrm{Spc}_{\vartriangle} \mathsf{Ho}(\mathcal{C}).$$

*Proof.* One direction is clear. For the other direction, note that by the fundamental adjunction of higher Zariski geometry in [ABC+25], there is a fully faithful embedding

$$\mathrm{Spec} : \mathsf{2CAlg}_{\mathsf{rig}} \hookrightarrow \{\text{Locally 2-Ringed Spaces}\}^{\mathsf{op}}$$

where $\mathsf{2CAlg}_{\mathsf{rig}}$ denotes the $\infty$-category of rigid 2-rings (see also [ABC+25, Observation 5.12, Remark 5.13]). In particular, if $(\mathcal{C}, \otimes)$ is rigid and the locally 2-ringed space $\mathrm{Spec}(\mathcal{C}, \otimes)$ is equivalent to the locally 2-ringed space given by a (proper) variety $X$, then we have an equivalence

$$(\mathcal{C}, \otimes) \simeq (\mathsf{Perf}\ X, \otimes_X) \quad \text{in } \mathsf{2CAlg}_{\mathsf{rig}}. \qquad \square$$

*Remark* 4.3.23. Let $(\mathcal{C}, \otimes)$ be a rigid 2-ring. Note that by [Che25, Theorem E] (see also loc. cit. for notations/terminology), if the locally spectrally ringed space $(\mathrm{Spc}_{\otimes} \mathcal{C}, \mathcal{R}_{\mathcal{O}_{\mathrm{Spc}_{\otimes} \mathcal{C}}})$ is a (classical) variety $X$ (which implies $\mathrm{Spec}_{\otimes} \mathsf{Ho}(\mathcal{C}) \cong X$), then there is a symmetric monoidal fully faithful embedding

$$(\mathsf{Perf}\ X, \otimes_X) \hookrightarrow (\mathcal{C}, \otimes).$$

In particular, if $\mathcal{C} = \mathsf{Perf}\ Y$ for some smooth proper variety $Y$, then the embedding above implies there is in particular a fully faithful embedding $\mathsf{Perf}\ X \hookrightarrow \mathsf{Perf}\ Y$, which gives a restriction on possible variety $X$. For example, if we further require $\dim X = \dim Y$, then such varieties seem to be closely related to birational geometry (e.g. pull-backs along blow-ups). In such a case, Conjecture 4.3.21 may fail in an interesting way due to the phenomenon described in Example 2.2.4.

Finally, let us note that the classification of tt-structure up to spectral identity is quite coarse by the results of Ito–Nolan (cf. [IN25, Proposition 6.1, Proposition 6.11], Proposition 6.17).

**Theorem 4.3.24** (Ito–Nolan)**.** *Let $A$ be a nonzero finite-dimensional commutative $k$-algebra.*

*(i) There is a canonical tt-structure $\star'_A$ on $\mathsf{Perf}\ \mathbb{P}(A)$.*

*(ii) The construction $A \mapsto \star'_A$ is functorial in surjective maps of finite-dimensional commutative $k$-algebras and essentially injective.*

*(iii)* $\mathrm{Spec}_{\star'_A} \mathsf{Perf}\ \mathbb{P}(A) = \mathrm{Spec}_{\star'_{k^{\dim A}}} \mathsf{Perf}\ \mathbb{P}(A) \subset \mathrm{Spec}_{\vartriangle} \mathsf{Perf}\ \mathbb{P}(A)$ *for any finite-dimensional commutative algebra $A$. Here, $\star_{k^{\dim A}}$ is tt-equivalent to $\otimes_{\mathsf{rep}}$ of the Beilinson quiver for $\mathbb{P}(A)$.*

*In particular, for any non-isomorphic finite-dimensional commutative $k$-algebras $A, A'$ of same dimension, we see that $\star'_A$ and $\star'_{A'}$ are not tt-equivalent but spectrally identical.*

# Chapter 5

# Applications of polarizations on a triangulated category

In this chapter, we observe several situations where polarizations appear naturally.

## 5.1 Reconstruction of varieties

First, let us introduce the following notions to conveniently formulate/give proofs of several reconstructions.

**Definition 5.1.1.** Let $\mathcal{T}$ be a triangulated category and let $X$ be a noetherian scheme.

- A polarization $\tau$ is said to be $X$**-reconstructive** (resp. **weakly** $X$**-reconstructive**) if
  - (i) $\operatorname{Spc}^\tau \mathcal{T}$ has a unique maximal-dimensional connected component $(\operatorname{Spc}^\tau \mathcal{T})^*$, and
  - (ii) there exists a tt-structure $\otimes$ on $\mathcal{T}$ with tt-equivalence $(\mathcal{T}, \otimes) \simeq (\mathsf{Perf}\ X, \otimes^{\mathbb{L}}_{\mathcal{O}_X})$ such that
  $$\operatorname{Spc}_\otimes \mathcal{T} = (\operatorname{Spc}^\tau \mathcal{T})^* \subset \operatorname{Spc}_\vartriangle \mathcal{T} \quad (\text{resp. } \operatorname{Spc}_\otimes \mathcal{T} \subset (\operatorname{Spc}^\tau \mathcal{T})^* \subset \operatorname{Spc}_\vartriangle \mathcal{T}).$$
  Note, by [Mat25b, Theorem 2.11], that the inclusion $\operatorname{Spc}_\otimes \mathcal{T} \subset (\operatorname{Spc}^\tau \mathcal{T})^*$ is open.
- A line bundle $\mathscr{L}$ on $X$ is said to be **reconstructive** if the polarization $\tau_{\mathscr{L}}$ on $\mathsf{Perf}\ X$ is $X$-reconstructive. Note that if $\tau_{\mathscr{L}}$ is (weakly) $Y$-reconstructive, then $X \cong Y$ by Theorem 5.1.5, so the notion should not cause any confusion. Also, note that $\tau_{\mathscr{L}}$ is always weakly $X$-reconstructive.

We define the analogous notions for semi-polarizations and multi-polarizations.

**Example 5.1.2.** Let $X$ be a noetherian scheme.

(i) If $\tau$ is (weakly) $X$-reconstructive, then so is any conjugation $\eta \circ \tau \circ \eta^{-1}$ for $\eta \in \operatorname{Auteq} \mathsf{Perf}\ X$. Similarly, if $\tau$ is (weakly) $X$-reconstructive, then so is any polarization $\eta \circ \tau \circ \eta^{-1}$ on a triangulated category $\mathcal{T}$ together with an exact equivalence $\eta : \mathsf{Perf}\ X \simeq \mathcal{T}$.

(ii) A $\otimes$-generating line bundle $\mathscr{L}$ on $X$ is reconstructive by Proposition 4.2.3.

(iii) On an elliptic curve, a non-torsion line bundle of degree 0 is reconstructive, but not $\otimes$-generating by Example 4.2.12. This is a motivating example for why we do not simply ask for $\operatorname{Spc}^{\tau} \mathcal{T} = \operatorname{Spc}_{\otimes} \mathcal{T}$ in the definition.

(iv) If $X$ is a Gorenstein proper variety, then the Serre functor is weakly $Y$-reconstructive for every variety $Y$ with $\mathsf{Perf}\, Y \simeq \mathsf{Perf}\, X$.

Now, let us state a generalization of the reconstruction of Bondal–Orlov and Ballard from the perspective of polarizations.

**Theorem 5.1.3.** *Let $X$ be a Gorenstein proper variety of dimension $n$ and suppose* $\operatorname{Spec}^{\mathsf{Ser}} \mathsf{Perf}\, X$ *is a separated scheme whose $n$-dimensional connected components (with reduced structure) are all isomorphic to each other. Then, the following assertions hold:*

*(i) $X$ can be reconstructed as an $n$-dimensional connected component of* $\operatorname{Spec}^{\mathsf{Ser}} \mathsf{Perf}\, X$ *with reduced structure.*

*(ii) If there exists a variety $Y$ with* $\mathsf{Perf}\, X \simeq \mathsf{Perf}\, Y$*, then $X \cong Y$.*

Let us note that thanks to [Mat25b, Theorem 2.11] there is no need for any projectivity assumption and that Theorem 5.1.3 is indeed a strict generalization of the reconstruction of Bondal–Orlov and Ballard by the following example (ii).

**Example 5.1.4.**

(i) A Gorenstein proper variety with reconstructive canonical bundle satisfies the suppositions of Theorem 5.1.3 by definition.

(ii) More specifically, we can still apply Theorem 5.1.3 to any Gorenstein proper variety with $\otimes$-generating canonical bundle that is neither ample nor anti-ample (cf. Theorem 3.3.5). We can indeed construct a non-projective proper Gorenstein toric variety with $\otimes$-generating canonical bundle, so the supposition of properness is a strict generalization as well.

(iii) By [HO22, Theorem 4.11] (cf. Lemma 4.1.9), elliptic curves over algebraically closed field of characteristic zero also satisfy the suppositions.

Before giving a proof of Theorem 5.1.3, let us make the following simple yet effective observation.

**Theorem 5.1.5.** *Let $\mathcal{T}$ be a triangulated category and suppose that there are proper, connected and reduced schemes $X, Y$ with* $\mathsf{Perf}\, X \simeq \mathsf{Perf}\, Y \simeq \mathcal{T}$*. Suppose there is a $X$-reconstructive and weakly $Y$-reconstructive polarization $\tau$ on $\mathcal{T}$. Then, $\tau$ is $Y$-reconstructive and $X \cong Y$. The same claim follows for weak polarizations and multi-polarizations.*

*Proof.* By supposition, there exist tt-structures $\otimes_X$ and $\otimes_Y$ such that

- $(\mathcal{T}, \otimes_X)$ (resp. $(\mathcal{T}, \otimes_Y)$) is tt-equivalent to $(\mathsf{Perf}\ X, \otimes^{\mathbb{L}}_{\mathcal{O}_X})$ (resp. $(\mathsf{Perf}\ Y, \otimes^{\mathbb{L}}_{\mathcal{O}_Y})$) and
- $(\mathrm{Spec}^\tau \mathcal{T})^* = \mathrm{Spec}_{\otimes_X} \mathcal{T} \cong X$ and $Y \cong \mathrm{Spec}_{\otimes_Y} \mathcal{T} \underset{\text{open}}{\subset} (\mathrm{Spec}^\tau \mathcal{T})^*$.

Since $Y$ is proper and $X$ is separated, $Y$ is a closed subscheme of $X$ as well. Therefore, $Y$ is an open and closed subvariety of the connected scheme $X$ and hence $Y \cong \mathrm{Spec}_{\otimes_Y} \mathcal{T} = (\mathrm{Spec}^\tau \mathcal{T})^* \cong X$. □

Let us note that Theorem 5.1.5 gives quick proofs for variants of [Fav12, Theorem 3.8].

**Corollary 5.1.6.** *Let $\mathcal{T}$ be a triangulated category and suppose that there are proper, connected and reduced schemes $X, Y$ with $\mathsf{Perf}\ X \simeq \mathsf{Perf}\ Y \simeq \mathcal{T}$. Let $\mathscr{L}$ be a reconstructive line bundle on $X$ and let $\mathscr{M}$ be any line bundle on $Y$. Suppose there exists a pt-equivalence*

$$F : (\mathsf{Perf}\ X, \tau_{\mathscr{L}}) \xrightarrow{\sim} (\mathsf{Perf}\ Y, \tau_{\mathscr{M}}).$$

*Then, $\mathscr{M}$ is a reconstructive line bundle on $Y$ and $X \cong Y$. The same claims follow for multi-polarizations by replacing $\mathscr{L}$ with an amply family on $X$ (assuming $X$ is moreover divisorial) and $\mathscr{M}$ with a collection of any line bundles on $Y$.*

*Proof.* Since $F \circ \tau_{\mathscr{L}} \circ F^{-1}$ is a $X$-reconstructive and weakly $Y$-reconstructive polarization on $\mathsf{Perf}\ Y$, we are done by Theorem 5.1.5. □

*Remark* 5.1.7. In the original result by Favero, varieties are not assumed to be proper (only divisorial), but for our arguments to work, properness is essential for the existence of Serre functors. It is interesting to see if there is a way to show Favero's original result from the perspective of the triangular spectrum.

Now, with similar ideas, we give a short proof of the reconstruction theorem.

*Proof of Theorem 5.1.3.* For part (i), first note that $X \cong \mathrm{Spec}_{\otimes_X} \mathsf{Perf}\ X$ is a connected, closed and open subscheme of $\mathrm{Spec}^{\mathsf{Ser}} \mathsf{Perf}\ X$ by the same arguments as before. Hence, $\mathrm{Spec}_{\otimes_X} \mathsf{Perf}\ X$ is an $n$-dimensional connected component of $\mathrm{Spec}^{\mathsf{Ser}} \mathsf{Perf}\ X$ with reduced structure. Thus, by taking any $n$-dimensional connected component of $\mathrm{Spec}^{\mathsf{Ser}} \mathsf{Perf}\ X$ with reduced structure, we reconstruct $X$.

For part (ii), since $Y$ is automatically proper and Gorenstein (e.g., [dSdS12, Proposition 1.5]), the argument of part (i) shows $Y$ is also isomorphic to any $n$-dimensional connected component of $\mathrm{Spec}^{\mathsf{Ser}} \mathsf{Perf}\ X$ with reduced structure, and hence $X \cong Y$. □

Finally, let us generalize yet another reconstruction result of Favero [Fav12, Lemma 4.1]. This result clarifies the interplay of automorphisms and line bundles as polarizations, which lead to certain rigidity.

**Theorem 5.1.8.** *Let $X$ be a proper variety and let $\tau$ be a polarization on $\mathsf{Perf}\ X$. Suppose there is a variety $Y$ and a standard autoequivalence*

$$\sigma = f_* \circ \tau_{\mathscr{M}}[i] \in \mathrm{Auteq}^{\mathsf{st}}(\mathsf{Perf}\ Y) = \mathrm{Aut}\, Y \ltimes \mathrm{Pic}\, Y \times \mathbb{Z}[1]$$

*together with a pt-equivalence*

$$\Phi : (\mathsf{Perf}\ X, \tau_{\mathscr{L}}) \simeq (\mathsf{Perf}\ Y, \sigma)$$

*given by a Fourier–Mukai transform* $\Phi = \Phi_{\mathscr{K}}$ *with Fourier–Mukai kernel* $\mathscr{K} \in \mathsf{D}_{\mathsf{qc}}(X \times Y)$. *Then,*

*(i) If* $\tau = \tau_{\mathscr{L}}$ *for a line bundle* $\mathscr{L}$ *on* $X$ *such that* $\mathrm{H}^l(X, \mathscr{L}^{\otimes m}) \neq 0$ *for some* $l, m \in \mathbb{Z}$, *then* $f^m$ *has a fixed point.*

*(ii) If* $f^m$ *has a fixed point for some* $m \in \mathbb{Z}$ *and if every connected component of* $\mathrm{Spec}^{\tau^m}\ \mathsf{Perf}\ X$ *is isomorphic to* $X$, *then we have* $f^m = id_Y$ *and* $X \cong Y$.

*In particular, if* $\tau = \tau_{\mathscr{L}}$ *for a* $\otimes$*-generating line bundle* $\mathscr{L}$, *then* $f^m = id_Y$ *for some* $m$ *and* $X \cong Y$.

*Proof.* Let us first recall some computations from [Fav12]. Take $\mathscr{K}' \in \mathsf{D}_{\mathsf{qc}}(X \times Y)$ so that $\Phi^{-1} = \Phi_{\mathscr{K}'}$. Then, we get an equivalence $\Phi_{\mathscr{K} \boxtimes \mathscr{K}'} : \mathsf{D}_{\mathsf{qc}}(X \times X) \simeq \mathsf{D}_{\mathsf{qc}}(Y \times Y)$ such that if $\Phi : (\mathsf{Perf}\ X, \Phi_{\mathscr{S}}) \to (\mathsf{Perf}\ Y, \Phi_{\mathscr{T}})$ is a pt-equivalence, then $\Phi_{\mathscr{K} \boxtimes \mathscr{K}'}(\mathscr{S}) \cong \mathscr{T}$.

(i) We proceed by mimicking the proof of [Fav12, Lemma 4.1]. Fix $l, m \in \mathbb{Z}$ so that

$$\mathrm{H}^l(X, \mathscr{L}^{\otimes m}) = \mathrm{Hom}_{\mathsf{D}_{\mathsf{qc}}}(\mathscr{O}_X, \mathscr{L}^{\otimes m}[l]) \neq 0.$$

Then, since $\tau_{\mathscr{L}}^m = \Phi_{\Delta_* \mathscr{L}^{\otimes m}}$ and $\sigma^m = \Phi_{(\mathrm{id} \times f^m)_* \mathscr{M}_m[mi]}$ for $\mathscr{M}_m := f_* \mathscr{M} \otimes \cdots \otimes f_*^m \mathscr{M}$, we get

$$\begin{aligned} 0 \neq \mathrm{Hom}_{\mathsf{D}_{\mathsf{qc}}(X)}(\mathscr{O}_X, \mathscr{L}^{\otimes m}[l]) &\hookrightarrow \mathrm{Hom}_{\mathsf{D}_{\mathsf{qc}}(X \times X)}(\Delta_* \mathscr{O}_X, \Delta_* \mathscr{L}^{\otimes m}[l]) \\ &\cong \mathrm{Hom}_{\mathsf{D}_{\mathsf{qc}}(Y \times Y)}(\Phi_{\mathscr{K} \boxtimes \mathscr{K}'}(\Delta_* \mathscr{O}_X), \Phi_{\mathscr{K} \boxtimes \mathscr{K}'}(\Delta_* \mathscr{L}^{\otimes m})[l]) \\ &\cong \mathrm{Hom}_{\mathsf{D}_{\mathsf{qc}}(Y \times Y)}(\Delta_* \mathscr{O}_Y, (\mathrm{id}_Y \times f^m)_* \mathscr{M}_m[mi + l]) \\ &\cong \mathrm{Hom}_{\mathsf{D}_{\mathsf{qc}}(Y)}(\mathscr{O}_Y, \Delta^!(\mathrm{id}_Y \times f^m)_* \mathscr{M}_m[mi + l]). \end{aligned}$$

Here, the first inclusion follows since the diagonal map $\Delta$ has a left inverse (given by a projection) and hence $\Delta_* = \mathbb{R}\Delta_*$ is faithful on $\mathsf{D}_{\mathsf{qc}}$. Now, since the last term is the cohomology of a complex supported on the fixed locus $Y^{f^m}$ of $f^m$, we see that $f^m$ has a fixed point.

(ii) We may assume $m = 1$ since $\sigma^m = f_*^m \circ \tau_{\mathscr{M}_m}[mi]$. Now, since $\mathrm{Spc}_{\otimes_Y^{\mathbb{L}}}\ \mathsf{Perf}\ Y \subset \mathrm{Spc}_{\vartriangle}\ \mathsf{Perf}\ Y$ is an open inclusion, by taking fixed points of $\sigma$, we obtain

$$\emptyset \neq (\mathrm{Spc}_{\otimes_Y^{\mathbb{L}}}\ \mathsf{Perf}\ Y)^{\sigma} \underset{\text{open}}{\subset} \mathrm{Spc}^{\sigma}\ \mathsf{Perf}\ Y \cong \mathrm{Spc}^{\tau}\ \mathsf{Perf}\ X.$$

Thus, by supposition $\dim(\mathrm{Spc}_{\otimes_Y^{\mathbb{L}}}\ \mathsf{Perf}\ Y)^{\sigma} = \dim Y$. Now, note

$$(\mathrm{Spc}_{\otimes_Y^{\mathbb{L}}}\ \mathsf{Perf}\ Y)^{\sigma} = (\mathrm{Spc}_{\otimes_Y^{\mathbb{L}}}\ \mathsf{Perf}\ Y)^{f_* \circ \tau_{\mathscr{M}}} = (\mathrm{Spc}_{\otimes_Y^{\mathbb{L}}}\ \mathsf{Perf}\ Y)^{f_*} \cong Y^f$$

since $\tau_{\mathscr{M}}$ fixes every point in the Balmer spectrum. Therefore, we have $\dim Y^f = \dim Y$ and hence we necessarily have $f = \mathrm{id}_Y$. Then, we have $X \cong Y$ by Theorem 5.1.5.

The last claim follows from part (i) and (ii), noting that for a $\otimes$-generating line bundle $\mathscr{L}$, there exists $m \neq 0$ such that $\mathrm{H}^0(X, \mathscr{L}^{\otimes m}) \neq 0$ by Theorem 3.3.3 (as a large enough power of a big line bundle has a global section by definition) and $\mathscr{L}^{\otimes m}$ is $\otimes$-generating by [IO25, Lemma 3.14]. □

There are some simplifications we can make to suppositions.

*Remark* 5.1.9. Use the same notations as in Theorem 5.1.8.

(i) If $X$ is projective, then $\Phi$ and $\tau$ are automatically of Fourier–Mukai type by [Bal09, Theorem 1.2].

(ii) If $X$ is a variety all of whose automorphisms have a fixed point, then part (iii) simplifies. For example, any smooth proper complex variety $X$ with $\mathrm{H}^i(X, \mathscr{O}_X) = 0$ for all $i > 0$ falls into this category by the holomorphic Lefschetz fixed point formula. This includes smooth projective rationally connected varieties (e.g., smooth weak Fano varieties over an algebraically closed field of characteristic zero), Enriques surfaces and any smooth proper varieties birationally equivalent to those varieties.

(iii) On the other hand, the equidimensionality in part (iii) cannot be removed with our current proof, so we cannot apply this theorem with $\tau = \tau_{\mathscr{L}}$ for a reconstructive line bundle $\mathscr{L}$ with $(\mathrm{Spc}^{\tau_{\mathscr{L}}} \mathsf{Perf}\ X)^* \subsetneq \mathrm{Spc}^{\tau_{\mathscr{L}}} \mathsf{Perf}\ X$.

Finally, let us leave another observation suggesting a further interplay between "positivity" of line bundles and fixed-point properties of automorphisms (see also [Fav12, Remark 4.2]).

**Example 5.1.10.** Let $E$ be an elliptic curve over an algebraically closed field of characteristic zero and take a line bundle $\mathscr{L}$ on $E$ of degree 0 with $\mathscr{L}^{\otimes n} \not\cong \mathscr{O}_E$ for any $n > 0$. In particular, $\mathscr{L}$ is reconstructive. Then, as we observed in Example 4.2.12, $\mathscr{L}$ is quite close to cutting out the Balmer spectrum while $\mathrm{H}^l(E, \mathscr{L}^{\otimes m}) = 0$ for all $l, m \in \mathbb{Z}$. Now, using the same notations as in Example 4.2.12, consider an equivalence

$$\Phi_{\mathscr{P}} : \mathsf{Perf}\ \hat{E} \to \mathsf{Perf}\ E,$$

where $\hat{E} = M_{1,0} = \mathrm{Pic}^0(E)$ is the dual of $E$ and $\mathscr{P} = \mathscr{U}_{1,0}$ is the Poincaré bundle. Then, it is easy to check that $\Psi := \Phi_{\mathscr{P}}^{-1} \circ \tau_{\mathscr{L}} \circ \Phi_{\mathscr{P}} : \mathsf{Perf}\ \hat{E} \xrightarrow{\sim} \mathsf{Perf}\ \hat{E}$ sends skyscraper sheaves at closed points to skyscraper sheaves at closed points and hence we can write $\Psi = f_* \circ \tau_{\mathscr{M}}$ for some $f \in \mathrm{Aut}\ \hat{E}$ and $\mathscr{M} \in \mathrm{Pic}\ \hat{E}$. Therefore, we have a pt-equivalence

$$\Phi_{\mathscr{P}} : (\mathsf{Perf}\ \hat{E}, f_* \circ \tau_{\mathscr{M}}) \simeq (\mathsf{Perf}\ E, \tau_{\mathscr{L}}).$$

Note it is moreover clear that we have $f$ equals the translation $t_{[\mathscr{L}]}$ on $\hat{E}$ by $[\mathscr{L}] \in \hat{E}$, i.e.,

$$f = t_{[\mathscr{L}]} : \hat{E} \xrightarrow{\sim} \hat{E}, \quad [\mathscr{E}] \mapsto [\mathscr{E} \otimes_E \mathscr{L}].$$

Hence, we have observed that the reconstructive line bundle $\mathscr{L}$ fails to satisfy the supposition of Theorem 5.1.8 (ii) and indeed $f = t_{[\mathscr{L}]}$ has no periodic point. On the other hand, if we take $\mathscr{L}$ to be a line bundle of degree 0 with $\mathscr{L}^{\otimes n} \cong \mathscr{O}_E$ for some $0 \neq n \in \mathbb{Z}$, then $f^n = t_{[\mathscr{L}^n]} = t_{[\mathscr{O}_E]} = \mathrm{id}_E$ and $\mathrm{Spc}^{\tau_{\mathscr{L}^{\otimes n}}} \mathsf{Perf}\ E = \mathrm{Spc}_{\vartriangle} \mathsf{Perf}\ E$ is equidimensional of dimension 1, so we can apply Theorem 5.1.8 (iii) while $\mathscr{L}$ is not reconstructive.

## 5.2 Geometric realization of noncommutative projective schemes and Iitaka fibrations

We observe that fixed loci can be thought of as geometric realization of (derived) noncommutative projective schemes in the sense of [AZ94]. Along the way, we will also make some observations on relations with birational geometry. First, let us quickly recall materials from [AZ94].

**Definition 5.2.1.** Let $A$ be a right noetherian $\mathbb{N}$-graded algebra (over $k$). The **noncommutative projective scheme** $\mathrm{Proj}^{\mathsf{nc}} A$ associated to $A$ is a triple $(\mathsf{qgr}\, A, s_A, A)$, where

- the symbol $\mathsf{qgr}\, A$ denotes the ($k$-linear) abelian category obtained as the quotient of the abelian category of finitely generated $\mathbb{Z}$-graded (right) $A$-modules by the Serre subcategory of finitely generated torsion $\mathbb{Z}$-graded $A$-modules,
- the symbol $s_A$ denotes the autoequivalence of $\mathsf{qgr}\, A$ given by shifting gradings by one, and
- the symbol $A$ denotes the image of $A$ in $\mathsf{qgr}\, A$ by abuse of notation.

Similarly, let $\mathsf{QGr}\, A$ denote the quotient category of the abelian category of $\mathbb{Z}$-graded (right) $A$-modules by the Serre subcategory of torsion $\mathbb{Z}$-graded $A$-modules. Note that $\mathsf{QGr}\, A$ is the ind-completion of $\mathsf{qgr}\, A$.

The following classical results are their guiding examples (see for example [Har77, Exercise II.5.9]):

**Theorem 5.2.2.** *Let $A$ be a finitely generated $\mathbb{N}$-graded commutative algebra generated in degree 1.*

*(i) The functor*

$$\Gamma : \mathsf{Coh}\, \mathrm{Proj}\, A \to \mathsf{qgr}\, A, \quad \mathscr{M} \mapsto \oplus_{n\in\mathbb{Z}} \Gamma(\mathrm{Proj}\, A, \mathscr{M}(n))$$

*is an equivalence of abelian categories, where $\mathsf{Coh}\, \mathrm{Proj}\, A$ denotes the abelian category of coherent sheaves on $\mathrm{Proj}\, A$. Under this equivalence, the autoequivalence $- \otimes_{\mathscr{O}_{\mathrm{Proj}\, A}} \mathscr{O}_{\mathrm{Proj}\, A}(1)$ on $\mathsf{Coh}(\mathrm{Proj}\, A)$ corresponds to the autoequivalence $s_A$ on $\mathsf{qgr}\, A$. Furthermore, $\Gamma$ naturally extends to an equivalence*

$$\Gamma : \mathsf{QCoh}\, \mathrm{Proj}\, A \simeq \mathsf{QGrA},$$

*where $\mathsf{QCoh}\, \mathrm{Proj}\, A$ denotes the abelian category of quasi-coherent sheaves on $\mathrm{Proj}\, A$.*

*(ii) The functor $\Gamma$ above induces an isomorphism*

$$\mathrm{Proj}\, A = \mathrm{Proj}\left(\oplus_{n\geq 0} \Gamma(\mathrm{Proj}\, A, \mathscr{O}_{\mathrm{Proj}\, A}(n))\right) \cong \mathrm{Proj}\left(\oplus_{n\geq 0} \mathrm{Hom}_{\mathsf{qgr}\, A}\left(A, s_A^n A\right)\right)$$

*of schemes. In particular, note that the right hand side is constructed precisely by the data of $\mathrm{Proj}^{\mathsf{nc}} A$.*

Therefore, for a general right noetherian $\mathbb{N}$-graded algebra $A$, we may think of $\mathrm{Proj}^{\mathsf{nc}} A$ as a "(noncommutative) space" whose abelian category of coherent sheaves (resp. quasi-coherent sheaves) is $\mathsf{qgr}\, A$ (resp. $\mathsf{QGr}\, A$). Now, to make comparisons of fixed loci/pt-spectra with noncommutative projective schemes and to see some relations with birational geometry, let us introduce some notions and recall some facts on the Iitaka fibration.

**Notation 5.2.3.** For a pt-category $(\mathcal{T}, \tau)$ and a fixed object $c \in \mathcal{T}$, define the $\mathbb{N}$-graded (not necessarily commutative) algebra

$$\mathcal{R}_{\tau,c} := \oplus_{n\geq 0} \mathrm{Hom}_{\mathcal{T}}(c, \tau^n(c)),$$

where multiplications are given by compositions under identifications

$$\tau^m : \mathrm{Hom}_{\mathcal{T}}(c, \tau^n(c)) \cong \mathrm{Hom}_{\mathcal{T}}(\tau^m(c), \tau^{m+n}(c)).$$

For a pt-category $(\mathsf{Perf}\, X, \tau_{\mathscr{L}})$ given by a line bundle $\mathscr{L}$ on a normal projective variety $X$, note that this construction gives the **section ring**

$$\mathcal{R}_{\mathscr{L}} := \mathcal{R}_{\tau_{\mathscr{L}}, \mathcal{O}_X} \cong \oplus_{n\geq 0} \Gamma(X, \mathscr{L}^{\otimes n})$$

of $\mathscr{L}$, where multiplications are given by multiplication of sections and hence are commutative. Note if $\mathcal{R}_{\mathscr{L}}$ is a finitely generated $k$-algebra (e.g., if $\mathscr{L}$ is semi-generating ([Laz17, Example 2.1.30]) or if $\mathscr{L} = \omega_X$ for a smooth projective variety $X$ ([BCHM10])) and if the Iitaka dimension of $\mathscr{L}$ is positive, then there is a rational map

$$\phi_{\mathscr{L}} : X \dashrightarrow \mathrm{Proj}\, \mathcal{R}_{\mathscr{L}}$$

that is birationally equivalent to the natural rational maps

$$\phi_{|\mathscr{L}^{\otimes n}|} : X \dashrightarrow Y_n := \phi_{|\mathscr{L}^{\otimes n}|}(X) \subset \mathbb{P}\mathrm{H}^0(X, \mathscr{L}^{\otimes n})$$

for all large enough $n$ with $\mathscr{L}^{\otimes n}$ being effective, in which case $Y_n \cong \mathrm{Proj}\, \mathcal{R}_{\mathscr{L}}$ by finite generation of $\mathcal{R}_{\mathscr{L}}$ (cf. [Laz17, Theorem 2.1.33]). The rational map $\phi_{\mathscr{L}}$ is called the **Iitaka fibration** and $\dim \mathrm{Proj}\, \mathcal{R}_{\mathscr{L}}$ equals the Iitaka dimension of $\mathscr{L}$. Furthermore, if $\mathscr{L}$ is moreover big (resp. semi-generating), then $\phi_{\mathscr{L}}$ is birational (resp. defined everywhere) (cf. [Laz17, Definition 2.2.1. (resp. Theorem 2.1.27.)]).

As a special case, when $X$ is a smooth projective variety, the construction above gives the **canonical ring** $\mathcal{R}_X := \mathcal{R}_{\omega_X}$ of $X$ and the **canonical model** $X^{\mathsf{can}} := \mathrm{Proj}\, \mathcal{R}_X$ of $X$. By the results above, if the Kodaira dimension of $X$ (i.e., the Iitaka dimension of $\omega_X$) is positive, we have a natural rational map

$$\phi_X : X \dashrightarrow X^{\mathsf{can}}$$

where $\dim X^{\mathsf{can}}$ equals the Kodaira dimension of $X$ and $\phi_X$ is birational when $X$ is **of general type** (i.e., $\omega_X$ is big). The canonical ring and the canonical model are important in birational geometry since they are birational invariants of smooth projective varieties.

Now, let us make our guiding observation.

**Proposition 5.2.4.** *Let $X$ be a projective scheme and take an ample line bundle $\mathscr{L}$. Then, we have isomorphisms*

$$\operatorname{Spec}^{\tau_{\mathscr{L}}} \mathsf{Perf}\ X \cong X \cong (\operatorname{Proj} \mathcal{R}_{\mathscr{L}})_{\mathsf{red}}.$$

*Moreover, we have a pt-equivalence*

$$(\mathsf{Perf}\ X, \tau_{\mathscr{L}}) \simeq (\mathsf{Perf}\ \operatorname{Proj}^{\mathsf{nc}} \mathcal{R}_{\mathscr{L}}, s_{\mathcal{R}_{\mathscr{L}}})$$

*where* $\mathsf{Perf}\ \operatorname{Proj}^{\mathsf{nc}} \mathcal{R}_{\mathscr{L}}$ *is the subcategory on compact objects in the unbounded derived category* $\mathsf{D}(\mathsf{QGr}\,\mathcal{R}_{\mathscr{L}})$ *of* $\mathsf{QGr}\,\mathcal{R}_{\mathscr{L}}$ *and* $s_{\mathcal{R}_{\mathscr{L}}}$ *denotes the restriction of the derived functor of* $s_{\mathcal{R}_{\mathscr{L}}}$ *on* $\mathsf{D}(\mathsf{QGr}\,\mathcal{R}_{\mathscr{L}})$ *to compact objects by slight abuse of notation.*

*Proof.* For the first claim, the first isomorphism is Proposition 4.2.1 and the second isomorphism is classical (e.g., cf. [GW10, Corollary 13.75]). The last claim follows from Theorem 5.2.2 (i) and the fact that the compact objects in the unbounded derived category of quasi-coherent sheaves on a quasi-compact quasi-separated scheme agrees with perfect complexes (e.g., cf. [Sta21, Tag 09M8]). □

Therefore, given a noncommutative projective scheme $\operatorname{Proj}^{\mathsf{nc}} A$, we may think of the ringed space

$$\operatorname{Spec}^{s_A} \mathsf{Perf}\ \operatorname{Proj}^{\mathsf{nc}} A$$

as its geometric realization with reduced structure. The following is one obvious future direction, which we do not pursue further in this paper.

**Question 5.2.5.** How much information of $A$ and $\mathsf{Perf}\ \operatorname{Proj}^{\mathsf{nc}} A$ is retained in $\operatorname{Spec}^{s_A} \mathsf{Perf}\ \operatorname{Proj}^{\mathsf{nc}} A$. Moreover, how rich is the geometry of $\operatorname{Spec}^{s_A} \mathsf{Perf}\ \operatorname{Proj}^{\mathsf{nc}} A$ when $A$ is noncommutative?

*Remark* 5.2.6. We can learn several more observations from Proposition 5.2.4.

(i) It is natural to ask if we can also recover the non-reduced structure sheaf on $\operatorname{Spc}^{\tau_{\mathscr{L}}} \mathsf{Perf}\ X$, corresponding to $\mathscr{O}_X$. Not surprisingly, it is possible if we use the extra data of an object $c := \mathscr{M}[n] \in \mathsf{Perf}\ X$ for any line bundle $\mathscr{M}$ on $X$ and any $n \in \mathbb{Z}$. Indeed, the same construction as the structure sheaf of the tt-spectrum where we use $c$ instead of the tensor unit object (cf. the last paragraph of Construction 2.1.8), we can define the sheaf $\mathscr{O}_{\tau_{\mathscr{L}},c} := \mathscr{O}_{\operatorname{Spc}^{\tau_{\mathscr{L}}} \mathsf{Perf}\ X, c}$ of rings on $\operatorname{Spc}^{\tau_{\mathscr{L}}} \mathsf{Perf}\ X$ and moreover by construction we have an isomorphism

$$\operatorname{Spec}_c^{\tau_{\mathscr{L}}} \mathsf{Perf}\ X := (\operatorname{Spc}^{\tau_{\mathscr{L}}} \mathsf{Perf}\ X, \mathscr{O}_{\tau_{\mathscr{L}},c}) \cong \operatorname{Spec}_{\otimes_X} \mathsf{Perf}\ X \cong \operatorname{Proj} \mathcal{R}_{\mathscr{L}}.$$

Note on the right hand side, we construct a ringed space using both $\tau_{\mathscr{L}}$ and $\mathscr{O}_X$ simultaneously while on the left hand side, we construct the underlying topological space and the structure sheaf separately, which indicates different nature of those two constructions.

Now, since a noncommutative projective scheme $\operatorname{Proj}^{\mathsf{nc}} A$ gives $(\mathsf{Perf}\ \operatorname{Proj}^{\mathsf{nc}} A, s_A, A)$, we can construct a ringed space

$$\operatorname{Spec}_A^{s_A} \mathsf{Perf}\ \operatorname{Proj}^{\mathsf{nc}} A := (\operatorname{Spc}^{s_A} \mathsf{Perf}\ \operatorname{Proj}^{\mathsf{nc}} A, \mathscr{O}_{s_A,A})$$

as above, which may be thought of as a more natural geometric realization of $\operatorname{Proj}^{\mathsf{nc}} A$. Note on the other hand that $\mathcal{O}_{s_A,A}$ is in general a sheaf of noncommutative algebras.

(ii) For a line bundle $\mathscr{L}$ on a projective variety $X$, $\operatorname{Spec}^{\tau_{\mathscr{L}}} \mathsf{Perf}\, X$ and $\operatorname{Proj} \mathcal{R}_{\mathscr{L}}$ do not necessarily agree. For example, if $X$ is a smooth projective variety, then we have

$$\operatorname{Spec}^{\tau_{\omega_X}} \mathsf{Perf}\, X = \operatorname{Spec}^{\mathsf{Ser}} \mathsf{Perf}\, X$$

while we have $\operatorname{Proj} \mathcal{R}_{\omega_X} = \operatorname{Proj} \mathcal{R}_X = X^{\mathsf{can}}$. To be more concrete, let $S'$ be a blow-up of a smooth projective surface $S$ with ample canonical bundle at a point (which has a $\otimes$-generating canonical bundle by Theorem 3.3.5 (ii)). Since the canonical model is birational invariant, we may relate them via the Iitaka fibration

$$\operatorname{Spec}^{\tau_{\omega_{S'}}} \mathsf{Perf}\, S' \cong S' \dashrightarrow S \cong \operatorname{Proj} \mathcal{R}_S \cong \operatorname{Proj} \mathcal{R}_{S'}.$$

Therefore, in general, the fixed loci/pt-spectra are supposed to remember more birational geometric data than the Proj construction.

Combining (i) and (ii) for a smooth projective variety $X$ with a good enough line bundle $\mathscr{L}$, the Iitaka fibration provides a way of naturally comparing two ringed spaces $\operatorname{Spec}^{\tau_{\mathscr{L}}}_{\mathcal{O}_X} \mathsf{Perf}\, X$ and $\operatorname{Proj} \mathcal{R}_{\mathscr{L}}$ constructed from the data of the same triple $(\mathsf{Perf}\, X, \tau_{\mathscr{L}}, \mathcal{O}_X)$ via the rational map

$$\operatorname{Spec}^{\tau_{\mathscr{L}}}_{\mathcal{O}_X} \mathsf{Perf}\, X \supset \operatorname{Spec}_{\otimes_X} \mathsf{Perf}\, X \cong X \dashrightarrow \operatorname{Proj} \mathcal{R}_{\mathscr{L}} = \operatorname{Spec}^{s_{\mathcal{R}_{\mathscr{L}}}}_{\mathcal{R}_{\mathscr{L}}} \mathsf{Perf}\, \operatorname{Proj}^{\mathsf{nc}} \mathcal{R}_{\mathscr{L}}.$$

Now, a natural question is if we can do this process purely categorically. Namely:

**Question 5.2.7.** Let $X$ be a smooth projective variety and $\mathscr{L}$ be a nice enough (e.g., big) line bundle on $X$. Can we construct a pt-functor

$$(\mathsf{Perf}\, \operatorname{Proj}^{\mathsf{nc}} \mathcal{R}_{\mathscr{L}}, s_{\mathcal{R}_{\mathscr{L}}}) \to (\mathsf{Perf}\, X, \tau_{\mathscr{L}})$$

(or the other way around) that induces a "rational" map

$$\operatorname{Spec}^{\tau_{\mathscr{L}}}_{\mathcal{O}_X} \mathsf{Perf}\, X \dashrightarrow \operatorname{Proj} \mathcal{R}_{\mathscr{L}}$$

agreeing with the Iitaka fibration? For $\mathcal{L} = \omega_X$, is the corresponding map $\operatorname{Spec}^{\mathsf{Ser}} \mathsf{Perf}\, X \dashrightarrow X^{\mathsf{can}}$ universal in the sense that for any Fourier–Mukai partner $Y$ that is birationally equivalent to $X$, it is birationally equivalent to $\operatorname{Spec}^{\mathsf{Ser}} \mathsf{Perf}\, Y \dashrightarrow Y^{\mathsf{can}}$?

*Remark* 5.2.8. Such a map can be understood as an analogue of comparison maps in tensor triangulated geometry (cf. [Bal10a]). Indeed, we can construct such a comparison map from the pt-spectrum for the Serre functor to the homogeneous prime spectrum of the canonical ring, which is a work in progress.

## 5.3 Constructions of mirror partners as spectra of Fukaya categories

Homological mirror symmetry predicts that there is an equivalence between the bounded derived category $\mathsf{D}^b_{\mathsf{coh}}(X)$ of coherent sheaves on a variety $X$ over $\mathbb{C}$ and a certain category $\mathsf{Fuk}(M,\omega)$ associated to a symplectic manifold $(M,\omega)$ possibly with additional data. The notation $\mathsf{Fuk}(M,\omega)$ is based on the Fukaya category, but it could denote the Fukaya-Seidel category, the (partially) wrapped Fukaya category, the category of constructible sheaves with prescribed microsupport, etc. If there is such an equivalence $\mathsf{D}^b_{\mathsf{coh}}(X) \simeq \mathsf{Fuk}(M,\omega)$, we say $X$ and $(M,\omega)$ are **mirror partners**. On the other hand, we know that in general there are non-isomorphic Fourier–Mukai partners, so we have the following natural questions.

**Question 5.3.1.** Given a symplectic manifold $(M,\omega)$, is there a canonical choice of an algebraic mirror partner? What kind of data of the symplectic manifold is responsible for determining its mirror partner(s)?

If we suppose that a mirror partner of $(M,\omega)$ is a divisorial variety $X$ (e.g., a smooth variety or a quasi-projective variety), then from our perspective those questions are equivalent to asking if we can canonically associate to $\mathsf{Fuk}(M,\omega)$ a multi-polarization corresponding to an ample family on $X$ (cf. Theorem 4.2.18), using purely symplecto-geometric data of $(M,\omega)$. In other words:

**Question 5.3.2.** Is there a symplecto-geometric way of constructing a (multi)polarization $\tau$ on $\mathsf{Fuk}(M,\omega)$ such that $\operatorname{Spec}^\tau \mathsf{Fuk}(M,\omega)$ is an algebraic mirror partner of $(M,\omega)$, possibly using additional data? How canonical is such a mirror partner?

Indeed, in some cases, the answers are yes. In this section, we will follow the philosophy of the SYZ picture in the sense that Lagrangian sections of the torus fibration $M \to B$ are supposed to correspond to line bundles on (the compactification of) $X$ (cf. [Abo09]). Before looking into examples, let us mention relations with existing works.

- Note that with our approach, we can explicitly construct algebraic mirror partners as fixed loci/pt-spectra, so our picture can lead to predictions of algebraic mirror partners given symplectic data. Note the construction of symplectic mirrors as certain spectra of algebraic mirrors is done for log Calabi–Yau surfaces in [GHK15]. We will discuss more on homological mirror symmetry for log Calabi–Yau surfaces later.

- It is also natural to ask if we can geometrically construct a symmetric monoidal structure $\otimes_M$ on $\mathsf{Fuk}(M,\omega)$ so that under homological mirror symmetry, we have
$$X \cong \operatorname{Spec}_{\otimes_X} \mathsf{Perf}\, X \cong \operatorname{Spec}_{\otimes_M} \mathsf{Fuk}(M,\omega).$$

Let us mention constructions of two such monoidal structures which are related to the context of polarizations.

**Example 5.3.3.**

(i) In [Sub10, Theorem 5.0.34], a symmetric monoidal structure $\hat{\otimes}_M$ is constructed on the generalized Donaldson-Fukaya category $\mathsf{Don}^\sharp(M)$ (c.f. [Sub10, Definition 4.0.27]) for a smooth Lagrangian torus fibration $\pi : M \to B$ over a compact connected base, which restricts to fiberwise addition on the level of objects that are Lagrangian sections of the fibration. It is also shown that when $M$ is a 2-torus, homological mirror symmetry ([PZ98]) gives a monoidal equivalence

$$(\mathsf{Perf}\ X, \otimes_X) \simeq (\mathsf{Don}^+(M), \hat{\otimes}_M)$$

where $\mathsf{Don}^+(M)$ denotes the extended Donaldson-Fukaya category (see also [Sub10, Theorem 7.0.45]). It is recently announced in [ABN24] that the construction of this monoidal structure can be extended to singular cases, more precisely, to almost toric fibrations only with focus-focus singularities in the sense of [Sym03, Definition 4.2, Definition 4.5], while it seems to be still unknown if those monoidal structures correspond to $\otimes_X$ on $\mathsf{Perf}\ X$ under homological mirror symmetry. We will later mention when such a monoidal structure $\hat{\otimes}_M$ can be expected to be at least spectrally identical to $\otimes_X$ on $\mathsf{Perf}\ X$ (Proposition 5.3.15 and Remark 5.3.16 (iv)) under homological mirror symmetry, i.e.,

$$X \cong \mathrm{Spec}_{\otimes_X} \mathsf{Perf}\ X \cong \mathrm{Spec}_{\hat{\otimes}_M} \mathsf{Don}^+(M).$$

(ii) In [Han19] Hanlon introduced the **monomially admissible Fukaya-Seidel category** $\mathcal{F}_\Delta(W_\Sigma)$ for the Landau-Ginzburg model $((\mathbb{C}^*)^n, W_\Sigma)$, where $W_\Sigma$ is the Hori-Vafa superpotential associated to a fan $\Sigma \subset \mathbb{R}^n$ defining a smooth proper toric variety $X_\Sigma$ and $\Delta$ is additional data called a monomial division associated to a fixed toric Kähler form on $(\mathbb{C}^*)^n$ and a fixed moment map $\mu : (\mathbb{C}^*)^n \to \mathbb{R}^n$ ([Han19, Definition 2.1, Definition 3.18]). Roughly speaking, the objects are Lagrangians in $(\mathbb{C}^*)^n$ subject to admissibility conditions given by $\Delta$. Now, for the subcategory

$$\mathcal{F}^s_\Delta(W_\Sigma) \subset \mathcal{F}_\Delta(W_\Sigma)$$

of Lagrangian sections of the torus fibration $\mu$, Hanlon constructed a symmetric monoidal structure $\otimes_{\Delta,\Sigma}$ on the homotopy category $\mathsf{Ho}(\mathcal{F}^s_\Delta(W_\Sigma))$ so that there is a symmetric monoidal equivalence

$$(\mathsf{Lin}\ X_\Sigma, \otimes_{X_\Sigma}) \simeq (\mathsf{Ho}(\mathcal{F}^s_\Delta(W_\Sigma)), \otimes_{\Delta,\Sigma})$$

where $(\mathsf{Lin}\ X_\Sigma, \otimes_{X_\Sigma})$ denotes the symmetric monoidal full subcategory of $(\mathsf{Perf}\ X_\Sigma, \otimes_{X_\Sigma})$ on line bundles [Han19, Corollary 4.6]. Since $X_\Sigma$ is assumed to be smooth (in particular, divisorial) and hence $\mathsf{Lin}\ X$ split-generates $\mathsf{Perf}\ X = \mathsf{D}^b_{\mathsf{coh}}(X)$ by Proposition 4.2.17, any extension $\hat{\otimes}_{\Delta,\Sigma}$ of $\otimes_{\Delta,\Sigma}$ to $\mathsf{D}^\pi\mathcal{F}^s_\Delta(W_\Sigma)$ agrees with $\otimes_X$ on objects under homological mirror symmetry, i.e., we have

$$X_\Sigma \cong \mathrm{Spec}_{\otimes_{X_\Sigma}} \mathsf{Perf}\ X_\Sigma \cong \mathrm{Spec}_{\hat{\otimes}_{\Delta,\Sigma}} \mathsf{D}^\pi\mathcal{F}^s_\Delta(W_\Sigma).$$

In fact, the symmetric monoidal structure $\otimes_{\Delta,\Sigma}$ on $\mathsf{Ho}(\mathcal{F}^s_\Delta(W_\Sigma)) \simeq \mathsf{Lin}\ X$ is constructed by identifying multi-polarizations on $\mathcal{F}_\Delta(W_\Sigma)$ corresponding to Pic $X$ under the equivalence, so it is indeed more direct to construct $X_\Sigma$ as a fixed locus instead of as a tt-spectrum.

In the rest of this section, we will consider spherical twists and some other classes of autoequivalences together with their corresponding fixed loci. We will particularly focus on homological mirror symmetry for log Calabi–Yau surfaces as in [HK21] and [HK23]. Our main claim in this section is Proposition 5.3.15, which reformulates results in [HK21] and realizes algebraic mirror partners as fixed loci of Fukaya categories. Readers with familiarity in [HK21] and [HK23] can safely skip to Proposition 5.3.14.

**Notation 5.3.4.** In the rest of this section, we will assume an algebraic mirror partner $X$ is smooth so that homological mirror symmetry means an equivalence $\mathsf{Perf}\ X = \mathsf{D}^b_{\mathsf{coh}}(X) \simeq \mathsf{Fuk}(M,\omega)$ for simplicity. However, even without the assumption, we can actually generalize the arguments below to any (divisorial) variety $X$ by recalling $\mathsf{Perf}\ X$ can be constructed from $\mathsf{D}^b_{\mathsf{coh}}(X)$ as locally finite homological functors on $\mathsf{D}^b_{\mathsf{coh}}(X)$ by [Nee22, Theorem 6.6.(2)] (or by [Rou08, Corollary 7.51] if $X$ is moreover projective). Hence, in those cases, we can replace $\mathsf{Fuk}(M,\omega)$ with the corresponding triangulated category $\mathsf{Fuk}(M,\omega)^\vee$ of locally finite homological functors so that we have $\mathsf{Perf}\ X \simeq \mathsf{Fuk}(M,\omega)^\vee$. Indeed, by construction, any pt-equivalence

$$(\mathsf{D}^b_{\mathsf{coh}}(X), \tau) \simeq (\mathsf{Fuk}(M,\omega), \sigma)$$

restricts to a pt-equivalence

$$(\mathsf{Perf}\ X, \tau|_{\mathsf{Perf}\ X}) \simeq (\mathsf{Fuk}(M,\omega)^\vee, \sigma|_{\mathsf{Fuk}(M,\omega)^\vee}).$$

Now, let us first begin with geometrically understanding some classes of spherical twists on $\mathsf{Fuk}(M,\omega)$.

**Example 5.3.5** (Dehn twists and spherical twists)**.** To a **framed exact Lagrangian sphere** $S$ in an exact symplectic manifold $(M,\omega)$ (i.e., an exact Lagrangian submanifold with a fixed diffeomorphism to a sphere), we can associate an automorphism $\tau_S$ of $(M,\omega)$, called the **Dehn twist** along $S$ (cf. e.g., [Sei08, (16c)]), which induces the autoequivalence $\tau_{S*}$ on the Fukaya category $\mathsf{Fuk}(M,\omega)$. In [ST01], it is observed that a Lagrangian sphere is a spherical object in the split-closed derived Fukaya category $\mathsf{D}^\pi\,\mathsf{Fuk}(M,\omega)$ ([ST01, 1c]) and thus we can construct the associated spherical twists $T_S$. Now, by [Sei08, Corollary 17.17], we see that over $\mathsf{D}^\pi\,\mathsf{Fuk}(M,\omega)$, the autoequivalence $\tau_{S*}$ agrees with the spherical twist $T_S$ on objects and hence, independently of choices of a framing on $S$, we have

$$\mathrm{Spec}^{\tau_{S*}}\,\mathsf{D}^\pi\,\mathsf{Fuk}(M,\omega) = \mathrm{Spec}^{T_S}\,\mathsf{D}^\pi\,\mathsf{Fuk}(M,\omega).$$

On the other hand, since fixed loci given by spherical twists can behave in various ways (cf. Example 4.2.12 and Example 4.2.20), we cannot expect an arbitrary choice of Dehn twists give a multi-polarization that constructs an algebraic mirror. In particular, note that over a variety of dimension $\geq 2$, spherical twists will never agree with tensoring with line bundles by combining [Ito23, Corollary 5.15] with the fact that skyscraper sheaves on a (smooth) variety of dimension $\geq 2$ are not spherical. Nevertheless, it will be interesting to understand the relation of the geometry of fixed loci and Lagrangian spheres in the case of elliptic curves, where we already have explicit computations.

As we just observed above, we are interested in automorphisms of a symplectic manifold since they are supposed to induce autoequivalences of $\mathsf{Fuk}(M, \omega)$. In [HK21], more classes of automorphisms, called Lagrangian translations and nodal slide recombinations, are defined and studied for Weinstein 4-manifolds that are mirrors to log Calabi–Yau surfaces. Since Lagrangian translations correspond to tensoring with line bundles and thus are particularly relevant to us, let us explain them more in detail.

**Definition 5.3.6.** A smooth quasi-projective surface $U$ is a **log Calabi–Yau surface with maximal boundary** if there exist a smooth rational projective surface $Y$ and a singular nodal curve $D \in |-K_Y|$ such that $U = Y \setminus D$. In particular, $D$ needs to be either an irreducible rational nodal curve or a cycle of smooth rational curves. We may also refer to such a pair $(Y, D)$ as a log Calabi–Yau surface with maximal boundary.

**Example 5.3.7.** A most basic example is when $Y$ is a smooth projective toric surface and $D$ is its torus boundary divisor consisting of a cycle of rational curves. Note in this case we have $U = Y \setminus D$ is a torus $\mathbb{G}_m^2$. On the other hand, when a smooth projective toric surface $Y$ contains a $(-2)$-curve $C$ and $-K_Y$ is nef (e.g., $Y = \mathbb{F}_2$), we can find an irreducible rational nodal curve $D \in |-K_Y|$ so that $C \subset Y \setminus D$ as follows. First, since $-K_Y$ is nef (and hence base-point-free on a toric surface), we can take distinct general smooth irreducible members $D_1, D_2 \in |-K_Y|$ not containing $C$ (and hence disjoint from $C$ as $K_Y \cdot C = 0$). If we consider the pencil formed by $D_1$ and $D_2$, then a generic member is an elliptic curve by the adjunction formula for arithmetic genus. Now, if we blow up finitely many base points, then we obtain a rational elliptic surface, as we may assume base points are simple and hence exceptional divisors give sections. Since a generic singular fiber $D$ is of Kodaira type $I_1$ (i.e., an irreducible rational nodal curve as $g_a(D) = 1$), we obtain a desired rational curve $D \in |-K_Y|$ by taking the pencil to be general enough, as singular fibers are characterized by vanishing of the discriminant and a fiber of type $I_1$ corresponds to a simple zero (see [SS19] for an exposition of (rational) elliptic surfaces). In particular, $(Y, D)$ is a log Calabi–Yau surface with maximal boundary and $Y \setminus D$ is not quasi-affine as it contains a $(-2)$-curve.

We also have more examples of non-quasi-affine log Calabi–Yau surfaces with maximal boundary.

**Example 5.3.8.** Let $Y$ be a rational elliptic surface with a section and a singular fiber (e.g., the blow-up of $\mathbb{P}^2$ at the 9 base points of a cubic pencil). Note in this case $-K_X = F$ where $F$ is a fiber class. In particular, if $|F|$ has a fiber $D$ of Kodaira type $I_n$ $(n \geq 1)$, then $(Y, D)$ is a log Calabi–Yau surface with maximal boundary. Also note $U = Y \setminus D$ is not quasi-affine in this case since $U$ contains other fibers.

**Notation 5.3.9.** Let $(Y, D)$ be a log Calabi–Yau surface with maximal boundary. Within its complex deformation class, there exists a distinguished class in the sense that it induces a split mixed Hodge structure. See [HK23, Section 2.2] for more details. In the sequel, we assume that a log Calabi–Yau surface with maximal boundary is equipped with this distinguished complex structure.

In this setting, we have the following homological mirror symmetry ([HK23, Theorem 1.1]).

**Theorem 5.3.10** (Hacking-Keating)**.** *Let $(Y, D)$ be a log Calabi–Yau surface with maximal boundary (and with distinguished complex structure). Then, there is a Weinstein 4-fold $M$ together with a Lefschetz fibration $w : M \to \mathbb{C}$ such that there are compatible equivalences*

$$\mathsf{Perf}\ D = \mathsf{D}^{\pi}\,\mathsf{Fuk}(\Sigma) \quad \mathsf{Perf}\ Y \simeq \mathsf{D}^{b}\mathsf{Fuk}(w) \quad \mathsf{Perf}\ U \simeq \mathsf{D}^{b}\mathcal{W}\mathsf{Fuk}(M),$$

*where $\mathsf{Fuk}(\Sigma)$ denotes the Fukaya category of a smooth fiber $\Sigma$ of $w$ near infinity, $\mathsf{Fuk}(w)$ denotes the directed Fukaya category of $w$ and $\mathcal{W}\mathsf{Fuk}(M)$ denotes the wrapped Fukaya category of $M$.*

Now, let us recall constructions of Lagrangian translations introduced in [HK21].

**Construction 5.3.11** (Lagrangian translations)**.** Suppose $M$ is an exact symplectic 4-manifold that is the total space of an almost toric fibration

$$\pi : M \to B$$

only with focus-focus singularities (cf. e.g., [Sym03, Definition 4.2]), where $B$ is an integral affine manifold homeomorphic to a disk. Further suppose there is a fixed Lagrangian section $L_0$ for $\pi$. Those settings are applicable when $M$ is a mirror to a log Calabi–Yau surface. Now, for the smooth locus $M^{\mathsf{sm}}$ of $\pi$, there are so called global action-angle coordinates

$$\mathfrak{p}_{L_0} : T^*B \to M^{\mathsf{sm}}$$

so that over the zero section of $T^*B$, it functions as the fixed Lagrangian section $L_0$. Roughly speaking, $\mathfrak{p}_{L_0}$ takes $(q, p) \in T^*B$ to a point in $M^{\mathsf{sm}}$ given by transporting $L_0(q) \in M$ along the time-one Hamiltonian flow prescribed by $p \in T^*_qB$. Now, for a Lagrangian section $L : B \to M$, we define

$$\sigma_L : M^{\mathsf{sm}} \to M^{\mathsf{sm}}, \quad x \mapsto \mathfrak{p}_{L_0}\left(\tilde{x} + (L(\pi(x)))^{\sim}\right),$$

where $\tilde{y} \in T^*B$ denote any preimage of $y$ under $\mathfrak{p}_{L_0}$. Intuitively, we are adding $L - L_0$ at each fiber using the linear structure. By [HK21, Proposition 4.11], $\sigma_L$ is well-defined and extends to a symplectomorphism of $M$, which is called the **Lagrangian translation** by $L$ (with respect to $L_0$).

The following are the main results in [HK21, Corollary 4.21, Theorem 5.5].

**Theorem 5.3.12** (Hacking–Keating)**.** *Let $(Y, D)$ be a log Calabi–Yau surface with maximal boundary and set $U := Y \setminus D$. Define the subgroup $Q := \{\mathscr{L} \in \operatorname{Pic} Y \mid \mathscr{L}|_D \cong \mathscr{O}_D\} \subset \operatorname{Pic} Y$. Fix a toric model of $(Y, D)$ and consider the corresponding almost toric fibration $\pi : M \to B$ for the mirror partner $(M, w)$ (cf.* [HK23, Section 1.1]*). Further, fix a Lagrangian section $L_0$ of $\pi$. Then, the following assertions hold.*

*(i) There exists a canonical diffeomorphism $\iota : U \to M$ (cf.* [HK21, Lemma 4.20]*) that induces bijections*

$$\mathrm{Pic}(U) \xrightarrow{\sim} \left\{ \begin{array}{l} \textit{Lagrangian sections of } \pi \textit{ up to fiber-} \\ \textit{preserving Hamiltonian isotopy} \end{array} \right\} =: \mathsf{Lag}_\pi$$

$$Q \xrightarrow{\sim} \left\{ \begin{array}{l} \textit{Lagrangian sections of } \pi \textit{ equal to } L_0 \textit{ near} \\ \partial M \textit{ up to fiber preserving Hamiltonian} \\ \textit{isotopy} \end{array} \right\} =: \mathsf{Lag}_{\pi,L_0}$$

(ii) *Take a line bundle $\mathscr{L} \in Q$ and the corresponding Lagrangian section $L \in \mathsf{Lag}_{\pi,L_0}$ as above. Then, $\sigma_L$ induces an autoequivalence $\sigma_{L*}$ of $\mathsf{D}^b\mathcal{W}\mathsf{Fuk}(M)$ and under homological mirror symmetry (Theorem 5.3.10), $\tau_{\mathscr{L}|_U}$ corresponds to $\sigma_L$, i.e., we have a pt-equivalence*

$$(\mathsf{Perf}\, U, \tau_{\mathscr{L}|_U}) \simeq (\mathsf{D}^b\mathcal{W}\mathsf{Fuk}(M), \sigma_{L*}).$$

*Remark* 5.3.13. Use the same notations as in Theorem 5.3.12. The action of $Q$ on $\mathsf{Perf}\, U$ only depends on the image $\bar{Q} \subset \mathrm{Pic}\, U$ of $Q$ under the restriction $\mathrm{Pic}\, Y \to \mathrm{Pic}\, U$. We have $Q \cong \bar{Q}$ if and only if $D$ is negative definite or indefinite, i.e., the intersection form associated to $D_i$ is negative definite or indefinite when we write $D = D_1 + \cdots + D_n$ where $D_i$ are irreducible components that are necessarily rational curves (cf. [HK21, Section 2.3] for more discussions). In such a case, by [HK21, Remark 5.6], we have a pt-equivalence

$$(\mathsf{Perf}\, Y, \tau_{\mathscr{L}}) \simeq (\mathsf{D}^b\mathsf{Fuk}(w), \sigma_{L*}).$$

As a consequence, we get the following result.

**Proposition 5.3.14.** *Use the same notations as in Theorem 5.3.12. Then, the diffeomorphism $\iota : U \to M$ and homological mirror symmetry induces a multi-pt-equivalence*

$$(\mathsf{Perf}\, U, \tau_Q) \simeq (\mathsf{D}^b\mathcal{W}\mathsf{Fuk}(M), \tau_{\pi,\partial}),$$

*where we set the multi-polarizations $\tau_Q$ and $\tau_{\pi,\partial}$ to be*

$$\tau_Q := \{\tau_{\mathscr{L}|_U} \mid \mathscr{L} \in Q\} \subset \mathrm{Auteq}\, \mathsf{Perf}\, U, \quad \tau_{\pi,\partial} := \{\sigma_{L*} \mid L \in \mathsf{Lag}_{\pi,L_0}\} \subset \mathrm{Auteq}\, \mathsf{D}^b\mathcal{W}\mathsf{Fuk}(M).$$

*In particular, the multi-polarization $\tau_{\pi,\partial}$ does not depend on the choice of $L_0$.*

*Proof.* By Theorem 5.3.12, homological mirror symmetry induces the following commutative diagram:

$$\begin{tikzcd} Q \arrow[r, two heads] \arrow[d, "\cong"'] & \tau_Q \arrow[r, hook] \arrow[d] & \mathrm{Auteq}\, \mathsf{Perf}\, U \arrow[d, "\cong"] \\ \mathsf{Lag}_{\pi,L_0} \arrow[r, two heads] & \tau_{\pi,\partial} \arrow[r, hook] & \mathrm{Auteq}\, \mathsf{D}^b\mathcal{W}\mathsf{Fuk}(M) \end{tikzcd}$$

which proves the first claim. Since homological mirror symmetry functor does not depend on the choice of $L_0$, the multi-polarization $\tau_{\pi,\partial}$ is determined as the image of $\tau_Q$ under homological mirror symmetry and does not depend on the choice of $L_0$. □

As a further consequence, we can obtain a construction of a mirror algebraic partner as a fixed locus, which may be viewed as a partial converse of the main result of [GHK15].

**Proposition 5.3.15.** *With the same notations as in Theorem 5.3.12, we have*

$$U \cong \mathrm{Spec}^{\tau_{\pi,\partial}} \mathsf{D}^b\mathcal{W}\mathsf{Fuk}(M).$$

*Moreover, if there exists an $M$-compatible tt-structure $\otimes_M$ on $\mathsf{D}^b\mathcal{W}\mathsf{Fuk}(M)$ such that its unit $\mathbb{1}_M$ corresponds to the structure sheaf $\mathcal{O}_U$ under homological mirror symmetry and that for any $L, L' \in \mathsf{Lag}_{\pi,L_0}$, we have*

$$\sigma_L(\sigma_{L'}(\mathbb{1}_M)) \cong \sigma_L(\mathbb{1}_M) \otimes_M \sigma_{L'}(\mathbb{1}_M),$$

*then we have*

$$U \cong \mathrm{Spec}^{\tau_{\pi,\partial}} \mathsf{D}^b\mathcal{W}\mathsf{Fuk}(M) \cong \mathrm{Spec}_{\otimes_M} \mathsf{D}^b\mathcal{W}\mathsf{Fuk}(M).$$

*Proof.* First, we claim $\bar{Q}$ contains an ample line bundle. Indeed, take a line bundle $\mathscr{L}$ on $Y$ that restricts to an ample line bundle on $U$ (which exists as the restriction $\mathrm{Pic}\, Y \to \mathrm{Pic}\, U$ is surjective). If $\mathscr{L}|_D$ is not trivial on some irreducible components of the boundary divisor, then we can twist by divisors corresponding to such components so that the restriction to $D$ is trivial without changing the restriction to $U$. So, we may assume $\mathscr{L} \in Q$ and hence we have an ample line bundle $\mathscr{L}|_U \in \bar{Q}$. Now, since we have $\mathrm{Spec}_{\otimes_U} \mathsf{Perf}\, U \subset \mathrm{Spec}^{\tau_{\mathscr{L}}} \mathsf{Perf}\, U$ for any $\tau_{\mathscr{L}} \in \tau_Q \subset \mathrm{Pic}\, U$ and $\bar{Q}$ contains an ample line bundle on $U$, we get

$$U \cong \mathrm{Spec}^{\tau_Q} \mathsf{Perf}\, U \cong \mathrm{Spec}^{\tau_{\pi,\partial}} \mathsf{D}^b\mathcal{W}\mathsf{Fuk}(M).$$

Moreover, since under homological mirror symmetry, we see $\langle \sigma(\mathbb{1}_M) \mid \sigma \in \tau_{\pi,\partial}\rangle = \mathsf{D}^\pi\mathcal{W}\mathsf{Fuk}(\mathsf{M})$ and hence by the same argument as in Theorem 4.2.18, we have

$$\mathrm{Spec}^{\tau_{\pi,\partial}} \mathsf{D}^b\mathcal{W}\mathsf{Fuk}(M) \cong \mathrm{Spec}_{\otimes_M} \mathsf{D}^b\mathcal{W}\mathsf{Fuk}(M).$$ □

*Remark* 5.3.16. Use the same notations as in Theorem 5.3.12.

(i) If $U$ is quasi-affine, then we already have $\mathrm{Spec}_\vartriangle \mathsf{D}^b\mathcal{W}\mathsf{Fuk}(M) \cong U$ by [Mat23, Corollary 4.7]. However, as we have seen in examples, there are plenty of cases where $U$ is not quasi-affine.

(ii) Recall from Remark 5.3.13 (cf. [HK21, Remark 5.6, Remark 5.7])) that if $D$ is negative-definite or indefinite, then for any $\mathscr{L} \in Q$ and the corresponding $L \in \mathsf{Lag}_{\pi,L_0}$, we have a pt-equivalence

$$(\mathsf{Perf}\, Y, \tau_{\mathscr{L}}) \simeq (\mathsf{D}^b\mathsf{Fuk}(w), \sigma_{L*}).$$

Thus, we know that we have

$$Y \cong \mathrm{Spec}_{\otimes_Y} \mathsf{Perf}\, Y \underset{\text{open}}{\subset} \mathrm{Spec}^{\tau_Q} \mathsf{Perf}\, Y \cong \mathrm{Spec}^{\tau_{\pi,\partial}} \mathsf{D}^b\mathsf{Fuk}(w),$$

where $\tau_Q$ and $\tau_{\pi,\partial}$ are naturally extended to $\mathsf{Perf}\, Y$ and $\mathsf{D}^b\mathcal{W}\mathsf{Fuk}(w)$. A natural question is if we can add more polarizations to $\tau_{\pi,\partial}$ to cut out $Y$ while only looking at the symplectic side. Note, by passing to $\mathsf{D}^b\mathsf{Fuk}(w) \simeq \mathsf{Perf}\, Y$, we "compactify" $\mathsf{D}^b\mathcal{W}\mathsf{Fuk}(M) \simeq \mathsf{Perf}\, U$ so that we have the Serre functor $\mathbb{S}_w = \mathbb{S}_Y$. In particular, we have a multi-polarization $\bar{\tau}_{\pi,\partial} := \tau_{\pi,\partial} \cup \{\mathbb{S}_w\}$ on $\mathsf{D}^b\mathsf{Fuk}(w)$. If an algebraic mirror $Y$ is a toric surface (or more generally a surface of general type), we conjecture

$$Y \cong \mathrm{Spec}^{\bar{\tau}_{\pi,\partial}} \mathsf{D}^b\mathsf{Fuk}(w)$$

(if and) only if $\omega_Y = \mathcal{O}_Y(-D)$ is reconstructive. In particular, unless the compactification is nice, we expect to need more polarizations to cut out the tt-spectrum.

(iii) In general, suppose that we have an $M$-compatible tensor triangulated structure $\otimes_M$ on the Fukaya-like category constructed from an almost torus fibration. We expect we can apply similar strategies as above to show $\otimes_M$ is spectrally identical to the tt-structures $\otimes_X$ of its algebraic mirror $X$ if there are enough Lagrangian sections (so that they correspond to an ample family) and tensor product of those sections are given by certain modifications of fiber-wise additions.

Let us finish this subsection with plenty of natural questions arising from our observations.

**Question 5.3.17.** Let $M$ be a Weinstein 4-fold and suppose there is an almost-torus fibration $\pi : M \to B$ with a fixed Lagrangian section $L_0$.

(i) Can we characterize an almost toric fibration $\pi' : M \to B$ for which we have an isomorphism

$$\mathrm{Spec}^{\tau_{\pi,\partial}} \mathsf{D}^b\mathcal{W}\mathsf{Fuk}(M) \cong \mathrm{Spec}^{\tau_{\pi',\partial}} \mathsf{D}^b\mathcal{W}\mathsf{Fuk}(M)?$$

More generally, given a multi-polarization $\mathrm{Pic}(X)$ on $\mathsf{Perf}\, X \simeq \mathsf{Fuk}(M)$, can we construct the corresponding Lagrangian fibration of $M$?

(ii) If $\mathrm{Spec}^{\tau_{\pi,\partial}} \mathsf{D}^b\mathcal{W}\mathsf{Fuk}(M)$ is a log Calabi–Yau surface with maximal boundary, then is it a mirror partner of $M$? In other words, can we predict mirror partners of $M$ using fixed loci?

(iii) Can we give symplecto-geometric formulae for prime thick subcategories (or even prime $\otimes$-ideals) in $\mathrm{Spec}^{\tau_{\pi,\partial}} \mathsf{D}^b\mathcal{W}\mathsf{Fuk}(M)$?

(iv) Using such formulae, can we construct $\mathrm{Spec}^{\tau_{\pi,\partial}} \mathsf{D}^b\mathcal{W}\mathsf{Fuk}(M)$ more directly from $M$ without fully going through the wrapped Fukaya category?

## 5.4 Polarizations as local systems of stable $\infty$-categories on $S^1$

In order to make the comparison of pt-spectra with relative Matsui spectra precise, let us set up the $\infty$-category version of pt-categories.

**Notation 5.4.1.** Let $k$ be a (discrete) field (while many results work over any $\mathbb{E}_\infty$-ring spectrum). Let $\mathcal{S}$ denotes the $\infty$-category of (small) spaces and let $\mathsf{Cat}_\infty$ denote the $\infty$-category of (small) $\infty$-categories.

(i) Let $\mathsf{Pr}^{\mathsf{L}}$ denote the $\infty$-category of presentable $\infty$-categories and colimit-preserving functors, which has a natural symmetric monoidal structure (cf. [Lur17, Proposition 4.8.1.15]), and let $\mathsf{Cat}^{\mathsf{perf}}$ denote the $\infty$-category of small **perfect** (i.e., stable and idempotent-complete) $\infty$-categories with exact functors. By [BGT14, Lemma 4.5], the presentable $\infty$-category $\mathsf{Cat}^{\mathsf{perf}}$ has a natural symmetric monoidal structure that preserves small colimits in each variable, i.e., $\mathsf{Cat}^{\mathsf{perf}} \in \mathsf{CAlg}(\mathsf{Pr}^{\mathsf{L}})$. From [BGT13, §3.1], for example, recall that we have the symmetric monoidal equivalence

$$\mathsf{Ind} : \mathsf{Cat}^{\mathsf{perf}} \leftrightarrows \mathsf{Pr}^{\mathsf{L}}_{\mathsf{st},\omega} : (-)^{\omega}$$

given by the Ind-completion and taking compact objects, where $\mathsf{Pr}^{\mathsf{L}}_{\mathsf{st},\omega}$ denotes the $\infty$-category of stable compactly generated $\infty$-categories together with functors that preserve colimits and compact objects (i.e., $\mathsf{Pr}^{\mathsf{L}}_{\mathsf{st},\omega}$ is not a full subcategory of $\mathsf{Pr}^{\mathsf{L}}$).

(ii) Let

$$\mathsf{Cat}^{\mathsf{perf}}_k := \mathsf{Mod}_{\mathsf{Perf}(k)}(\mathsf{Cat}^{\mathsf{perf}})$$

denote the $\infty$-category of $k$-linear small perfect $\infty$-categories. Since $\mathsf{Perf}\, k \in \mathsf{CAlg}(\mathsf{Cat}^{\mathsf{perf}})$, we see that $\mathsf{Cat}^{\mathsf{perf}}_k$ is presentable by [Lur17, Corollary 4.2.3.7]. Moreover, since $\mathsf{Cat}^{\mathsf{perf}} \in \mathsf{CAlg}(\mathsf{Pr}^{\mathsf{L}})$ satisfies the supposition of [Lur17, Theorem 4.5.2.1] (as the geometric realization is a small colimit), $\mathsf{Cat}^{\mathsf{perf}}_k$ is a symmetric monoidal $\infty$-category with the natural relative monoidal structure. Again since $\mathsf{Cat}^{\mathsf{perf}} \in \mathsf{CAlg}(\mathsf{Pr}^{\mathsf{L}})$ satisfies the supposition of [Lur17, Corollary 4.4.2.16] for each small simplicial set $K$, we see that the relative monoidal structure preserves all small colimits in each variable. Therefore, we naturally have

$$\mathsf{Cat}^{\mathsf{perf}}_k \in \mathsf{CAlg}(\mathsf{Pr}^{\mathsf{L}}).$$

Letting $\mathsf{Pr}^{\mathsf{L}}_{k,\omega} = \mathsf{Mod}_{\mathsf{IndPerf}(k)}(\mathsf{Pr}^{\mathsf{L}}_{\mathsf{st},\omega})$ denote the $\infty$-category of stable $k$-linear compactly generated $\infty$-categories with morphisms given by $k$-linear functors that preserve colimits and compact objects, we, by construction, have the equivalence

$$\mathsf{Ind} : \mathsf{Cat}^{\mathsf{perf}}_k \leftrightarrows \mathsf{Pr}^{\mathsf{L}}_{k,\omega} : (-)^{\omega}.$$

We view $\mathsf{Pr}^{\mathsf{L}}_{k,\omega}$ as a symmetric monoidal $\infty$-category through this equivalence. **In the sequel, perfect $\infty$-categories are meant to be $k$-linear small perfect $\infty$-categories.** Set $\mathsf{Mod}\, k := \mathsf{Ind}\,\mathsf{Perf}\, k$.

(iii) Applying [Lur17, Proposition 3.4.2.1] to the symmetric monoidal $\infty$-category $\mathsf{Pr}^{\mathsf{L}}$ with its unit $\mathcal{S}$ ([Lur17, Corollary 3.2.1.9, Example 4.8.1.20]), the forgetful functor induces a symmetric monoidal equivalence

$$\mathsf{Mod}_{\mathcal{S}}(\mathsf{Pr}^{\mathsf{L}}) \simeq \mathsf{Pr}^{\mathsf{L}}.$$

Moreover, by [Lur17, Corollary 3.4.1.7] (see also the proof of [PPS25, Lemma 2.2]), we have the equivalence

$$\mathsf{CAlg}(\mathsf{Pr}^{\mathsf{L}}) \simeq \mathsf{CAlg}(\mathsf{Mod}_{\mathcal{S}}(\mathsf{Pr}^{\mathsf{L}})) \simeq \mathsf{CAlg}(\mathsf{Pr}^{\mathsf{L}})_{\mathcal{S}/}.$$

Therefore, given a presentably symmetric monoidal $\infty$-category $\mathcal{A}$, there uniquely (up to contractible choices) exists a symmetric monoidal and colimit-preserving functor

$$- \otimes 1_{\mathcal{A}} : \mathcal{S} \to \mathcal{A}$$

sending $\{*\}$ to $1_{\mathcal{A}}$, where $\otimes$ denotes the canonical $\mathcal{S}$-module action on $\mathcal{A}$ coming from the underlying equivalence of $\mathsf{Mod}_{\mathcal{S}}(\mathsf{Pr}^{\mathsf{L}}) \simeq \mathsf{Pr}^{\mathsf{L}}$. For any presentable $\infty$-category $\mathcal{C}$, equipped with the canonical left $\mathcal{S}$-module structure (see also [Lur17, Proposition 4.2.4.9]), and for $A \in \mathsf{Alg}_{\mathbb{A}_\infty}(\mathcal{S}) \simeq \mathsf{Alg}_{\mathbb{E}_1}(\mathcal{S})$, we define the $\infty$-category $\mathsf{LMod}_A(\mathcal{C})$ of left $A$-modules in $\mathcal{C}$.

(iv) For $n \in \mathbb{Z}_{\geq 1}$, recall that by [GH15, Theorem 6.3.2] together with discussions in [GH15, Definition 6.3.10] applied to $\mathcal{V} = \mathcal{S}$ (noting $\mathsf{Cat}_\infty^{\mathcal{S}} \simeq \mathsf{Cat}_\infty$, e.g., by [GH15, Theorem 5.4.6]), we have the monoidal fully faithful embedding

$$\mathsf{B} : \mathsf{Alg}_{\mathbb{E}_n}(\mathcal{S}) \hookrightarrow \mathsf{Alg}_{\mathbb{E}_{n-1}}(\mathsf{Cat}_\infty)$$

where $\mathsf{B}M$ is the single-object $\infty$-category (with the object denoted by $*$) together with $\mathrm{Hom}_{\mathsf{B}M}(*,*) = M$ (cf. [GH15, Remark 6.3.5]). Furthermore, if $M$ is **group-like**, i.e., $\pi_0(M)$ is a group (cf. [Lur17, Example 5.2.6.4]), then the $\infty$-groupoid $\mathsf{B}M$ is equivalent to (the fundamental $\infty$-groupoid of) the classifying space $BM$ (cf. [Lur09, Remark 1.2.5.2]).

**Definition 5.4.2.** For $\infty$-categories $X$ and $\mathcal{C}$, define the $\infty$-category of **local systems** on $X$ with coefficients in $\mathcal{C}$ to be

$$\mathsf{Loc}(X;\mathcal{C}) := \mathsf{Fun}(X,\mathcal{C}),$$

where $\mathsf{Fun}$ denotes the $\infty$-category of $\infty$-functors. For an $\mathbb{E}_n$-monoid $M$ in $\mathcal{S}$ (for $n \geq 1$), define the $\infty$-category of **$M$-polarized perfect $\infty$-categories** to be

$$\mathsf{pCat}_k^{\mathsf{perf}}(M) := \mathsf{Loc}(\mathsf{B}M;\mathsf{Cat}_k^{\mathsf{perf}}).$$

Note that $\mathsf{pCat}_k^{\mathsf{perf}}(M)$ is naturally $k$-linear presentably $\mathbb{E}_{n-1}$-monoidal, i.e.,

$$\mathsf{pCat}_k^{\mathsf{perf}}(M) \in \mathsf{Alg}_{\mathbb{E}_{n-1}}(\mathsf{Pr}_k^{\mathsf{L}})$$

by (the opposite of) the Day convolution (cf. [PPS25, Proposition 2.8, Remark 2.9]).

Let us first do a sanity check for discrete free monoids.

**Example 5.4.3** (Semi-polarized $\infty$-categories)**.** Let $F_E$ be the (discrete) free monoid generated by a set $E$. Then, by [Lur26, Tag 00JD] (or [Lur26, Tag 00JC] when $F_E = \mathbb{Z}_{\geq 0}$), the $\infty$-category $\mathsf{pCat}_k^{\mathsf{perf}}(M)$ is equivalent to the $\infty$-category of perfect $\infty$-categories $\mathcal{C}$ together with endofunctors $\{\tau_e : \mathcal{C} \to \mathcal{C}\}_{e\in E}$, under the functor sending $\Phi \in \mathsf{pCat}_k^{\mathsf{perf}}(M)$ to $(\Phi(*), \{\Phi(e)\}_{e\in E})$. In other words, such a pair $(\mathcal{C}, \{\tau_e\}_{e\in E})$ admits a unique extension (up to contractible choices) to a functor $\mathsf{B}F_E \to \mathsf{Cat}_k^{\mathsf{perf}}$. In particular, we call $\mathsf{pCat}_k^{\mathsf{perf}}(\mathbb{Z}_{\geq 0})$ the $\infty$-category of **semi-polarized** perfect $\infty$-categories, whose objects will be denoted by pairs $(\mathcal{C}, \tau)$ of a perfect $\infty$-category $\mathcal{C}$ and its endofunctor $\tau$ using the equivalence above. This is a natural $\infty$-categorical enhancement of semi-polarized triangulated categories (cf. Definition 4.2.13). Note that under the equivalence, we also have the expected equivalence of $\infty$-groupoids:

$$\mathsf{Map}_{\mathsf{pCat}_k^{\mathsf{perf}}(\mathbb{Z}_{\geq 0})}((\mathcal{C}_1, \tau_1), (\mathcal{C}_2, \tau_2)) \simeq \mathsf{Eq}\left(\mathsf{Map}_{\mathsf{Cat}_k^{\mathsf{perf}}}(\mathcal{C}_1, \mathcal{C}_2) \underset{\tau_2\circ -}{\overset{-\circ\tau_1}{\rightrightarrows}} \mathsf{Map}_{\mathsf{Cat}_k^{\mathsf{perf}}}(\mathcal{C}_1, \mathcal{C}_2)\right).$$

The following interpretation is quite well-known to many people, but let us record precise arguments for readers' convenience.

**Lemma 5.4.4.** *The natural functor*

$$\mathsf{B}\mathbb{Z}_{\geq 0} \to \mathsf{B}\mathbb{Z}$$

*induced by the monoid inclusion $\mathbb{Z}_{\geq 0} \hookrightarrow \mathbb{Z}$ exhibits $\mathsf{B}\mathbb{Z}$ as the localization $\mathsf{B}\mathbb{Z}_{\geq 0}$ with respect to $\{1\}$ (in the sense of* [Lur26, Tag 01MP]*). In particular, the induced functor*

$$\mathsf{pCat}_k^{\mathsf{perf}}(\mathbb{Z}) \hookrightarrow \mathsf{pCat}_k^{\mathsf{perf}}(\mathbb{Z}_{\geq 0})$$

*is fully faithful and the essential image is equivalent to the $\infty$-category of pairs $(\mathcal{C}, \tau)$ of a perfect $\infty$-category $\mathcal{C}$ and its autoequivalence $\tau$ under the equivalence of Example 5.4.3.*

*Proof.* First, note that by definition (cf. [Lur26, Tag 01MF, Tag 01MP]), the localization of $\mathsf{B}\mathbb{Z}_{\geq 0}$ with respect to $\{1\}$ is the same as the localization with respect to all edges, as a functor $\mathsf{B}\mathbb{Z}_{\geq 0} \to \mathcal{C}$ sends 1 to an equivalence if and only if it sends all $n \in \mathbb{Z}_{\geq 0}$ to equivalences. Now, we see $\mathsf{B}\mathbb{Z}_{\geq 0} \to \mathsf{B}\mathbb{Z}$ is a weak homotopy equivalence. By Quillen's theorem A [Lur26, Tag 02P0], it suffices to show the nerve of $\mathsf{Ho}(\mathsf{B}\mathbb{Z}_{\geq 0}) \times_{\mathsf{Ho}(\mathsf{B}\mathbb{Z})} \mathsf{Ho}(\mathsf{B}\mathbb{Z})_{*/}$ is weakly contractible. Indeed, $\mathsf{Ho}(\mathsf{B}\mathbb{Z}_{\geq 0}) \times_{\mathsf{Ho}(\mathsf{B}\mathbb{Z})} \mathsf{Ho}(\mathsf{B}\mathbb{Z})_{*/}$ is equivalent to the (ordinary) poset $(\mathbb{Z}, \geq)$ and hence its nerve is a filtered $\infty$-category, i.e., weakly contractible by [Lur26, Tag 02PH]. Since $\mathsf{B}\mathbb{Z}$ is a Kan complex, we conclude that $\mathsf{B}\mathbb{Z}_{\geq 0} \to \mathsf{B}\mathbb{Z}$ is a localization by [Lur26, Tag 01MY].

Now, by the definition of localizations, the induced functor $\mathsf{pCat}_k^{\mathsf{perf}}(\mathbb{Z}) \hookrightarrow \mathsf{pCat}_k^{\mathsf{perf}}(\mathbb{Z}_{\geq 0})$ is fully faithful and the essential image consists of functors $\mathsf{B}\mathbb{Z}_{\geq 0} \to \mathsf{Cat}_k^{\mathsf{perf}}$ that send the morphism $1 : * \to *$ to an equivalence. Under the equivalence of Example 5.4.3, such functors correspond to pairs $(\mathcal{C}, \tau)$ of a perfect $\infty$-category $\mathcal{C}$ and its autoequivalence $\tau$. $\square$

**Example 5.4.5** (Polarized $\infty$-categories)**.** Let $M$ be the discrete monoid $\mathbb{Z}$. By Lemma 5.4.4, objects of $\mathsf{pCat}_k^{\mathsf{perf}}(\mathbb{Z})$ can be thought of being given by pairs $(\mathcal{C}, \tau)$ of a perfect $\infty$-category $\mathcal{C}$ and its autoequivalence $\tau$ and such a pair $(\mathcal{C}, \tau)$ uniquely extends to a functor $\mathsf{B}\mathbb{Z} \to \mathsf{Cat}_k^{\mathsf{perf}}$ such that $* \mapsto \mathcal{C}$ and $1 \mapsto \tau$ up to contractible choices. Also, we have an equivalence

$$\mathsf{Map}_{\mathsf{pCat}_k^{\mathsf{perf}}(\mathbb{Z})}((\mathcal{C}_1, \tau_1), (\mathcal{C}_2, \tau_2)) \simeq \mathsf{Eq}\left(\mathsf{Map}(\mathcal{C}_1, \mathcal{C}_2) \underset{\tau_2\circ -}{\overset{-\circ\tau_1}{\rightrightarrows}} \mathsf{Map}(\mathcal{C}_1, \mathcal{C}_2)\right)$$

of $\infty$-groupoids. In particular, an object of $\mathsf{Map}_{\mathsf{pCat}_k^{\mathsf{perf}}(\mathbb{Z})}((\mathcal{C}_1, \tau_1), (\mathcal{C}_2, \tau_2))$ is a functor $f : \mathcal{C}_1 \to \mathcal{C}_2$ together with an equivalence $\alpha : f \circ \tau_1 \simeq \tau_2 \circ f$. Therefore, we define

$$\mathsf{pCat}_k^{\mathsf{perf}} := \mathsf{pCat}_k^{\mathsf{perf}}(\mathbb{Z})$$

to be the $\infty$-category of **polarized perfect $\infty$-categories**, which is also the $\infty$-category of local systems of perfect $\infty$-categories on $B\mathbb{Z} \simeq S^1$ by definition.

**Example 5.4.6** (Multi-polarized categories)**.** Let $M$ be the (discrete) free group $\mathbb{Z}^{*E}$ generated by a set $E$. By the same arguments as above, we define $\mathsf{pCat}_k^{\mathsf{perf}}(\mathbb{Z}^{*E})$ to be the $\infty$-category of **multi-polarized perfect $\infty$-categories** indexed by $E$. Note on the other hand that a $\mathbb{Z}^E$-polarized perfect $\infty$-category corresponds to a perfect $\infty$-category $\mathcal{C}$ equipped with a (homotopy-coherently) **commuting** family $\{\tau_e\}_{e\in E}$ of autoequivalences of $\mathcal{C}$.

Now, we will observe that when $M$ is a group-like $\mathbb{E}_n$-monoid in spaces, we can view $M$-polarized categories as $k$-linear presentably $\mathbb{E}_{n-1}$-monoidal categories of left modules in $\mathsf{Cat}_k^{\mathsf{perf}}$ over the $k$-linear stably $\mathbb{E}_n$-monoidal category

$$\mathsf{Perf}^M(k) := \mathsf{Loc}(M; \mathsf{Mod}\, k)^\omega,$$

where $\mathsf{Loc}(M; \mathsf{Mod}\, k)$ is equipped with a $k$-linear presentably $\mathbb{E}_n$-monoidal structure as in (the proof of) [PPS25, Proposition 2.8, Remark 2.9]. To see this, let us note the following quick observation.

**Lemma 5.4.7.** *Let $M$ be an $\mathbb{E}_n$-monoid in $\mathcal{S}$. Then, there is a canonical $k$-linear stably $\mathbb{E}_n$-monoidal equivalence*

$$M \otimes 1_{\mathsf{Cat}_k^{\mathsf{perf}}} \simeq \mathsf{Perf}^M(k).$$

*Proof.* By [PPS25, Proposition 2.8], we have an equivalence of $k$-linear presentably $\mathbb{E}_n$-monoidal categories: $M \otimes 1_{\mathsf{Pr}^{\mathsf{L}}_{k,\omega}} \simeq \mathsf{Loc}(M; \mathsf{Mod}\, k)$ as $1_{\mathsf{Pr}^{\mathsf{L}}_{k,\omega}} \simeq \mathsf{Mod}\, k$ is presentably symmetric monoidal. Thus, by construction, we have the equivalence of $k$-linear stably $\mathbb{E}_n$-monoidal categories:

$$M \otimes 1_{\mathsf{Cat}_k^{\mathsf{perf}}} \simeq M \otimes (1_{\mathsf{Pr}^{\mathsf{L}}_{k,\omega}})^\omega \simeq (M \otimes 1_{\mathsf{Pr}^{\mathsf{L}}_{k,\omega}})^\omega \simeq \mathsf{Perf}^M(k). \qquad \square$$

**Example 5.4.8.** Let $M$ be a discrete monoid. Then, $\mathsf{Perf}^M(k)$ is equivalent to the $\infty$-category of $M$-graded perfect complexes over $k$ in the usual sense. Indeed, a functor $\Phi : M \to \mathsf{Mod}\, k$ is compact if and only if $\Phi$ lands in $\mathsf{Perf}\ k$ and $\Phi(m) = 0$ for all but finitely many $m \in M$.

Now, let us recall the following monodromy equivalence (cf. Beardsley–Péroux [BP23, Lemma 3.9] for the underlying equivalence and Pascaleff–Pavia–Sibilla [PPS25, Proposition 2.10] as a monoidal equivalence):

**Theorem 5.4.9** (Monodromy equivalence)**.** *Let $M$ be a group-like $\mathbb{E}_n$-monoid in $\mathcal{S}$ and suppose that the classifying space $BM$ is connected. Then, there is the $k$-linear presentably $\mathbb{E}_{n-1}$-monoidal monodromy equivalence:*

$$\mathsf{pCat}_k^{\mathsf{perf}}(M) \simeq \mathsf{LMod}_{\mathsf{Perf}^M(k)}(\mathsf{Cat}_k^{\mathsf{perf}}).$$

*Proof.* Since the notations slightly differ, let us comment on how this version follows. First, since $\mathsf{Cat}_k^{\mathsf{perf}}$ is a presentably symmetric monoidal $\infty$-category, [PPS25, Proposition 2.10] gives a $k$-linear presentably $\mathbb{E}_{n-1}$-monoidal equivalence:

$$\mathsf{pCat}_k^{\mathsf{perf}}(M) = \mathsf{Loc}(\mathsf{B}M; \mathsf{Cat}_k^{\mathsf{perf}}) \simeq \mathsf{LMod}_{\Omega_*\mathsf{B}M}(\mathsf{Cat}_k^{\mathsf{perf}}) \simeq \mathsf{LMod}_M(\mathsf{Cat}_k^{\mathsf{perf}}).$$

Now, by [PPS25, Lemma 2.2] and Lemma 5.4.7, we have $k$-linear presentably $\mathbb{E}_n$-monoidal equivalences:

$$\mathsf{LMod}_M(\mathsf{Cat}_k^{\mathsf{perf}}) \simeq \mathsf{LMod}_{M\otimes 1_{\mathsf{Cat}_k^{\mathsf{perf}}}}(\mathsf{Cat}_k^{\mathsf{perf}}) \simeq \mathsf{LMod}_{\mathsf{Perf}^M(k)}(\mathsf{Cat}_k^{\mathsf{perf}}). \qquad \square$$

As a special case, we obtain the following (cf. [Tel14, Remark 4.6], [GHMG22, Example 0.4]):

**Proposition 5.4.10.** *We have a $k$-linear presentably symmetric monoidal equivalence*

$$\mathsf{pCat}_k^{\mathsf{perf}}(\mathbb{Z}^n) \simeq \mathsf{LMod}_{\mathsf{Perf}(B\mathbb{G}^n_{m,k})}(\mathsf{Cat}_k^{\mathsf{perf}}).$$

*By viewing a polarized $\infty$-category $(\mathcal{C}, \tau)$ as a $\mathsf{Perf}(B\mathbb{G}_{m,k})$-module in $\mathsf{Cat}_k^{\mathsf{perf}}$, we have the following identification, where the right hand side denotes the relative triangular spectrum (*[Mat25b]*):*

$$\mathrm{Spc}_\vartriangle(\mathsf{Ho}(\mathcal{C}), \mathsf{Ho}(\tau)) = \mathrm{Spc}_\vartriangle(\mathcal{C}, \mathsf{Perf}(B\mathbb{G}_{m,k}), \mathsf{Perf}\ k).$$

*Proof.* First, recall that there is a natural $k$-linear symmetric monoidal equivalence (by [Mou21, Theorem 4.1] for $n = 1$ and [BH25, Theorem 3.3.10,Remark 1.1.1] for general $n$):

$$\mathsf{Perf}(B\mathbb{G}^n_{m,k}) \simeq \mathsf{Perf}^{\mathbb{Z}^n}(k).$$

Thus, the monodromy equivalence (Theorem 5.4.9) applied to $M = \mathbb{Z}^n$ gives a desired equivalence. Now, assume $n = 1$. Then, on objects, the monodromy equivalence takes $(\mathcal{C}, \tau)$ to a left $\mathsf{Perf}(B\mathbb{G}_{m,k})$-module structure on $\mathcal{C}$ with the action $\mathcal{O}_{B\mathbb{G}_{m,k}}(l) \otimes M := \tau^l(M)$ for $M \in \mathcal{C}$. By construction, a thick subcategory of $\mathcal{C}$ is a left $\mathsf{Perf}(B\mathbb{G}_{m,k})$-submodule of $\mathcal{C}$ if and only if it is fixed by $\tau$. Therefore, we have the identification

$$\mathrm{Spc}_\vartriangle(\mathsf{Ho}(\mathcal{C}), \mathsf{Ho}(\tau)) = \mathrm{Spc}_\vartriangle(\mathcal{C}, \mathsf{Perf}(B\mathbb{G}_{m,k}), \mathsf{Perf}\ k)$$

of the pt-spectrum and the relative Matsui spectrum of the corresponding $\mathsf{Perf}(B\mathbb{G}_{m,k})$-module. $\square$

From this observation, we define the following.

**Definition 5.4.11.** Let $M$ be a group-like $\mathbb{E}_1$-monoid in $\mathcal{S}$ and let $(\mathcal{C}, \tau)$ be an $M$-polarized perfect $\infty$-category, which can be naturally viewed as a $\mathsf{Perf}^M(k)$-module in $\mathsf{Cat}_k^{\mathsf{perf}}$ by the monodromy equivalence. Define the **$M$-polarized triangular spectrum** of $(\mathcal{C}, \tau)$ to be

$$\mathrm{Spec}_\vartriangle(\mathcal{C}, \tau) := \mathrm{Spec}_\vartriangle(\mathcal{C}, \mathsf{Perf}^M(k), \mathsf{Perf}\ k).$$

## 5.5 Functoriality of spectra and topological lattices

In Remark 4.2.6, we mentioned that the association taking pt-categories to their pt-spectra does not define a functor due to lack of well-definedness on morphisms. The following observations let us remedy this feature to some degree.

**Lemma 5.5.1.** *Let $F : (\mathcal{T}_1, \tau_1) \to (\mathcal{T}_2, \tau_2)$ be a pt-functor. Then, a map*

$$F^{-1} : \mathrm{Th}^{\tau_2} \mathcal{T}_2 \to \mathrm{Th}^{\tau_1} \mathcal{T}_1, \quad \mathcal{I} \mapsto F^{-1}(\mathcal{I})$$

*is well-defined, continuous (with respect to the subspace topology of the Balmer topology) and preserving inclusions of categories.*

*Proof.* First, for $\mathcal{I} \in \mathrm{Th}^{\tau_2} \mathcal{T}_2$, we have $F^{-1}(\mathcal{I}) \in \mathrm{Th}^{\tau_1} \mathcal{T}_1$ since

$$\tau_1^{-1}(F^{-1}(\mathcal{I})) = F^{-1}(\tau_2^{-1}(\mathcal{I})) = F^{-1}(\mathcal{I}).$$

Hence, $F^{-1}$ is well-defined. To show continuity, take a collection $\mathcal{E}$ of objects in $\mathcal{T}_1$. Then, we have

$$(F^{-1})^{-1}(V(\mathcal{E})) = \{\mathcal{I} \in \mathrm{Th}\, \mathcal{T}_2 \mid F^{-1}(\mathcal{I}) \cap \mathcal{E} = \emptyset\} = \{\mathcal{I} \in \mathrm{Th}\, \mathcal{T}_2 \mid \mathcal{I} \cap F(\mathcal{E}) = \emptyset\} = V(F(\mathcal{E})),$$

i.e., $F^{-1} : \mathrm{Th}\, \mathcal{T}_2 \to \mathrm{Th}\, \mathcal{T}_1$ is continuous, which restricts to a continuous map $\mathrm{Th}^{\tau_2} \mathcal{T}_2 \to \mathrm{Th}^{\tau_1} \mathcal{T}_1$ since we put the subspace topology on $\mathrm{Th}^{\tau_i} \mathcal{T}_i$. The fact that $F^{-1}$ preserves inclusions is clear. □

Now, let us introduce the following notion as a target category.

**Definition 5.5.2.** Define a **topological complete lattice** to be a complete lattice $(L, \leq)$ together with topology defined on the set $L$, where no compatibility is required. Let

$$\mathsf{TopCLat}_{\mathsf{meet}}$$

denote the category of topological complete lattices whose morphisms are poset maps that preserve arbitrary meets and that are continuous.

*Remark* 5.5.3. Since a lattice can already be thought of as a generalization of topological spaces, a topological (complete) lattice can be thought of as essentially having two different topologies. Compatibility of two topologies indicates certain geometricity. Indeed, for a noetherian scheme $X$ with a $\otimes$-generating line bundle $\mathscr{L}$, the lattice structure on $\mathrm{Th}^{\tau_{\mathscr{L}}} \mathsf{Perf}\, X$ recovers the Balmer topology on $\mathrm{Spc}_{\vartriangle}(\mathsf{Perf}\, X, \tau_{\mathscr{L}}) = \mathrm{Spc}_{\otimes_X} \mathsf{Perf}\, X$ by the Hochster duality (cf. [KP17]).

It is natural to ask how much the converse holds. Namely, if topology on $\mathrm{Spc}_{\vartriangle}(\mathsf{Perf}\, X, \tau_{\mathscr{L}})$ constructed by the Hochster duality, if applicable, agrees with the Balmer topology, then is it isomorphic to (a Fourier–Mukai partner of) $X$?

**Construction 5.5.4.** Let $\mathsf{ptCat}$ denote the category of pt-categories with pt-functors. Define the **thick spectrum functor** to be the functor

$$\mathrm{Th} : \mathsf{ptCat}^{\mathsf{op}} \to \mathsf{TopCLat}_{\mathsf{meet}}$$

sending a pt-category $(\mathcal{T}, \tau)$ to the sublattice $\mathrm{Th}^{\tau}\,\mathcal{T} \subset \mathrm{Th}\,\mathcal{T}$ of fixed points by $\tau$ equipped with the subspace topology of the Balmer topology on $\mathrm{Th}\,\mathcal{T}$ (cf. Notation 2.1.1 (i) and (viii)) and sending a pt-functor $F : (\mathcal{T}_1, \tau_1) \to (\mathcal{T}_2, \tau_2)$ to the continuous poset map $F^{-1} : \mathrm{Th}^{\tau_2}\,\mathcal{T}_2 \to \mathrm{Th}^{\tau_1}\,\mathcal{T}_1$ (cf. Lemma 5.5.1). For well-definedness, we need to note the following two facts:

- For any non-empty subset $S \subset \mathrm{Th}^{\tau}\,\mathcal{T}$, we see that $\bigcap_{\mathcal{P}\in S} \mathcal{P}$ is a thick subcategory and that

$$\tau\left(\bigcap_{\mathcal{P}\in S} \mathcal{P}\right) = \bigcap_{\mathcal{P}\in S} \tau(\mathcal{P}) = \bigcap_{\mathcal{P}\in S} \mathcal{P},$$

  i.e., $\bigcap_{\mathcal{P}\in S} \mathcal{P} \in \mathrm{Th}^{\tau}\,\mathcal{T}$. Note we also have $\mathcal{T} \in \mathrm{Th}^{\tau}\,\mathcal{T}$, which is a unique largest element, i.e., the empty meet. Since a poset that admits an arbitrary meet is a complete lattice, we have $\mathrm{Th}^{\tau}\,\mathcal{T} \in \mathsf{TopCLat}_{\mathsf{meet}}$.

- For any subset $S \subset \mathrm{Th}^{\tau_2}\,\mathcal{T}_2$, we have

$$F^{-1}\left(\bigcap_{\mathcal{P}\in S} \mathcal{P}\right) = \bigcap_{\mathcal{P}\in S} F^{-1}(\mathcal{P})$$

  and hence $F^{-1}$ is meet-preserving, i.e., we have shown

$$F^{-1} \in \mathrm{Hom}_{\mathsf{TopCLat}_{\mathsf{meet}}}(\mathrm{Th}^{\tau_2}\,\mathcal{T}_2, \mathrm{Th}^{\tau_1}\,\mathcal{T}_1).$$

Since the pt-spectrum $\mathrm{Spec}_{\vartriangle}(\mathcal{T}, \tau)$ is (non-functorially) constructed from $\mathrm{Th}^{\tau}\,\mathcal{T}$ viewed as a topological complete lattice, we may focus on $\mathrm{Th}^{\tau}\,\mathcal{T}$ to achieve a functorial theory.

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