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Deformations of the Scalar Curvature and the Mean Curvature

DISSERTATION

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by

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ABSTRACT OF THE DISSERTATION

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On a compact manifold with boundary, the map consisting of the scalar curvature in the interior and the mean curvature on the boundary is a local surjection at generic metrics [19]. We prove that this result may be localized to compact subdomains in an arbitrary Riemannian manifold with boundary, as motivated by an attempt to generalize the Riemannian Penrose inequality in dimension 8. This result is a generalization of Corvino's result [15] about localized scalar curvature deformations; however, the existence part needs to be handled delicately since the problem is non-variational. For non-generic cases, we give a classification theorem for domains in space forms and Schwarzschild manifolds, and show the connection with positive mass theorems.

Chapter 1

Introduction

In special relativity, a flat spacetime is modeled by Minkowski spacetime $\mathbb{R}^{1,3}$ endowed with the non-degenerate symmetric quadratic form

$$\bar{g}_0 = -dt^2 + \sum_{i=1}^{3}(dx^i)^2,$$

where $t = x^0$ is the temporal coordinate, and x^i's $(i = 1, 2, 3)$ are the spatial coordinates. We say that the metric $\bar{g}_0$ has signature $(-, +, +, +)$, for $\bar{g}_0$ has one negative eigenvalue and three positive eigenvalues.

Einstein's general relativity is a theory of gravity compatible with special relativity. Unlike Newtonian physics, gravity is a consequence of the curvature of the spacetime rather than being considered as a force. There are three fundamental hypotheses in the theory of general relativity (cf. [55, Section 4.3]):

(H1) The spacetime is a 4-dimensional time-orientable Lorentzian manifold $(N^4, \bar{g})$;

(H2) A freely falling test massive body travels along time-like geodesics;

(H3) Einstein's equation holds:

$$G := \mathrm{Ric}(\bar{g}) - \frac{1}{2}R(\bar{g})\bar{g} = 8\pi T, \tag{1.1}$$

where G is called the Einstein curvature tensor, T is a symmetric $(0,2)$-tensor, called the stress-energy-momentum tensor, representing a continuous matter distribution in the spacetime, $\mathrm{Ric}(\bar{g})$ is the Ricci curvature tensor, and $R(\bar{g})$ is the scalar curvature of $\bar{g}$.

When $T = 0$, (1.1) is called the vacuum Einstein equation, and can be reduced to $\mathrm{Ric} = 0$. Historically, Einstein discovered the vacuum equation before writing down the full equation. The beauty of general relativity is that this simple formula explains gravity better than Newton's theory and is completely consistent with large scale experiments.

If we consider (M^3, g) as a space-like hypersurface of $(N^4, \bar{g})$ with second fundamental form k, then (1.1) together with the Gauss and Codazzi equations implies

$$\mu := \frac{1}{16\pi}\left(R_g - |k|_g^2 + (\mathrm{tr}_g\, k)^2\right),$$

$$J := \frac{1}{8\pi}\,\mathrm{div}\left(k - (\mathrm{tr}_g\, k)g\right),$$

where R_g is the scalar curvature of the metric g, μ is the local energy density, and J is the local current density. These two equations are called the constraint equations for M^3 in N^4. There is also a very natural condition in general relativity called dominant energy condition, which says that the speed of energy flow of matter is always less than the speed of light. This assumption of nonnegative energy density everywhere in N^4 implies that we must have

$$\mu \geq |J|_g \tag{1.2}$$

at all points on M^3 [44]. An important special case is when M is totally geodesic, i.e., $k = 0$, and such M is called a time-symmetric slice. Then the dominant energy condition (1.2) becomes a positivity condition on scalar curvature

$$R_g \geq 0. \tag{1.3}$$

1.1 Schwarzschild spacetime

The first solution of the Einstein equation (1.1) (with $T = 0$) was obtained by Schwarzschild in 1916. The Schwarzschild solution is an important example to consider when discussing the notion of total mass and its related properties, e.g., the positive mass theorem and the Penrose inequality. For $m > 0$, define the Schwarzschild spacetime metric with mass m to be

$$\bar{g}_m := -\left(1 - \frac{2m}{r}\right) dt^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2 d\xi^2_{\mathbb{S}^2} \tag{1.4}$$

in standard spherical coordinates. Note that the singularity at $r = 2m$ is only a coordinate singularity, and the metric can be extended to all of $\mathbb{R} \times (\mathbb{R}_+ \times \mathbb{S}^2)$, but the spacetime does contain the first example of a black hole singularity at $r = 0$. In the weak field regime ($r \to \infty$), the behavior of a test mass in the Schwarzschild spacetime agrees with the behavior of a test mass in the Newtonian theory of gravity of an isolated point mass m at the origin (cf. [55, Section 6.2]). Thus, the parameter m is interpreted as the total mass of the Schwarzschild spacetime. If $m < 0$, the metric $\bar{g}_m$ is incomplete; if $m = 0$, $\bar{g}_m = \bar{g}_0$ is simply the Minkowski metric, which can be viewed as a special case of the Schwarzschild solution.

The spacetime is spherically symmetric, and the induced Riemannian metric g_m on the time-slice $\{t = 0\}$, which is often called the Riemannian Schwarzschild metric, is a time-symmetric solution to the vacuum constraints. This spacelike Schwarzschild slice is actually a complete Riemannian manifold, as near $r = 0$ the metric is asymptotically flat. It can also be shown that the only spherically symmetric, asymptotically flat solutions to the constraint $R(g) = 0$ are isometric to these Schwarzschild slices [32]. Under the coordinate transformation $r = \rho(1 + \frac{m}{2\rho})^2$, we can see that the Riemannian Schwarzschild metric is conformally flat

$$g_m = \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2 d\xi^2_{\mathbb{S}^2} = \left(1 + \frac{m}{2\rho}\right)^4 \left(d\rho^2 + \rho^2 d\xi^2_{\mathbb{S}^2}\right), \tag{1.5}$$

where $d\rho^2 + \rho^2 d\xi^2_{\mathbb{S}^2}$ is the Euclidean metric in spherical coordinates.

Actually this discussion extends to $\mathbb{R}^n$ ($n \geq 3$). Note that the Schwarzschild metric has an analogue in higher dimensions, given by metrics which are conformal to the flat metric on $\mathbb{R}^n \backslash \{\mathbf{c}\}$:

$$g^S_{(m,\mathbf{c})}(\mathbf{x}) = \left(1 + \frac{m}{2r^{n-2}}\right)^{\frac{4}{n-2}} \delta$$

in standard coordinates with $r = |\mathbf{x} - \mathbf{c}|$. Here $\mathbf{c}$ is the center-of-mass parameter. As above, we again find that these are the only spherically symmetric, asymptotically flat solutions to the vacuum constraint equations.

1.2 Asymptotic flatness and ADM mass

In general relativity, one often is interested in studying the properties of isolated systems. Since gravity is attractive, it is physically reasonable to believe that matter is concentrated in some bounded regions, e.g., galaxies. When studying the structure of a condensed star far away from others, we may approximate it by an isolated system and study the problem as if the star were suited in a spacetime which becomes flat. Asymptotic flatness characterizes the property that in an isolated system the gravitational field becomes weak and thus the spacetime is asymptotic to the flat Minkowski spacetime near infinity.

In particular, we are interested in Riemannian manifolds that are asymptotically flat.

Definition 1.1. *A manifold M^n is called asymptotically flat (with ℓ ends) if there is a compact subset $K \subset M$ such that $M \backslash K$ consists of finite number of connected components $M_1, \ldots, M_\ell$, called infinite ends, each of which is diffeomorphic to $\mathbb{R}^n \backslash \bar{B}$ for a closed ball $\bar{B}$ in $\mathbb{R}^n$ such that under these diffeomorphisms*

$$g_{ij} = \delta_{ij} + O\left(|x|^{-p}\right), \quad |x|\,|\partial g_{ij}| + |x|^2 \left|\partial^2 g_{ij}\right| = O\left(|x|^{-p}\right)$$

for some $p > \frac{n-2}{2}$. We also require the scalar curvature R to satisfy

$$|R| = O\left(|x|^{-q}\right)$$

for some $q > n$.

For example, the Riemannian Schwarzschild manifold is asymptotically flat.

In general relativity, one is also interested in understanding the nature of the behavior of mass and energy. However, defining an energy satisfying the conservation law in general relativity is very different from pre-relativistic theories. The strategy of integrating local energy density over the background space does not work, and the primary reason is that gravitational field $\bar{g}$ describes both the spatial property and the dynamical aspect of the spacetime. Einstein's equivalence principle asserts that there is no observer who can be insulated by the influence of gravity, and thus there is no canonical gauge-free decomposition of $\bar{g}$ into a background part and a dynamical part. This leads to lack of local energy in general relativity. Moreover, integrating the local energy of matter T over a space-like hypersurface is not enough, since the gravitational field also contributes to the total energy. For instance, in the Riemannian Schwarzschild manifold with metric g_m, T is everywhere zero, but the total mass should be m.

Thanks to the definition of asymptotic flatness, it is now possible to define the notion of total mass of an isolated system measured by an observer at infinity. Motivated by the comparison between Schwarzschild spacetime and Newtonian model in weak field regime, if the Riemannian metric g on time-slice is asymptotic to Schwarzschild at an infinite end, i.e.,

$$g_{ij} = \left(1 + \frac{m}{2|x|}\right)^4 \delta_{ij} + O(|x|^{-2}),$$

one may expect the total energy measured at this infinite end to be m. More generally, for an asymptotically flat manifold, R. Arnowitt, S. Deser and C.W. Misner [3] introduced the total mass at any infinite end M_p, now often called the ADM mass, which is defined in terms of how fast the metric becomes flat at infinity:

$$m_{\mathrm{ADM}}\left(M_p, g\right) = \frac{1}{16\pi} \lim_{\sigma \to \infty} \sum_{i,j=1}^{3} \int_{S_\sigma} \left(\partial_{x^i} g_{ij} - \partial_{x^j} g_{ii}\right) \nu_j d\mu,$$

where S_σ is the euclidean sphere of radius σ in the x coordinates, and the unit normal ν and volume integral are with respect to the euclidean metric. Such total energy is gauge invariant [5]. Similarly, we can define the ADM mass for n-dimensional asymptotically flat manifold:

$$m_{\mathrm{ADM}}\left(M_p, g\right) = \frac{1}{2(n-1)\omega_{n-1}} \lim_{\sigma \to \infty} \sum_{i,j=1}^{n} \int_{S_\sigma} \left(\partial_{x^i} g_{ij} - \partial_{x^j} g_{ii}\right) \nu_j d\mu,$$

where $\omega_{n-1} = \mathrm{Vol}(\mathbb{S}^{n-1})$.

1.3 The positive mass theorem

The positive mass theorem can be thought of as a first attempt at understanding the relationship between the local energy density of a space-time and its total mass. In physical terms, the positive mass theorem states that an isolated gravitational system with nonnegative local energy density must have nonnegative total energy. The idea is that nonnegative energy densities must "add up" to something nonnegative, which is very natural. For simplicity, let us first consider the Riemannian case where the manifold M^3 has only one infinite end. In 1979, Schoen and Yau [43] proved the following positive mass theorem for Riemannian manifolds (M^3, g) using minimal surfaces:

Theorem 1.2 (The Riemannian Positive Mass Theorem, Schoen-Yau). *Let (M^3, g) be an asymptotically flat Riemannian manifold satisfying $R_g \geq 0$. Then $m_{ADM} \geq 0$ and equality holds if and only if (M^3, g) is isometric to $(\mathbb{R}^3, \delta)$.*

Recall that the dominant energy condition is equivalent to $R_g \geq 0$ in time-symmetric slice. In fact, the proof in [43] would also give the Riemannian positive mass theorem in dimensions up to 7. Then in 1981, Witten [56] gave another proof using spinors.

In 1981, Schoen and Yau [45] showed the space-time version of the positive energy theorem:

Theorem 1.3 (The Positive Energy Theorem, Schoen-Yau). *Let (M^3, g, k) be an asymptotically flat initial data set satisfying the dominant energy condition. Then $E_{ADM} \geq 0$ and equality holds if and only if (M^3, g, k) can be embedded in Minkowski spacetime $\mathbb{R}^{1,3}$.*

Focusing on the Riemannian case, there are also a lot of generalizations of the positive mass theorem later on. For example, Miao [39] as well as Shi-Tam [47] showed the positive mass theorem for compact manifolds with boundary, Almaraz-Barbosa-de Lima [1] proved a positive mass theorem for asymptotically flat manifolds with a non-compact boundary, and many people have made contributions to varirous versions of positive mass theorems for asymptotically hyperbolic Riemannian manifolds (see e.g. [33, 35, 57]). We will take a closer look at some of these generalizations in Section 3.2.

Recently in 2017, Schoen and Yau [46] proved the Riemannian positive mass theorem in any dimensions:

Theorem 1.4 (The Riemannian Positive Mass Theorem, Schoen-Yau)**.** *Let* (M^n, g) *be an asymptotically flat Riemannian manifold satisfying* $R_g \geq 0$*. Then* $m_{ADM} \geq 0$ *and equality holds if and only if* (M^n, g) *is isometric to* $(\mathbb{R}^n, \delta)$*.*

1.4 The Penrose inequality

The Penrose inequality can be thought of as a second attempt at understanding the relationship between the local energy density of a space-time and its total mass. It states that if an isolated gravitational system with nonnegative local energy density contains a black hole of mass m, then the total energy of the system must be at least m. In some sense, the Penrose inequality can also be viewed as a refinement of the positive mass theorem.

In 1973, Roger Penrose proposed the Penrose inequality as a test of the cosmic censor hypothesis [42]. The cosmic censor hypothesis states that naked singularities do not develop starting with physically reasonable nonsingular generic initial conditions for the Cauchy problem in general relativity. If naked singularities did typically develop from generic initial conditions, then this would be a serious problem for general relativity since it would not be possible to solve the Einstein equations uniquely past these singularities. Singularities such as black holes do develop but are shielded from observers at infinity by their horizons so that the Einstein equations can still be solved from the point of view of an observer at infinity.

Let us assume M^3 has zero second fundamental form in N^4, then apparent horizons of black holes in N^4 correspond to outermost minimal surfaces of M^3. For a chosen end of M^3, an outermost minimal surface is a minimal surface which is not contained entirely inside another minimal surface with respect to this end. As an example, consider the Schwarzschild manifold $(\mathbb{R}^3 \backslash \{\mathbf{0}\}, g^S)$. It has an outermost minimal sphere at $r = m/2$.

Conjecture 1.5 (Penrose)**.** *The total mass of a space-time which contains black holes with event horizons of total area A should be at least $\sqrt{\frac{A}{16\pi}}$.*

An important special case in Riemannian geometry is now known as the Riemannian Penrose inequality. This inequality was first established by G. Huisken and T. Ilmanen [36] in 1997 using the inverse mean curvature flow for a single black hole and then by H. Bray [8] in 1999 for any number of black holes, using the technique of a conformal flow.

Theorem 1.6 (The Riemannian Penrose inequality, Bray)**.** *Let (M^3, g) be a complete, smooth, asymptotically flat 3-manifold with nonnegative scalar curvature, total ADM mass m, and an outer minimizing horizon (with one or more components) of total area A. Then*

$$m \geq \sqrt{\frac{A}{16\pi}}$$

with equality if and only if (M^3, g) is isometric to a Schwarzschild manifold outside their respective outermost horizons.

In Bray's proof, he constructed a conformal flow (M^3, g_t). The horizon is the boundary of an exterior Dirichlet Problem and the solution of the Dirichlet Problem gives the conformal factor. Along the flow, the horizon is moving towards infinite, the area of the horizon is a constant, and the mass is decreasing. Note that the positive mass theorem [43] plays a very important role in proving the mass is decreasing along the flow. Finally, the flow will converge to the Schwarzschild manifold, and this gives us the Riemannian Penrose inequality.

Later in 2009, H. Bray and D. Lee [9] generalized Bray's result to dimension up to 7.

Theorem 1.7 (The Riemannian Penrose inequality in dimensions less than eight, Bray-Lee)**.** *Let (M^n, g) be a complete asymptotically flat manifold with nonnegative scalar curvature, where $n < 8$. Fix one end. Let m be the mass of that end, and let A be the area of an outer-minimizing horizon (with one or more components). Let ω_{n-1} be the area of the standard* *unit* $(n-1)$*-sphere. Then*

$$m \geq \frac{1}{2}\left(\frac{A}{\omega_{n-1}}\right)^{(n-2)/(n-1)}.$$

Furthermore, if we also assume that M *is spin, then equality occurs if and only if the part of* (M, g) *outside the horizon is isometric to a Riemannian Schwarzschild manifold outside its unique outer-minimizing horizon.*

Later, the spin assumption in the rigidity part was removed by combining results of Bray-Lee [9] and McFeron-Székelyhidi [38].

As for now, the Riemannian Penrose inequality in dimensions higher than 8 is still open.

1.5 Minimizing hypersurface

The Plateau's problem investigates those surfaces of least area spanning a given contour. It is one of the most classical problems in the calculus of variations.

Problem 1.8 (Plateau Problem)**.** *Given a closed* $(m-1)$*-dimensional surface* $\Gamma \subset \mathbb{R}^N$ *, find an* m*-dimensional surface* S *of least area among surfaces with* $\partial S = \Gamma$*.*

The original formulation is attributed to the Belgian physicist Plateau, although it was considered earlier by Lagrange. The problem itself and its various generalizations have found fundamental and interesting applications in several mathematical and scientific branches.

Federer and Fleming's theory of integral currents [25] successfully solved the existence problems in all dimensions and codimensions. On the other hand, we are particularly interested in the regularity problems of area-minimizing hypersurfaces in Riemannian manifolds. In fact, many people, including Fleming, De Giorgi, Almgren, Federer, Simons (see e.g. [2, 20, 21, 23, 24, 26, 49, 52]), have made contributions to the following famous result:

Theorem 1.9. *Let $U \subset \mathbb{R}^{n+k}$ be open and N be an $(n+1)$-dimensional oriented embedded C^2 submanifold of $\mathbb{R}^{n+k}$ with $(\bar{N}\backslash N) \cap U = \varnothing$, $N \cap U \neq \varnothing$. Suppose $T = \partial[[E]] \in \mathcal{D}_n(U)$ is an integer multiplicity mass minimizing current in $N \cap U$, with $E \subset N \cap U$ an $\mathcal{H}^{n+1}$-measurable subset. Then $\operatorname{sing} T = \varnothing$ for $n \leq 6$, $\operatorname{sing} T$ is locally finite in U for $n = 7$, and $\mathcal{H}^{n-7+\alpha}(\operatorname{sing} T) = 0$ $\forall \alpha > 0$ in case $n > 7$.*

In particular, singularities on hypersurfaces are isolated in dimension 8 and may be perturbed away [31]. Later Smale [54] constructed a local deformation to perturb away the singularities. To be more precise, the set of metrics g_0 such that (M^8, g_0) has a unique smooth embedded area minimizing hypersurface Σ in its homology class, is dense in the space of smooth metrics, thus any metric can be perturbed by a local deformation to such a metric. This means, for generic metrics on M, the minimizing hypersurface Σ in its homology class is smooth.

Theorem 1.10 (Smale)**.** *Let N be a smooth, compact, 8-dimensional manifold with non-trivial $H_7(N, \mathbb{Z})$. For $k = 3, 4, \cdots$, let $\mathcal{M}^k$ denote the class of C^k metrics on N. For $\alpha \in H_7(N, \mathbb{Z})$, $\alpha \neq 0$, define the subclass $\mathcal{F}^k_\alpha \subset \mathcal{M}^k$ to be the set of metrics such that $g \in \mathcal{F}^k_\alpha$ if and only if there is a smooth area minimizing (relative to g) 7-dimensional, integer multiplicity current T homologous to α. Then $\mathcal{F}^k_\alpha$ is generic in $\mathcal{M}^k$.*

1.6 Localized deformation

Localized deformations, in contrast to global deformations (e.g. conformal deformations), play an important role in gluing constructions and have already been studied in many different settings. In 2000, Corvino [15] used a variational approach to prove a local surjectivity result for the scalar curvature operator.

Theorem 1.11 (Corvino)**.** *Let $\Omega \subset M$ be a compact domain in a Riemannian manifold (M^n, g_0). Assuming certain generic conditions, for any compactly supported deformation S*

of $R(g_0)$ in Ω, there is a metric g such that $R(g) = S$ in Ω, and $g \equiv g_0$ outside Ω.

Here by generic condition, we mean $\ker L^*$ is trivial in H^2_{loc}, where L is the linearization of the scalar curvature operator and L^* is its formal L^2-adjoint. This localized scalar curvature deformation was obtained much in the spirit of the Fischer-Marsden's paper [27], where they used a Hodge-type elliptic splitting theorem along with the implicit function theorem. In fact, L^* has injective symbol, so by elliptic theory, the appropriate Sobolev function space splits as $\operatorname{ran} L \oplus \ker L^*$. Hence the generic condition which is sufficient to show local surjectivity is that L^* has trivial kernel. However, Corvino's proof required a bit more delicate analysis to produce a solution with compact support.

Later, Corvino and Schoen [18] proved a local surjectivity result for the full constraint map. Chruściel and Delay [14] introduced finer weighted spaces and derived a systematic approach to localized deformations for the constraint map in various settings. Recently, Corvino and Huang [17] presented localized deformations without assuming that initial data sets are either vacuum or have the strict dominant energy condition, using a new modified constraint operator, and obtained new gluing applications. However, to the best of our knowledge, nobody has yet considered localized deformations which simultaneously prescribe the interior and the boundary. In Chapter 2, we will introduce our main result of localized deformations of the scalar curvature and the mean curvature, which is a generalization of Corvino's result.

1.7 Motivation of the main theorem

Finally, we would like to introduce some motivations of the main theorem, which will be discussed in Chapter 2.

As for now, the Riemannian Penrose inequality in dimensions higher than 8 is still open. One of the main difficulties in constructing the generalized Bray's conformal flow in all dimensions is that the outer minimizing hypersurfaces may not be smooth when $n \geq 8$, as mentioned in Theorem 1.9. However, singularities on hypersurfaces are isolated in dimension 8 and may be perturbed away. So we first tried to generalize the conformal flow in dimension 8, using Smale's result to smooth out the singularities on the horizon.

Even though Smale's result is very useful, it cannot be directly applied to the conformal flow. To be more precise, let (M^8, g) be a smooth, complete, eight-dimensional manifold. Smale [54] constructed a local conformal perturbation of the metric which pushed the new minimizing hypersurface to one side of the previous one, thus by Hardt-Simon's result [31], the new hypersurface is smooth. The problem comes, however, that the perturbation will also change the scalar curvature in a small region near the hypersurface. So even though the scalar curvature was non-negative everywhere at first, it may not be the case after the perturbation, which makes it hard to apply the result of the positive mass theorem [44]. In order to overcome this problem, we modify Smale's perturbation and construct a smooth outer-minimizing hypersurface, and at the same time, ensure that the scalar curvature is non-negative everywhere outside the hypersurface.

Let us start with a simpler case where (M^8, g_0) is a smooth, complete, eight-dimensional manifold with a uniquely area minimizing boundary Σ_0, and g_0 is asymptotically flat and scalar flat. First we use Smale's result [54, Lemma 1.4] to get a smooth hypersurface Σ_1 in (M, g_1).

Lemma 1.12 (Smale)**.** *Let $n = 7$. Given $g_0 \in \mathcal{M}^k$, with unique minimizing current T_0 homologous to α, and given $\epsilon > 0$, there exists $g \in \mathcal{F}^k_\alpha$ with $\|g - g_0\|_k < \epsilon$.*

Because Smale's perturbation is local and small, Σ_1 is also nearby. But after the perturbation, the scalar curvature could be negative somewhere. We used a localized scalar curvature and mean curvature deformation to overcome this problem: we construct a new metric g_2 nearby such that the corresponding scalar curvature is 0 in a local region Ω and the mean curvature

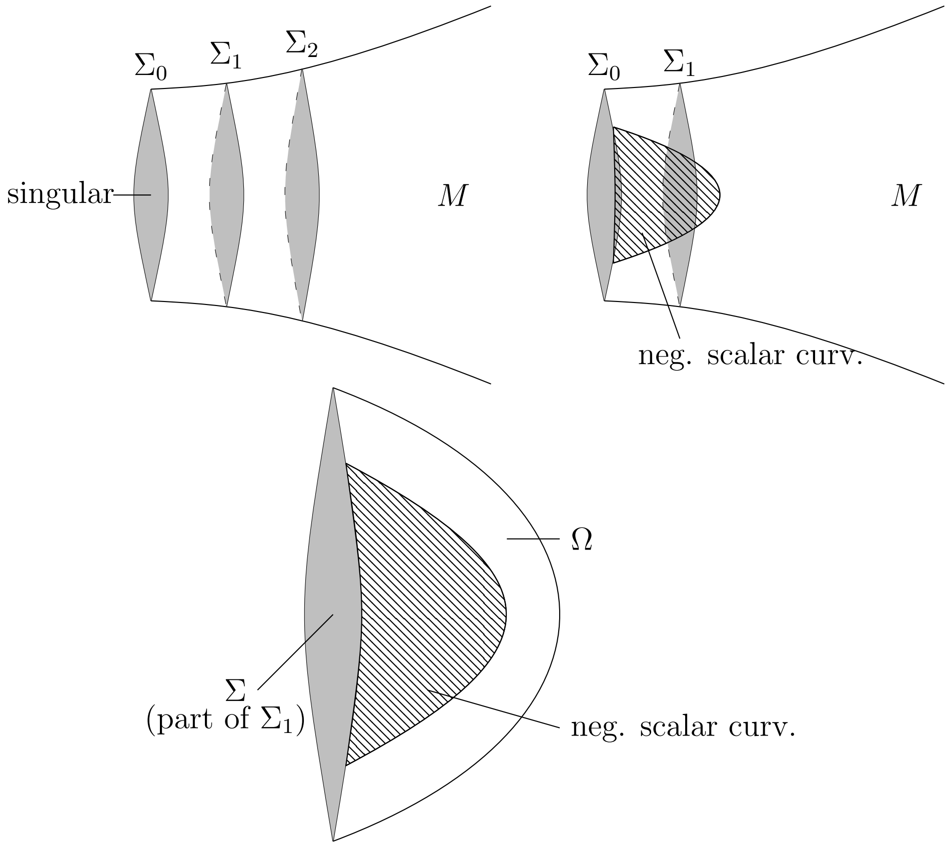

Figure 1.1: Smale's perturbation and localized deformation

of Σ_1 is also 0. The mean curvature condition tells us that Σ_1 is mean convex, which ensures that the new minimizing hypersurface Σ_2 in (M, g_2) lies outside of Σ_1. Finally, because Σ_2 is also very close to Σ_0, we can apply Hardt-Simon's result [31] again to conclude that Σ_2 is smooth. At this point, we have constructed a smooth minimizing hypersurface Σ_2 of M and meanwhile, g_2 is asymptotically flat and scalar flat outside of Σ_2.

This deformation technique allows us to perturb a singular horizon to a non-singular horizon and to construct Bray's conformal flow in dimension 8, which will eventually lead to a proof of the Riemanninan Penrose inequality in dimension 8. Additionally, the deformation technique itself is also very useful and has some further applications in general relativity. We will take a closer look at it in Chapter 2.

Chapter 2

Preliminaries

2.1 Basic notation

Let Ω denote a compactly contained subdomain with boundary of a smooth manifold M^n ($n \geq 3$). We will specify regularity conditions on the boundary as needed, and we will generally assume $\partial\Omega$ is at least C^2. We list here some notation and function spaces.

- $\mathrm{Ric}(g) = R_{ij}$ and $R(g) = g^{ij}R_{ij}$ denote the Ricci and scalar curvatures, respectively, of a Riemannian metric g on M; we use the Einstein summation convention throughout.
- Let $d\mu_g$ denote the volume measure induced by g, and $d\sigma_g$ the induced surface measure on submanifolds.
- Let D and ∇ denote the Levi-Civita connections of (Ω, g) and $(\partial\Omega, \hat{g})$, respectively, with $\hat{g}$ the induced metric on the boundary.
- $\mathcal{S}^{(0,2)}$ denotes the space of symmetric $(0,2)$-tensor fields.
- $\mathcal{H}^k$ denotes the subspace of $\mathcal{S}^{(0,2)}$ consisting of those measurable tensors which are square

integrable along with the first k weak covariant derivatives; with the standard $\mathcal{H}^k$-inner product induced by the metric g, $\mathcal{H}^k$ becomes a Hilbert space. H^k is defined similarly for functions, and the spaces $\mathcal{H}^k_{\text{loc}}(H^k_{\text{loc}})$ are defined as the spaces of tensors (functions) which are in $\mathcal{H}^k(H^k)$ on each compact subset.

- $\mathcal{M}^k(k > \frac{n}{2})$ denotes the open subset of $\mathcal{H}^k$ of Riemannian metrics, and $\mathcal{M}^{k,\alpha}$ denotes the open subset of metrics in $\mathcal{C}^{k,\alpha}$.

- Let ρ be a smooth positive function on Ω. Define $L^2_\rho(\Omega)$ to be the set of locally-L^2 functions f such that $f\rho^{\frac{1}{2}} \in L^2(\Omega)$. The pairing

$$\langle f, g\rangle_{L^2_\rho(\Omega)} = \left\langle f\rho^{1/2}, g\rho^{1/2}\right\rangle_{L^2(\Omega)}$$

makes $L^2_\rho(\Omega)$ a Hilbert space. Define $\mathcal{L}^2_\rho(\Omega)$ similarly for tensor fields.

- Given $k \in \mathbb{N}$. Let $H^k_\rho(\Omega)$ be the Hilbert space of $L^2_\rho(\Omega)$ functions whose covariant derivatives up through order k are also $\mathcal{L}^2_\rho(\Omega)$, i.e. the following norm is finite

$$\|u\|_{H^k_\rho(\Omega)} := \sum_{j=0}^{k} \left\|D^j u\right\|_{\mathcal{L}^2_\rho(\Omega)}$$

Define $\mathcal{H}^k_\rho$ similarly for tensor fields.

2.2 Linearized equations and integration by parts formula

Let Ω be a compact smooth n-manifold with smooth boundary Σ. Now let us consider the variation of metrics $g_t = g_0 + ta$ on Ω. We note the linearization $\dot{R}$ of the scalar curvature operator is given by [7, 27]:

$$\dot{R}(a) = -\Delta_{g_0}(\mathrm{tr}_{g_0} a) + \mathrm{div}_{g_0}(\mathrm{div}_{g_0} a) - \langle a, \mathrm{Ric}_{g_0} \rangle_{g_0}.$$

Now we want to calculate the linearization of the mean curvature of Σ. Consider the inclusion map

$$F = i : \Sigma \to \Omega; \qquad x = (x^1, \cdots, x^{n-1}) \mapsto F(x) = (F^1(x), \cdots, F^n(x))$$

Note that the inclusion map and local coordinates are invariant under the variation. Then the induced metric on Σ is

$$\hat{g}_{ij}(t) = g_{\alpha\beta}(t) \frac{\partial F^\alpha}{\partial x^i} \frac{\partial F^\beta}{\partial x^j}$$

where $i, j = 1, \cdots, n-1$; $\alpha, \beta = 1, \cdots, n$; and we have

$$\dot{\hat{g}}_{ij} = \left. \frac{d}{dt} \right|_{t=0} (g_0 + ta)_{\alpha\beta} \frac{\partial F^\alpha}{\partial x^i} \frac{\partial F^\beta}{\partial x^j} = a_{\alpha\beta} \frac{\partial F^\alpha}{\partial x^i} \frac{\partial F^\beta}{\partial x^j}$$

Let $\nu(t)$ be the outer unit normal (w.r.t. g_t) to Σ. We choose a local field of frames $e_1, \cdots, e_n$ in Ω such that restricted to Σ, we have $e_i = \partial F / \partial x^i, e_n(t) = \nu(t)$. Then

$$\begin{cases} \|\nu(t)\|_{g_t}^2 = g_{\alpha\beta}(t) \nu^\alpha(t) \nu^\beta(t) \equiv 1 \\ \langle \nu(t), e_i \rangle_{g_t} = g_{\alpha\beta}(t) \nu^\alpha(t) e_i^\beta \equiv 0 \quad (i = 1, \cdots, n-1) \end{cases}$$

Differentiating them on both sides and evaluating at $t = 0$, we will get

$$\begin{cases} 2 g^0_{\alpha\beta} \dot{\nu}^\alpha \nu^\beta + a_{\alpha\beta} \nu^\alpha \nu^\beta = 0 \\ g^0_{\alpha\beta} \dot{\nu}^\alpha e_i^\beta + a_{\alpha\beta} \nu^\alpha e_i^\beta = 0 \end{cases} \tag{2.1}$$

If we write

$$\dot{\nu} = c\nu + c^i e_i,$$

then from (2.1) we will get

$$\begin{cases} -\frac{1}{2}a(\nu,\nu) = \langle \dot{\nu}, \nu \rangle_{g_0} = \langle c\nu, \nu \rangle_{g_0} = c \\ -a(\nu, e_i) = \langle \dot{\nu}, e_i \rangle_{g_0} = \left\langle c^j e_j, e_i \right\rangle_{g_0} = c^j \hat{g}_{ij} \end{cases}$$

So

$$\dot{\nu} = -\frac{1}{2}a(\nu,\nu)\,\nu - \hat{g}^{ij}a(\nu, e_j)\,e_i$$

The second fundamental form h_{ij} is

$$h_{ij}(t) = -\left\langle \nu(t), D_{e_i}e_j \right\rangle_{g_t} = -g_{\alpha\theta}(t)\nu^\theta(t)\left(\frac{\partial^2 F^\alpha}{\partial x^i \partial x^j} + \Gamma^\alpha_{\beta\gamma}(t)\frac{\partial F^\beta}{\partial x^i}\frac{\partial F^\gamma}{\partial x^j}\right)$$

where the Levi-Civita connection $\Gamma^\alpha_{\beta\gamma} = \frac{1}{2}g^{\alpha\sigma}\left(\partial_\beta g_{\gamma\sigma} + \partial_\gamma g_{\beta\sigma} - \partial_\sigma g_{\beta\gamma}\right)$. Without loss of generality, we may assume $\partial_\alpha g^0_{\beta\gamma} \equiv 0$, then

$$\dot{\Gamma}^\alpha_{\beta\gamma} = \frac{1}{2}g^{\alpha\sigma}\left(\partial_\beta a_{\gamma\sigma} + \partial_\gamma a_{\beta\sigma} - \partial_\sigma a_{\beta\gamma}\right)$$

Thus,

$$\begin{aligned} \dot{h}_{ij} &= -a(\nu, D_{e_i}e_j) - \left\langle -\frac{1}{2}a(\nu,\nu)\nu - \hat{g}^{kl}a(\nu,e_l)e_k, D_{e_i}e_j \right\rangle_{g_0} \\ &\quad - g^0_{\alpha\theta}\,\nu^\theta \frac{1}{2}g^{\alpha\sigma}\left(\partial_\beta a_{\gamma\sigma} + \partial_\gamma a_{\beta\sigma} - \partial_\sigma a_{\beta\gamma}\right)e_i^\beta e_j^\gamma \\ &= -a(\nu, D_{e_i}e_j) - \frac{1}{2}a(\nu,\nu)h_{ij} + \hat{g}^{kl}a(\nu,e_l)\left\langle e_k, D_{e_i}e_j \right\rangle_{g_0} \\ &\quad - \frac{1}{2}\nu^\sigma\left(\partial_\beta a_{\gamma\sigma} + \partial_\gamma a_{\beta\sigma} - \partial_\sigma a_{\beta\gamma}\right)e_i^\beta e_j^\gamma \end{aligned}$$

By the Gauss-Weingarten Formulae,

$$D^{\alpha}_{e_i} e_j - \nabla^{\alpha}_{e_i} e_j = \frac{\partial^2 F^{\alpha}}{\partial x^i \partial x^j} + \Gamma^{\alpha}_{\beta\gamma} \frac{\partial F^{\beta}}{\partial x^i} \frac{\partial F^{\gamma}}{\partial x^j} - \hat{\Gamma}^{k}_{ij} \frac{\partial F^{\alpha}}{\partial x^k} = -h_{ij} \nu^{\alpha}$$

$$D^{\alpha}_{e_i} \nu = \frac{\partial \nu^{\alpha}}{\partial x^i} + \Gamma^{\alpha}_{\beta\gamma} \frac{\partial F^{\beta}}{\partial x^i} \nu^{\gamma} = h_{ij} \hat{g}^{jl} \frac{\partial F^{\alpha}}{\partial x^l}$$

we can rewrite the first three terms of $\dot{h}_{ij}$ as

$$\begin{aligned}
\mathrm{I} + \mathrm{II} + \mathrm{III} &= -a(\nu, D_{e_i} e_j) - \frac{1}{2} a(\nu,\nu) h_{ij} + \hat{g}^{kl} a(\nu, e_l) \left\langle e_k, \hat{\Gamma}^{p}_{ij} e_p - h_{ij} \nu \right\rangle_{g_0} \\
&= -a(\nu, D_{e_i} e_j) - \frac{1}{2} a(\nu,\nu) h_{ij} + a(\nu, e_l) \hat{\Gamma}^{l}_{ij} \\
&= a(\nu, e_l \hat{\Gamma}^{l}_{ij} - D_{e_i} e_j) - \frac{1}{2} a(\nu,\nu) h_{ij} \\
&= a(\nu, h_{ij} \nu) - \frac{1}{2} a(\nu,\nu) h_{ij} \\
&= \frac{1}{2} a(\nu,\nu) h_{ij}
\end{aligned}$$

So

$$\dot{h}_{ij} = \frac{1}{2} a(\nu,\nu) h_{ij} - \frac{1}{2} \nu^{\sigma} \left(\partial_{\beta} a_{\gamma\sigma} + \partial_{\gamma} a_{\beta\sigma} - \partial_{\sigma} a_{\beta\gamma} \right) e^{\beta}_i e^{\gamma}_j$$

Finally, the linearization $\dot{H}$ of the mean curvature operator is

$$\begin{aligned}
\dot{H} &= \dot{h}_{ij} \hat{g}^{ij} - h_{ij} \hat{g}^{ik} \dot{\hat{g}}_{kl} \hat{g}^{lj} \\
&= \left(\frac{1}{2} a(\nu,\nu) h_{ij} - \frac{1}{2} \nu^{\sigma} \left(\partial_{\beta} a_{\gamma\sigma} + \partial_{\gamma} a_{\beta\sigma} - \partial_{\sigma} a_{\beta\gamma} \right) e^{\beta}_i e^{\gamma}_j \right) \hat{g}^{ij} - a(e_k, e_l) h^{kl} \\
&= -\frac{1}{2} \left(2 \partial_{\beta} a_{\gamma\sigma} - \partial_{\sigma} a_{\beta\gamma} \right) \nu^{\sigma} e^{\beta}_i e^{\gamma}_j \hat{g}^{ij} + \frac{1}{2} a(\nu,\nu) H - \langle a, h \rangle_{\hat{g}}
\end{aligned}$$

We note that

$$\nu(\hat{g}_{ij}) = \nu \langle e_i, e_j \rangle_{g_0} = \langle D_{\nu} e_i, e_j \rangle_{g_0} + \langle e_i, D_{\nu} e_j \rangle_{g_0} = \langle D_{e_i} \nu, e_j \rangle_{g_0} + \left\langle e_i, D_{e_j} \nu \right\rangle_{g_0} = 2 h_{ij}$$

and

$$\nu(\hat{g}^{ij}) = -\hat{g}^{ik}\nu(\hat{g}_{kl})\hat{g}^{lj} = -\hat{g}^{ik}(2h_{kl})\hat{g}^{lj} = -2h^{ij}$$

So

$$\begin{aligned}
\frac{1}{2}\partial_\sigma a_{\beta\gamma}\nu^\sigma e_i^\beta e_j^\gamma \hat{g}^{ij} &= \frac{1}{2}\nu(a_{\beta\gamma}e_i^\beta e_j^\gamma)\hat{g}^{ij} - a_{\beta\gamma}\nu(e_i^\beta)e_j^\gamma \hat{g}^{ij} \\
&= \frac{1}{2}\nu(a(e_i, e_j))\hat{g}^{ij} - a(D_\nu e_i, e_j)\hat{g}^{ij} \\
&= \frac{1}{2}\nu(a(e_i, e_j)\hat{g}^{ij}) - \frac{1}{2}a(e_i, e_j)\nu(\hat{g}^{ij}) - a(h_{ik}\hat{g}^{kl}e_l, e_j)\hat{g}^{ij} \\
&= \frac{1}{2}\nu(\operatorname{tr}_{\hat{g}} a) + a(e_i, e_j)h^{ij} - a(e_l, e_j)h^{jl} \\
&= \frac{1}{2}\nu(\operatorname{tr}_\Sigma a)
\end{aligned}$$

and

$$\begin{aligned}
-\partial_\beta a_{\gamma\sigma}\nu^\sigma e_i^\beta e_j^\gamma \hat{g}^{ij} &= -e_i(a_{\gamma\sigma}\nu^\sigma e_j^\gamma)\hat{g}^{ij} + a_{\gamma\sigma}\nu^\sigma e_i(e_j^\gamma)\hat{g}^{ij} + a_{\gamma\sigma}e_i(\nu^\sigma)e_j^\gamma \hat{g}^{ij} \\
&= -e_i(a(\nu, e_j))\hat{g}^{ij} + a(\nu, D_{e_i}e_j)\hat{g}^{ij} + a(D_{e_i}\nu, e_j)\hat{g}^{ij} \\
&= -e_i(a(\nu, e_j))\hat{g}^{ij} + a(\nu, \nabla_{e_i}e_j)\hat{g}^{ij} - a(\nu, h_{ij}\nu)\hat{g}^{ij} + a(e_l, e_j)h^{jl} \\
&= -\operatorname{div}_\Sigma a(\nu, \cdot) - a(\nu, \nu)H + \langle a, h\rangle_{\hat{g}}
\end{aligned}$$

where $\operatorname{div}_\Sigma a(\nu, \cdot) = \langle \nabla_{e_i} a(\nu, \cdot), e_j\rangle \hat{g}^{ij}$. Thus,

$$\begin{aligned}
\dot{H} &= -\operatorname{div}_\Sigma a(\nu, \cdot) - a(\nu, \nu)H + \langle a, h\rangle_{\hat{g}} + \frac{1}{2}\nu(\operatorname{tr}_\Sigma a) + \frac{1}{2}a(\nu, \nu)H - \langle a, h\rangle_{\hat{g}} \\
&= \frac{1}{2}\nu(\operatorname{tr}_\Sigma a) - \operatorname{div}_\Sigma a(\nu, \cdot) - \frac{1}{2}a(\nu, \nu)H
\end{aligned}$$

As a summary, we have the following proposition:

Proposition 2.1. *The scalar curvature map is a smooth map of Banach manifolds, as a map $R : \mathcal{M}^{l+2}(\Omega) \to H^l(\Omega)\,(l + 2 > \frac{n}{2} + 1)$, or $R : \mathcal{M}^{k+2,\alpha}(\Omega) \to C^{k,\alpha}(\Omega)\,(k \geq 0)$. The* *linearization L_g of the scalar curvature operator is given by*

$$L_g(a) = -\Delta_g\,(\operatorname{tr}_g a) + \operatorname{div}_g(\operatorname{div}_g a) - \langle a, \operatorname{Ric}_g\rangle_g$$

in the above spaces.

The mean curvature map is a smooth map of Banach manifolds, as a map $H : \mathcal{M}^{l+2}(\Omega) \to H^{l+\frac{1}{2}}(\Sigma)\,(l+2 > \frac{n}{2}+1)$, or $H : \mathcal{M}^{k+2,\alpha}(\Omega) \to C^{k,\alpha}(\Sigma)\,(k \geq 0)$. The linearization $\dot{H}$ of the mean curvature operator of Σ is given by

$$\dot{H}(a) = \frac{1}{2}\nu(\operatorname{tr}_\Sigma a) - \operatorname{div}_\Sigma a(\nu, \cdot) - \frac{1}{2}a(\nu,\nu)H$$

in the above spaces.

The next step is to derive an integration by parts formula between L and its formal L^2-adjoint L^*, i.e.

$$\int_\Omega L(a)u = \int_\Omega \langle a, L^*u\rangle + \text{Bdry}$$

where

$$L^*(u) = -\left(\Delta_g u\right) g + \operatorname{Hess}(u) - u\operatorname{Ric}_g$$

and

$$\begin{aligned}
\text{Bdry} &= \int_\Sigma \sum_\alpha u(D_\alpha a_{n\alpha} - \partial_n(\operatorname{tr} a)) - \int_\Sigma \sum_\alpha (a_{n\alpha}\partial_\alpha u - u_n \operatorname{tr} a) \\
&= \int_\Sigma \sum_i u(D_i a_{ni} + D_n a_{nn} - \partial_n(\operatorname{tr}_\Sigma a) - \partial_n a_{nn}) - \int_\Sigma \sum_i (a_{ni}\partial_i u - u_n \operatorname{tr}_\Sigma a)
\end{aligned}$$

Note that in Fermi coordinates $D_n e_n = 0$, so

$$D_n a_{nn} = \partial_n a_{nn} - 2a(e_n, D_n e_n) = \partial_n a_{nn}$$

Since

$$\begin{aligned}
\sum_i D_i a_{ni} &= \sum_i \left(\partial_i a_{ni} - a(e_n, D_i e_i) - a(D_i e_n, e_i)\right) \\
&= \sum_i \left(\partial_i a_{ni} - a(e_n, \nabla_{e_i} e_i)\right) + \sum_i a(e_n, h_{ii} e_n) - \sum_{i,j} a(h_{ij} e_j, e_i) \\
&= \operatorname{div}_\Sigma a(\nu, \cdot) + a(\nu,\nu) H - \langle a, h\rangle_{\hat{g}}
\end{aligned}$$

we then have

$$\begin{aligned}
\text{Bdry} &= \int_\Sigma \sum_i u(D_i a_{ni} - \nu(\operatorname{tr}_\Sigma a)) + \int_\Sigma u \operatorname{div}_\Sigma a(\nu, \cdot) + \int_\Sigma u_\nu \operatorname{tr}_\Sigma a \\
&= \int_\Sigma 2u \left(-\frac{1}{2}\nu(\operatorname{tr}_\Sigma a) + \operatorname{div}_\Sigma a(\nu, \cdot) + \frac{1}{2} a(\nu,\nu) H\right) - \int_\Sigma u \langle a, h\rangle_{\hat{g}} + \int_\Sigma u_\nu \operatorname{tr}_\Sigma a \\
&= -\int_\Sigma 2u\dot{H}(a) - \int_\Sigma u \langle a, h\rangle_{\hat{g}} + \int_\Sigma u_\nu \operatorname{tr}_\Sigma a
\end{aligned}$$

Proposition 2.2. *We have the following integration by parts formula:*

$$\int_\Omega L(a) u \, d\mu_g = \int_\Omega \langle a, L^* u\rangle \, d\mu_g + \int_\Sigma \left(-2u\dot{H}(a) - u\langle a, h\rangle_{\hat{g}} + u_\nu \operatorname{tr}_\Sigma a\right) d\sigma_g.$$

2.3 Weight function

Let Ω denote a compactly contained subdomain with boundary of a smooth manifold M^n. Unless noted, we assume the boundary is smooth. Let $\Sigma \neq \varnothing$ be a smooth $(n-1)$-submanifold, where Σ is a part of $\partial\Omega$. For example, one can think of Ω as a ball and Σ as the upper hemi-sphere.

We would like to define a weight function ρ on $\Omega \cup \Sigma$ whose behavior near $\partial\Omega \setminus \Sigma$ is determined by the distance to $\partial\Omega \setminus \Sigma$. Let $d_g(x) = d_g(x, \partial\Omega \setminus \Sigma)$ be the distance to $\partial\Omega \setminus \Sigma$ with respect to g; the boundary is assumed to be a smooth hypersurface, so near $\partial\Omega \setminus \Sigma$, d is as regular

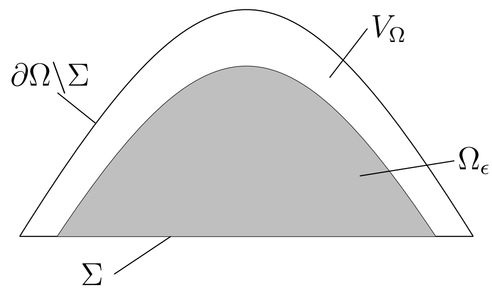

Figure 2.1: Weight function

as g. We will work with uniformly equivalent metrics in a bounded open set $\mathcal{U}_0$ in the space of $C^m(\bar{\Omega})(m \geq 2)$ Riemannian metrics such that $\|d_g\|_{C^m}$ is uniformly bounded near $\partial\Omega \setminus \Sigma$.

Let $V_\Omega = \{x \in \Omega \cup \Sigma : d_g(x) < r_0 \text{ for some } g \in \mathcal{U}_0\}$ be a thin regular collar neighborhood of $\partial\Omega \setminus \Sigma$. There is $r_0 \in (0, \frac{1}{2})$ sufficiently small so that a neighborhood of V_Ω is foliated by smooth (as regular as the metric g is) level sets of d_g and that $d_g(x) \leq \frac{1}{2}$ for all $x \in V_\Omega$ and $g \in \mathcal{U}_0$.

We will define $0 \leq \rho \leq 1$ to tend monotonically to zero with decreasing distance to the boundary $\partial\Omega \setminus \Sigma$, and use the weight ρ^{-1} which blows up at the boundary $\partial\Omega \setminus \Sigma$ to form weighted L^2-spaces of tensors, which by design will be forced to decay suitably at $\partial\Omega \setminus \Sigma$.

Let us follow Corvino-Huang [17] to define an exponential weight function. Let $0 < r_1 < r_0$ be fixed. Define a smooth positive monotone function $\tilde{\rho} : (0, \infty) \to \mathbb{R}$ such that $\tilde{\rho}(t) = e^{-1/t}$ for $t \in (0, r_1)$ and $\tilde{\rho}(t) = 1$ for $t > r_0$. For $N > 0$, let ρ_g be the positive function on Ω defined by

$$\rho_g(x) = (\tilde{\rho} \circ d_g(x))^N .$$

We will eventually fix N to be a suitably large number.

We also denote $\Omega_\epsilon \subset \Omega$ as the region where $d_g > \epsilon$ for some small $\epsilon \geq 0$, so that $\Omega_{r_0} \subset \Omega$ is a region where $\rho_g = 1$.

2.4 Weighted Hölder spaces

In this section, we define the weighted Hölder space and discuss some related properties. Here we follow the idea of [14] to consider weighted Hölder norms in small balls $B_{\phi(x)}(x)$ that cover Ω. The weight function $\phi = \phi_g$ satisfies the following properties with uniform estimates across g in a $\mathcal{C}^m(\Omega)$ neighborhood $\mathcal{U}_0$. Recall the neighborhood V_Ω defined in the previous section, and suppose we have chosen a suitable N.

Proposition 2.3 (Corvino-Huang [17]). *For $g \in \mathcal{U}_0$, we define $\phi(x) = (d(x))^2$ in V_Ω. There exists a constant $C > 0$, uniform across $\mathcal{U}_0$, such that we can extend ϕ to Ω with $0 < \phi < 1$ and with the following properties.*

(i) ϕ has a positive lower bound on $\Omega \backslash V_\Omega$ uniformly in $g \in \mathcal{U}_0$, and for each x, $\phi(x) < d(x)$, so that $\overline{B_{\phi(x)}(x)} \subset \Omega$.

(ii) For $x \in \Omega$ and $k \leq m$, we have $\left|\phi^k \rho^{-1} \nabla^k \rho\right| \leq C$.

(iii) For $x \in \Omega$ and for $y \in B_{\phi(x)}(x)$, we have

$$\begin{aligned} C^{-1}\rho(y) \leq \rho(x) \leq C\rho(y) \\ C^{-1}\phi(y) \leq \phi(x) \leq C\phi(y). \end{aligned} \tag{2.2}$$

Let $r, s \in \mathbb{R}$ and $\varphi = \phi^r \rho^s$. For $u \in C^{k,\alpha}_{\mathrm{loc}}(\Omega)$, we define the weighted Hölder norms by

$$\|u\|_{C^{k,\alpha}_{\phi,\varphi}(\Omega)} = \sup_{x\in\Omega} \left(\sum_{j=0}^{k} \varphi(x)\phi^j(x) \left\|\nabla^j_g u\right\|_{C^0\left(B_{\phi(x)}(x)\right)} + \varphi(x)\phi^{k+\alpha}(x) \left[\nabla^k_g u\right]_{0,\alpha;B_{\phi(x)}(x)} \right).$$

Note that ϕ is to make the norm scaling invariant with respect to the size of the ball. The weighted Hölder space $C^{k,\alpha}_{\phi,\varphi}(\Omega)$ consists of $C^{k,\alpha}_{\mathrm{loc}}(\Omega)$ functions or tensor fields with finite $C^{k,\alpha}_{\phi,\varphi}(\Omega)$ norm. If $u \in C^{k,\alpha}_{\phi,\varphi}(\Omega)$, then u is dominated by φ^{-1} in the sense that $u = O(\varphi^{-1})$ and $\nabla^j u = O(\varphi^{-1}\phi^{-j})$ near the boundary.

Proposition 2.4 (Corvino-Huang [17]). *We have some properties for weighted Hölder spaces:*

(i) The differentiation is a continuous map:

$$\nabla : C^{k,\alpha}_{\phi,\varphi}(\Omega) \to C^{k-1,\alpha}_{\phi,\phi\varphi}(\Omega).$$

(ii) For $u \in C^{k,\alpha}_{\phi,\varphi}(\Omega)$*,* $v \in C^{k,\alpha}(\bar{\Omega})$*, we have* $uv \in C^{k,\alpha}_{\phi,\varphi}(\Omega)$ *and* $C = C(k)$ *with*

$$\|uv\|_{C^{k,\alpha}_{\phi,\varphi}(\Omega)} \leq C\|u\|_{C^{k,\alpha}_{\phi,\varphi}(\Omega)}\|v\|_{C^{k,\alpha}(\bar{\Omega})}.$$

(iii) The multiplication by ρ *is a continuous map from* $C^{k,\alpha}_{\phi,\varphi}$ *to* $C^{k,\alpha}_{\phi,\varphi\rho^{-1}}$*.*

We will use the following Banach spaces $\mathcal{B}_k(\Omega)$ (for functions or tensor fields):

$$\begin{aligned}
\mathcal{B}_0(\Omega) &= C^{0,\alpha}_{\phi,\phi^{4+\frac{n}{2}}\rho^{-\frac{1}{2}}}(\Omega) \cap H^{-2}_{\rho^{-1}}(\Omega)\\
\mathcal{B}_1(\Omega) &= C^{1,\alpha}_{\phi,\phi^{3+\frac{n}{2}}\rho^{-\frac{1}{2}}}(\Omega) \cap H^{-1}_{\rho^{-1}}(\Omega)\\
\mathcal{B}_2(\Omega) &= C^{2,\alpha}_{\phi,\phi^{2+\frac{n}{2}}\rho^{-\frac{1}{2}}}(\Omega) \cap L^{2}_{\rho^{-1}}(\Omega)\\
\mathcal{B}_3(\Omega) &= C^{3,\alpha}_{\phi,\phi^{1+\frac{n}{2}}\rho^{\frac{1}{2}}}(\Omega) \cap H^{1}_{\rho}(\Omega)\\
\mathcal{B}_4(\Omega) &= C^{4,\alpha}_{\phi,\phi^{\frac{n}{2}}\rho^{\frac{1}{2}}}(\Omega) \cap H^{2}_{\rho}(\Omega),
\end{aligned}$$

with the Banach norms:

$$\begin{aligned}
\|u\|_{\mathcal{B}_0(\Omega)} &= \|u\|_{C^{0,\alpha}_{\phi,\phi^{4+\frac{n}{2}}\rho^{-\frac{1}{2}}}(\Omega)} + \|u\|_{H^{-2}_{\rho^{-1}}(\Omega)}\\
\|u\|_{\mathcal{B}_1(\Omega)} &= \|u\|_{C^{1,\alpha}_{\phi,\phi^{3+\frac{n}{2}}\rho^{-\frac{1}{2}}}(\Omega)} + \|u\|_{H^{-1}_{\rho^{-1}}(\Omega)}\\
\|u\|_{\mathcal{B}_2(\Omega)} &= \|u\|_{C^{2,\alpha}_{\phi,\phi^{2+\frac{n}{2}}\rho^{-\frac{1}{2}}}(\Omega)} + \|u\|_{L^{2}_{\rho^{-1}}(\Omega)}\\
\|u\|_{\mathcal{B}_3(\Omega)} &= \|u\|_{C^{3,\alpha}_{\phi,\phi^{1+\frac{n}{2}}\rho^{\frac{1}{2}}}(\Omega)} + \|u\|_{H^{1}_{\rho}(\Omega)}\\
\|u\|_{\mathcal{B}_4(\Omega)} &= \|u\|_{C^{4,\alpha}_{\phi,\phi^{\frac{n}{2}}\rho^{\frac{1}{2}}}(\Omega)} + \|u\|_{H^{2}_{\rho}(\Omega)}.
\end{aligned}$$

It is clear that these Banach spaces contain the smooth functions with compact supports in Ω. We can similarly define Banach spaces $\mathcal{B}_k(\Sigma)$, but note that the weight may be different on the boundary. For example, $\mathcal{B}_0(\Sigma) = C^{0,\alpha}_{\phi,\phi^{3+\frac{n}{2}}\rho^{-\frac{1}{2}}}(\Sigma) \cap \mathcal{D}^*$, which will be defined later.

2.5 Statement of the main theorem

The main theorems are stated as follows:

Theorem 2.5. *In a Riemannian manifold M^n with $n \geq 2$ and a $C^{4,\alpha}$-metric g_0, let $\Omega \subset M$ be a compactly contained C^3-domain with smooth boundary, and let $\Sigma \neq \varnothing$ be a smooth $(n-1)$-submanifold, where Σ is a part of $\partial\Omega$. Assuming certain generic conditions, there is an $\epsilon > 0$ such that*

(i) for any $S \in C^{0,\alpha}(\Omega)$ for which $(S - R(g_0)) \in \mathcal{B}_0(\Omega)$ with the support of $(S - R(g_0))$ contained in $\bar{\Omega}$;

(ii) and any $H' \in C^{0,\alpha}(\Sigma)$ for which $(H' - H(g_0)) \in \mathcal{B}_0(\Sigma)$ with the support of $(H' - H(g_0))$ contained in $\bar{\Sigma}$ and

$$\|S - R(g_0)\|_{\mathcal{B}_0(\Omega)} + \|H' - H(g_0)\|_{\mathcal{B}_0(\Sigma)} < \epsilon,$$

there is a $C^{2,\alpha}$-metric g with $R(g) = S$ in Ω, $H(g) = H'$ in Σ and $g \equiv g_0$ outside Ω.

Our proofs also give the following version of Theorem 2.5 that includes higher.

Theorem 2.6. *In a Riemannian manifold M^n with $n \geq 2$ and a $C^{k+4,\alpha}$-metric g_0, let $\Omega \subset M$ be a compactly contained C^{k+3}-domain with smooth boundary, and let $\Sigma \neq \varnothing$ be a smooth $(n-1)$-submanifold, where Σ is a part of $\partial\Omega$. Assuming certain generic conditions, there is an $\epsilon > 0$ such that*

(i) for any $S \in C^{k,\alpha}(\Omega)$ for which $(S - R(g_0)) \in \mathcal{B}_0(\Omega)$ with the support of $(S - R(g_0))$ contained in $\bar{\Omega}$;

(ii) and any $H' \in C^{k,\alpha}(\Sigma)$ for which $(H' - H(g_0)) \in \mathcal{B}_0(\Sigma)$ with the support of $(H' - H(g_0))$

contained in $\bar{\Sigma}$ *and*

$$\|S - R(g_0)\|_{\mathcal{B}_0(\Omega)} + \|H' - H(g_0)\|_{\mathcal{B}_0(\Sigma)} < \epsilon,$$

there is a $C^{k+2,\alpha}$*-metric* g *with* $R(g) = S$ *in* Ω*,* $H(g) = H'$ *in* Σ *and* $g \equiv g_0$ *outside* Ω*.*
If, in addition, $g_0 \in C^\infty(\bar{\Omega})$ *and* $(S, H') \in C^\infty(\Omega) \times C^\infty(\Sigma)$*, then we can achieve* $g \in C^\infty(\bar{\Omega})$*.*

We will discuss generic conditions in more detail in Chapter 3.

We note that Cruz and Vitório [19] had the following local surjectivity theorem for compact manifolds with boundary:

Theorem 2.7 (Cruz-Vitório)**.** *Let* $f = (f_1, f_2) \in L^p(M) \oplus W^{\frac{1}{2},p}(\partial M)$*,* $p > n$*. Assume certain generic conditions, then there is an* $\epsilon > 0$ *such that if*

$$\|f_1 - R_{g_0}\|_{L^p(M)} + \|f_2 - H_{\gamma_0}\|_{W^{1/2,p}(\partial M)} < \epsilon,$$

then there is a metric $g_1 \in \mathcal{M}^{2,p}$ *such that* $R(g_1) = f_1, H(g_1) = f_2$*. Moreover,* g *is smooth in any open set where* f *is smooth.*

Their result can now be viewed as a special case of our main theorem, where $M = \Omega$ and $\Sigma = \partial\Omega$.

2.6 A basic lemma

Finally, we would like to introduce a basic lemma, which will be used frequently in later discussions.

Lemma 2.8. *There is no non-zero solution in $H^2_{\mathrm{loc}}(\Omega)$ to the following equations:*

$$\begin{cases} L^*u = 0 & \textit{in } \Omega \\ u = u_\nu = 0 & \textit{on } \Sigma. \end{cases}$$

Proof. First of all, it is obvious that $u \equiv 0$ is a trivial solution.

Now fix an open neighborhood $O \subset \Sigma$. For each $x \in O$, $u_\nu(x) = u(x) = 0$. Let $\gamma_x(t)$ be a geodesic starting at x and $\gamma_x'(t)$ normal to Σ, and let $f(t) = u(\gamma_x(t))$. Then $f(t)$ satisfies the linear second order differential equation

$$\begin{aligned} f''(t) &= \mathrm{Hess}_{\gamma_x(t)}\, u\,(\gamma_x'(t), \gamma_x'(t)) \\ &= \left[\left(\mathrm{Ric}(g) - \frac{R(g)}{n-1} g\right)(\gamma_x'(t), \gamma_x'(t))\right] f(t) \end{aligned}$$

with $f(0) = u(x) = 0$ and $f'(0) = du(x) \cdot \gamma_x'(0) = u_\nu(x) = 0$. Thus u is zero along $\gamma_x(t)$ for every $x \in O$. Since Σ is smooth, this would imply that there is $y \in \Omega$ in the interior such that u is identically zero in a neighborhood $\tilde{O} \subset \Omega$ of y. Then as pointed out in [15], u satisfies the elliptic equation $(n-1)\Delta_g u = -R(g)u$ in Ω (by taking the trace of the equation $L^*u = 0$), so we have by Aronszajn's unique continuation theorem [4] that u must vanish on all of Ω. □

Chapter 3

Generic Conditions and Non-generic Domains

3.1 Generic conditions

As we mentioned previously, we will need to assume certain generic conditions in order to perform localized deformations. Now let us take a closer look at the generic conditions, and some examples when generic conditions are not satisfied.

Let (Ω, Σ) be a domain in $(M^n, g)\,(n \geq 3)$ where Σ is the boundary of Ω. Denote $\tilde{\mathcal{S}}^{(0,2)}(\Omega)$ as the space of symmetric $(0,2)$-tensor fields that are smooth. Define $\Phi : \tilde{\mathcal{S}}^{(0,2)}(\Omega) \longrightarrow C^\infty(\Omega) \oplus C^\infty(\Sigma)$ to be

$$\Phi(a) = (L(a), B(a)),$$

where $B(a) = 2\dot{H}(a)$.

Then for any $u \in C^\infty(\Omega)$,

$$
\begin{aligned}
\langle \Phi(a), u\rangle &\triangleq \langle (L(a), B(a)), (u, u|_\Sigma)\rangle \\
&= \int_\Omega L(a) u \, d\mu_g + \int_\Sigma 2\dot{H}(a) u \, d\sigma_g \\
&= \int_\Omega \langle a, L^* u\rangle \, d\mu_g + \int_\Sigma \langle a, u_\nu \hat{g} - uh\rangle_{\hat{g}} \, d\sigma_g \triangleq \langle a, \Phi^*(u)\rangle
\end{aligned}
$$

So the formal L^2-adjoint of Φ is the operator $\Phi^* : C^\infty(\Omega) \longrightarrow \tilde{\mathcal{S}}^{(0,2)}(\Omega) \oplus \tilde{\mathcal{S}}^{(0,2)}(\Sigma)$ given by

$$\Phi^*(u) = (L^*(u), u_\nu \hat{g} - uh)$$

with norm $\|\Phi^* u\|_{\mathcal{L}^2} = \|L^* u\|_{\mathcal{L}^2(\Omega)} + \|u_\nu \hat{g} - uh\|_{\mathcal{L}^2(\Sigma)}$.

The kernel of Φ^* is then the solutions to the following system of equations:

$$
\begin{cases}
L^* u = 0 & \text{in } \Omega \\
u_\nu \hat{g} = uh & \text{on } \Sigma
\end{cases}
\tag{3.1}
$$

We say that generic conditions are satisfied on (Ω, Σ) when $\ker \Phi^*$ is trivial. Notice here Ω may not be compact.

If $x_0 \in \Sigma$ is an umbilical point, then $h_{ij} = \lambda(x_0)\hat{g}_{ij} = \frac{H}{n-1}\hat{g}_{ij}$. This means

$$u_\nu = \frac{H}{n-1} u \qquad \text{at } x_0.$$

And if $x_0 \in \Sigma$ is not an umbilical point, then we must have

$$u_\nu = u = 0 \qquad \text{at } x_0.$$

Moreover, if $x_0 \in \Sigma$ is a non-umbilical point, then there is an open neighborhood $O \subset \Sigma$ of x_0 such that any $x \in O$ is a non-umbilical point. For each $x \in O$, $u_\nu(x) = u(x) = 0$. Then the same proof as Lemma 2.8 tells us that u vanishes on all of Ω.

In [15], Corvino found that $\ker L^* \neq 0$ is equivalent to the metric g being static, and it would then imply the scalar curvature is constant in Ω [27]. In fact, it is related to static spacetimes in general relativity. Recall that a static spacetime is a four-dimensional Lorentzian manifold which possesses a timelike Killing field and a spacelike hypersurface which is orthogonal to the integral curves of this Killing field. Corvino found that

Proposition 3.1 (Corvino)**.** *Let (M^n, g) be a Riemannian manifold. Then $0 \neq f \in \ker L_g^*$ if and only if the warped product metric $\bar{g} \equiv -f^2 dt^2 + g$ is Einstein.*

Combined with the above discussion, we have

Theorem 3.2. *Let (Ω, Σ) be a domain in (M^n, g). If the metric g is non-static, or there is any non-umbilical point on Σ, then $\ker \Phi^*$ is trivial.*

Many authors call a non-trivial element $u \in \ker L^*$ a static potential. With the boundary term, however, we will call a non-trivial element $u \in \ker L^*$ a **possible** static potential, and if it satisfies the boundary condition on Σ as well, we then call it a static potential. That is, a non-trivial element $u \in \ker \Phi^*$ is called a static potential in our setting. So in this case, generic conditions are not satisfied. Let us now study the static potentials more carefully and derive some geometric properties of Ω and Σ. In fact, we have the following result:

Theorem 3.3. *If $\ker \Phi^*$ is non-trivial, then the scalar curvature of Ω is constant, the boundary Σ is umbilic, and the mean curvature is constant on Σ.*

Proof. From the above discussion, we know Σ is umbilic and $h = \frac{H}{n-1}\hat{g}$ on Σ. So under an orthonormal basis $\{e_1, \cdots, e_{n-1}, \nu\}$, where e_i's are tangent to Σ and ν is the outward unit normal, we can take the covariant derivative in e_i of the second equation of (3.1),

$$
\begin{aligned}
0 &= \nabla_i \left(u_\nu \hat{g}^i_j - u h^i_j\right) \\
&= \nabla_i \left(u_\nu \hat{g}^i_j - u\frac{H}{n-1}\hat{g}^i_j\right) \\
&= \nabla_i \left(u_\nu - u\frac{H}{n-1}\right)\hat{g}^i_j \\
&= \left(u_{i\nu} + (D_i\nu)u - u_i\frac{H}{n-1} - \frac{u}{n-1}\nabla_i H\right)\hat{g}^i_j \\
&= \left(u_{i\nu} + h^k_i u_k - u_i\frac{H}{n-1} - \frac{u}{n-1}\nabla_i H\right)\hat{g}^i_j \\
&= u_{j\nu} + h^k_j u_k - u_i h^i_j - \frac{u}{n-1}\nabla_j H \\
&= u_{j\nu} - \frac{u}{n-1}\nabla_j H
\end{aligned}
$$

And by looking at the $i\nu$-th component of $0 = L^* u = -\left(\Delta_g u\right) g + \mathrm{Hess}(u) - u\,\mathrm{Ric}_g$, we have

$$0 = u_{i\nu} - uR_{i\nu}$$

On the other hand, by the Codazzi equation,

$$
\begin{aligned}
\nabla_i H = \nabla_i(h_{jk}\hat{g}^{jk}) &= \nabla_i h_{jk}\hat{g}^{jk} = \left(\nabla_j h_{ik} - R_{\nu kij}\right)\hat{g}^{jk} \\
&= \nabla_j\left(\frac{H}{n-1}\delta^j_i\right) - R_{i\nu} = \frac{1}{n-1}\nabla_i H - R_{i\nu}
\end{aligned}
$$

So by the above three equations, we get

$$\frac{1}{n-1}u\nabla_i H = u_{i\nu} = uR_{i\nu} = -\frac{n-2}{n-1}u\nabla_i H$$

and this means $u\nabla_i H = 0$.

We claim that $\nabla H = 0$ on Σ, which implies the mean curvature is constant. Otherwise, if there exists $x_0 \in \Sigma$ such that $\nabla H(x_0) \neq 0$, then there is an open neighborhood $O \subset \Sigma$ of x_0 such that at any $x \in O$, $\nabla H(x) \neq 0$. By the above condition, $u(x) = 0$ for any $x \in O$. Then since Σ is umbilic, $u_\nu(x) = u(x) = 0$ for any $x \in O$. Finally the same proof as Lemma 2.8 tells us that u vanishes on all of Ω. □

3.2 Non-generic domains

If generic conditions are not satisfied, then (Ω, Σ) is called a non-generic domain in (M^n, g). We have particular interest in non-generic domains and the space of static potentials on them. One of the reasons is that, non-generic domains are often related to certain kinds of positive mass theorem (rigidity theorem) (e.g. [1,33,35,39,47]).

In general, for non-generic domains (Ω, Σ), we have the following results as a direct consequence of [15, Corollary 2.4, Proposition 2.5]:

Proposition 3.4. *For non-generic domains* (Ω, Σ) *in* (M^n, g)*, we have* $\dim \ker \Phi^* \leq n$ *in* $H^2_{\text{loc}}(\Omega)$.

As we will see later, the upper bound is achieved by simple non-generic domains in space forms.

Proposition 3.5. H^2_{loc} *elements in* $\ker \Phi^*$ *are actually in* $C^2(\bar{\Omega})$.

3.2.1 Compact non-generic domains

Let us first focus on the case where non-generic domains are compact.

Taking the trace of (3.1), we get

$$\begin{cases} \Delta_g u + \dfrac{R_g}{n-1} u = 0 & \text{in } \Omega \\ u_\nu - \dfrac{H}{n-1} u = 0 & \text{on } \Sigma \end{cases} \tag{3.2}$$

This means, for any static potential u, they must first satisfy (3.2). It is very useful when we want to get some global properties of compact non-generic domains.

From Theorem 3.3 we know that the scalar curvature R of Ω and the mean curvature H of Σ are constant on non-generic domains. Thus all the non-generic domains may be divided into 9 cases, according to whether R and H are positive, negative or zero. However, some of the cases are not possible. Cruz and Vitório [19] found that:

Proposition 3.6 (Cruz-Vitório)**.** *Assume* $\ker \Phi^*$ *is non-trivial.*

(i) If $R = 0$*, then* $H \geq 0$*; and if* $H = 0$ *as well, then* $h \equiv 0$*.*

(ii) If $H = 0$*, then* $R \geq 0$*; and if* $R = 0$ *as well, then* $Ric_g \equiv 0$*.*

Note that:

(i) If $R = 0$ and $H > 0$, then $\frac{H}{n-1}$ is an Steklov eigenvalue; in this case, $\ker \Phi^*$ lies in the eigenspace corresponding to the Steklov eigenvalue $\frac{H}{n-1}$.

(ii) If $R > 0$ and $H = 0$, then $\frac{R}{n-1}$ is an eigenvalue of the Neumann boundary value problem; in this case, $\ker \Phi^*$ lies in the eigenspace corresponding to the eigenvalue $\frac{R}{n-1}$.

(iii) If $R = H = 0$, then g is Ricci flat and Σ is totally geodesic. On the other hand, Ho and Huang [34] found that:

Proposition 3.7 (Ho-Huang)**.** *If g is Ricci flat and Σ is totally geodesic, then* $\ker \Phi^* \neq 0$*. In fact,* $\ker \Phi^*$ *consists of constant functions in* Ω*.*

Thus, for case (iii) we have the following result:

Theorem 3.8. *Assuming* $\ker \Phi^*$ *is non-trivial, if* $R = H = 0$*, then* g *is Ricci flat,* Σ *is totally geodesic, and* $\ker \Phi^*$ *consists of constant functions in* Ω*.*

We may further rule out some more cases:

Theorem 3.9. *Assuming* $\ker \Phi^*$ *is non-trivial, if* $R < 0$*, then* $H > 0$*; if* $H < 0$*, then* $R > 0$*.*

Proof. If $0 \neq u \in \ker \Phi^*$, then u must satisfy (3.2). Multiplying u to the first equation and integrating it over Ω, we get

$$\begin{aligned} 0 &= \int_\Omega \left(u\Delta_g u + \frac{R_g}{n-1} u^2 \right) d\mu_g \\ &= \int_\Omega \left(-|Du|^2 + \frac{R_g}{n-1} u^2 \right) d\mu_g + \int_\Sigma u u_\nu \, d\sigma_g \\ &= \int_\Omega \left(-|Du|^2 + \frac{R_g}{n-1} u^2 \right) d\mu_g + \int_\Sigma \frac{H}{n-1} u^2 \, d\sigma_g \end{aligned}$$

Hence, if $R < 0$, the first integral is strictly negative, then H must be strictly positive. Similarly, if $H < 0$, then $R > 0$. □

As a summary, we have the following diagram for compact non-generic domains:

H \ R	+	0	−
+	?	Steklov	?
0	Neumann	*	×
−	?	×	×

Table 3.1: Cases of compact non-generic domains

In what follows, we will see some more examples that will fill up the diagram above.

3.2.2 General non-generic domains

Now we would like to study non-generic domains which may not be compact. Among them we are particularly interested in non-generic domains in a space form and in the Schwarzschild manifold.

The umbilic hypersurface in a space form has constant mean curvature [13], thus the region bounded by an umbilic hypersurface in a space form satisfies all the necessary conditions of a non-generic domain. We will see that the regions bounded by umbilic hypersurfaces in a space form are indeed non-generic domains.

The unit sphere $\mathbb{S}^n$

Consider the standard metric $g_{\mathbb{S}^n}$ on the unit sphere $\mathbb{S}^n$. In this case, the sectional curvature $K = 1$, the Ricci curvature $\mathrm{Ric} = (n-1)g_{\mathbb{S}^n}$, and the scalar curvature $R = n(n-1)$.

Cruz and Vitório [19] found that the upper hemisphere with its boundary is a non-generic domain in $\mathbb{S}^n$. Later Ho and Huang [34] had the following result:

Proposition 3.10 (Ho-Huang). *Let $(\mathbb{S}^n_+, \partial\mathbb{S}^n_+)$ be the n-dimensional upper hemisphere equipped with the standard metric $g_{\mathbb{S}^n}$, where $n \geq 3$. Then $\ker \Phi^*$ is non-trivial. Moreover,*

$$\ker \Phi^* = \mathrm{span}\{x_1, \cdots, x_n\},$$

where $(x_1, \cdots, x_{n+1})$ are the coordinates of $\mathbb{S}^n \subset \mathbb{R}^{n+1}$.

In fact, for $x_i\,(i = 1, \cdots, n+1)$, we have

$$\mathrm{Hess}_{g_{\mathbb{S}^n}} x_i + x_i g_{\mathbb{S}^n} = 0.$$

Thus,

$$
\begin{aligned}
L^*(x_i) &= -\left(\Delta x_i\right) g_{\mathbb{S}^n} + \mathrm{Hess}_{g_{\mathbb{S}^n}} x_i - (n-1)x_i g_{g_{\mathbb{S}^n}} \\
&= -\left(\Delta x_i\right) g_{\mathbb{S}^n} - x_i g_{\mathbb{S}^n} - (n-1)x_i g_{g_{\mathbb{S}^n}} \\
&= -\left(\Delta x_i + n x_i\right) g_{\mathbb{S}^n} \\
&= 0
\end{aligned}
$$

Theorem 3.11. *The linear combinations of the coordinates* $x_i\,(i = 1, \cdots, n+1)$ *are the only possible static potentials on non-generic domains* $\Omega \subset \mathbb{S}^n\,(n \geq 3)$.

Proof. From the non-generic condition on Ω

$$
L^* u = -\left(\Delta u\right) g + \mathrm{Hess}\, u - (n-1)ug = 0,
$$

we get

$$
\mathcal{L}_{Du} g = 2\,\mathrm{Hess}\, u = 2\left(\Delta u + (n-1)u\right) g.
$$

Therefore, Du is a conformal killing vector field.

From Liouville's theorem on conformal mappings [6], every conformal diffeomorphism between two domains of $\mathbb{R}^n\,(n \geq 3)$ has the form

$$
x \mapsto \lambda A i(x) + b
$$

where $\lambda > 0, A \in O(n)$, i is either the identity or an inversion, and $b \in \mathbb{R}^n$.

Moreover, for conformal groups of $\mathbb{S}^n\,(n \geq 3)$, we have [37]

$$
\mathrm{Conf}(\mathbb{S}^n)/SO(n+1,1) \cong \mathrm{B}^{n+1},
$$

and similar results hold for open subsets of $\mathbb{S}^n$.

However, a smooth gradient flow cannot be a family of inversions, and it cannot be a family of rotations either. This is because, if Du is a generator in $\mathrm{Isom}(\mathbb{S}^n)$, then

$$0 = \mathcal{L}_{Du} g = 2\,\mathrm{Hess}\, u,$$

but

$$2\left(\Delta u + (n-1)u\right) g = 0 + 2(n-1)ug \neq 0,$$

which contradicts the equation of the non-generic condition.

Thus Du can only lie in the Lie algebra of $F(\mathrm{B}^{n+1}) \subset \mathrm{Conf}(\mathbb{S}^n)$[37]. Any point $0 \neq a \in \mathrm{B}^{n+1}$ can be written as $a = c_1 e_1 + \cdots + c_{n+1} e_{n+1}$, and the associated conformal vector field is given by $V_a = \frac{1}{\|a\|}(c_1 e_1^\top + \cdots + c_{n+1} e_{n+1}^\top)$. Here $\{e_1, \cdots, e_{n+1}\}$ is an orthonormal basis of $\mathbb{R}^{n+1}$, and $e_i^\top$'s are the projections onto $T\mathbb{S}^n$. This means,

$$Du = c_1 e_1^\top + \cdots + c_{n+1} e_{n+1}^\top,$$

which implies

$$u = c_1 x_1 + \cdots + c_{n+1} x_{n+1}.$$

As a conclusion, the static potential u can only be a linear combination of the coordinate functions x_i's. □

On the other hand, it is a well-known fact that a compact (without boundary) totally umbilical hypersurface of a simply connected real space form is a geodesic sphere. So the only possible non-generic domains in $\mathbb{S}^n$ are spherical caps.

Theorem 3.12. *Let (Ω, Σ) be the n-dimensional spherical cap equipped with the standard metric $g_{\mathbb{S}^n}$, where $n \geq 3$. Then $\ker \Phi^*$ is non-trivial. Moreover, if the north pole N is the* *center of the spherical cap, then*

$$\ker \Phi^* = \operatorname{span}\{x_1, \cdots, x_n\}.$$

In fact, the spherical caps are the only non-generic domains (with only one boundary component) in $\mathbb{S}^n$.

Proof. As we discussed above, the coordinates $x_i\,(i = 1, \cdots, n)$ are the only possible static potentials, and they satisfy the first equation of (3.1).

Let us now check x_i's also satisfy the boundary equation of (3.1). If the north pole N is the center of the spherical cap, then Σ may be parametrized as $\{x \in \mathbb{S}^n : x_{n+1} = \cos\theta\}$, so that any point $x \in \Sigma$ can be written as $x = (\sin\theta\,\xi, \cos\theta)$, where $\xi \in \mathbb{S}^{n-1}$, and the induced metric on Σ is $\hat{g} = \sin^2\theta\, g_{\mathbb{S}^{n-1}}$.

In this case, the outward unit normal $\nu = \frac{\partial}{\partial\theta}$, and the mean curvature may be calculated as:

$$\begin{aligned} H &= \frac{\partial}{\partial\theta} \ln\sqrt{\det \hat{g}} \\ &= \frac{\partial}{\partial\theta} \ln\sqrt{\left(\sin^2\theta\right)^{n-1} \det g_{\mathbb{S}^{n-1}}} \\ &= \frac{\partial}{\partial\theta} \Big((n-1)\ln(\sin\theta) + \ln\sqrt{\det g_{\mathbb{S}^{n-1}}} \Big) \\ &= (n-1)\frac{\cos\theta}{\sin\theta} \end{aligned}$$

This means,

$$(x_i)_\nu - \frac{H}{n-1} x_i = \cos\theta\,\xi_i - \frac{\cos\theta}{\sin\theta}\sin\theta\,\xi_i = 0 \qquad \text{on } \Sigma$$

Thus,

$$\ker \Phi^* = \operatorname{span}\{x_1, \cdots, x_n\}.$$

□

For spherical caps, the scalar curvature $R > 0$; when θ varies between 0 and π, the range of the mean curvature H is the set of all real numbers $\mathbb{R}$.

We would like to mention here that non-generic domains in $\mathbb{S}^n$ are related to the famous Min-Oo's Conjecture [41]:

Conjecture 3.13 (Min-Oo)**.** *Let (M, g) be an n-dimensional compact Riemannian manifold with boundary and the scalar curvature $R \geq n(n-1)$. The boundary is isometric to the standard sphere $\mathbb{S}^{n-1}$ and is totally geodesic. Then (M, g) is isometric to the hemisphere $\mathbb{S}^n_+$.*

Min-Oo's conjecture has been verified in many special cases (see e.g. [10, 22, 29, 30, 40]), however, counterexamples were also constructed [11].

The Euclidean space $\mathbb{R}^n$

Consider the flat metric $g_{ij} = \delta_{ij}$ on the Euclidean space $\mathbb{R}^n$.

Cruz and Vitório [19] found that the Euclidean ball of radius r with its boundary is a non-generic domain in $\mathbb{R}^n$. Later Ho and Huang [34] had the following result:

Proposition 3.14 (Ho-Huang)**.** *Let $(\mathrm{B}^n, \partial\, \mathrm{B}^n)$ be the n-dimensional unit ball equipped with the flat metric. Then $\ker \Phi^*$ is non-trivial. Moreover,*

$$\ker \Phi^* = \mathrm{span}\{x_1, \cdots, x_n\},$$

where $(x_1, \cdots, x_n)$ are the coordinates of $\mathrm{B}^n \subset \mathbb{R}^n$.

In fact, the same proof as in [34] would tell us that, if $(\Omega, \Sigma) = (B_r, S_r)$ is the Euclidean ball of radius r with its boundary, then

$$\ker \Phi^* = \mathrm{span}\{x_1, \cdots, x_n\}.$$

Thus, the interior region of a Euclidean sphere is a non-generic domain in $\mathbb{R}^n$.

Let us try to find other non-generic domains in $\mathbb{R}^n$. First of all, notice that we have the following result:

Theorem 3.15. *The linear combinations of the coordinates* $x_i\,(i = 1, \cdots, n)$ *and constant functions are the only possible static potentials on non-generic domains* $\Omega \subset \mathbb{R}^n$*.*

Proof. The coordinates x_i as well as constant functions are possible static potentials, because

$$L^*(x_i) = -\left(\Delta x_i\right) g + \operatorname{Hess} x_i - x_i \operatorname{Ric}_g = 0,$$

$$\text{and} \quad L^*(c) = -\left(\Delta c\right) g + \operatorname{Hess} c - c \operatorname{Ric}_g = 0.$$

On the other hand, they are the only possible static potentials. Since for any static potential u,

$$L^* u = -\left(\Delta u\right) g + \operatorname{Hess} u = 0.$$

From (3.2) we know $\Delta u = 0$. So this means $\operatorname{Hess} u = 0$, which implies u is a linear combination of the coordinates x_i and constant functions. □

As for totally umbilical hypersurfaces $\Sigma \subset \mathbb{R}^n$, they can only be spheres or planes [13]. So let us check that the exterior region of a Euclidean sphere and the upper half plane (with their boundary) are also non-generic domains in $\mathbb{R}^n$.

If Ω is the exterior region of a sphere of radius r and $\Sigma = S_r$, then

$$(x_i)_\nu - \frac{H}{n-1} x_i = D x_i \cdot \left(-\frac{x}{r}\right) + \frac{1}{r} x_i = -\frac{x_i}{r} + \frac{x_i}{r} = 0 \qquad \text{on } \Sigma$$

Thus, the exterior region of a sphere is a non-generic domain in $\mathbb{R}^n$, and

$$\ker \Phi^* = \operatorname{span}\{x_1, \cdots, x_n\}.$$

If Ω is the upper half space $\{x_n > 0\}$ and $\Sigma = \{x_n = 0\}$, then for $1 \leq i \leq n-1$,

$$(x_i)_\nu - \frac{H}{n-1} x_i = \frac{\partial x_i}{\partial x_n} + 0 = 0 \qquad \text{on } \Sigma$$

$$\text{and} \quad (c)_\nu - \frac{H}{n-1} c = 0 + 0 = 0 \qquad \text{on } \Sigma$$

Thus, the upper half space is a non-generic domain in $\mathbb{R}^n$, and

$$\ker \Phi^* = \operatorname{span}\{1, x_1, \cdots, x_{n-1}\}.$$

As a summary, we have the following classification result:

Theorem 3.16 (Classification of boundaries of non-generic domains in $\mathbb{R}^n$)**.** *There are two types of boundaries of non-generic domains in* $\mathbb{R}^n$*:*
(i) the sphere; *(ii) the hyperplane.*

In 2016, Almaraz-Barbosa-de Lima [1] had the following conjecture of a positive mass theorem for asymptotically flat manifolds with a non-compact boundary, and they were able to prove the case when either $3 \leq n \leq 7$ or $n \geq 3$ and M is spin.

Conjecture 3.17 (Almaraz-Barbosa-de Lima)**.** *If* (M, g) *is asymptotically flat and satisfies* $R_g \geq 0$ *and* $H_g \geq 0$*, then* $m(M, g) \geq 0$*, with the equality occurring if and only if* (M, g) *is isometric to* $(\mathbb{R}^n_+, \delta)$*.*

They also had the following corollary regarding the rigidity part:

Corollary 3.18 (Almaraz-Barbosa-de Lima)**.** *Let (M, g) be asymptotically flat with $3 \leq n \leq 7$ or $n \geq 3$ and M spin, and satisfy $R_g \geq 0$ and $H_g \geq 0$. Assume further that there exists a compact subset $K \subset M$ such that $(M \setminus K, g)$ is isometric to $(\mathbb{R}^n_+ \setminus \bar{B}^+_1(0), \delta)$. Then (M, g) is isometric to $(\mathbb{R}^n_+, \delta)$.*

Thanks to the above corollary, we can now better understand why the half space is a non-generic domain in $\mathbb{R}^n$. Suppose not, then the half space with $R = 0$ and $H = 0$ satisfies generic conditions. This means we would be able to perform a local deformation on a compact region $K \subset \mathbb{R}^n_+$ such that $R > 0$ or $H > 0$ in K. However, this is a contradiction to the rigidity theorem.

There is another generalized version of the positive mass theorem [39, 47] that will imply the following rigidity theorem for the unit ball in $\mathbb{R}^n$:

Theorem 3.19. *Let (M, g) be an n-dimensional compact Riemannian manifold with boundary and the scalar curvature $R \geq 0$. The boundary is isometric to the standard sphere $\mathbb{S}^{n-1}$ and has mean curvature $n-1$. Then (M, g) is isometric to the unit ball in $\mathbb{R}^n$. (If $n > 7$, we also assume M is spin.)*

Notice that it can also give a good explanation of why the unit ball is a non-generic domain in $\mathbb{R}^n$.

The hyperbolic space $\mathbb{H}^n$

Consider the hyperboloid model for the hyperbolic space $\mathbb{H}^n$. To be more precise, let us consider the Minkowski quadratic form defined on $\mathbb{R}^{1,n}$ by

$$Q\left(x_0, x_1, \ldots, x_n\right) = -x_0^2 + x_1^2 + \cdots + x_n^2.$$

Then the hyperbolic space is defined by

$$\mathbb{H}^n = \{x \in \mathbb{R}^{1,n} : Q(x) = -1, x_0 > 0\}$$

with induced metric $g_{\mathbb{H}^n}$. In this case, the sectional curvature $K = -1$, the Ricci curvature $\mathrm{Ric} = -(n-1)g_{\mathbb{H}^n}$, and the scalar curvature $R = -n(n-1)$.

As a part of the unit sphere in the Minkowski space, the hyperbolic space has many properties that are similar to the unit sphere $\mathbb{S}^n$ in the Euclidean space.

Theorem 3.20. *The linear combinations of the coordinates $x_i\,(i = 0, 1, \cdots, n)$ are the only possible static potentials on non-generic domains $\Omega \subset \mathbb{H}^n$. Here $(x_0, x_1, \cdots, x_n)$ are the coordinates of $\mathbb{H}^n \subset \mathbb{R}^{1,n}$.*

Proof. Let us consider the hyperbolic space as a graph in $\mathbb{R}^{1,n}$. To be more precise, any point $(x_0, \mathbf{x}) = (x_0, x_1, \ldots, x_n) \in \mathbb{H}^n$ satisfies

$$x_0 = \sqrt{1 + \|\mathbf{x}\|^2}$$

Thus the hyperbolic space can be characterized as

$$\begin{aligned} F : \mathbb{R}^n &\to \mathbb{R}^{1,n}; \\ \mathbf{x} &\mapsto (\sqrt{1 + \|\mathbf{x}\|^2}, \mathbf{x}) \end{aligned}$$

Denote $e_i \in T\mathbb{H}^n\,(i = 1, \cdots, n)$ as

$$e_i = dF\left(\frac{\partial}{\partial x_i}\right) = \left(\frac{x_i}{\sqrt{1 + \|\mathbf{x}\|^2}}, 0, \cdots, \underset{i\text{-th}}{1}, \cdots, 0\right) = \frac{x_i}{x_0}\frac{\partial}{\partial x_0} + \frac{\partial}{\partial x_i}$$

Then the induced metric is

$$g_{ij} = \langle e_i, e_j \rangle = \delta_{ij} - \frac{x_i x_j}{x_0^2} \qquad \text{and} \qquad g^{ij} = \delta^{ij} + x_i x_j.$$

For $k = 1, \cdots, n$

$$\begin{aligned}
\operatorname{Hess}_{ij} x_k &= e_i(e_j x_k) - \Gamma^l_{ij} e_l x_k \\
&= e_i(\delta_{jk}) - \Gamma^l_{ij}\delta_{lk} \\
&= -\Gamma^k_{ij} \\
&= -\frac{1}{2} g^{kl} \left(\frac{\partial}{\partial x^j} g_{li} + \frac{\partial}{\partial x^i} g_{lj} - \frac{\partial}{\partial x^l} g_{ij} \right)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial}{\partial x^j} g_{li} &= \frac{\partial}{\partial x^j} \left(\delta_{li} - \frac{x_l x_i}{x_0^2} \right) \\
&= -\frac{\frac{\partial}{\partial x^j}(x_l x_i) x_0^2 - 2 x_l x_i x_0 \frac{\partial}{\partial x^j} x_0}{x_0^4} \\
&= -\frac{\delta_{jl} x_i + \delta_{ji} x_l}{x_0^2} + 2\frac{x_l x_i x_j}{x_0^4}
\end{aligned}$$

So

$$\begin{aligned}
\operatorname{Hess}_{ij} x_k = \frac{1}{2} g^{kl} &\left(\frac{\delta_{jl} x_i + \delta_{ji} x_l}{x_0^2} - 2\frac{x_l x_i x_j}{x_0^4} + \frac{\delta_{il} x_j + \delta_{ji} x_l}{x_0^2} - 2\frac{x_l x_i x_j}{x_0^4} \right) \\
&+ \frac{1}{2} g^{kl} \left(-\frac{\delta_{jl} x_i + \delta_{li} x_j}{x_0^2} + 2\frac{x_l x_i x_j}{x_0^4} \right) \\
&= (\delta^{kl} + x^k x^l) \left(\frac{\delta_{ji} x_l}{x_0^2} - \frac{x_l x_i x_j}{x_0^4} \right) \\
&= \frac{\delta_{ji} x_k}{x_0^2} + \frac{\sum_l x_k \delta_{ji} x_l^2}{x_0^2} - \frac{x_k x_i x_j}{x_0^4} - \frac{\sum_l x_k x_i x_j x_l^2}{x_0^4} \\
&= x_k \delta_{ij} - \frac{x_k x_i x_j}{x_0^2} \\
&= x_k g_{ij}
\end{aligned}$$

and

$$
\begin{aligned}
\operatorname{Hess}_{ij} x_0 &= e_i(e_j x_0) - \Gamma_{ij}^l e_l x_0 \\
&= e_i\left(\frac{x_j}{x_0}\right) - \Gamma_{ij}^l \frac{x_l}{x_0} \\
&= \frac{\delta_{ij} x_0 - x_j \frac{x_i}{x_0}}{x_0^2} + \sum_l x_l g_{ij} \frac{x_l}{x_0} \\
&= \frac{1}{x_0} g_{ij} + \frac{1}{x_0} g_{ij} \sum_l x_l^2 \\
&= x_0 g_{ij}
\end{aligned}
$$

Thus for $i = 0, 1, \cdots, n$

$$
\operatorname{Hess}_{g_{\mathbb{H}^n}} x_i = x_i g_{\mathbb{H}^n},
$$

and we have

$$
\begin{aligned}
L^*(x_i) &= -(\Delta x_i)\, g + \operatorname{Hess} x_i - x_i \operatorname{Ric}_g \\
&= -(\Delta x_i)\, g + x_i g + (n-1) x_i g \\
&= (-\Delta x_i + n x_i)\, g \\
&= 0.
\end{aligned}
$$

On the other hand, by Corvino [15, Corollary 2.4] we have $\dim \ker L^* \leq n+1$ in $H^2_{\mathrm{loc}}(\Omega)$. This means the linear combinations of the coordinates $x_i\, (i = 0, 1, \cdots, n)$ are the only possible static potentials on Ω. □

As for totally umbilical hypersurfaces $\Sigma \subset \mathbb{H}^n$, they are of the form $P \cap \mathbb{H}^n$, where P is some affine hyperplane of $\mathbb{R}^{1,n}$ [13]. Alternatively, if we consider the upper half-space model, then Σ can only be the geodesic spheres, the horospheres, the hyperspheres and the intersection with $\mathbb{H}^n$ of hyperplanes of $\mathbb{R}^n$ [13]. To be more precise, following do Carmo, let $\Sigma = S \cap \mathbb{H}^n$ be the intersection of $\mathbb{H}^n$ with a Euclidean $(n-1)$-sphere $S \subset \mathbb{R}^n$ of radius 1 and center in

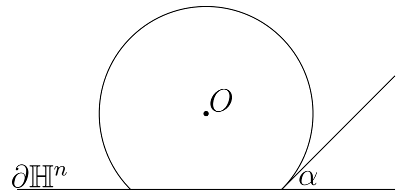

Figure 3.1: A hypersphere in the half-space model

$\mathbb{H}^n$. Then the mean curvature $\frac{H}{n-1}$ of Σ is

(i) 1, if S is tangent to $\partial\mathbb{H}^n$ (horosphere);

(ii) $\cos\alpha$, if S makes an angle α with $\partial\mathbb{H}^n$ (hypersphere);

(iii) the height h of the Euclidean center of S relative to $\partial\mathbb{H}^n$, if $S \subset \mathbb{H}^n$ (geodesic sphere).

Now let us check one by one whether each of the above umbilical hypersurfaces will give us a boundary of the non-generic domain in $\mathbb{H}^n$. However, Theorem 3.20 tells us that the coordinates of $\mathbb{H}^n \subset \mathbb{R}^{1,n}$ in the hyperboloid model are the only possible static potentials, so we would like to first rewrite these coordinates x_i's in terms of the coordinates in the upper half-space model.

Denote $\mathbb{H}^n_{-1}$ as the hyperboloid model, D^n as the unit ball model, and $\mathbb{H}^n$ as the upper half-space model of the hyperbolic space. The inverse of the stereographic projection

$$f : D^n \to \mathbb{H}^n_{-1};$$
$$(0, u_1, \cdots, u_n) \mapsto (x_0, x_1, \cdots, x_n)$$

is given by

$$x_\alpha = \frac{2u_\alpha}{1 - \sum_\alpha u_\alpha^2}, \quad \alpha = 1, \cdots, n$$
$$x_0 = \frac{2}{1 - \sum_\alpha u_\alpha^2} - 1.$$

And the conformal mapping between the unit ball D^n and the upper half-space $\mathbb{H}^n$

$$g : \mathbb{H}^n \to D^n;$$

$$(y_1, \cdots, y_n) \mapsto (u_1, \cdots, u_n)$$

is given by

$$u_k = \frac{2y_k}{\sum_k y_k^2 + (y_n + 1)^2}, \qquad k = 1, \cdots, n-1$$
$$u_n = 1 - \frac{2(y_n + 1)}{\sum_k y_k^2 + (y_n + 1)^2}.$$

Combining them together, we get the expression of the coordinates x_i's in the upper half-space $\mathbb{H}^n$:

$$x_0 = \frac{\sum_k y_k^2 + y_n^2 + 1}{2y_n},$$
$$x_k = \frac{y_k}{y_n}, \qquad k = 1, \cdots, n-1$$
$$x_n = \frac{\sum_k y_k^2 + y_n^2 - 1}{2y_n}.$$

Theorem 3.21. *Consider the upper half-space model of $\mathbb{H}^n$. Let $\Sigma = S \cap \mathbb{H}^n$ be the intersection of $\mathbb{H}^n$ with a Euclidean $(n-1)$-sphere $S \subset \mathbb{R}^n$ and center in $\mathbb{H}^n$. Suppose S makes an angle α with $\partial\mathbb{H}^n$. Then both the interior region and the exterior region with boundary Σ are non-generic domains in $\mathbb{H}^n$. Moreover,*

$$\ker \Phi^* = \operatorname{span}\{x_1, \cdots, x_{n-1}, (2 - \cos^2\alpha)x_0 + \cos^2\alpha\, x_n\}.$$

Here $(x_0, x_1, \cdots, x_n)$ are the coordinates of $\mathbb{H}^n \subset \mathbb{R}^{1,n}$ in the hyperboloid model.

Proof. We will just verify that the interior region bounded by Σ is a non-generic domain in $\mathbb{H}^n$; the same calculation will give us that the exterior region bounded by Σ is a non-generic domain as well.

Let us use y_i's for the coordinates in $\mathbb{R}^n$. The metric of the upper half-space model of $\mathbb{H}^n$

is $g = y_n^{-2}\delta$, where δ is the Euclidean metric. Without loss of generality, we may assume the Euclidean radius of S is 1, and that the center of S is $O = (0, \cdots, 0, \cos\alpha)$. Suppose $\vec{e} = \sum a_i \frac{\partial}{\partial y_i}$ is any vector with Euclidean norm 1. Then we may parametrize Σ as $\{\vec{y} = O + \vec{e}\}$, which means the points on Σ satisfy

$$\begin{aligned} y_k &= a_k \qquad (k = 1, \cdots, n-1) \\ y_n &= \cos\alpha + a_n \end{aligned}$$

Denote the outward unit normal at $\vec{y}$ as $\nu = c\vec{e}$, then

$$1 = g(\nu, \nu) = y_n^{-2}\delta(c\vec{e}, c\vec{e}) = c^2 y_n^{-2}$$

So

$$\nu = y_n \vec{e} = (\cos\alpha + a_n) \sum a_i \frac{\partial}{\partial y_i}$$

Note that the mean curvature with respect to this normal is $H = (n-1)\cos\alpha$. Then

$$\begin{aligned} (x_0)_\nu - \frac{H}{n-1} x_0 &= y_n \sum a_i \frac{\partial}{\partial y_i} x_0 - \cos\alpha\, x_0 = -\frac{1}{2}\cos^2\alpha \\ (x_k)_\nu - \frac{H}{n-1} x_k &= y_n \sum a_i \frac{\partial}{\partial y_i} x_k - \cos\alpha\, x_k = a_k - y_k = 0 \quad (k = 1, \cdots, n-1) \\ (x_n)_\nu - \frac{H}{n-1} x_n &= y_n \sum a_i \frac{\partial}{\partial y_i} x_n - \cos\alpha\, x_n = 1 - \frac{1}{2}\cos^2\alpha > 0 \end{aligned}$$

where we make use of the relation $\sum a_i^2 = 1$. Notice that the boundary equation is linear, and we have

$$\big((2 - \cos^2\alpha)x_0 + \cos^2\alpha\, x_n\big)_\nu - \frac{H}{n-1}\big((2 - \cos^2\alpha)x_0 + \cos^2\alpha\, x_n\big) = 0.$$

So the region with boundary Σ is a non-generic domain in $\mathbb{H}^n$, and

$$\ker\Phi^* = \operatorname{span}\{x_1, \cdots, x_{n-1}, (2 - \cos^2\alpha)x_0 + \cos^2\alpha\, x_n\}.$$

□

Remark. Note the above theorem tells us that both the horospheres and the hyperspheres are boundaries of non-generic domains in $\mathbb{H}^n$.

Theorem 3.22. *Consider the upper half-space model of $\mathbb{H}^n$. Let $\Sigma = S \cap \mathbb{H}^n$ be the intersection of $\mathbb{H}^n$ with a Euclidean $(n-1)$-sphere $S \subset \mathbb{R}^n$ and center in $\mathbb{H}^n$. Suppose $S \subset \mathbb{H}^n$ is a geodesic sphere with Euclidean radius 1 and the Euclidean center of S relative to $\partial\mathbb{H}^n$ is at height h. Then both the interior region and the exterior region with boundary Σ are non-generic domains in $\mathbb{H}^n$. Moreover,*

$$\ker \Phi^* = \mathrm{span}\{x_1, \cdots, x_{n-1}, (2-h^2)x_0 + h^2\, x_n\}.$$

Here $(x_0, x_1, \cdots, x_n)$ are the coordinates of $\mathbb{H}^n \subset \mathbb{R}^{1,n}$ in the hyperboloid model.

Proof. Without loss of generality, we may assume the center of S is $O = (0, \cdots, 0, h)$. Then by replacing $\cos\alpha$ by h, the same calculation as above will give us that

$$\begin{aligned}
(x_0)_\nu - \frac{H}{n-1}x_0 &= y_n \sum a_i \frac{\partial}{\partial y_i} x_0 - h x_0 = -\frac{1}{2}h^2 < 0 \\
(x_k)_\nu - \frac{H}{n-1}x_k &= y_n \sum a_i \frac{\partial}{\partial y_i} x_k - h x_k = a_k - y_k = 0 \quad (k = 1, \cdots, n-1) \\
(x_n)_\nu - \frac{H}{n-1}x_n &= y_n \sum a_i \frac{\partial}{\partial y_i} x_n - h x_n = 1 - \frac{1}{2}h^2
\end{aligned}$$

The result follows immediately. □

Theorem 3.23. *Consider the upper half-space model of $\mathbb{H}^n$. Let $\Sigma = P \cap \mathbb{H}^n$ be the intersection of $\mathbb{H}^n$ with a Euclidean $(n-1)$-plane $P \subset \mathbb{R}^n$ that is parallel to $\partial\mathbb{H}^n$. Then the region with boundary Σ is a non-generic domain in $\mathbb{H}^n$. Moreover,*

$$\ker \Phi^* = \mathrm{span}\{x_1, \cdots, x_{n-1}, x_0 - x_n\}.$$

Here $(x_0, x_1, \cdots, x_n)$ *are the coordinates of* $\mathbb{H}^n \subset \mathbb{R}^{1,n}$ *in the hyperboloid model.*

Proof. We will just verify that the lower region bounded by Σ is a non-generic domain in $\mathbb{H}^n$; the same calculation will give us that the upper region bounded by Σ is a non-generic domain as well.

We may parametrize Σ as $\{y_n = c > 0\}$, with the outward unit normal $\nu = y_n \frac{\partial}{\partial y_n}$. So the mean curvature with respect to this normal is

$$H = y_n \left(H_0 + (n-1) \left\langle \nabla(-\ln y_n), \frac{\partial}{\partial y_n} \right\rangle_\delta \right) = -(n-1).$$

Then

$$\begin{aligned} (x_0)_\nu - \frac{H}{n-1} x_0 &= y_n \frac{\partial}{\partial y_n} x_0 + x_0 = y_n > 0 \\ (x_k)_\nu - \frac{H}{n-1} x_k &= y_n \frac{\partial}{\partial y_n} x_k + x_k = -\frac{y_k}{y_n} + \frac{y_k}{y_n} = 0 \quad (k = 1, \cdots, n-1) \\ (x_n)_\nu - \frac{H}{n-1} x_n &= y_n \frac{\partial}{\partial y_n} x_n + x_n = y_n > 0 \end{aligned}$$

The result follows immediately. □

Theorem 3.24. *Consider the upper half-space model of* $\mathbb{H}^n$*. Let* $\Sigma = P \cap \mathbb{H}^n$ *be the intersection of* $\mathbb{H}^n$ *with a Euclidean* $(n-1)$*-plane* $P \subset \mathbb{R}^n$ *that makes an angle* $\alpha \in (0, \pi)$ *with* $\partial\mathbb{H}^n$*. Then the region with boundary* Σ *is a non-generic domain in* $\mathbb{H}^n$*. Moreover, if we parametrize* Σ *as* $\{\cos\alpha\, y_n = \sin\alpha\, y_1 : y_n > 0\}$*, then*

$$\ker \Phi^* = \operatorname{span}\{x_0, x_2, \cdots, x_n\}.$$

Here $(x_0, x_1, \cdots, x_n)$ *are the coordinates of* $\mathbb{H}^n \subset \mathbb{R}^{1,n}$ *in the hyperboloid model.*

Proof. We will just verify that the lower region bounded by Σ is a non-generic domain in $\mathbb{H}^n$; the same calculation will give us that the upper region bounded by Σ is a non-generic domain as well.

Without loss of generality, we may parametrize Σ as $\{\cos\alpha\, y_n = \sin\alpha\, y_1 : y_n > 0\}$, with the outward unit normal $\nu = y_n(-\sin\alpha\frac{\partial}{\partial y_1} + \cos\alpha\frac{\partial}{\partial y_n})$. So the mean curvature with respect to this normal is

$$H = y_n\left(H_0 + (n-1)\left\langle \nabla(-\ln y_n), -\sin\alpha\frac{\partial}{\partial y_1} + \cos\alpha\frac{\partial}{\partial y_n}\right\rangle_\delta\right) = -(n-1)\cos\alpha.$$

Then

$$\begin{aligned}
(x_0)_\nu - \frac{H}{n-1}x_0 &= y_n(-\sin\alpha\frac{\partial}{\partial y_1} + \cos\alpha\frac{\partial}{\partial y_n})x_0 + \cos\alpha\, x_0 = 0\\
(x_1)_\nu - \frac{H}{n-1}x_1 &= y_n(-\sin\alpha\frac{\partial}{\partial y_1} + \cos\alpha\frac{\partial}{\partial y_n})x_1 + \cos\alpha\, x_1 = -\sin\alpha < 0\\
(x_k)_\nu - \frac{H}{n-1}x_k &= y_n(-\sin\alpha\frac{\partial}{\partial y_1} + \cos\alpha\frac{\partial}{\partial y_n})x_k + \cos\alpha\, x_k = 0 \ (k = 2, \cdots, n-1)\\
(x_n)_\nu - \frac{H}{n-1}x_n &= y_n(-\sin\alpha\frac{\partial}{\partial y_1} + \cos\alpha\frac{\partial}{\partial y_n})x_n + \cos\alpha\, x_n = 0
\end{aligned}$$

The result follows immediately. □

As a summary, we have the following classification result:

Theorem 3.25 (Classification of boundaries of non-generic domains in $\mathbb{H}^n$)**.** *There are four types of boundaries of non-generic domains in $\mathbb{H}^n$:*

(i) the horosphere; *(ii) the hypersphere;*

(iii) the geodesic sphere; *(iv) the intersection with $\mathbb{H}^n$ of hyperplanes of $\mathbb{R}^n$.*

We notice Huang-Jang-Martin [35] had the following rigidity result for hyperbolic manifolds:

Theorem 3.26 (Huang-Jang-Martin)**.** *Let $n \geq 3$ and (M, g) an n-dimensional asymptotically hyperbolic manifold with scalar curvature $R_g \geq -n(n-1)$ and with the equality $p_0 = \sqrt{p_1^2 + \cdots + p_n^2}$. Then (M, g) is isometric to hyperbolic space.*

Here they are considering conformally compact, asymptotically hyperbolic manifolds (X^n, g)

whose conformal boundary is the unit round sphere $(\mathbb{S}^{n-1}, h)$ with the following expansion:

$$g = \frac{1}{(\sinh\rho)^2}\left(d\rho^2 + h + \frac{\rho^n}{n}\kappa + O\left(\rho^{n+1}\right)\right)$$

where ρ is a boundary defining function and κ is a symmetric $(0,2)$-tensor defined on $\mathbb{S}^{n-1}$. And the mass $(p_0, p_1, \cdots, p_n)$ of g is defined by

$$p_0 = \int_{\mathbb{S}^{n-1}} \operatorname{tr}_h \kappa \, d\mu_h, \quad p_i = \int_{\mathbb{S}^{n-1}} x_i \operatorname{tr}_h \kappa \, d\mu_h \quad \text{for } i = 1, \cdots, n$$

where $(x_1, ..., x_n)$ are the Cartesian coordinates of $\mathbb{R}^n$ restricted on $\mathbb{S}^{n-1}$.

We observe that the mass here consists of $(n+1)$ numbers $(p_0, p_1, \cdots, p_n)$, instead of a single number, the ADM mass, as for the asymptotically flat manifolds.

On the other hand, Hijazi-Montiel-Raulot [33] had the following rigidity results for asymptotically hyperbolic manifolds with inner boundary:

Theorem 3.27 (Hijazi-Montiel-Raulot)**.** *Let (M^3, g) be a three-dimensional complete AH manifold with scalar curvature satisfying $R_g \geq -6$ and compact inner boundary Σ homeomorphic to a 2-sphere whose mean curvature is such that*

$$H \leq 2\sqrt{\frac{4\pi}{\text{Area}_g(\Sigma)} + 1}.$$

Then the energy-momentum vector $\mathbf{p}_g$ is time-like future-directed or zero. Moreover, if it vanishes then (M^3, g) is isometric to the complement of a geodesic ball in $\mathbb{H}^3$.

Theorem 3.28 (Hijazi-Montiel-Raulot)**.** *Let (M^n, g) be an n-dimensional $(n \geq 4)$ complete AH spin manifold with scalar curvature satisfying $R_g \geq -n(n-1)$, compact inner boundary*

Σ *of positive Yamabe invariant* $Y(\Sigma)$ *and of mean curvature*

$$H \leq (n-1)\sqrt{\frac{Y(\Sigma)}{(n-1)(n-2)}\operatorname{Vol}_g(\Sigma)^{-\frac{2}{n-1}}+1}.$$

Then the energy-momentum vector $\mathbf{p}_g$ *is time-like future-directed or zero. If it is zero, then* (M^n, g) *has an imaginary Killing spinor field and* Σ *is a totally umbilical hypersurface with constant mean curvature carrying a real Killing spinor.*

These results together give us a very good explanation of non-generic domains in $\mathbb{H}^n$.

The Schwarzschild manifold

Finally, we would like to determine the non-generic domains in the Schwarzschild manifold. We have already known that the umbilic hypersurfaces in $\mathbb{R}^n$ are spheres and hyperplanes, and that umbilic hypersurfaces remain umbilic after the conformal change [13]. Since the n-dimensional Schwarzschild manifold is conformal to $\mathbb{R}^n$, we get the umbilic hypersurfaces in the Schwarzschild manifold are Euclidean spheres and hyperplanes.

Theorem 3.29. *Consider the Schwarzschild metric* $g^S = \left(1+\frac{m}{2r^{n-2}}\right)^{\frac{4}{n-2}}\delta$ *on* $\mathbb{R}^n\backslash\{\mathbf{0}\}$*, where* $r = |\vec{x}|$*. Denote* Σ *as the Euclidean sphere centered at* $\mathbf{0}$ *with radius* $r_\pm = \left(\frac{(n-1)\pm\sqrt{n^2-2n}}{2}m\right)^{\frac{1}{n-2}}$*. Then the region with boundary* Σ *is a non-generic domain in the Schwarzschild manifold. Moreover,*

$$\ker \Phi^* = \operatorname{span}\left\{\frac{1-\frac{m}{2r^{n-2}}}{1+\frac{m}{2r^{n-2}}}\right\}.$$

Proof. As mentioned in [15], the function $u_0 = \frac{1-\frac{m}{2r}}{1+\frac{m}{2r}} \in \ker L^*_{g^S}$ is a possible static potential in 3-dimensional Schwarzschild manifold. Analogously, the function $u = \frac{1-\frac{m}{2r^{n-2}}}{1+\frac{m}{2r^{n-2}}} \in \ker L^*_{g^S}$ is a possible static potential in n-dimensional Schwarzschild manifold. (This can also be found using the method introduced in [12].) Indeed, it is the only possible static potential in connected open domains, as found by Corvino [16]. Now let us verify whether the function satisfies the boundary condition on Euclidean spheres centered at $\mathbf{0}$.

Suppose $\vec{e}$ is any vector with Euclidean norm 1. Then we may parametrize the Euclidean sphere as $\{\vec{x} = r\vec{e}\}$. Denote the outward unit normal at $\vec{x}$ as $\nu = c\vec{e}$, then

$$1 = g^S(\nu,\nu) = \left(1 + \frac{m}{2r^{n-2}}\right)^{\frac{4}{n-2}} \delta(c\vec{e}, c\vec{e}) = c^2 \left(1 + \frac{m}{2r^{n-2}}\right)^{\frac{4}{n-2}}$$

So

$$\nu = \left(1 + \frac{m}{2r^{n-2}}\right)^{-\frac{2}{n-2}} \vec{e}$$

Note that the mean curvature with respect to this normal is

$$\begin{aligned} H &= \left(1 + \frac{m}{2r^{n-2}}\right)^{-\frac{2}{n-2}} \left(H_0 + (n-1)\left\langle \nabla \left(\frac{2}{n-2}\ln(1 + \frac{m}{2r^{n-2}})\right), \vec{e}\right\rangle_\delta\right) \\ &= \left(1 + \frac{m}{2r^{n-2}}\right)^{-\frac{2}{n-2}} \left(\frac{n-1}{r} - (n-1)\left\langle \frac{m\vec{x}}{r^n(1+\frac{m}{2r^{n-2}})}, \vec{e}\right\rangle_\delta\right) \\ &= \left(1 + \frac{m}{2r^{n-2}}\right)^{-\frac{2}{n-2}} \left(\frac{n-1}{r} - \frac{(n-1)m}{r^{n-1}(1+\frac{m}{2r^{n-2}})}\right) \\ &= (n-1)\left(1 + \frac{m}{2r^{n-2}}\right)^{-\frac{2}{n-2}} \frac{1 - \frac{m}{2r^{n-2}}}{r(1+\frac{m}{2r^{n-2}})} \\ &= (n-1)\left(1 + \frac{m}{2r^{n-2}}\right)^{-\frac{2}{n-2}} r^{-1}u \end{aligned}$$

Then

$$\begin{aligned} u_\nu - \frac{H}{n-1}u &= \left(1 + \frac{m}{2r^{n-2}}\right)^{-\frac{2}{n-2}} \vec{e}\left(\frac{1 - \frac{m}{2r^{n-2}}}{1 + \frac{m}{2r^{n-2}}}\right) - \left(1 + \frac{m}{2r^{n-2}}\right)^{-\frac{2}{n-2}} r^{-1}u^2 \\ &= \left(1 + \frac{m}{2r^{n-2}}\right)^{-\frac{2}{n-2}} \left\langle \frac{(n-2)m\vec{x}}{r^n(1+\frac{m}{2r^{n-2}})^2}, \vec{e}\right\rangle_\delta - \frac{(1-\frac{m}{2r^{n-2}})^2}{r(1+\frac{m}{2r^{n-2}})^{2+\frac{2}{n-2}}} \\ &= \frac{(n-2)m}{r^{n-1}(1+\frac{m}{2r^{n-2}})^{2+\frac{2}{n-2}}} - \frac{\frac{m^2}{4r^{n-2}} - m + r^{n-2}}{r^{n-1}(1+\frac{m}{2r^{n-2}})^{2+\frac{2}{n-2}}} \\ &= \frac{-\frac{m^2}{4r^{n-2}} + (n-1)m - r^{n-2}}{r^{n-1}(1+\frac{m}{2r^{n-2}})^{2+\frac{2}{n-2}}} \end{aligned}$$

Thus $u_\nu - \frac{H}{2}u = 0$ if and only if

$$r^{n-2} = \frac{(n-1) \pm \sqrt{n^2 - 2n}}{2} m.$$

This means the region with boundary Σ is a non-generic domain in $(\mathbb{R}^n \backslash \{\mathbf{0}\}, g^S)$. □

Remark. Even though the Euclidean spheres not centered at $\mathbf{0}$ are umbilic hypersurfaces in the Schwarzschild manifold, they are not boundaries of non-generic domains. This is mainly because the function r is no longer a constant on the sphere. More precisely, we may assume the sphere is centered at $\vec{e}$ ($\vec{e}$ has Euclidean norm 1) with radius ρ. Then let us look at two different points on the sphere, namely $x_1 = \vec{e} + \rho\vec{e}$ and $x_2 = \vec{e} - \rho\vec{e}$. Following the same calculation as above, we will see the two ρ's satisfying the boundary condition are different.

Theorem 3.30. *Denote Σ as the Euclidean hyperplane containing $\mathbf{0}$. Then the region with boundary Σ is a non-generic domain in the Schwarzschild manifold. Moreover,*

$$\ker \Phi^* = \operatorname{span}\left\{ \frac{1 - \frac{m}{2r^{n-2}}}{1 + \frac{m}{2r^{n-2}}} \right\}.$$

Proof. First note that Σ is complete and has two ends in the Schwarzschild manifold. Now let us verify whether the static potential mentioned above satisfies the boundary condition.

Without loss of generality, we may parametrize Σ as $\{x_n = c\}$. Then the outward unit normal is

$$\nu = \left(1 + \frac{m}{2r^{n-2}}\right)^{-\frac{2}{n-2}} \frac{\partial}{\partial x_n}$$

and the mean curvature is

$$\begin{aligned} H &= \left(1 + \frac{m}{2r^{n-2}}\right)^{-\frac{2}{n-2}} \left(H_0 + (n-1) \left\langle \nabla \left(\frac{2}{n-2} \ln(1 + \frac{m}{2r^{n-2}}) \right), \frac{\partial}{\partial x_n} \right\rangle_\delta \right) \\ &= -\left(1 + \frac{m}{2r^{n-2}}\right)^{-\frac{2}{n-2}} \left((n-1) \frac{m x_n}{r^n (1 + \frac{m}{2r^{n-2}})} \right) \\ &= -(n-1) \frac{m x_n}{r^n (1 + \frac{m}{2r^{n-2}})^{1 + \frac{2}{n-2}}} \end{aligned}$$

Then

$$
\begin{aligned}
u_\nu - \frac{H}{n-1}u &= \Big(1+\frac{m}{2r^{n-2}}\Big)^{-\frac{2}{n-2}} \frac{(n-2)mx_n}{r^n(1+\frac{m}{2r^{n-2}})^2} + \frac{mx_n}{r^n(1+\frac{m}{2r^{n-2}})^{1+\frac{2}{n-2}}}\left(\frac{1-\frac{m}{2r^{n-2}}}{1+\frac{m}{2r^{n-2}}}\right) \\
&= \frac{(n-2)mx_n}{r^n(1+\frac{m}{2r^{n-2}})^{2+\frac{2}{n-2}}} + \frac{mx_n(1-\frac{m}{2r^{n-2}})}{r^n(1+\frac{m}{2r^{n-2}})^{2+\frac{2}{n-2}}} \\
&= \frac{mc((n-1)-\frac{m}{2r^{n-2}})}{r^n(1+\frac{m}{2r^{n-2}})^{2+\frac{2}{n-2}}}
\end{aligned}
$$

Thus $u_\nu - \frac{H}{2}u = 0$ if and only if $c = 0$. This means the region with boundary Σ is a non-generic domain in $(\mathbb{R}^n\backslash\{\mathbf{0}\}, g^S)$. □

As a summary, we have the following classification result:

Theorem 3.31 (Classification of boundaries of non-generic domains in the Schwarzschild manifold)**.** *There are two types of boundaries of non-generic domains in the Schwarzschild manifold:*

(i) the Euclidean sphere centered at $\mathbf{0}$ *with radius* $r_\pm$*;*

(ii) the Euclidean hyperplane containing $\mathbf{0}$*.*

Remark. For the above examples in space forms and in the Schwarzschild manifold, we are just considering non-generic domains with only one boundary component. We call them simple non-generic domains. If a non-generic domain has multiple boundary components, then it can be viewed as the intersection of simple non-generic domains, as long as their boundaries do not intersect with each other. In this case, the space of static potentials is the intersection of those corresponding to each simple non-generic domain. But of course we will need to make sure the eventual space of static potentials has dimension at least 1.

Chapter 4

Local Deformation

In this chapter, we give the proof of our main theorem. We first solve the linearized equations using Fredholm theory, and then use iteration for the nonlinear problem. The proof is contained in several sections.

4.1 Poincaré-type inequalities

The following two Poincaré-type inequalities for the operators L^* and Φ^* will be useful for later proof.

We will need a basic estimate proved in [15], using only the overdeterminedness of the operator L^*:

Lemma 4.1 (Corvino)**.** *For $u \in H^2_{\mathrm{loc}}(\Omega)$,*

$$\|u\|_{H^2(\Omega)} \leq C(n, g, \Omega) \left(\|L^* u\|_{\mathcal{L}^2(\Omega)} + \|u\|_{L^2(\Omega)} \right).$$

In fact, for any $\epsilon \geq 0$ small we have the following estimate, where the constant C is independent of ϵ:

$$\|u\|_{H^2(\Omega_\epsilon)} \leq C(n, g, \Omega, \Sigma) \left(\|L^* u\|_{\mathcal{L}^2(\Omega_\epsilon)} + \|u\|_{L^2(\Omega_\epsilon)} \right).$$

Then by a standard integration technique discussed in [15], we have:

Corollary 4.2. *There is a constant $C = C(n, g, \Omega, \Sigma, \rho)$, uniform for metrics $\mathcal{C}^2$-near g, so that for $u \in H^2_{\mathrm{loc}}(\Omega)$,*

$$\|u\|_{H^2_\rho(\Omega)} \leq C \left(\|L^* u\|_{\mathcal{L}^2_\rho(\Omega)} + \|u\|_{L^2_\rho(\Omega)} \right).$$

Theorem 4.3. *There is a constant $C = C(n, g, \Omega, \Sigma, \rho)$, uniform for metrics $\mathcal{C}^2$-near g, so that for $u \in H^2_{\mathrm{loc}}(\Omega)$ with $u = u_\nu = 0$ on Σ, we have*

$$\|u\|_{H^2_\rho(\Omega)} \leq C \|L^* u\|_{\mathcal{L}^2_\rho(\Omega)}.$$

Proof. If $\|u\|_{L^2_\rho(\Omega)} = 0$, then the inequality follows directly from Cor. 4.2. Now let us assume $\|u\|_{L^2_\rho} > 0$. We may further assume $\|u\|_{L^2_\rho} = 1$ by normalization. Consider the functional $\mathcal{F}_0$ defined on $H^2_\rho(\Omega)$ by

$$\mathcal{F}_0(u) = \int_\Omega \|L^* u\|^2 \rho \, d\mu_g$$

and minimize it over

$$\mathcal{A} = \{u \in H^2_\rho(\Omega) : \|u\|_{L^2_\rho} = 1 \text{ and } u = u_\nu = 0 \text{ on } \Sigma\}.$$

By Cor. 4.2, $\|L^* u\|_{\mathcal{L}^2_\rho} \geq C\|u\|_{H^2_\rho} - 1$, so when $\|u\|_{H^2_\rho} >> 1$ this implies

$$\mathcal{F}_0(u) \geq \left(C\|u\|_{H^2_\rho} - 1 \right)^2.$$

Thus $\mathcal{F}_0$ satisfies the coercivity condition, and attains its infimum in $\mathcal{A}$.

Suppose the functional is minimized by $\mathcal{F}_0(u_0) = \lambda$ over $\mathcal{A}$. It is obvious that $\lambda \geq 0$.

Furthermore, if $\lambda = 0$, we would have $\|L^* u_0\|^2_{\mathcal{L}^2_\rho} = 0$; by Lemma 2.8, $u_0 \equiv 0$, which is a contradiction to $\|u_0\|_{L^2_\rho} = 1$. Thus we must have $\lambda > 0$. So,

$$\|u\|_{H^2_\rho} \leq C\left(\|L^* u\|_{\mathcal{L}^2_\rho} + 1\right) \leq C\left(\|L^* u\|_{\mathcal{L}^2_\rho} + \frac{1}{\sqrt{\lambda}}\|L^* u\|_{\mathcal{L}^2_\rho}\right) \leq C\|L^* u\|_{\mathcal{L}^2_\rho}$$

□

The following theorem uses the no-kernel condition. The proof is similar to that of [15] with some modifications.

Theorem 4.4. *Suppose the kernel of Φ^* is trivial in Ω. Then there is a constant $C = C(n, g, \Omega, \Sigma, \rho)$, uniform for metrics $\mathcal{C}^2$-near g, so that for $u \in H^2_{\text{loc}}(\Omega)$,*

$$\|u\|_{H^2_\rho(\Omega)} \leq C\|\Phi^* u\|_{\mathcal{L}^2_\rho}.$$

In fact, we only need to show the following proposition, then the result follows directly from the integration technique discussed in [15]:

Proposition 4.5. *Suppose the kernel of Φ^* is trivial in Ω. For $\epsilon \geq 0$ sufficiently small, and for any $u \in H^2_{\text{loc}}(\Omega)$,*

$$\|u\|_{H^2(\Omega_\epsilon)} \leq C\|\Phi^* u\|_{\mathcal{L}^2(\Omega_\epsilon)}$$

where C is independent of ϵ.

Proof. Since Φ^* has trivial H^2_{loc}-kernel in Ω, we claim that there is an $\epsilon > 0$ small enough so that Φ^* has no H^2-kernel in Ω_ϵ. The proof is the same as that in Corvino's result [15].

By a standard application of Rellich's lemma, we have for all $\epsilon \geq 0$ small enough and $u \in H^2(\Omega_\epsilon)$,

$$\|u\|_{H^2(\Omega_\epsilon)} \leq C\|\Phi^* u\|_{\mathcal{L}^2(\Omega_\epsilon)}.$$

We claim furthermore that the constant C above is independent of ϵ small. Suppose the contrary. Then there is a sequence $\epsilon_i \downarrow 0$ and $u_i \in H^2(\Omega_i)$ such that

$$\|u_i\|_{H^2(\Omega_i)} > i \|\Phi^*(u_i)\|_{\mathcal{L}^2(\Omega_i)}.$$

By normalizing $\|u_i\|_{H^2(\Omega_i)}$ to 1, we have $\|\Phi^*(u_i)\|_{\mathcal{L}^2(\Omega_i)} < 1/i$. Since $\partial\Omega_i$ is $C^{1,1}$, we can extend u_i to $H^2(\Omega)$ in such a way that for all i, $\|u_i\|_{H^2(\Omega)} \le C_1$.

Since the inclusion $H^2(\Omega) \hookrightarrow H^1(\Omega)$ is compact, by relabeling indices we have a function $\varphi \in H^1(\Omega)$ with $u_i \to \varphi$ in $H^1(\Omega)$; similarly, up to a subsequence $\{u_i|_\Sigma\}$ also converges in $H^1(\Sigma)$, and by the uniqueness, it converges to $\varphi|_\Sigma$. Moreover, for any $\eta \in \mathcal{C}_c^\infty(\Omega \cup \Sigma)$,

$$\begin{aligned} 0 = \lim_{i\to\infty} \langle \eta, \Phi^*(u_i)\rangle_{\mathcal{L}^2} &= \lim_{i\to\infty} \left(\int_\Omega L(\eta) u_i \, d\mu_g + \int_\Sigma 2u_i \dot{H}(\eta) d\sigma_g \right) \\ &= \int_\Omega L(\eta)\varphi \, d\mu_g + \int_\Sigma 2\varphi \dot{H}(\eta) d\sigma_g \\ &= \langle \eta, \Phi^*(\varphi)\rangle_{\mathcal{L}^2} \end{aligned}$$

we have $\Phi^*(\varphi) = 0$ weakly. By elliptic regularity, $\varphi \in H^2_{\text{loc}}(\Omega)$. Then by the no-kernel condition of Φ^*, φ must be zero. On the other hand, we show that φ is not identically zero, which is a contradiction.

By Lemma 4.1,

$$\|u\|_{H^2(\Omega_\epsilon)} \le C \left(\|L^* u\|_{\mathcal{L}^2(\Omega_\epsilon)} + \|u\|_{L^2(\Omega_\epsilon)} \right), \tag{4.1}$$

where C is independent of ϵ small and uniform for metrics $\mathcal{C}^2$-near g. By the H^1-convergence of $\{u_i\}$, there is an i_1 so that for $i \ge i_1$,

$$\|u_i - \varphi\|_{L^2(\Omega)} < \frac{1}{4C}.$$

Moreover, we can choose i_1 large enough so that $i \geq i_1$ implies

$$\|L^*(u_i - \varphi)\|_{\mathcal{L}^2(\Omega_i)} \leq \|\Phi^*(u_i - \varphi)\|_{\mathcal{L}^2(\Omega_i)} = \|\Phi^*(u_i)\|_{\mathcal{L}^2(\Omega_i)} < \frac{1}{4C}.$$

Plugging $(u_i - \varphi)$ into the above estimate (4.1) we get for i large

$$\|u_i - \varphi\|_{H^2(\Omega_i)} < \frac{1}{2}.$$

Note we have strongly used the independence of C on i large. This shows that φ cannot be zero by the normalization of the u_i. □

4.2 The Dirichlet problem

We would like to first solve the following problem:

$$\begin{cases} L(\rho L^* u) = 0 & \text{in } \Omega \\ B(\rho L^* u) = \psi & \text{on } \Sigma \end{cases}$$

Theorem 4.6. *For each $f \in H^{-2}_{\rho^{-1}}(\Omega)$ (the dual space of $H^2_\rho(\Omega)$), there is a unique solution $u \in H^2_\rho(\Omega)$ to the following system of equations:*

$$\begin{cases} L(\rho L^* u) = f & \text{in } \Omega \\ u = 0 & \text{on } \Sigma \\ u_\nu = 0 & \text{on } \Sigma \end{cases} \tag{4.2}$$

and moreover,

$$\|u\|_{H^2_\rho(\Omega)} \leq C\|f\|_{H^{-2}_{\rho^{-1}}(\Omega)}.$$

Proof. For the uniqueness, assume $u_1, u_2 \in H^2_\rho(\Omega)$ are both solutions. Then $w = u_1 - u_2$ satisfies

$$\begin{cases} L(\rho L^* w) = 0 & \text{in } \Omega \\ w = 0 & \text{on } \Sigma \\ w_\nu = 0 & \text{on } \Sigma \end{cases}$$

Since

$$0 = \int_\Omega L(\rho L^* w) w \, d\mu_g = \int_\Omega \|L^* w\|^2 \rho \, d\mu_g$$

we have $L^* w = 0$ in Ω. Then by Lemma 2.8, $w \equiv 0$.

We will use the standard variational method to show the existence, which is similar to the proof of Theorem 4.3. Consider the functional $\mathcal{F}_1$ defined on $H^2_\rho(\Omega)$ by

$$\mathcal{F}_1(u) = \int_\Omega \left(\frac{1}{2} \|L^* u\|^2 \rho - f u \right) d\mu_g$$

and minimize it over

$$\mathcal{A} = \{u \in H^2_\rho(\Omega) : u = u_\nu = 0 \text{ on } \Sigma\}.$$

By Theorem 4.3, $\|L^* u\|_{\mathcal{L}^2_\rho} \geq \frac{1}{C} \|u\|_{H^2_\rho}$, so we have

$$\mathcal{F}_1(u) \geq \frac{1}{2C^2} \|u\|^2_{H^2_\rho} - \|f\|_{H^{-2}_{\rho^{-1}}} \cdot \|u\|_{H^2_\rho}$$

Thus $\mathcal{F}_1$ satisfies the coercivity condition and attains its infimum. Suppose the functional is minimized by $\mathcal{F}_1(u_0) = \lambda$, and let $\eta \in H_0^2(\Omega)$. Then

$$
\begin{aligned}
0 &= \frac{d}{dt}\bigg|_{t=0} \mathcal{F}_1(u_0 + t\eta) \\
&= \frac{d}{dt}\bigg|_{t=0} \left[\int_\Omega \left(\frac{1}{2}\|L^*(u_0 + t\eta)\|^2\rho - f(u_0 + t\eta)\right) d\mu_g\right] \\
&= \frac{d}{dt}\bigg|_{t=0} \left[\int_\Omega \frac{1}{2}\sum_{\alpha,\beta} (L^*u_0 + tL^*\eta)^2_{\alpha\beta}\,\rho\, d\mu_g\right] - \int_\Omega f\eta\, d\mu_g \\
&= \int_\Omega \langle \rho L^* u_0, L^*\eta\rangle\, d\mu_g - \int_\Omega f\eta\, d\mu_g \\
&= \int_\Omega L(\rho L^* u_0)\eta\, d\mu_g - \int_\Omega f\eta\, d\mu_g
\end{aligned}
$$

Thus for all $\eta \in H_0^2(\Omega)$, $\int_\Omega L(\rho L^* u_0)\eta\, d\mu_g = \int_\Omega f\eta\, d\mu_g$, which means $u_0 \in H_\rho^2(\Omega)$ is a weak solution.

Finally, the inequality comes from elliptic regularity. □

Let us define the domain $\mathcal{D}$ to be the set of functions on Σ which extend to Ω to be in $H_\rho^2(\Omega)$. The following Dirichlet problem tells us that we only need to focus on the domain $\mathcal{D}$, given $L(\rho L^* u) = 0$ in Ω.

Theorem 4.7 (The Dirichlet problem)**.** *For each $\hat{u} \in \mathcal{D}$, there is a unique solution $u \in H_\rho^2(\Omega)$ to the following system of equations:*

$$
\begin{cases}
L(\rho L^* u) = 0 & \text{in } \Omega \\
u = \hat{u} & \text{on } \Sigma \\
u_\nu = 0 & \text{on } \Sigma
\end{cases}
\tag{4.3}
$$

Proof. Let us consider an equivalent system of equations with zero boundary data on Σ. Let $u_0 \in H_\rho^2(\Omega)$ be any extension of $\hat{u}$ such that $(u_0)_\nu|_\Sigma = 0$, and let $v = u - u_0$. Then

$v \in H^2_\rho(\Omega)$ solves the following equations:

$$\begin{cases} L(\rho L^* v) = -L(\rho L^* u_0) & \text{in } \Omega \\ v = 0 & \text{on } \Sigma \\ v_\nu = 0 & \text{on } \Sigma \end{cases} \tag{4.4}$$

if and only if $u \in H^2_\rho(\Omega)$ solves (4.3). Also, the existence and uniqueness of the solution u is equivalent to those of v. Denote $f = -L(\rho L^* u_0) \in H^{-2}_{\rho^{-1}}(\Omega)$, then from Theorem 4.6, we get a unique solution $v \in H^2_\rho(\Omega)$. □

From now on, for $\hat{u} \in \mathcal{D}$ we will denote u to be the unique extension as a solution of the Dirichlet Problem (4.3).

Assuming the kernel of Φ^* is trivial in Ω, let us define the inner product on $\mathcal{D}$ by

$$\langle \hat{u}, \hat{v} \rangle_{\mathcal{D}} \triangleq \int_\Omega \langle L^* u, L^* v \rangle \, \rho \, d\mu_g + C_0 \int_\Sigma \hat{u}\hat{v}\rho \, d\sigma_g$$

where $C_0 = \sup_\Sigma \|h\|^2 < \infty$. Note that by the Trace Theorem $\hat{u}, \hat{v} \in H^1_\rho(\Sigma)$, so the boundary term is well-defined.

Let us assume $C_0 > 0$ for now; otherwise if $C_0 = 0$, Σ is totally geodesic and

$$\Phi^*(u) = (L^*(u), 0).$$

Then we would have $\ker \Phi^* = \ker L^*$ and $\|\Phi^*\| = \|L^*\|$. So all the discussion about the space $\mathcal{D}$ is still valid without the boundary term.

Proposition 4.8. *Assuming the kernel of Φ^* is trivial in Ω, $\mathcal{D}$ is a Hilbert space.*

Proof. For any Cauchy sequence $\{\hat{u_j}\}$ in $\mathcal{D}$, u_j's are in $H^2_\rho(\Omega)$, and we have

$$\begin{aligned}
\|\Phi^*(u_j - u_k)\|^2_{\mathcal{L}^2_\rho} &= \|L^*(u_j - u_k)\|^2_{\mathcal{L}^2_\rho(\Omega)} + \int_\Sigma \|(u_j - u_k)_\nu \hat{g} - (\hat{u_j} - \hat{u_k})h\|^2_{\hat{g}} \rho \, d\sigma_g \\
&= \|L^*(u_j - u_k)\|^2_{\mathcal{L}^2_\rho(\Omega)} + \int_\Sigma \|h\|^2 (\hat{u_j} - \hat{u_k})^2 \rho \, d\sigma_g \\
&\leq \|L^*(u_j - u_k)\|^2_{\mathcal{L}^2_\rho(\Omega)} + C_0 \|\hat{u_j} - \hat{u_k}\|^2_{L^2_\rho(\Sigma)} \\
&= \|\hat{u_j} - \hat{u_k}\|^2_{\mathcal{D}}
\end{aligned}$$

On the other hand, by Theorem 4.4

$$\|u_j - u_k\|_{H^2_\rho(\Omega)} \leq C \, \|\Phi^*(u_j - u_k)\|_{\mathcal{L}^2_\rho}$$

Thus,

$$\|u_j - u_k\|_{H^2_\rho(\Omega)} \leq C \|\hat{u_j} - \hat{u_k}\|_{\mathcal{D}}$$

and $\{u_j\}$ is a Cauchy sequence in $H^2_\rho(\Omega)$. Since $H^2_\rho(\Omega)$ is a Hilbert space, there is $u \in H^2_\rho(\Omega)$ such that $u_j \to u$ in $H^2_\rho(\Omega)$. Define $\hat{u} = u|_\Sigma$, then $\hat{u} \in \mathcal{D}$. It is easy to see that u is the unique solution of the Dirichlet Problem (4.3) with boundary data $\hat{u}$, and $\hat{u_j} \to \hat{u}$ in $H^1_\rho(\Sigma)$ by the Trace Theorem. Finally,

$$\begin{aligned}
\|\hat{u_j} - \hat{u}\|^2_{\mathcal{D}} &= \|L^*(u_j - u)\|^2_{\mathcal{L}^2_\rho(\Omega)} + C_0 \|\hat{u_j} - \hat{u}\|^2_{L^2_\rho(\Sigma)} \\
&\leq C_1 \|u_j - u\|^2_{H^2_\rho(\Omega)} + C_2 \|u_j - u\|^2_{H^2_\rho(\Omega)} \\
&= C \|u_j - u\|^2_{H^2_\rho(\Omega)} \longrightarrow 0
\end{aligned}$$

This means $\mathcal{D}$ is a Hilbert space. $\square$

4.3 The self-adjoint problem

Define an operator $P : \mathcal{D} \to \mathcal{D}^*$ by

$$P(\hat{u}) = 2\dot{H}(\rho L^* u) + \langle \rho L^* u, h \rangle_{\hat{g}} + C_0 \hat{u} \rho$$

and

$$\begin{aligned} (P\hat{u})(\hat{v}) &= 2\int_\Sigma \hat{v}\dot{H}(\rho L^* u) + \int_\Sigma \hat{v}\,\langle \rho L^* u, h\rangle_{\hat{g}} + C_0 \int_\Sigma \hat{v}\hat{u}\rho \\ &= \int_\Omega \langle L^* u, L^* v\rangle\,\rho + C_0 \int_\Sigma \hat{v}\hat{u}\rho \\ &= (P\hat{v})(\hat{u}) \end{aligned} \tag{4.5}$$

Proposition 4.9. *The operator P is unitary.*

Proof.

$$\begin{aligned} \|P(\hat{u})\|_{\mathcal{D}^*} &= \sup_{0 \neq \hat{v} \in \mathcal{D}} \frac{|\,(P\hat{u})(\hat{v})\,|}{\|\hat{v}\|_{\mathcal{D}}} \\ &= \left(\frac{\left(\int_\Omega \langle L^* u, L^* v\rangle\,\rho + C_0 \int_\Sigma \hat{v}\hat{u}\rho\right)^2}{\|L^* v\|^2_{\mathcal{L}^2_\rho(\Omega)} + C_0\|\hat{v}\|^2_{L^2_\rho(\Sigma)}} \right)^{1/2} \\ &\leq \left(\frac{\left(\|L^* u\|_{\mathcal{L}^2_\rho(\Omega)}\|L^* v\|_{\mathcal{L}^2_\rho(\Omega)} + C_0\|\hat{u}\|_{L^2_\rho(\Sigma)}\|\hat{v}\|_{L^2_\rho(\Sigma)}\right)^2}{\|L^* v\|^2_{\mathcal{L}^2_\rho(\Omega)} + C_0\|\hat{v}\|^2_{L^2_\rho(\Sigma)}} \right)^{1/2} \\ &\leq \left(\frac{\left(\|L^* u\|^2_{\mathcal{L}^2_\rho(\Omega)} + C_0\|\hat{u}\|^2_{L^2_\rho(\Sigma)}\right)\left(\|L^* v\|^2_{\mathcal{L}^2_\rho(\Omega)} + C_0\|\hat{v}\|^2_{L^2_\rho(\Sigma)}\right)}{\|L^* v\|^2_{\mathcal{L}^2_\rho(\Omega)} + C_0\|\hat{v}\|^2_{L^2_\rho(\Sigma)}} \right)^{1/2} \\ &= \|\hat{u}\|_{\mathcal{D}} \end{aligned}$$

On the other hand, for $\hat{u} \in \mathcal{D}$, $\hat{u} \neq 0$,

$$\|P(\hat{u})\|_{\mathcal{D}^*} \geq \frac{|\,(P\hat{u})\,(\hat{u})\,|}{\|\hat{u}\|_{\mathcal{D}}} = \frac{\left(\|L^*u\|^2_{\mathcal{L}^2_\rho(\Omega)} + C_0\|\hat{u}\|^2_{L^2_\rho(\Sigma)}\right)}{\left(\|L^*u\|^2_{\mathcal{L}^2_\rho(\Omega)} + C_0\|\hat{u}\|^2_{L^2_\rho(\Sigma)}\right)^{1/2}} = \|\hat{u}\|_{\mathcal{D}}$$

This means $\|P(\hat{u})\|_{\mathcal{D}^*} = \|\hat{u}\|_{\mathcal{D}}$ and P is unitary. □

For $\psi \in \mathcal{D}^*$ and assuming the triviality condition on the kernel of Φ^*, we want to solve the self-adjoint problem

$$P(\hat{u}) = \psi.$$

Theorem 4.10 (The self-adjoint problem)**.** *Assume the kernel of Φ^* is trivial. For each $\psi \in \mathcal{D}^*$, there is a unique solution $\hat{u} \in \mathcal{D}$ such that*

$$P(\hat{u}) = \psi.$$

Moreover,

$$\|\hat{u}\|_{\mathcal{D}} \leq C\|\psi\|_{\mathcal{D}^*}.$$

Proof. Uniqueness follows directly from Proposition 4.9.

Let us consider the functional $\mathcal{F}_2$ defined on $H^2_\rho(\Omega)$ by

$$\mathcal{F}_2(v) = \frac{1}{2}\int_\Omega \|L^*v\|^2 \rho\, d\mu_g + \frac{C_0}{2}\|\hat{v}\|^2_{L^2_\rho(\Sigma)} - \int_\Sigma \psi\hat{v}\, d\sigma_g$$

and minimize it over

$$\mathcal{A} = \{u \in H^2_\rho(\Omega) : u_\nu = 0 \text{ on } \Sigma\}.$$

By Theorem 4.4 we have $\|v\|_{H^2_\rho(\Omega)} \leq C \|\Phi^* v\|_{\mathcal{L}^2_\rho(\Omega)}$, then

$$\begin{aligned}
\mathcal{F}_2(v) &= \frac{1}{2}\|\Phi^* v\|^2_{\mathcal{L}^2_\rho(\Omega)} - \frac{1}{2}\int_\Sigma \|h\|^2 \hat{v}^2 \rho \, d\sigma_g + \frac{C_0}{2}\|\hat{v}\|^2_{L^2_\rho(\Sigma)} - \int_\Sigma \psi \hat{v} \, d\sigma_g \\
&\geq \frac{1}{2C}\|v\|^2_{H^2_\rho(\Omega)} - \|\psi\|_{\mathcal{D}^*}\|\hat{v}\|_{\mathcal{D}} \\
&\geq \frac{1}{2C}\|v\|^2_{H^2_\rho(\Omega)} - C'\|\psi\|_{\mathcal{D}^*}\|v\|_{H^2_\rho(\Omega)}
\end{aligned}$$

Thus $\mathcal{F}_2(v)$ satisfies the coercivity condition and attains its infimum. We note that if $v_0 \in \mathcal{A}$ is a minimizer of $\mathcal{F}_2$, then $L(\rho L^* v_0) = 0$ in Ω, which means any minimizer v_0 is a solution to the Dirichlet problem with boundary data $\hat{v}_0$. This is because, take any $\tilde{\eta} \in C^\infty_c(\Omega)$, then

$$\begin{aligned}
0 &= \left.\frac{d}{dt}\right|_{t=0} \mathcal{F}_2(v_0 + t\tilde{\eta}) \\
&= \left.\frac{d}{dt}\right|_{t=0} \left[\int_\Omega \frac{1}{2}\|L^*(v_0 + t\tilde{\eta})\|^2 \rho \, d\mu_g + \frac{C_0}{2}\|\hat{v}_0\|^2_{L^2_\rho(\Sigma)} - \int_\Sigma \psi \hat{v}_0 \, d\sigma_g\right] \\
&= \int_\Omega \langle \rho L^* v_0, L^* \tilde{\eta}\rangle \, d\mu_g \\
&= \int_\Omega L(\rho L^* v_0)\tilde{\eta} \, d\mu_g
\end{aligned}$$

Suppose the functional is minimized by $\mathcal{F}_2(v_0) = \lambda$, and let $\hat{\eta} \in C^\infty_c(\Sigma) \subset \mathcal{D}$, then

$$\begin{aligned}
0 &= \left.\frac{d}{dt}\right|_{t=0} \mathcal{F}_2(v_0 + t\eta) \\
&= \left.\frac{d}{dt}\right|_{t=0} \left[\int_\Omega \frac{1}{2}\|L^*(v_0 + t\eta)\|^2 \rho \, d\mu_g + \frac{C_0}{2}\|\hat{v}_0 + t\hat{\eta}\|^2_{L^2_\rho(\Sigma)} - \int_\Sigma \psi(\hat{v}_0 + t\hat{\eta}) d\sigma_g\right] \\
&= \int_\Omega \langle \rho L^* v_0, L^* \eta\rangle \, d\mu_g + C_0 \int_\Sigma \hat{v}_0 \hat{\eta} \rho \, d\sigma_g - \int_\Sigma \psi \hat{\eta} \, d\sigma_g \\
&= \int_\Omega L(\rho L^* v_0)\eta \, d\mu_g + \int_\Sigma \hat{\eta}\mathcal{P}(\hat{v}_0) d\sigma_g - \int_\Sigma \eta_\nu \operatorname{tr}_\Sigma(\rho L^* v_0) d\sigma_g - \int_\Sigma \psi \hat{\eta} \, d\sigma_g \\
&= \int_\Sigma \hat{\eta} P(\hat{v}_0) d\sigma_g - \int_\Sigma \psi \hat{\eta} \, d\sigma_g
\end{aligned}$$

So for any $\hat{\eta} \in C^\infty_c(\Sigma)$, $\int_\Sigma \hat{\eta} P(\hat{v}_0) d\sigma_g = \int_\Sigma \psi \hat{\eta} \, d\sigma_g$, which means $v_0 \in H^2_\rho(\Omega)$, and thus $\hat{v}_0 \in \mathcal{D}$ is a weak solution.

Finally, the inequality comes from elliptic regularity. □

Corollary 4.11. *The operator P is an isometric isomorphism.*

4.4 Solving the linearized equation using Fredholm theory

Define an operator $\hat{B} : \mathcal{D} \to \mathcal{D}^*$ by

$$\hat{B}(\hat{u}) = B(\rho L^* u) = 2\dot{H}(\rho L^* u).$$

Then $\hat{B}$ differs from P by a lower order term $K = K_1 + K_2$.

Theorem 4.12. *The operator $K_1 : \mathcal{D} \to \mathcal{D}^*$ defined by $K_1(\hat{u}) = C_0 \hat{u}\rho$ is compact.*

Proof. Consider a bounded sequence $\{\hat{u}_j\}$ in $\mathcal{D}$ such that $\|\hat{u}_j\|_{\mathcal{D}} = 1$, then u_j's are in $H^2_\rho(\Omega)$, and we have

$$\|u_j\|_{H^2_\rho(\Omega)} \leq C\|\hat{u}_j\|_{\mathcal{D}} = C.$$

We notice that for $u_j \in H^2_\rho(\Omega)$ we have [17]

$$\left\| u_j \rho^{\frac{1}{2}} \right\|_{H^2(\Omega)} \leq C\|u_j\|_{H^2_\rho(\Omega)} \leq C,$$

where the constant C is uniform for metrics $\mathcal{C}^2$-near g.

From Rellich compactness theorem and the trace theorem, the inclusion $H^2(\Omega) \hookrightarrow H^1(\Sigma)$ is compact, so by relabeling indices we have that $\{\hat{u}_j \rho^{\frac{1}{2}}\big|_\Sigma\}$ converges to $\{\hat{u}\rho^{\frac{1}{2}}\big|_\Sigma\}$ in $H^1(\Sigma)$.

Thus for any $\hat{v} \in \mathcal{D}$,

$$
\begin{aligned}
K_1(\hat{u_j})(\hat{v}) &= C_0 \int_\Sigma \hat{v}\hat{u}_j\rho \\
&= C_0 \int_\Sigma \left(\hat{v}\rho^{\frac{1}{2}}\right)\left(\hat{u}_j\rho^{\frac{1}{2}}\right) \\
&\longrightarrow C_0 \int_\Sigma \left(\hat{v}\rho^{\frac{1}{2}}\right)\left(\hat{u}\rho^{\frac{1}{2}}\right) = C_0\hat{u}\rho(\hat{v})
\end{aligned}
$$

This means the operator K_1 is compact. □

Theorem 4.13. *The operator $K_2 : \mathcal{D} \to \mathcal{D}^*$ defined by $K_2(\hat{u}) = \langle \rho L^* u, h\rangle_{\hat{g}}$ is compact.*

Proof. Consider a bounded sequence $\{\hat{u_j}\}$ in $\mathcal{D}$ such that $\|\hat{u_j}\|_{\mathcal{D}} = 1$, and thus $\|u_j\|_{H^2_\rho(\Omega)} \leq C$. Notice that $L^*(u) = -\left(\Delta_g u\right) g + \mathrm{Hess}(u) - u\,\mathrm{Ric}_g$, so we want to analyze each term one by one. For any $\hat{v} \in \mathcal{D}$,

$$
\mathrm{III} = \int_\Sigma \langle \rho u_j \,\mathrm{Ric}_g, h\rangle_{\hat{g}}\,\hat{v} = \int_\Sigma \langle \mathrm{Ric}_g, h\rangle_{\hat{g}}\,\rho\hat{u}_j\hat{v}
$$

Because $\langle \mathrm{Ric}_g, h\rangle_{\hat{g}}$ is bounded, we have up to a subsequence III is convergent, which is similar to Theorem 4.12.

$$
\mathrm{II} = \int_\Sigma \langle \rho\,\mathrm{Hess}(u_j), h\rangle_{\hat{g}}\,\hat{v} = \int_\Sigma \langle \mathrm{Hess}_\Sigma(u_j) + (u_j)_\nu h, h\rangle_{\hat{g}}\,\rho\hat{v} = \int_\Sigma \langle \mathrm{Hess}_\Sigma(\hat{u}_j), h\rangle_{\hat{g}}\,\rho\hat{v}
$$

Consider the local Fermi coordinate, where $k, l = 1, \cdots, n-1$, then

$$
\begin{aligned}
\mathrm{II} &= \int_\Sigma \nabla_k\nabla_l\hat{u}_j h^{kl}\rho\hat{v} = -\int_\Sigma \nabla_l\hat{u}_j\nabla_k\left(h^{kl}\rho\hat{v}\right) \\
&= -\int_\Sigma \nabla_l\hat{u}_j\nabla_k h^{kl}\rho\hat{v} - \int_\Sigma \nabla_l\hat{u}_j h^{kl}\rho\nabla_k\hat{v} - \int_\Sigma \nabla_l\hat{u}_j h^{kl}\nabla_k\rho\hat{v}
\end{aligned}
$$

Denote $w_j = \nabla u_j$, so that

$$
\|w_j\|^2_{\mathcal{H}^1_\rho(\Omega)} = \int_\Omega |\nabla u_j|^2\rho + |\nabla^2 u_j|^2\rho \leq \|u_j\|^2_{H^2_\rho(\Omega)} \leq C.
$$

From [17] we get $\|w_j\rho^{\frac{1}{2}}\|_{\mathcal{H}^1(\Omega)} \le C\|w_j\|_{\mathcal{H}^1_\rho(\Omega)} \le C$. Then from Rellich compactness theorem and the trace theorem, the inclusion $\mathcal{H}^1(\Omega) \hookrightarrow \mathcal{L}^2(\Sigma)$ is compact, so by relabeling indices we have that $\{\hat{w}_j\rho^{\frac{1}{2}}\big|_\Sigma\}$ converges to $\{\hat{w}\rho^{\frac{1}{2}}\big|_\Sigma\}$ in $\mathcal{L}^2(\Sigma)$, and thus $\{\nabla\hat{u}_j\rho^{\frac{1}{2}}\big|_\Sigma\}$ converges to $\{\nabla\hat{u}\rho^{\frac{1}{2}}\big|_\Sigma\}$ in $\mathcal{L}^2(\Sigma)$. This means the first two integrals of II are convergent.

For the third integral of II, notice that in [17] we have for $\hat{v} \in H^1_\rho(\Sigma)$,

$$\int_\Sigma \hat{v}^2 d^{-4}\rho \le C\|\hat{v}\|^2_{H^1_\rho(\Sigma)}.$$

Thus,

$$\begin{aligned}\left|\int_\Sigma \nabla_l\hat{u}_j h^{kl}\nabla_k\rho\hat{v} - \int_\Sigma \nabla_l\hat{u}h^{kl}\nabla_k\rho\hat{v}\right| &\le C\int_\Sigma |\nabla\hat{u}_j - \nabla\hat{u}|\,\rho d^{-2}\hat{v} \\ &\le C\left(\int_\Sigma |\nabla\hat{u}_j - \nabla\hat{u}|^2\rho\right)^{\frac{1}{2}}\left(\int_\Sigma d^{-4}\hat{v}^2\rho\right)^{\frac{1}{2}} \\ &\le C\|\hat{v}\|_{H^1_\rho(\Sigma)}\left(\int_\Sigma \left|\nabla\hat{u}_j\rho^{\frac{1}{2}} - \nabla\hat{u}\rho^{\frac{1}{2}}\right|^2\right)^{\frac{1}{2}} \\ &\longrightarrow 0.\end{aligned}$$

As a result, II is convergent up to a subsequence.

$$\mathrm{I} = \int_\Sigma \langle\rho\,(\Delta_g u_j)\,g, h\rangle_{\hat{g}}\,\hat{v} = \int_\Sigma \Delta_g u_j H\rho\hat{v}$$

Now choose $w \in L^2_\rho(\Omega)$ with $w = 0$ on Σ and $w_\nu = H\hat{v}$ on Σ. Then since $\nabla w|_\Sigma \in \mathcal{D}$, $\nabla w \in H^2_\rho(\Omega)$ and thus $w \in H^3_\rho(\Omega)$. Notice that

$$\begin{aligned}0 &= \int_\Omega L(\rho L^* u)w \\ &= \int_\Omega \langle L^*u, L^*w\rangle\,\rho + \int_\Sigma w_\nu\,\mathrm{tr}_\Sigma(\rho L^* u) \\ &= \int_\Omega \langle L^*u, \rho L^*w\rangle + \int_\Sigma H\hat{v}\,(-(n-1)\Delta_g u\rho + \rho\Delta_\Sigma u - \hat{u}\rho\,\mathrm{tr}_\Sigma\,\mathrm{Ric}_g)\end{aligned}$$

Thus,

$$(n-1)\mathrm{I} = \int_\Omega \langle L^* u_j, \rho L^* w\rangle + \int_\Sigma H\rho\hat{v}\Delta_\Sigma u_j - \int_\Sigma H \operatorname{tr}_\Sigma \operatorname{Ric}_g \hat{v}\hat{u}_j\rho$$

where

$$\begin{aligned} H \operatorname{tr}_\Sigma \operatorname{Ric}_g &= H(R_g - \operatorname{Ric}(\nu,\nu)) \\ &= \frac{1}{2}H(R_g + R_\Sigma + \|h\|^2 - H^2) \end{aligned}$$

is bounded. So similar to Theorem 4.12, we get a convergent subsequence for the third integral of I. On the other hand,

$$\int_\Sigma H\rho\hat{v}\Delta_\Sigma u_j = -\int_\Sigma \nabla_\Sigma(H\rho\hat{v})\cdot\nabla_\Sigma u_j$$

So the second integral of I is also convergent, which is similar to II. Similarly, since $L^*w \in H^1_\rho(\Omega)$, we may use integration by parts to move one derivative from L^*u to $\rho L^* w$ and get the convergent subsequence. As a result, I is convergent up to a subsequence.

In conclusion, we get up to a subsequence,

$$K_2(\hat{u_j})(\hat{v}) = \int_\Sigma \langle \rho L^* u_j, h\rangle_{\hat{g}}\,\hat{v} \longrightarrow \int_\Sigma \langle \rho L^* u, h\rangle_{\hat{g}}\,\hat{v} = K_2(\hat{u})(\hat{v})$$

This means the operator K_2 is compact. □

Theorem 4.14. *The operator $\hat{B}$ is Fredholm and* $\operatorname{ind}(\hat{B}) = \operatorname{ind}(P) = 0$.

Proof. From the above propositions we have $\hat{B} = P + K$, where $K : \mathcal{D} \to \mathcal{D}^*$ is a compact operator. Then the result follows directly from the basic properties of the Fredholm operators [28,48]. □

Corollary 4.15. *$\hat{B}(\mathcal{D})$ is closed and has finite codimension in $\mathcal{D}^*$.*

Now let us consider the space

$$S = \{a \in \mathcal{S}^{(0,2)} \cap \mathcal{L}^2_{\rho^{-1}}(\Omega) : L(a) = 0\}$$

and the operator $B : S \to \mathcal{D}^*$ given by

$$B(a) = 2\dot{H}(a).$$

Proposition 4.16. *The space $S' = \{\rho L^* u : \hat{u} \in \mathcal{D}\}$ has infinite codimension in S.*

Proof. Let us first notice that for $\hat{u} \in \mathcal{D}$, $\rho L^* u \in \mathcal{S}^{(0,2)} \cap \mathcal{L}^2_{\rho^{-1}}(\Omega)$ and $L(\rho L^* u) = 0$ in Ω, thus $S' \subset S$.

By definition,

$$\mathcal{S}^{(0,2)} \cap \mathcal{L}^2_{\rho^{-1}}(\Omega) \cong L^2_{\rho^{-1}}(\Omega) \oplus L^2_{\rho^{-1}}(\Omega)$$

thus

$$S \cong L^2_{\rho^{-1}}(\Omega) \oplus L^2_{\rho^{-1}}(\Omega)/\{L(a) = 0\}.$$

On the other hand, $S' \subset L^2_{\rho^{-1}}(\Omega)$, and in particular

$$S' \subset L^2_{\rho^{-1}}(\Omega)/\{L(a) = 0\}.$$

So

$$L^2_{\rho^{-1}}(\Omega) \subset S/S'$$

and thus S' has infinite codimension in S. □

Then

$$\hat{B}(\mathcal{D}) = B(S') \subset B(S) \subset \mathcal{D}^*.$$

Since $\hat{B}(\mathcal{D})$ has finite codimension in $\mathcal{D}^*$, $B(S)$ also has finite codimension in $\mathcal{D}^*$ and thus is closed.

On the other hand, we know the kernel of Φ^* is trivial by the generic condition, then Φ is surjective, where

$$\Phi(a) = (L(a), B(a)).$$

This means B is surjective. As a result, we get $B(S) = \mathcal{D}^*$. So for any $\psi \in \mathcal{D}^*$, there is some $a \in \mathcal{S}^{(0,2)} \cap \mathcal{L}^2_{\rho^{-1}}(\Omega)$ such that

$$\begin{cases} L(a) = 0 & \text{in } \Omega \\ B(a) = \psi & \text{on } \Sigma \end{cases} \tag{4.6}$$

Here we would like to emphasize that we are not claiming the uniqueness of the solution. Now that we have already got the existence of the weak solution, we would like to study its structure more carefully. Note as a corollary of Corollary 4.15, we have

Corollary 4.17. $\hat{B}^{-1} : \hat{B}(\mathcal{D}) \to \mathcal{D}/\ker \hat{B}$ *is a bounded linear operator.*

Proof. As a closed subspace of $\mathcal{D}^*$, $\hat{B}(\mathcal{D})$ is also a Hilbert space. Then it follows directly from the Bounded Inverse Theorem. □

For simplicity, we will always choose the representing element $\hat{u}$ to be in the orthogonal complement of $\ker \hat{B}$, so that $\hat{B}^{-1} \in \mathcal{L}(\hat{B}(\mathcal{D}), (\ker \hat{B})^{\perp})$ and

$$\psi = \hat{B}(\hat{u}) \xmapsto{\hat{B}^{-1}} \hat{u}.$$

On the other hand, by Theorem 4.14, denote $\dim(\hat{B}(\mathcal{D}))^{\perp} = \dim \ker \hat{B} = p < \infty$. Now we are going to deal with this finite dimensional subspace of $\mathcal{D}^*$. Suppose $\{\psi_1, \cdots, \psi_p\}$ is a basis of $(\hat{B}(\mathcal{D}))^{\perp}$. For each ψ_j, there is a solution $a_j \in S$ of (4.6), and we may choose $a_j \in (S')^{\perp}$ to be linearly independent. This is possible because of Proposition 4.16. Denote this bounded linear map as $T_0 : (\hat{B}(\mathcal{D}))^{\perp} \to (S')^{\perp}$.

Then we can construct a bounded linear map $T : \mathcal{D}^* \to S$ by

$$\begin{array}{ccccc} \mathcal{D}^* & = & \hat{B}(\mathcal{D}) & \oplus & (\hat{B}(\mathcal{D}))^{\perp} \\ \big\downarrow{\scriptstyle T} & & \big\downarrow{\scriptstyle \rho L^* \circ \hat{B}^{-1}} & & \big\downarrow{\scriptstyle T_0} \\ S & = & S' & \oplus & (S')^{\perp} \end{array}$$

More precisely, for any $\psi \in \hat{B}(\mathcal{D}) \subset \mathcal{D}^*$, the map T is defined in the following way, where **DP** indicates the Dirichlet problem (4.3):

$$\psi = \hat{B}(\hat{u}) \xmapsto{\hat{B}^{-1}} \hat{u} \xmapsto{\text{DP}} u \xmapsto{\rho L^*} \rho L^* u,$$

and it is easy to see that $L(\rho L^* u) = 0, B(\rho L^* u) = \hat{B}(\hat{u}) = \psi$.

For any $\psi \in (\hat{B}(\mathcal{D}))^{\perp} \subset \mathcal{D}^*$, the map T behaves like T_0,

$$\psi = \sum c_j \psi_j \xmapsto{T_0} \sum c_j a_j,$$

and it is easy to see that $L(\sum c_j a_j) = 0, B(\sum c_j a_j) = \psi$.

This means, $B \circ T = \text{Id.}$ on $\mathcal{D}^*$ and T is a right inverse for B. As a result, for any $\psi \in \mathcal{D}^*$, $T\psi \in S$ is a solution to (4.6).

Finally, we are ready to solve the general case: for any $f \in H^{-2}_{\rho^{-1}}(\Omega)$ and any $\psi \in \mathcal{D}^*$, we would like to get a solution $a \in \mathcal{S}^{(0,2)} \cap \mathcal{L}^2_{\rho^{-1}}(\Omega)$ such that

$$\begin{cases} L(a) = f & \text{in } \Omega \\ B(a) = \psi & \text{on } \Sigma \end{cases} \tag{4.7}$$

We can get the existence in this way: let $a_0 \in \mathcal{S}^{(0,2)} \cap \mathcal{L}^2_{\rho^{-1}}(\Omega)$ be any solution of $L(a_0) = f$ (e.g., the solution $a_0 = \rho L^* u_0$ of the Dirichlet problem with zero boundary data (4.2)) and let us first solve

$$\begin{cases} L(a) = 0 & \text{in } \Omega \\ B(a) = \psi - B(a_0) & \text{on } \Sigma \end{cases} \tag{4.8}$$

using the result we get in (4.6). Suppose $a_1 = T(\psi - B(a_0)) \in S$ is a solution of (4.8). Then we claim $a_2 = a_0 + a_1 \in \mathcal{S}^{(0,2)} \cap \mathcal{L}^2_{\rho^{-1}}(\Omega)$ is a solution of (4.7). In fact, by linearity

$$\begin{aligned} L(a_2) &= L(a_0) + L(a_1) = f + 0 = f, \\ B(a_2) &= B(a_0) + B(a_1) = B(a_0) + (\psi - B(a_0)) = \psi. \end{aligned}$$

In conclusion, for any $f \in H^{-2}_{\rho^{-1}}(\Omega)$ and any $\psi \in \mathcal{D}^*$,

$$a_2 = \rho L^* u_0 + T\left(\psi - B(\rho L^* u_0)\right) \in \mathcal{S}^{(0,2)} \cap \mathcal{L}^2_{\rho^{-1}}(\Omega)$$

is a solution of (4.7), where u_0 is the unique solution of the Dirichlet problem with zero boundary data (4.2).

4.5 The nonlinear problem and iteration

In this section, we improve the regularity of the weak solution and get Schauder estimates. Then we use Picard's iteration to get a solution for the nonlinear problem.

Let $x \in \Omega$ be fixed. Consider the ball $B_{\phi(x)}(x)$ centered at x of radius $\phi(x)$, where $\phi(x)$ is the weight function defined in Section 2.4. We blur the distinction between $B_{\phi(x)}(x)$ and its coordinate image, and we consider the diffeomorphism $F_x : B_1(0) \to B_{\phi(x)}(x)$ by

$z \mapsto x + \phi(x)z = y$, where $B_1(0)$ is the unit ball in $\mathbb{R}^n$ centered at the origin. For any function f defined on $B_{\phi(x)}(x)$, let

$$\tilde{f}(z) = F_x^*(f)(z) = f \circ F_x(z)$$

denote the pull-back of f on $B_1(0)$.

With a minor abuse of notation, we denote for $a \in (0,1]$,

$$\|f\|_{C^{k,\alpha}_{\phi,\varphi}(B_{a\phi(x)}(x))} = \sum_{j=0}^{k} \varphi(x)\phi^j(x) \left\|\nabla^j f\right\|_{C^0\left(B_{a\phi(x)}(x)\right)} + \varphi(x)\phi^{k+\alpha}(x)\left[\nabla^k f\right]_{0,\alpha;B_{a\phi(x)}(x)}.$$

We have the following useful lemma.

Lemma 4.18 (Corvino-Huang [17]). *Let f and g be functions defined on $B_{\phi(x)}(x)$. The following properties hold.*
(i) $\widetilde{f+g} = \tilde{f} + \tilde{g}$ and $\widetilde{fg} = \tilde{f}\tilde{g}$.
(ii) $\widetilde{\partial_y^\beta f} = (\phi(x))^{-|\beta|}\partial_z^\beta \tilde{f}$, where $\beta = (\beta_1, \cdots, \beta_k)$ is a multi-index.
(iii) For any $a \in (0,1]$,

$$\|\varphi(x)\tilde{f}\|_{C^{k,\alpha}(B_a(0))} = \|f\|_{C^{k,\alpha}_{\phi,\varphi}(B_{a\phi(x)}(x))}$$
$$\|\varphi(x)\tilde{f}\|_{L^2(B_a(0))} = \|f\|_{L^2_{\phi^{-n}\varphi^2}(B_{a\phi(x)}(x))}.$$

Following Schauder estimates by scaling [53] and using the above properties as well as the properties discussed in Section 2.4, we have the interior and boundary Schauder estimates.

Theorem 4.19 (Interior Schauder estimates). *For any $k \in \mathbb{N}$ and any $r, s \in \mathbb{R}$, there is a constant C such that for any $x \in \Omega$,*

$$\|u\|_{C^{k+4,\alpha}_{\phi,\phi^r\rho^s}(B_{\phi(x)/2}(x))} \leq C\left(\|L(\rho L^* u)\|_{C^{k,\alpha}_{\phi,\phi^{r+4}\rho^{s-1}}(B_{\phi(x)}(x))} + \|u\|_{L^2_{\phi^{-n}(\phi^{2r}\rho^{2s})}(B_{\phi(x)}(x))}\right).$$

Theorem 4.20 (Boundary Schauder estimates)**.** *For any $k \in \mathbb{N}$ and any $r, s \in \mathbb{R}$, there is a constant C such that for any $y \in \Sigma$,*

$$\|u\|_{C^{k+4,\alpha}_{\phi,\phi^r\rho^s}(B^+_{\phi(y)/2}(y))} \leq C\left(\|L(\rho L^* u)\|_{C^{k,\alpha}_{\phi,\phi^{r+4}\rho^{s-1}}(B^+_{\phi(y)}(y))} + \|u\|_{L^2_{\phi^{-n}(\phi^{2r}\rho^{2s})}(B^+_{\phi(y)}(y))}\right)$$

where $B^+_{\phi(y)}(y) = B_{\phi(y)}(y) \cap \bar{\Omega}$.

Now we may take the supremum over $x \in \Omega$ and $y \in \Sigma$ and combine the above two theorems.

Theorem 4.21 (Global Schauder estimates)**.** *For any $k \in \mathbb{N}$ and any $r, s \in \mathbb{R}$, there is a constant $C = C(\Omega, \Sigma, g, h, k, n, \alpha, r, s)$ such that*

$$\|u\|_{C^{k+4,\alpha}_{\phi,\phi^r\rho^s}(\Omega\cup\Sigma)} \leq C\left(\|L(\rho L^* u)\|_{C^{k,\alpha}_{\phi,\phi^{r+4}\rho^{s-1}}(\Omega\cup\Sigma)} + \|u\|_{L^2_{\phi^{-n}(\phi^{2r}\rho^{2s})}(\Omega)}\right). \tag{4.9}$$

In particular, if u_0 is the unique solution of the Dirichlet problem with zero boundary data (4.2) with $f \in C^{k,\alpha}_{\phi,\phi^{r+4}\rho^{s-1}}(\Omega \cup \Sigma)$, then (4.9) becomes

$$\|u_0\|_{C^{k+4,\alpha}_{\phi,\phi^r\rho^s}(\Omega\cup\Sigma)} \leq C\|f\|_{C^{k,\alpha}_{\phi,\phi^{r+4}\rho^{s-1}}(\Omega\cup\Sigma)}.$$

Then from Proposition 2.4 we have the following estimates.

$$\|\rho L^* u_0\|_{C^{k+2,\alpha}_{\phi,\phi^{r+2}\rho^{s-1}}(\Omega\cup\Sigma)} \leq C\|f\|_{C^{k,\alpha}_{\phi,\phi^{r+4}\rho^{s-1}}(\Omega\cup\Sigma)}.$$

$$\|B(\rho L^* u_0)\|_{C^{k,\alpha}_{\phi,\phi^{r+3}\rho^{s-1}}(\Sigma)} \leq C\|f\|_{C^{k,\alpha}_{\phi,\phi^{r+4}\rho^{s-1}}(\Omega\cup\Sigma)}.$$

Take $k = 0, r = \frac{n}{2}, s = \frac{1}{2}$, and use the fact that $T : \mathcal{B}_0(\Sigma) \to \mathcal{B}_2(\Omega)$ is a continuous map, we get the following theorem.

Theorem 4.22. *Let g_0 be a $C^{4,\alpha}$-metric such that the operator Φ^* has trivial kernel in $H^2_{\mathrm{loc}}(\Omega)$. Then there is a constant C uniform for metrics near g_0 in $C^{4,\alpha}(\bar{\Omega})$ such that for*

$(f,\psi) \in \mathcal{B}_0(\Omega) \times \mathcal{B}_0(\Sigma)$, *if* $a \in \mathcal{S}^{(0,2)} \cap \mathcal{L}^2_{\rho^{-1}}(\Omega)$ *is a weak solution of (4.7), then* $a \in \mathcal{B}_2(\Omega)$ *and*

$$\|a\|_{\mathcal{B}_2(\Omega)} \leq C\|(f,\psi)\|_{\mathcal{B}_0(\Omega)\times\mathcal{B}_0(\Sigma)}. \tag{4.10}$$

The following estimate is similar to Corvino's.

Theorem 4.23. *Let* g_0 *be a* $C^{4,\alpha}$*-metric such that the operator* Φ^* *has trivial kernel in* $H^2_{\text{loc}}(\Omega)$. *Then there is a constant* C *uniform for metrics near* g_0 *in* $C^{4,\alpha}$, *and an* $\epsilon > 0$ *(sufficiently small) so that if*

(i) $S \in C^{0,\alpha}(\Omega)$ *with* $(S - R(g_0)) \in \mathcal{B}_0(\Omega)$*;*

(ii) and $H' \in C^{0,\alpha}(\Sigma)$ *with* $(H' - H(g_0)) \in \mathcal{B}_0(\Sigma)$ *and with*

$$\|S - R(g_0)\|_{\mathcal{B}_0(\Omega)} + \|H' - H(g_0)\|_{\mathcal{B}_0(\Sigma)} < \epsilon,$$

then upon solving $\Phi(h_0) = (S - R(g_0), 2(H' - H(g_0)))$ *via the previous method, and letting* $g_1 = g_0 + h_0$, *we have*

$$\|h_0\|_{\mathcal{B}_2(\Omega)} \leq C(\|S - R(g_0)\|_{\mathcal{B}_0(\Omega)} + \|H' - H(g_0)\|_{\mathcal{B}_0(\Sigma)}) < C\epsilon$$

and

$$\|S - R(g_1)\|_{\mathcal{B}_0(\Omega)} + \|H' - H(g_1)\|_{\mathcal{B}_0(\Sigma)} \leq C(\|S - R(g_0)\|_{\mathcal{B}_0(\Omega)} + \|H' - H(g_0)\|_{\mathcal{B}_0(\Sigma)})^2 < C\epsilon^2.$$

Moreover, the metric g_1 *is* $C^{2,\alpha}$.

We later iterate the procedure of linear correction. We only linearize about g_0 due to the apparent loss in differentiability [15]. Having found h_0 as above, we can repeat the procedure to find a symmetric tensor h_1 so that $\Phi_{g_0}(h_1) = (S - R(g_1), 2(H' - H(g_1)))$. Let $g_2 = g_1 + h_1$. By the preceding estimates, we see that for small enough ϵ, g_2 will indeed be a $C^{2,\alpha}$-metric.

In fact, we have

$$\begin{aligned}\|h_1\|_{\mathcal{B}_2(\Omega)} &\le C(\|S-R(g_1)\|_{\mathcal{B}_0(\Omega)} + \|H'-H(g_1)\|_{\mathcal{B}_0(\Sigma)})\\ &\le C^2(\|S-R(g_0)\|_{\mathcal{B}_0(\Omega)} + \|H'-H(g_0)\|_{\mathcal{B}_0(\Sigma)})^2.\end{aligned}$$

We then define h_m and g_m recursively by taking $g_m = g_0 + \sum_{l=0}^{m-1} h_l = g_{m-1} + h_{m-1}$, and letting h_m be a solution of $\Phi_{g_0}(h_m) = (S - R(g_m), 2(H' - H(g_m)))$; this assumes we have enough control to keep g_m a metric; we then show we can control h_m so that g_{m+1} is a metric, and moreover the sequence $\{g_m\}$ converges in $\mathcal{M}^{2,\alpha}$.

Theorem 4.24. *Suppose that in the above iteration procedure we have recursively obtained $h_0, \cdots, h_{m-1}$ and that $g_0, \cdots, g_m$ are $C^{2,\alpha}$-metrics. Let C be as in Theorem 4.23 and suppose that there is a constant K and a δ with $0 < \delta < 1$ so that for all $l < m$,*

$$\|h_l\|_{\mathcal{B}_2(\Omega)} \le CK(\|S-R(g_0)\|_{\mathcal{B}_0(\Omega)} + \|H'-H(g_0)\|_{\mathcal{B}_0(\Sigma)})^{1+l\delta},$$

and for all $j \le m$,

$$\|S-R(g_j)\|_{\mathcal{B}_0(\Omega)} + \|H'-H(g_j)\|_{\mathcal{B}_0(\Sigma)} \le K(\|S-R(g_0)\|_{\mathcal{B}_0(\Omega)} + \|H'-H(g_0)\|_{\mathcal{B}_0(\Sigma)})^{1+j\delta}.$$

Then for sufficiently small $\|S-R(g_0)\|_{\mathcal{B}_0(\Omega)} + \|H'-H(g_0)\|_{\mathcal{B}_0(\Sigma)}$ (independent of m), the iteration can proceed to the next step and the above inequalities persist for $l = m$ and $j = m+1$.

Theorem 4.24 show that the series $\sum_{l=0}^{\infty} h_l$ converges geometrically to some "small" $h \in \mathcal{B}_2(\Omega)$,and hence g_m converges in $\mathcal{M}^{2,\alpha}$ to $g = g + h$ with $R(g) = S$ and $H(g) = H'$. This completes the proof of our main theorem 2.5.

One can also argue similarly for higher regularity, and thus give a proof for main theorem 2.6.

Bibliography

[1] Sérgio Almaraz, Ezequiel Barbosa, and Levi Lopes de Lima, *A positive mass theorem for asymptotically flat manifolds with a non-compact boundary*, Comm. Anal. Geom. **24** (2016), no. 4, 673–715, DOI 10.4310/CAG.2016.v24.n4.a1. MR3570413

[2] F. J. Almgren Jr., *Some interior regularity theorems for minimal surfaces and an extension of Bernstein's theorem*, Ann. of Math. (2) **84** (1966), 277–292, DOI 10.2307/1970520. MR200816

[3] R. Arnowitt, S. Deser, and C. W. Misner, *The dynamics of general relativity*, Gravitation: An introduction to current research, Wiley, New York, 1962, pp. 227–265. MR0143629

[4] Nachman Aronszajn, *A unique continuation theorem for solutions of elliptic partial differential equations or inequalities of second order*, J. Math. Pures Appl. (9) **36** (1957), 235–249. MR92067

[5] Robert Bartnik, *The mass of an asymptotically flat manifold*, Comm. Pure Appl. Math. **39** (1986), no. 5, 661–693, DOI 10.1002/cpa.3160390505. MR849427

[6] Riccardo Benedetti and Carlo Petronio, *Lectures on hyperbolic geometry*, Universitext, Springer-Verlag, Berlin, 1992. MR1219310

[7] Arthur L. Besse, *Einstein manifolds*, Classics in Mathematics, Springer-Verlag, Berlin, 2008. Reprint of the 1987 edition. MR2371700

[8] Hubert L. Bray, *Proof of the Riemannian Penrose inequality using the positive mass theorem*, J. Differential Geom. **59** (2001), no. 2, 177–267. MR1908823

[9] Hubert L. Bray and Dan A. Lee, *On the Riemannian Penrose inequality in dimensions less than eight*, Duke Math. J. **148** (2009), no. 1, 81–106, DOI 10.1215/00127094-2009-020. MR2515101

[10] Simon Brendle and Fernando C. Marques, *Scalar curvature rigidity of geodesic balls in* S^n, J. Differential Geom. **88** (2011), no. 3, 379–394. MR2844438

[11] Simon Brendle, Fernando C. Marques, and Andre Neves, *Deformations of the hemisphere that increase scalar curvature*, Invent. Math. **185** (2011), no. 1, 175–197, DOI 10.1007/s00222-010-0305-4. MR2810799

[12] Gary L. Bunting and A. K. M. Masood-ul-Alam, *Nonexistence of multiple black holes in asymptotically Euclidean static vacuum space-time*, Gen. Relativity Gravitation **19** (1987), no. 2, 147–154, DOI 10.1007/BF00770326. MR876598

[13] Manfredo Perdigão do Carmo, *Riemannian geometry*, Mathematics: Theory & Applications, Birkhäuser Boston, Inc., Boston, MA, 1992. Translated from the second Portuguese edition by Francis Flaherty. MR1138207

[14] Piotr T. Chruściel and Erwann Delay, *On mapping properties of the general relativistic constraints operator in weighted function spaces, with applications*, Mém. Soc. Math. Fr. (N.S.) **94** (2003), vi+103, DOI 10.24033/msmf.407 (English, with English and French summaries). MR2031583

[15] Justin Corvino, *Scalar curvature deformation and a gluing construction for the Einstein constraint equations*, Comm. Math. Phys. **214** (2000), no. 1, 137–189, DOI 10.1007/PL00005533. MR1794269

[16] Justin Corvino, Michael Eichmair, and Pengzi Miao, *Deformation of scalar curvature and volume*, Math. Ann. **357** (2013), no. 2, 551–584, DOI 10.1007/s00208-013-0903-8. MR3096517

[17] Justin Corvino and Lan-Hsuan Huang, *Localized deformation for initial data sets with the dominant energy condition*, Calc. Var. Partial Differential Equations **59** (2020), no. 1, Paper No. 42, 43, DOI 10.1007/s00526-019-1679-9. MR4062040

[18] Justin Corvino and Richard M. Schoen, *On the asymptotics for the vacuum Einstein constraint equations*, J. Differential Geom. **73** (2006), no. 2, 185–217. MR2225517

[19] Tiarlos Cruz and Feliciano Vitório, *Prescribing the curvature of Riemannian manifolds with boundary*, Calc. Var. Partial Differential Equations **58** (2019), no. 4, Paper No. 124, 19, DOI 10.1007/s00526-019-1584-2. MR3977557

[20] Ennio De Giorgi, *Frontiere orientate di misura minima*, Editrice Tecnico Scientifica, Pisa, 1961 (Italian). Seminario di Matematica della Scuola Normale Superiore di Pisa, 1960-61. MR0179651

[21] ———, *Una estensione del teorema di Bernstein*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (3) **19** (1965), 79–85 (Italian). MR178385

[22] Michael Eichmair, *The size of isoperimetric surfaces in 3-manifolds and a rigidity result for the upper hemisphere*, Proc. Amer. Math. Soc. **137** (2009), no. 8, 2733–2740, DOI 10.1090/S0002-9939-09-09789-5. MR2497486

[23] Herbert Federer, *Geometric measure theory*, Die Grundlehren der mathematischen Wissenschaften, Band 153, Springer-Verlag New York, Inc., New York, 1969. MR0257325

[24] ———, *The singular sets of area minimizing rectifiable currents with codimension one and of area minimizing flat chains modulo two with arbitrary codimension*, Bull. Amer. Math. Soc. **76** (1970), 767–771, DOI 10.1090/S0002-9904-1970-12542-3. MR260981

[25] Herbert Federer and Wendell H. Fleming, *Normal and integral currents*, Ann. of Math. (2) **72** (1960), 458–520, DOI 10.2307/1970227. MR123260

[26] Wendell H. Fleming, *On the oriented Plateau problem*, Rend. Circ. Mat. Palermo (2) **11** (1962), 69–90, DOI 10.1007/BF02849427. MR157263

[27] Arthur E. Fischer and Jerrold E. Marsden, *Deformations of the scalar curvature*, Duke Math. J. **42** (1975), no. 3, 519–547. MR380907

[28] Gerd Grubb, *Distributions and operators*, Graduate Texts in Mathematics, vol. 252, Springer, New York, 2009. MR2453959

[29] Fengbo Hang and Xiaodong Wang, *Rigidity and non-rigidity results on the sphere*, Comm. Anal. Geom. **14** (2006), no. 1, 91–106. MR2230571

[30] ———, *Rigidity theorems for compact manifolds with boundary and positive Ricci curvature*, J. Geom. Anal. **19** (2009), no. 3, 628–642, DOI 10.1007/s12220-009-9074-y. MR2496569

[31] Robert Hardt and Leon Simon, *Area minimizing hypersurfaces with isolated singularities*, J. Reine Angew. Math. **362** (1985), 102–129, DOI 10.1515/crll.1985.362.102. MR809969

[32] S. W. Hawking and G. F. R. Ellis, *The large scale structure of space-time*, Cambridge Monographs on Mathematical Physics, No. 1, Cambridge University Press, London-New York, 1973. MR0424186

[33] Oussama Hijazi, Sebastián Montiel, and Simon Raulot, *A positive mass theorem for asymptotically hyperbolic manifolds with inner boundary*, Internat. J. Math. **26** (2015), no. 12, 1550101, 17, DOI 10.1142/S0129167X15501013. MR3432532

[34] Pak Tung Ho and Yen-Chang Huang, *Deformation of the scalar curvature and the mean curvature*, arXiv preprint arXiv:2008.11893 (2020).

[35] Lan-Hsuan Huang, Hyun Chul Jang, and Daniel Martin, *Mass rigidity for hyperbolic manifolds*, Comm. Math. Phys. **376** (2020), no. 3, 2329–2349, DOI 10.1007/s00220-019-03623-0. MR4104551

[36] Gerhard Huisken and Tom Ilmanen, *The inverse mean curvature flow and the Riemannian Penrose inequality*, J. Differential Geom. **59** (2001), no. 3, 353–437. MR1916951

[37] Peter Li and Shing Tung Yau, *A new conformal invariant and its applications to the Willmore conjecture and the first eigenvalue of compact surfaces*, Invent. Math. **69** (1982), no. 2, 269–291, DOI 10.1007/BF01399507. MR674407

[38] Donovan McFeron and Gábor Székelyhidi, *On the positive mass theorem for manifolds with corners*, Comm. Math. Phys. **313** (2012), no. 2, 425–443, DOI 10.1007/s00220-012-1498-8. MR2942956

[39] Pengzi Miao, *Positive mass theorem on manifolds admitting corners along a hypersurface*, Adv. Theor. Math. Phys. **6** (2002), no. 6, 1163–1182 (2003), DOI 10.4310/ATMP.2002.v6.n6.a4. MR1982695

[40] Pengzi Miao and Luen-Fai Tam, *Scalar curvature rigidity with a volume constraint*, Comm. Anal. Geom. **20** (2012), no. 1, 1–30, DOI 10.4310/CAG.2012.v20.n1.a1. MR2903099

[41] Maung Min-Oo, *Scalar curvature rigidity of certain symmetric spaces*, Geometry, topology, and dynamics (Montreal, PQ, 1995), CRM Proc. Lecture Notes, vol. 15, Amer. Math. Soc., Providence, RI, 1998, pp. 127–136. MR1619128

[42] R. Penrose, *Naked singularities*, Annals N. Y. Acad. Sci. **224** (1973), 125–134, DOI 10.1111/j.1749-6632.1973.tb41447.x.

[43] Richard Schoen and Shing Tung Yau, *On the proof of the positive mass conjecture in general relativity*, Comm. Math. Phys. **65** (1979), no. 1, 45–76. MR526976

[44] ———, *Positivity of the total mass of a general space-time*, Phys. Rev. Lett. **43** (1979), no. 20, 1457–1459, DOI 10.1103/PhysRevLett.43.1457. MR547753

[45] ———, *Proof of the positive mass theorem. II*, Comm. Math. Phys. **79** (1981), no. 2, 231–260. MR612249

[46] ———, *Positive Scalar Curvature and Minimal Hypersurface Singularities*, arXiv preprint arXiv:1704.05490 (2017).

[47] Yuguang Shi and Luen-Fai Tam, *Positive mass theorem and the boundary behaviors of compact manifolds with nonnegative scalar curvature*, J. Differential Geom. **62** (2002), no. 1, 79–125. MR1987378

[48] Barry Simon, *A Comprehensive Course in Analysis, Part 4: Operator Theory*, American Mathematical Society, Providence, RI, 2015.

[49] James Simons, *Minimal varieties in riemannian manifolds*, Ann. of Math. (2) **88** (1968), 62–105, DOI 10.2307/1970556. MR233295

[50] Leon Simon, *Lectures on geometric measure theory*, Proceedings of the Centre for Mathematical Analysis, Australian National University, vol. 3, Australian National University, Centre for Mathematical Analysis, Canberra, 1983. MR756417

[51] ———, *A strict maximum principle for area minimizing hypersurfaces*, J. Differential Geom. **26** (1987), no. 2, 327–335. MR906394

[52] ———, *Rectifiability of the singular sets of multiplicity* 1 *minimal surfaces and energy minimizing maps*, Surveys in differential geometry, Vol. II (Cambridge, MA, 1993), Int. Press, Cambridge, MA, 1995, pp. 246–305. MR1375258

[53] ———, *Schauder estimates by scaling*, Calc. Var. Partial Differential Equations **5** (1997), no. 5, 391–407, DOI 10.1007/s005260050072. MR1459795

[54] Nathan Smale, *Generic regularity of homologically area minimizing hypersurfaces in eight-dimensional manifolds*, Comm. Anal. Geom. **1** (1993), no. 2, 217–228, DOI 10.4310/CAG.1993.v1.n2.a2. MR1243523

[55] Robert M. Wald, *General relativity*, University of Chicago Press, Chicago, IL, 1984. MR757180

[56] Edward Witten, *A new proof of the positive energy theorem*, Comm. Math. Phys. **80** (1981), no. 3, 381–402. MR626707

[57] Xiao Zhang, *A definition of total energy-momenta and the positive mass theorem on asymptotically hyperbolic 3-manifolds. I*, Comm. Math. Phys. **249** (2004), no. 3, 529–548, DOI 10.1007/s00220-004-1056-0. MR2084006