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Generation and Optimization of High Quality Multi-GeV Electron Beams in Plasma
Wakefield Accelerators

A dissertation submitted in partial satisfaction
of the requirements for the degree
Doctor of Philosophy in Physics

by

Thamine N. Dalichaouch

2021

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ABSTRACT OF THE DISSERTATION

Generation and Optimization of High Quality Multi-GeV Electron Beams in Plasma Wakefield Accelerators

by

Thamine N. Dalichaouch

Doctor of Philosophy in Physics

University of California, Los Angeles, 2021

Professor Warren B. Mori, Chair

In this dissertation, a new method for producing ultra-bright electron beams in nonlinear plasma wave wakes driven by an electron beam driver is explored using particle-in-cell simulations and analytic theory. In order to understand this process an accurate description of a nonlinear wakefield is required. These nonlinear wakefields are excited by intense particle beams or lasers pushing plasma electrons radially outward, creating an ion bubble surrounded by a sheath of electrons characterized by the source term $S \equiv -\frac{1}{en_p}(\rho - J_z/c)$, where e is the electron charge, n_p is the plasma number density, ρ is the charge density, and J_z is the axial current density. Previously, the sheath source term was described phenomenologically with a positive-definite function thereby resulting in a positive definite wake potential. In reality, the wake potential is negative at the rear of the ion column, which is important for self-injection and accurate beam loading models.

To account for this, in the first part of this dissertation a multi-sheath model in which the source term, S , of the plasma wake can be negative in regions outside the ion bubble is introduced. Using this model, a new expression for the wake potential and a modified

differential equation for the bubble radius is obtained. Numerical results obtained from these equations are validated against particle-in-cell simulations for unloaded and loaded wakes. The new model provides accurate predictions of the shape and duration of trailing bunch current profiles that flatten plasma wakefields. It is also used to design a trailing bunch for a desired longitudinally varying loaded wakefield. The multi-sheath model is also applied to beam loading in laser wakefields. Areas where the multi-sheath model can be improved for laser drivers in future work are discussed.

In the second part of this dissertation, a new method of controllable injection to generate high quality electron bunches in the nonlinear blowout regime driven by electron beams is proposed and demonstrated using particle-in-cell simulations. Injection is facilitated by decreasing the wake phase velocity through focusing the drive beam spot size. Two regimes are examined. In the first, the spot size is focused according to the vacuum Courant-Snyder (CS) beta function while, in the second, it is self-focused by the plasma ion column. The effects of the driver intensity and vacuum CS parameters on the wake velocity and injected beam parameters are examined via theory and simulations. For plasma densities of $\sim 10^{19} \text{ cm}^{-3}$, particle-in-cell (PIC) simulations demonstrate that peak normalized brightnesses $\gtrsim 10^{20} \text{ A/m}^2/\text{rad}^2$ can be obtained with projected energy spreads of $\lesssim 1\%$ within the middle section of the injected beam and with normalized slice emittances as low as $\sim 10 \text{ nm}$.

In the last part of the dissertation, a predictive model for injection using the self-evolving driver method in the plasma focusing regime is developed. The model is used to characterize how the wake evolution and final injected beam parameters scale with the driver parameters. Parameter scans of PIC simulations using different drivers are performed and compared with the model predictions. In particular, the dependence of the injected beam parameters with the diffraction length, energy, intensity, spot size, and duration of the driver is examined. It is found that injection and optimal beam loading can be simultaneously achieved. The multi-sheath model is also used to study the beam loading effects from the injected bunch in this case. PIC simulation results indicate that the injected beam can be efficiently accelerated

to 18.27 GeV with a projected energy spread of 0.49% and peak normalized brightness of $B_n \sim 10^{20}$ A/m²/rad² for a plasma density of $\sim 10^{19}$ cm⁻³.

The dissertation of Thamine N. Dalichaouch is approved.

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University of California, Los Angeles

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CHAPTER 1

Introduction

1.1 Background

Particle collider experiments of high energy electron and positron beams have provided tremendous insight into the fundamental physics governing our universe. Conventional accelerators use radio-frequency (RF) technology to generate the kinds of TeV-class energies needed for these experiments. However, the acceleration gradients provided by the vacuum structures are limited to 100 MV/m due to material breakdown of the accelerator walls. Therefore, conventional accelerators based on RF-technology needed to study high energy physics are large and very costly to build. Two prominent examples are the Large Hadron Collider – LHC (27 km), a circular accelerator capable of TeV beam energies, and the Stanford Linear Accelerator – SLAC (3.2 km), a linear accelerator that was capable of generating up to 50 GeV beams. SLAC is now home to the LINAC Coherent Light Source – LCLS, which generates a 4 GeV electron beam to drive a free electron laser that produces 5 – 25 KeV coherent x-ray pulses. Not only are these machines large and expensive, but they also require a significant amount of maintenance to keep them operational.

Plasma-based accelerators have been proposed as alternatives to conventional accelerators because they are capable of providing much higher accelerating gradients. It has long been known that plasmas can sustain electric fields on the order of nonrelativistic wavebreaking field $E \simeq mc\omega_p/e \simeq 0.96\sqrt{n_p[cm^{-3}]} \text{ V/m}$ [1, 2], where n_p is the plasma electron density and $\omega_p = \sqrt{4\pi n_p e^2/m_e}$ is the plasma frequency. For a plasma density of $n_p = 10^{18} \text{ cm}^{-3}$,

accelerating fields of ~ 100 GV/m can be achieved, which is several orders of magnitude higher than the breakdown limit of RF-accelerators. Therefore, the size and cost of future linear collider facilities could be dramatically reduced if plasma-based accelerator technology can be successfully developed.

There are two main approaches to plasma-based acceleration (PBA). In 1979, Tajima and Dawson proposed using a short-pulse laser to excite a plasma wave wakefield traveling at near the speed of light [3]. This idea is commonly referred to as Laser Wakefield Acceleration (LWFA). In LWFA, the laser ponderomotive force pushes the plasma electrons outward, leaving behind background ions. Due to the space-charge separation, the electrons are then attracted back to the axis by the ions, thereby forming the plasma wake. Using 1D computer simulations, Tajima and Dawson demonstrated that electrons could be trapped and accelerated by the wake due to the ultra-high electric fields of 100 GV/m produced for a plasma density of 10^{18} cm $^{-3}$. Their work outlined many of the basic concepts of plasma-based acceleration. Since short pulse laser systems were not available at the time, numerous variations of the LWFA concept were proposed and investigated in the following years, such as the Plasma Beat Wave Accelerator (PBWA) [4, 5, 6] and the Self-Modulation Laser Wakefield Accelerator (SMLWFA) [7, 8, 9]. In PBWA, two lasers with different frequencies are used to generate a beat-wave of short pulses at the plasma frequency. In SMLWFA, a single pulse extending over many plasma wavelengths self-modulates as it propagates in the plasma due to Raman forward scattering [7, 10]. Using these schemes, many important experimental results were obtained in the years following the work of Tajima and Dawson [3]. In 1985, it was confirmed experimentally that relativistic plasma wakefields with high electric fields of up to 1 GV/m could be produced using PBWA [11]. Electron trapping and acceleration in plasma wakefields were also observed in experiments using PBWA in 1993 [12] and 1994 [13] and SMLWFA in 1995 [9, 10]. With the advent of short pulse tabletop laser systems, many subsequent LWFA experiments demonstrated that GeV-class beam energies could be produced over centimeter length scales [14, 15, 16, 17, 18, 19, 20, 21, 22, 23].

In lieu of a laser pulse, a relativistic electron beam can also be used to excite a plasma wave wakefield. This concept was proposed by Chen et al. [24] in 1985 and is commonly referred to as Plasma wakefield Acceleration (PWFA). The main difference between PWFA and LWFA is that the plasma electrons are initially displaced by the space charge force of the particle beam instead of the ponderomotive force of the laser. Nevertheless, a similar wake structure is produced as electrons are attracted back to the axis by the background ions. Naturally, beam-driven wakefields are also capable of producing ultra-high accelerating fields similar to those seen in LWFA. In initial experiments performed at the Stanford Linear Accelerator Laboratory (SLAC) using a single relativistic positron bunch, accelerating gradients of 56 MeV/m were demonstrated by Blue et al. [25] in 2003. In subsequent experiments using a single relativistic electron drive bunch, accelerating gradients of 280 MeV/m were demonstrated by Muggli et al. [26] in 2004, 27 GeV/m by Hogan et al. [27] in 2005, and 52 GeV/m by Blumenfeld et al. [28] in 2007. Notably, the results of Blumenfeld et al. [28] showed that the 50 GeV/m gradients could be sustained for over 85 cm, resulting in the energy doubling of some electrons from 42 GeV to 85 GeV. In 2014, efficient high gradient acceleration (4.4 GeV/m) of a trailing bunch was demonstrated by Litos et al. [29] using a drive bunch to excite the wake and a separate trailing bunch to extract energy from it. In 2015, it was shown that the tail of a positron bunch could form a mono-energetic beam in a > 3 GeV/m wakefield excited by the main body of the bunch [30].

Theory and simulation have played a critical role in our understanding of the structure and fields of plasma wakefields. In the linear regime, a wide driver excites a small amplitude plasma wakefield. For this regime, the necessary linear fluid theory has existed for over thirty years [24, 31]. However, a second regime also exists where the plasma response is nonlinear and electrons can be completely cavitated in regions near the driver. Cavitation from a long laser pulse was first pointed out by Sun et al. [32] in 1987 when they showed theoretically that for an intense and narrow laser with power above the critical self-focusing limit, P_c , the plasma electrons could be completely blown out by the ponderomotive force

of the laser. Using 2D cylindrical fluid and particle-in-cell (PIC) simulations, Rosenzweig et al. [33] demonstrated in 1991 that a wakefield based on complete electron cavitation could also be produced with a narrow relativistic electron driver. Due to the blowout, a bubble-like cavity is produced in which only ions remain. The electrons which are initially blown out are attracted back to the axis by the ions, thereby creating a thin plasma sheath around the bubble and a wakefield. For obvious reasons, this regime is now often referred to as the “bubble” or “blowout” regime. In the work of Rosenzweig et al. [33], it was also pointed out that, within the bubble region, the focusing force varies linearly with the radius, while the accelerating field does not vary transversely. These features, which are not present in the linear regime, are ideal for preserving emittance and maintaining low energy spreads during acceleration. In 2002, Pukhov and Meyer-ter-vehn demonstrated through full-3D PIC simulations of LWFA that a similar bubble-like cavity could be produced using a highly intense, short-pulse laser [34]. In the laser case, the phase velocity of the wake is typically smaller than for the particle beam cases. Thus, in their simulation results, they found that some of the plasma electrons that were blown out were eventually trapped at the rear of the bubble and accelerated by the wakefield. They found that this process occurred continuously throughout the laser propagation, generating quasi-monoenergetic beams as the electrons rotated in phase space.

The work for Rosenzweig et al. [33] and Pukhov and Meyer-ter-vehn [34] relied on simulations. At the time, theoretical analysis of nonlinear wakes was limited to 1D nonlinear fluid theory [1, 2]. In 2006, Lu et al. [35, 36] introduced a nonlinear theory for 3D plasma wave wakefields in the blowout regime. Lu et al. showed that the shape of the bubble and longitudinal electric field could be completely described by the driver and plasma wake potential. The theory showed that the 3D wakefields were electromagnetic in nature and was applicable for moderate to large amplitude wakes. Using a single-sheath model for the plasma, the authors were also able to predict the bubble radius and electric field along most of the wake. They found that their results were largely insensitive to the phenomenological profiles of

their single-sheath model. However, they could not reproduce the electric field divergence at the rear of the bubble as their model did not account for negative wake potentials. In 2007, Lu et al. [37] provided a phenomenological theory and scaling laws for LWFA in a nonlinear self-guided regime. The scaling laws were compared against 3D particle-in-cell simulations.

Using 3D nonlinear theory in the blowout regime, Tzoufras et al. [38, 39] then showed that beam loading could be viewed as a modification to the shape of the bubble. Assuming a large blowout radius and ignoring the plasma electron source terms, they developed analytical expressions for the shape and fields of the bubble. They found that a beam load with a trapezoidal current profile could flatten the wakefield. In addition, they also examined beam loading from gaussian bunches. However, while their analytical results could reasonably flatten the wake, they often underestimated the amount of charge that could be loaded [38, 39]. Accurate modeling of beam loading by injected electrons was also not possible using their approach. Nevertheless, their work was an important milestone as it identified the theoretical approach to analyzing and offered the first predictive tool for beam loading in the nonlinear blowout regime.

In addition to the linear collider applications, PBA can also be used to generate beams for x-ray free-electron-lasers (XFELs). While the generation of ultra-relativistic electron beams through self-injection has been demonstrated in LWFA experiments [18, 19, 20, 21], the beams produced to date do not exhibit the sufficiently low energy spreads σ_γ and high normalized brightnesses $B_n = 2I/\epsilon_n^2$ required to drive XFEL devices [40]. Therefore, numerous controllable injection schemes have been proposed and investigated to generate high quality relativistic beams. We note that very recently LWFA was used to generate beams to drive an FEL at 27 nm [41].

The most promising injection schemes for generating high quality beams largely fall into two categories. In the first method, an intense laser pulse or particle beam ionizes electrons inside the plasma bubble close to where the wake potential is maximum to reduce the trapping threshold [42, 43, 44]. By tuning the ionizing pulse and selecting an appropriate high-Z

gas, injection can be confined near the radial axis where the ionizing field is largest, which can produce trapped electrons with low transverse momenta and low normalized emittances [45]. Numerous variations of this concept using multiple pulses to separate wake excitation and injection have been proposed [45, 46, 47, 48]. In the second method, sheath electrons from the background plasma are injected at the rear of the wake by reducing the wake phase velocity γ_ϕ using a plasma density down ramp (DDR) [49, 50, 51, 52, 53, 54, 55, 56] or an evolving driver [57, 58, 59].

1.2 Particle-in-Cell Simulation

PIC simulations [60, 61] can provide tremendous insight into the physics of plasma wave wakefields. Indeed, many important experiments and theoretical models are often supported by PIC simulations. For problems such as LWFA and PWFA, which are fully electromagnetic in nature, the physics is described by Maxwell's equations. In a PIC simulation, the electromagnetic fields are defined on a grid and macroparticles (which can represent many individual charged particles) can move within the grid. The basic iteration loop for each time step of the simulation is as follows. First, the macroparticles are pushed by the force equation, where the forces are evaluated by interpolating the electromagnetic fields from the grid to the locations of the particles. The updated charge and current densities of each macroparticle species are then interpolated onto the grid. Finally, the electromagnetic fields are updated on the grid using a discretized version of Maxwell's equations and the new current density (and in some cases charge density) sources. The grid resolution must be high enough to resolve the minimum length scales of interest or important physics will be lost. In PWFA, the plasma skin depth usually needs to be resolved in both directions while, in LWFA, the laser wavelength needs to be resolved longitudinally. However, for processes such as injection, even higher fidelity simulations may need to be performed to accurately model electron trapping and resolve the injected bunch, which can be quite small compared

to the wake length. Therefore, while technological advances in computing over the past few decades have made high-fidelity full-3D simulations feasible, the latter are still computationally expensive.

Several reduced models have been developed to efficiently simulate PBA. In situations where the plasma wake does not evolve over the time it takes for the driver to pass by a plasma particle, the quasi-static method [62, 63, 64] can be used to reduce the three-dimensional electromagnetic field equations to two-dimensional Poisson equations. Due to the assumptions of the quasi-static approximation, the full electromagnetic fields of the wake can be solved for using very large time steps. As a result, this method can greatly reduce the computational time needed to simulate PBA by several orders of magnitude without loss of accuracy for highly nonlinear problems [63, 64]. However, the quasi-static method cannot model self-trapping.

A quasi-3D algorithm [65, 66, 67] can be used to model three-dimensional LWFA problems with computational costs similar to a 2D cylindrical ($r - z$) code. In this approach, the fields and currents are expanded into azimuthal Fourier harmonics (modes) in ϕ with respect to the laser propagation axis. Substituting this expansion into Maxwell's equations, a set of equations governing the complex coefficients of the fields for each m mode can be obtained. The fields are therefore represented using an array of 2D $r - z$ grids (one for each real and imaginary component of a given m mode), while the plasma particles are allowed to move freely in three-dimensions. The quasi-3D algorithm can therefore be viewed as a hybrid approach which uses PIC grids in $r - z$ and a gridless approach in ϕ [68]. One of the main advantages of the quasi-3D algorithm is that the plasma wakefield can be fully described by the $m = 0$ mode while a circularly or linearly polarized laser with a round spot size can be completely described by the $m = 1$ mode. Thus, many LWFA problems can be accurately simulated using the quasi-3D algorithm with only the first two modes $m \leq 1$, which can dramatically reduce the computational time when compared to full 3D cartesian simulations. In addition, the natural separation of the wakefield ($m = 0$) and laser field

($m = 1$) in the quasi-3D algorithm can be useful in studying effects such as direct laser acceleration (DLA) [69, 70]. While the quasi-3D algorithm was originally envisioned for laser drivers (LWFA), it can also be used to simulate plasma wakefields produced by particle beams (PWFA) with small transverse asymmetries. The quasi-3D approach has recently been applied to the quasi-static approximation approach [71].

Another method that can speed up LWFA simulations is the Lorentz boosted frame technique [72, 73]. This approach relies on performing an LWFA simulation in a Lorentz boosted frame so that the plasma length is Lorentz contracted while the laser wavelength is Lorentz expanded. If the boosted frame velocity v_b is chosen appropriately, the length of the laser pulse and plasma column in the boosted frame can be comparable. Boosted frame simulations can therefore use much larger spatial and temporal resolutions than the equivalent lab frame simulations, providing computational savings on the order γ_b^2 . It works well if there is no reflected laser light, i.e., if the phase velocity of all waves are sufficiently close to v_b . However, one of the main challenges to performing Lorentz boosted frame simulations is the Numerical Cerenkov Instability (NCI) [74, 75, 76, 77] that occurs due to the unphysical coupling between the Langmuir modes and electromagnetic modes of the relativistically drifting plasma from the numerical dispersion relation of the simulation. In recent years, numerous techniques have been developed to mitigate this instability using a pseudo-spectral analytical time-domain (PSATD) PIC algorithm [78], a modified electromagnetic dispersion relation with an artificial bump [79, 80, 81], a customized finite difference solver [82], or Galilean coordinates [83].

An important framework widely used in the PBA community is the massively parallel, full relativistic PIC code OSIRIS [84, 85]. The code supports 1D, 2D, and 3D cartesian (x, y, z) geometries, as well as 2D cylindrical geometry for symmetrical PWFA problems. The quasi-3D (r, ϕ, z) algorithm using an azimuthal mode expansion of Maxwell's equations is also implemented into OSIRIS with a charge conserving current deposition [67]. In addition, OSIRIS features a customized finite-difference solver that can reduce numerical effects from

relativistic particles [82, 86]. Without these solvers, many PWFA and LWFA simulations would not have been possible. Also implemented into OSIRIS is the boosted frame algorithm with a bump solver to mitigate NCI instabilities for both 3D and quasi-3D geometries [79, 80, 81]. The code also features a recently developed dynamic load balancing algorithm [87] that can be used to significantly speed up PIC simulations of LWFA and PWFA problems by repartitioning the simulation box to evenly distribute particles.

1.3 Dissertation Overview

As outlined in this chapter, the important goals of PBA are high-gradient, high efficiency, and high quality beam generation. In many problems within these research areas, the interesting physics usually occurs in regions near the rear of the wake that are not well-described by the simple sheath model of Lu et al. [35, 36] and the analytical beam loading theory of Tzoufras et al. [38, 39] which uses the simple sheath model. To shed light on the important physics in these regions, we present in this dissertation an improved description of highly nonlinear plasma wakefields in the blowout regime. We also explore new ideas for generating and optimizing high quality beams using self-injection.

The remainder of the dissertation is organized as follows. In chapter 2, we start by reviewing the basic equations governing highly nonlinear plasma wave wakefields in the blowout regime. In chapter 3, we present an improved multi-sheath model for the plasma wake potential in highly nonlinear plasma wakefields excited by intense particle beams or lasers. The model accounts for the negative wake potentials at the rear of the wake through the plasma source term and is compared with fully kinetic PIC simulations. Beam loading scenarios are also considered. In chapter 4, a new method of controllable injection is proposed based on an evolving wake caused by a decreasing spot size of a relativistic electron beam driver. This new method is illustrated and demonstrated using PIC simulations. Two regimes are examined, in one the evolution of the spot size is dominated by its vacuum

Courant-Snyder [88] parameters and in the second it is controlled through self-focusing from the plasma wave wakefield itself. We refer to the second as the plasma self-focusing regime. The multi-sheath model from chapter 3 is also used to characterize beam loading effects due to injection. In chapter 5, a predictive framework is developed for injection using the evolving driver method in the plasma focusing regime. We investigate how the final injected beam parameters vary with the driver parameters and compare the results against parameter scans from PIC simulations. High quality beam generation and optimal beam loading are also demonstrated using PIC simulations and the multi-sheath model. Finally, in chapter 6 we provide a summary as well as directions for future work.

CHAPTER 2

A Review of Nonlinear Theory for Plasma Wakefields in the Blowout Regime

This dissertation is focused on developing a new method for producing very high normalized brightness electron beams through synchronized self-injection in nonlinear plasma wakefields. In order to understand and analyze self-injection and acceleration it is important to have an accurate theory for the field structure in nonlinear wakefields including the ability to understand how these fields are loaded by high current electron beams. To set the stage for the rest of the dissertation, in this chapter, we will review the nonlinear theory for plasma wakefields in the blowout regime developed by Lu et al. [35, 36]. These plasma wave wakefields can be excited by short pulse lasers or relativistic particle beams with phase velocities near the speed of light. The purpose of this chapter is to introduce the basic equations needed to describe the nonlinearities and complexities of plasma wakefields in the blowout regime. The derivations presented in this chapter also provide important context and motivation for the work presented in the following chapters.

2.1 Basic Equations

To describe wakefield problems in multi-dimensions, we start with Maxwell's equations in the Lorentz gauge and the equations of motion for an arbitrary number of laminar fluids using cgs units,

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right) \begin{bmatrix} \mathbf{A} \\ \phi \end{bmatrix} = 4\pi \begin{bmatrix} \mathbf{J}/c \\ \rho \end{bmatrix}, \quad (2.1)$$

$$\frac{1}{c}\frac{\partial\phi}{\partial t} + \nabla \cdot \mathbf{A} = 0, \quad (2.2)$$

$$\left(\frac{\partial}{\partial t} + \mathbf{V}_i \cdot \nabla\right) \mathbf{P}_i = q_i \left[\mathbf{E} + \frac{\mathbf{V}_i}{c} \times \mathbf{B}\right], \quad (2.3)$$

where $\rho = \sum_i q_i n_i$ and $\mathbf{J} = \sum_i q_i n_i \mathbf{V}_i$. The current and charge densities for each fluid satisfy the charge conservation equation given by

$$\frac{\partial\rho}{\partial t} + \nabla \cdot \mathbf{J} = 0. \quad (2.4)$$

Beyond this point, approximations are needed. Since the plasma wakefield moves with a phase velocity near the speed of light, it is assumed that the fields of the wake can be formulated in the moving frame described by the coordinate $\xi \equiv ct - z$. This can be represented mathematically by a coordinate transformation from (x, y, z, t) to $(x, y, \xi \equiv ct - z, s \equiv z)$. Under this transformation, we note that $\frac{\partial}{\partial z} = \frac{\partial}{\partial s} - \frac{\partial}{\partial \xi}$ and $\frac{\partial}{\partial t} = c\frac{\partial}{\partial \xi}$. In the first approximation, it is assumed that the driver does not evolve over the length scale of the plasma wake, i.e., the plasma skin depth c/ω_p . Thus, the wake response in the moving frame, which is described by the coordinate ξ , can be considered static within the time it takes for the driver to pass by a plasma electron. This is referred to as the quasi-static approximation [89] or frozen field approximation [90] and is represented mathematically by $\frac{\partial}{\partial s} \ll \frac{\partial}{\partial \xi}$. This implicitly precludes situations where there are trapped electrons.

This approximation naturally requires different conditions for laser and particle beams. For a relativistic electron beam, the driver evolves on the length scale of the betatron wavelength $2\pi\sqrt{2\gamma_b}c/\omega_p$. Therefore, the quasi-static approximation is valid if $\gamma_b \gg 1$. For a laser pulse, the envelope evolves on the length scale of the Rayleigh length $Z_R \equiv \pi W_0^2/\lambda_0$, where W_0 is the laser spot size and λ_0 is the laser wavelength. For a W_0 on the order of

the plasma skin depth, the quasi-static approximation is valid if $\omega_0/\omega_p \gg 1$. Recall that the group velocity of light in a plasma is $v_g = c\sqrt{1 - \omega_p^2/\omega_0^2}$ so $\gamma_g = \omega_0/\omega_p$. Thus, $\gamma_b \gg 1$ or $\gamma_g \gg 1$ is required for the quasistatic approximation. When the laser frequency is much higher than the plasma frequency, we can make what is called the ponderomotive guiding center (PGC) approximation. In the PGC approximation, the smooth motion of a cold plasma fluid element can be described by time averaging over a laser oscillation

$$\frac{d\mathbf{P}_{i,s}}{dt} = \left(\frac{\partial}{\partial t} + \mathbf{V}_{i,s} \cdot \nabla \right) \mathbf{P}_{i,s} = q_i \left[\mathbf{E}_s + \frac{\mathbf{V}_{i,s}}{c} \times \mathbf{B}_s \right] + \mathbf{F}_p, \quad (2.5)$$

where the subscript “s” denotes the smooth quantities and $\mathbf{F}_p$ is the ponderomotive force of the laser,

$$\mathbf{F}_p = -\frac{q_i}{\bar{\gamma}_i m_c c^2} \nabla \left| \frac{\hat{A}_L}{2} \right|^2, \quad (2.6)$$

where the laser field is given by $A_{laser} = \hat{A}_L/2e^{-i\omega_0\xi/c} + \text{c.c.}$, $a \equiv e\hat{A}_L/mc^2$, and $\bar{\gamma}_i = \sqrt{1 + P_i^2/(mc)^2 + |a|^2/2}$ [62]. The evolution of $\hat{A}_L$ can be obtained in terms of the wakefield [62]. We note here that for a relativistic particle beam the smooth fields $\mathbf{E}_s$ and $\mathbf{B}_s$ describe both the driver and the wake response.

Henceforth, for convenience we will employ normalized units, where charge is normalized to electron charge e , mass to electron mass m , velocity to c , charge density to en_p , current density to $en_p c$, length to c/ω_p , time to ω_p^{-1} , electric fields to $m c \omega_p / e$, and potentials to $m c^2 / e$. We will also drop the subscripts “i” and “s”.

Using the new coordinates $(x, y, \xi = ct - z, s = z)$ and the quasi-static approximation $\frac{\partial}{\partial s} \ll \frac{\partial}{\partial \xi}$, Maxwell’s equations can be written as

$$-\nabla_{\perp}^2 \begin{bmatrix} \mathbf{A} \\ \phi \end{bmatrix} = \begin{bmatrix} \mathbf{J}/c \\ \rho \end{bmatrix}, \quad (2.7)$$

$$\nabla_{\perp} \cdot \mathbf{A}_{\perp} = -\frac{\partial \psi}{\partial \xi}, \quad (2.8)$$

where $\psi \equiv \phi - A_z$, $\nabla_{\perp} \equiv \hat{x} \frac{\partial}{\partial x} + \hat{y} \frac{\partial}{\partial y}$, and $\mathbf{A}_{\perp} \equiv \hat{x} A_x + \hat{y} A_y$. Thus, in the quasi-static approximation, the fields are described by a set of 2D-like Poisson equations. We can therefore use 2D electrostatics to solve them. It is worth noting that ψ is often referred to as the “wake potential” or “pseudo potential”. Noting $d\xi = dt - dz = (1 - v_z)dt$, the equation of motion of a plasma electron can also be expressed as

$$\frac{d}{d\xi} P_{\perp} = \frac{1}{1 - v_z} \left[-[\mathbf{E}_{\perp} + (\mathbf{V} \times \mathbf{B})_{\perp}] - \frac{1}{\bar{\gamma}} \nabla_{\perp} \frac{|a|^2}{4} \right], \quad (2.9)$$

where $\bar{\gamma} = \sqrt{1 + P^2 + |a|^2/2}$ and we are assuming that s can be viewed as constant. The perpendicular and longitudinal fields are described by

$$E_z = -\frac{\partial \phi}{\partial z} - \frac{\partial A_z}{\partial t} = \frac{\partial \psi}{\partial \xi}, \quad (2.10)$$

$$B_z = (\nabla_{\perp} \times \mathbf{A}_{\perp}) \cdot \hat{z}, \quad (2.11)$$

$$\mathbf{E}_{\perp} = -\nabla_{\perp} \phi - \frac{\partial \mathbf{A}_{\perp}}{\partial \xi}, \quad (2.12)$$

$$\mathbf{B}_{\perp} = (\nabla_{\perp} \times \mathbf{A}_z) + (\nabla_z \times \mathbf{A}_{\perp}). \quad (2.13)$$

It is worth noting that the electric field E_z can be written purely in terms of wake potential $E_z = \frac{\partial \psi}{\partial \xi}$. However, the other fields depend explicitly on the scalar potential ϕ and vector potential A_z which are described by source terms ρ and J_z in Maxwell’s equations [Eq. (2.7)]. In the following section, we will show how the problem can be further simplified by considering the profiles of these source terms in the blowout regime.

2.2 The Blowout Regime

In the nonlinear blowout regime, the plasma electrons are completely expelled within some region by the ponderomotive force of the laser or space-charge force of the electron beam leaving behind a column of ions. Due to the space-charge separation, the electrons, which are initially blown out, are then attracted back to the axis forming a thin plasma electron sheath covering a nearly spherical ion bubble. In Fig. 2.1, we plot the electron charge density and on-axis electric field from a PWFA simulation in the blowout regime using the PIC code OSIRIS [84]. The channel radius $r_b(\xi)$ is delineated by the trajectory of the innermost sheath electron. In Refs. 35 and 36, it was shown that for short bi-Gaussian electron drivers ($k_p\sigma_z \sim 1$), complete cavitation can occur as long as the spot size σ_r much less than the maximum radius of the channel $r_m \equiv \max(r_b)$. This condition can also be represented by the requirement that the peak beam density is much greater than the plasma density $n_b \gg n_p$. In such cases, there are no electrons remaining inside the bubble $r < r_b(\xi)$ and the much more massive ions can be considered stationary. Therefore, in the ion channel, the charge and current densities from the plasma electrons are described by $\rho_e = J_{ze} = 0$ while the plasma ions are given by $\rho_{ion} = 1$ and $J_{z,ion} = 0$. For a laser driver, we note that the conditions for complete blowout and the features of the wake are more complicated and will be addressed later in chapter 3. Therefore, we will assume here that a plasma wakefield is excited in the blowout regime using a bi-Gaussian electron driver with a density profile of

$$n_b(r, \xi) = \frac{N}{(2\pi)^{3/2}\sigma_r^2\sigma_z} e^{-r^2/(2\sigma_r^2)} e^{-\xi^2/(2\sigma_z^2)}. \quad (2.14)$$

In the channel, i.e., $r \leq r_b$, the functional form of the potentials can be determined by integrating the quasistatic form of Maxwell's equations [Eq. 2.7]. Assuming azimuthal symmetry, this leads to

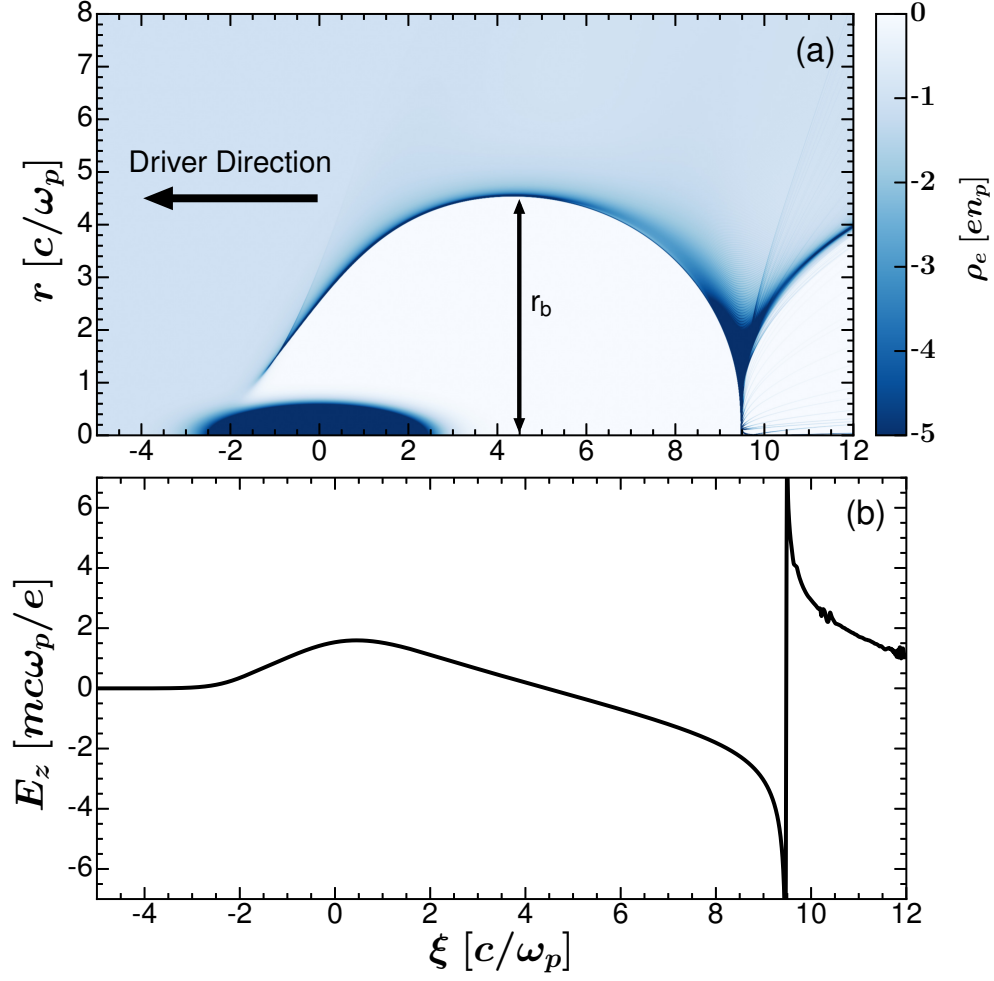


Figure 2.1: (a) Electron density distribution ρ_e and (b) on-axis electric field $E_z(\xi)$ of a plasma wake excited in the nonlinear blowout regime. The electron drive beam parameters are $\Lambda_d = 6$, $\gamma_b = 20000$, $k_p\sigma_z = 1$, $k_p\sigma_r = 0.245$, and $k_p\xi_c = 0$.

$$\phi(r, \xi) = \phi_0(\xi) - \frac{r^2}{4} + \lambda(\xi) \ln r, \quad (2.15)$$

$$A_z(r, \xi) = A_{z0}(\xi) + \lambda(\xi) \ln r, \quad (2.16)$$

where the $\lambda(\xi) = \int_0^{r \gg \sigma_r} n_b r dr$ denotes the charge per unit length of the driver. The factors $\phi_0(\xi) \equiv \phi(r=0, \xi)$ and $A_{z0}(\xi) \equiv A_z(r=0, \xi)$ are the on-axis potentials and depend on the structure of the source terms ρ and J_z . The wake potential ψ and longitudinal electric field E_z can therefore be written as [35, 36]

$$\psi(r, \xi) = \psi_0(\xi) - \frac{r^2}{4}, \quad (2.17)$$

$$E_z(\xi) = \frac{\partial \psi}{\partial \xi} = \frac{d\psi_0}{d\xi}, \quad (2.18)$$

where $\psi_0(\xi) = \phi_0(\xi) - A_{z0}(\xi)$. From the Lorentz gauge Eq. (2.8), it can also be shown that

$$A_r(r, \xi) = -\frac{1}{2} \frac{d\psi_0(\xi)}{d\xi} r. \quad (2.19)$$

Using Eqs. (2.17) and (2.19), the transverse force equation of a plasma or beam electron can now be expressed as

$$\begin{aligned} F_\perp &= -(E_r - v_z B_\phi) \\ &= \frac{\partial \phi}{\partial r} - v_z \frac{\partial A_z}{\partial r} + (1 - v_z) \frac{\partial A_r}{\partial \xi} \\ &= \frac{\partial \psi}{\partial r} + (1 - v_z) \frac{\partial A_z}{\partial r} + (1 - v_z) \frac{\partial A_r}{\partial \xi} \end{aligned} \quad (2.20)$$

$$= -\frac{1}{2} r + (1 - v_z) \frac{\lambda(\xi)}{r} + (1 - v_z) \left[-\frac{1}{2} \frac{d^2 \psi_0}{d\xi^2} r \right]. \quad (2.21)$$

The first term is the focusing force of the ions, the second term is the space-charge force of the particle beam driver, and the last term is the force from the radial plasma currents. For

a beam electron $v_z \rightarrow 1$, it is straightforward to show that the transverse force is perfectly linear $F_\perp = -\frac{1}{2}r$. Upon inspection of Eq. (2.20), the transverse force on a beam electron can also be written purely in terms of the wake potential $F_\perp = \frac{\partial\psi}{\partial r}$. In addition, recall the longitudinal force is given by $F_z = -E_z = -\frac{\partial\psi}{\partial\xi}$. Therefore, all of the forces on a beam electron are completely described by gradients of the wake potential ψ . Using Eq. (2.20), it can be shown that for a beam particle, $F_z = -\partial_\xi\psi$ and $\mathbf{F}_\perp = \nabla_\perp\psi$ from which it follows that

$$\nabla_\perp F_z = -\frac{\partial}{\partial\xi}\mathbf{F}_\perp, \quad (2.22)$$

which is called the Panofsky Wenzel Theorem for the plasma wake potential [91]. For a beam electron $v_z \rightarrow 1$, this theorem essentially dictates that the the linear focusing force $F_\perp$ is constant along ξ while the longitudinal accelerating force F_z is constant along r . These properties are ideal for preserving the emittance and energy spread of an electron beam accelerated by the wakefield.

We are interested in solving the transverse equation of motion [Eq. (2.21)] for the innermost electron trajectory $r_b(\xi)$. To do this, we will use the constant of motion for a plasma electron $\gamma - P_z = 1 + \psi$ [62] where $\gamma = \sqrt{1 + P_\perp^2 + P_z^2}$. Recall from Eq. (2.9) that the force equation can be formulated as

$$\frac{dP_\perp}{d\xi} = \frac{1}{1 - v_z}F_\perp. \quad (2.23)$$

Using $\gamma - P_z = 1 + \psi$, the left-hand side can also be written as

$$\frac{dP_\perp}{d\xi} = \frac{d\gamma v_\perp}{d\xi} = \frac{d}{d\xi}\gamma(1 - v_z)\frac{dr}{d\xi} = \frac{d}{d\xi}(1 + \psi)\frac{dr}{d\xi}. \quad (2.24)$$

By direct observation of Eq. (2.24), we note that the transverse momentum can be written as $P_\perp = (1 + \psi)\frac{dr}{d\xi}$. Using Eqs. (2.21), (2.23), and (2.24), it can be shown that

$$\frac{d}{d\xi} \left[(1 + \psi) \frac{dr}{d\xi} \right] = \frac{1}{1 - v_z} \left(-\frac{1}{2} r \right) + \frac{\lambda(\xi)}{r} + \left(-\frac{1}{2} \frac{d^2 \psi_0}{d\xi^2} r \right). \quad (2.25)$$

The constant of motion $\gamma - P_z = 1 + \psi$ can be used to solve algebraically for $1 - v_z$ in terms of the wake potential ψ and transverse momentum $P_\perp$. It is straightforward to show that

$$\begin{aligned} \gamma^2 &= (1 + \psi + P_z)^2, \\ 1 + P_\perp^2 + P_z^2 &= (1 + \psi)^2 + 2P_z(1 + \psi) + P_z^2, \\ P_z &= \frac{1 + P_\perp^2 - (1 + \psi)^2}{2(1 + \psi)}, \end{aligned} \quad (2.26)$$

from which it follows that

$$\gamma = 1 + \psi + P_z = \frac{1 + P_\perp^2 + (1 + \psi)^2}{2(1 + \psi)}, \quad (2.27)$$

$$1 - v_z = \frac{1 + \psi}{\gamma} = \frac{2(1 + \psi)^2}{1 + P_\perp^2 + (1 + \psi)^2}. \quad (2.28)$$

Using Eqs. (2.25) and (2.28), the equation of motion of the innermost sheath electron that traces the bubble trajectory $r_b(\xi)$ can finally be written as

$$\begin{aligned} \frac{d}{d\xi} \left[(1 + \psi) \frac{d}{d\xi} r_b \right] &= r_b \left\{ -\frac{1}{4} \left[1 + \frac{1}{(1 + \psi)^2} + \left(\frac{dr_b}{d\xi} \right)^2 \right] \right. \\ &\quad \left. - \frac{1}{2} \frac{d^2 \psi_0}{d\xi^2} + \frac{\lambda(\xi)}{r_b^2} \right\} \end{aligned} \quad (2.29)$$

Eq. (2.29) is important because it shows that the shape of the bubble $r_b(\xi)$ can be completely described by the wake potential ψ and the driver profile $\lambda(\xi)$. This result first appeared in the nonlinear theory presented by Lu et al. [35, 36]. At the time, the authors also pointed out that the wakefield E_z (which is a function of the wake potential) can be viewed in the context of the shape of the bubble $r_b(\xi)$. Their work eventually led to the development of a nonlinear beam loading theory in the blowout regime by Tzoufras et al. [38, 39].

2.3 The Sheath Model

To use their theory as a predictive tool, Lu et al. [35, 36] developed a “sheath” model for the source term of the plasma wakefield in the nonlinear blowout regime. From Maxwell’s equations [Eq. (2.15)], it follows that the wake potential obeys a Poisson-like equation

$$\nabla_{\perp}^2 \psi = S \equiv -(\rho - J_z) \quad (2.30)$$

where the source term is $S \equiv -(\rho - J_z)$. The advantage of formulating the problem in terms of ψ is now apparent since only a single effective source term S is needed to describe the plasma wake. This source term S is constrained by charge conservation [Eq. (2.4)], which dictates that the integral of the source term from $r = \infty$ to $r = 0$ vanishes

$$\begin{aligned} \frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} &= 0 \\ \frac{\partial \rho}{\partial \xi} - \frac{\partial J_z}{\partial \xi} &= -\nabla_{\perp} \cdot \mathbf{J}_{\perp} \\ \frac{d}{d\xi} \int (\rho - J_z) r dr &= 0 \\ \int (\rho - J_z) r dr &= 0. \end{aligned} \quad (2.31)$$

To solve for ψ , Lu et al. modeled the source term as two regions: $S = -1$ inside the ion bubble ($r < r_b$) and $S = n_{\Delta} > 0$ inside a finite width plasma sheath outside the bubble ($r_b < r < r_b + \Delta$). We stress that the formalism of Lu et al. permits the use of more complicated source term profiles. These regions were represented mathematically by step functions in each region such that

$$S = \begin{cases} -1, & \text{if } r < r_b \\ n_{\Delta}, & \text{if } r_b < r < r_b + \Delta \\ 0, & \text{otherwise} \end{cases} \quad (2.32)$$

and $n_\Delta = \frac{r_b^2}{(r_b + \Delta)^2 - r_b^2}$ is constrained by Eq. (2.31). Using this model, they obtained an expression for the wake potential

$$\begin{aligned}\psi(r, \xi) &= \psi_0(\xi) - \frac{r^2}{4}, \\ &= \frac{r_b^2(\xi)}{4}(1 + \beta) - \frac{r^2}{4}\end{aligned}\tag{2.33}$$

for $r \leq r_b$, where

$$\beta = \frac{(1 + \alpha)^2 \ln(1 + \alpha^2)}{(1 + \alpha)^2 - 1} - 1, \quad \alpha = \frac{\Delta}{r_b}.$$

Assuming a phenomenological sheath profile which only depends on the bubble radius, i.e., $d\beta/d\xi = (dr_b/d\xi)\partial\beta/\partial r_b$, it was shown in Refs. 35 and 36 that Eq. (2.29) can be expressed as

$$A(r_b)\frac{d^2 r_b}{d\xi^2} + B(r_b)r_b\left(\frac{dr_b}{d\xi}\right)^2 + C(r_b)r_b = \frac{\lambda(\xi)}{r_b}\tag{2.34}$$

where the coefficients $A(r_b)$, $B(r_b)$, and $C(r_b)$ are

$$A(r_b) = 1 + \left[\frac{1}{4} + \frac{\beta}{2} + \frac{1}{8}r_b \frac{d\beta}{dr_b} \right] r_b^2,$$

$$B(r_b) = \frac{1}{2} + \frac{3}{4}\beta + \frac{3}{4}r_b \frac{d\beta}{dr_b} + \frac{1}{8}r_b^2 \frac{d^2\beta}{dr_b^2},$$

$$C(r_b) = \frac{1}{4} \left[1 + \frac{1}{\left(1 + \frac{\beta r_b^2}{4} \right)^2} \right].$$

Once $r_b(\xi)$ is determined from Eq. (2.34), the wake potential $\psi(r, \xi)$ and electric field

$$E_z(\xi) = \frac{d\psi_0}{d\xi} = \frac{d}{d\xi} \left[\frac{1}{4} r_b^2 (1 + \beta) \right] \quad (2.35)$$

are known.

Using a simple phenomenological model for the thickness of the plasma sheath $\Delta = 1 + \epsilon r_b$, Lu et al. [35, 36] found that their model for ψ [Eq. (2.33)] could reproduce many of the features of the plasma wake. For example, it could predict the channel radius $r_b(\xi)$ and slope $dr_b/d\xi$ along most of the bubble. It could also capture the electric field slope $E_z \simeq -1/2$ within the middle of the bubble. However, it could not account for the negative spike in the electric field at the rear of the bubble as seen in Fig. 2.1(b). Fundamentally, their use of the simple source term profile given by Eq. (2.32) led to a form for ψ that missed important physics at the rear of the wake. For example, the model breakdown at the rear of the wake meant that they could not describe important processes such as trapping of test charges. This can be seen by the fact that the on-axis wake potential of the simple sheath model $\psi_0 = (1 + \beta)r_b^2/4$ [Eq. (2.33)] is positive definite while trapping requires ψ_0 approach -1 at the rear of the wake [44]. In the following chapter, we will propose an improved sheath model for the plasma wakefield source term that accurately describes the wakefield at the rear of the bubble. It will be shown that the proposed model leads to important improvements to the beam loading theory developed by Tzoufras et al. [38, 39]. The new beam loading results will also be validated against PIC simulations. Laser-driven wakefields will also be considered.

CHAPTER 3

A Multi-Sheath Model for Highly Nonlinear Plasma Wakefields

3.1 Introduction

In PBA, a critical process is beam loading where a witness or trailing beam of particles is located at an appropriate phase of the wake where it is accelerated and focused. As it is accelerated it absorbs energy from the wake and can distort, i.e., load, it. Developing an accurate beam loading theory is important in order to accurately understand and control the energy spread and emittance of the witness beam. In some cases, this needs to be understood even as the beam phase slips inside the wakefield. The witness beam can be externally or self-injected. Self-injection has advantages as it leads to synchronized injection which can be difficult to achieve for external injection due to the short periods and wavelengths of the plasma wakefields; however, self-injection may not produce the beam charge required for a linear collider.

Another application driving research in PBA is a compact XFEL. For this application, high currents and normalized brightnesses are required. Self-injection may provide the parameters needed to drive an XFEL. Recently, there have been many self-injection schemes proposed to generate high quality electron beams with low energy spread σ_γ and normalized emittance ϵ_n . The most promising ideas typically involve decreasing the phase velocity γ_ϕ of the plasma wake using either a plasma density down ramp [49, 50, 51, 52, 53, 54, 55, 56] through the accordion effect [92] or an evolving driver [57, 58, 59]. In each of these instances,

plasma electrons are injected at the very rear of the first bucket of the wake where they can then be accelerated over long periods of time.

In order to characterize how the injected beams alter the wakefield and effect the energy spread, a theoretical model for the wakefield that is accurate in the rear of the region is required. If the model is accurate enough it can also be used to design experiments and simulations capable of generating injected beams that can flatten a wakefield or provide the necessary slope in the acceleration gradient to compensate for an initial energy chirp inherent in self-injection after some acceleration distance [52].

In the linear regime, the necessary beam loading theory has existed for over thirty years [31]. The theory works for both electron and positron bunches. However, in the nonlinear regime the theory is significantly more complicated. In Rosenzweig et al. [33], it was shown that the focusing fields for nonlinear wakefields are ideal for accelerating electrons. Conversely, these fields are defocusing for positrons. In reality, there is a very small phase of the wake that provides focusing and acceleration for positrons. Furthermore, developing a beam loading theory for electron bunches requires a new theoretical formalism.

In nonlinear wakefields the plasma electrons are expelled by the space-charge force of a particle beam (PWFA) or radiation pressure of a laser pulse (LWFA) leaving behind a column of ions. These electrons, which are initially blown-out, are attracted back to the axis due to the space-charge force from the ions, forming a plasma sheath covering a nearly spherical ion channel radius $r_b(\xi)$. For convenience, this structure is shown again in Fig. 3.1(a), where the electron density from a PWFA simulation using the particle-in-cell (PIC) code OSIRIS [84] is plotted. A non-evolving driver with a peak normalized charge per unit length $\Lambda_d \equiv 4\pi r_e \int_0^{r \gg \sigma_r} n_b r dr = 6$, energy $\gamma_b = 20000$, spot size $k_p \sigma_r = 0.245$, duration $k_p \sigma_z = 1$, and centroid $k_p \xi_c = 0$ was used, where $k_p = \omega_p/c$ is the plasma wavenumber, $\omega_p^2 = \frac{4\pi e^2 n_p}{m}$ is the plasma frequency, and n_b is the drive beam density.

All of the forces on a particle moving close to the speed of light can be obtained from the normalized wake potential $\psi = \frac{e}{mc^2}(\phi - A_z)$, where ϕ is the electric potential and A_z is the

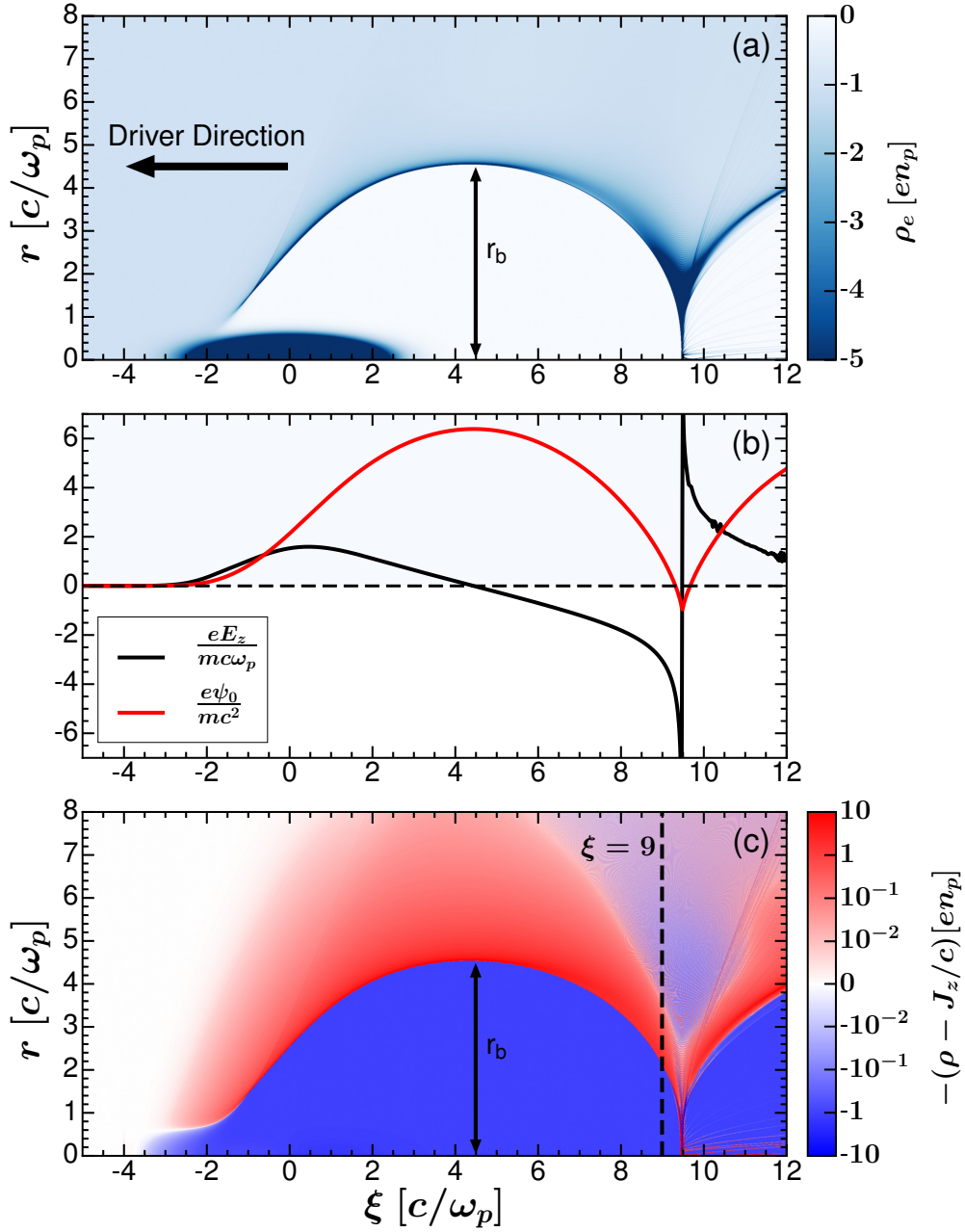


Figure 3.1: (a) Electron density distribution of a plasma wake excited by an electron drive beam with parameters $\Lambda_d = 6$, $\gamma_b = 20000$, $k_p\sigma_z = 1$, $k_p\sigma_r = 0.245$, and $k_p\xi_c = 0$. The maximum bubble radius is $k_pr_m \simeq 4.53$. (b) The on-axis electric field E_z and wake potential ψ_0 . (c) Contour plot of the source term $-(\rho - J_z/c)$.

vector potential in the $\hat{z}$ direction. In Refs. 35 and 36, it was shown that the wake potential can be obtained from a two dimensional Poisson equation

$$\nabla_{\perp}^2 \psi = S \equiv -\frac{1}{en_p}(\rho - J_z/c) \quad (3.1)$$

where the integral of the source term over the transverse coordinates in each ξ slice vanishes.

For azimuthally symmetric beams or laser drivers, the solutions to this equation for radii inside the ion column have the form $\psi = \frac{1}{4}(1 + \beta)r_b^2 - r^2$ where β depends on integrals over transverse gradients of ψ written in terms of S from $r = \infty$ to 0. It was shown in Refs. 35 and 36 that for a source term comprised of two regions where $S = -1$ inside the ion bubble ($r < r_b$) and $S \geq 0$ in a finite width plasma sheath outside the bubble ($r_b < r < r_b + \Delta$), $\beta = \frac{(1+\alpha)^2 \ln(1+\alpha)^2}{(1+\alpha)^2 - 1} - 1$, where $\alpha = \frac{\Delta}{r_b}$. Using this expression for β a differential equation for r_b was then obtained [see Eq. (2.34)]. It was shown in [35] that these equations could explain many of the features observed in particle-in-cell simulations.

However, this simple model for the plasma sheath and hence the wake potential has its limitations. For example, a direct consequence of using such a model is that β is positive definite for each ξ slice and hence ψ is positive definite at all locations within the ion column.

To develop some intuition about the behavior of ψ consider the one dimensional limit. In one dimension, wavelike analysis to the cold fluid equations show that solutions exist until wavebreaking occurs. This can be physically interpreted as the limit where the plasma density compression approaches infinity, the electric field fully steepens (its slope approaches infinity), two plasma sheets cross, particle trapping of a background electron occurs, i.e, a particle moves with the phase velocity of the wave [93]. In this case, the minimum wake potential becomes -1 which is the threshold for particle trapping. Even in multi-dimensions, the trapping condition for background electrons, $\psi \leq -1 + \frac{\sqrt{1+(P_{\perp}/mc)^2}}{\gamma_{\phi}}$ [44], also requires that the wake potential approach -1. PIC simulations show empirically that the wake potential is negative in the rear of the first bucket as shown in Fig. 3.1(b). In this region of the wake,

plasma sheath electrons can be accelerated to large forward velocities $\gamma_z \equiv (1 - \beta_z^2)^{-1/2} \gg 1$ as they return back to the ξ -axis due to the large accelerating fields $E_z(\xi)$ at the rear of the wake. Near the axis, the wake potential ψ must approach -1 if v_z is to approach c as can be seen from the constant of motion equation $\gamma - P_z/mc = 1 + \psi$ [62]. In fact, many self-injection schemes [49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 92] rely on ψ approaching -1 at the rear of the wake in order to satisfy the electron trapping condition $\gamma_z > \gamma_\phi \gg 1$. To accurately derive the trapping threshold, we do not let the $v_\phi = c$. The constant of motion is $\gamma - v_\phi P_z/mc^2 = 1 + \psi$ and $\psi = \phi - v_\phi A_z/c$.

Although the constant of motion is strictly valid for uniform plasmas and nonevolving drivers, it is still a useful approximation for slowly evolving wakes. Therefore, while the model used by Lu et al. [35, 36] can predict the bubble trajectory $r_b(\xi)$ and longitudinal electric field $E_z(\xi)$ in regions where the wake potential $\psi(\xi)$ is sufficiently positive, it will not be accurate near the rear of the bubble where the wake potential becomes negative.

As noted in Ref. 52, if the source term S is negative in some region outside the bubble $r > r_b$ then wake potential ψ can be negative inside the ion column. This is illustrated in Fig. 3.1(c) where three distinct regions are evident from the contour plot of S . In addition to the ion channel and plasma sheath regions included in the simple single-sheath model employed in Refs. 35 and 36, it is clear that there is a third region of finite width where $S < 0$ at the rear of the wake outside the bubble $r > r_b$. This negative region is highly localized to the rear of the bubble and drops off rapidly in terms of amplitude at ξ where the bubble radii is larger.

Thus, we propose to use a multi-sheath model for the source term S comprised of three regions to obtain a new expression for the wake potential $\psi(\xi)$. Using the proposed model for $\psi(\xi)$ in conjunction with the nonlinear blowout theory presented by Lu et al. [35, 36], we will calculate the trajectory of the bubble radius $r_b(\xi)$ and electric field $E_z(\xi)$ while using the constants of motion to constrain the variables. We find that numerical results obtained using the proposed model agree well with PIC simulation results throughout the entire ion column.

We also compare the results for the multi-sheath model to the those from the single sheath model employed in Refs. 35 and 36. We also show the importance of using the multi-sheath model when studying beam loading of nonlinear wakes from witness electron beams. To accurately analyze beam loading in the nonlinear regime it is essential to have an accurate equation for $r_b(\xi)$ and for $\psi(r < r_b)$. In the original work of Tzoufras et al. [38, 39], beam loading was analyzed by determining how $r_b(\xi)$ is modified by the electromagnetic forces of the witness beam. To obtain analytical results, Tzoufras et al. applied the large r_b limit to the differential equation for r_b . We show that the multi-sheath model provides better agreement and apply it to linear collider and self-injection parameters. Current profiles of witness beams that flatten the wakes are provided. We will show numerical results obtained for the multi-sheath model for wakes excited by intense lasers. For laser wakefield parameters considered here, where matched spot sizes are used, we find that the theory does not work as well as electrons are not completely blown out by the driver and the sheath structures are more complicated. Last, a detailed discussion on the differences between the results for the multi-sheath and single sheath models will be given.

3.2 The Plasma Wake Potential

Our goal is to obtain a more accurate expression for the differential equation for $r_b(\xi)$ and the fields inside the ion column. We begin by concentrating on the wake potential from which the accelerating and focusing fields for a witness beam are derived. The differential equation for $r_b(\xi)$ can therefore be completely described if the wake potential is known. As mentioned above, the simple sheath model used in Refs. 35 and 36 cannot accurately describe $r_b(\xi)$ or the wake potential unless $r_b(\xi)$ is sufficiently large. For such a simple sheath model, the wake potential is positive definite; however, it is known empirically from PIC simulations that the wake potential approaches -1 at the rear of nonlinear multi-dimensional wakes. In subsequent sections, we show that in order to get accurate predictions for r_b and beam

loading it is essential that the potential approach -1 .

As noted above, it is straightforward to show that in order for ψ to be negative the source term must be negative for some $r > r_b$ beyond the sheath. We thus propose a phenomenological source term S that extends the single-sheath model from Refs. 35 and 36 by introducing a second plasma sheath Δ_2 in which the source term is negative, i.e., $S \equiv n_2 < 0$, outside the ion bubble $r > r_b$. This is shown schematically in Fig. 3.2(a). For comparison, we also show the simple model utilized by Lu et al. [35, 36]. While two sheaths are usually enough to model the source term for wakes created by electron beams (PWFA problems), the formalism can be extended to include an arbitrary number of sheaths. This may be needed for accurate descriptions of nonlinear wakes created by laser drivers which will be discussed later. Therefore, we refer to the proposed model as the “multi-sheath model” since it can employ two or more sheaths while we refer to model used by Lu et al. [35, 36] as the “single-sheath model” since it employs only one sheath.

As we will show in this section, this second sheath region Δ_2 is needed to describe both the physics and mathematics of the plasma wake features at the rear of the bubble. Once an expression for ψ is obtained using the proposed model for S , we can solve for the trajectory of the bubble radius r_b , wake potential ψ , and accelerating field E_z . We note that others [94, 95] have proposed different phenomenological sheath models than the one presented here. These authors were motivated to obtain accurate descriptions for the fields inside the sheath in order to study self-injection and hosing. In Ref. 94, an exponentially decaying source term was used for S with a separate decay coefficient for J_z . For the fields inside the ion cavity, such a model does not offer improvement and it still provides a positive definite form for ψ . Thus, they cannot be used to study the trapping of test charges in static wakes. Furthermore, the relative importance of ρ and J_z varies with ξ so that the viewing of ρ and J_z as having two distinct decay coefficients is not consistent.

In addition, others [96] have proposed using a specific functional form for $E_z(\xi)$ to address the field divergence and describe the plasma forces in unloaded plasma wakes. While

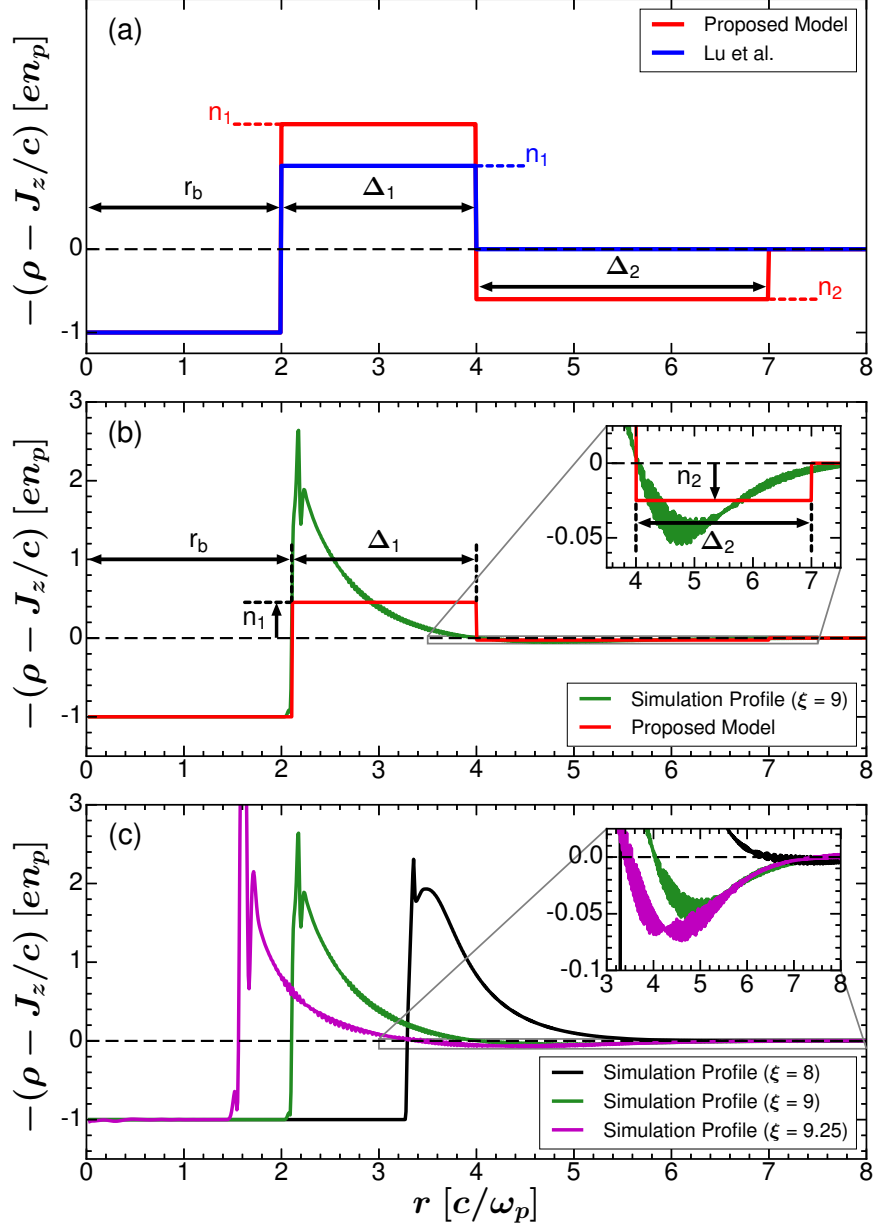


Figure 3.2: (a) Comparison of the proposed multi-sheath model profile (red) and single-sheath model [35, 36] (blue). (b) The proposed model profile (red) and simulation profile (green) for the transverse slice at $k_p\xi = 9$ (dashed black) from Fig. 3.1(c). (c) Transverse slices of the source term profiles from Fig. 3.1(c) at different $k_p\xi$.

this can provide improvements, it does not provide an explanation for why this scaling is necessary. Therefore, it is not predictive and cannot be used to study beam loading. On the other hand, in the multi-sheath model, the plasma forces determine the bubble trajectory $r_b(\xi)$ from which the electric field $E_z[r_b(\xi)]$ is calculated. The multi-sheath model thus structurally addresses the field divergence through the plasma source term without constraining the functional form of the electric field. Therefore, beam loading with arbitrary current profiles can be self-consistently described by the force equation as a modification to $r_b(\xi)$ and $E_z[r_b(\xi)]$. And this cannot be done with the model from Ref. 96.

Henceforth, we will employ normalized units, where charge is normalized to electron charge e , mass to electron mass m , velocity to c , charge density to en_p , current density to $en_p c$, length to c/ω_p , time to ω_p^{-1} , electric fields to $mc\omega_p/e$, and potentials to mc^2/e . We will also assume that the wake is excited by a bi-Gaussian electron bunch with a density profile $n_b \sim e^{-r^2/(2\sigma_r^2)}e^{-\xi^2/(2\sigma_z^2)}$ and a spot size σ_r much smaller than the blowout radius r_m . Following the convention used in Refs. 35 and 36, we will use step functions to model the source term S as illustrated in Fig. 3.2(a) with

$$S = \begin{cases} -1, & \text{if } r < r_b \\ n_1, & \text{if } r_b < r < r_b + \Delta_1 \\ n_2, & \text{if } r_b + \Delta_1 < r < r_b + \Delta_1 + \Delta_2 \\ 0, & \text{otherwise.} \end{cases} \quad (3.2)$$

In Fig. 3.2(b), we show how the multi-sheath model compares to the actual source term profile at $k_p\xi = 9$ obtained from the PIC simulation in Fig. 3.1. In order to understand the physics represented in each region we write out the electron and ion source terms, $S = -(\rho - J_z) = -\rho_{ion} - \rho_e(1 - v_z)$. Inside the bubble, $r < r_b$ (region I), $\rho_e = 0$ and $\rho_{ion} = 1$ so $S(r < r_b) = -1$ as shown in Fig. 3.1(a). Due to space-charge separation from blowout, plasma electrons are attracted back to the ξ -axis by the ion channel, thereby forming a plasma sheath (region II) with a large negative density spike $\rho_e \ll -1$ at the bubble interface $r_b(\xi)$.

From the constant of motion for a plasma particle $\gamma - P_z = 1 + \psi$ [62], it can be shown that $v_z = \frac{1+P_z^2-(1+\psi)^2}{1+P_z^2+(1+\psi)^2}$ [35, 36]. Therefore, the innermost sheath electrons at the top of bubble, where $P_z \simeq 0$, propagate backwards $v_z < 0$. Since $(1 - v_z) > 1$ and $\rho_e \ll -1$ in this region, the innermost sheath is characterized by a positive source term $S \equiv n_1 > 0$ within a finite width region denoted by Δ_1 . As these innermost sheath electrons return back to the axis where the wake potential ψ can approach -1 , they can propagate in the forward direction at nearly the speed of light, i.e., $v_z \sim 1$. In this region, the source term of the electrons is reduced by the factor $(1 - v_z) \ll 1$. Despite this, the sheath width Δ_1 remains finite at the rear of the wake because the electron density spike along the bubble interface $r_b(\xi)$ is large enough to offset the ion term, i.e., $-\rho_e > \frac{1}{(1-v_z)} \gg 1$.

Near the back of the bubble, there exists a second plasma sheath (region III) of width Δ_2 bordering the first in which the source term $S \equiv n_2 < 0$. In this region, the electron density is of the order of unity, i.e., $\rho_e \lesssim -1$, and plasma electrons are still propagating forward, i.e., $v_z > 0$. Therefore, $-\rho_e(1 - v_z) < 1$, resulting in a negative source term $S < 0$.

As can be seen in Fig. 3.2(c), the simulation profile of the second sheath varies with the bubble radius $r_b(\xi)$ and ξ . For $r_b(\xi)$ close to the maximum radius $r_m \simeq 4.53$, the second sheath can generally be neglected since $n_2 \approx 0$. However, the amplitude of n_2 rapidly increases as $r_b(\xi)$ decreases, which is observed for transverse slices located at $k_p \xi = 9, 9.25$ in Fig. 3.1(c). In most simulations, peak n_2 values of $\sim O(-\frac{1}{10})$ can be observed in the second sheath region at the very rear of the wake where $r_b(\xi)$ goes to zero. Despite the fact that n_2 is small, the width Δ_2 usually extends over several plasma skin depths. Therefore, the source term of the second sheath actually contributes the most to the negative pseudopotentials observed at the rear of the wake when integrating Eq. (3.1). In regions $r \gg r_b$ far from the blowout, both Δ_1 and Δ_2 connect to the linear regime where electron perturbation $|\delta\rho_e| \ll 1$ is small and the electron velocities $|v_z| \ll 1$ are non-relativistic. In this limit (region), electrons oscillate at the plasma frequency ω_p .

The parameters defined in Eq. (3.2), n_1, n_2, Δ_1 , and Δ_2 , are related by the requirement

that charge is conserved in each slice as derived in Refs. 35 and 36,

$$\int_0^\infty (\rho - J_z) r dr = 0. \quad (3.3)$$

Integrating Eq. (3.3), we obtain

$$\begin{aligned} & -1 + n_1 \left[\left(1 + \frac{\Delta_1}{r_b} \right)^2 - 1 \right] + \\ & n_2 \left[\left(1 + \frac{\Delta_1 + \Delta_2}{r_b} \right)^2 - \left(1 + \frac{\Delta_1}{r_b} \right)^2 \right] = 0. \end{aligned} \quad (3.4)$$

We can rewrite Eq. (3.4) to solve for n_1 in terms of n_2 , α_1 , and α_2 ,

$$n_1 = \frac{1 - n_2 (\alpha_2^2 + 2\alpha_2\alpha_1 + 2\alpha_2)}{(1 + \alpha_1)^2 - 1} \quad (3.5)$$

where $\alpha_1 \equiv \frac{\Delta_1}{r_b}$ and $\alpha_2 \equiv \frac{\Delta_2}{r_b}$. If $n_2 = 0$, we can recover the expression $n_1 = n_\Delta \equiv \frac{1}{(1+\alpha_1)^2-1}$ from the single-sheath model.

We next calculate the wake potential $\psi(r, \xi)$. To do this, we first need to determine the on-axis potential $\psi_0(\xi) \equiv \psi(0, \xi)$. Once $\psi_0(\xi)$ is known, the wake potential $\psi(r, \xi) = \psi_0(\xi) - \frac{r^2}{4}$ is defined everywhere inside the bubble. To obtain $\psi_0(\xi)$, we integrate Eq. (3.1) across all three regions defined in Eq. (3.2)

$$\begin{aligned} \psi_0 &= \int_0^\infty \frac{dr}{r} \int_0^r (\rho - J_z) r' dr' \\ &= \left[\int_0^{r_b} + \int_{r_b}^{r_b+\Delta_1} + \int_{r_b+\Delta_1}^{r_b+\Delta_1+\Delta_2} \right] \frac{dr}{r} \int_0^r (-S) r' dr' \\ &= \Psi_I + \Psi_{II} + \Psi_{III} \end{aligned} \quad (3.6)$$

where $\Psi_I = \frac{r_b^2}{4}$ is the contribution from the ion bubble, $\Psi_{II} = \frac{r_b^2}{4} \left\{ 2(1 + n_1) \ln(1 + \alpha_1) - n_1 \left[(1 + \alpha_1)^2 - 1 \right] \right\}$ is the contribution innermost plasma sheath of width Δ_1 , and $\Psi_{III} =$

$\frac{r_b^2}{4} \left\{ 2n_2(1 + \alpha_1 + \alpha_2)^2 \ln \left(1 + \frac{\alpha_2}{1 + \alpha_1} \right) - n_2 \left[(1 + \alpha_1 + \alpha_2)^2 - (1 + \alpha_1)^2 \right] \right\}$ is the contribution from second plasma sheath of width Δ_2 .

Summing the expressions of all the three regions and simplifying with Eq. (3.5), we obtain a final expression for the wake potential inside the bubble ($r \leq r_b(\xi)$), similar to the one derived in Refs. 35 and 36,

$$\begin{aligned} \psi(r, \xi) &= \psi_0(\xi) - \frac{r^2}{4} \\ &= \frac{r_b^2(\xi)}{4} (1 + \beta') - \frac{r^2}{4} \end{aligned} \quad (3.7)$$

where

$$\begin{aligned} \beta' &= 2(1 + n_1) \ln(1 + \alpha_1) - 1 \\ &\quad + 2n_2(1 + \alpha_1 + \alpha_2)^2 \ln \left(1 + \frac{\alpha_2}{1 + \alpha_1} \right). \end{aligned} \quad (3.8)$$

Eqs. (3.7) and (3.8) contain the key differences between the present work and that in Refs. 35 and 36. Naturally, these differences also affect the bubble trajectory r_b since the plasma forces depend on ψ . Thus, it is worth comparing and discussing the differences between β and β' . First, if n_2 is set to zero then it is trivial to see that β' reduces to $\beta \equiv \frac{(1 + \alpha_1)^2 \ln(1 + \alpha_1)^2}{(1 + \alpha_1)^2 - 1} - 1$ (using the conservation of charge to express n_1 in terms of α_1) which is only a function of α_1 , and we recover the on-axis wake potential $\psi_0 = (1 + \beta)r_b^2/4$. Furthermore, it is important to note that $r_b^2\beta \rightarrow 0$ as $r_b \rightarrow 0$ and therefore the minimum ψ at rear of the bubble is 0 for the single-sheath model.

On the other hand β' is a function of four parameters $\beta'(n_1, n_2, \alpha_1, \alpha_2)$ where each of these parameters are unknown functions of r_b . The goal is to use a combinations of physics constraints and phenomenological arguments to reduce β' to be a known function of r_b . First, we use the conservation of charge constraint, Eq. (3.5), which gives $n_1(n_2, \alpha_1, \alpha_2)$ to eliminate n_1 from β' . Next, we will use empirical observations regarding $\psi_0(r_b = 0)$ and

phenomenological arguments for the dependence of n_2 , Δ_1 , and Δ_2 to obtain an expression for β' in terms of r_b .

We assume that the sheath widths are finite as r_b approaches 0 and can thus be written as $\Delta_1(r_b) = \Delta_{10}c_1(r_b)$, $\Delta_2(r_b) = \Delta_{20}c_2(r_b)$, and $n_2 = n_{20}h(r_b)$ where c_1 , c_2 , and h are functions of r_b that approach 1 as $r_b \rightarrow 0$. We next take the limit of $\psi(r_b)$ as $r_b \rightarrow 0$ to obtain a relationship between the empirical value of $\psi_{min} \equiv \lim_{r_b \rightarrow 0} \psi(r_b(\xi))$, and Δ_{10} , Δ_{20} , and n_{20} ,

$$\begin{aligned}\psi_{min} &\equiv \lim_{r_b(\xi) \rightarrow 0} \beta' \frac{r_b^2}{4} \\ &= \lim_{r_b(\xi) \rightarrow 0} \left[(1 + n_1) \frac{r_b^2}{2} \ln \left(1 + \frac{\Delta_1}{r_b} \right) - \frac{r_b^2}{4} \right. \\ &\quad \left. + n_2 (r_b + \Delta_1 + \Delta_2)^2 \ln \left(1 + \frac{\Delta_2}{r_b + \Delta_1} \right) \right] \\ &= \frac{n_{20}}{2} (\Delta_{10} + \Delta_{20})^2 \ln \left(1 + \frac{\Delta_{20}}{\Delta_{10}} \right).\end{aligned}\tag{3.9}$$

where we have used the fact that it is straightforward to show that the first term in the limit vanishes since $\lim_{r_b \rightarrow 0} r_b^2 \ln(1 + \Delta_1/r_b) = 0$ and that the amplitude of the innermost plasma sheath remains finite $n_{10} \equiv n_1(r_b = 0) = \frac{-n_{20}(\Delta_{20}^2 + 2\Delta_{20}\Delta_{10} + 2\Delta_{20})}{\Delta_{10}^2}$ due to charge conservation with the second sheath from Eq. (3.5). Rearranging the terms in Eq. (3.9) to solve for n_{20} , we obtain

$$n_{20} = \frac{2\psi_{min}}{(\Delta_{10} + \Delta_{20})^2 \ln \left(1 + \frac{\Delta_{20}}{\Delta_{10}} \right)}\tag{3.10}$$

where ψ_{min} is the minimum value of the on-axis potential ψ_0 . Empirically, it is known that ψ_{min} is negative, from which it follows from Eq. (3.10) that n_{20} must also be negative, which had been argued above when we motivated the need for the multi-sheath model. Additionally, from Eq. (3.5), it also follows that n_{10} must be positive if n_{20} is negative.

Eq. (3.10) is important because it constrains the parameter n_{20} for given value of Δ_{10} , Δ_{20} , and ψ_{min} . While PIC simulations can be used to determine the exact value of ψ_{min} in the

nonlinear blowout regime, ψ_{min} can be well approximated by -1 when the maximum bubble radius is sufficiently large, i.e., $r_m \gtrsim 3$. Under these conditions, sheath electrons that trace the bubble $r_b(\xi)$ travel near the speed of light, $v_z \sim 1$, with finite transverse momentum $P_\perp$ at the rear of the wake [52, 59]. From the constant of motion for a plasma particle $\gamma - P_z = 1 + \psi$ [62], it can be shown that $1 - v_z = \frac{2(1+\psi)^2}{1+P_\perp^2+(1+\psi)^2}$ [35, 36] so that $v_z \rightarrow 1$ when $\psi_{min} \rightarrow -1$.

Until this point, we have not specified $n_1(r_b)$, $n_2(r_b)$, $\Delta_1(r_b)$ and $\Delta_2(r_b)$. In general, since the phenomenological model for S employs simple step functions in each region, we will not be able to fit these parameters exactly to empirical wake structures across all regions. Instead, the goal is to use profiles for n_1 , n_2 , Δ_1 and Δ_2 that can reproduce the on-axis pseudopotential $\psi_0(\xi)$ and innermost electron trajectory $r_b(\xi)$ for nonlinear plasma wakes. Once these quantities are determined, the wake potential $\psi(r, \xi) = \psi_0(\xi) - \frac{r^2}{4}$ would be correct everywhere inside the bubble $r \leq r_b(\xi)$, which is the region of interest in the nonlinear blowout regime.

Following the single-sheath model [35, 36], we use a profile for the first plasma sheath width of $\Delta_1 = \Delta_{10} + \Delta_s$, where $\Delta_{10} \sim O(1)$ and $\Delta_s = \epsilon r_b$ such that $c_1 = 1 + \epsilon r_b$. This profile is consistent with the physical picture described earlier in which Δ_1 remains finite near the axis due to the large electron density spike $|\rho_e| \gg 1$ along $r_b(\xi)$. It is worth pointing out that the values of Δ_{10} and ϵ can be varied slightly to adjust the length of the plasma wake. We use $\Delta_{10} = 1$ and $\epsilon = .05$ in most cases and explicitly assume they do not depend on ξ .

To model the $S < 0$ region outside the bubble, we use a Gaussian profile $n_2 = n_{20}e^{-sr_b^2/r_m^2}$ such that $h = e^{-sr_b^2/r_m^2}$ and a constant width $\Delta_2 = \Delta_{20}$ for simplicity, where r_m is the maximum value of r_b . While other profiles can be used to model this region, the Gaussian profile is largely motivated by the observed behavior of n_2 in PIC simulations [see Fig. 3.2(c)] where it reaches a negative minimum when $r_b = 0$ and approaches 0 as r_b approaches the blowout radius r_m . We note that super Gaussians can be used, $h = e^{-sr_b^t/r_m^t}$ where tuning t can improve the accuracy. The value of n_{20} is obtained using Eq. (3.10), where the minimum

wake potential ψ_{min} also needs to be provided as input. For nonlinear wakes in the blowout regime, i.e., $r_m \gtrsim 3$, $\psi_{min} \approx -1$ can be used. For all cases presented in this paper, we use $s = 3$ for the Gaussian coefficient ($t = 2$) and $\Delta_{20} = 3$ for the second sheath so that the n_{20} values calculated from Eq. (3.10) are in agreement with values observed in simulations that are typically $\sim O(-\frac{1}{10})$. However, the results are largely insensitive to values of s and Δ_{20} ranging from 2 to 4.

We can now use $\beta'(n_{20}, \Delta_{10}, \Delta_{20}, h(s, r_b), c_1(\epsilon, r_b)/r_b)$ (note that $c_2 = 1$) to obtain the trajectory of the bubble r_b for a given r_m . In a plasma wake, the bubble radius r_b starts at 0 reaches a maximum value of r_m and then returns to 0 at the rear. In the single-sheath model, $\beta(r_b)$ is single valued (symmetric) inside the bubble and the sheath structure looks the same in the front and back half of the bubble. On the other hand, in the multi-sheath model (and in real wakes), the sheath structure looks different between the front and back half (there is a single sheath in the front half). While it may be possible to merge the two models, in what follows we concentrate on examining the back half of the bubble for both loaded and unloaded wakes starting from the maximum blowout radius $r_b = r_m$.

As noted above much of the formalism in [35, 36] is independent of the choice for the sheath model. Following the same procedure, the differential equation describing the trajectory of the innermost plasma electron tracing the ion channel $r_b(\xi)$ can be shown to be,

$$\begin{aligned} \frac{d}{d\xi} \left[(1 + \psi) \frac{d}{d\xi} r_b \right] = r_b \left\{ -\frac{1}{4} \left[1 + \frac{1}{(1 + \psi)^2} + \left(\frac{dr_b}{d\xi} \right)^2 \right] \right. \\ \left. - \frac{1}{2} \frac{d^2 \psi_0}{d\xi^2} + \frac{\lambda(\xi)}{r_b^2} \right\} \end{aligned} \quad (3.11)$$

where the current profile of the drive and/or trailing bunch is given by $\lambda(\xi) = \int_0^{r \gg \sigma_r} n_b r dr$. By expressing $\psi(r, \xi)$ in the form shown in Eq. (3.7), we can obtain a differential equation for the innermost particle trajectory $r_b(\xi)$

$$A'(r_b) \frac{d^2 r_b}{d\xi^2} + B'(r_b) r_b \left(\frac{dr_b}{d\xi} \right)^2 + C'(r_b) r_b = \frac{\lambda(\xi)}{r_b} \quad (3.12)$$

where the coefficients $A'(r_b)$, $B'(r_b)$, and $C'(r_b)$ are

$$A'(r_b) = 1 + \left[\frac{1}{4} + \frac{\beta'}{2} + \frac{1}{8} r_b \frac{d\beta'}{dr_b} \right] r_b^2,$$

$$B'(r_b) = \frac{1}{2} + \frac{3}{4} \beta' + \frac{3}{4} r_b \frac{d\beta'}{dr_b} + \frac{1}{8} r_b^2 \frac{d^2 \beta'}{dr_b^2},$$

$$C'(r_b) = \frac{1}{4} \left[1 + \frac{1}{\left(1 + \frac{\beta' r_b^2}{4} \right)^2} \right].$$

This is identical to Eq. (2.34) from chapter 2 except β is replaced by β' . It is worth recalling that an underlying assumption of Eq. (3.12) is that the r_b dependence in β' arises from h , c_1 and c_2 .

Once $r_b(\xi)$ is calculated by integrating Eq. (3.12) starting from r_m , the wake potential described by Eq. (3.7) can be used to obtain the longitudinal electric field in the back half of the bubble ($r \leq r_b(\xi)$)

$$E_z(\xi) = \frac{d}{d\xi} \psi_0(\xi) = D'(r_b) r_b \frac{dr_b}{d\xi} \quad (3.13)$$

where $D'(r_b) \equiv \frac{1}{2} + \frac{\beta'}{2} + \frac{1}{4} r_b \frac{d\beta'}{dr_b}$. The slope of the electric field follows directly from Eq. (3.13)

$$\frac{dE_z}{d\xi} = D'(r_b)r_b\frac{d^2r_b}{d\xi^2} + F'(r_b)\left(\frac{dr_b}{d\xi}\right)^2 \quad (3.14)$$

where $F'(r_b) \equiv D'(r_b) + \frac{3}{4}r_b\frac{d\beta'}{dr_b} + \frac{1}{4}r_b^2\frac{d^2\beta'}{dr_b^2}$.

In Fig. 3.3(a), we plot the bubble radius $r_b(\xi)$ numerically obtained by integrating Eq. (3.12) and the sheath widths $\Delta_1(r_b)$ and $\Delta_2(r_b)$ on top of the actual source term S from the simulation shown in Fig. 3.1 using $\Delta_{10} = 1$, $\Delta_s = \epsilon r_b = 0.05r_b$, and $\Delta_{20} = 3$. We note that Eq. (3.12) is numerically integrated using the 4th order Runge-Kutta method (RK4). The maximum bubble radius is $r_m \simeq 4.53$. We also plot $n_1(r_b)$ and $n_2(r_b)$ where the n_{20} parameter is calculated from Eq. (3.10) using $\psi_{min} = -1$ and n_1 is calculated from Eq. (3.5). Excellent agreement is observed in the bubble trajectory $r_b(\xi)$ calculated using the multi-sheath model and simulation results as seen in Fig. 3.3(a).

It can be seen in Fig. 3.3(a) that the model for Δ_1 captures the most important regions of the innermost plasma sheath ($S > 0$) along the bubble interface. While the constant width profile for Δ_2 sufficiently characterizes the $S < 0$ region at the rear of the wake, it does not precisely track the empirical second sheath width at the top of the bubble. However, as shown in Fig. 3.3(b), the profile of n_2 (red) decays exponentially to zero near r_m and, therefore, the exact profile of Δ_2 is irrelevant in this region. Near the top of the bubble, n_1 (black) can also be well-approximated by $\frac{1}{(1+\alpha_1)^2-1}$ because n_2 approaches zero. When r_b goes to 0, n_1 remains finite because of the negative n_2 term in the continuity equation [Eq. (3.5)], which is consistent with the physical picture described earlier and shown in Fig. 3.3(a). However, in the single-sheath model, $n_1 = n_\Delta \rightarrow 0$ for $r_b \rightarrow 0$.

The limiting contributions from each of the three regions can also be characterized by the respective wake potential terms defined in Eq. (3.6) and plotted in Fig. 3.3(c). When r_b approaches r_m , the ion term Ψ_I (black) clearly dominates, the sheath term Ψ_{II} (red) is on the order of unity, and Ψ_{III} (blue) can be neglected because n_2 goes to zero. However, when $r_b/r_m \ll 1$, the order of importance is reversed, where Ψ_{III} is now the most negative

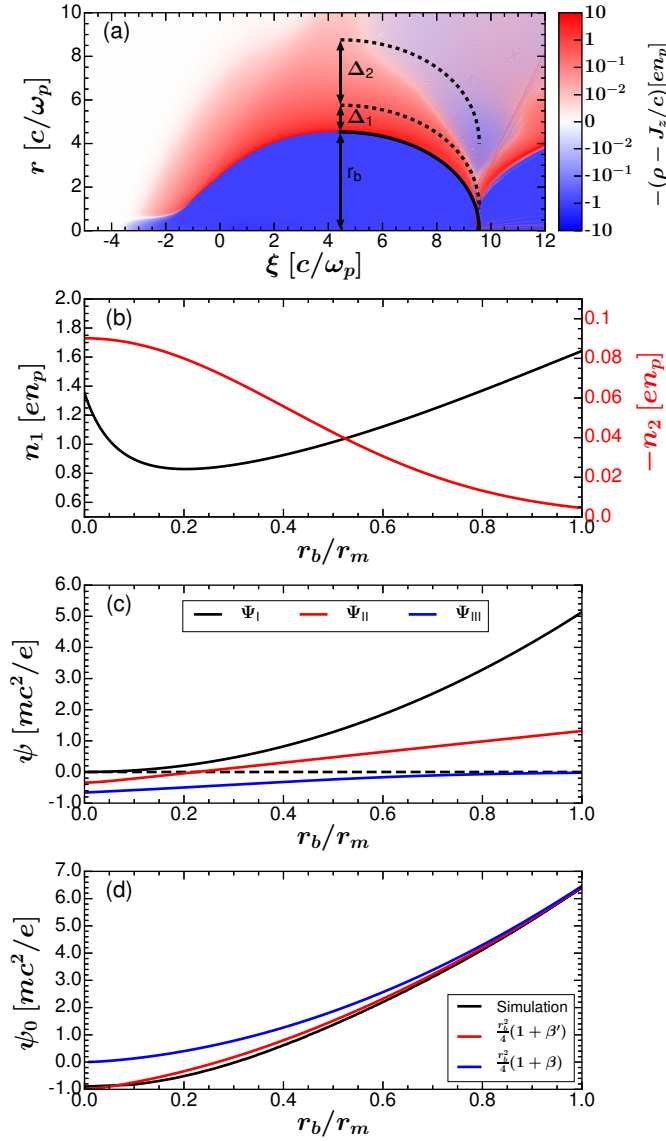


Figure 3.3: (a) Contour plot of source term S for a driver with $\Lambda_d = 6$, $\gamma_b = 20000$, $k_p\sigma_z = 1$, and $k_p\xi_c = 0$. The maximum bubble radius is $k_pr_m \simeq 4.53$. The bubble radius r_b [Eq. (3.12)] is calculated using the multi-sheath model β' with sheath widths $\Delta_1 = \Delta_{10} + \Delta_s = 1 + 0.05r_b$ and $\Delta_2 = \Delta_{20} = 3$ annotated in black. We use $n_2 = n_{20}e^{-sr_b^2/r_m^2}$ where $s = 3$ and n_{20} is calculated from Eq. (3.10) using $\psi_{min} = -1$. (b) n_1 [Eq. (3.5)] (black) and n_2 (red), (c) Ψ_I (black), Ψ_{II} (red), Ψ_{III} (blue) from Eq. (3.6), and (d) ψ_0 [Eq. (3.7)] (red) and simulation data (black) plotted as a function of r_b/r_m . The wake potential from the single-sheath model [35, 36] (blue) is obtained using Eq. (3.7) with $n_{20} = 0$ [Eq. (3.10)] and $\psi_{min} = 0$.

component, Ψ_{II} is less negative, and Ψ_I approaches zero since $r_b \rightarrow 0$.

Whereas in Refs. 35 and 36 the sheath potential term Ψ_{II} was modeled as positive definite, it can now flip sign because n_1 remains finite at the rear of the wake instead of going to zero. Combining all three terms, we observe strong agreement between ψ_0 calculated from Eq. (3.6) (red) and the on-axis wake potential obtained from the simulation data (black) in Fig. 3.3(d). It is also worth noting that we can recover the single-sheath model, which assumed that $S \geq 0$ outside of the ion bubble $r > r_b(\xi)$, by setting $n_2 = 0$ everywhere. Under this assumption, n_1 becomes $n_\Delta = \frac{1}{(1+\alpha_1)^2-1}$ and β' becomes $\beta = \frac{(1+\alpha_1)^2 \ln(1+\alpha_1)^2}{(1+\alpha_1)^2-1} - 1$ [35, 36]. It is clear from Eq. (3.10) that $\psi_{min} = 0$ for such a model. We plot ψ_0 (blue) in Fig. 3.3(d) obtained by reintegrating Eq. (3.12) to obtain r_b with $n_{20} = 0$ and all other parameters kept the same. It is clear that the result from the single-sheath model begins to deviate from the simulation results for $r_b/r_m \lesssim 0.7$. This shows that although the single-sheath model is reasonable for such values, ψ_0 still deviates because it is connecting to an incorrect value for $r_b \rightarrow 0$.

In Fig. 3.4(a), we plot numerical calculations of the bubble trajectory $r_b(\xi)$ using the multi-sheath model β' with $\psi_{min} = -1$ (red) and single-sheath model β with $\psi_{min} = 0$ (blue) along with $r_b(\xi)$ obtained from PIC simulation results (black). It can be readily seen that the addition of a second plasma sheath acts to bend the electron trajectories toward the axis sooner, thus shortening the predicted wavelength. As a result, the multi-sheath model demonstrates improved agreement with the simulation results over the single-sheath model. The progressively more negative slope $\frac{dr_b}{d\xi}$ observed in the simulation results is due to the fact that sheath electrons copropagate with the wake, i.e., $v_z \sim 1$, as they approach the axis where $\psi_{min} \approx -1$. Thus, they exhibit virtually no phase slippage $d\xi = (1 - v_z)dt \approx 0$ for a given change in bubble radius dr_b in this region. In fact, we can show mathematically why this also occurs in the multi-sheath model by rewriting $\frac{dr_b}{d\xi}$. Using the constant of motion $\gamma - P_z = 1 + \psi$ [62], we find that

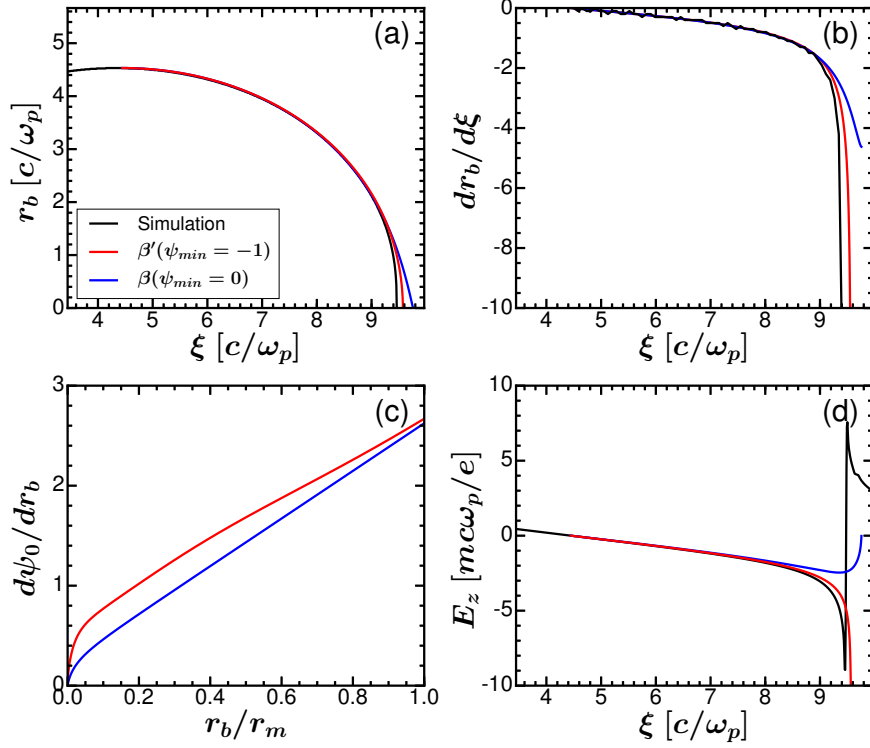


Figure 3.4: Comparisons of (a) r_b , (b) $\frac{dr_b}{d\xi}$, (c) $\frac{d\psi_0}{dr_b}$ and (d) E_z obtained from PIC simulation results (black) and numerical calculations (red, blue) for the wakefield shown in Fig. (3.3). Integration parameters are $\Delta_1 = 1 + 0.05r_b$, $\Delta_2 = 3$, and $n_2 = n_{20}e^{-sr_b^2/r_m^2}$, where $s = 3$ and n_{20} is calculated from Eq. (3.10) using multi-sheath model β' with $\psi_{min} = -1$ (red) and single-sheath model β with $\psi_{min} = 0$ (blue).

$$\frac{dr_b}{d\xi} = \frac{dr_b/dt}{d\xi/dt} = \frac{v_\perp}{1 - v_z} = \frac{P_\perp}{1 + \psi}. \quad (3.15)$$

For the innermost sheath electrons returning back the axis, the numerator $P_\perp$ is known to be negative and finite. However, the denominator will depend on what kind of model is used for ψ . For the single-sheath model wherein $\psi_{min} = 0$, the slope of the trajectory is limited by $\frac{dr_b}{d\xi} \approx P_\perp$ near the axis. However, for the multi-sheath model which employs $\psi_{min} = -1$, the slope $\frac{dr_b}{d\xi}$ approaches $-\infty$ near the axis where the denominator $(1 + \psi)$ approaches zero.

As shown in Fig. 3.4(b), the asymptotic behavior of $\frac{dr_b}{d\xi}$ predicted by the multi-sheath

model (red) at the rear of the wake is also borne out in the PIC simulation results (black) where the observed minimum wake potential is close to -1 . This is an important point because the derived expression for the electric field $E_z(\xi)$ [Eq. (3.13)] not only depends on r_b but also on the slope of the trajectory $\frac{dr_b}{d\xi}$. Since both the multi-sheath and single-sheath models for $\psi(r_b)$ are functions of only r_b , we can express the electric field as $E_z = \frac{d\psi_0}{dr_b} \frac{dr_b}{d\xi}$. As can be seen in Fig. 3.4(c), the slope of the wake potential $\frac{d\psi_0}{dr_b}$ is larger for the multi-sheath model across all r_b than it is for the single-sheath model due to the larger peak to trough amplitude of ψ_0 when using $\psi_{min} = -1$ rather than $\psi_{min} = 0$. It is this term $\frac{d\psi_0}{dr_b}$ which is initially responsible for the more negative electric fields obtained using the multi-sheath model in the range $8 \lesssim k_p \xi \lesssim 9$ seen in Fig. 3.3(d) where the slope of the trajectories $\frac{dr_b}{d\xi}$ are largely similar for both models and simulation results. At the rear of the wake where $r_b/r_m \ll 1$, the slope of the potential $\frac{d\psi_0}{dr_b}$ is small and approaches zero near the axis for both single and multi-sheath models. Since $\frac{dr_b}{d\xi}$ is finite for the single-sheath model, the calculated electric field increases to zero at the rear of the wake $k_p \xi \gtrsim 9$ as $\frac{d\psi_0}{dr_b}$ decreases. In contrast, $\frac{dr_b}{d\xi}$ approaches $-\infty$ for the multi-sheath model near the axis resulting in the characteristic negative spike in the electric field at the rear of the wake observed in PIC simulations results. Therefore, for highly nonlinear plasma wakes, the multi-sheath model employing negative ψ_{min} is needed to predict the electric fields at the back of the bubble, which is a region of interest for accelerating self-injected and trailing bunches.

3.3 Comparisons of Plasma Wakefield Theory and Simulations

In the work of Tzoufras et al. [38, 39], it was shown that beam loading in nonlinear plasma wakes can be viewed as a modification to the trajectory of $r_b(\xi)$ due to the presence of a witness beam with a normalized charge per unit length $\lambda(\xi)$. Implicit in such an analysis is the assumption that the theory of Lu et al. [35, 36] provides a reasonable prediction for $r_b(\xi)$ (and hence the fields) due to the drive beam. However, the single-sheath model used by Lu

et al. does not do as well in the second half of the bubble particularly where a witness beam would be loaded into it.

In this section, we examine the predictions of Eqs. (3.12)-(3.14) for witness beams with specified $\lambda(\xi)$ and compare the numerical results with simulation results obtained using the PIC code OSIRIS [84] for various examples of nonlinear plasma wakefields in the blowout regime. We also show how the multi-sheath model improves upon previous results by comparing it to the single-sheath model from Refs. 35 and 36. The purpose of these comparisons is to show that the new multi-sheath model can be used to accurately predict the wake potential ψ and electric field E_z at the rear of an unloaded plasma wake and in a loaded wake with a known trailing bunch profile $\lambda(\xi)$. In a subsequent section, we discuss how to use the multi-sheath model to determine a profile $\lambda(\xi)$ of a witness beam that leads to a desired loaded wakefield $E_z(\xi)$ and compare the results to those in Tzoufras et al. [38, 39].

3.3.1 Unloaded Plasma Wakes

We first examine several cases where an electron drive bunch is used to excite an unloaded plasma wake. As mentioned previously, we are interested in plasma wakes where $r_m \gtrsim 3$. For a bi-Gaussian driver with $k_p \sigma_z \sim 1$, this corresponds to $\Lambda_d \gtrsim 2$ since $r_m \approx 2\sqrt{\Lambda_d}$ [35, 36]. For these parameters, sheath electrons that trace $r_b(\xi)$ can be accelerated to high velocities $v_z \sim 1$ as they approach the ξ -axis where ψ_{min} can be well-approximated by -1 [52, 59]. This regime is important because most injection schemes rely on accelerating sheath electrons into the plasma wake at the back of the bubble by temporarily decreasing the phase velocity of the wake. Once injected, these electrons can be accelerated to GeV energies with ultra-high gradients. This region is also interesting because the accelerating fields E_z and transformer ratios are largest for electron bunches at the rear of the wake. However, in order to model the effects of beam loading in this region, we must first be able to capture the behavior of the plasma wake in the absence of any externally injected or trailing bunch.

In Fig. 3.5, we plot the numerical calculations of the bubble trajectory $r_b(\xi)$, potential

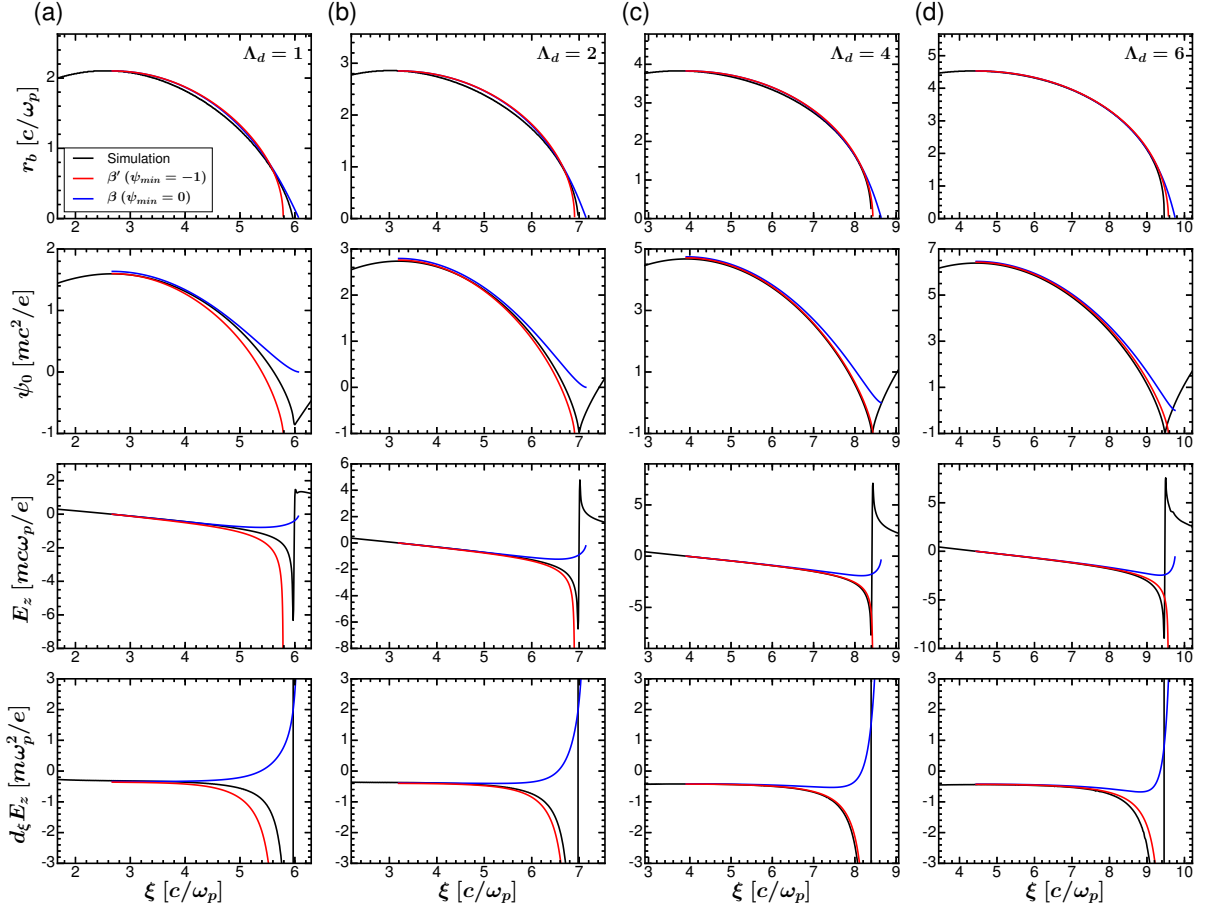


Figure 3.5: Comparisons of simulations (black) and numerical calculations (red, blue) of r_b , ψ_0 , E_z , and $d_\xi E_z$ using Eqs. (3.7)-(3.10) for plasma wakes excited by bi-Gaussian electron drivers. The peak charge per unit length Λ_d is 1, 2, 4 and 6 and the blowout radius $k_p r_m$ is 2.10, 2.85, 3.83 and 4.53 in figures (a)-(d), respectively. Nonevolving electron drivers with energy $\gamma_b = 20000$, centroid $k_p \xi_c = 0$, $k_p \sigma_z = 1$, and $k_p \sigma_r = \frac{1}{10} \sqrt{\Lambda_d}$ were used. The integration parameters $\Delta_{10} = 1$, $\Delta_s = 0.05 r_b$, $\Delta_{20} = 3$ and $s = 3$ are used for all numerical calculations. The multi-sheath model β' (red) calculates n_{20} [Eq. (3.10)] using $\psi_{min} = -1$ and single-sheath model β [35, 36] (blue) uses $\psi_{min} = 0$ and $n_2 = 0$ everywhere. n_1 is determined from Eq. (3.5).

$\psi_0(\xi)$, electric field $E_z(\xi)$, and electric field slope $d_\xi E_z(\xi)$ from Eqs. (3.7)-(3.10) using the multi-sheath model β' (red) and single-sheath model β (blue) together with the cylindrical PIC simulations results (black curve) for electron drivers with different Λ_d ranging from 1 to 6. The PIC simulations used high resolution grids with $\Delta z = \Delta r = \frac{1}{128} \frac{c}{\omega_p}$, $\Delta t = \frac{1}{512} \frac{1}{\omega_p}$ and 4 macro-particles per cell. Nonevolving drivers were used with $\gamma_b = 20000$, $k_p \sigma_z = 1$, and $k_p \sigma_r = \frac{\sqrt{\Lambda_d}}{10} \approx \frac{r_m}{20}$. For each case, the “simulation” results plotted in Fig. 3.5 are generated from PIC grid diagnostics after the driver has propagated $500 c/\omega_p$ into the plasma at which point the wake response has reached steady-state. The simulated bubble radius $r_b(\xi)$ is determined by the contour of the source term $S = -1$ using PIC diagnostics for ρ and J_z . The simulated electric field $E_z(\xi)$ and wake potential $\psi_0(\xi)$ are also obtained directly from PIC diagnostics. The multi-sheath model results are obtained using the same profiles $\Delta_1 = 1 + 0.05r_b$, $\Delta_2 = 3$, $n_2 = n_{20}e^{-sr_b^2/r_m^2}$ and $s = 3$ for all calculations. The multi-sheath model calculates n_{20} [Eq. (3.10)] using $\psi_{min} = -1$ while single-sheath model uses $n_2 = 0$ everywhere and, therefore, $\psi_{min} = 0$. The multi-sheath model is integrated starting at the PIC simulation grid point where the axial electric field E_z is closest to zero in the bubble and bubble radius is closest to its maximum. Therefore, the values for $k_p r_m$ and $k_p \xi$ used by the multi-sheath model are accurate to within a grid cell spacing, i.e., 7.8×10^{-3} .

In each case, strong agreement is observed between the calculated bubble radius $r_b(\xi)$ and the simulation results along regions where r_b is close to the maximum blowout radius r_m and where n_2 can be neglected due to its exponential profile. It is only at the rear of the wake that the trajectories $r_b(\xi)$ of the single-sheath and multi-sheath models begin to deviate due to inclusion of the negative source term n_2 which allows for $\psi_{min} = -1$.

As noted previously, the negative wake potential near the axis employed by the multi-sheath model and observed in PIC simulations is responsible for the bubble trajectories bending back to the axis with large negative slopes $\frac{dr_b}{d\xi}$ from Eq. (3.15) and hence large negative values of $E_z(\xi) = \frac{d\psi}{d\xi}$. This leads to the multi-sheath model providing better agreement with the simulation results at the rear of the wake when compared to the single-sheath

model. Although not shown for $\Lambda_d = 1$, the multi-sheath still works well if a less negative value for ψ_{min} is used. From simulation results, it can be seen that $\psi_{min} \approx -0.85$ for $\Lambda_d = 1$. Therefore, for these smaller amplitude wakes, the slope of the trajectory $\frac{dr_b}{d\xi}$ from the PIC simulation does not bend as much as predicted by Eq. (3.15) using the multi-sheath model near the axis (with $\psi_{min} = -1$). Thus, it is possible to improve the results by tailoring ψ_{min} from PIC simulation data for drivers with $\Lambda_d \lesssim 1$.

By construction, the on-axis wake potentials $\psi_0(\xi)$ differ at the very rear of the wake. Both sheath models predict nearly identical peak potentials $\psi_0(r_m)$, i.e., $\beta(r_m) \approx \beta'(r_m)$. However, the values of $\psi_0(r_b)$ differ between the two sheath models for $r_b \lesssim 0.7r_m$. Since the multi-sheath model covers a larger range of potentials from peak, $\psi_0(r_m)$, to trough, $\psi_{min} \approx -1$, it also exhibits larger $\frac{d\psi_0}{dr_b}$ at all r_b when compared to the single-sheath model. The difference between the two models is more pronounced at lower Λ_d since the peak potential scales roughly with the blowout radius squared $\psi_0(r_m) \sim r_m^2$ from Eq. (3.7) while the minimum wake potentials connect to $\psi_{min} = 0$ for the single-sheath model and $\psi_{min} = -1$ for the multi-sheath model.

In each case, the multi-sheath model produces a monotonically decreasing electric field with a characteristic negative spike near the axis, which is also borne out in PIC simulation results. However, this characteristic spike is absent in the single-sheath model, wherein the electric field actually increases at the rear of the wake in every case. This is also noted in the positive electric field slope $d_\xi E_z$ predicted by the single-sheath model at the rear of the wake. By comparison, the multi-sheath model and simulation results indicate that $d_\xi E_z$ should remain negative and monotonically decreasing until the innermost electrons reach the ξ -axis.

3.3.2 Gaussian Trailing Bunches

We now present several cases in which short bi-Gaussian trailing bunches are placed at the rear of the plasma wakefields shown in Fig. 3.5. The goal is to show that the multi-sheath

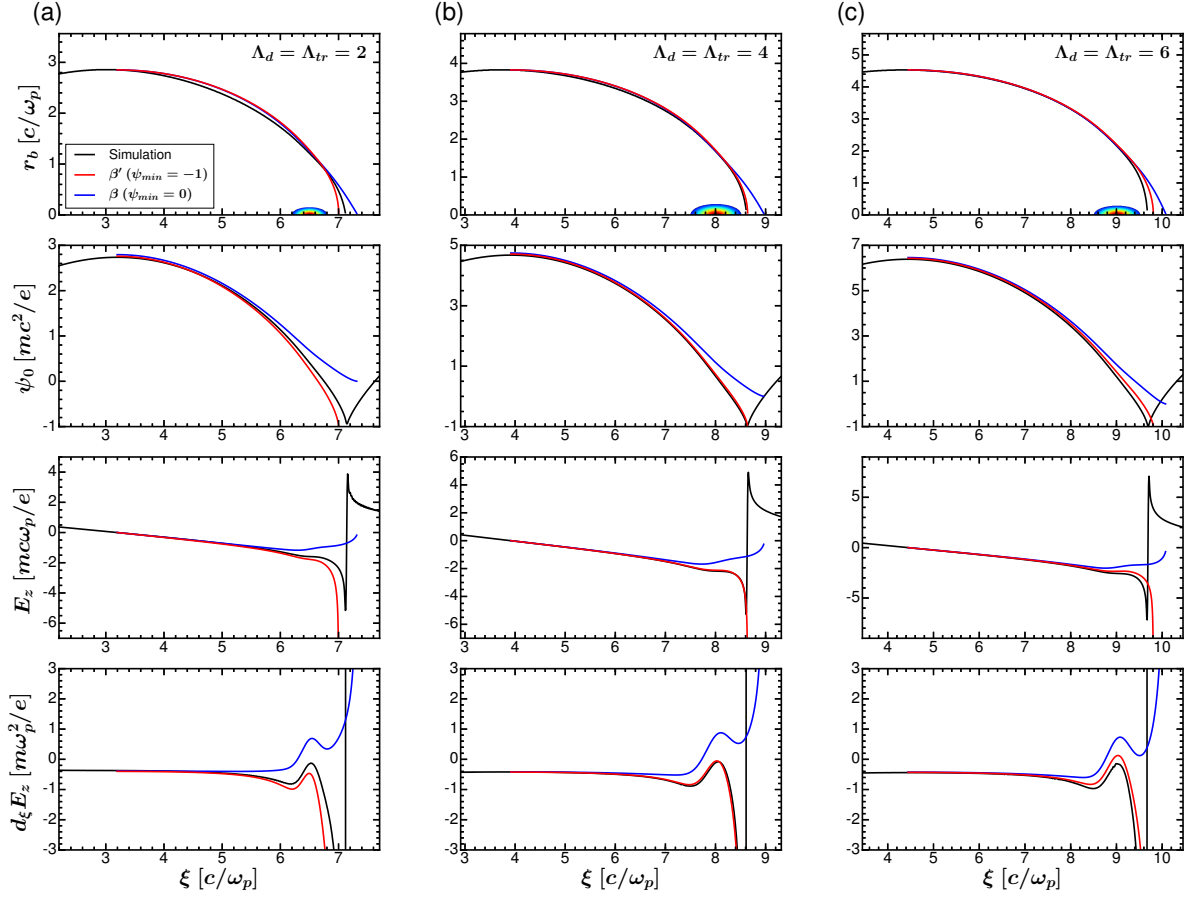


Figure 3.6: Comparisons of simulations (black) and numerical calculations (red, blue) of r_b , ψ_0 , E_z , and $d_\xi E_z$ using Eqs. (3.7)-(3.10) with bi-Gaussian drive and trailing bunches. The drivers used are identical to those described in Fig. 3.5 for each Λ_d . The trailing bunch parameters are $k_p \xi_{c2} = 6.5, 8, 9$, $k_p \sigma_{r2} = 0.08, 0.14, 0.14$ and $k_p \sigma_{z2} = 0.15, 0.25, 0.25$ in figures (a)-(c), respectively. All bunches are nonevolving with $\gamma_b = 20000$ and the peak charge per unit length of the driver Λ_d and trailing bunch Λ_{tr} are the same in each case. The integration parameters $\{\Delta_{10}, \Delta_s, \Delta_{20}, s\}$ are the same as those used in Fig. 3.5. The multi-sheath model β' (red) calculates n_{20} [Eq. (3.10)] using $\psi_{min} = -1$ and the single-sheath model β (blue) uses $\psi_{min} = 0$ and $n_2 = 0$ everywhere. For all calculations, n_1 is determined from Eq. (3.5).

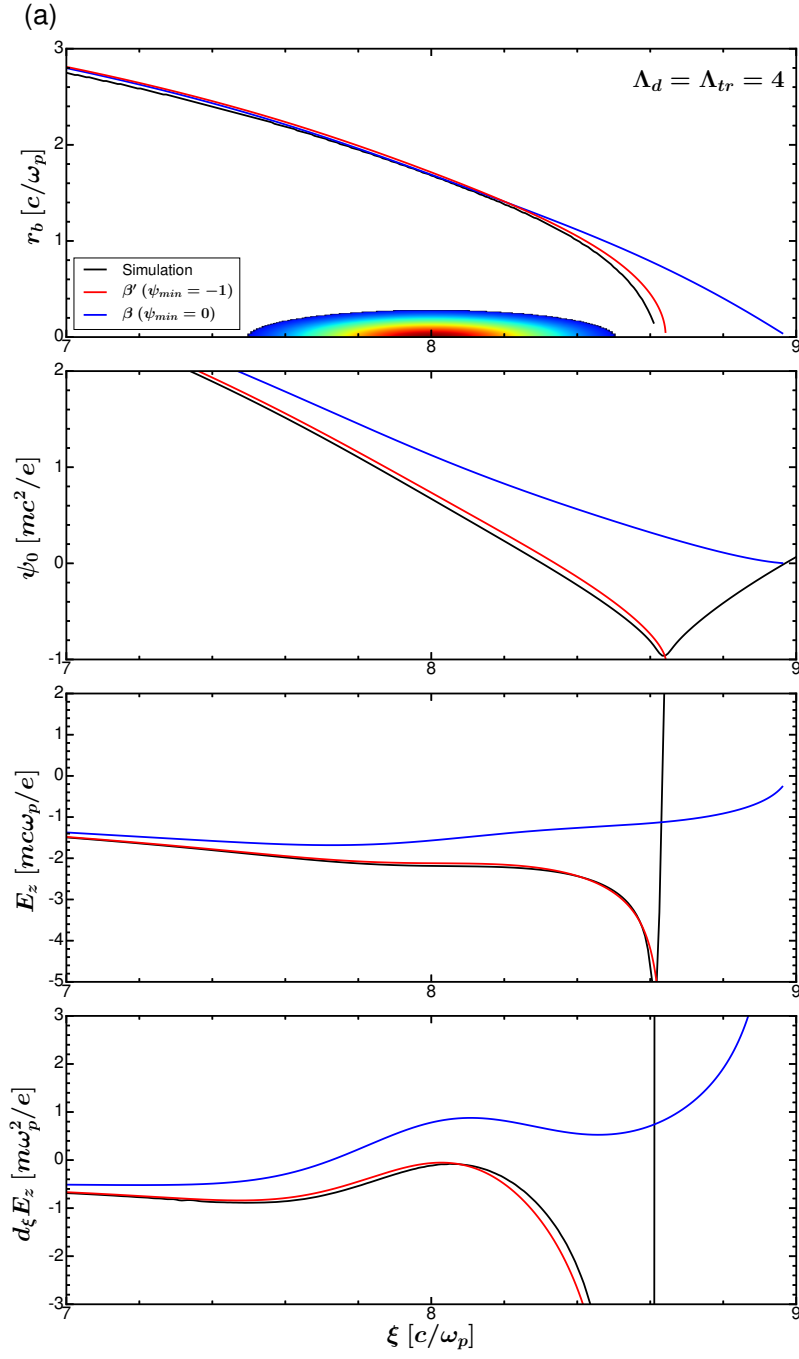


Figure 3.7: Zoomed in version of Fig. 3.6(b).

model provides accurate predictions for beam loading including regions where the wake potential ψ is negative. In Fig. 3.6, we examine several examples in which trailing bunches were added at the back of the same ion channels with centroids located at $k_p \xi_{c2} = 6.5, 8, 9$ and bunch lengths $k_p \sigma_{z2} = 0.15, 0.15, 0.2$. The density profile contours of the narrow bunches are also shown in plots of $r_b(\xi)$ (top row of Fig. 3.6). In each case, non-evolving drive and trailing bunches with the same energy $\gamma_b = 20000$ and peak charge per unit length $\Lambda_{tr} = \Lambda_d$ are used. We note that the two-bunch configurations required longer propagation distances of $1000 c/\omega_p$ in the PIC simulations to obtain steady-state wakefields. The multi-sheath model using $\psi_{min} = -1$ is shown in red while the single-sheath model using $\psi_{min} = 0$ is shown in blue. It is clear that the wake potentials $\psi_0(\xi)$ and electric fields $E_z(\xi)$ of the two models diverge at the back of the wake. In every case, the single-sheath model overestimates the electric field in regions where the beam load is present. In contrast, the multi-sheath model accurately captures the behavior of the nearly constant electric field in the center of each beam and exhibits strong agreement with the simulation results. The difference between the two models is also illustrated in plots of the electric field slope $d_\xi E_z$. To further highlight the differences between the multi-sheath model and single-sheath model, we plot in Fig. 3.7 the results from Fig. 3.6(b) zoomed in at the rear of the wakefield.

3.3.3 Phenomenological Parameter Optimization for Beam Loading

In the results presented so far, we have shown that the multi-sheath model reproduces the qualitative plasma wake features in a various cases using a fixed set of phenomenological parameters $\{\Delta_{10} = 1, \Delta_s = 0.05r_b, \Delta_{20} = 3, s = 3, \psi_{min} = -1\}$. In some instances, however, there can be a slight mismatch between the length of the wake predicted by the multi-sheath model and observed in PIC simulations. The underlying reason is that the expression for ψ employed by the multi-sheath model does not perfectly match the empirical wake potential. Therefore, the trajectory [Eq. (3.12)] can slightly undershoot or overshoot the PIC simulation results.

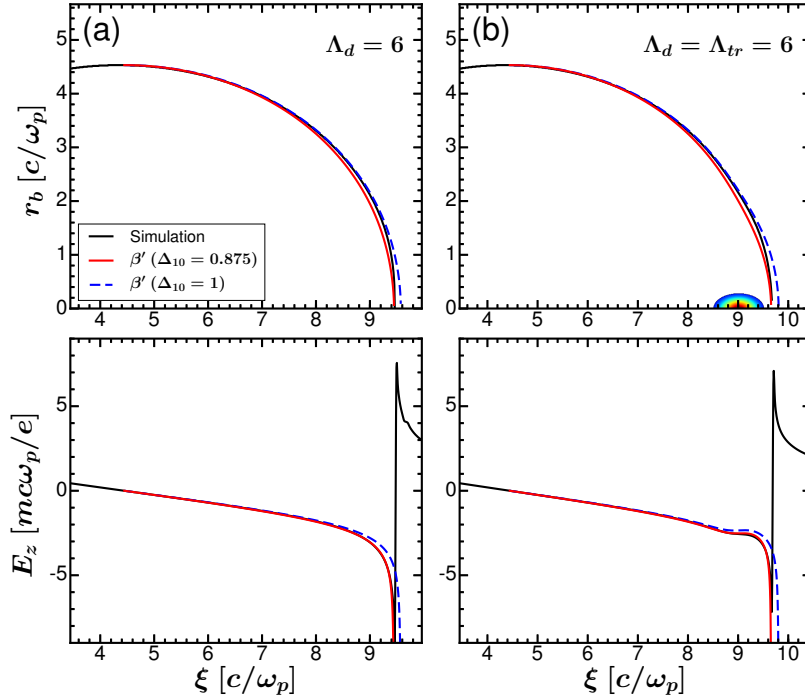


Figure 3.8: Comparisons of simulations (black) and numerical calculations (red, blue) of r_b (top) and E_z (bottom) using Eqs. (3.7)-(3.10) for an (a) unloaded and (b) loaded plasma wake. The driver and witness beam parameters are the same as those used for Figs. 3.5(d) and 3.6(c). Results are shown using the multi-sheath model β' with $\Delta_{10} = 0.875$ (red) and $\Delta_{10} = 1$ (dashed blue). The parameters $\{\Delta_s = 0.05r_b, \Delta_{20} = 3, s = 3, \text{ and } \psi_{min} = -1\}$ are the same as those used in Figs. 3.5 and 3.6.

In Fig. 3.8(a), we show how the trajectory obtained using the multi-sheath model can be adjusted by tuning the parameter Δ_{10} for the plasma wake shown in Fig. 3.5(d). Using $\Delta_{10} = 1$ (blue), the multi-sheath model overestimates the the plasma wake length and, therefore, the negative spike in the electric field occurs at a larger ξ when compared to the simulations results. This disagreement can be addressed by reducing the first sheath width Δ_{10} to decrease the length of the wake for improved numerical results. By using $\Delta_{10} = 0.875$ (solid red), it can be seen that the calculated trajectory now crosses the ξ -axis sooner resulting in improved agreement with the simulated bubble length. As a result, the

calculated electric field exhibits nearly perfect agreement with the PIC simulation results at the rear of the wake. While Δ_{10} was lowered to reduce the wake length in this example, it is worth noting that higher values of Δ_{10} can be used to increase the wake length in other cases. Once the parameter Δ_{10} is optimized for a particular driver, it can be used for any beam loading calculations involving trailing bunches. In Fig. 3.8(b), we show how beam loading results are improved by using the optimized Δ_{10} with a Gaussian trailing bunch. It can be readily seen that numerical results using $\Delta_{10} = 0.875$ provide better agreement with the simulation results for the loaded wakefield and trajectory crossing with the ξ -axis.

3.4 Designing Beam Loads for Nonlinear Plasma Wakes

In the previous section, we showed that the multi-sheath model accurately predicts the bubble trajectories and fields in the second half of unloaded and loaded plasma wakes in the nonlinear blowout regime. We considered situations where the current profile of the trailing bunch $\lambda(\xi)$ was specified beforehand. In this section, we show how to design a beam load $\lambda(\xi)$ using Eqs. (3.10)-(3.12) to produce a specified plasma wakefield $E_z(\xi) = f(\xi)$ for the axial wake potential $\psi_0 = (1 + \beta')r_b^2/4$. The beam profiles designed using the multi-sheath model will be validated against PIC simulations using OSIRIS [84]. Simulation results using the multi-sheath model are compared to results obtained from Tzoufras et al. [38, 39]. In the subsequent section, we discuss the differences between this model and Refs. 38 and 39.

3.4.1 Exact Solutions for Loading Arbitrary Wakefields

We consider here a general methodology for loading wakefields of arbitrary profiles $E_z(\xi_t \leq \xi \leq \xi_f) = f(\xi)$ when the current profile of the bunch has a well-defined beginning at $\xi = \xi_t$ and end at $\xi = \xi_f$. The current $\lambda(\xi)$ profile required to produce the specified wakefield $f(\xi)$ can be obtained using a simple two-step process. In the first step, the unloaded bubble trajectory $r_b(\xi)$ is calculated by integrating Eq. (3.12) starting from $r_b = r_m$. Once $r_b(\xi)$ is

obtained, the unloaded wakefield $E_z(\xi)$ is determined from Eq. (3.13). In the second step, the order of operations is reversed. Since the desired loaded wakefield $E_z(\xi_t \leq \xi \leq \xi_f) = f(\xi)$ is known, the loaded bubble trajectory $\tilde{r}_b(\xi)$ can be reversed engineered from Eq. (3.13) by numerically integrating the following

$$\frac{d\tilde{r}_b}{d\xi} = \frac{f(\xi)}{D'(\tilde{r}_b)\tilde{r}_b} \quad (3.16)$$

from $\tilde{r}_b(\xi_t) = r_b(\xi_t) \equiv r_t$ to either $\tilde{r}_b(\xi_f)$ or $\tilde{r}_b = 0$, whichever comes first. The function $f(\xi)$ is constrained by boundary conditions at $\xi = \xi_t$, which require wakefield continuity $f(\xi_t) = \lim_{\epsilon \rightarrow 0^-} E_z(\xi_t + \epsilon) \equiv -E_t$. Once $\tilde{r}_b(\xi)$ is calculated, the corresponding wake potential $\psi_0(\tilde{r}_b) = (1 + \beta'(\tilde{r}_b))\tilde{r}_b^2/4$ can be determined. Although Eqs. (3.12)-(3.14) were derived to solve for $r_b(\xi)$ given $\lambda(\xi)$, conversely they can instead be used to solve for $\lambda(\xi)$ for a given trajectory $\tilde{r}_b(\xi)$. Expressing the derivatives of $\tilde{r}_b(\xi)$ in terms of $f(\xi)$ and $\frac{df}{d\xi}$ using Eqs. (3.13)-(3.14), we can rewrite Eq. (3.12) as

$$\lambda(\xi) = C'\tilde{r}_b^2 + \left(\frac{B'}{D'^2} - \frac{A'F'}{D'^3\tilde{r}_b^2} \right) f(\xi)^2 + \left(\frac{A'}{D'} \right) \frac{df(\xi)}{d\xi}. \quad (3.17)$$

$f(\xi)$ is the desired loaded electric field and $A'(\tilde{r}_b)$, $B'(\tilde{r}_b)$, $C'(\tilde{r}_b)$, $D'(\tilde{r}_b)$, and $F'(\tilde{r}_b)$ are specified in Sec. 3.2. Since the left-hand side must be positive definite ($\lambda(\xi) \geq 0$) for an electron bunch, the wakefield slope $d_\xi f$ is naturally constrained by Eq. (3.17). Physically, this means that the slope of the loaded wakefield $f(\xi)$ cannot be more negative than the corresponding slope in the absence of any load. In cases where the slope $d_\xi f$ is sufficiently negative, the current profile calculated from Eq. (3.17) would flip sign ($\lambda(\xi) < 0$) which would require positive charge densities, e.g., positrons, along portions of the beam load which cannot be focused. Although we have not discussed the transverse force, we note that for azimuthally symmetric wakes the focusing force remains perfectly linear even for loaded wakefields.

3.4.2 Ultrarelativistic Blowout Regime

In the ultrarelativistic limit, where the bubble radius is large $r_b \gg 1$, the sheath terms are small, i.e., $\beta' \ll 1$ and $\beta' r_b^2/4 \gtrsim 1$, relative to the ion term $r_b^2/4 \gg 1$. Therefore, Eqs. (3.16) and (3.17) can be approximated by the leading terms of the coefficients $[A'(r_b), B'(r_b), C'(r_b), D'(r_b), F'(r_b), \beta'(r_b)] \rightarrow (r_b^2/4, 1/2, 1/4, 1/2, 1/2, 0)$. In this limit, the loaded bubble trajectory can be expressed as

$$\begin{aligned} \frac{d\tilde{r}_b}{d\xi} &= \frac{f(\xi)}{\tilde{r}_b/2} \\ \tilde{r}_b(\xi)^2 &= r_t^2 + 4 \int_{\xi_t}^{\xi} f(\xi) d\xi. \end{aligned} \quad (3.18)$$

In the ultrarelativistic limit, the corresponding current profile of the beam load from Eq. (3.17) becomes

$$\lambda(\xi) = \frac{\tilde{r}_b^2}{4} + f(\xi)^2 + \left(\frac{\tilde{r}_b^2}{2} \right) \frac{df(\xi)}{d\xi}. \quad (3.19)$$

In the absence of a beam load ($\lambda(\xi) = 0$), it can be seen from Eq. (3.19) that the slope of the electric field inside the bubble $df/d\xi = dE_z/d\xi \simeq -1/2$ when the bubble radius is maximum $r_b = r_m$ and $f = E_z \simeq \frac{1}{2} r_b \frac{dr_b}{d\xi} \simeq 0$. As noted in the previous section, the slope of the loaded wakefield is naturally constrained by Eq. (3.19). For an electron (or anti-proton) beam $\lambda \geq 0$,

$$\frac{\tilde{r}_b^2}{4} + f(\xi)^2 + \left(\frac{\tilde{r}_b^2}{2} \right) \frac{df(\xi)}{d\xi} \geq 0 \quad (3.20)$$

from which it follows,

$$\frac{df(\xi)}{d\xi} \geq -\frac{1}{2} \left(1 + \frac{4f(\xi)^2}{\tilde{r}_b^2} \right). \quad (3.21)$$

Eqs. (3.18) and (3.19) can be solved analytically for many functions $f(\xi)$. However, any such functions $f(\xi)$ must satisfy Eq. (3.21) for all $\xi \in [\xi_t, \xi_f]$ along the beam load and the continuity constraint at the head of the bunch $f(\xi_t) = -E_t$. In Refs. 38 and 39, analytic solutions to Eqs. (3.18) and (3.19) were derived for a beam load designed to produce a constant wakefield $f(\xi \geq \xi_t) = -E_t$ and $df/d\xi = 0$ extending from the head of the bunch ξ_t all the way to the rear of the wake $\tilde{r}_b(\xi_f) = 0$. Such a wakefield can be used to accelerate a trailing bunch to multi-GeV energies while maintaining the kinds of low energy spreads needed for next-generation linear collider and XFEL applications.

For a constant wakefield $df/d\xi = 0$, it is trivial to show that Eq. (3.21) is always satisfied and the solution to Eq. (3.18) is a parabola $\tilde{r}_b^2 = r_t^2 - 4E_t(\xi - \xi_t)$. It follows directly from the loaded bubble trajectory that the maximum length of the beam load $\Delta\xi_{tr} \equiv \xi_f - \xi_t = \frac{r_t^2}{4E_t}$ is limited by length of the bubble $\tilde{r}_b(\xi_f) = 0$. Substituting the loaded trajectory $\tilde{r}_b$ into Eq. (3.19), the underlying current profile is given by

$$\lambda(\xi) = E_t^2 + \frac{\tilde{r}_b(\xi)^2}{4} = E_t^2 + \frac{r_t^2}{4} - E_t(\xi - \xi_t). \quad (3.22)$$

In Refs. 38 and 39, it was shown that this trapezoidal current profile could be written as

$$\lambda(\xi) = \sqrt{E_t^4 + \frac{r_m^4}{2^4}} - E_t(\xi - \xi_t) \quad (3.23)$$

by solving for $r_t^2/4 = \sqrt{E_t^4 + r_m^4/2^4} - E_t^2$ in terms of E_t and r_m in the ultrarelativistic limit. In the following sections, we will compare the analytic result [Eq. (3.23)] derived by Tzoufras et al. [38, 39] in the ultrarelativistic limit ($\beta, \beta' \rightarrow 0, 0$) with the exact beam profiles obtained by numerically integrating Eq. (3.16) for both the multi-sheath β' and single-sheath β models.

3.4.3 Comparisons of Theory and Simulation Results for Loading Constant Wakefields

We next use the methodology outlined above to design beam loads that produce constant electric fields extending to the very rear of the bubble. Exact profiles will be calculated numerically from Eqs. (3.16) and (3.17) for the multi-sheath β' and single-sheath models β . We also present results for beam loads calculating using the analytic theory [Eq. (3.23)] in the relativistic limit ($\beta, \beta' \rightarrow 0, 0$). Beam profiles obtained for each model will be simulated using the PIC code OSIRIS [84]. Finally, we will present examples of loading longitudinally varying electric fields using the multi-sheath model and compare the numerical results to PIC simulations.

We use $\Delta_{20} = 3$, $n_2 = n_{20}e^{-sr_b^2/r_m^2}$, and $s = 3$ when numerically integrating Eqs. (3.16) and (3.17). For each case, Δ_{10} is first optimized for the multi-sheath model using the unloaded plasma wake. For the multi-sheath model, n_{20} is determined from Eq. (3.10), which depends on the ψ_{min} used. For the single-sheath model, $n_{20} = 0$ and, therefore, $n_2 = 0$ everywhere. Reasonable estimates can be obtained for a large parameter space if $\Delta_{10} = 1$.

In Fig. 3.9, we compare results for an electron drive bunch with parameters $\Lambda_d = 6$, $\gamma_b = 20000$, $k_p\sigma_r = 0.245$, $k_p\sigma_z = 1$ and $k_p\xi_c = 0$. The maximum blowout radius is $r_m \simeq 4.53$. For this driver, we found that a value of $\Delta_{10} = 0.875$ gave the best results (see Fig. 3.8). We use $\psi_{min} = -1$ to determine n_{20} . The unloaded bubble trajectory $r_b(\xi)$ (dashed red) obtained from integrating Eq. (3.12) is also shown in Fig. 3.9(a). We are interested in calculating the current profile of the beam load that can produce a constant wakefield starting at $\xi_t = 8.5$ using the multi-sheath model. To self-consistently solve for the loaded bubble trajectory $\tilde{r}_b$ using the multi-sheath model, we numerically integrate Eq. (3.16) using $r_t \simeq 2.73$ and $E_t \simeq 2.26$ from the unloaded calculations. The loaded trajectory $\tilde{r}_b$ obtained using the multi-sheath model is plotted (solid red) in Fig. 3.9(a) and exhibits strong agreement with the simulated wake trajectory produced by the underlying trailing bunch. The loaded trajectory

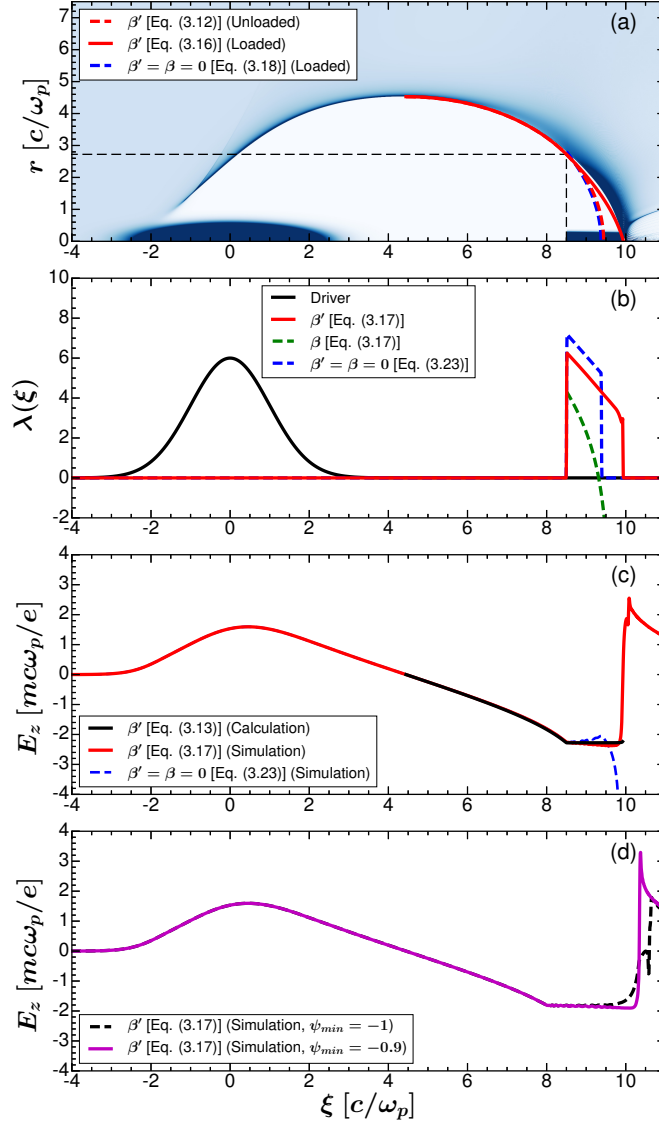


Figure 3.9: (a) Plasma wake excited by a driver $\{\Lambda_d = 6, k_p \sigma_r = 0.245, k_p \sigma_z = 1, k_p \xi_c = 0\}$ with a load [Eq. (3.17)] placed at $\xi_t = 8.5$ designed to produce a constant wakefield $E_t = 2.26$ using the multi-sheath model β' with $\psi_{min} = -1$. The unloaded (loaded) trajectory is shown in dashed (solid) red. (b) Predicted current profiles $\lambda(\xi)$ for the multi-sheath model β' (red), single-sheath model β (green) with $\psi_{min} = 0$, and analytic theory $\beta' = \beta = 0$ (blue). (c) Simulated $E_z(\xi)$ using the profiles shown in (b). $E_z(\xi)$ calculated from Eq. (3.13) using the β' current profile is in black. (d) Simulated $E_z(\xi)$ using multi-sheath model β' [Eq. (3.17)] to load the wake at $\xi_t = 8$, $E_t = 1.81$ with $\psi_{min} = -0.9$ (purple) and -1 (black).

in the ultrarelativistic limit $\tilde{r}_b(\xi)^2 = r_t^2 - 4E_t(\xi - \xi_t)$ is also plotted (dashed blue) using $r_t \simeq 2.81$ and $E_t \simeq 2.26$ from the PIC simulation data. It can be seen that the loaded parabolic trajectory (analytic result for ultra-relativistic regions of r_b) underestimates the length of the plasma wake as it crosses the axis much sooner than expected when compared to the loaded multi-sheath trajectory.

The current profile $\lambda(\xi)$ predicted by the multi-sheath model [Eq. (3.17)] is plotted (solid red) in Fig. 3.9(b). For comparison, we also plot the current profile $\lambda(\xi) \simeq 7.22 - 2.26 \times (\xi - 8.5)$ [Eq. (3.23)] obtained in the ultrarelativistic limit (dashed blue line). While the profiles are both trapezoidal, the multi-sheath model predicts a bunch length $\Delta\xi_{tr} \simeq 1.44$ which is $\sim 63\%$ longer than that of the analytic beam loading theory $\Delta\xi_{tr} = \frac{r_t^2}{4E_t} \simeq 0.88$. On the other hand, the analytic theory predicts larger currents along the load compared to the multi-sheath model. Integrating the current profiles, the total loaded charge predicted by the multi-sheath model $Q_{tr}/en_pk_p^{-3} \simeq 41.8$ (solid red) is $\sim 21\%$ higher than the loaded charge predicted by the analytic beam loading theory $Q_{tr}/en_pk_p^{-3} \simeq 34.5$ (dashed blue).

We also plot the beam profile calculated using the single-sheath model (dashed green) by integrating Eqs. (3.16)-(3.17) with $n_2 = 0$. For the single-sheath results, we self-consistently sample the bubble radius $r_t \simeq 2.7$ and electric field $E_t \simeq 2.05$ at $\xi_t = 8.5$ from the unloaded wake trajectory integrated from Eq. (3.12) using β instead of β' . The single-sheath model significantly underestimates the slice currents of the beam load at all longitudinal positions. The disagreement is largely attributed to the fact that the unloaded electric field predicted by the single-sheath model does not capture the characteristic negative spike observed in the multi-sheath model and simulation results in Figs. 3.4 and 3.5. Instead, the slope of the unloaded electric field $d_\xi E_z$ predicted by the single-sheath model flips signs from negative to positive at the rear of the wake. As a result, the current profile predicted by the single-sheath model also flips sign ($\lambda < 0$) at the rear of the wake which corresponds to positive charge densities, i.e. positrons, along portions of the beam load.

In Fig. 3.9(c), we show the electric fields from PIC simulations using the currents profiles

predicted by the multi-sheath model and analytic theory shown in Fig. 3.9(b). For reference, we also plot the expected electric field from the multi-sheath model using Eq. (3.13). The simulation results clearly show that the profile delineated by Eq. (3.23) (dashed blue) does not extend to the very rear of the wake. The simulation results using the multi-sheath model (black) produces a nearly constant electric field E_z over almost the entire length of the load extending all the way to the back of the bubble. The electric field in the simulation is in agreement with the expected field calculated using the multi-sheath model shown in red. The deviation between these two curves is small (the red curve has a slight negative slope) and can be attributed to the minimum wake potential ψ_{min} not being exactly equal to -1 .

In cases with longer loads $\Delta\xi_{tr} \gtrsim r_m/2$, higher values of ψ_{min} may be needed to correctly load the wake, since the loads themselves can modify the electron momenta at the back of the bubble and, therefore, alter the wake potential described by the constant of motion $1 + \psi = \gamma - P_z$ [62]. In Fig. 3.9(d), we show electric field simulation results using beam loads [Eq. (3.17)] designed to produce a constant wakefield $E_t \simeq 1.81$ starting at $\xi_t = 8$. The bubble radius is $r_t \simeq 3.31$ at ξ_t from the unloaded PIC simulation. Two different cases are presented for trailing beams where n_{20} is determined from Eq. (3.10) using $\psi_{min} = -0.9$ (solid purple) and $\psi_{min} = -1$ (dashed black). While the black curve increases at the back of the bubble ($9.5 < \xi < 10.5$), the purple curve remains flat over virtually the entire beam load. The difference can be attributed to the fact that the underlying beam profile used to load the wakefield in black is calculated using $\psi_{min} = -1$ which is more negative than the empirical value of $\psi_{min} \simeq -0.65$ observed in the PIC simulation with the beam load as predicted from the model. As a result, the current profile overestimates the length $\Delta\xi_{tr}$ over which the wake can be loaded as well as the ion channel radius r_b , thus leading to larger currents from Eq. (3.17).

This issue can be addressed by incrementally increasing the value of ψ_{min} used by the multi-sheath model until it matches the empirical ψ_{min} from the PIC simulation results with the beam load. For $\psi_{min} = -0.9$, we see that the plasma wakefield is nearly perfect

loaded over a reduced bunch length in purple. The value $\psi_{min} = -0.9$ is now in good agreement with the minimum wake potential $\psi_{min} \simeq -0.87$ found in the PIC simulation results in purple. Integrating the current profiles of the underlying beam loads, the total loaded charge $Q_{tr}/en_pk_p^{-3} \simeq 55.4$ obtained using the multi-sheath model with $\psi_{min} = -0.9$ is only marginally lower ($\sim 7.5\%$) than the loaded charge $Q_{tr}/en_pk_p^{-3} \simeq 59.9$ calculated when using $\psi_{min} = -1$.

As in the previous example, the bunch length $\Delta\xi_{tr} \simeq 2.31$ predicted by the multi-sheath model using $\psi_{min} = -0.9$ is $\sim 53\%$ longer than the optimal bunch length predicted by analytic beam loading theory $\Delta\xi_{tr} = \frac{r_t^2}{4E_t} \simeq 1.51$. As a result, the loaded charge $Q_{tr}/en_pk_p^{-3} \simeq 55.4$ predicted by the multi-sheath model is $\sim 23\%$ higher than the loaded charge $Q_{tr}/en_pk_p^{-3} \simeq 45$ predicted by integrating Eq. (3.23) of the analytic theory. As in the previous example shown in Fig. 3.9(b), the gain in the loaded charge Q_{tr} is primarily driven by the longer bunch length $\Delta\xi_{tr}$. The increase in Q_{tr} is lower than that of $\Delta\xi_{tr}$ because the slice currents are also lower.

In the next example, we revisit a beam loading result published by Tzoufras et al. [39] in which a bi-Gaussian driver $\{n_b(r, \xi) = [N_b/(2\pi)^{3/2}\sigma_r^2\sigma_z]e^{-r^2/(2\sigma_r^2)}e^{-\xi^2/(2\sigma_z^2)}, k_p\sigma_r = 0.5, k_p\sigma_z = 1.41, N_b = 139(c/\omega_p)^3\}$ is used to excite a plasma wakefield with a blowout radius $r_m \simeq 5$ in Fig. 3.10(a). For these parameters, the peak charge per unit length of the driver is $\Lambda_d \simeq 6.24$.

Using an optimized sheath width $\Delta_{10} = 0.825$ and $\psi_{min} = -1$, the unloaded bubble trajectory (dashed red) is calculated using the multi-sheath model [Eq. (3.12)] and plotted in Fig. 3.10(a). We are interested in generating a beam profile that can load a constant wakefield $E_t \simeq 1.75$, as previously done in Ref. 39. From the unloaded PIC simulation results, this electric field occurs at $\xi_t \simeq 8.27$ where the simulated bubble radius is $r_t \simeq 3.91$. Using the multi-sheath model, we self-consistently solve for the loaded bubble trajectory $\tilde{r}_b$ (solid red) by integrating Eq. (3.16) starting at $\xi_t \simeq 8.27$. We use $\psi_{min} = -0.9$ in this example since the bunch length is long, i.e., $\Delta\xi_{tr} \gtrsim r_m/2$. Strong agreement is observed

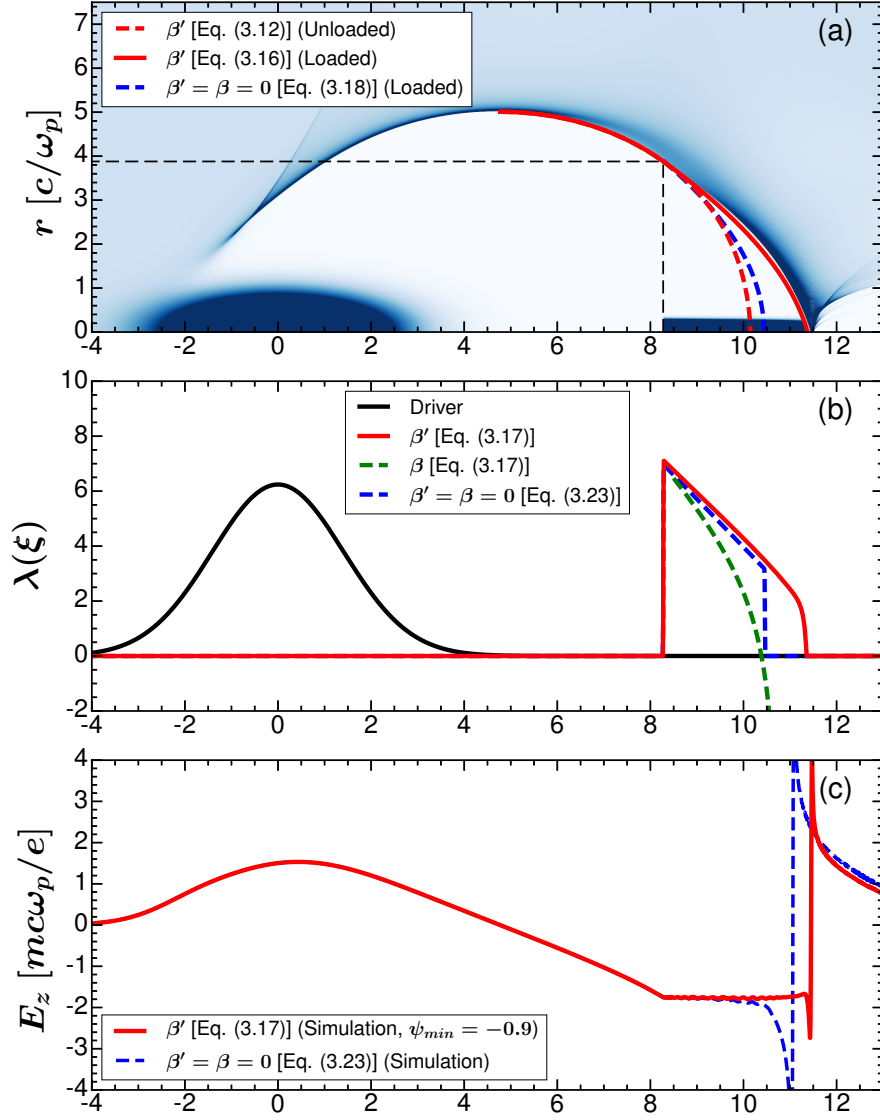


Figure 3.10: (a) Electron density distribution of a plasma wake excited by an electron driver $\{\Lambda = 6.24, k_p \sigma_r = 0.5, k_p \sigma_z = 1.41, k_p \xi_c = 0\}$ with a load [Eq. (3.17)] placed at $\xi_t = 8.27$ designed to produce a constant wakefield $E_t = 1.75$ using the multi-sheath model β' . Integration parameters are $\Delta_{10} = 0.825$, $\Delta_s = 0.05 r_b$, $\Delta_{20} = 3$, and $s = 3$. The unloaded (dashed red) and loaded (solid red) trajectories are calculated using $\psi_{min} = -1$ and $\psi_{min} = -0.9$, respectively. (b) Current profiles $\lambda(\xi)$ for the multi-sheath model β' (red) with $\psi_{min} = -0.9$, single-sheath model β (green) with $\psi_{min} = 0$, and analytic theory $\beta' = \beta = 0$ (blue). (c) Simulated $E_z(\xi)$ using the profiles shown in (b).

between the loaded trajectory calculated from the multi-sheath model and the trajectory from the PIC simulation with the underlying bunch. For reference, we also plot the loaded trajectory in the ultrarelativistic limit $\tilde{r}_b^2 = r_t^2 - 4E_t(\xi - \xi_t)$ using r_t and E_t from PIC simulation data. Like in the previous example, the parabolic trajectory underestimates the length of the wake when compared to the loaded trajectory of the multi-sheath model.

The current profile obtained using the multi-sheath model [Eq. (3.17)] is shown in Fig. 3.10(b) in solid red. For reference, we also plot the beam profile $\lambda(\xi) \simeq 6.96 - 1.75 \times (\xi - 8.27)$ [Eq. (3.23)] in the ultrarelativistic limit in dashed blue. While both models predict trapezoidal profiles with similar slice currents, the bunch length predicted by the multi-sheath model $\Delta\xi_{tr} \simeq 3.09$ is nearly $\sim 42\%$ longer than that of the analytic theory $\Delta\xi_{tr} = \frac{r_t^2}{4E_t} \simeq 2.18$. Despite the fact that most of the charge is front-loaded in both profiles, the total charge predicted by the multi-sheath model $Q_{tr}/en_pk_p^{-3} \simeq 87.1$ is still $\sim 25\%$ more than the total charge predicted by the analytic theory $Q_{tr}/en_pk_p^{-3} \simeq 69.4$.

The beam profile obtained using the single-sheath model (dashed green) with $n_2 = 0$ is also shown for qualitative comparisons. Since the head of the bunch is situated at $r_t/r_m \simeq 0.78$ where the second sheath can be largely neglected, the single-sheath model will initially predict slice currents similar to those obtained using the multi-sheath model. However, at lower $\tilde{r}_b$, the multi-sheath model and single-sheath begin to diverge as the second sheath comes into play. Eventually, the current profile predicted by the single-sheath model turns negative ($\lambda < 0$), similar to what can be seen in Fig. 3.9(b), due to the absence of the characteristic spike in the electric field when using the single-sheath model.

In Fig. 3.10(c), we plot the PIC simulation results using the profiles given by the multi-sheath model and analytic theory shown in Fig. 3.10(b). It can be readily seen that the multi-sheath model provides improved accuracy over the analytic theory in terms of flattening the wakefield. Furthermore, the beam load predicted by the multi-sheath extends all the way to the very rear of the wake while the beam load predicted by the analytic theory does not.

3.4.4 Total Accelerating Force

When loading a constant wakefield $f(\xi \geq \xi_t) = -E_t$, the interplay between the maximum loaded charge Q_{tr} and the accelerating field E_t can be characterized by examining the total accelerating force $Q_{tr}E_t$. In Ref. 39, the total accelerating force was found to be

$$\frac{Q_{tr}}{e} \frac{eE_t}{mc^2/r_e} = \frac{1}{4^3} (k_p r_m)^4, \quad (3.24)$$

in the ultrarelativistic limit by integrating the analytic theory described by Eq. (3.23). An exact calculation for $Q_{tr}E_t$ can be obtained by numerically integrating loaded trajectory $\tilde{r}_b(\xi)$ from Eq. (3.16) and, then, integrating the current profile $\lambda(\xi)$ described by Eq. (3.17) using

$$\frac{Q_{tr}}{e} \frac{eE_t}{mc^2/r_e} = \frac{1}{2} \int_{\xi_t}^{\xi_f} \lambda(\xi) E_t d\xi \quad (3.25)$$

for a specified constant wakefield E_t until the very rear of the wake defined by $\tilde{r}_b(\xi_f) = 0$.

In Fig. 3.11, we plot the accelerating force $Q_{tr}E_t$ as a function of the accelerating field E_t for the plasma wakefield shown in Fig. 3.9(a) with a blowout radius $r_m \simeq 4.53$. The blue line corresponds to the analytic theory described by Eq. (3.24) while the red line is obtained by numerically integrating Eq. (3.25) using the multi-sheath model starting at different positions in the wake. For simplicity, we use $\psi_{min} = -1$ to calculate $Q_{tr}E_t$ using the multi-sheath model rather than tailoring ψ_{min} for cases with long beam loads, i.e., $\Delta\xi_{tr} \gtrsim r_m/2$. As shown in the previous section, adjusting ψ_{min} to account for self-consistent beam loading effects can decrease the predicted charge by $\lesssim O(10\%)$.

As can be seen in Fig. 3.11, Eq. (3.24) predicts a constant accelerating force regardless of where the load is placed while the multi-sheath model predicts an accelerating force that decreases as the amplitude of the accelerating field E_t increases. For values of $E_t < 2.73$, the multi-sheath model predicts more loaded charge Q_{tr} than the analytic theory due to its

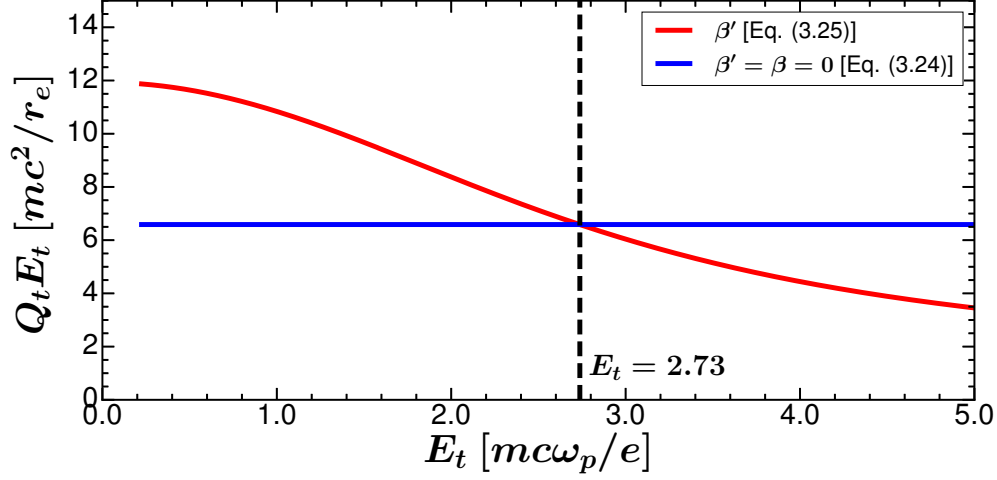


Figure 3.11: The accelerating force $Q_t E_t$ as a function of the accelerating field E_t for the plasma wakefield from Fig. 3.9 with maximum bubble radius $r_m \simeq 4.53$. The blue curve corresponds to Eq. (3.24). The red curve is numerically integrated from Eq. (3.25) using the multi-sheath model β' [Eq. (3.17)] to load constant wakefields, i.e., $E_z(\xi \geq \xi_t) = -E_t$, at different positions in the wake. The same integration parameters $\{\Delta_{10} = 0.875, \Delta_{20} = 3, s = 3, \psi_{min} = -1\}$ from Fig. 3.9 are used for each calculation.

longer beam loads with comparable slice currents. It is worth pointing out that all examples of beam loading presented in Figs. 3.9 and 3.10 were operating in this range. On the other hand, the multi-sheath model predicts less charge can be loaded for larger accelerating fields $E_t > 2.73$ because the lower slice currents now outweigh the differences between the predicted bunch lengths. While the value of E_t at the crossing point will vary on a case by case basis, the qualitative features of the accelerating force $Q_t E_t$ predicted by multi-sheath model will be largely similar for nonlinear wakes with different r_m . Since the analytic theory is also a limit of the multi-sheath model, the accelerating force $Q_{tr} E_t$ obtained using the multi-sheath model also scales with r_m^4 when $r_m \gg 1$.

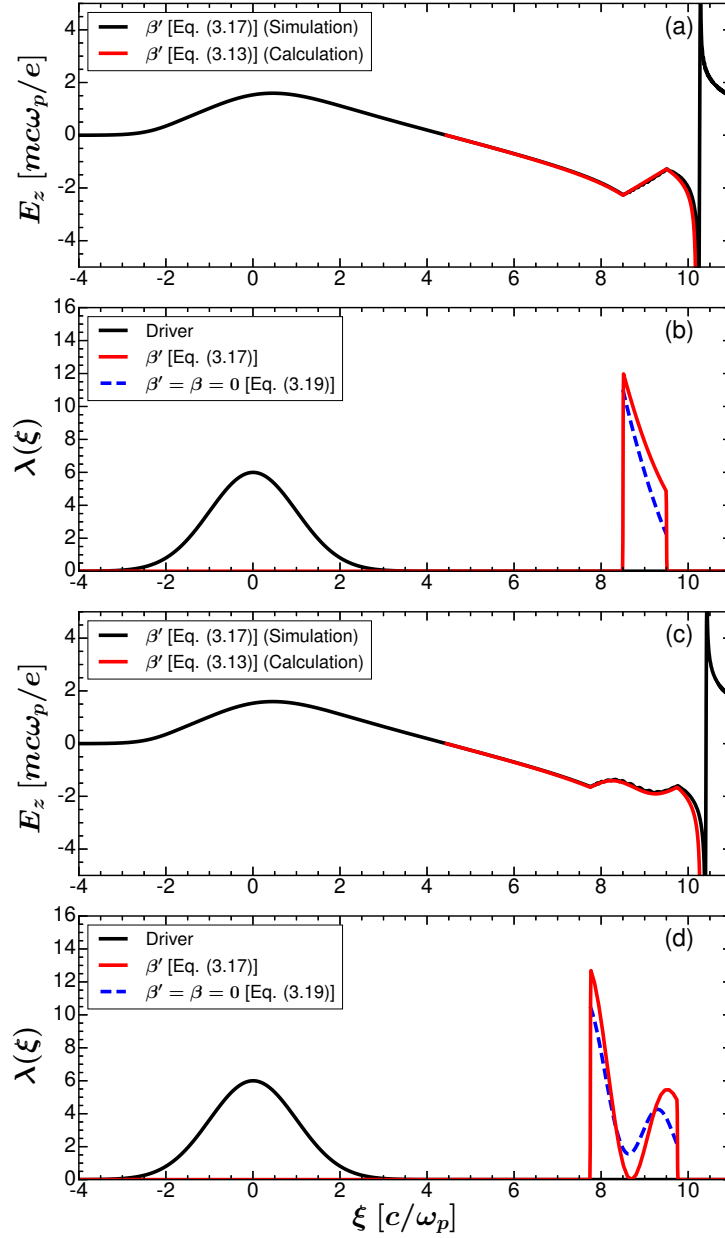


Figure 3.12: Simulation results (black) using the multi-sheath model [Eqs. (3.16)-(3.17)] with $\psi_{min} = -1$ for two different electric field profiles: (a) $f(\xi_t \leq \xi \leq \xi_f) = -E_t + (\xi - \xi_t)$ from $\xi_t = 8.5$ to $\xi_f = 9.5$ and (c) $f(\xi_t \leq \xi \leq \xi_f) = -E_t + E_1 \sin(k(\xi - \xi_t))$ from $\xi_t = 7.75$ to $\xi_f = 9.75$ where $E_1 = \frac{1}{4}$ and $k = \pi$. The solid red lines correspond to the electric field calculated from Eq. (3.13). The corresponding current profiles $\lambda(\xi)$ [Eq. (3.17)] from (a) and (c) are plotted in (b) and (d), respectively. Analytic current profiles $\lambda(\xi)$ calculated from Eq. (3.19) are plotted in dashed blue.

3.4.5 Beam Loading Longitudinally Varying Wakefields

While much of the discussion has focused on loading constant wakefields, we will now present several examples in which the loaded wakefield varies along ξ using the formalism described by Eqs. (3.16)-(3.17) of the multi-sheath model. Designing beam loads for tailored $\frac{dE_z}{d\xi}$ is of interest to self-injection or collider designs where initial energy chirps are present in the witness beam. Beam profiles will also be presented in the ultrarelativistic limit ($\beta', \beta \rightarrow 0, 0$) using Eqs. (3.18) and (3.19). In each case, an electron drive bunch $\{\Lambda_d = 6, \gamma_b = 20000, k_p\sigma_r = 0.245, k_p\sigma_z = 1, k_p\xi_c = 0\}$ is used to excite a plasma wake with a blowout radius $r_m \simeq 4.53$. Since the driver is identical to the one used in Fig. 3.9, we will also use the same parameters $\{\Delta_{10} = 0.875, \Delta_s = 0.05r_b, \Delta_{20} = 3, s = 3\}$ to obtain results using the multi-sheath model β' . n_{20} is calculated from Eq. (3.10) using $\psi_{min} = -1$ since the trailing bunches are short. For this driver, we refer to Fig. 3.9(a) for the unloaded wake trajectory (dashed red) calculated using the multi-sheath model.

In Figs. 3.12(a)-(b), we design a beam profile that provides a plasma wakefield with a positive slope $df/d\xi > 0$ that is constant. Such a wakefield can be used to dechirp a beam with a positive energy chirp $d_\xi\gamma > 0$ while still maintaining an accelerating field $E_z < 0$ over the electron load. In this example, we choose to load the function $f(\xi) = -E_t + (\xi - \xi_t)$ from $\xi_t = 8.5$ to $\xi_f = 9.5$ such that $df/d\xi = 1$. From the unloaded trajectory calculated from Eq. (3.12), the bubble radius $r_t \simeq 2.73$ and electric field $E_t \simeq 2.26$ at the head of the bunch are known.

To calculate the beam load profile, we must first numerically integrate Eq. (3.16) to obtain the loaded bubble trajectory $\tilde{r}_b(\xi)$ from ξ_t to ξ_f . After the beam load $\xi > \xi_f$, the remaining unloaded trajectory is calculated by integrating Eq. (3.12) starting at $\tilde{r}_b(\xi_f)$. The electric field is then calculated from Eq. (3.13) across all regions (unloaded and loaded) and plotted (solid red) in Fig. 3.12(a). From the results, it is clearly evident that the loaded region does not extend to the very rear of the wake since the characteristic negative spike in

the electric field is still present. In Fig. 3.12(b), the underlying current profile $\lambda(\xi)$ calculated from Eq. (3.17) is shown (solid red). Using this current profile, PIC simulation results (solid black) shown in Fig. 3.12(a) indeed confirm that the desired wakefield $f(\xi)$ is produced along the bunch. Strong agreement is also observed between the simulated and calculated electric fields in regions before and after the beam load.

In the ultrarelativistic limit $\beta' \rightarrow 0$, it is straightforward to show that the analytic solution to Eq. (3.18) along the beam load is a hyperbola $\tilde{r}_b(\xi)^2 = r_t^2 - 4E_t(\xi - \xi_t) + 2(\xi - \xi_t)^2$. Substituting $\tilde{r}_b$ into Eq. (3.19), the analytic current profile is a parabola $\lambda(\xi) = \frac{3}{4}r_t^2 + E_t^2 - 5E_t(\xi - \xi_t) + \frac{5}{2}(\xi - \xi_t)^2$. This profile is evaluated using $r_t \simeq 2.81$ and $E_t \simeq 2.26$ from the PIC simulation data and plotted (dashed blue) in Fig. 3.12(b). While this analytic profile captures the general trend of the multi-sheath results, we note that disagreement is still observed between the two profiles along portions of the load.

In Figs. 3.12(c)-(d), we design a beam profile that loads a sinusoidal plasma wakefield to highlight the limitations of the magnitude of the loaded plasma wakefield slope $df/d\xi$. For this case, we choose to load the function $f(\xi) = -E_t + E_1 \sin(k(\xi - \xi_t))$ from $\xi_t = 7.75$ to $\xi_f = 9.75$ where $E_1 = \frac{1}{4}$ and $k = \pi$. From the unloaded bubble trajectory calculated from Eq. (3.12), we use $r_t \simeq 3.45$ and $E_t \simeq 1.66$ to numerically integrate the loaded bubble trajectory $\tilde{r}_b(\xi)$ [Eq. (3.16)] from ξ_t to ξ_f . After the beam load $\xi > \xi_f$, the remaining unloaded trajectory is calculated from Eq. (3.12) starting at $\tilde{r}_b(\xi_f)$. In Fig. 3.12(c), we plot the electric field calculated from Eq. (3.13) using the multi-sheath model in solid red. The current profile of the load $\lambda(\xi)$ calculated from Eq. (3.17) is plotted (solid red) in Fig. 3.12(d). Using this current profile, the electric field from PIC simulation results (solid black) exhibits strong agreement with the multi-sheath results in all regions.

As pointed out previously, $\lambda(\xi)$ is defined to be positive definite for an electron load, which limits how negative $df/d\xi$ can be in Eq. (3.17). As shown in Figs. 3.12(c)-(d), the current profile approaches zero around $\xi \approx 8.7$ where $df/d\xi$ is near its minimum. In this case, increasing the amplitude or the frequency of the sinusoidal oscillation would result in

a more negative slope $df/d\xi$, which would require a positive (positron) charge density along regions of the load to attract the sheath electrons that trace the bubble trajectory back to the axis more quickly.

A similar analysis can be done in the ultrarelativistic limit where the analytic solution to Eq. (3.18) is $\tilde{r}_b(\xi)^2 = r_t^2 - 4E_t(\xi - \xi_t) + \frac{4E_1}{k} [1 - \cos(k(\xi - \xi_t))]$. Using this analytic trajectory, the current profile of the underlying bunch $\lambda(\xi) = \frac{\tilde{r}_b^2}{4} + f(\xi)^2 + \frac{\tilde{r}_b^2}{2} \frac{df}{d\xi}$ [Eq. (3.19)] can now be completely expressed in terms of ξ where $\frac{df}{d\xi} = kE_1 \cos(k(\xi - \xi_t))$. The slope $df/d\xi$ is naturally constrained since it is the only term which can be negative and $\lambda \geq 0$ for an electron driver by definition. In Fig. 3.12(c), this profile is evaluated using $r_t \simeq 3.49$ and $E_t \simeq 1.63$ from unloaded PIC simulation results and plotted (dashed blue). While the analytic current profile qualitatively reproduces the oscillations observed in the multi-sheath profile, it is still an approximation of the multi-sheath model and, therefore, deviates from it along portions of the beam load. For example, near $\xi \approx 8.7$, the analytic current profile dips to $\lambda \approx 1.56$ whereas the multi-sheath profile approaches $\lambda \approx 0$. In Sec. 3.6, we provide more detailed comparisons between the single and multi-sheath models and the analytic results. Explanations for these differences are also given.

3.5 Beam Loading in Laser Wakefields

Up to this point, we assumed that the wakefields are excited by electron drivers. However, the formalism described in Sec. 3.2 can be easily extended to a laser driver specified by the vector potential $A_{laser} = \Re\{A_\perp e^{i\omega_0/c\xi}\}$ where ω_0 is the laser frequency and $a \equiv eA_\perp/mc^2$ is the normalized vector potential envelope. To do this, we use the same source term profile for S described by Eq. (3.2). Therefore, the expressions for the wake potential $\psi = (1 + \beta')r_b^2/4 - r^2/4$ obtained by integrating Eq. (3.1) and the electric field $E_z = \frac{d\psi_0}{d\xi} = D'(r_b)r_b \frac{dr_b}{d\xi}$ are identical to the those derived in Sec. 3.2. The trajectory of the sheath electron for a laser driver, as derived by Lu et al. [36], can be written as

$$\begin{aligned}
A'(r_b) \frac{d^2 r_b}{d\xi^2} + B'(r_b) r_b \left(\frac{dr_b}{d\xi} \right)^2 + C'_L(r_b) r_b \\
= \frac{\lambda(\xi)}{r_b} - G'_L(r_b) \nabla_\perp |a|^2
\end{aligned} \tag{3.26}$$

where the new coefficients for the laser case denoted with the subscript “L” are defined as

$$\begin{aligned}
C'_L(r_b) &= \frac{1}{4} \left[1 + \frac{1 + |a|^2/2}{\left(1 + \frac{\beta' r_b^2}{4} \right)^2} \right], \\
G'_L(r_b) &= \frac{1}{4} \left[\frac{1}{\left(1 + \frac{\beta' r_b^2}{4} \right)} \right].
\end{aligned}$$

It is worth mentioning that $A'(r_b)$ and $B'(r_b)$ are the same coefficients specified for the beam-driven wake in Sec. 3.2. While the equation of motion describing the trajectory of the sheath electron is slightly different in the case of a laser driver, the general procedure for modeling the wakefield remains the same. For the laser wakefields in this section, we use the same profiles for $\Delta_1(r_b)$, $\Delta_2(r_b)$ and $n_2(r_b)$ as described in Sec. 3.2. As in the beam-driven case, $n_1(r_b)$ is calculated from Eq. (3.5) and n_{20} is constrained by Eq. (3.10). Using these quantities, we can calculate $\beta'(r_b)$ from Eq. (3.8) and, then, numerically integrate Eq. (3.26) to obtain the trajectory of the sheath electron that traces $r_b(\xi)$ starting at the maximum bubble radius r_m . Once $r_b(\xi)$ is known, the wake potential $\psi(r, \xi)$ and electric field $E_z(\xi)$ can be obtained everywhere inside the bubble $r < r_b(\xi)$.

For a laser that is well-guided in the plasma with a matched spot size $W_0 \sim r_m \sim 2\sqrt{a_0}$ [36], determining the exact value of the blowout radius r_m is generally more difficult because the electrons are not completely blown out by the ponderomotive force of the laser, which

is largely localized to W_0 . Complete blowout occurs for spot sizes W_0 much less than the matched spot size $2\sqrt{a_0}$ or for $a_0 \gg 1$. In the absence of complete blowout, the particle tracing “ r_b ” is no longer the innermost electron but the characteristic sheath electron with the largest forward velocity $v_z = 1 - \frac{2(1+\psi)^2}{1+P_\perp^2+|a|^2/2+(1+\psi)^2}$ near the axis where ψ is minimum. Like in the beam-driven cases, the minimum wake potential ψ_{min} can be well-approximated by values close to -1 when $r_m \gtrsim 3$ for the trajectory traced out by this electron. This regime typically corresponds to lasers with normalized vector potentials $a_0 \gtrsim 2$ since $r_m \approx 2\sqrt{a_0}$.

The methodology for loading a wakefield $E_z(\xi_t \leq \xi \leq \xi_f) = f(\xi)$ also remains largely unchanged from the procedure described in Sec. 3.4. By integrating Eq. (3.16), we can then obtain the modified sheath electron trajectory $\tilde{r}_b(\xi)$ corresponding to a loaded wakefield $f(\xi)$ starting at the beam head located at ξ_t . The only difference is that the corresponding current profile for the beam load is now given by

$$\begin{aligned} \lambda(\xi) = & C'_L \tilde{r}_b^2 + \left(\frac{B'}{D'^2} - \frac{A'F'}{D'^3 \tilde{r}_b^2} \right) f(\xi)^2 \\ & + \left(\frac{A'}{D'} \right) \frac{df(\xi)}{d\xi} + G'_L \tilde{r}_b \nabla_\perp |a|^2 \end{aligned} \quad (3.27)$$

where the last term corresponds to the ponderomotive force from Eq. (3.26). Typically, the laser pulse length is sufficiently short enough so that the ponderomotive term in Eq. (3.26) can be dropped in the back half of the wake.

In Figs. 3.13(a)-(c), we show the results for the bubble trajectory $r_b(\xi)$, potential $\psi_0(\xi)$, and electric field $E_z(\xi)$ obtained from calculations using the multi-sheath model (red) and OSIRIS (quasi-3D) PIC simulation results (black) for an unloaded wake excited by a 40 fs (FWHM) $0.8 \mu\text{m}$ laser driver after propagating into a constant plasma density $n_p = 1.5 \times 10^{18} \text{ cm}^{-3}$. The PIC simulation used a customized finite-difference solver to reduce numerical effects from relativistic particles [86, 82], a high resolution grid with $k_p \Delta r = 0.0073$, $k_p \Delta z = 0.0059$ and $\omega_p \Delta t = 0.0029$, and 32 particles per cell (2x2x8). The laser is initially focused at the plasma entrance with a normalized vector potential $a_0 = 4$ and

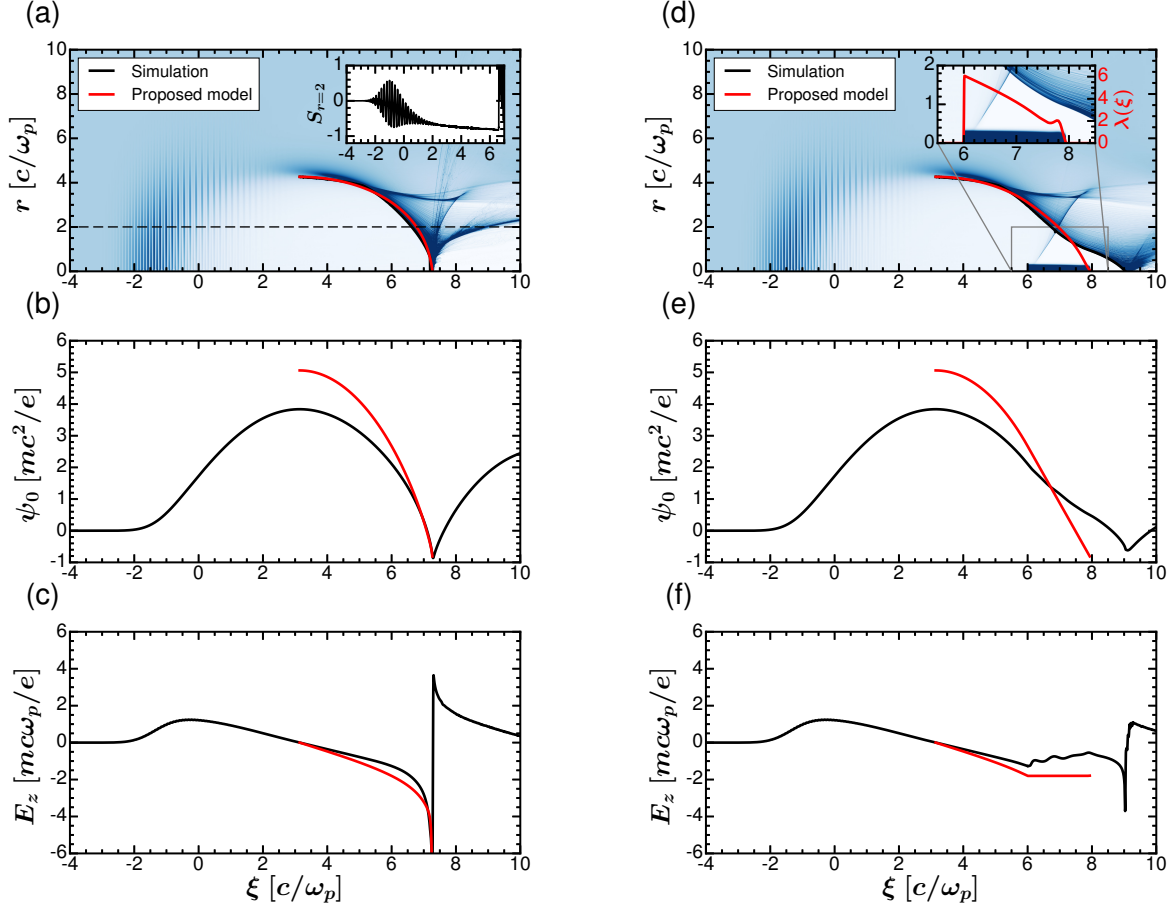


Figure 3.13: Comparisons of simulations (black) and numerical calculations (red) of r_b , ψ_0 and E_z using Eqs. (3.7), (3.13), and (3.26) for wakefields excited by intense lasers $\{a_0 = 4, k_p W_0 = 4, \lambda = 0.8\mu\text{m}, \tau = 40\text{fs (FWHM)}\}$ propagating in a constant plasma density $n_p = 1.5 \times 10^{18} \text{ cm}^{-3}$. At $z = 0.32 \text{ mm}$, the electron density distribution is shown for two cases with (a) no load and (d) a beam load designed to produce a constant wakefield starting at $k_p \xi = 6$ using Eqs. (3.16) and (3.27). The insets show the lineout of the source term S at $k_p r = 2$ (dashed black) for the unloaded case and the trailing bunch profile $\lambda(\xi)$ (red) obtained using the multi-sheath model for the loaded case. The corresponding $\psi_0(\xi)$ and $E_z(\xi)$ are shown in (b)-(c) and (e)-(f) for the unloaded and loaded cases, respectively. The integration parameters are $\Delta_{10} = 0.3$, $\Delta_{20} = 3$, and $s = 3$. n_{20} is calculated from Eq. (3.10) using $\psi_{\min} = -0.85$ and n_1 is calculated from Eq. (3.5).

a transverse gaussian envelope having a matched spot size $k_p W_0 = 2\sqrt{a_0} = 4$ [36]. The electron density distribution in the $r - \xi$ plane is shown in Fig. 3.13(a) after a propagation distance $z = 0.32$ mm into the plasma at which point the blowout radius is $r_m \simeq 4.26$. It can readily be seen that while the multi-sheath model generates a sheath electron trajectory $r_b(\xi)$ in Fig. 3.13(a) that is in good agreement with the bubble trajectory from the simulation results, it overestimates (underestimates) the potential (electric field) over most of the wake in Figs. 3.13(b)-(c). The disagreement is primarily due to the fact that the plasma electrons are not completely blown out by the ponderomotive force of the laser. Therefore, plasma electrons can now propagate inside the bubble, i.e., $r < r_b$, resulting in spatially varying charge densities and currents in the bubble.

As a result, the expressions for $\psi(r, \xi)$ [Eq. (3.7)] and $E_z(\xi)$ [Eq. (3.13)] obtained using the multi-sheath model [Eq. (3.2)] break down because the source term inside the channel $S(r < r_b)$ is no longer exactly -1 as shown in the inset plot of S along $r = 2$ in Fig. 3.13(a). The presence of plasma electrons inside the bubble is also important because the focusing force is no longer perfectly linear. In addition, these electrons can move from the inside ($r < r_b$) to the outside of the bubble ($r > r_b$) effectively splitting the plasma sheath into two. This effect can be seen from the simulation results in the Fig. 3.13(a) inset near the bubble radius $r_b(\xi)$ at $\xi \approx 6$. This sheath splitting phenomenon can typically produce more than 3 distinct regions in which S has alternating signs, which differs from the model assumed in Eq. (3.2).

In Figs. 3.13(d)-(f), we show the electron density distribution of the laser-driven wakefield at $z = 0.32$ mm with a beam load designed to produce a constant wakefield starting at $\xi_t = 6$. To self-consistently load the wake using Eqs. (3.16) and (3.27), the electric field at the head of the bunch is sampled from the multi-sheath model $f(\xi > \xi_t) = -E_t = -1.74$ rather than the simulation results. The current profile $\lambda(\xi)$ of the beam load calculated from Eq. (3.27) is shown in the inset of Fig. 3.13(d).

It is clear from the simulation results that the multi-sheath model fails to capture the

behavior of the modified sheath electron trajectory $\tilde{r}_b$ [Eq. (3.16)], wake potential ψ_0 , and electric field E_z in Figs. 3.13(d)-(f). The underlying reason is that the multi-sheath model predicts an electric field at the head of the bunch that is more negative than the simulated electric field. Therefore, the currents calculated from Eq. (3.27) are larger than needed due to the $f(\xi)^2$ term on the right-hand side. The simulation results show that the current profile produces an electric field that actually increases along the beam load rather than remaining constant.

It is also worth noting that the beam load blows out the remaining electrons inside the channel $r < r_b$ and forms another thin plasma sheath as can be seen in the electron density phase space and inset plots in Fig. 3.13(d). As these electrons are being blown out, the source term inside the bubble becomes more negative until only ions remain and $S(r < r_b) = -1$. This effect also contributes to the positive slope of the loaded wakefield E_z near the head of the bunch ξ_t .

From the results presented in this section, it is evident that the model for S described by Eq. (3.2) is not sufficient for modeling unloaded and loaded laser wakefields. Plasma electrons propagating inside the “bubble,” sheath splitting, and blowout of remaining electrons by the beam load are some of the features making it difficult to apply the multi-sheath model, as is, to cases with a laser driver. As a result, the structure of the plasma source term S for an unloaded laser wakefield can be more complex as shown in Fig. 3.14(a). For these very same reasons, electron beams are ideal for driving high-quality plasma wakefields in which electrons are completely blown out and the focusing force is perfectly linear. While the multi-sheath model can be adapted to laser drivers by using a source term model in which $S(r < r_b) \equiv n_0$ is no longer constant [see Fig. 3.14(b)], the force equation will also need to be modified due to the fields from the plasma currents inside the bubble. Such an analysis will also require assumptions about the electron currents inside the channel. This is an area for future work.

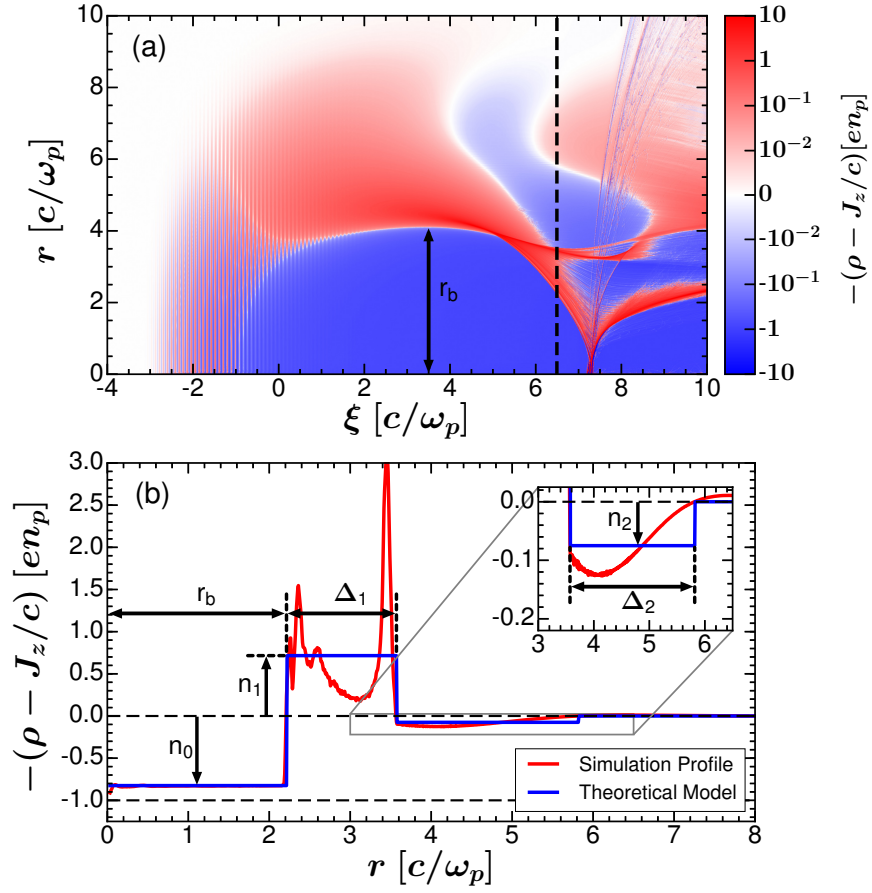


Figure 3.14: (a) Contour plot of the source term $-(\rho - J_z/c)$ for the unloaded laser wakefield produced in Fig. 3.13. (b) Transverse slice of the source term profiles at different $k_p \xi = 6.5$.

3.6 Differences Between the Sheath Models and Analytic Theory for Beam Loading

In this section, we provide details regarding the differences in the predictions between the sheath and analytical models. These details also show why the analytic model provides reasonable agreement for the witness beam current but poor predictions for r_b .

From the results presented in Sec. 3.4, it is clear that the analytic theory can be a useful tool for predicting the general form of the current profiles for beam loading. However, as it is an approximation of the multi-sheath (and single-sheath) model, it is generally not as

accurate even for $r_b \sim r_m$. For beam loads designed to produce constant wakefields, the resulting parabolic trajectory $\tilde{r}_b(\xi)^2 = r_t^2 - 4E_t(\xi - \xi_t)$ predicted by Eq. (3.18) can also deviate significantly from that of the multi-sheath model as seen in Figs. (3.9)-(3.10). As a result, the analytic theory can underestimate the maximum length of the beam load $\Delta\xi_{tr}$ and, thus, the total charge Q_{tr} when compared to the multi-sheath results for these cases. Despite this, the slice currents predicted by the analytic theory are still comparable to those obtained using the multi-sheath model.

To understand why this occurs, we revisit the differential equation for the bubble trajectory from Eq. (3.12). In the ultrarelativistic limit $(\beta', \beta \rightarrow 0, 0)$, Eq. (3.12) describing the innermost particle trajectory $r_b(\xi)$ was found to be [38, 39]

$$r_b \frac{d^2 r_b}{d\xi^2} + 2 \left(\frac{dr_b}{d\xi} \right)^2 + 1 = \frac{4\lambda(\xi)}{r_b^2} \quad (3.28)$$

where the wake potential is now

$$\psi_0(\xi) \approx \Psi_I = \frac{r_b^2}{4} \quad (3.29)$$

and the electric field is $E_z(\xi) = \frac{d\psi_0}{d\xi} = \frac{1}{2}r_b \frac{dr_b}{d\xi}$. As shown in Refs. 38 and 39, Eq. (3.28) can be integrated starting at the blowout radius r_m to obtain the following expression for the bubble trajectory for an unloaded plasma wake ($\lambda = 0$)

$$\frac{\xi}{r_m} = 2E \left[\arccos \left(\frac{r_b}{r_m} \right) \middle| \frac{1}{2} \right] - F \left[\arccos \left(\frac{r_b}{r_m} \right) \middle| \frac{1}{2} \right] \quad (3.30)$$

where F and E are incomplete elliptical integrals of the first and second kind. The corresponding electric field can also be calculated analytically to be [38, 39]

$$E_z(\xi) = \frac{1}{2}r_b \frac{dr_b}{d\xi} = -\frac{r_b}{2\sqrt{2}} \sqrt{\frac{r_m^4}{r_b^4} - 1}. \quad (3.31)$$

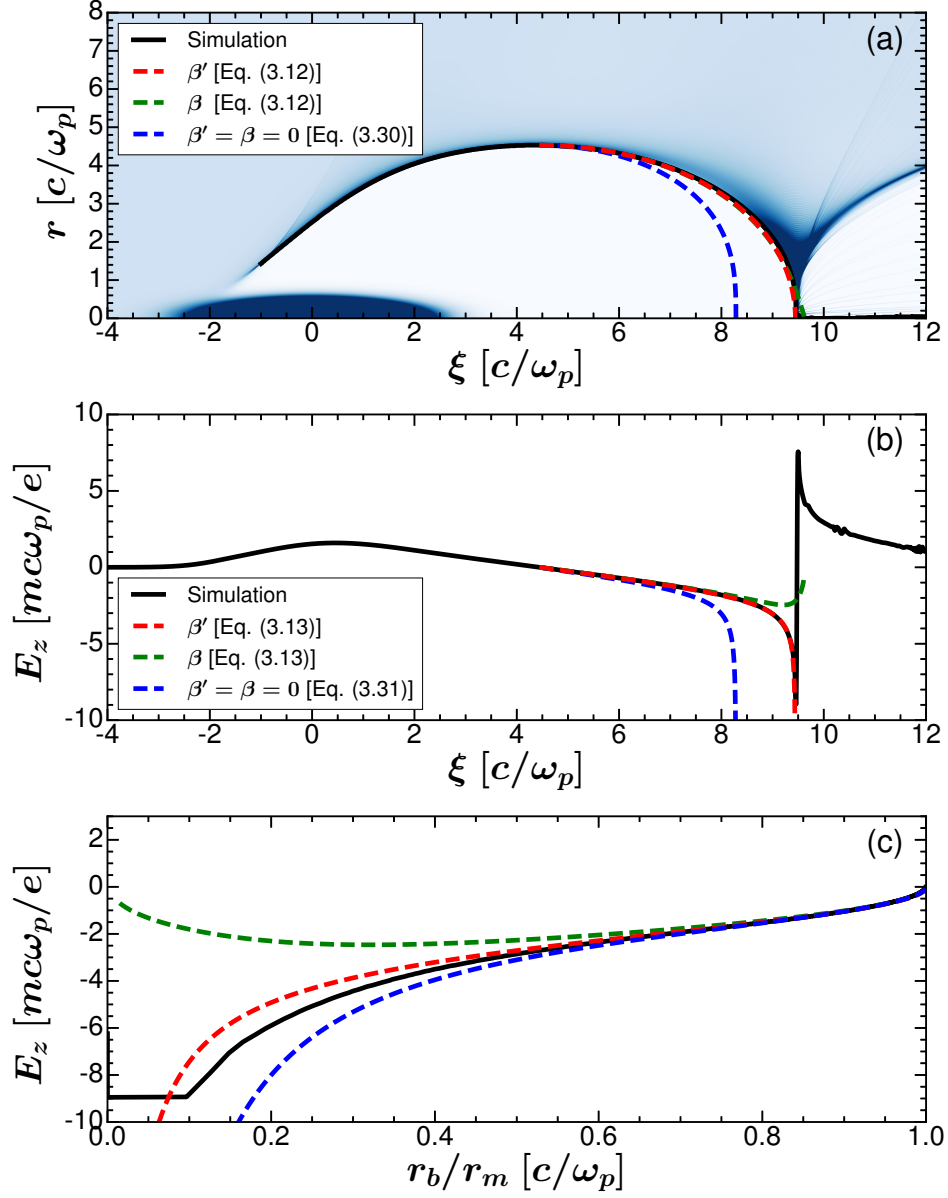


Figure 3.15: (a) Electron density distribution and electric field E_z as a function of (b) ξ and (c) $r_b(\xi)$ for a plasma wake excited by a nonevolving driver with parameters $\Lambda_d = 6$, $\gamma_b = 20000$, $k_p \xi_c = 0$, $k_p \sigma_r = 0.245$, and $k_p \sigma_z = 1$. The maximum bubble radius is $k_p r_m \simeq 4.53$. Simulation data is shown in black. Numerically integrated results are shown for the multi-sheath model β' (dashed red) with $\psi_{min} = -1$ and the single-sheath model β (dashed green) with $\psi_{min} = 0$ using Eqs. (3.7)-(3.10) with $\Delta_{10} = 0.875$, $\Delta_s = 0.05r_b$, $\Delta_{20} = 3$, and $s = 3$. Analytic results using Eqs. (3.30) and (3.31) are shown in dashed blue.

In Fig. 3.15, we compare the analytic theory [Eqs. (3.30)-(3.31)] with the multi-sheath model (β'), single-sheath model (β), and simulation results for an unloaded plasma wake excited by an electron driver with $\Lambda_d = 6$, $k_p\sigma_z = 1$, $\gamma_b = 20000$, and $k_p\sigma_r = 0.245$. For the multi-sheath and single-sheath calculations, the integration parameters are specified in the figure caption. From Fig. 3.15(a), it is clear that the bubble trajectory $r_b(\xi)$ (blue dashed) described by Eq. (3.30) deviates significantly from the simulation results (black), multi-sheath model (dashed red), and single-sheath model (dashed green). In fact, Eq. (3.30) will always predict an ion channel with a half-length $L_h \equiv \xi(r_b = 0) - \xi(r_b = r_m) \approx 0.85r_m$ whereas the bubble actually traces a nearly spherical shape $L_h \approx r_m$ in this case.

In Ref. 36, it was pointed out that the deviation between the analytic expression for $r_b(\xi)$ in Eq. (3.30) and the actual wake trajectory from PIC simulations could be largely attributed to the additional $(dr_b/d\xi)^2$ term in Eq. (3.28) which caused the particle trajectories to bend toward the ξ -axis sooner than expected. Since the analytic theory underestimates the length of the ion channel, it naturally follows that as shown in Fig 3.15(b) the electric field predicted by Eq. (3.31) deviates from the empirical wakefield in a similar fashion. However, Eq. (3.31) still captures the negative spike in the electric field near the axis since the slope of the trajectory $\frac{dr_b}{d\xi} = -\sqrt{\frac{r_m^4}{2r_b^4} - \frac{1}{2}}$ approaches $-\infty$ as $r_b \rightarrow 0$.

Upon inspection of Eq. (3.15), this behavior arises because $P_\perp$ [Eq. (3.15)] asymptotes to $-\infty$ since $\psi_{min} = 0$ for the wake potential $\psi_0 \approx r_b^2/4$. Thus, although its underlying approximations break down as $r_b \rightarrow 0$, the analytic model still predicts a spike because $P_\perp \rightarrow -\infty$ while the multi-sheath model predicts a spike because $(1 + \psi) \rightarrow 0$. On the other hand the single-sheath model cannot predict a spike because $P_\perp$ remains finite and $\psi_{min} = 0$. This is perhaps the most important distinction between the analytic theory and single-sheath model in which the electric field is not a monotonically decreasing function of ξ despite the fact that it also employs $\psi_{min} = 0$. As we have shown in Sec. 3.4, this limitation of the single-sheath model at the rear of the wake is the primary reason why it cannot be used to design beam loads that produce constant wakefields, i.e., $\frac{dE_z}{d\xi} \simeq 0$.

Despite the fact that Eqs. (3.30) and (3.31) cannot accurately model the bubble radius and electric field as a function of ξ , the (E_z, r_b) phase space predicted by Eq. (3.31) agrees well with the simulation results and multi-sheath model for values of $r_b \gtrsim 2$ as depicted in Fig. 3.15(c). This is important because the analytic current profile described by Eq. (3.22) for loading a constant wakefield $E_z(\xi \geq \xi_t) = -E_t$ samples the phase space of E_t and r_t at the head of the load ξ_t . While Eqs. (3.30)-(3.31) do not accurately predict r_t and E_t as a function of ξ_t , Eq. (3.23) can be evaluated using the simulation data instead. Sampling the parameters this way will still produce self-consistent results in regions where the analytic theory is assumed to be valid ($r_b \gtrsim 3$) since we are only shifting our initial position up the phase space curve (E_z, r_b) .

For the profile described by Eq. (3.23), the loaded bubble trajectory obtained from Eq. (3.28) is parabolic $r_b^2 = r_t^2 - 4E_t(\xi - \xi_t)$. As we will show below, this expression underestimates the loaded wake length in the same manner as with Eq. (3.30). Therefore, in many cases, the current profile predicted by analytic theory [Eq. (3.23)] does not produce a perfectly constant wakefield over the entire bunch length. This can be seen in several examples provided in Ref. 39 where wakefields loaded using Eq. (3.23) still exhibit marginally nonzero slopes.

The disagreement between the analytic theory ($\beta, \beta' \rightarrow 0, 0$) and multi-sheath model for $r_b(\xi)$ stems directly from the underlying assumption that the wake potential contributions from regions outside the bubble can be neglected, i.e., $\psi_0(\xi) \approx \Psi_I = \frac{r_b^2}{4}$. In Fig. 3.16(a), we compare the ion term $\frac{r_b^2}{4}$ (dashed blue) to ψ_0 obtained from the simulation results (black), the single-sheath model $\frac{r_b^2}{4}(1 + \beta)$ (dashed green), and the multi-sheath model $\frac{r_b^2}{4}(1 + \beta')$ (dashed red) for the driver specified in Fig. 3.15.

From the results, it is clear that the analytic curve ($\beta' \rightarrow 0$) is only close to the simulation and multi-sheath model in a small region around $r_b \sim 0.45r_m$. This is because the sheath components $\Psi_{II} + \Psi_{III} = \beta' r_b^2/4$ cancel each other nearly exactly in this region as seen in Fig. 3.3(c). In contrast, the term $\beta r_b^2/4$ from the single-sheath model is positive-definite and

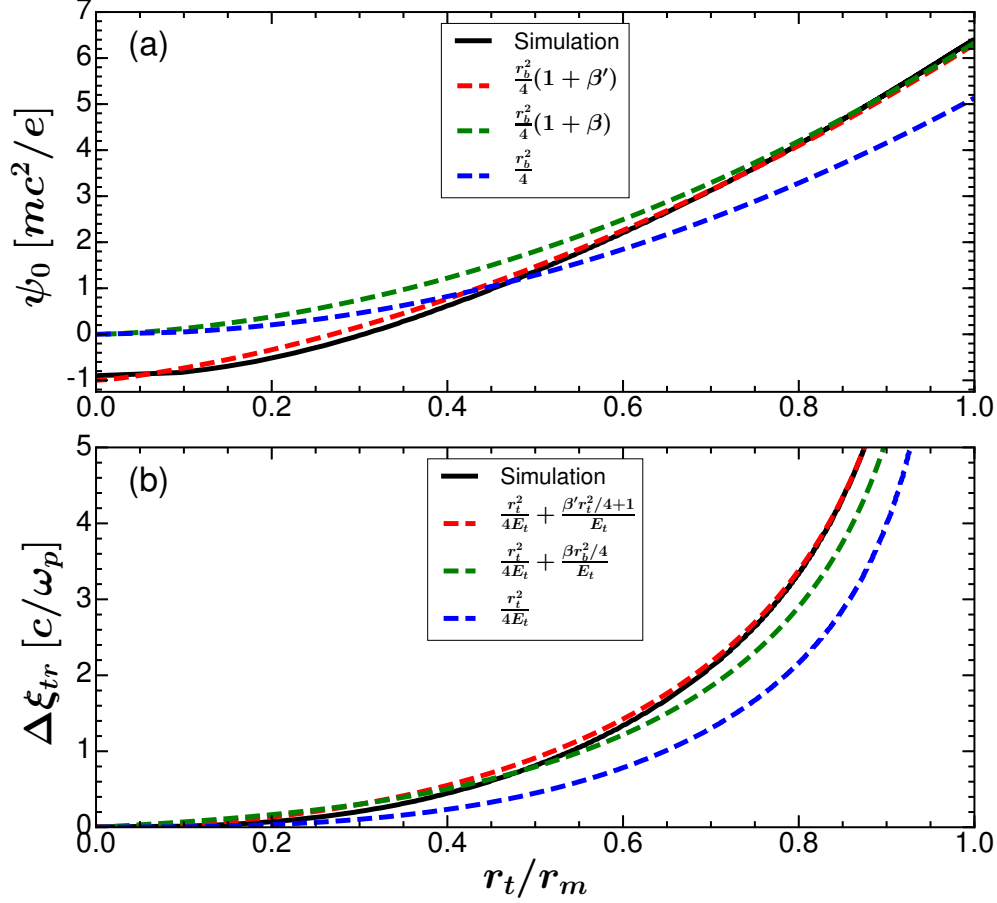


Figure 3.16: (a) The on-axis wake potential ψ_0 and (b) maximum bunch length $\Delta\xi_{tr}$ [Eq. (3.33)] for loading a constant accelerating field $E_z(r_b \leq r_t) \simeq \text{const} \equiv -E_t$ as a function of r_t/r_m for the plasma wake shown in Fig. 3.15. The black curves use simulation data for $\psi(r_t)$ and $E_z(r_t)$ to evaluate Eq. (3.33). Numerical calculations are shown for the multi-sheath model β' (dashed red) with $\psi_{min} = -1$ and the single-sheath model β (dashed green) with $\psi_{min} = 0$ using Eqs. (3.7)-(3.10) with $\Delta_{10} = 0.875$, $\Delta_s = 0.05r_b$, $\Delta_{20} = 3$, and $s = 3$. The analytic theory (dashed blue) for $\psi_0 \approx \frac{r_b^2}{4}$ and $\Delta\xi_{tr} = \frac{r_t^2}{4E_t}$ is evaluated using simulation data for r_t and E_t .

only goes to zero when r_b is zero. While the ion contribution $\Psi_I = r_b^2/4$ is the leading term when $r_b \approx r_m$, it underestimates the wake potential since the sheath term $\Psi_{II}(r_m)$ is on the order of unity while $\Psi_{III}(r_m)$ is approximately zero, and can be neglected (since $n_2 \approx 0$) as shown in Figs. 3.3(b)-(c). In this region, the wake potentials of the single-sheath and multi-sheath models are in agreement with the simulation results since $\beta' \approx \beta$.

Even though the blowout radius $r_m \simeq 4.53$ is large, the ion term $r_b^2/4$ still underestimates the simulated wake potential $\frac{\psi_0(r_m) - r_m^2/4}{\psi_0(r_m)}$ by approximately ($\sim 20\%$) at the top of the bubble. Therefore, from Eq. (3.15), the bubble trajectory predicted by the analytic theory should initially bend toward the axis with a more negative slope resulting in a shorter wake length.

On the other hand, when $r_b \ll r_m$, the ion term Ψ_I can be neglected while Ψ_{II} and Ψ_{III} from the multi-sheath model are both negative at the rear of the wake and, when combined, capture the limiting behavior of the wake potential $\psi_{min} \equiv \lim_{r_b \rightarrow 0} \beta' r_b^2/4 \approx -1$ in Fig. 3.3(c). However, the analytic theory using $\psi_0 \approx r_b^2/4$ and single-sheath model using $\psi_0 = (1 + \beta)r_b^2/4$ result in $\psi_{min} = 0$ at the axis. In addition, it is evident that the sheath term $\beta' \rightarrow -\infty$ near the axis which violates the underlying assumption of the analytic theory that β' can be neglected. When including the source terms outside the ion channel described by Eq. (3.2) of the multi-sheath model, the wake potential $\psi_0 = (1 + \beta')r_b^2/4$ obtained in Eq. (3.7) exhibits significantly improved agreement with the empirical simulation results across all values of r_b . As seen in Fig. 3.16(a), the curves for ψ_0 from the single and multi-sheath models deviate from each other at r_t/r_m as large as 0.7. This occurs because the curves must diverge such that $\psi_{min} = 0$ (single-sheath) rather than $\psi_{min} = -1$ (multi-sheath) for $r_t \rightarrow 0$.

The maximum theoretical length $\Delta\xi_{tr}$ over which the plasma wake can be loaded also depends on the profiles used for the on-axis wake potentials $\psi_0(r_b(\xi))$. In general, an expression for $\Delta\xi_{tr}$ can be obtained by integrating the electric field starting from the head of the bunch $r_b(\xi_t) = r_t$ to the rear of the bubble where the innermost particles cross the axis

$r_b(\xi_f) = 0$ as follows

$$\psi_0(r_t) - \psi_0(0) = - \int_{\xi_t}^{\xi_f} E_z d\xi \quad (3.32)$$

where $\xi_f = \xi_t + \Delta\xi_{tr}$. For a constant loaded wakefield $E_z(\xi_t \leq \xi \leq \xi_f) \equiv -E_t$, we obtain

$$\Delta\xi_{tr} = \frac{\psi_0(r_t) - \psi_{min}}{E_t} \quad (3.33)$$

where the minimum wake potential is defined by $\psi_{min} \equiv \psi_0(0)$. While the presence of the load will not modify the potential at the head $\psi_0(r_t)$ due to continuity with the unloaded region, it can alter the exact value of potential ψ_{min} at the back of the bubble in some cases. For the purpose of this analysis, we will use $\psi_{min} \simeq -1$ to obtain an upper bound on $\Delta\xi_{tr}$ for the multi-sheath model.

We can now calculate the maximum bunch length $\Delta\xi_{tr}$ over which a constant wakefield can be loaded for the potential profiles specified in the analytic theory, single-sheath and multi-sheath models. For the wake potential $\psi_0(r_b) \approx \Psi_I = \frac{r_b^2}{4}$ from Eq. (3.29), we recover the expression $\Delta\xi_{tr} = \frac{r_t^2}{4E_t}$. This expression was also derived in Sec. 3.4 by solving the loaded parabolic trajectory $\tilde{r}_b(\xi)^2 = r_t^2 - 4E_t(\xi - \xi_t)$ for $\tilde{r}_b = 0$ of the analytic theory. For the potential used in the single-sheath model $\psi_0(r_b) = \frac{r_b^2}{4}(1 + \beta)$ where $\psi_{min} = 0$, the maximum bunch length is $\Delta\xi_{tr} = \frac{r_t^2}{4E_t} + \frac{\beta(r_t)r_t^2/4}{E_t}$. For the multi-sheath potential $\psi_0(r_b) = \frac{r_b^2}{4}(1 + \beta')$, where β' satisfies the condition $\psi_{min} = -1$, the maximum bunch length is $\Delta\xi_{tr} = \frac{r_t^2}{4E_t} + \frac{\beta'(r_t)r_t^2/4+1}{E_t}$.

In Fig. 3.16(b), we plot $\frac{r_t^2}{4E_t}$ (dashed blue) calculated by extracting (E_t, r_t) from simulation data along with $\Delta\xi_{tr}$ calculated from Eqs. (3.7)-(3.10) for the single-sheath β (dashed green) and multi-sheath model β' (dashed red). For reference, we also plot Eq. (3.33) using values of ψ_0 and E_t from simulation data (black). Since the potentials $\psi_0(r_t)$ in Fig. 3.16(a) are monotonically increasing, the maximum bunch length $\Delta\xi_{tr}$ that can be loaded increases with r_t in each case. The limiting behavior $\Delta\xi_{tr} \rightarrow \infty$ is also observed in each case at the top of the bubble ($r_t = r_m$) where $\frac{dr_b}{d\xi} = 0$ and $E_z(r_m) = -E_t = 0$.

From the results displayed in Figs. 3.16(a)-(b), it can be readily seen that the multi-sheath model generates values of $\Delta\xi_{tr}$ that agree well with those calculated from simulation data while the model for ψ_0 used by Tzoufras et al. [38, 39] underestimates the maximum bunch length for all values of r_t . The underlying reason is that the wake potential contributions Ψ_{II} and Ψ_{III} from source terms outside the ion channel are monotonically increasing with r_b as shown in Fig. 3.3(c) and, therefore, add to the potential difference between any two points in the back half of the bubble. This potential difference manifests itself in the term $\frac{\beta'(r_t)r_t^2/4-\psi_{min}}{E_t}$, which is positive definite since $\beta'(r_t)r_t^2/4 = \Psi_{II} + \Psi_{III} \geq \psi_{min}$ for all r_t as depicted in Fig. 3.3(c). Since the ion channel ends at the back of the bunch, i.e., $\xi_f = \xi_t + \Delta\xi_{tr}$, it also follows that the analytic expression for the bubble trajectory $r_b^2 = r_t^2 - 4E_t(\xi - \xi_t)$ also underestimates the length of the ion channel L_h regardless of where the load is placed. While the single-sheath model predicts longer bunch lengths than the analytic theory due to the additional sheath term $\frac{\beta(r_t)r_t^2/4}{E_t}$ which is positive definite, it still falls short of the multi-sheath model since it does not account for the negative wake potential $\psi_{min} \approx -1$ near the axis.

3.7 Summary

We have proposed a multi-sheath phenomenological model for describing the source term profile $S \equiv -\frac{1}{en_p}(\rho - J_z/c)$ of plasma wakefields excited by relativistic electron drivers in the nonlinear blowout regime. Using the multi-sheath model and following the formalism of Lu et al. [36], a new expression for the wake potential $\psi(r, \xi)$ is obtained and then used to solve for the trajectory of the innermost sheath electron r_b by integrating the equation of motion from the nonlinear blowout theory [36]. In cases with and without trailing bunches, we have shown that the bubble radius r_b , wake potential ψ_0 , and electric field E_z predicted by the multi-sheath model are in much closer agreement with simulations results at the rear of the wake when compared to the results from the simple sheath model used by Lu et al. [36].

In addition, the model demonstrates the capability to predict plasma wakefields in cases where electrons are injected at the rear of the bubble. We have shown how the multi-sheath model can be used to design beams that can load a constant wakefield and have discussed differences between the predictions for beam loading based on the multi-sheath model and single-sheath model in the ultrarelativistic limit used by Tzoufras et al. [39]. Two examples are also provided in which the multi-sheath model is used to load longitudinally varying wakefields. Finally, we examined the shortcomings of the multi-sheath model in cases with laser drivers and briefly outlined how the model can be better adapted for laser drivers in future work.

CHAPTER 4

Generating High Quality Ultra-Relativistic Electron Beams using an Evolving Electron Beam Driver

4.1 Overview and Proof-of-Concept Simulations

While the generation of ultra-relativistic electron beams through self-injection in an evolving plasma wake has been observed in LWFA experiments [18, 19, 20, 21] and demonstrated in simulations [58, 97, 98, 37], the beams produced to date do not exhibit the sufficiently low energy spreads σ_γ and high normalized brightnesses $B_n = 2I/\epsilon_n^2$ required to drive XFEL devices [40] where I and ϵ_n represent the current and normalized emittance, respectively. In recent years, electron injection schemes involving field ionization [43, 44, 45, 46, 47, 48] or the use of a plasma density down ramp (DDR) [49, 50, 52, 53, 92] have shown tremendous potential for high quality beam generation for XFEL applications.

In this chapter, we propose and demonstrate a new method of controllable injection using an electron beam driver whose spot size is decreasing in the nonlinear blowout regime to control the wake phase velocity and hence induce electron trapping. As when using a DDR, this proposed method relies on gradually elongating the ion column or cavity length as the drive beam propagates. In this scheme, injection can be achieved by focusing the electron beam driver over spot sizes, σ_r , ranging from the scale length of the blowout radius $\sim r_m$ to spot sizes much less than r_m . A schematic of this process is shown in Fig. 4.1(a). For spot sizes in this range, we will show below that the ion column length and blowout radius slightly increase as σ_r decreases and become insensitive to variations in σ_r when $\sigma_r \ll r_m$.

Therefore, self-injection can be induced by controlling the focusing of the drive beam. This new approach is also a physically simpler alternative to injection methods such as DDR and Ionization, which require sharp density gradients or multiple drivers to produce high quality electron beams. Using OSIRIS simulations, we find for parameters expected to be achievable at FACET II [99] that beams with $B_n \sim 10^{20}$ A/m²/rad² could be produced.

The parameters that control this self-injection process are the peak normalized charge per unit length $\Lambda(\xi) \equiv 4\pi r_e \int_0^{r \gg \sigma_r} n_b(\xi) r dr$ where $\xi \equiv (ct - z)$ for a driver moving in the $\hat{z}$ direction, and the Courant-Snyder (CS) parameters, β , α , and γ [88], where $\beta = \langle x^2 \rangle / \epsilon$, $\alpha = -\langle xx' \rangle / \epsilon$, $\gamma = -\langle x'^2 \rangle / \epsilon$, and $\epsilon = \sqrt{\langle x^2 \rangle \langle x'^2 \rangle - \langle xx' \rangle^2}$ is the transverse geometric emittance. The diffraction length for a particle beam in vacuum is $\beta^* \equiv \frac{\sigma_0^2}{\epsilon}$ where σ_0 is the focal spot size.

The proposed injection process relies on tuning how the spot size evolves. The initial evolution of the spot size, $\sqrt{\epsilon\beta}$, can be estimated from Taylor expanding about its initial value as it enters the plasma,

$$\sigma_r = \sqrt{\epsilon\beta} = \sigma_i + \sigma'_i \Delta z + \frac{\sigma''_i}{2} \Delta z^2 \dots \quad (4.1)$$

where Δz is the distance of the beam from the entrance of the plasma, $\sigma'_i = \sqrt{\epsilon_i \gamma_i} \frac{-\alpha_i}{\sqrt{1+\alpha_i^2}}$ and σ''_i for the part of the beam that resides in the fully formed wake can be obtained from $\sigma''_r = \frac{\epsilon_n^2}{\gamma_b^2 \sigma_r^3} \left(1 - \frac{\gamma_b^2 k_\beta^2 \sigma_r^4}{\epsilon_n^2}\right)$ where $k_\beta \equiv \frac{k_p}{\sqrt{2}\gamma_b}$, $k_p \equiv \frac{\omega_p}{c}$, $\omega_p^2 \equiv \frac{4\pi e^2 n_0}{m}$, $\epsilon_n \equiv \gamma_b \epsilon$ is the normalized emittance and γ_b is used for the relativistic factor of the beam to differentiate it from the CS parameter, γ . Two distinct regimes exist depending on whether σ'_i or σ''_i terms dominate respectively. These regimes can also be defined as when $k_\beta \beta^*$ either $\ll$ or $\gg$ than unity.

These two regimes are illustrated in Fig. 4.1 where results from OSIRIS simulations are shown. Simulations results presented here use the quasi-3D algorithm implemented in OSIRIS which expands the fields and currents into an arbitrary number of azimuthal modes m on an r - z grid [65, 67]. We also use a new customized finite-difference solver to reduce numerical effects induced by the rapid rise in current during injection [82, 86]. The

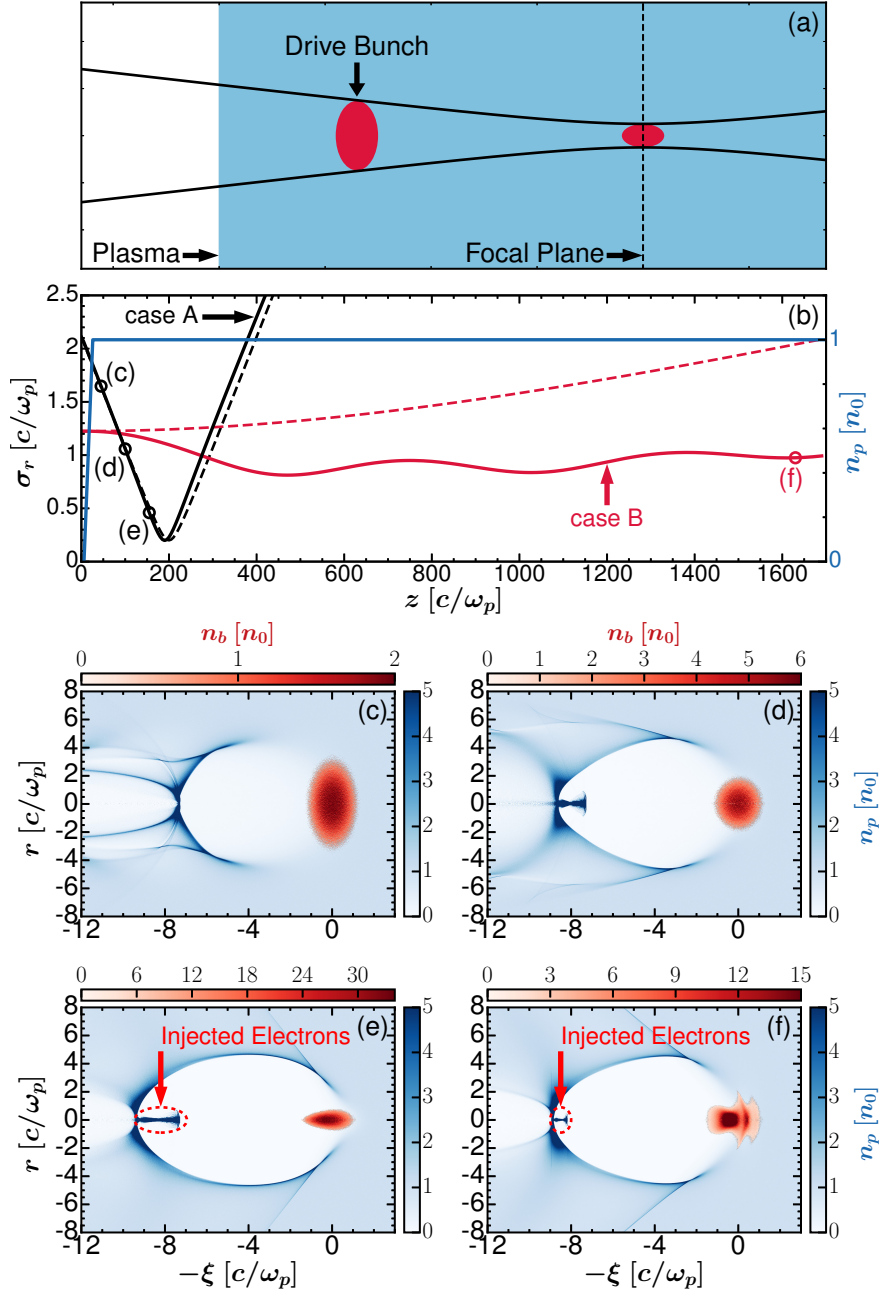


Figure 4.1: (a) Schematic of self-injection in PWFA. Simulation results are shown in two regimes for for electron drivers with $\Lambda = 6$ focused into a plasma. (b) The projected spot size σ_r of the driver is calculated from the simulation data (solid) and for vacuum propagation (dashed) for case A (black) and case B (red). The plasma density profile is shown in blue. The plasma wake is shown at propagation distances into the plasma of (c) $z = 45 c/\omega_p$, (d) $z = 100 c/\omega_p$, (e) $z = 155 c/\omega_p$ for case A and (f) $z = 1630 c/\omega_p$ for case B.

simulations used relatively high resolution with $\Delta z = \Delta r = \frac{1}{128} \frac{c}{\omega_p}$, $\Delta t = \frac{1}{512} \frac{1}{\omega_p}$ and 4 to 128 macro-particles per cell depending on the species and the number of azimuthal modes used.

In case A, a 10 GeV ($\gamma_b = 20000$) driver with a peak current of 51 kA ($\Lambda = 6$) is initialized with a centroid at $\xi_c = 0$, $\sigma_z = 0.7 c/\omega_p$, a normalized emittance $\epsilon_n \equiv \gamma_b \epsilon = 41.9 c/\omega_p$. For this case, it is assumed that the driver is focused by a conventional optic, such as a permanent magnet quadrupole (PMQ), so that the CS parameters at the plasma entrance are given by $\beta_i \approx 2115 c/\omega_p$ and $\alpha_i \approx 10.5$. For these parameters $k_\beta \beta^* \approx .1$ and the evolution of the spot size is dominated by vacuum diffraction as seen by the agreement between the spot size evolution in the plasma from the simulation and the predicted vacuum evolution (dashed black) in Fig. 4.1(b). As shown in Fig. 4.1(c), the wake is being excited in the nonlinear blowout regime when the beam enters the plasma. As the driver spot size decreases, in Fig. 4.1(c)-(e) the bubble length also increases, thereby reducing the wake velocity γ_ϕ allowing the highest energy sheath electrons with $\gamma_z \equiv (1 - \beta_z^2)^{-1/2} > \gamma_\phi$ to be trapped at the rear of the wake. At spot sizes much less than the maximum blowout radius $r_m \approx 2\sqrt{\Lambda}c/\omega_p$ [36], the wake wavelength and blowout radius approach their maximum values in Figs. 4.1(e) and further focusing does not produce any injection.

In case B, an electron drive bunch with the same length, centroid, and energy is focused to the entrance of the plasma with $\epsilon_n \equiv \gamma_b \epsilon = 24.5 c/\omega_p$, and CS parameters $\beta_i = \beta^* \approx 1225 c/\omega_p$ and $\alpha_i \approx 0$. For these parameters $k_\beta \beta^* = 6.125$ and as seen in Fig 4.1(b) the spot size evolution (solid red) is now dominated by the focusing force of the plasma ion column. Since the driver is not matched, it is self-focused by the plasma and oscillates at the scale length of the betatron wavelength $2\pi\sqrt{2\gamma_b}c/\omega_p$. During the first betatron period, plasma electrons are injected at the rear of the bubble as the spot size of the driver decreases and the wake expands as seen in Fig. 4.1(f). The spot size evolution $\sigma_r(z)$ therefore deviates significantly from the vacuum curve (dashed red). This regime is more complicated because the ion channel is not fully formed along the entire driver and therefore only the rear of the beam is fully focused. Each slice of the front of the beam oscillates at different betatron

frequencies leading to the scalloping [28] seen in Fig. 4.1(f). Each scallop corresponds to a full betatron oscillation within the ion channel.

While the spot size continues to oscillate after the initial injection in case B, the bubble remains fully expanded due to beam loading effects and spot size damping (which will be discussed in chapter 5). This can be seen in Fig. 4.2(a) where we plot the electron density distribution of the plasma wake, driver, and injected beam after the driver has propagated a distance $k_p z = 1630$ into the constant density plasma. The blowout radius at this point is $r_m \simeq 3.9$ and each driver scallop corresponds to a full betatron oscillation [28]. As seen in the inset plot of Fig. 4.2(a), the current profile of the injected bunch $\lambda(\xi)$ varies from 2 (~ 17 kA) to 4 (~ 34 kA) over the core of the bunch.

Since the driver evolves over a length scale much greater than the wavelength of the wake in case B, we can use the multi-sheath model [100] to characterize the effects of beam loading. In Fig. 4.2, we compare numerical calculations (dashed red, solid red, and solid blue) of the bubble trajectory $r_b(\xi)$, potential $\psi_0(\xi)$, and electric field $E_z(\xi)$ from Eqs. (3.7)-(3.10) to simulation results (black). The numerical calculations use $\lambda(\xi)$ of the injected bunch taken from the simulation. The integration parameters $\{\Delta_{10}, \Delta_{20}, s\}$ are identical to those used in Figs. 3.5 and 3.6. The solid red lines correspond to the loaded wake calculated using the multi-sheath model with $\psi_{min} = -1$ in Eq. (3.10), while the solid blue lines correspond to the loaded wake calculated using the single-sheath model with $\psi_{min} = 0$. For reference, we also plot the unloaded wake (dashed red) obtained from the multi-sheath model to illustrate the effects of beam loading from the self-injected bunch. Based on the results, it is clear that the presence of the injected beam modifies the bubble trajectory and extends the wavelength of the wake [Fig. 4.2(a)]. The injected beam, therefore, loads the wakefield $E_z(\xi)$ as illustrated by the numerical results from the multi-sheath model in Fig. 4.2(c). At the rear of the wake, the multi-sheath model also agrees very well with the PIC simulation results while the single-sheath model does not. This agreement is significant because it shows that it is now possible to model precisely how self-injected beams load the plasma wake.

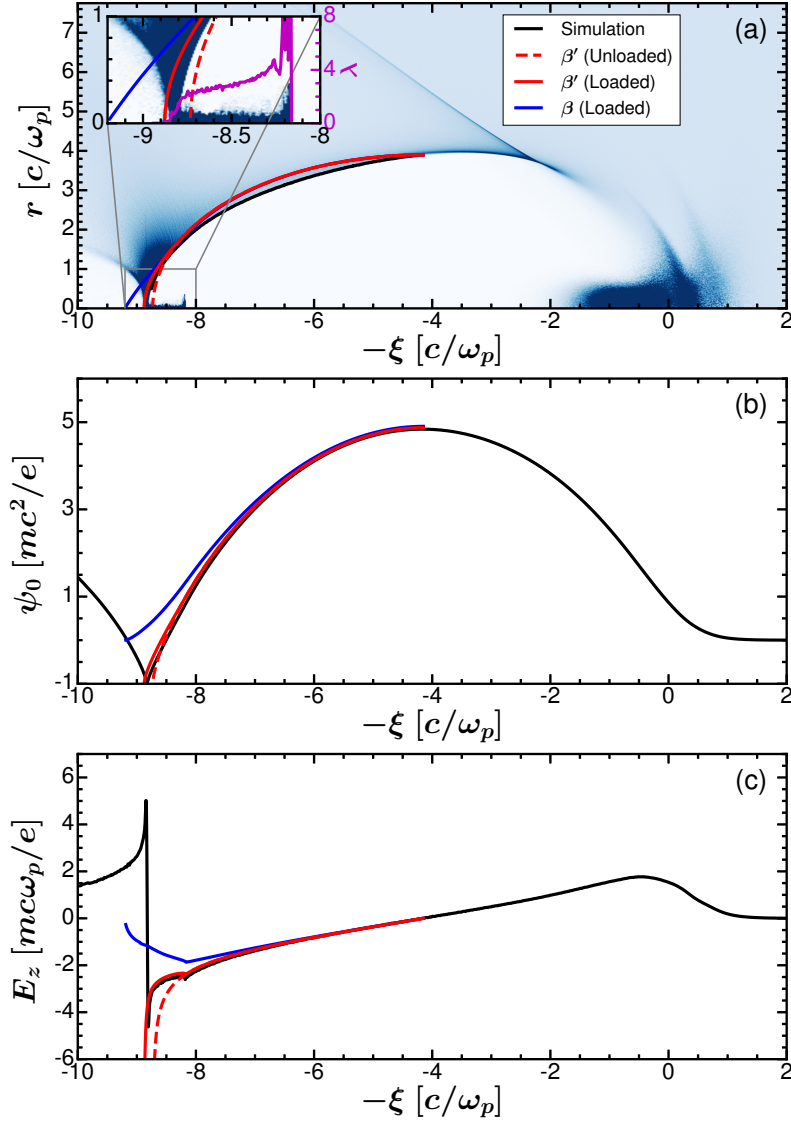


Figure 4.2: (a) Electron density distribution of the plasma wake from case B at $z = 1630 c/\omega_p$. The blowout radius is $r_m \simeq 3.9$ at this point. The current profile $\lambda(\xi)$ of the injected electrons is shown in purple in the inset plot. Comparisons of simulation results (black) and numerical calculations (dashed red, solid red, solid blue) of (a) r_b , (b) ψ_0 , and (c) E_z using Eqs. (3.7)-(3.10). For all calculations, the integration parameters $\{\Delta_{10} = 1, \Delta_s = 0.05r_b, \Delta_{20} = 3, s = 3\}$ are the same as those used in Figs. 3.5 and 3.6. The dashed and solid red lines correspond to the unloaded and loaded wake calculated using the multi-sheath model β' with $\psi_{min} = -1$. The blue lines correspond to the loaded wake calculated using the single-sheath model β with $\psi_{min} = 0$.

4.2 Electron Dynamics in Quasistatic Wakes with Different Driver Spot Sizes

To understand the injection dynamics, we start by examining the electron trajectories within quasistatic wakes excited by drivers with different spot sizes σ_r . For beam-driven wakefields (PWFA), electrons are initially expelled by the space-charge force of the driver leaving behind ions which attract the electrons back the axis. For a bi-Gaussian driver with a density profile $n_b(\xi, r) = n_{b0}e^{-r^2/(2\sigma_r^2)}e^{-\xi^2/(2\sigma_z^2)}$, the transverse force it exerts on a plasma electron is $F_d = mc\omega_p(1 - \beta_z)\frac{\Lambda(\xi, r)}{k_p r}$, where $\Lambda(\xi, r) \equiv \Lambda(\xi)[1 - e^{-r^2/(2\sigma_r^2)}]$ denotes the charge per unit length seen by an electron. In Refs. 35 and 36, it was shown that in the nonlinear blowout regime the plasma wake forms a nearly spherical shape with a maximum blowout radius $r_m \approx 2\sqrt{\Lambda}c/\omega_p$ based on the peak charge per unit length $\Lambda = n_{b0}\sigma_r^2$. Therefore, the electrons of interest reside within characteristic scale length of the wake $r_i \lesssim O(\sqrt{\Lambda}c/\omega_p)$.

In the large spot size limit $\sigma_r \gtrsim \sqrt{\Lambda}c/\omega_p$, the transverse force of the driver on these plasma electrons is small since only a fraction of the total charge per unit length of the driver is seen $\Lambda(\xi, r) \ll \Lambda(\xi)$. Therefore, a linear wake is produced by the driver where the plasma electron perturbation is small. This can be seen in Fig. 4.3(a), where the electron trajectories from a quasistatic PWFA simulation using the PIC code OSIRIS are plotted. A nonevolving driver with peak normalized charge per unit length $\Lambda = 6$ (51 kA), energy $\gamma_b = 20000$ (10 GeV), spot size $k_p\sigma_r = \sqrt{\Lambda}$, duration $k_p\sigma_z = 0.7$, and centroid $k_p\xi_c = 0$ was used.

As the driver spot size σ_r decreases, the wake response transitions from the linear regime to the nonlinear blowout regime as the effective charge per unit length $\Lambda(\xi, r)$ and the transverse force increase within $r \lesssim \sqrt{\Lambda}c/\omega_p$. This transition is typically signaled by the formation of a thin, high density plasma sheath and electron trajectory crossing. This is illustrated in Fig. 4.3(b) for a driver spot size of $k_p\sigma_r = 0.75\sqrt{\Lambda}$ corresponding to the particle crossing threshold $n_b \geq 1.792n_p$ [36]. As in the linear regime, the transverse force exerted on

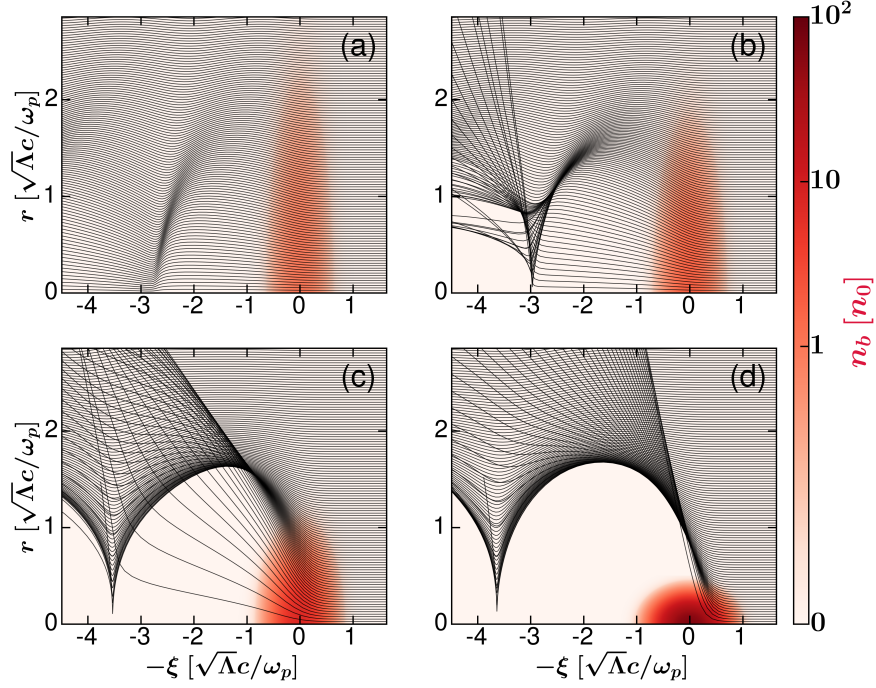


Figure 4.3: Electron trajectories obtained from simulations of quasistatic plasma wakes triggered by nonevolving electron drivers with $\Lambda = 6$, $\gamma_b = 20000$, $k_p\sigma_z = 0.7$, and $k_p\xi_c = 0$. The spot sizes $k_p\sigma_r$ of the drivers are (a) $\sqrt{\Lambda}$, (b) $\frac{3}{4}\sqrt{\Lambda}$, (c) $\frac{3}{8}\sqrt{\Lambda}$, and (d) $\frac{1}{8}\sqrt{\Lambda}$.

the plasma electrons within $r \lesssim \sqrt{\Lambda}c/\omega_p$ continues to increase as σ_r decreases in the blowout regime. This leads to the larger blowout radii and bubble lengths seen in Figs. 4.3(c)-(d) until complete electron cavitation occurs when $\sigma_r \ll \sqrt{\Lambda}c/\omega_p$.

In order to understand and quantify how the wake expansion depends on the evolution of the spot size, we performed numerous simulations using non-evolving drivers with varying σ_r and Λ . In each quasi-static simulation, a 10 GeV ($\gamma_b = 20000$) driver with $\sigma_z = 0.7 c/\omega_p$ was used. We tracked the trajectories of the sheath electrons to determine the blowout radius r_m , the ion column length L_b , and the highest forward velocity $\gamma_{z,m}$ of sheath electrons at the rear of the wake. In Fig. 4.4(a), we plot the trajectories of $r(\xi)$ for the highest energy sheath electron for $\Lambda = 6$ and two different driver spot sizes $k_p\sigma_r = \frac{1}{32}\sqrt{\Lambda}$ and $\frac{3}{4}\sqrt{\Lambda}$ respectively. These tracks clearly show that the ion channel size varies for these two spot sizes. For each

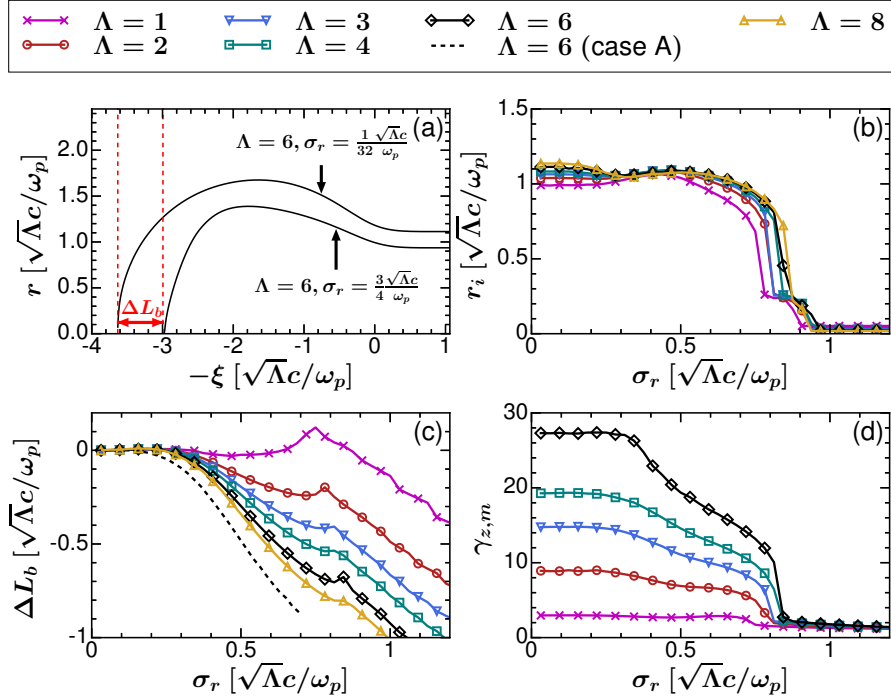


Figure 4.4: (a) Trajectories of the most energetic sheath electrons for $\sigma_r = \frac{1}{32}\sqrt{\Lambda}c/\omega_p$ and $\frac{3}{4}\sqrt{\Lambda}c/\omega_p$ when $\Lambda = 6$. The change in the bubble length ΔL_b is shown by the red dashed lines. (b) The initial position of the plasma sheath electrons r_i , (c) the change in bubble length ΔL_b , and (d) the maximum sheath electron forward velocity $\gamma_{z,m}$ as a function of the spot size σ_r for quasi-static plasma wakes. Lengths are normalized by $\sqrt{\Lambda}c/\omega_p$. Non-evolving electron drivers with energy $\gamma_b = 20000$ and $\sigma_z = 0.7 c/\omega_p$ were used. ΔL_b is also calculated from the evolving wake in case A [see Fig. 4.1] and plotted in (c) with dashed black lines.

simulation in Figs. 4.4(b)-(d), we also plot the initial radius r_i , deviation from the maximum bubble length $\Delta L_b(\sigma_r) \equiv L_b(\sigma_r) - L_b(\sigma_r \ll \sqrt{\Lambda}c/\omega_p)$, and maximum sheath electron forward velocity $\gamma_{z,m} \equiv (1 - \beta_{z,m}^2)^{-1/2}$ for the most energetic electron for a variety of spot sizes and Λ . ΔL_b calculated from case A (dashed black) is also plotted in Fig. 4.4(c) which is in reasonable agreement with the non-evolving results. The deviation can be explained from beam loading effects from injected electrons.

As the spot size approaches the particle crossing condition, typically around $\sim 0.75 -$

0.85 $\sqrt{\Lambda}c/\omega_p$ depending on Λ , the wake response transitions from the linear regime to the blowout regime. This transition is signaled by a local maximum in ΔL_b and a rapid rise in $\gamma_{z,m}$ and r_i due to formation of a thin plasma sheath. The most energetic sheath electrons originate from initial positions $r_i \approx \sqrt{\Lambda}c/\omega_p$ [Fig. 4.4(b)] which are insensitive to Λ . As in the linear regime, the transverse repelling force exerted on these electrons increases with the effective charge per unit length $\Lambda(\xi, r_i)$ as σ_r decreases, creating larger blowout radii and bubble lengths as shown in Figs. 4.4(a) and 4.4(c). The sheath electrons also exhibit higher $\gamma_{z,m}$ [Fig. 4.4(d)] as more background electrons are blown out by the driver, increasing the density spike and magnitude of E_z at the rear of the wake. Although $\gamma_{z,m}$ increases significantly it is still orders of magnitude below γ_b so trapping cannot occur. We also note that both $\left| \frac{d(\Delta L_b)}{d\sigma_r} \right|$ and $\gamma_{z,m}$ increase with Λ as seen in Fig. 4.4(c)-(d). When $\sigma_r \ll \sqrt{\Lambda}c/\omega_p$, the transverse driver force on the sheath electrons at r_i peaks as $\Lambda(\xi, r_i) \rightarrow \Lambda(\xi)$, producing complete electron cavitation within the wake. In this limit, the bubble length L_b , blowout radius r_m , and $\gamma_{z,m}$ saturate at their maximum values and are insensitive to $k_p\sigma_r$.

4.3 Electron Trapping and Acceleration in Evolving Wakes

For an evolving relativistic drive bunch ($\gamma_b \gg 1$), the wake phase velocity is defined by

$$\beta_\phi \approx 1 - \frac{dL_b}{dz} = 1 - \frac{dL_b}{d\sigma_r} \frac{d\sigma_r}{dz}. \quad (4.2)$$

To calculate γ_ϕ , we need to determine $\frac{dL_b}{d\sigma_r}$ and $\frac{d\sigma_r}{dz}$. We assume that $\frac{dL_b}{d\sigma_r} = \frac{d(\Delta L_b)}{d\sigma_r}$ depends only on the instantaneous spot size σ_r and, therefore, can be calculated directly from Fig. 4.4(c). The second term σ_r' is straightforward to obtain for case A from Fig. 4.1. However, for case B, σ_r' is more complicated to determine because the spot size evolution varies along ξ . The rear evolves as in a fully formed wake while the slices in the front have betatron frequencies that depend on the amount of blowout which leads to a scalloping behavior described above.

For simplicity, we analyze case A in which σ_r evolves only as a function z and $\sigma_r(z)$ is well approximated by the vacuum CS parameters, $\sigma'_r = \frac{-\alpha\sqrt{\epsilon}}{\sqrt{\beta}}$. Using the beta function and the CS relation $\beta\gamma = 1 + \alpha^2$, we obtain an expression for the spot size evolution $\sigma'_r = -\sqrt{\epsilon\gamma}\frac{\alpha}{\sqrt{1+\alpha^2}} \approx -\sqrt{\epsilon\gamma_i}$, since γ is constant in vacuum and $\alpha \gg 1$ during most of the injection.

We next approximate $\gamma_\phi \approx \sqrt{\frac{1}{2(1-\beta_\phi)}}$ and use Eq. (4.2) to obtain,

$$\gamma_\phi \approx \sqrt{\frac{1}{2} \frac{1}{\frac{dL_b}{d\sigma_r}} \frac{1}{\sigma'_r}} = \sqrt{\frac{\gamma_b}{-\frac{dL_b}{d\sigma_r}} \frac{\sigma_0}{2\epsilon_n}}. \quad (4.3)$$

From Fig. 4.4(c) we can see that $\frac{dL_b}{d\sigma_r} \approx -1.76$ for $k_p\sigma_r = 0.5\sqrt{\Lambda}$ and $\Lambda = 6$; therefore, γ_ϕ (≈ 5.2) can be significantly reduced from γ_b making injection possible. In Fig. 4.5(a), we plot γ_ϕ (dashed black) calculated from Eq. (4.3) for case A against $\gamma_{z,m}$ (solid black) when $\Lambda = 6$ from Fig 4.4(d). The vertical blue dashed lines enclose the region where the injection condition is satisfied $\gamma_{z,m} > \gamma_\phi$. As the driver focuses, self-injection begins near $\sigma_r \approx 0.81\sqrt{\Lambda}c/\omega_p$ as $\gamma_{z,m}$ rapidly increases and crosses γ_ϕ . When $\sigma_r \approx 0.21\sqrt{\Lambda}c/\omega_p$, the bubble length saturates $\frac{dL_b}{d\sigma_r} \rightarrow 0$ and injection terminates as γ_ϕ increases above $\gamma_{z,m}$.

Since L_b is a monotonically decreasing function of σ_r in the blowout regime, there is a one-to-one mapping between the initial position z_i and final position ξ_f of injected electrons as seen in Fig. 4.5(b). This feature can lead to low absolute slice energy spreads as pointed out in [52]. The dependance of ξ_f on z_i is given by

$$\frac{d\xi_f}{dz_i} \approx \frac{dL_b}{dz_i} \approx \kappa \frac{d\sigma_r}{dz_i} \quad (4.4)$$

where κ is the average of $\frac{d(\Delta L_b)}{d\sigma_r}$ over the range $0.3 - 0.7\sqrt{\Lambda}c/\omega_p$ when $\Lambda = 6$ in Fig. 4.4(c).

Whereas $\sigma'_r \approx -\sqrt{\epsilon\gamma_i}$ is determined from the vacuum beta function in case A, an lower bound for σ'_r in case B can be estimated by integrating $\sigma''_r = \frac{\epsilon_n^2}{\gamma_b^2\sigma_r^3} \left(1 - \frac{\gamma_b\sigma_r^4}{2\epsilon_n^2}\right)$ to obtain

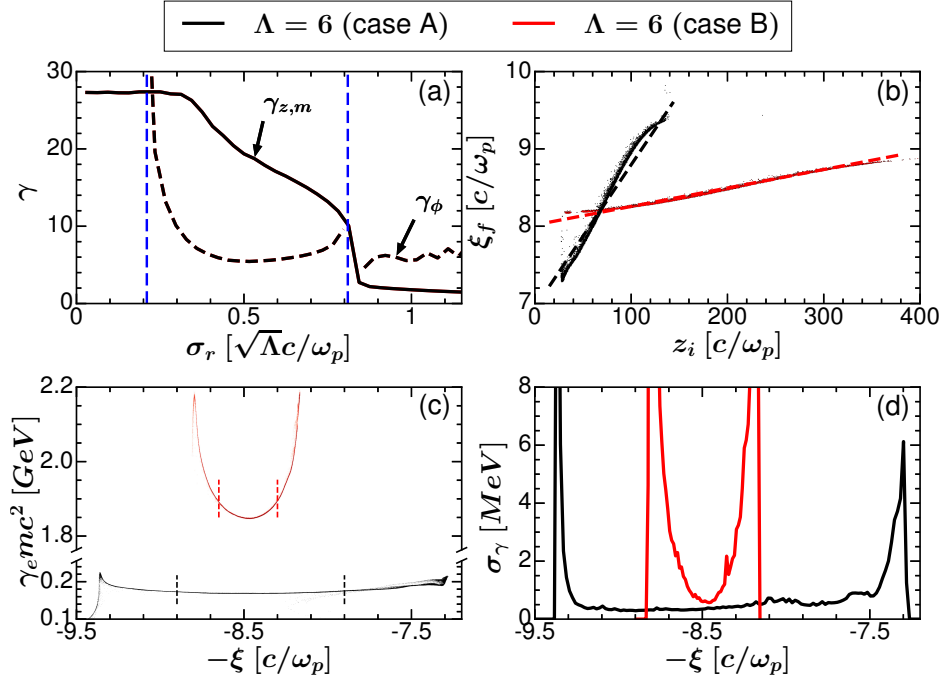


Figure 4.5: (a) $\gamma_{z,m}$ (solid) and γ_ϕ (dashed) calculated from Eq. (4.2) as a function of σ_r for case A. The blue dashed lines enclose regions of injection where $\gamma_{z,m} > \gamma_\phi$. (b) The phase-space of injected electron charge in the $\xi_f - z_i$ plane for case A (black) and case B (red). The dashed lines are calculated from Eq. (4.4). (c) The phase space $\gamma - \xi$ and (d) slice energy spread σ_γ of injected electrons at position $z = 260$ (1630) c/ω_p for case A (B). The injected beams are subdivided into 128 (64) slices for cases A (B), respectively, in (d).

$$\int \sigma'_r \sigma''_r dz = \int \frac{\epsilon_n^2}{\gamma_b^2 \sigma_r^3} \left(1 - \frac{\gamma_b \sigma_r^4}{2 \epsilon_n^2} \right) \sigma'_r dz$$

$$\sigma_r'^2 = -\frac{\epsilon_n^2}{\gamma_b^2 \sigma_r^2} - k_\beta^2 \sigma_r^2 + Const. \quad (4.5)$$

Since the driver in case B is initially focused at the plasma entrance with a focal spot size $\sigma_r(z=0) = \sigma_0$ such that $\sigma'_r(z=0) = 0$, Eq. (4.5) can be expressed as

$$\sigma_r'^2 = -\frac{\epsilon_n^2}{\gamma_b^2} \left(\frac{1}{\sigma_r^2} - \frac{1}{\sigma_0^2} \right) - k_\beta^2 (\sigma_r^2 - \sigma_0^2). \quad (4.6)$$

to obtain a lower bound for σ'_r , we evaluate Eq. (4.6) at the matched condition $\sigma_r = \sigma_m = \left(\frac{2\epsilon_n^2}{\gamma_b}\right)^{\frac{1}{4}}$ where $\sigma''_r = 0$

$$\sigma'_r(\sigma_m) = -\sqrt{-\frac{\epsilon_n^2}{\gamma_b^2} \left(\frac{1}{\sigma_m^2} - \frac{1}{\sigma_0^2} \right) - k_\beta^2 (\sigma_m^2 - \sigma_0^2)}. \quad (4.7)$$

To account for axial variations in the betatron frequency in case B, we also assume a simple model in which the rear half of the drive bunch (which is fully contained within the ion channel) undergoes betatron oscillations while the front half is approximately non-evolving during injection. A more detailed analysis of the spot size evolution in the $k_\beta \beta^* \gg 1$ regime (i.e., case B) will be provided in chapter 5.

In Fig. 4.5(b), excellent agreement is observed between the injected electron phase space (ξ_f, z_i) and Eq. (4.4) in both cases. After some optimal acceleration distances, flat phase space distributions in (γ_e, ξ) are achieved over some regions of the injected beams as shown in Fig. 4.5(c). Within the regions enclosed by dashed lines in Fig. 4.5(c), average energies of 169.7 MeV and 1.86 GeV and projected energy spreads of 1.1% and 0.7% for cases A and B, respectively. In both cases, slice energy spreads as low as ~ 0.5 MeV are also observed along significant regions of the injected electron beams in Fig. 4.5(d).

The slice currents and normalized brightnesses of the final injected beams are calculated and plotted in Fig. 4.6(a)-(b). While the injected beam duration in case A is nearly three times that of case B, larger slice currents of $I \gtrsim 17 - 34$ kA are observed across most of the injected beam in case B. The large variations in the current profile at the front of the injected beam in case B can be attributed to a more slowly evolving wake and beam loading effects from the initial electron injection as noted in [52]. In Fig. 4.6(b), peak normalized brightnesses of ~ 16 and ~ 27 [$n_0 \text{ cm}^{-3}$] A/m²/rad² are found for cases A and B, respectively. In Figs. 4.6(c)-(d), peak slice brightnesses and corresponding slice currents are plotted for various simulations ranging from Λ of 2 to 6. In the results shown, peak brightnesses of at least $\gtrsim 10^{21}$ A/m²/rad² are achieved across all values of Λ for plasma

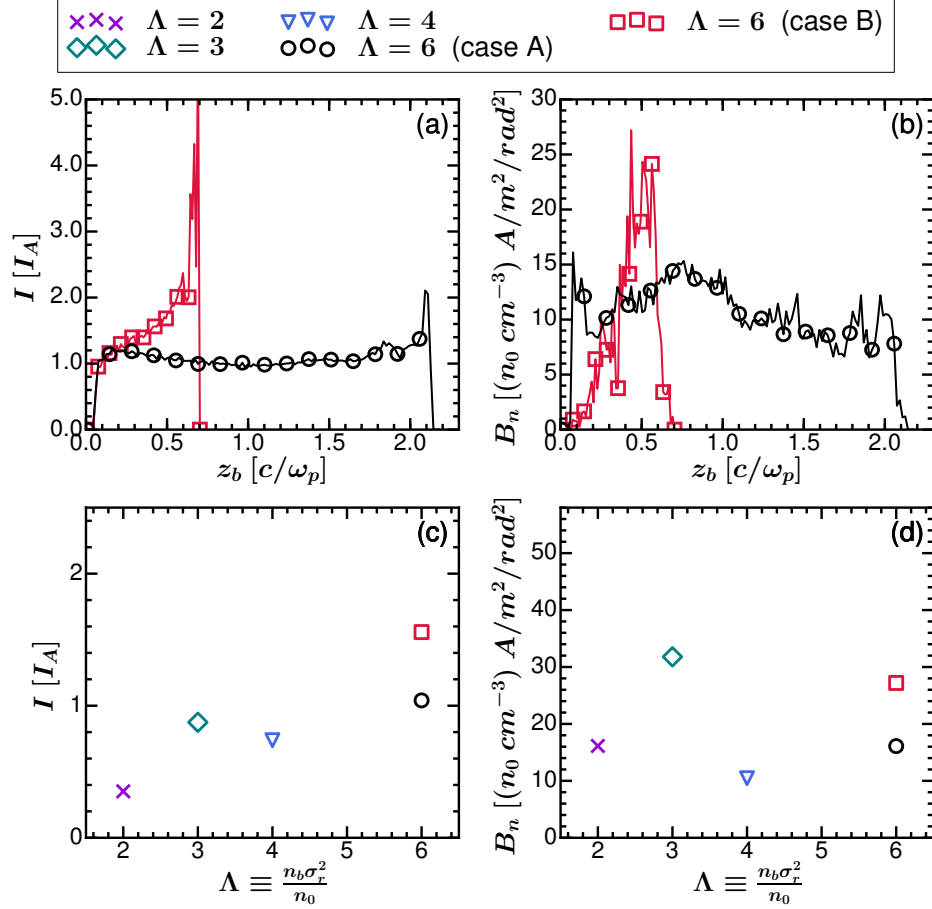


Figure 4.6: The beam slice parameters of the injected electrons for case A and case B: (a) the current, in units of the Alfvén current I_A and (b) the normalized brightness B_n in units of $[(n_0 \text{ cm}^{-3}) \text{ A/m}^2/\text{rad}^2]$. The injected beams are subdivided into 128 (64) slices for cases A (B), respectively. (c) Slice current corresponding to the (d) peak normalized brightness of the injected electron beam as a function of the driver parameter Λ . Each marker corresponds to a unique simulation. Approximately 95 % of injected electrons are used when calculating the beam parameters.

densities of $n_0 \sim 10^{20} \text{ cm}^{-3}$ with slice currents ranging from 6 – 26 kA.

At FACET II, it is anticipated that drive beams with currents in the range of 50-150 kA and durations of $\sim 3 \text{ fs}$ ($1 \text{ }\mu\text{m}$) will be available [99]. These parameters match the simulations presented here for $n_0 \sim 10^{19} \text{ cm}^{-3}$, indicating that peak brightnesses higher than $10^{20} \text{ A/m}^2/\text{rad}^2$ and normalized emittances $\epsilon_n = \sqrt{\frac{2I}{B_n}}$ as low as $\sim 10 \text{ nm}$ can be produced as shown in Figs. 4.6(c)-(d). In the next chapter, we present a detailed parameter scan of what is possible in the plasma focusing regime.

4.4 Summary

In this chapter, we proposed and demonstrated a new method of controllable injection to generate high quality electron bunches. The concept relies on focusing an electron drive bunch in the nonlinear blowout regime to expand the wavelength of the wake and induce electron trapping. Injection was demonstrated in two different regimes using PIC simulations. In the $k_\beta \beta^* \ll 1$ regime (case A), the driver is focused according to the vacuum CS beta function, while in the $k_\beta \beta^* \gg 1$ regime (case B), it is self-focused by the plasma ion channel. The multi-sheath model was used to illustrate the effects of beam loading due to self-injection in case B. The wake evolution and injected beam parameters were also characterized via theory and simulations. Injected beams with average energies of 169.7 MeV and 1.86 GeV and projected energy spreads of 1.1% and 0.7% were demonstrated in PIC simulations of cases A and B, respectively. High quality beams were produced with peak normalized brightnesses in excess of $10^{20} \text{ A/m}^2/\text{rad}^2$ and normalized emittances as low as $\sim 10 \text{ nm}$ for plasma densities of $n_0 \sim 10^{19} \text{ cm}^{-3}$.

CHAPTER 5

Characterizing the Injected Beam Parameters using Plasma Focusing

5.1 Advantages of using the Plasma as the Focusing Optic

There are several advantages to operating in the plasma self-focusing regime ($k_\beta \beta^* \gg 1$) rather than the vacuum diffraction regime ($k_\beta \beta^* \ll 1$). Perhaps, the most obvious advantage is that much higher focusing gradients can be achieved using the plasma rather than a conventional permanent quadrupole magnet (PMQ) as the focusing optic. In fact, the concept of a passive plasma lens was originally proposed to improve beam luminosity for future linear collider experiments [101]. For a typical plasma density of 10^{17} cm^{-3} , the focusing strength of the plasma can reach $\sim 3 \text{ MT/m}$ [102], which is several orders of magnitude higher than the focusing strength of state-of-the-art quadrupoles $\sim 560 \text{ T/m}$ [103, 104]. Therefore, beams focused by PMQs are generally not capable of operating in the $k_\beta \beta^* \ll 1$ regime (case A) for plasma densities which can provide high gradient acceleration. As a result, for typical parameters injection must be triggered in the $k_\beta \beta^* \gg 1$ regime (case B). In this regime, the spot size is initially unmatched and will therefore oscillate. Injection occurs during the first oscillation.

In reality, for short drivers $k_p \sigma_z \sim 1$, the plasma density and, thus, the focusing force will vary along the driver in regions where plasma electrons are not completely blown out. As we will show later, this can lead to different beam slices in ξ coming into focus at different times. As a result, the driver can remain well-guided by the plasma with a reduced spot size long

after the initial injection [see Fig. 4.1(b)]. This allows for much longer acceleration distances and higher energies in the $k_\beta\beta^* \gg 1$ regime (case B) compared to the regime $k_\beta\beta^* \ll 1$ (case A), where the driver defocuses as quickly as it focuses. This is ideal for beam loading scenarios in which sufficient charge is injected, the plasma wakefield is flattened, i.e., $d_\xi E_z \approx 0$, and it is possible to take advantage of the longer propagation distances to produce multi-GeV injected beam energies with low projected energy spreads. In addition, the injected beams produced in the $k_\beta\beta^* \gg 1$ regime also exhibit the kinds of trapezoidal current profiles [see Fig. 4.6(a)] needed to load constant wakefields [38, 39].

While the underlying mechanics of injection are the same in each regime – the driver focuses, the wake expands, and energetic sheath electrons are injected – it is clear that there are numerous advantages to operating in the $k_\beta\beta^* \gg 1$ regime as outlined above. Previously in chapter 4, the evolution of the spot size of the drive beam, σ'_r , was evaluated at the matched spot size. In this chapter, we will develop a more robust model for characterizing the injection process in the plasma self-focusing regime. The model will utilize the electron sheath dynamics [Fig. 4.4] to predict how the final injected beam parameters vary with the driver diffraction length β^* , initial energy γ_b , initial spot size σ_r and peak current Λ . Parameter scans of PIC simulations using the code OSIRIS will also be presented and compared with the theory. We will also show how to flatten the plasma wakefield by tuning injected currents through the drive beam length σ_z .

5.2 A Theoretical Framework for Injection in the Plasma Focusing Regime

We are interested in characterizing the wake evolution and, hence, injection due to spot size focusing within the first betatron oscillation in the plasma focusing regime where $k_\beta\beta^* \gg 1$. As shown previously, the expansion rate of the wake can be written as

$$\frac{dL_b}{dz} = \frac{dL_b}{d\sigma_r} \frac{d\sigma_r}{dz} = \kappa \sigma'_r \quad (5.1)$$

where κ can be calculated directly from Fig. 4.4(c). To evaluate Eq. (5.1), we need to determine σ'_r . In the previous chapter, the spot size evolution σ'_r was estimated by integrating $\sigma''_r = \frac{\epsilon_n^2}{\gamma_b^2 \sigma_r^3} \left(1 - \frac{\gamma_b \sigma_r^4}{2\epsilon_n^2}\right)$ and evaluating it at the matched condition $\sigma_r = \sigma_m = \left(\frac{2\epsilon_n^2}{\gamma_b k_p^2}\right)^{\frac{1}{4}}$. Implicit in our analysis was the assumption that a nonlinear plasma wakefield is excited in the blowout regime using a driver that is focused at the plasma entrance with a focal spot size $k_p \sigma_0 \lesssim 0.8\sqrt{\Lambda}c/\omega_p$. For such a driver, the diffraction length is given by the initial CS parameter $\beta_i \approx \beta^* = \sigma_0^2/\epsilon$ and the matched spot size can be written as $\sigma_m = \frac{\sigma_0}{\sqrt{k_\beta \beta^*}}$. As shown in the previous chapter, injection can only be triggered by focusing the driver over a limited range of spot sizes $0.3\sqrt{\Lambda}c/\omega_p \lesssim \sigma_r \lesssim 0.8\sqrt{\Lambda}c/\omega_p$. Since we are interested in determining σ'_r during injection, it does not make much sense to evaluate σ'_r at the matched spot size in the asymptotic limit $k_\beta \beta^* \rightarrow \infty$ where $\sigma_m \rightarrow 0$. Therefore, in the following section, we will develop a more robust model for σ'_r based on the periodicity of the betatron oscillations rather than the matched spot size. Using this expression, we then obtain a formula for the wake evolution $\frac{dL_b}{dz}$ from which the wake phase velocity γ_ϕ can be calculated. Finally, we develop expressions for the optimal acceleration distance and beam energy of the injected beam. The theoretical predictions are then compared with cylindrical PIC simulations using OSIRIS for different driver parameters.

5.2.1 Drive Beam Evolution

We are interested in characterizing the evolution of the driver within the first betatron oscillation where injection occurs. We consider here an axially symmetric and ultrarelativistic ($\gamma_b \gtrsim 10^4$) electron driver propagating along the $\hat{z}$ direction in a plasma with a constant background density n_0 . The driver is assumed to be operating in the nonlinear blowout regime $\sigma_r \lesssim 0.8\sqrt{\Lambda}c/\omega_p$ producing an ion channel wherein $\rho \approx \rho_{ion} = en_0$. In chapter 2, it

was shown that the focusing force is linear $F_\perp \approx -\frac{1}{2}r$ for a beam electron ($v_z \rightarrow 1$). For the parameters considered here $\Lambda \lesssim 10$ and $\gamma_b \gtrsim 10^4$, the energy loss of the beam electron over a betatron oscillation can be neglected since $|\Delta\gamma_b| \sim eE_{z,m}k_\beta^{-1}/mc^2 \approx \sqrt{2\gamma_b\Lambda} \ll \gamma_b$. In addition, radiative damping can also be ignored as it typically occurs over length scales much longer than the betatron wavelength [105]. Thus, the beam energy γ_b can be considered constant over a betatron wavelength. As a result, the transverse equation of motion for a beam electron can be written as

$$\frac{d^2x}{dz^2} = -k_\beta^2 x \quad (5.2)$$

where x is the transverse coordinate, $z \simeq ct$ is the propagation distance, and $k_\beta = k_p/\sqrt{2\gamma_b}$ is the betatron wavenumber corresponding to the background plasma density. The solution to this simple harmonic oscillator is

$$x(z) = x_i \cos(k_\beta z) + \frac{x'_i}{k_\beta} \sin(k_\beta z) \quad (5.3)$$

where x_i is the initial transverse position and $x'_i = dx_i/dz_i = p_{xi}/p_{zi}$ is the initial divergence of the particle. Assuming the driver is fully contained within the ion channel, we take the ensemble average to solve for the spot size in terms of the initial CS parameters and obtain

$$\begin{aligned} \sigma_x^2 &= \langle x^2 \rangle \\ &= \epsilon \left[\beta_i \cos^2(k_\beta z) + \frac{\gamma_i}{k_\beta^2} \sin^2(k_\beta z) - \frac{\alpha_i}{k_\beta} \sin(2k_\beta z) \right] \end{aligned} \quad (5.4)$$

where $\beta_i = \frac{\langle x_i^2 \rangle}{\epsilon}$, $\gamma_i = \frac{\langle x_i'^2 \rangle}{\epsilon}$, and $\alpha_i = -\frac{\langle x_i x_i' \rangle}{\epsilon}$. It is evident from Eq. (5.4) that these parameters, which are related by $\beta\gamma = 1 + \alpha^2$ [88], determine the evolution of the driver spot size. For a driver that is initially focused such that $\beta_i \approx \beta^* = \frac{\sigma_0^2}{\epsilon}$, $\gamma_i = \frac{1}{\beta^*}$ and $\alpha_i \approx 0$,

we can simplify further to obtain an expression for the spot size in terms of the betatron wavenumber k_β and diffraction length β^*

$$\sigma_x^2 = \sigma_0^2 \left[\cos^2(k_\beta z) + \frac{1}{(k_\beta \beta^*)^2} \sin^2(k_\beta z) \right]. \quad (5.5)$$

For axially symmetric beams considered here, we note that the spot size evolution is the same in both planes $\sigma_r = \sigma_x = \sigma_y$.

Eq. (5.5) is useful in that it describes the various operating modes of the driver. In the absence of any plasma, i.e., $k_\beta \rightarrow 0$, it is evident from Eq. (5.5) that the driver diffracts exactly according to the vacuum CS evolution $\sigma_r^2 = \sigma_0^2 \left(1 + \frac{z^2}{\beta^{*2}}\right)$. The driver evolution can also be well-approximated by the vacuum solution when the diffraction length is much smaller than the betatron wavelength, i.e., $k_\beta z \sim k_\beta \beta^* \ll 1$. In such cases, the driver usually diffracts sufficiently enough that the spot size becomes large $\sigma_r > \sqrt{\Lambda}c/\omega_p$, causing the wake response to transition from the blowout to the linear regime. In this regime, the electron perturbation is small, and the driver diffracts as it would in vacuum.

For larger values of $k_\beta \beta^*$, the plasma focusing force plays an increasingly important role. Upon inspection of Eq. (5.5), it can be seen that the driver spot size is perfectly matched, i.e., $\sigma_r(z) = \sigma_0$, when $k_\beta \beta^* = 1$ as beam diffraction is balanced by the plasma focusing force. However, a matched beam is not of interest since wake expansion and injection rely on an evolving (non-matched) driver. When the betatron wavelength is much smaller than the diffraction length, i.e., $k_\beta \beta^* \gg 1$, the spot size evolution is dominated by the plasma focusing force and from Eq. (5.5) can be approximated by $\sigma_r^2(z) \approx \sigma_0^2 \cos^2(k_\beta z) = \frac{\sigma_0^2}{2} [1 + \cos(2k_\beta z)]$. In this limit, individual beam electrons that comprise the driver oscillate at the betatron wavelength $\lambda_\beta = 2\pi/k_\beta = 2\pi\sqrt{2\gamma_b}c/\omega_p$. However, since the driver spot size is positive-definite by definition, it oscillates at $2k_\beta$ corresponding to half the betatron wavelength $\lambda_\beta/2 = \pi\sqrt{2\gamma_b}c/\omega_p$.

In reality, the spot size evolution of a short driver ($k_p \sigma_z \sim 1$) propagating in a plasma

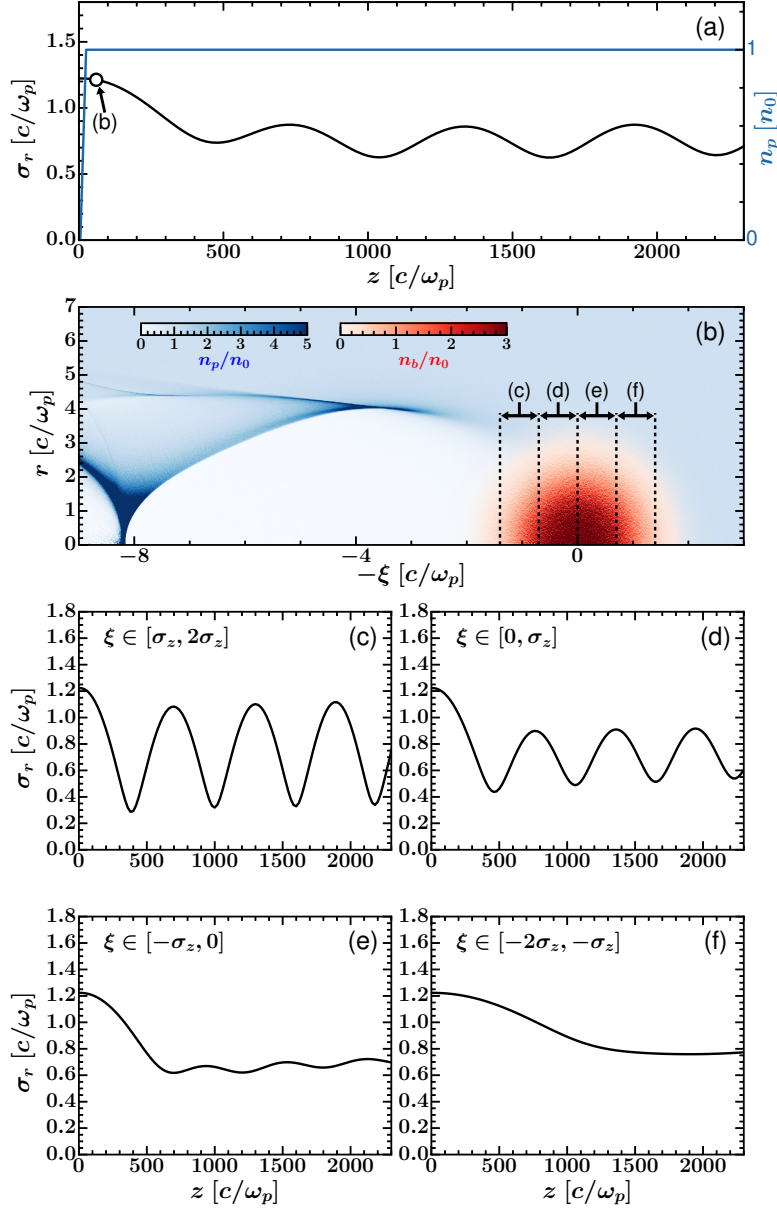


Figure 5.1: (a) The spot size evolution $\sigma_r(z)$ of an electron drive bunch (solid black) with $\Lambda = 6$, $\gamma_b = 20000$, and $k_p \sigma_r = k_p \sigma_0 = 1.225$ focused at the plasma entrance with CS parameters $\beta_i = \beta^* \approx 3200 c/\omega_p$ and $\alpha_i \approx 0$. The longitudinal plasma density profile is shown in solid blue. (b) The plasma wake produced near the plasma entrance at $z = 60 c/\omega_p$. The spot size evolution is shown for slices in ξ corresponding to (c) $[\sigma_z, 2\sigma_z]$, (d) $[0, \sigma_z]$, (e) $[-\sigma_z, 0]$, and (f) $[-2\sigma_z, -\sigma_z]$.

can be very complex as beam slices along ξ can evolve differently. It takes time for the ion channel to form so the front of the beam feels a weaker focusing force. In Fig. 5.1(a), we show how the projected spot size evolves from a PIC simulation of a 10 GeV driver ($\gamma_b = 20000$) with a 51 kA peak current ($\Lambda = 6$), centroid $k_p \xi_c = 0$, and duration $k_p \sigma_z = 0.7$ propagating in a constant shelf density plasma. The driver is initially focused at the plasma entrance with a spot size $k_p \sigma_0 = 0.5\sqrt{\Lambda} \simeq 1.225$ which is much larger than the matched spot size so it operates in the plasma focusing regime $k_\beta \beta^* = 16$ with a diffraction length $\beta^* = 3200 c/\omega_p$. From the simulation results in Fig. 5.1(a), it is clear that a portion of the driver oscillates at nearly half the betatron wavelength $\lambda_\beta/2 = \pi\sqrt{2\gamma_b}c/\omega_p \simeq 628 c/\omega_p$ corresponding to the background plasma shelf density $n_p = n_0$. However, it is also evident that some portions approach an equilibrium spot size near the matched spot size long after the first oscillation, as the spot size does not return its original value. As shown in the electron density plot taken near the plasma entrance in Fig. 5.1(b), this occurs because the plasma wake (blue) is not fully formed along the entire driver (red). In regions where the plasma electrons are not completely blown out, the effective charge density of the ion channel is reduced below en_0 . Thus, the effective plasma focusing force and betatron wavelength will vary spatially for beam electrons at different ξ and r and temporally in z/c as the wake evolves during focusing.

For the reasons stated above, solving a Hill's-like equation $x'' + k_\beta^2(z)x = 0$ is generally not possible for the front of the beam since the plasma forces depend on the evolution of the spot size. Instead, our goal is to develop a simple model for the driver spot size by examining the empirical evolution of beam slices (each σ_z thick) along ξ from the PIC simulation in Figs. 5.1(c)-(f). These slices are annotated in Fig. 5.1(b). We start with the slice at the rear of the driver $\xi \in [\sigma_z, 2\sigma_z]$ in Fig. 5.1(c) because its motion is easiest to understand. It can be readily seen that the spot size of this slice oscillates at roughly half the betatron wavelength $\lambda_\beta/2$ since the ion channel is fully formed in this region. Examining the next slice forward along the driver $\xi \in [0, \sigma_z]$ (shown in Fig. 5.1(d)), we see that the focal distance

increases as the ion channel is no longer fully formed. In this slice, the plasma electrons are not completely blown out by the driver. It can be seen in Figs. 5.1(e)-(f) that this effect becomes more pronounced for slices closer to the head of the beam as there is progressively less expelled electrons. Even in these regions, the beam slices are still focused given that the diffraction length is sufficiently long. However, at the very front of the driver where the plasma is approximately neutral (not shown here), head erosion still occurs due to the absence of a focusing force. Thus, the local wavenumber $k(\xi)$ essentially ramps from 0 to $k_{\beta 0}$ from the front to the middle of the beam. Each slice therefore oscillates at different k_{β} such that after each $2\pi/k_{\beta 0}$ the front of the beam develops an additional focus forming the scalloping feature alluded to earlier.

The driver evolution described above bares some similarities to the concept of a flying focus [106, 107] as different beam slices in ξ come into focus at different positions z in the plasma. In PWFA, this occurs naturally because the plasma density and focusing force vary across the driver. Since the focal length of a given beam slice depends on its longitudinal position ξ , the overall projected spot size can remain below its initial value over scale lengths much greater than the betatron wavelength. Consider two beam slices $\xi \in [0, \sigma_z]$ [Fig. 5.1(d)] and $\xi \in [-\sigma_z, 0]$ [Fig. 5.1(e)] in regions where electrons are not fully blown out. As the front slice begins to focus, it blows out more electrons in the regions behind it and the rear slice sees the focusing force increase due to the higher effective ion channel density. As a result, the spot size oscillation of the rear slice dampens after the first oscillation and σ_r remains below its initial value. A similar process plays out for the slices in Figs. 5.1(e)-(f) and, in general, for any two consecutive beam slices in regions where electrons are not fully blown out. This cascading effect from one slice to the next leads to the total projected spot size remaining focused below its initial value over distances much greater than the betatron wavelength as seen in Fig. 5.1(a). Thus, the driver can remain well-guided in the plasma until head erosion or pump depletion occurs.

In the previous chapter, it was shown that injection occurs as beam electrons are attracted

to the axis within the first betatron oscillation. At the rear of the driver where plasma electrons are mostly blown out, beam electrons are focused over the quarter period of the betatron motion $\lambda_\beta/4 \simeq (\pi/2)\sqrt{2\gamma_b}c/\omega_p$. However, as the effective betatron wavelength is stretched for electrons in the front, spot size focusing and injection can be sustained over longer distances. Even though a fraction of the driver continues to oscillate at nearly $2k_\beta$ after the initial focusing [Fig. 5.1], the wake can remain fully expanded due to a combination of beam loading effects [Fig. 4.2], scalloping, and damping of the spot size oscillation. As a result, the injected electrons can remain accelerated at the rear of the wake over many betatron wavelengths, enabling multi-GeV injected beam energies.

While it is difficult to capture all the physics of σ_r , we argue that σ'_r can be approximated by the fastest moving components over the injection length scale $L_{inj} \sim \lambda_\beta/4$. Based on the simulation results in Fig. 5.1, we propose a simple model in which the back half of the driver oscillates at $2k_\beta$ while the front half is considered constant over length scales $z \lesssim L_{inj} \simeq 314 c/\omega_p$. Under this model, the total projected spot size can be written as

$$\sigma_r(z) = \sqrt{\frac{\sigma_{osc}^2 + \sigma_0^2}{2}} \quad (5.6)$$

where $\sigma_{osc}^2 \equiv \frac{\sigma_0^2}{2} [1 + \cos(2k_\beta z)]$. Simplifying, the spot size can be expressed as

$$\sigma_r(z) = \frac{\sigma_0}{2} \sqrt{3 + \cos(2k_\beta z)}. \quad (5.7)$$

Taking the derivative of Eq. (5.7), we obtain

$$\sigma'_r(z) = -\frac{k_\beta \sigma_0 \sin(2k_\beta z)}{2\sqrt{3 + \cos(2k_\beta z)}}. \quad (5.8)$$

In Figs. 5.2(a)-(b), we plot the analytic model for σ_r [Eqs. (5.7)-(5.8)] (red) along with the PIC simulation results (black) for the driver shown in Fig. 5.1. From the results, it is

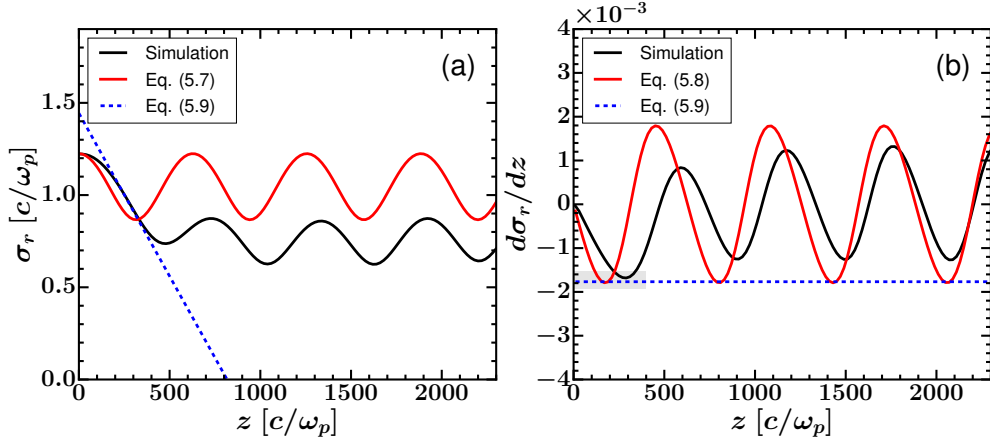


Figure 5.2: (a) $\sigma_r(z)$ and (b) $\sigma'_r(z)$ for the driver depicted in Fig. 5.1. Simulation results are shown in black. Eqs. (5.5) and (5.8) are plotted in red and Eq. (5.9) is plotted in dashed blue.

evident that Eq. (5.7) does not account for the longer wavelength components of the driver and spot size damping after the initial oscillation. Despite this, Eq. (5.8) provides a good lower-bound estimate of σ'_r calculated from the PIC simulation, as highlighted by the shaded region in Fig. 5.2(b). This is important because the wake evolution from Eq. (5.1) depends on σ'_r . A lower-bound estimate for σ'_r can therefore be approximated by evaluating Eq. (5.8) at $k_\beta z = \pi/4$ to obtain

$$\sigma'_r \approx -\frac{k_\beta \sigma_0}{2\sqrt{3}}. \quad (5.9)$$

This expression is plotted (dashed blue) in Figs. 5.2(a)-(b) and exhibits strong agreement with the driver evolution from the PIC simulation over a range of spot sizes. Eq. (5.9) contains the betatron wavenumber k_β and the amplitude of the spot size focusing σ_0 , which are the main features of σ'_r within the first oscillation. Thus, even though it is a crude approximation, it can give reasonable results for short drivers ($k_p \sigma_z \sim 1$) operating in the blowout regime ($k_p \sigma_r \lesssim \sqrt{\Lambda}$) as we will show in the following sections.

5.2.2 Wake Evolution and Injection

Using our model for σ'_r , we can now characterize the wake evolution for a driver in the plasma focusing regime ($k_\beta\beta^* \gg 1$). Substituting Eq. (5.9) into Eq. (5.1), we find

$$\frac{dL_b}{dz} \approx \kappa\sigma'_r \approx -\frac{\kappa k_\beta\sigma_0}{2\sqrt{3}}. \quad (5.10)$$

Implicit in Eq. (5.10) is the assumption that the driver enters the plasma at a focus ($\beta_i \approx \beta^*, \alpha_i \approx 0$) with a spot size $k_p\sigma_0 \lesssim 0.8\sqrt{\Lambda}$ such that a nonlinear plasma wake is excited in the blowout regime. It is also assumed that the spot size is sufficiently large $k_p\sigma_0 \gtrsim 0.3\sqrt{\Lambda}$ so that the wake expands through spot size focusing. Recall that the relationship between L_b and σ_r is approximately linear (κ is constant for this range of $k_p\sigma_r$) in the blowout regime as illustrated in Fig. 4.4(c). Therefore, $\kappa = \frac{dL_b}{d\sigma_r}$ can be estimated numerically by calculating the slope directly from Fig. 4.4(c) at $k_p\sigma_r = 0.5\sqrt{\Lambda}$ for a given Λ . It is worth noting that the magnitude of the slope $|\kappa|$ increases with Λ and is typically on the order of unity in the ultrarelativistic limit $\Lambda \gtrsim 2$. Thus, κ is an implicit function of Λ , where $\frac{d|\kappa|}{d\Lambda} > 0$.

Eq. (5.10) is important because it can be used to determine a lower bound for the relativistic factor of the wake, $\gamma_\phi \approx \sqrt{\frac{1}{2(1-\beta_\phi)}}$. Recall from Eq. (4.2), $\beta_\phi = 1 - \frac{dL_b}{dz}$, which can be used to approximate γ_ϕ by

$$\gamma_\phi \approx \sqrt{\frac{1}{2} \frac{1}{\frac{dL_b}{dz}}} \approx \sqrt{\frac{\sqrt{3}}{-\kappa k_\beta\sigma_0}}. \quad (5.11)$$

Normalizing the focal spot size σ_0 to the scale length of the wake $\sqrt{\Lambda}c/\omega_p$ and expressing the betatron wavenumber in terms of the driver energy $k_\beta = k_p/\sqrt{2\gamma_b}$, Eq. (5.11) can be written in a more convenient form

$$\gamma_\phi \approx \gamma_b^{1/4} \left(\frac{1}{\Lambda}\right)^{1/4} \left(\frac{\sqrt{\Lambda}}{k_p\sigma_0}\right)^{1/2} \left(\frac{\sqrt{6}}{-\kappa}\right)^{1/2}. \quad (5.12)$$

From Eq. (5.12), it can be seen that the phase velocity γ_ϕ depends on the driver energy γ_b , focal spot size σ_0 , and peak charge per unit length Λ (κ is implicitly a function of Λ).

A higher drive beam energy γ_b naturally leads to a higher γ_ϕ as the betatron wavelength is stretched. Assuming $k_p\sigma_0/\sqrt{\Lambda}$ is kept fixed, higher peak current Λ results in a lower γ_ϕ as both the wake expansion rate $|\kappa| = |\frac{dL_b}{d\sigma_r}|$ and scale length $\sqrt{\Lambda}c/\omega_p$ increase. A larger normalized spot size $k_p\sigma_0/\sqrt{\Lambda}$ can also be used to reduce γ_ϕ but $k_p\sigma_0/\sqrt{\Lambda}$ must be in the range between 0.3 and 0.8. Since the wake velocity scales as $\gamma_\phi \sim \gamma_b^{1/4}$, it can be dramatically lower than γ_b . In fact, for $|\kappa| \sim O(1)$, $\sigma_0 \sim O(\sqrt{\Lambda}c/\omega_p)$, and $\Lambda \lesssim 10$, the wake velocity can be well-approximated by $\gamma_\phi \approx \gamma_b^{1/4}$. Thus, for multi-GeV drivers $\gamma_b \sim O(10^4)$, we can expect phase velocities $\gamma_\phi \sim O(10)$, which is several orders of magnitude lower than γ_b .

In order for injection to occur, sheath electrons must travel faster than the phase velocity of the wake as they approach the axis. Thus, the trapping condition is given by

$$\gamma_{z,m} > \gamma_\phi \quad (5.13)$$

where $\gamma_{z,m}$ is the maximum forward velocity of the sheath electrons determined empirically from Fig. 4.4(d) and γ_ϕ is wake velocity given by Eq. (5.12). For the driver $\{\gamma_b = 20000, \Lambda = 6, k_p\sigma_0 = 0.5\sqrt{\Lambda}, k_p\sigma_z = 0.7\}$ specified in Fig. 5.1, $\kappa = \frac{dL_b}{d\sigma_r} \approx -1.76$ from Fig. 4.4(c) and, in turn, $\gamma_\phi \approx 12.7$ from Eq. (5.12). Upon inspection of Fig. 4.4(d), it is evident that Eq. (5.13) is indeed satisfied as $\gamma_{z,m} \gtrsim 20$ for spot sizes $k_p\sigma_r \leq 0.5\sqrt{\Lambda}$ when $\Lambda = 6$.

In Fig. 5.3(a), we show the electron density distribution of the drive beam (red) and plasma (blue) after the driver from Fig. 5.1(a) has propagated a distance of $z = 1000 c/\omega_p$ into the plasma. At this point, the injected beam is fully formed and located at the rear of the wake. To illustrate the injection process, we plot the trajectories, $r(\xi)$, of injected electrons with colors indicating the original position, z_i . It can be seen that plasma electrons originating from initial positions $z_i \lesssim 100 c/\omega_p$ (light blue) trace out the initially unexpanded bubble radius $r_b(\xi)$ as the spot size begins to focus. As they come back to the axis, they are injected due to the reduced gamma factor, γ_ϕ , of the wake and begin to form the head of the injected beam. Injection continues to occur for electrons deeper into the plasma with larger initial positions $z_i \gtrsim 100 c/\omega_p$ (dark blue/purple) as they trace out longer wake lengths due

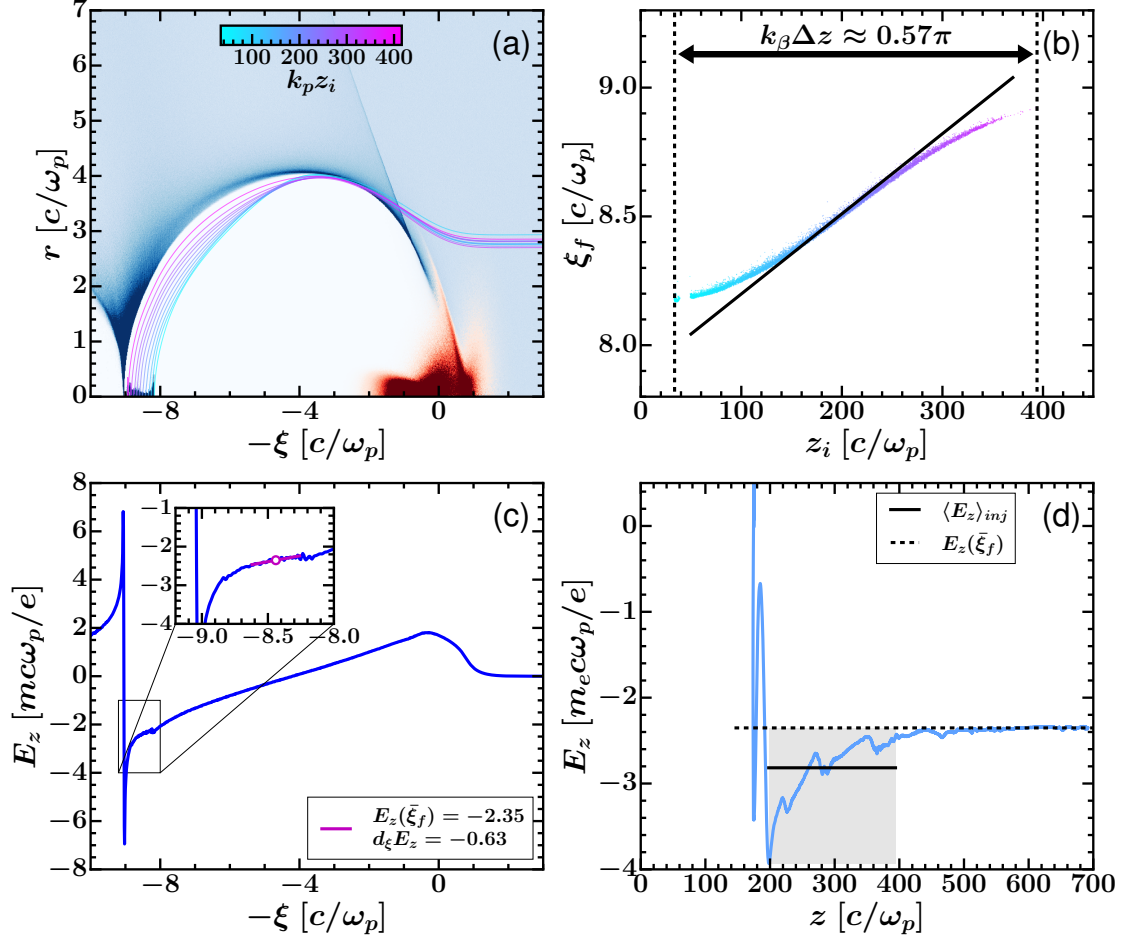


Figure 5.3: (a) The plasma wake produced by the drive bunch shown in Fig. 5.1 at $z = 1000 c/\omega_p$. Injected electron trajectories are shown with colors indicating the original longitudinal position z_i . (b) The phase space (ξ_f, z_i) of the injected electron charge. Eq. (5.10) is evaluated and plotted (solid black). The injection region is enclosed by the dashed black lines. (c) The on-axis electric field E_z produced by the plasma wake shown in (a). The purple line is a linear fit around the injected beam centroid, which is marked by an opaque circle in the inset plot. (d) The electric field, E_z , experienced by an electron that is mapped to the injected beam centroid $\bar{\xi}_f \approx 8.44$ (solid blue). $\langle E_z \rangle_{inj}$ (solid black) is the average electric field experienced by the electron as the wake continues to expand after the initial injection (shaded region). $E_z(\bar{\xi}_f)$ (dashed black) is the asymptotic accelerating field felt by the electron when the wake is fully expanded.

to spot size focusing, thereby forming the rest of the injected beam. Eventually, this process terminates when the spot size further decreases to the point that the wake is fully expanded.

The final position in the plasma wake, ξ_f , for an injected electron originating from a given initial position, z_i , can be approximated by the instantaneous wake length L_b at the “time” of injection $z/c = z_i/c$. Since L_b is a monotonically decreasing function of σ_r in the blowout regime, there is a one-to-one mapping between the injection time z_i/c and final position in the wake ξ_f of injected electrons as shown in Fig. 5.3(b). The relationship between ξ_f and z_i is characterized by the wake evolution described by Eq. (5.10) which is plotted in solid black on top of the phase space (ξ_f, z_i) in Fig. 5.3(b). Good agreement is observed between Eq. (5.10) and the simulation results over the middle of the injected beam. It can also be seen that the longitudinal distance over which injection occurs, enclosed by dashed black lines, can be well-approximated by the quarter betatron period $L_{inj} \sim \lambda_\beta/4 = \pi/2k_\beta$.

Once injected, electrons remain near the axis due to the focusing force of the ion channel F_i and are accelerated by the large negative electric field at the rear of the wake as seen in Fig. 5.3(c). As the length of the injected bunch $\Delta\xi_{tr}$ (recall that $\Delta\xi_{tr} = \frac{dL_b}{dz}\Delta z_i \sim \frac{1}{2\gamma_\phi^2}\Delta z_i$) is much smaller than the wake scale length $\sqrt{\Lambda}c/\omega_p$, the electric field near the injected beam centroid $\bar{\xi}_f \simeq 8.44 c/\omega_p$ can be approximated by a linear fit (purple) with $eE_z(\bar{\xi}_f)/mc\omega_p \simeq -2.35$ and $d(eE_z(\bar{\xi}_f)/mc\omega_p)/d(k_p\xi) \simeq -0.63$. While Fig. 5.3(c) provides a snapshot of the final steady state accelerating field $E_z(\xi)$ at $z = 1000 c/\omega_p$, it does not capture the electric field evolution during injection.

To illustrate the difference between the injection and acceleration phases, we plot the electric field E_z experienced by an electron mapped to the final beam centroid $\xi_f \approx 8.44$ as it is injected and accelerated along the longitudinal direction z in Fig. 5.3(d). The shaded region indicates the range of z over which injection occurs in Fig. 5.3(b). When the electron is injected around $z = 200 c/\omega_p$, it experiences a large accelerating field $eE_z/mc\omega_p \simeq -4$ as it is initially situated near the electric field spike at the rear of the wake. As the wake continues to expand, this characteristic spike moves backward to larger ξ and the magnitude of the

electric field felt by the electron at ξ_f decreases. Eventually, it asymptotically approaches the final accelerating field $eE_z(\bar{\xi}_f)/mc\omega_p \simeq -2.35$ (dashed black) when the wake is fully expanded (but no injection occurs then). However, it can be seen in Fig. 5.3(d) that the average electric field in the injection phase $e\langle E_z \rangle_{inj}/mc\omega_p \simeq -2.82$ (solid black) is nearly $\sim 20\%$ more negative than the asymptotic electric field. The initial energy chirp is thus larger than the simple estimate. This is an important distinction as the dynamics of injection and acceleration are both critical to determining the optimal injected beam parameters as we will show in the following section.

5.2.3 Optimal Driver Propagation Distance and Injected Beam Energy

Due to the electron mapping shown in Figs. 5.3(a)-(b), electrons at the head of the injected beam initially have higher energies than those at the tail since they are injected first. As a result, the trailing bunch usually acquires an energy chirp from injection. This is also true for downramp injection [52]. This can be seen in Fig. 5.4(a), where we show the plasma wakefield (top) and injected electron energy distribution (bottom) at $z = 500 c/\omega_p$ for the driver used in Figs. 5.1-5.3. In the bottom frame of Fig. 5.4(a), it is clear that $\frac{d\gamma_e}{d\xi} > 0$ just after injection. As the injected beam is accelerated in the plasma, electrons at the rear experience a stronger accelerating field than electrons at the front (as seen in the top frame of Fig. 5.4(a)) due to the negative chirp in the electric field, $d_\xi E_z < 0$. As a result, the energy-position phase space is rotated and eventually flattened after some acceleration distance as shown in Fig. 5.4(b). Thus, injection and acceleration in the plasma can naturally produce GeV-class beams with low energy spreads [52].

Using the theoretical framework described in the previous section, we are interested in determining the optimal acceleration distance at which point the energy spread is minimized within the center of the injected beam. To do this, we consider the energy difference between two points $\xi_1 \equiv \bar{\xi}_f - \frac{\Delta\xi}{2}$ and $\xi_2 \equiv \bar{\xi}_f + \frac{\Delta\xi}{2}$ equidistant from the injected beam centroid $\bar{\xi}_f$, where $\Delta\xi \equiv \xi_2 - \xi_1$ is used to differentiate from the total beam length $\Delta\xi_{tr}$. To visualize

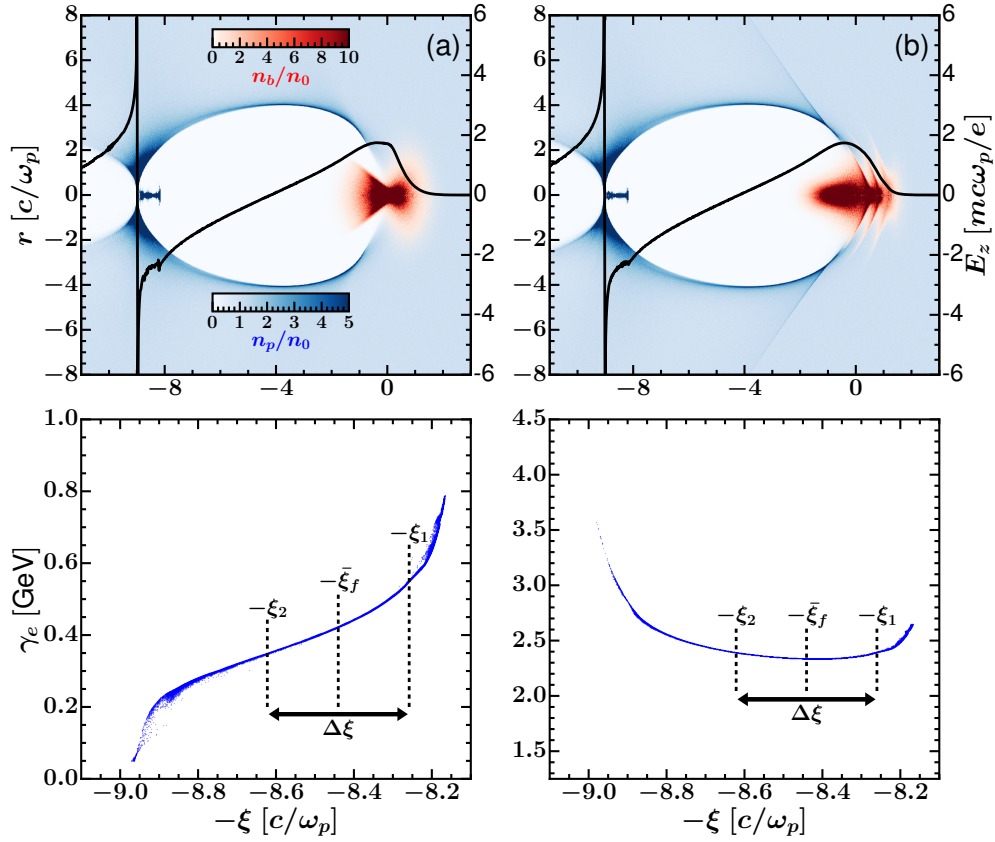


Figure 5.4: The electron density (top) and injected electron phase space γ_e - ξ (bottom) generated by the driver in Fig. (4.1) at (a) $z = 500 c/\omega_p$, and (b) $z = 2100 c/\omega_p$. The on-axis electric field E_z is plotted in black. A window of length $\Delta\xi \simeq 0.36 c/\omega_p$ is annotated around the injected beam centroid $\bar{\xi}_f \simeq 8.44 c/\omega_p$ containing 50% of the total injected charge.

this problem, we annotate ξ_1 and ξ_2 in Fig. 5.4 for a width $\Delta\xi \simeq 0.36 c/\omega_p$ around the beam centroid $\xi_f \simeq 8.44 c/\omega_p$ enclosing 50% of the total charge injected beam charge.

Henceforth, we will employ normalized units, where charge is normalized to electron charge e , mass to electron mass m , velocity to c , charge density to en_p , current density to $en_p c$, length to c/ω_p , time to ω_p^{-1} , electric fields to $mc\omega_p/e$, and potentials to mc^2/e . Using Eq. (5.10), the longitudinal distance L_{inj} over which electrons are mapped to final positions in $\Delta\xi$ is given by

$$L_{inj} = \frac{\Delta\xi}{\frac{dL_b}{dz}} \cong 2\gamma_\phi^2 \Delta\xi = \left(\frac{2\sqrt{3}}{-\kappa} \right) \left(\frac{\Delta\xi}{k_\beta \sigma_0} \right). \quad (5.14)$$

Since the electrons at ξ_1 have been accelerated over a distance L_{inj} while the electrons at ξ_2 have just been injected, the initial energy difference between these two points is given by

$$\Delta\gamma \approx -\langle E_z(\bar{\xi}_f) \rangle_{inj} L_{inj} \quad (5.15)$$

where $\langle \rangle_{inj}$ denotes an average over injection and we make the approximation $\langle E_z(\xi_1) \rangle_{inj} \approx \langle E_z(\bar{\xi}_f) \rangle_{inj}$. Substituting Eq. (5.14) into Eq. (5.15), we find

$$\Delta\gamma \approx -\langle E_z(\bar{\xi}_f) \rangle_{inj} \left(\frac{2\sqrt{3}}{-\kappa} \right) \left(\frac{\Delta\xi}{k_\beta \sigma_0} \right). \quad (5.16)$$

From Eq. (5.16), it can be seen that the initial energy difference between two points ξ_1 and ξ_2 depends linearly on the distance between them $\Delta\xi$. This is indeed borne out in the simulation results shown in Fig. 5.4(a). After injection, the beam is accelerated over some distance L_{acc} until its initial energy chirp is flattened by the wake. We can solve for this optimal acceleration distance L_{acc} by equating the initial energy chirp $\Delta\gamma$ to the difference in energy gain between ξ_1 and ξ_2 during acceleration

$$-\langle E_z(\xi_2) - E_z(\xi_1) \rangle_{acc} L_{acc} = \Delta\gamma \quad (5.17)$$

where $\langle \rangle_{acc}$ denotes an average over acceleration. Using Eq. (5.15), the acceleration length can also be written in terms of the injection length

$$L_{acc} = \frac{\langle E_z(\bar{\xi}_f) \rangle_{inj}}{\langle E_z(\xi_2) - E_z(\xi_1) \rangle_{acc}} L_{inj}. \quad (5.18)$$

Assuming $\Delta\xi$ is much smaller than the scale length of the wake $\sqrt{\Lambda}c/\omega_p$, the difference between $E_z(\xi_2)$ and $E_z(\xi_1)$ can be estimated by Taylor expanding E_z to first order around the beam centroid $\bar{\xi}_f$ to obtain

$$E_z(\xi_2) - E_z(\xi_1) \approx \left. \frac{dE_z}{d\xi} \right|_{\xi=\bar{\xi}_f} (\Delta\xi). \quad (5.19)$$

Using Eqs. (5.16)-(5.19), we solve for the optimal acceleration distance

$$L_{acc} = \frac{\langle E_z(\bar{\xi}_f) \rangle_{inj}}{\langle d_\xi E_z(\bar{\xi}_f) \rangle_{acc}} \left(\frac{2\sqrt{3}}{-\kappa} \right) \left(\frac{1}{k_\beta \sigma_0} \right) \quad (5.20)$$

It is worth noting that $\Delta\xi$ does not appear in Eq. (5.20). Indeed, if the energy distribution is flat for a width of $\Delta\xi$, it should also be flat for any two points within $\Delta\xi$. Therefore, the optimal acceleration distance should not depend on the exact value of $\Delta\xi$. For a width $\Delta\xi$ comparable to the total beam length $\Delta\xi_{tr}$, we can approximate the injection length scale by $L_{inj} \sim \lambda_\beta/4$. Combining the injection and acceleration distances, we obtain an expression for the total driver propagation distance z_d in the plasma where the energy spread of the injected bunch is minimized,

$$\begin{aligned} z_d &= L_{inj} + L_{acc} \\ &= \frac{1}{k_\beta} \left[\frac{\pi}{2} + \left(\frac{\langle E_z \rangle_{inj}}{\langle d_\xi E_z \rangle_{acc}} \right) \left(\frac{1}{\sqrt{\Lambda}} \right) \left(\frac{2\sqrt{3}}{-\kappa} \right) \left(\frac{\sqrt{\Lambda}}{\sigma_0} \right) \right] \end{aligned} \quad (5.21)$$

where averages over injection $\langle \rangle_{inj}$ and acceleration $\langle \rangle_{acc}$ are evaluated for an electron mapped to the injected beam centroid $\bar{\xi}_f$. As electrons are mapped to the beam centroid around the midpoint of injection $z \sim L_{inj}/2$, the average energy of the trailing bunch at z_d can be approximated by

$$\begin{aligned}\bar{\gamma}_{tr} &\approx - \left[\langle E_z \rangle_{inj} \left(\frac{L_{inj}}{2} \right) + \langle E_z \rangle_{acc} L_{acc} \right] \\ &= \frac{-\langle E_z \rangle_{inj}}{k_\beta} \left[\frac{\pi}{4} + \left(\frac{\langle E_z \rangle_{acc}}{\langle d_\xi E_z \rangle_{acc}} \right) \left(\frac{1}{\sqrt{\Lambda}} \right) \left(\frac{2\sqrt{3}}{-\kappa} \right) \left(\frac{\sqrt{\Lambda}}{\sigma_0} \right) \right].\end{aligned}\quad (5.22)$$

By direct observation, it can be seen that z_d and $\bar{\gamma}_{tr}$ both scale with the betatron wavelength $\lambda_\beta = 2\pi/k_\beta$, which is the underlying length scale of the problem. If the plasma wakefield is loaded with enough charge such that $\langle d_\xi E_z \rangle_{acc} \approx 0$, then $z_d \rightarrow \infty$ and $\bar{\gamma}_{tr} \rightarrow \infty$ and the injected beam can be accelerated until driver head erosion or pump depletion occurs. While it is straightforward to show that larger normalized spot sizes $k_p\sigma_0/\sqrt{\Lambda}$ can lead to lower z_d and $\bar{\gamma}_{tr}$, it is more difficult to predict exactly how z_d and $\bar{\gamma}_{tr}$ scale for drivers with higher peak currents Λ as E_z , $d_\xi E_z$, and κ can all vary. In subsequent sections, we will evaluate Eqs. (5.21) and (5.22) for driver parameters and compare the predictions to PIC simulation results. As will be shown in the following sections, Eqs. (5.21) and (5.22) can give reasonable estimates within 10 – 20% of the optimal beam parameters from PIC simulations for various different driver parameters.

5.2.4 Final Beam Parameters

Using Eqs. (5.21) and (5.22), we can now predict the optimal propagation distance and energy of the injected bunch for the driver $\{\gamma_b = 20000, \Lambda = 6, k_p\sigma_0 = 0.5\sqrt{\Lambda}, k_p\sigma_z = 0.7\}$ used in Figs. 5.1-5.4. From results shown in Fig. 5.3(d), we approximate $\langle E_z \rangle_{inj} \simeq -2.82$, $\langle E_z \rangle_{acc} \simeq -2.35$, and $\langle d_\xi E_z \rangle_{acc} \simeq -0.63$. It is worth recalling that $\kappa \approx -1.76$ was calculated directly from Fig. 4.4(c) and $k_\beta^{-1} = \sqrt{2\gamma_b} = 200$ for $\gamma_b = 20000$. For these values, we find that $z_d \simeq 1752 \text{ } c/\omega_p$ and $\bar{\gamma}_{tr} mc^2 \simeq 1.96 \text{ GeV}$.

Eqs. (5.21) and (5.22) provide reasonable estimates when compared to the PIC simulation results which indicate an optimal driver propagation distance of $z_d = 2100 \text{ } c/\omega_p$. The energy distribution of the injected electrons from the PIC simulation is shown in Fig. 5.5(a) and the

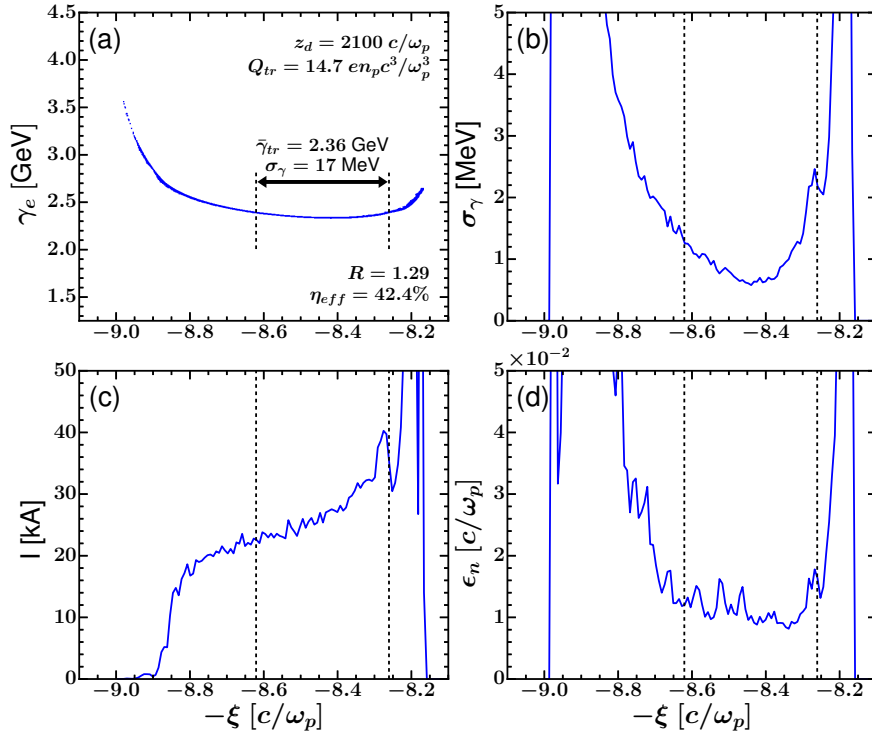


Figure 5.5: (a) The injected electron energy distribution at $z = 2100 c/\omega_p$ for the driver used in Figs. 5.1-5.4. The dashed black lines enclose 50% of total injected charge surrounding the centroid. The beam slice parameters of the injected electrons: (b) the energy spread σ_γ in units of MeV, (c) current I in units of kA, and (d) normalized emittance ϵ_n in units of c/ω_p .

total length of the injected beam is $\Delta\xi_{tr} \simeq 0.81 c/\omega_p$. Within a width $\Delta\xi \simeq 0.36 c/\omega_p$ that encloses 50% of the total injected charge around the beam centroid (dashed black), we find that the energy distribution is flattened with an average energy of 2.36 GeV and a projected energy spread of 17 MeV (0.72%). Since injected electrons are generally trapped at the rear of the wake where the accelerating field is large, the energy gain of the injected beam is usually larger than the energy loss of the drive beam. Defining the transformer ratio as the ratio of the average energy gain of the injected bunch to the maximum energy loss of the driver, we find $R \equiv \bar{\gamma}_{tr}/\max(|\Delta\gamma_b|) \simeq 1.27$ in this case.

Due to the one-to-one mapping between ξ_f and z_i , slice energy spreads as low as ~ 0.6 MeV are observed in Fig. 5.5(b). As pointed out in Ref. 52, this mapping also character-

izes the injected beam current. From Fig. 5.3(a), it can be seen that electrons mapped to a given slice $d\xi_f$ originate from cylindrical rings of radius r_i with finite width Δr_i from initial positions dz_i in the plasma. Therefore, the injected slice currents can be approximated by

$$I \approx (ec)2\pi r_i \Delta r_i n_p \frac{dz_i}{d\xi_f}. \quad (5.23)$$

We note that $\frac{d\xi_f}{dz_i} \approx \frac{dL_b}{dz_i}$ can be determined from Eq. (5.10). Alternatively, $\frac{d\xi_f}{dz_i} \approx (1 - \beta_\phi) \approx \frac{1}{2\gamma_\phi^2}$, where γ_ϕ is given by Eq. (5.11). Normalizing the slice currents to the Alfven current $I_A \approx mc^3/e \approx 17$ kA, Eq. (5.23) can be rewritten as

$$\frac{I}{I_A} \approx \gamma_\phi^2 (k_p r_i) (k_p \Delta r_i). \quad (5.24)$$

For a driver with $\{\gamma_b = 20000, \Lambda = 6, k_p \sigma_0 = 0.5\sqrt{\Lambda}, k_p \sigma_z = 0.7\}$, the wake velocity $\gamma_\phi \approx 12.7$ was previously determined from Eq. (5.11). We can approximate $k_p r_i \approx \sqrt{\Lambda} \approx 2.44$ from Fig. 4.4(b) and $k_p \Delta r_i \sim O(0.01)$ based on empirical observations from PIC simulations. For such values, the slice currents of the beam can easily exceed the Alfven current from Eq. (5.24). This is indeed the case as slice currents ranging from 20 to 40 kA are observed across most of the injected beam from the PIC Simulation results shown in Fig. 5.5(c). Upon integrating the current profiles, we obtain the total injected charge $Q_{tr}/en_p k_p^{-3} \simeq 14.7$, which is approximately 22% of the total driver charge $Q_d/en_p k_p^{-3} \simeq (2\pi)^{3/2} \Lambda \sigma_z \simeq 66.1$. As the transformer ratio exceeds unity and the decelerating field varies along the driver, the transfer efficiency from the drive to trailing bunch is $\eta_{eff} \equiv (\sum q_{tr} \gamma_{tr}) / (\sum q_d |\Delta \gamma_b|) \simeq 42.4\%$ in this case.

Inherent in Eq. (5.24) is a tradeoff between γ_ϕ and Δr_i , since Δr_i depends on the trapping condition $\gamma_{z,m} > \gamma_\phi$ and, therefore, is an implicit function of γ_ϕ . For lower wake velocities, v_ϕ , and relativistic factors, γ_ϕ , the trapping condition is eased and more plasma electrons from a given slice dz_i can be injected, leading to larger Δr_i . A similar effect can be achieved

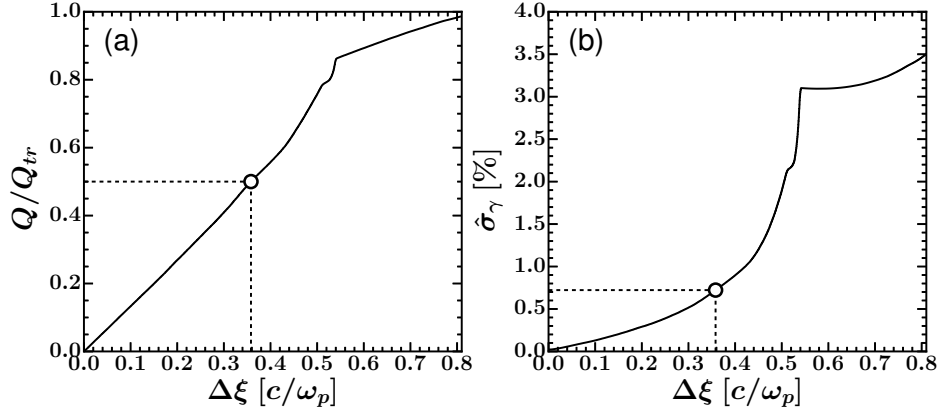


Figure 5.6: (a) The proportion of charge Q/Q_{tr} , and (b) the projected energy spread σ_γ inside a window of length $\Delta\xi$ centered on the injected beam centroid $\bar{\xi}_f$ from Fig. 5.5. The dashed lines indicate the window size $k_p\Delta\xi \simeq 0.36$ corresponding to $Q/Q_{tr} \simeq 0.5$.

by increasing $\gamma_{z,m}$ with higher driver peak currents Λ [Fig. 4.4(d)]. While it is difficult to quantify the exact dependence of Δr_i on γ_ϕ , we will use PIC simulations to provide additional insight into the scaling of the slice currents in subsequent sections.

As the injected electrons remain near the axis $k_p r \ll 1$ with a small transverse momentum $P_\perp/mc \sim O(1)$, ultra-low normalized emittances $\epsilon_n = \gamma\epsilon = \frac{1}{mc} \sqrt{\langle x^2 \rangle \langle p_x^2 \rangle - \langle xp_x \rangle^2}$ can be achieved as seen in Fig. 5.5(d). The reason for this low, transverse momentum is similar to downramp injection [52]. Within the middle section of the beam, we find normalized slice emittances as low as $\epsilon_n \sim 8 \times 10^{-3} c/\omega_p$, which corresponds to ~ 13 nm for a plasma density of $n_0 = 10^{19} \text{ cm}^{-3}$. For such parameters, the simulation results indicate that normalized slice brightnesses $B_n \equiv 2I/\epsilon_n^2$ greater than $10^{20} \text{ A/m}^2/\text{rad}^2$ can be achieved over most of the injected beam.

Up to this point, much of the discussion has focused on the core of the injected beam corresponding to a width $\Delta\xi$ around the beam centroid $\bar{\xi}_f$ where the energy distribution is flat. However, as seen in Fig. 5.5(a), the energy distribution at the tail of the beam is still chirped due to the electric field slope steepening at the very rear of the wake. For this reason, it is generally difficult to perfectly flatten the energy distribution of the entire

injected beam. Thus, the normalized projected energy spread $\hat{\sigma}_\gamma = \sigma_\gamma/\bar{\gamma}$ increases with the width $\Delta\xi$ as shown in Figs. 5.6(a)-(b). When considering the whole beam $\Delta\xi \simeq \Delta\xi_{tr} \simeq 0.81$, the normalized projected energy spread is $\sim 3.5\%$, which is nearly 5 times higher than the projected energy spread $\hat{\sigma}_\gamma \simeq 0.72\%$ within a width $\Delta\xi \simeq 0.36$ containing 50% of the injected charge as indicated in Fig. 5.5. In addition, slice energy spreads and normalized emittances also increase at positions further away from the center of the beam as seen in Figs. 5.5(b) and 5.5(d). As a result, slices at the head and tail may not satisfy the requirements for an XFEL. Therefore, in the following sections, our analysis will focus on the beam quality within the core ($\Delta\xi \lesssim 0.5\Delta\xi_{tr}$) of the bunch.

5.3 Parameter Scans of PIC Simulations with Different Drivers

In the previous section, we developed a theoretical framework to model the process of injection in the plasma focusing regime, $k_\beta\beta^* \gg 1$. We found that our expressions for the relativistic factor of the wake, γ_ϕ , optimal acceleration distance z_d , final trailing beam energy $\bar{\gamma}_{tr}$, and slice currents I provided reasonable estimates when compared to PIC simulation results. While the model was applied to a specific case, it is clear that injected beam parameters depend explicitly on the driver energy γ_b , peak current Λ , and focal spot size σ_0 . While not mentioned previously, the diffraction length β^* and duration σ_z of the driver can also influence the propagation distance and final energy of the injected beam. In this section, we will use our theoretical framework in conjunction with PIC simulations to characterize how injection and acceleration are affected by the various driver parameters. We will also compare the predictions of Eqs. (5.21) and (5.22) against PIC simulation results. Simulation results presented in this section use the 2D cylindrical ($r - z$) algorithm implemented into OSIRIS [84] with the customized finite-difference solver to reduce numerical effects induced by injection [82, 86]. The simulations also used $\Delta z = \Delta r = \frac{1}{128} \frac{c}{\omega_p}$, $\Delta t = \frac{1}{128} \frac{1}{\omega_p}$ with 4 to 16 macroparticles per cell depending on the species.

5.3.1 The Diffraction Length

As pointed out in Sec. 5.2, the propagation of the driver in the plasma focusing regime ($k_\beta\beta^* \gg 1$) is complicated due to the ξ -dependent focusing force at the front of the drive beam. As different slices at the front come into focus at different times, the total projected spot size can remain reduced over length scales much longer than the betatron wavelength. However, since the ion channel is not fully formed at the very front of the wake, the driver head still expands over the diffraction length β^* as if it were propagating in vacuum. This process, commonly referred to as head erosion, can cause the wake front to move backwards and the bubble length to contract as wake excitation is driven by a smaller portion of the driver. Depending on whether the diffraction length is comparable to the driver propagation distance, these two competing effects – wake front recession and bubble length contraction – will determine whether the injected beam remains accelerated by the electric field at the rear of the first plasma bucket or slips into the second bucket.

To illustrate the effects of head erosion, we compare the spot size evolution from PIC simulations of electron drivers $\{\gamma_b = 20000, \Lambda = 6, k_p\sigma_0 = 0.5\sqrt{\Lambda}, k_p\sigma_z = 0.7\}$, with different diffraction lengths $\beta^* = 800 c/\omega_p$ (red) and $\beta^* = 1600 c/\omega_p$ (black), propagating in a constant density plasma (blue) in Fig. 5.7(a). In normalized units, these parameters correspond to $k_\beta\beta^* = 4$ (red) and $k_\beta\beta^* = 8$ (black). From the results, it is evident that the spot size evolution for $k_\beta\beta^* = 4$ (red) expands more quickly than for $k_\beta\beta^* = 8$ (black) as the head of the driver diffracts over a smaller distance β^* for fixed σ_0 . Since the plasma focusing force plays an increasingly important role for higher $k_\beta\beta^*$, the spot size in black is initially focused to a smaller value than the spot size in red during the first betatron oscillation. This effect is captured in Eq. (5.5) as the minimum slice spot size scales with $1/(k_\beta\beta^*)$ and can result in more pronounced spot size focusing, wake expansion, and longer injected beams. However, this effect largely saturates for values $k_\beta\beta^* \gtrsim 10$ as slices at the rear of driver are focused to spot sizes much smaller than the length scale of the wake, $\sigma_r \ll \sqrt{\Lambda}c/\omega_p$, while slices in front evolve over longer time scales and do not contribute significantly to the injection.

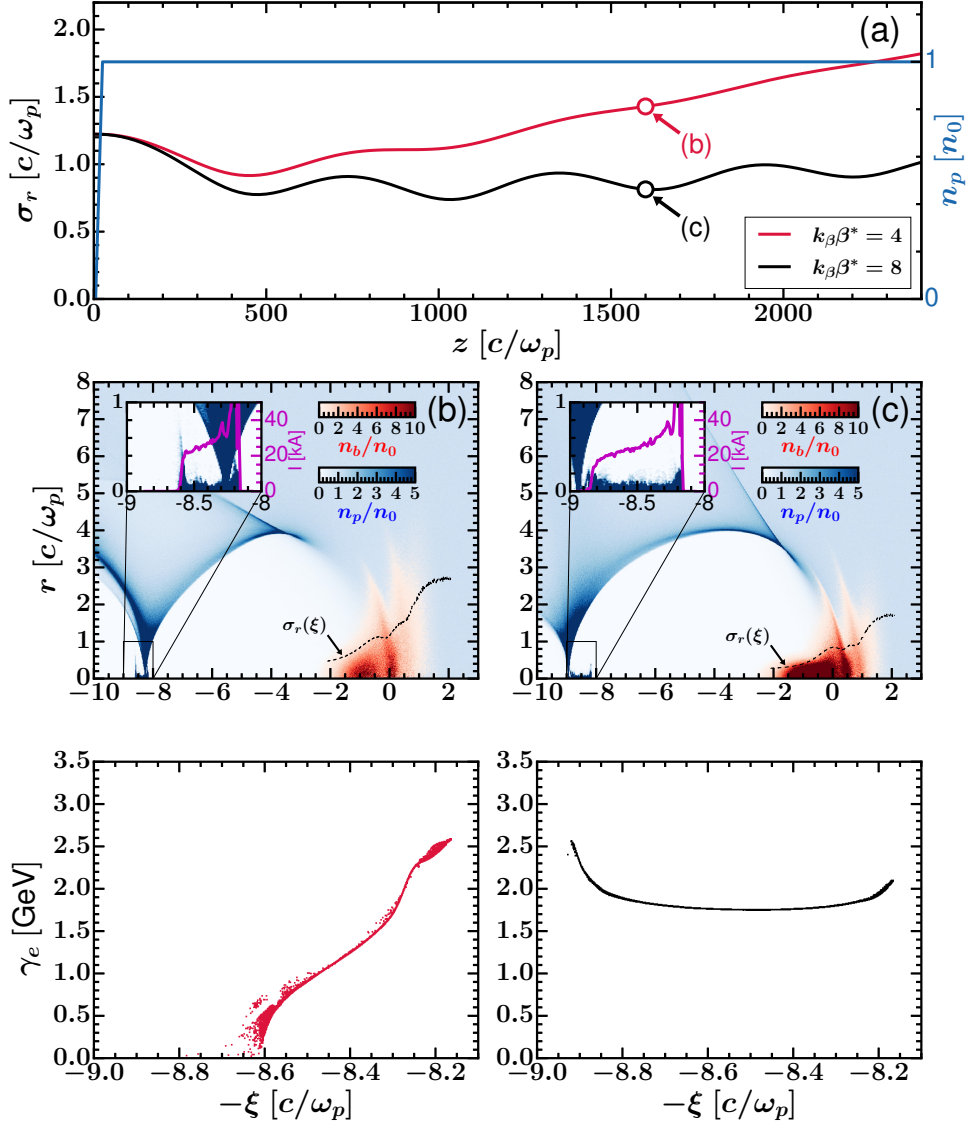


Figure 5.7: The projected spot size evolution $\sigma_r(z)$ of electron drivers $\{\gamma_b = 20000, \Lambda = 6, k_p\sigma_0 = 0.5\sqrt{\Lambda}, k_p\sigma_z = 0.7\}$ with different diffraction lengths $k_\beta\beta^* = 4$ (red) and $k_\beta\beta^* = 8$ (black) propagating inside a constant plasma density profile (blue). (Top) The electron density distribution of the driver and plasma and (bottom) energy distributions of the injected beams are plotted at $z = 1600 c/\omega_p$ for (b) $k_\beta\beta^* = 4$ and (c) $k_\beta\beta^* = 8$. In the top panels, we plot the injected beam current profiles (purple) in the insets and the slice spot sizes $\sigma_r(\xi)$ (dashed black).

In Fig. 5.7(b), we show the electron density distribution of the driver (red) and plasma (blue) at $k_p z = 1600$ for the $k_\beta \beta^* = 4$ case in the top panel. The slice spot size $\sigma_r(\xi)$ of the driver is also plotted (dashed black). Since the beam head propagates according to vacuum diffraction, the maximum slice spot size can be approximated by $\sigma_r \simeq \sigma_0 \sqrt{1 + \frac{z^2}{\beta^{*2}}} \simeq 2.73 \text{ } c/\omega_p$ for $\beta^* = 800 \text{ } c/\omega_p$ and $z = 1600 \text{ } c/\omega_p$, which is indeed observed at the head of the driver in Fig. 5.7(b). As the drive beam head expands, the wake front recedes and wake excitation is effectively driven by a smaller portion of the drive beam. Thus, the rear of the bubble moves forward and a portion of the injected beam slips into the second bucket as shown in the inset plot. As the electric field is positive at the front of the second bucket, these electrons transfer energy back to the wake as seen in the lower panel and, therefore, cannot reach the optimal acceleration distance in the plasma $z_d = 2100 \text{ } c/\omega_p$ determined from the PIC simulation results in Fig. 5.5.

By comparison, for the driver with $k_\beta \beta^* = 8$ the wake remains fully expanded at $k_p z = 1600$ as shown in Fig. 5.7(c). This can be attributed to the fact that head erosion is not as pronounced in this case considering that the diffraction length is twice as long for $k_\beta \beta^* = 8$ when compared to $k_\beta \beta^* = 4$. The maximum slice spot size at the head of the beam is approximately $\sigma_r \simeq \sigma_0 \sqrt{1 + \frac{z^2}{\beta^{*2}}} \simeq 1.73 \text{ } c/\omega_p$ at $z = \beta^* = 1600 \text{ } c/\omega_p$, which is significantly lower than the maximum slice spot size $\sigma_r \simeq 2.73 \text{ } c/\omega_p$ found in the case of $k_\beta \beta^* = 4$. As the total projected spot size of the driver now remains below its initial value during acceleration, the injected beam remains in the first bucket as shown in the inset plot. In addition, the injected beam in this case is noticeably longer because the driver is focused to smaller spot sizes during the initial betatron period. As it remains at the rear of the first bucket, the injected beam continues to be accelerated by the plasma wake until the energy spread is minimized at $z_d = 2100 \text{ } c/\omega_p$.

Based on simulations, we find that the injected beam can remain in the first bucket as long as the drive beam head does not erode significantly over the some propagation distance. For short drivers $k_p \sigma_z < 1$, this condition is generally satisfied if the diffraction length is

comparable to or larger than the optimal driver propagation distance

$$\beta^* \gtrsim z_d. \quad (5.25)$$

From Eq. (5.21), z_d scales with the betatron wavelength $\lambda_\beta \sim k_\beta^{-1}$, so it makes more sense to use the normalized parameter $k_\beta \beta^*$ in this context. Therefore, Eq. (5.25) can be expressed as

$$k_\beta \beta^* \gtrsim \left(\frac{\langle E_z \rangle_{inj}}{\langle d_\xi E_z \rangle_{acc}} \right) \left(\frac{1}{\sqrt{\Lambda}} \right) \left(\frac{2\sqrt{3}}{-\kappa} \right) \left(\frac{\sqrt{\Lambda}}{\sigma_0} \right) + \frac{\pi}{2}. \quad (5.26)$$

For driver parameters $\{\gamma_b = 20000, \Lambda = 6, k_p \sigma_0 = 0.5\sqrt{\Lambda}\}$, we determined $\kappa \approx -1.76$ from Fig. 4.4(c) as well as $\langle E_z \rangle_{inj} \simeq -2.82$, and $\langle d_\xi E_z \rangle_{acc} \simeq -0.63$ from Fig. 5.3(d). For these values, Eq. (5.26) predicts $k_\beta \beta^* \gtrsim 8.76$ is needed to reach the optimal acceleration distance.

In Figs. 5.8(a)-(d), we plot the optimal driver propagation distance z_d , as well as the charge Q_{tr} , core energy $\bar{\gamma}_{tr}$, and core normalized energy spread $\hat{\sigma}_\gamma$ of the injected beam from PIC simulations of drivers $\{\gamma_b = 20000, \Lambda = 6, k_p \sigma_0 = 0.5\sqrt{\Lambda}, k_p \sigma_z = 0.7\}$ with different diffraction lengths β^* propagating in a constant density plasma. Each data point corresponds to a unique PIC simulation for a total of 64 simulations with $k_\beta \beta^*$ ranging from 0.25 to 16. From the simulation results, it can be seen that there is no injection for $k_\beta \beta^* \leq 2$ as either the entire driver diffracts ($k_\beta \beta^* \ll 1$) or head erosion is significant over the betatron length scale ($1 \lesssim k_\beta \beta^* \lesssim 2$). At $k_\beta \beta^* \simeq 2.25$, we begin to see injection as the wake velocity can now be sufficiently reduced by spot size focusing. However, the driver propagation distance z_d and core energy $\bar{\gamma}_{tr}$ of the injected beam are limited by head erosion since the injected bunch needs to be extracted before it slips into the second bucket. In addition, the injected beam charge Q_{tr} is also limited as the spot size focusing is less pronounced at lower $k_\beta \beta^*$, leading to shorter injected beams [see Figs. 5.7(b)-(c)]. Because the injected beam is extracted before its energy distribution is flattened by the plasma wakefield, the normalized core energy spread $\hat{\sigma}_\gamma$ at z_d is large for $k_\beta \beta^* \simeq 2.25$.

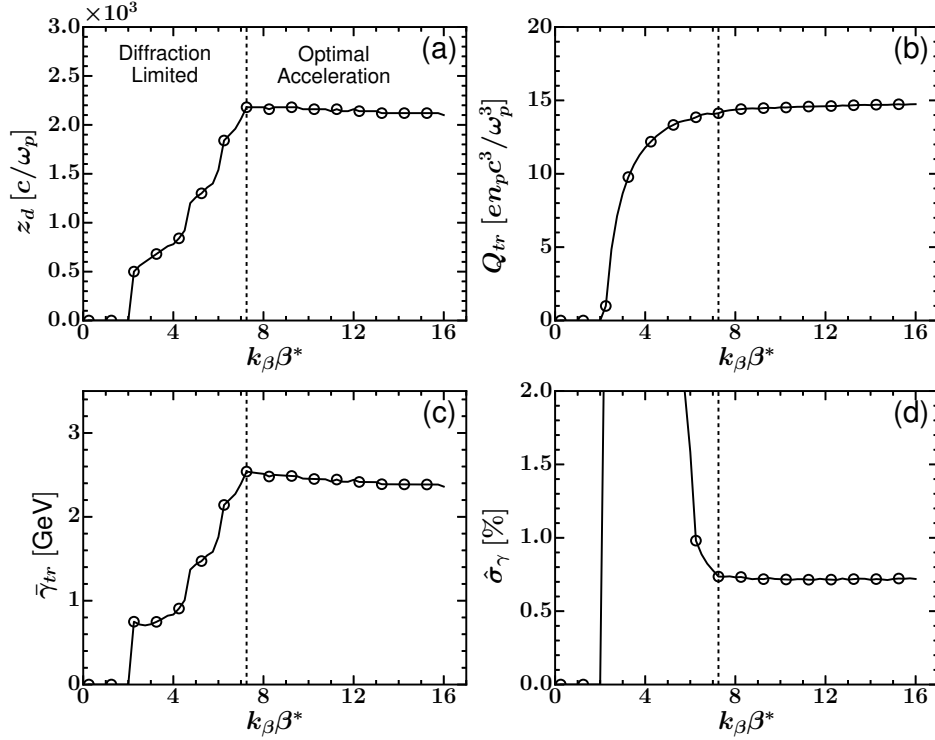


Figure 5.8: (a) The optimal driver propagation distance z_d in the plasma for different $k_\beta \beta^*$. Each data point corresponds to a unique PIC simulation. The transition from diffraction limited to optimal acceleration occurs at $k_\beta \beta^* \simeq 7.25$ and is annotated in dashed black. (b) Total charge Q_{tr} , (c) core energy $\bar{\gamma}_{tr}$, and (d) core energy spread $\hat{\sigma}_\gamma$ of the injected beam at z_d as a function of $k_\beta \beta^*$. We have kept $\Lambda = 6$ in all cases.

For drivers with longer diffraction lengths β^* and, hence, larger $k_\beta \beta^*$, head erosion takes place over longer distances. Thus, higher injected beam energies $\bar{\gamma}_{tr}$ and lower core energy spreads $\hat{\sigma}_\gamma$ can be achieved before it slips into the second bucket. The injected charge Q_{tr} also increases with $k_\beta \beta^*$, primarily driven by longer injected beam lengths $\Delta \xi_{tr}$ as the driver focuses to smaller spot sizes. Eventually around $k_\beta \beta^* \simeq 7.25$, the driver is no longer diffraction limited and can now propagate until the optimal acceleration distance $z_d = 2100 c/\omega_p$ determined in Fig. 5.5. Therefore, z_d , Q_{tr} , $\bar{\gamma}_{tr}$, and $\hat{\sigma}_\gamma$ all saturate at their optimal values and are insensitive to further increases in $k_\beta \beta^*$. We also find that the minimum diffraction length $k_\beta \beta^* \gtrsim 8.76$ predicted from Eq. (5.26) is in reasonable agreement with the simulation results.

By evaluating Eq. (5.26) for the driver $\{\gamma_b = 20000, \Lambda = 6, k_p\sigma_0 = 0.5\sqrt{\Lambda}, k_p\sigma_z = 0.7\}$, we find that $k_\beta\beta^* \gtrsim 10$ is generally sufficient for optimal acceleration of the injected beam in most cases.

5.3.2 The Driver Energy

In the plasma focusing regime ($k_\beta\beta^* \gg 1$), injection takes place on the length scale of the betatron wavelength $\lambda_\beta = 2\pi\sqrt{2\gamma_b}c/\omega_p$. As λ_β is a function of γ_b , the length scale of the problem can be varied by adjusting the drive beam energy. In Figs. 5.9(a)-(b), we examine the spot size evolution from PIC simulations of electron drivers with $\{\Lambda = 6, k_p\sigma_0 = 0.5\sqrt{\Lambda}, k_p\sigma_z = 0.7\}$ but with different energies $\gamma_b = 5000$ (red) and $\gamma_b = 10000$ (black) propagating in a constant density plasma. In each case, the diffraction length is scaled appropriately with the betatron wavelength where $k_\beta\beta^* = 16$. From the simulation results, it can be seen that the main difference between the evolution of the two drivers is the length scale of the betatron oscillations. As the minimum slice spot size scales with $1/(k_\beta\beta^*)$ from Eq. (5.5), the spot size evolution of the two drivers (as seen in Figs. 5.9(a)-(b)) is similar within the first few betatron oscillations. If the axis were plotted in $k_\beta z$ the curves would nearly line up. Recall that the injected bunch length can be related to the duration of injection and the bubble expansion rate, $\Delta\xi_{tr} = \frac{dL_b}{dz}\Delta z$. The duration of injection scales as k_β^{-1} so $\Delta\xi_{tr} = \frac{dL_b}{dk_\beta z}k_\beta\Delta z$ does not depend on k_β (or γ_b in this case). Therefore, $\Delta\xi_{tr}$ is similar for these two cases. However, it is worth noting that the energy loss of the driver over a betatron oscillation scales as $\frac{\Delta\gamma_b}{\gamma_b} \sim -\frac{E_z\lambda_\beta}{\gamma_b} \sim -\frac{\sqrt{\Lambda}}{\sqrt{\gamma_b}}$. Thus, pump depletion over betatron length scales is more pronounced for drivers with lower γ_b . From this, it also follows that the relative contraction of the betatron wavelength over a betatron oscillation $\frac{\Delta\lambda_\beta}{\lambda_\beta} \sim \frac{\Delta\gamma_b}{\gamma_b} \sim -\frac{\sqrt{\Lambda}}{\sqrt{\gamma_b}}$ is also more pronounced for lower γ_b . While such effects can be largely neglected for the driver parameters used here, they can come into play when γ_b is sufficiently small or Λ is sufficiently large.

In the top panels of Figs. 5.9(c)-(d), we show the electron density distribution of the

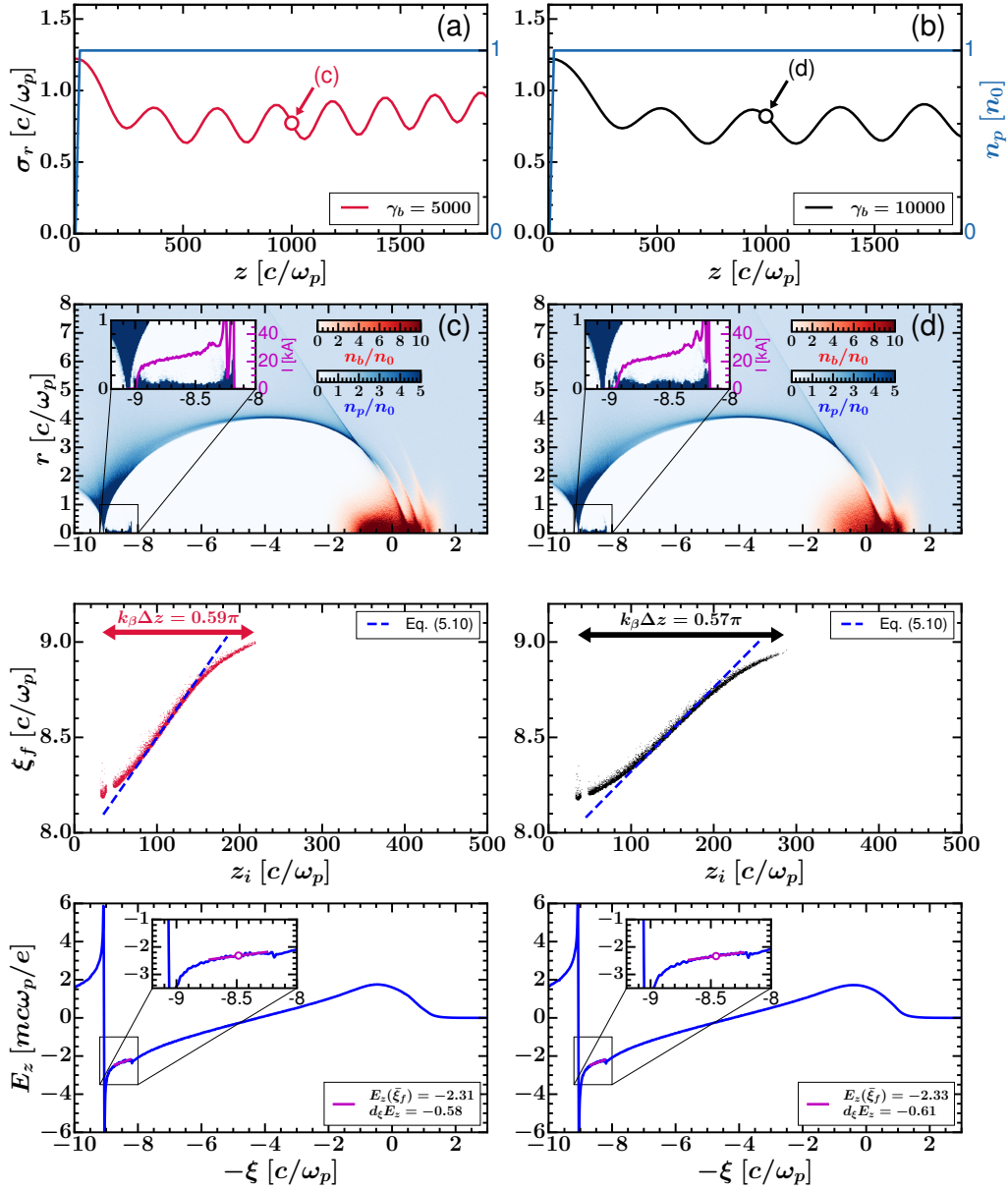


Figure 5.9: The spot size evolution $\sigma_r(z)$ of drivers $\{\Lambda = 6, k_p\sigma_0 = 0.5\sqrt{\Lambda}, k_p\sigma_z = 0.7, k_\beta\beta^*\}$ with (a) $\gamma_b = 5000$ (red) and (b) $\gamma_b = 10000$ (black) propagating in a constant plasma density (blue). (Top) The electron charge density, (middle) the phase space (ξ_f, z_i) of the injected charge, and (bottom) $E_z(\xi)$ at $z = 1000 c/\omega_p$ for (c) $\gamma_b = 5000$ and (d) $\gamma_b = 10000$. The injected beam current profiles (purple) are shown in the insets. Eq. (5.10) (dashed blue) and linear fits of the electric fields within the beam cores (purple) are plotted.

driver (red) and plasma (blue) at $z = 1000 c/\omega_p$ for $\gamma_b = 5000$ and $\gamma_b = 10000$, respectively. As the betatron wavelength scales as $\lambda_\beta \sim \sqrt{\gamma_b}$, the frequency of the betatron oscillation for the lower energy driver ($\gamma_b = 5000$) is nearly 1.41 times that of the higher energy driver ($\gamma_b = 10000$). Evidently, the lower energy driver has 3 scallops – one for each oscillation – while the higher energy driver has only 2 scallops at $z = 1000 c/\omega_p$. Despite the fact that the length scale of injection varies with the betatron wavelength $L_{inj} \sim \lambda_\beta/4$, the current profiles, $I = \frac{Q_{inj}}{\Delta\xi_{tr}}$, of the injected beams (purple) in the insets are very similar with slice currents ranging from 20 to 40 kA over the middle sections of the beams. Indeed, these profiles are also similar to the one shown in Fig. 5.5(c) for $\gamma_b = 20000$.

To understand why the current profiles are insensitive to the drive beam energy, we examine Eq. (5.24) where it was shown that the slice currents could be written in the form $I/I_A = \gamma_\phi^2(k_p r_i)(k_p \Delta r_i)$. As the wake velocity γ_ϕ scales with $\gamma_b^{1/4}$, the trapping threshold $\gamma_{z,m} > \gamma_\phi$ is easier to satisfy for lower γ_b . Indeed, from Eq. (5.12), we find that $\gamma_\phi \approx 9$ for $\gamma_b = 5000$ and $\gamma_\phi \approx 10.7$ for $\gamma_b = 10000$. Therefore, Δr_i increases for lower γ_b as more electrons are injected from a given longitudinal slice dz_i in the plasma. As the current profiles from the PIC simulations are insensitive to the drive beam energy, it can be inferred that $\Delta r_i \sim \gamma_b^{-1/2}$ for the drive beam parameters used here. However, it is worth noting that this scaling breaks down at sufficiently large γ_b where the trapping condition $\gamma_{z,m} > \gamma_\phi$ is no longer satisfied, $\Delta r_i \rightarrow 0$, and $I \rightarrow 0$. Conversely, if injection is not triggered for a given γ_b , the wake velocity can be further reduced by lowering the drive beam energy until the trapping threshold is satisfied.

As noted above, the injected beam length $\Delta\xi_{tr}$ does not depend on γ_b so long as the trapping threshold is satisfied and the diffraction length is scaled appropriately, i.e. $k_\beta \beta^* \gtrsim 10$ [see Fig. 5.8]. Indeed, this is borne out in the injected electron phase space $\xi_f - z_i$ shown in the middle panels of Figs. 5.9(c)-(d). We find that the injected beam lengths are $\Delta\xi_{tr} \simeq 0.86 c/\omega_p$ and $\Delta\xi_{tr} \simeq 0.84 c/\omega_p$ for $\gamma_b = 5000$ and $\gamma_b = 10000$, respectively. From these results, it can also be seen that the longitudinal length scale of injection is also well

approximated by $L_{inj} \sim \lambda_\beta/4 \sim \pi/2k_\beta$. The dependence of L_{inj} on the betatron wavelength λ_β is indeed accounted for in the wake evolution $\frac{dL_b}{dz}$ described by Eq. (5.10), which is evaluated for each driver and plotted in dashed blue. In both cases, strong agreement is observed between Eq. (5.10) and the PIC simulation results.

Since the current profiles of the injected bunches are insensitive to the γ_b , the accelerating fields are loaded in a similar manner as seen in the lower panels of Figs. 5.9(c)-(d). From the PIC simulation results, we find $\langle E_z \rangle_{acc} \simeq -2.31$ and $\langle d_\xi E_z \rangle_{acc} \simeq -0.58$ for $\gamma_b = 5000$ and $\langle E_z \rangle_{acc} \simeq -2.33$ and $\langle d_\xi E_z \rangle_{acc} \simeq -0.61$ at $z = 1000 c/\omega_p$. From Eq. (5.18), it also follows that the optimal acceleration distance to minimize the energy spread within the injected beam scales strongly with the injection length $L_{acc} \sim L_{inj}$. As a result, both the total propagation distance z_d of the driver in the plasma and injected beam energy $\bar{\gamma}_{tr}$ scale strongly with the betatron wavelength $z_d \sim \bar{\gamma}_{tr} \sim \lambda_\beta$ from Eqs. (5.21)-(5.22). Based on the results from Fig. 5.3(d), we approximate $\langle E_z \rangle_{inj} \approx 1.2 \times \langle E_z \rangle_{acc}$ to obtain $z_d \simeq 925 c/\omega_p$ and $\bar{\gamma}_{tr} mc^2 \simeq 1.02$ GeV for $\gamma_b = 5000$ and $z_d \simeq 1264 c/\omega_p$ and $\bar{\gamma}_{tr} mc^2 \simeq 1.4$ GeV for $\gamma_b = 10000$.

By comparison, the PIC simulation results indicate an optimal driver propagation distance of $1120 c/\omega_p$ for $\gamma_b = 5000$ and $1520 c/\omega_p$ for $\gamma_b = 10000$. The energy distributions of the injected beams at these positions are shown in the top panels of Figs. 5.10(a)-(b) for $\gamma_b = 5000$ (red) and $\gamma_b = 10000$ (black). The vertical dashed black lines delineate the injected beam cores containing 50% of the total injected charge. Within these regions, we find average energies of 1.22 GeV and 1.68 GeV and projected energy spreads of 9.9 MeV (0.81%) and 13.4 MeV (0.80%) for $\gamma_b = 5000$ and $\gamma_b = 10000$, respectively. In both cases, sub-MeV slice energy spreads and ultra-low normalized slice emittances of $\epsilon_n \sim O(0.01 c/\omega_p)$ are observed within the beam cores as shown in the lower panels. The total injected charge $Q_{tr}/en_p k_p^{-3} \simeq 17.3$ for $\gamma_b = 5000$ is approximately 5.5% higher than the injected charge $Q_{tr}/en_p k_p^{-3} \simeq 16.4$ for $\gamma_b = 10000$. The small increase in Q_{tr} can be attributed to the larger slice currents observed at the very rear of the injected beam. As the wakefields are loaded in a similar fashion, the transformer ratios are nearly identical, with $R = 1.26$ and

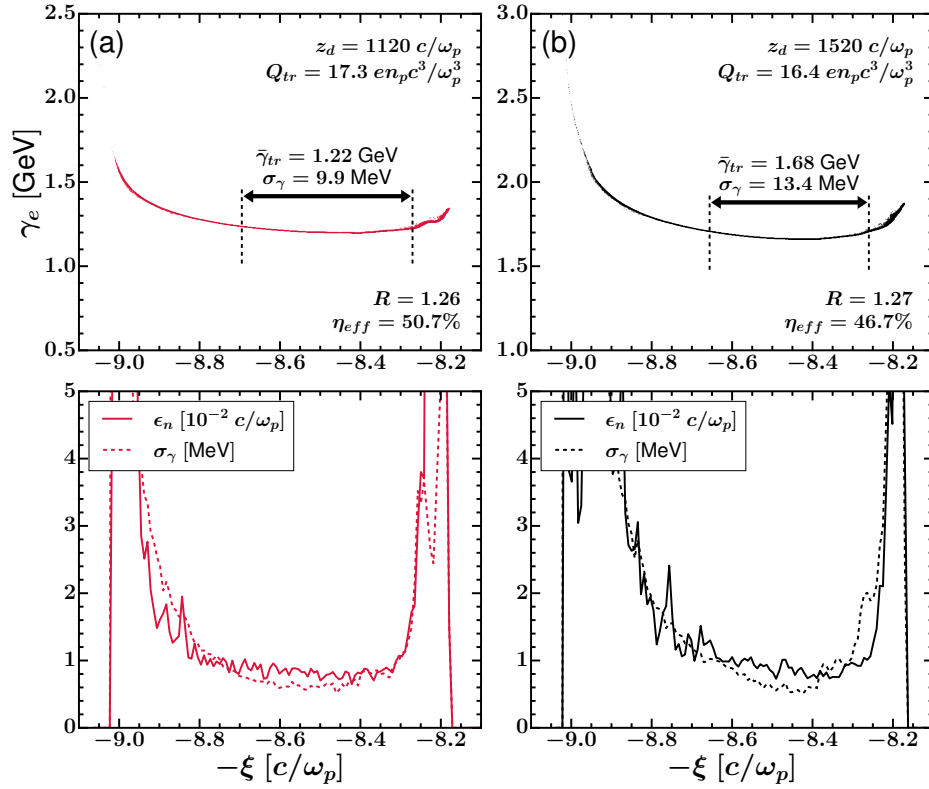


Figure 5.10: (Top) The energy distribution of the injected beams after optimal acceleration for drivers $\{\Lambda = 6, k_p\sigma_0 = 0.5\sqrt{\Lambda}, k_p\sigma_z = 0.7, k_\beta\beta^* = 16\}$ with different energies (a) $\gamma_b = 5000$ (red) and (b) $\gamma_b = 10000$ (black). The dashed black lines enclose 50% of total injected charge surrounding the centroid. (Bottom) The slice energy spread σ_γ in units of MeV (dashed) and normalized slice emittance ϵ_n in units of c/ω_p (solid).

$R = 1.27$ for $\gamma_b = 5000$ and $\gamma_b = 10000$, respectively. Therefore, the transfer efficiency $\eta_{eff} \simeq 50.7\%$ (46.7%) for $\gamma_b = 5000$ (10000) largely scales with the injected charge Q_{tr} .

In Figs. 5.11(a)-(b), we plot the optimal propagation distance z_d of the driver in the plasma and core energy, $\bar{\gamma}_{tr}$, of the injected beam determined from PIC Simulations (red) and calculations using Eqs. (5.21)-(5.22) (blue) for different driver energies $\gamma_b = 5000, 10000, 20000$. While the theoretical estimates do not perfectly match the simulation results, Eqs. (5.21) and (5.22) capture the general dependence of z_d and $\bar{\gamma}_{tr}$ on γ_b through the betatron wavelength, which scales as $\sqrt{\gamma_b}$. This scaling is also illustrated by linear fits of z_d and $\bar{\gamma}_{tr}$ from the

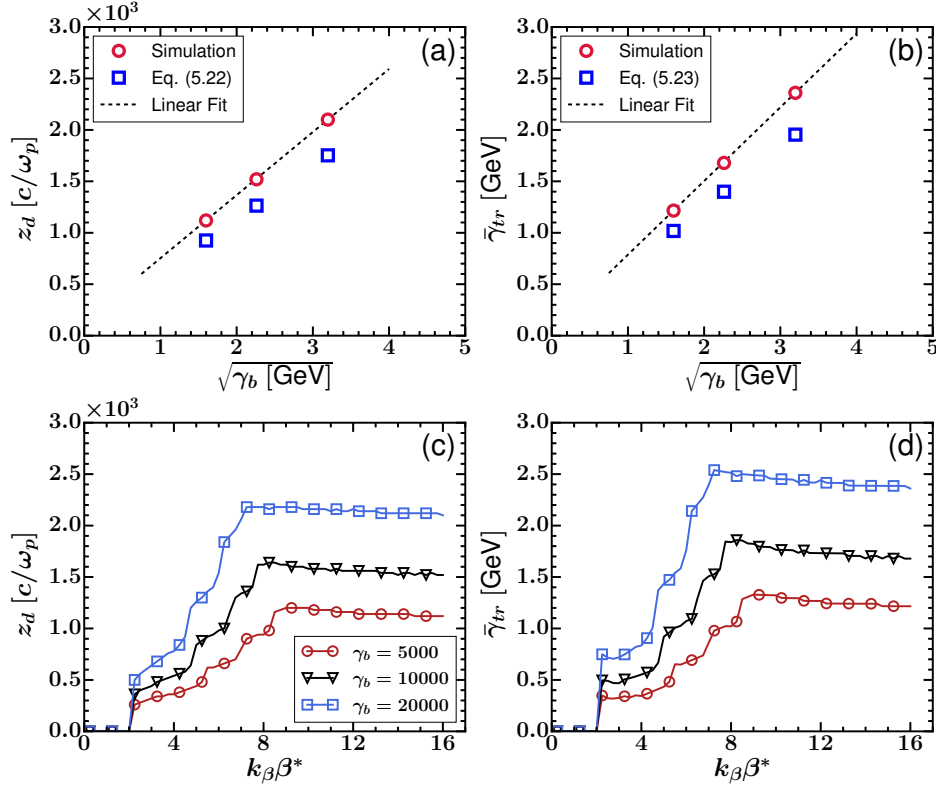


Figure 5.11: (a) The optimal driver propagation distance z_d and (b) core energy of the injected beam $\bar{\gamma}_{tr}$ as a function of $\sqrt{\gamma_b}$ for drivers $\{\Lambda = 6, k_p \sigma_0 = 0.5\sqrt{\Lambda}, k_p \sigma_z = 0.7, k_\beta \beta^* = 16\}$. PIC simulation results are shown in red. A linear fit of the simulation results is plotted in dashed black. Eqs. (5.21) and (5.22) are shown in blue. (c) z_d and (d) $\bar{\gamma}_{tr}$ for drivers with different $k_\beta \beta^*$ and energy γ_b . Each data point corresponds to a unique PIC simulation.

simulation results (dashed black). These results are important because they indicate that the optimal output beam energy can be tuned through the drive beam energy $\bar{\gamma}_{tr} \sim \sqrt{\gamma_b}$. To avoid the effects of head erosion for short drivers, the diffraction length β^* must also scale with the driver propagation distance z_d from Eq. (5.25) and, hence, betatron wavelength λ_β . In Figs. 5.11(c)-(d), we show how z_d and $\bar{\gamma}_{tr}$ vary for drivers with the different $k_\beta \beta^*$ and γ_b , where each data point corresponds to a unique PIC simulation. From the results, we find that the injected beam can be optimally accelerated as long as $k_\beta \beta^* \gtrsim 10$, even for drivers with different γ_b .

5.3.3 The Driver Intensity

The driver intensity Λ determines the scale length of the wake, $r_m \sim \sqrt{\Lambda}c/\omega_p$. Naturally, drivers with higher Λ are capable of driving more pronounced wake expansion ΔL_b [see Fig. 4.4(c)], leading to longer injected beams. As the magnitude of the wake expansion rate $|\kappa|$ increases for higher Λ , lower wake velocities $\gamma_\phi \sim \Lambda^{-1/4}\kappa^{-1/2}$ can be expected from Eq. (5.12). Moreover, the maximum forward velocity of the sheath electrons $\gamma_{z,m}$ increases with Λ [see Fig. 4.4(d)], resulting in higher injected charge and slice currents as the trapping threshold $\gamma_{z,m} > \gamma_\phi$ is eased. Thus, for drivers with higher Λ , it is possible to inject significantly more charge Q_{tr} and achieve higher transfer efficiencies η_{eff} .

For a bi-Gaussian driver with a peak current $I \propto n_{b0}\sigma_0^2 \propto \frac{n_{b0}}{n_0}k_{p0}^2\sigma_0^2 \equiv \Lambda$, there is a relationship between the beam density n_{b0} and focal spot size σ_0 . Injection relies on focusing a driver that starts from a spot size on the order of the wake scale length, $k_p\sigma_0 \sim \sqrt{\Lambda}$. Therefore, for any Λ , $n_b/n_0 \sim 1$ and $k_p\sigma_0 \sim \sqrt{\Lambda}$ when injection occurs. However, the rate of focusing depends on Λ , and as discussed above this leads to a more pronounced wake expansion. For $k_\beta\beta^* \gg 1$, the spot size evolves as $\sigma_r'' \sim \frac{k_p\sqrt{\Lambda}}{\gamma_b}$. Furthermore, as noted previously, pump depletion $\frac{\Delta\gamma_b}{\gamma_b} \sim -\frac{\sqrt{\Lambda}}{\sqrt{\gamma_b}}$ and betatron wavelength contraction $\frac{\Delta\lambda_\beta}{\lambda_\beta} \sim -\frac{\sqrt{\Lambda}}{\sqrt{\gamma_b}}$ over betatron length scales are both more pronounced for higher Λ . However, we can neglect these effects for the parameters considered here.

In Figs. 5.12(a), we plot the spot size evolution from PIC simulations of electron drivers $\{\gamma_b = 20000, k_p\sigma_0 = 0.5\sqrt{\Lambda}, k_p\sigma_z = 0.7\}$ with different peak currents $\Lambda = 4$ (red) and $\Lambda = 8$ (black) propagating in a constant density plasma. For these parameters, the initial focal spot sizes are $k_p\sigma_0 = 1$ and $k_p\sigma_0 = 1.41$ for $\Lambda = 4$ and $\Lambda = 8$, respectively. In both cases, the diffraction length is kept constant at $k_\beta\beta^* = 16$. From these results, it can be seen that the drivers are focused in a similar fashion, with the $\Lambda = 8$ case focusing slightly more rapidly. Over longer distances, each driver continues to oscillate at half the betatron wavelength $\lambda_\beta/2 = \pi/k_\beta \simeq 628 c/\omega_p$ as the rearmost slices are fully contained within the

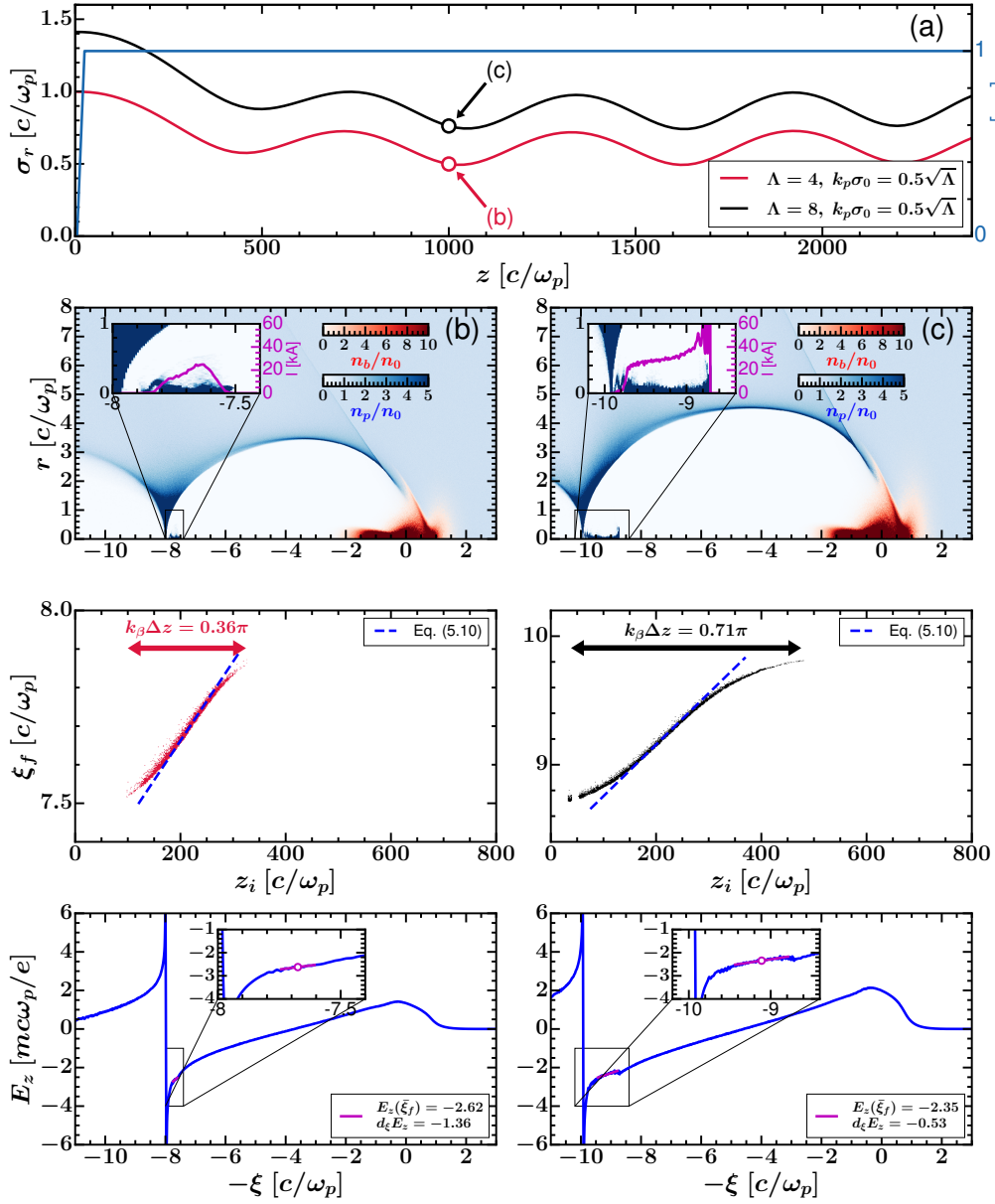


Figure 5.12: (a) $\sigma_r(z)$ of drivers $\{\gamma_b = 20000, k_p\sigma_0 = 0.5\sqrt{\Lambda}, k_p\sigma_z = 0.7, k_\beta\beta^* = 16\}$ with different peak currents $\Lambda = 4$ (red) and $\Lambda = 8$ (black) propagating in a constant plasma density (blue). (Top) The electron charge density, (middle) the phase space (ξ_f, z_i) of the injected charge and (bottom) $E_z(\xi)$ at $z = 1000 c/\omega_p$ for (b) $\Lambda = 4$ and (c) $\Lambda = 8$. The injected beam current profiles (purple) are shown in the insets. Eq. (5.10) (dashed blue) and linear fits of the electric fields within the beam cores (purple) are plotted.

plasma channel.

In the top panels of Figs. 5.12(b)-(c), we show the electron density distribution of the driver (red) and plasma (blue) at $z = 1000 \text{ } c/\omega_p$ for $\Lambda = 4$ and $\Lambda = 8$, respectively. Indeed, the features of both drivers are nearly identical as each has a single scallop and the charge density pinches in a similar fashion. The channel radius and wake length are noticeably larger in the case of $\Lambda = 8$ due to an increase in the wake scale length $\sqrt{\Lambda}c/\omega_p$. As shown in the inset plots, the current profiles of the injected beams (purple) differ meaningfully from each other in terms of shape, magnitude, and length.

The differences in the current profiles can be understood by examining the expressions for the trapping threshold $\gamma_{z,m} > \gamma_\phi$ and slice currents $I/I_A = \gamma_\phi^2(k_p r_i)(k_p \Delta r_i)$. To calculate γ_ϕ , we first approximate $\kappa = \frac{dL_b}{d\sigma_r} \approx -1.42$ (-1.96) for $\Lambda = 4$ (8) from the results in Fig. 4.4(c). It is worth noting here that κ largely scales as $\sqrt{\Lambda}$ for values of Λ ranging from 3 to 8. From Eq. (5.12), we find $\gamma_\phi \approx 15.6$ for $\Lambda = 4$ and $\gamma_\phi \approx 11.2$ for $\Lambda = 8$. In the case of $\Lambda = 4$, the trapping condition is not satisfied initially as $\gamma_{z,m} \approx 14.5 < \gamma_\phi$ for $k_p \sigma_r = 0.5\sqrt{\Lambda}$ [see Fig. 4.4(d)], resulting in $\Delta r_i \rightarrow 0$ and $I \rightarrow 0$. However, as the spot sizes continues to focus, injection is triggered when $\gamma_{z,m}$ increases above γ_ϕ around $\sigma_r \approx 0.45\sqrt{\Lambda}c/\omega_p$. The value of $\gamma_{z,m}$ eventually asymptotes to ~ 19 when $\sigma_r \ll \sqrt{\Lambda}c/\omega_p$. This is likely the reason why the current profile ramps smoothly from 0 to its maximum value $I \sim 25 \text{ kA}$. As Eq. (5.12) is a lower bound estimate for the wake velocity, γ_ϕ increases as the magnitude of $|\kappa| = |\frac{dL_b}{d\sigma_r}|$ starts to decrease when $k_p \sigma_r \lesssim 0.3\sqrt{\Lambda}c/\omega_p$ [Fig. 4.4(c)]. As a result, γ_ϕ increases above $\gamma_{z,m}$ well before the wake is fully expanded in the case of $\Lambda = 4$ and injection terminates early. This is indeed reflected in the injected electron phase space $\xi_f - z_i$ shown in the middle panel where it can be seen that $L_{inj} \approx 0.36\pi/k_\beta$.

It can be inferred from the general scaling in Fig. 4.4(d) that the trapping threshold $\gamma_{z,m} > \gamma_\phi$ is satisfied over virtually the entire wake expansion process. From the injected electron phase space $\xi_f - z_i$ shown in the middle panel of Fig. 5.12(c), it can be seen that injection is in fact sustained over a longer distance $L_{inj} \approx 0.71\pi/k_\beta$ for $\Lambda = 8$. As a result,

the current profile of the injected beam extends to the very rear of the expanded wake such that $\Delta\xi_{tr}$ is larger. Despite the fact that the wake evolves more quickly with $\gamma_\phi \approx 11.2$ for $\Lambda = 8$, larger slice currents $I \gtrsim 30 - 50$ kA are observed within the middle section of the injected beam due to larger $r_i \sim \sqrt{\Lambda}$ and Δr_i (larger Q_{tr}). From Eq. (5.24), it follows that Δr_i must increase in order to produce larger slice currents as $\gamma_\phi^2(k_p r_i) \sim \kappa^{-1}$ decreases for larger Λ .

Despite the fact that injection length L_{inj} varies in these two cases, the mapping of the injected electrons from z_i to ξ_f in each case is well-approximated by the wake evolution $\frac{dL_b}{dz}$ described by Eq. (5.10), which is evaluated and plotted in dashed blue for each driver. This can be largely attributed to the fact that the drivers evolve in a similar manner as can be seen in Fig. 5.12(a). In each case, the injected beam length $\Delta\xi_{tr}$ is determined by the amount of spot size focusing $\Delta\sigma_r \sim \sigma_r \sim \sqrt{\Lambda}c/\omega_p$ and $\kappa = \frac{dL_b}{d\sigma_r}$. As noted previously, $\kappa \sim \sqrt{\Lambda}$ for Λ ranging from 3 to 8, the scaling of the injected beam length can be well-approximated by $\Delta\xi_{tr} \sim \Delta L_b(\sigma_r) \sim \kappa\sigma_r \sim \Lambda$ for such parameters. While injection is limited in the case of $\Lambda = 4$, this scaling still captures the trend of the injected beam lengths $\Delta\xi_{tr} \simeq 0.44 c/\omega_p$ and $\Delta\xi_{tr} \simeq 1.18 c/\omega_p$ from the PIC simulation results using $\Lambda = 4$ and $\Lambda = 8$, respectively. As the injected beam length is very short in the case of $\Lambda = 4$, its centroid is situated at the very rear of the wake where $\langle E_z \rangle_{acc} \simeq -2.62$ and $\langle d_\xi E_z \rangle_{acc} \simeq -1.36$, which can be seen in the lower panel of Fig. 5.12(b). However, for $\Lambda = 8$, $\Delta\xi_{tr}$ is larger relative to the wake scale length and the beam centroid is further forward relative to the rear of the wake. Therefore, the electric field $\langle E_z \rangle_{acc} \simeq -2.35$ and electric field slope $\langle d_\xi E_z \rangle_{acc} \simeq -0.53$ at the beam centroid are less negative as seen in the lower panel of Fig. 5.12(c). Using Eqs. (5.21)-(5.22), we approximate $\langle E_z \rangle_{inj} \approx 1.2 \times \langle E_z \rangle_{acc}$ to obtain $z_d \simeq 1442 c/\omega_p$ and $\bar{\gamma}_{tr}mc^2 \simeq 1.76$ GeV for $\Lambda = 4$ and $z_d \simeq 1644 c/\omega_p$ and $\bar{\gamma}_{tr}mc^2 \simeq 1.82$ GeV for $\Lambda = 8$.

The predictions of Eqs. (5.21) and (5.22) are in reasonable agreement with the PIC simulation results which indicate optimal driver propagation distances of $1640 c/\omega_p$ for $\Lambda = 4$ and $1960 c/\omega_p$ for $\Lambda = 8$. The energy distributions of the injected beams at these positions

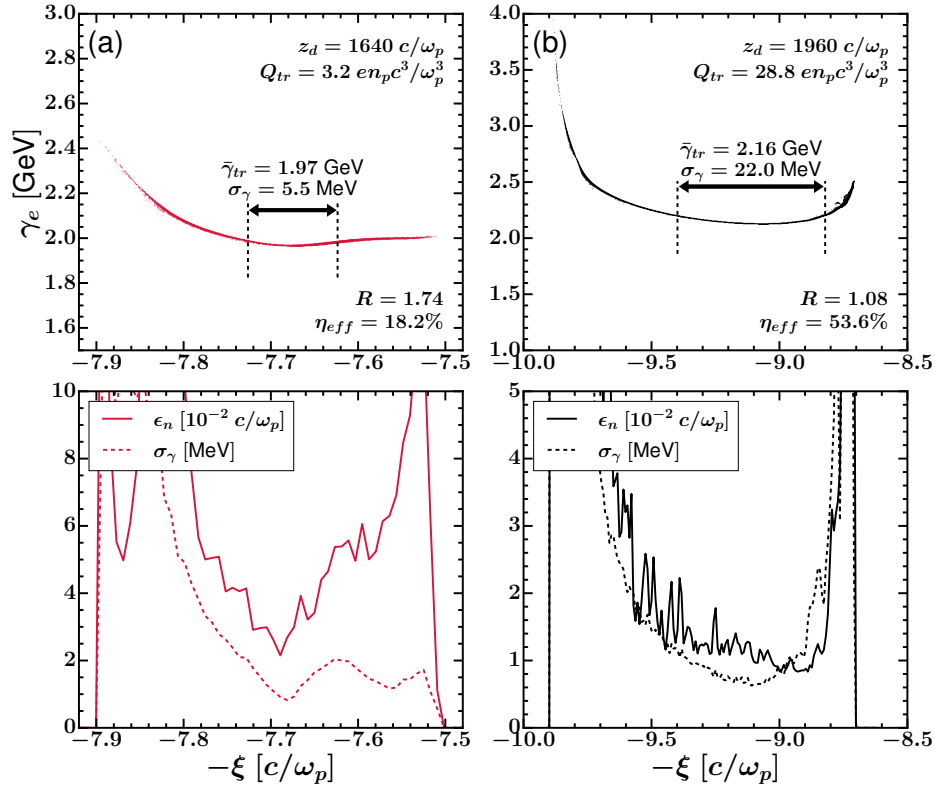


Figure 5.13: (Top) The injected electron energy distribution after optimal acceleration for drivers $\{\gamma_b = 20000, k_p\sigma_0 = 0.5\sqrt{\Lambda}, k_p\sigma_z = 0.7, k_\beta\beta^* = 16\}$ with different peak currents (a) $\Lambda = 4$ (red) and (b) $\Lambda = 8$ (black). The dashed black lines enclose 50% of total injected charge surrounding the centroid. (Bottom) The corresponding slice energy spread σ_γ in units of MeV (dashed) and normalized slice emittance ϵ_n in units of c/ω_p (solid).

are shown in the top panels of Figs. 5.13(b)-(c) for $\Lambda = 4$ and $\Lambda = 8$, respectively. The beam cores are delimited by vertical dashed black lines enclosing 50% of the total injected charge. Within these regions, we find average energies of 1.97 GeV and 2.16 GeV with projected energy spreads of 5.5 MeV (0.27%) and 22 MeV (1%) for $\Lambda = 4$ and $\Lambda = 8$. In addition, the lower panels show that sub-MeV slice energy spreads (dashed) and ultra-low normalized slice emittances (solid) $\epsilon_n \sim O(0.01 c/\omega_p)$ are achieved within the beam cores. Due to higher currents and a longer beam length, the total injected charge $Q_{tr}/en_pk_p^{-3} \simeq 28.8$ for $\Lambda = 8$ is nearly an order of magnitude higher than the total injected charge $Q_{tr}/en_pk_p^{-3} \simeq 3.2$ for

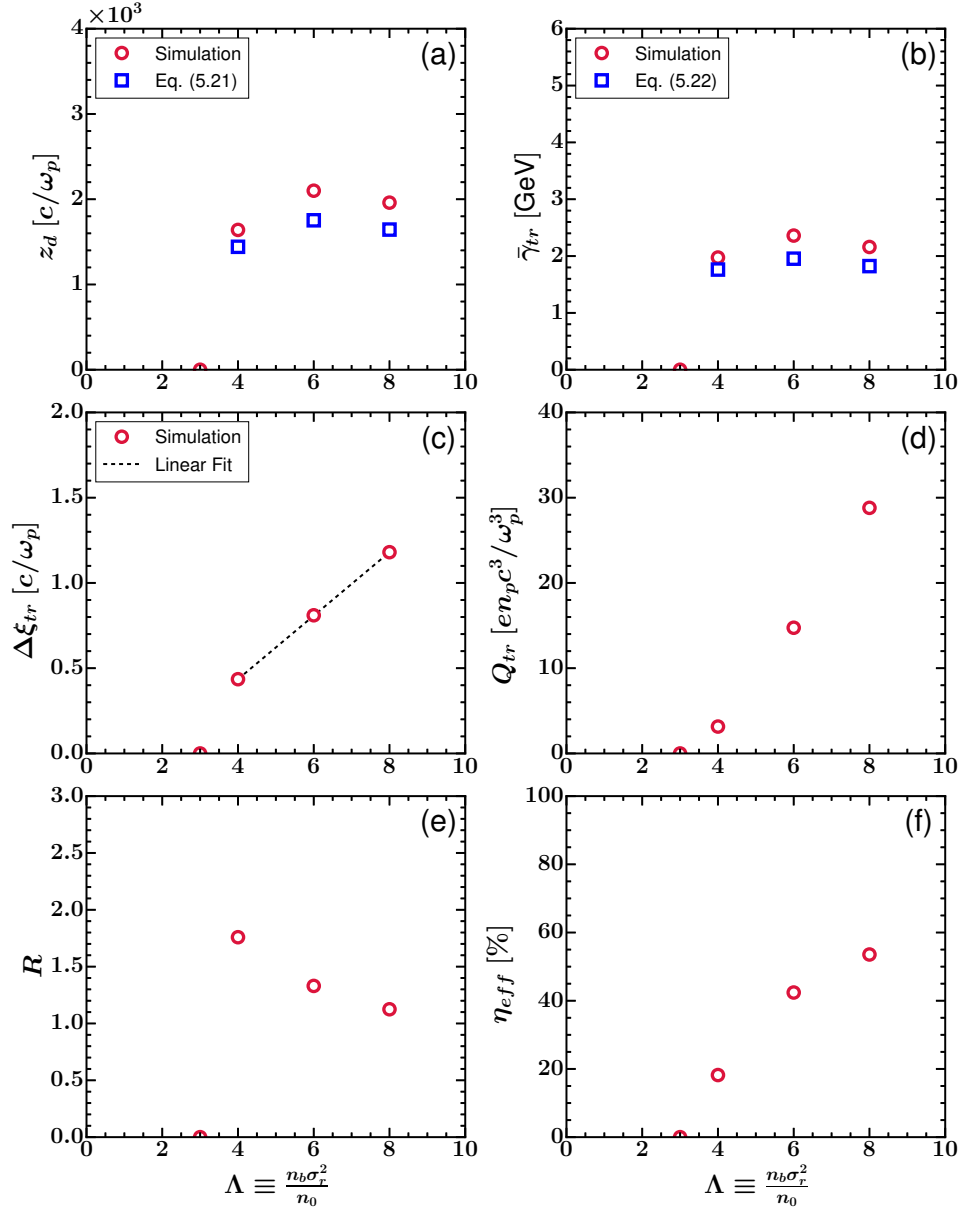


Figure 5.14: (a) The optimal driver propagation distance z_d , (b) injected beam core energy $\bar{\gamma}_{tr}$, (c) injected beam length $\Delta \xi_{tr}$, (d) injected charge Q_{tr} , (e) transformer ratio R , and (f) transfer efficiency η_{eff} as a function of Λ for drivers $\{\gamma_b = 20000, k_p \sigma_r = 0.5\sqrt{\Lambda}, k_p \sigma_z = 0.7, k_\beta \beta^* = 16\}$. PIC simulation results are shown in red. A linear fit of $\Delta \xi_{tr}$ determined from simulation results is plotted in dashed black. Eqs. (5.21) and (5.22) are shown in blue.

$\Lambda = 4$. However, the transformer ratio $R = 1.08$ for $\Lambda = 8$ is smaller than $R = 1.74$ for $\Lambda = 4$ due to both a lower accelerating field and higher decelerating field $E_{z,m} \sim \sqrt{\Lambda}$. Despite this, the transfer efficiency $\eta_{eff} \simeq 53.6\%$ for $\Lambda = 8$ is still significantly higher than $\eta_{eff} \simeq 18.2\%$ for $\Lambda = 4$ driven by the larger amount of injected charge Q_{tr} .

In Figs. 5.14(a)-(b), we plot the optimal driver propagation distance z_d and core energy of the injected beam $\bar{\gamma}_{tr}$ determined from PIC simulation results (red) using drivers $\{\gamma_b = 20000, k_p\sigma_r = 0.5\sqrt{\Lambda}, k_p\sigma_z = 0.7, k_\beta\beta^* = 16\}$ with different intensities $\Lambda = 3, 4, 6, 8$. Numerical calculations plotted in blue using Eqs. (5.21)-(5.22) are in good agreement with the simulation results. In the case of $\Lambda = 3$, we note that injection was not produced as the trapping condition $\gamma_{z,m} > \gamma_\phi$ was never satisfied. However, as pointed out in the previous section, the drive beam energy γ_b can be lowered in such cases to reduce the wake velocity $\gamma_\phi \sim \gamma_b^{1/4}$ until the trapping condition is met. In the cases where injection occurs, there is no clear correlation between z_d , $\bar{\gamma}_{tr}$ and Λ since the magnitude and slope of the accelerating field can depend on the length of the injected beam and beam loading effects from the injected slice currents. We pointed out previously that the injected beam length $\Delta\xi_{tr}$ varies linearly with Λ for the parameters considered in Fig. 5.14(c). Because the injected slice currents also increase with Λ , the total injected charge Q_{tr} is shown in Fig. 5.14(d) to increase at a faster rate than the drive beam charge $Q_d \sim \Lambda$ for higher values of Λ . Therefore, despite the lower transformer ratios observed for higher Λ in Fig. 5.14(e), higher transfer efficiencies can be achieved by increasing the driver intensity as shown in Fig. 5.14(f).

5.3.4 The Normalized Focal Spot Size

In the previous section, we examined how the injected beam parameters vary with the driver intensity Λ by scaling the driver parameters such that the normalized focal spot size $k_p\sigma_0/\sqrt{\Lambda}$ and peak beam density n_{b0} are kept constant. However, wake expansion and injection can vary depending on the amount of spot size focusing even for fixed Λ . Indeed, it was shown in Fig. 4.4(c) that deviation from the maximum wake length $\Delta L_b(\sigma_r)$ depends directly on

the normalized driver spot size $k_p\sigma_0/\sqrt{\Lambda}$ for a given Λ . Thus, one can tailor the length of the injected beam $\Delta\xi_{tr} \sim |\Delta L_b|$ by varying the focal spot size of the driver. In addition, the wake velocity γ_ϕ [Eq. (5.12)] and maximum sheath electron energy $\gamma_{z,m}$ [Fig. 4.4(d)] also depend on $k_p\sigma_0/\sqrt{\Lambda}$, which can lead to different current profiles for the injected beam from Eq. (5.24). In this section, we examine how the injected beam parameters vary with the normalized focal spot size $k_p\sigma_0/\sqrt{\Lambda}$ for fixed Λ .

In Figs. 5.15(a), we plot the projected spot size evolution $\sigma_r(z)$ from PIC simulations of electron drivers $\{\gamma_b = 20000, \Lambda = 6, k_p\sigma_z = 0.7, k_\beta\beta^* = 16\}$ with different focal spot sizes $k_p\sigma_0 = 0.4\sqrt{\Lambda}$ (red) and $k_p\sigma_0 = 0.6\sqrt{\Lambda}$ (black) propagating in a constant density plasma (blue). Since the driver intensity is fixed $\Lambda = 6$, the red and black curves correspond to $k_p\sigma_0 \simeq 0.98$ and $k_p\sigma_0 \simeq 1.47$, respectively. In the top panels of Figs. 5.15(b)-(c), we show the electron density distribution of the plasma (blue) and driver (red) at $z = 1000 c/\omega_p$ for $k_p\sigma_0 = 0.4\sqrt{\Lambda}$ and $k_p\sigma_0 = 0.6\sqrt{\Lambda}$, respectively.

Despite the fact that both drivers focus by a similar amount $k_p\Delta\sigma_r \sim 0.5$ within the first oscillation, the lengths of the injected beams are clearly different as shown in the inset plots. This is because wake expansion depends not only on the amount of spot size focusing $\Delta\sigma_r$ but also on the initial spot size σ_r . As seen in Fig. 4.4(c), the wake length approaches its maximum value when $k_p\sigma_r \lesssim 0.3\sqrt{\Lambda}c/\omega_p \simeq 0.73 c/\omega_p$. Therefore, in the case of $k_p\sigma_0 = 0.4\sqrt{\Lambda}c/\omega_p$, injection terminates early as the driver spot size focuses below this value. From the injected electron phase space $\xi_f - z_i$ depicted in the middle panel of Fig. 5.15(b), it can be seen that injection takes place over a limited distance $L_{inj} \approx 0.38\pi/k_\beta$, resulting in an injected beam length of $\Delta\xi_{tr} \simeq 0.38 c/\omega_p$. On the other hand, the initial deviation from the maximum wake length $\Delta L_b(\sigma_0)$ is larger for $k_p\sigma_0 = 0.6\sqrt{\Lambda}$. Thus, the wake can expand more in this case, resulting in a longer injected beam $\Delta\xi \simeq 1.22 c/\omega_p$. In addition, injection can now be sustained over longer distances $L_{inj} \approx 0.9\pi/k_\beta$ as seen in the injected electron phase space shown in the middle panel of Fig. 5.15(c). This is because electron blowout is less pronounced at larger spot sizes [Fig. 4.3], resulting a longer the effective betatron

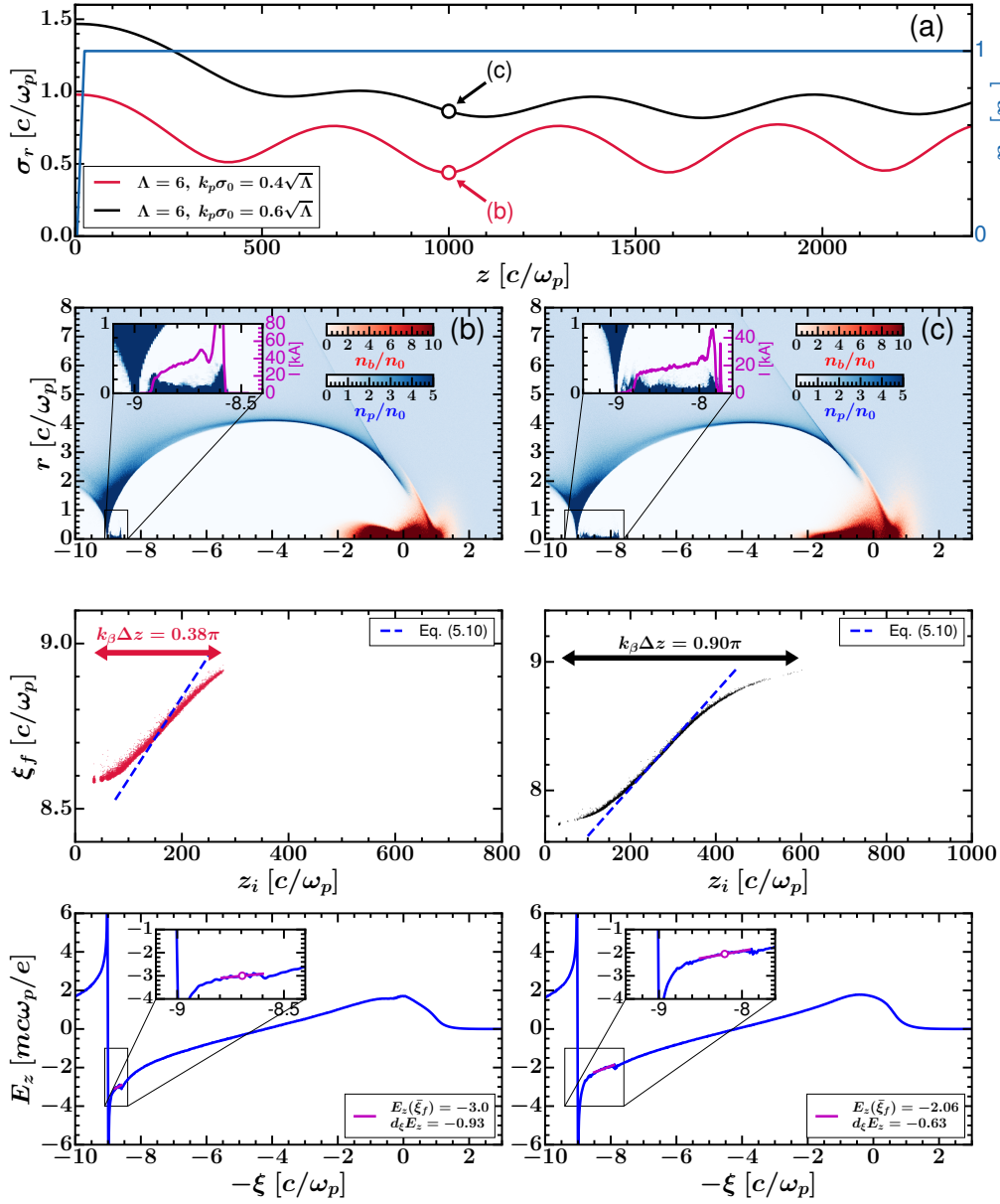


Figure 5.15: (a) $\sigma_r(z)$ of drivers $\{\gamma_b = 20000, \Lambda = 6, k_p\sigma_z = 0.7, k_\beta\beta^* = 16\}$ with $k_p\sigma_0 = 0.4\sqrt{\Lambda}$ (red) and $k_p\sigma_0 = 0.6\sqrt{\Lambda}$ (black) propagating in a constant plasma density (blue). (Top) The electron charge density, (middle) the phase space (ξ_f, z_i) of the injected charge and (bottom) $E_z(\xi)$ at $z = 1000 c/\omega_p$ for (b) $k_p\sigma_0 = 0.4\sqrt{\Lambda}$ and (d) $k_p\sigma_0 = 0.6\sqrt{\Lambda}$. The injected beam current profiles (purple) are shown in the insets. Eq. (5.10) (dashed blue) and linear fits of the electric fields within the beam cores (purple) are plotted.

wavelength. As both drivers are sufficiently focused, the wake length $k_p\xi \approx 9$ is the virutally the same in both cases.

For drivers operating in the nonlinear blowout regime, the relationship between ΔL_b and σ_r is approximately linear [Fig. 4.4(c)]. Previously, we determined $\kappa = \frac{dL_b}{d\sigma_r} \approx -1.76$ from Fig. 4.4(c) when $\Lambda = 6$. From Eq. (5.12), we also find $\gamma_\phi \approx 14.2$ for $k_p\sigma_0 = 0.4\sqrt{\Lambda}$ and $\gamma_\phi \approx 11.6$ for $k_p\sigma_0 = 0.6\sqrt{\Lambda}$. For $\gamma_{z,m}$ shown in Fig. 4.4(d), the injection condition $\gamma_{z,m} > \gamma_\phi$ is therefore satisfied for both drivers. In each case, the injected electron mapping from z_i to ξ_f is reasonably approximated by the wake evolution $\frac{dL_b}{dz}$ calculated from Eq. (5.10) and plotted (dashed blue) in the middle panels of Fig. 5.15(b)-(c). For $k_p\sigma_0 = 0.4\sqrt{\Lambda}$, we note that the disagreement between the simulation results and Eq. (5.10) can be attributed to the dropoff in the magnitude of κ when $k_p\sigma_r \lesssim 0.4\sqrt{\Lambda}$. As the wake expands more slowly for higher γ_ϕ , more charge can be deposited into a given slice $d\xi$. Indeed, this is the primary reason why larger slice currents $I/I_A = \gamma_\phi^2(k_p r_i)(k_p \Delta r_i)$ [Eq. (5.24)] are produced in the case of $k_p\sigma_0 = 0.4\sqrt{\Lambda}$.

Based on simulations, the expansion of the wake due to spot size focusing can be well-approximated by the initial deviation from the maximum wake length ΔL_b [Fig. 4.4(c)] for focal spot sizes σ_0 in the range $0.4 - 0.6\sqrt{\Lambda}c/\omega_p$. Thus, the injected beam length scales linearly as $\Delta\xi_{tr} \sim \Delta L_b \sim \kappa\sigma_0$ over such spot sizes and it follows that a shorter beam is produced from injection for $k_p\sigma_0 = 0.4\sqrt{\Lambda}$. As a result, the injected beam centroid is located close to the rear of the bubble where $\langle E_z \rangle_{acc} \simeq -3$ and $\langle d_\xi E_z \rangle_{acc} \simeq -0.93$ as seen in the lower panel of Fig. 5.15(b). Conversely, as a longer duration beam is injected in the case of $k_p\sigma_0 = 0.6\sqrt{\Lambda}$, the beam centroid is situated further forward relative to the rear of the bubble where $\langle E_z \rangle_{acc} \simeq -2.06$ and $\langle d_\xi E_z \rangle_{acc} \simeq -0.63$ as shown in the lower panel of Fig. 5.15(c).

In general, the magnitude and slope of the accelerating field within the injected beam varies depending on the length of the bunch $\Delta\xi_{tr}$, which is implicitly a function of the focal spot size σ_0 . Obviously, these quantities can effect the optimal driver propagation distance

z_d needed to dechirp the injected beam. In most cases where $\langle d_\xi E_z \rangle_{acc} < 0$, we find that z_d and, hence, the injected beam energy $\bar{\gamma}_{tr}$ can be largely characterized by the spot size scaling σ_0^{-1} in Eqs. (5.21)-(5.22). This scaling arises from the fact that the driver focuses more rapidly $\sigma_r' \sim k_\beta \sigma_0$ [Eq. (5.9)] for larger σ_0 . Thus, the wake expands more quickly as $\frac{dL_b}{dz} \sim \kappa \sigma_r'$ [Eq. (5.10)] increases with σ_0 , thereby reducing the energy chirp per unit length of the injected beam $d_\xi \gamma$ [Eq. (5.16)]. Consequently, the energy distribution of the injected beam can be flattened over a shorter acceleration distance for higher σ_0 . To estimate z_d and $\bar{\gamma}_{tr}$, we approximate $\langle E_z \rangle_{inj} \approx 1.2 \times \langle E_z \rangle_{acc}$ based on the results shown in Fig. 5.3(d). Using Eqs. (5.21)-(5.22), we calculate $z_d = 1869 \text{ c}/\omega_p$ and $\bar{\gamma}_{tr} m c^2 \simeq 2.67 \text{ GeV}$ for $k_p \sigma_0 = 0.4\sqrt{\Lambda}$ and $z_d = 1365 \text{ c}/\omega_p$ and $\bar{\gamma}_{tr} \simeq 1.3 \text{ GeV}$ for $k_p \sigma_0 = 0.6\sqrt{\Lambda}$.

From the PIC simulation results, we find that the optimal driver propagation distance is $2880 \text{ c}/\omega_p$ for $k_p \sigma_0 = 0.4\sqrt{\Lambda}$ and $1780 \text{ c}/\omega_p$ for $k_p \sigma_0 = 0.6\sqrt{\Lambda}$. The energy distributions of the injected beams at these positions are shown in the top panels of Figs. 5.16(a) and 5.16(b) for $k_p \sigma_0 = 0.4\sqrt{\Lambda}$ and $k_p \sigma_0 = 0.6\sqrt{\Lambda}$, respectively. The beam cores are delimited by dashed black lines enclosing 50% of the total injected charge. Within these regions, we find average energies of 4.26 GeV and 1.69 GeV and projected energy spreads of 19.6 MeV (0.46%) and 11.7 MeV (0.69%) for $k_p \sigma_0 = 0.4\sqrt{\Lambda}$ and $k_p \sigma_0 = 0.6\sqrt{\Lambda}$, respectively. While Eqs. (5.21)-(5.22) provide reasonable estimates for z_d and $\bar{\gamma}_{tr}$ in the case of $k_p \sigma_0 = 0.6\sqrt{\Lambda}$, the disagreement between the theoretical predictions and PIC simulation results for $k_p \sigma_0 = 0.4\sqrt{\Lambda}$ can be largely attributed to the dropoff in $|\kappa| = |\frac{dL_b}{d\sigma_r}|$ when $k_p \sigma_r \lesssim 0.4\sqrt{\Lambda}$ [Fig. 4.4(c)]. In both cases, high quality beams are produced with slice energy spreads of $\sigma_\gamma \sim O(\text{MeV})$ and normalized slice emittances of $\epsilon_n \sim O(0.01 \text{ c}/\omega_p)$ within the beam cores as shown in the lower panels of Figs. 5.16(a)-(b). The total injected charge $Q_{tr}/en_p k_p^{-3} \simeq 15.3$ for $k_p \sigma_0 = 0.6\sqrt{\Lambda}$ is nearly $\sim 34\%$ higher than the total injected charge $Q_{tr}/en_p k_p^{-3} \simeq 11.4$ for $k_p \sigma_0 = 0.4\sqrt{\Lambda}$. The gain in Q_{tr} is driven primarily by the significantly longer bunch length $\Delta \xi_{tr}$ while being partially offset by the lower slice currents. However, the transformer ratio $R = 1.11$ for $k_p \sigma_0 = 0.6\sqrt{\Lambda}$ is smaller than $R = 1.7$ for $k_p \sigma_0 = 0.4\sqrt{\Lambda}$ since the beam head is located further forward

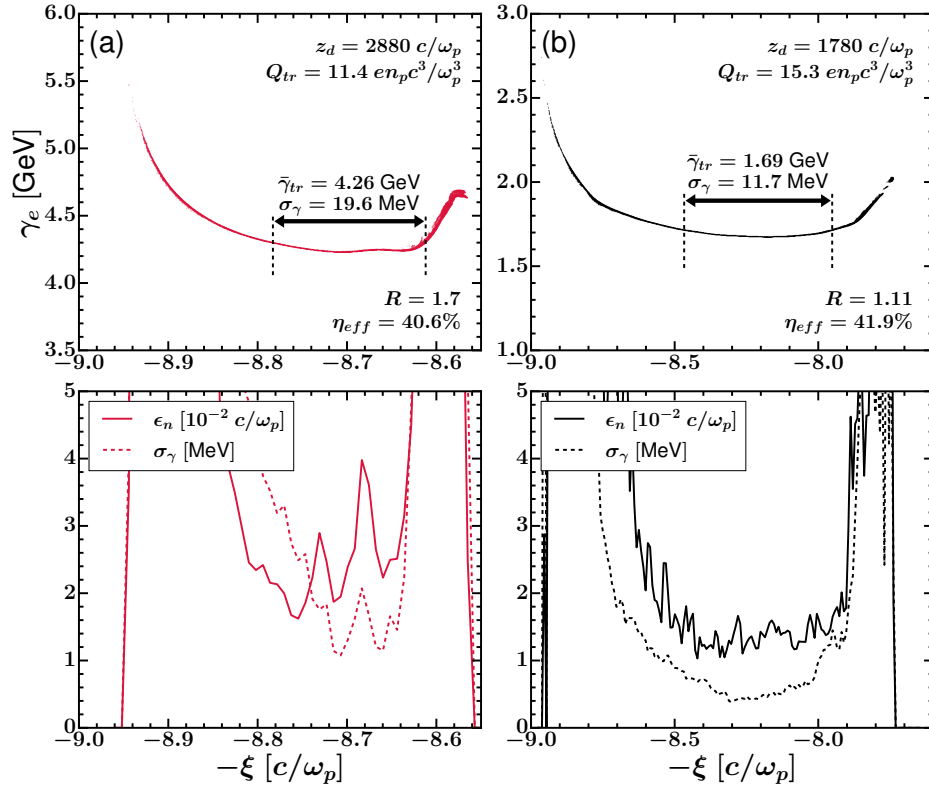


Figure 5.16: (Top) The injected electron energy distribution after optimal acceleration for drivers $\{\gamma_b = 20000, \Lambda = 6, k_p\sigma_z = 0.7, k_\beta\beta^* = 16\}$ with different focal spot sizes (a) $k_p\sigma_0 = 0.4\sqrt{\Lambda}$ (red) and (b) $k_p\sigma_0 = 0.6\sqrt{\Lambda}$ (black). The dashed black lines enclose 50% of total injected charge surrounding the centroid. (Bottom) The slice energy spread σ_γ in units of MeV (dashed) and normalized slice emittance ϵ_n in units of c/ω_p (solid).

relative to the rear of the wake. As the increase in Q_{tr} is almost completely offset by the decrease in R , similar transfer efficiencies $\eta_{eff} \simeq 40.6\%$ and $\eta_{eff} \simeq 41.9\%$ are achieved for $k_p\sigma_0 = 0.4\sqrt{\Lambda}$ and $k_p\sigma_0 = 0.6\sqrt{\Lambda}$, respectively.

In Figs. 5.17(a)-(b), we plot the optimal driver propagation distance z_d and injected beam energy $\bar{\gamma}_{tr}$ determined from PIC simulations (red) using drivers $\{\gamma_b = 20000, \Lambda = 6, k_p\sigma_z = 0.7, k_\beta\beta^* = 16\}$ with different focal spot sizes $k_p\sigma_0$ in the range of $0.4 - 1.0\sqrt{\Lambda}$. Numerical calculations using Eqs. (5.21)-(5.22) are also plotted (blue) for $k_p\sigma_0$ in the range of $0.4 - 0.8\sqrt{\Lambda}$ corresponding to the blowout regime. While the agreement is not perfect, we

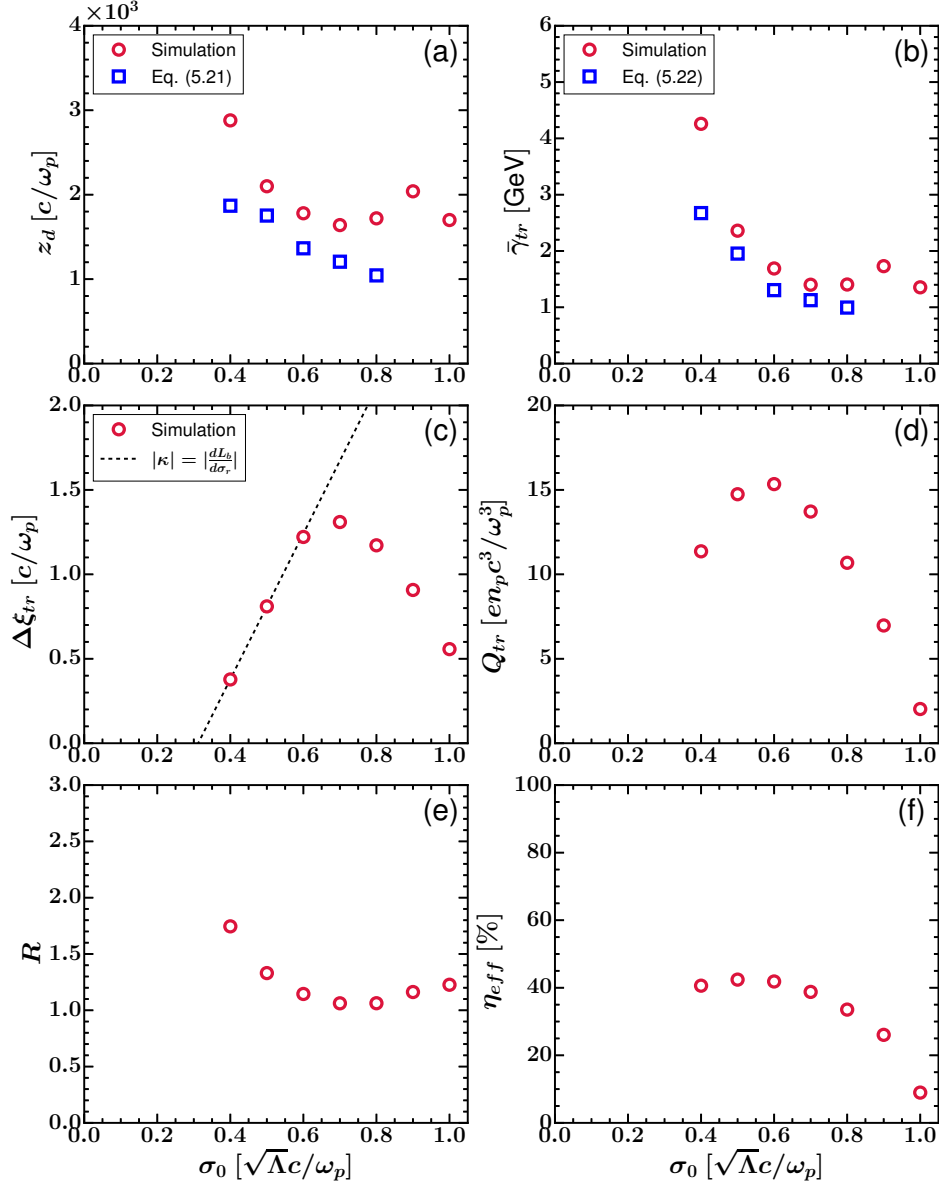


Figure 5.17: (a) The optimal driver propagation distance z_d , (b) injected beam energy $\bar{\gamma}_{tr}$, (c) injected beam length $\Delta\xi_{tr}$, (d) injected charge Q_{tr} , (e) transformer ratio R , and (f) transfer efficiency η_{eff} as a function of σ_0 for drivers $\{\gamma_b = 20000, \Lambda = 6, k_p\sigma_z = 0.7, k_\beta\beta^* = 16\}$. PIC simulation results are shown in red. A linear fit of $\Delta\xi_{tr}$ determined from simulation results is plotted in dashed black. Eqs. (5.21) and (5.22) are shown in blue.

find that Eqs. (5.21)-(5.22) capture the general trend of z_d and $\bar{\gamma}_{tr}$ from the PIC simulation results. These parameters scale strongly with σ_0^{-1} in the blowout regime. The linear scaling of the injected beam length $\Delta\xi_{tr} \sim \kappa\sigma_0$ argued above is also borne out by the PIC simulation results for spot sizes $k_p\sigma_0$ in the range of $0.4-0.6\sqrt{\Lambda}$ as shown in Fig. 5.17(c). For such values, the dependence of $\Delta\xi_{tr}$ on σ_0 is well-characterized by a linear fit with slope $|\frac{dL_b}{d\sigma_r}| = |\kappa| \approx 1.76$ (dashed black). These results are important because they indicate that the length of the injected beam can be tuned linearly through the focal spot size of the driver for $k_p\sigma_0$ in the range of $0.4-0.6\sqrt{\Lambda}$. For larger focal spot sizes $k_p\sigma_0 \gtrsim 0.7\sqrt{\Lambda}c/\omega_p$, there is a clear dropoff in $\Delta\xi_{tr}$ as the projected spot size of the driver no longer focuses sufficiently enough to produce maximum wake expansion in the blowout regime.

As shown in Fig. 5.17(d), the increase in $\Delta\xi_{tr}$ for $k_p\sigma_0$ in the range of $0.4-0.6\sqrt{\Lambda}$ more than offsets the lower slice currents, resulting in higher injected charge Q_{tr} . While $\Delta\xi_{tr}$ peaks at $k_p\sigma_0 = 0.7\sqrt{\Lambda}$, we note that Q_{tr} is maximum at $k_p\sigma_0 = 0.6\sqrt{\Lambda}$ because the lower slice currents at $k_p\sigma_0 = 0.7\sqrt{\Lambda}$ outweigh the increase in $\Delta\xi_{tr}$. For larger spot sizes $k_p\sigma_0 \gtrsim 0.7\sqrt{\Lambda}$, the injected charge Q_{tr} continues to decrease due to lower slice currents I and shorter beam lengths $\Delta\xi_{tr}$. It can be seen in Fig. 5.17(e) that the transformer ratio R varies inversely with the beam length $\Delta\xi_{tr}$. Thus, for focal spot sizes $k_p\sigma_0$ in the range of $0.4-0.6\sqrt{\Lambda}$, the increase in charge Q_{tr} is mostly balanced by the decrease in the transformer ratio R , which results in the transfer efficiency η_{eff} [Fig. 5.17(f)] being nearly constant over such values. However, for $k_p\sigma_0 \gtrsim 0.7\sqrt{\Lambda}$, there is a clear dropoff in η_{eff} due to lower Q_{tr} .

5.3.5 The Driver Duration

The current profile of the injected beam can also depend on the driver duration $k_p\sigma_z$. A longer driver can be used to excite a wake with larger blowout radius r_m at the same peak current Λ and focal spot size σ_0 . For larger r_m , the innermost sheath electrons typically originate from larger initial positions r_i and exhibit larger $\gamma_{z,m}$ near the axis due to the more pronounced blowout, leading to larger Δr_i . As a result, higher injected slice currents can be

achieved by focusing longer drivers with $k_p\sigma_z \sim 1$. Since injection in the plasma focusing regime $k_\beta\beta^* \gg 1$ naturally produces beams with trapezoidal current profiles, optimal loading of the plasma wakefield $d_\xi E_z \approx 0$ can be achieved by tuning the driver duration $k_p\sigma_z$. In such cases, the injected beam can be accelerated to multi-GeV energies while maintaining the kinds of low projected energy spreads for XFEL and linear collider applications.

In Fig. 5.18(a), we plot the spot size evolution $\sigma_r(z)$ from PIC simulations of electron drivers $\{\gamma_b = 20000, \Lambda = 6, k_p\sigma_0 = 0.5\sqrt{\Lambda}, k_\beta\beta^* = 16\}$ with different durations $k_p\sigma_z = 0.4$ (red) and $k_p\sigma_z = 1$ (black) propagating in a constant density plasma (blue). As the driver is shorter in the case of $k_p\sigma_z = 0.4$, a larger proportion of the beam resides in regions where the wake is not fully formed. Therefore, the focusing force along the driver is reduced and it takes longer for the projected spot size to be focused by the plasma. On the other hand, for $k_p\sigma_z = 1$, most of the plasma electrons are blown out further forward near the drive beam head, resulting in a larger focusing force along most of the driver. As a result, the projected spot size of the longer driver is focused over a shorter distance, which is well-approximated by $\lambda_\beta/4 \sim \pi/2k_\beta$.

The top panels of Figs. 5.18(b)-(c) show the electron density distribution of the plasma (blue) and driver (red) at $z = 1000 c/\omega_p$ for $k_p\sigma_z = 0.4$ and $k_p\sigma_z = 1$, respectively. Despite the fact that both drivers have the same peak current, it is clear that the longer driver produces a longer wake with a larger blowout radius. Due to the more pronounced blowout for $k_p\sigma_z = 1$, the sheath electrons that trace out the bubble radius also originate from larger initial radii r_i and exhibit higher $\gamma_{z,m}$ as they return back the axis. Thus, the current profiles of the injected bunches (purple) shown in the inset plots differ meaningfully in terms of amplitude and shape.

For the shorter driver $k_p\sigma_z = 0.4$, the injected current profile shown in Fig. 5.18(b) ramps up from 0 to ~ 7 kA within the center of the beam and then down to 0. This kind of profile is usually produced when the trapping threshold is not initially satisfied as the driver begins to focus [Fig. 5.12(b)]. In such cases, $\gamma_{z,m}$ starts below γ_ϕ and then ramps up above γ_ϕ

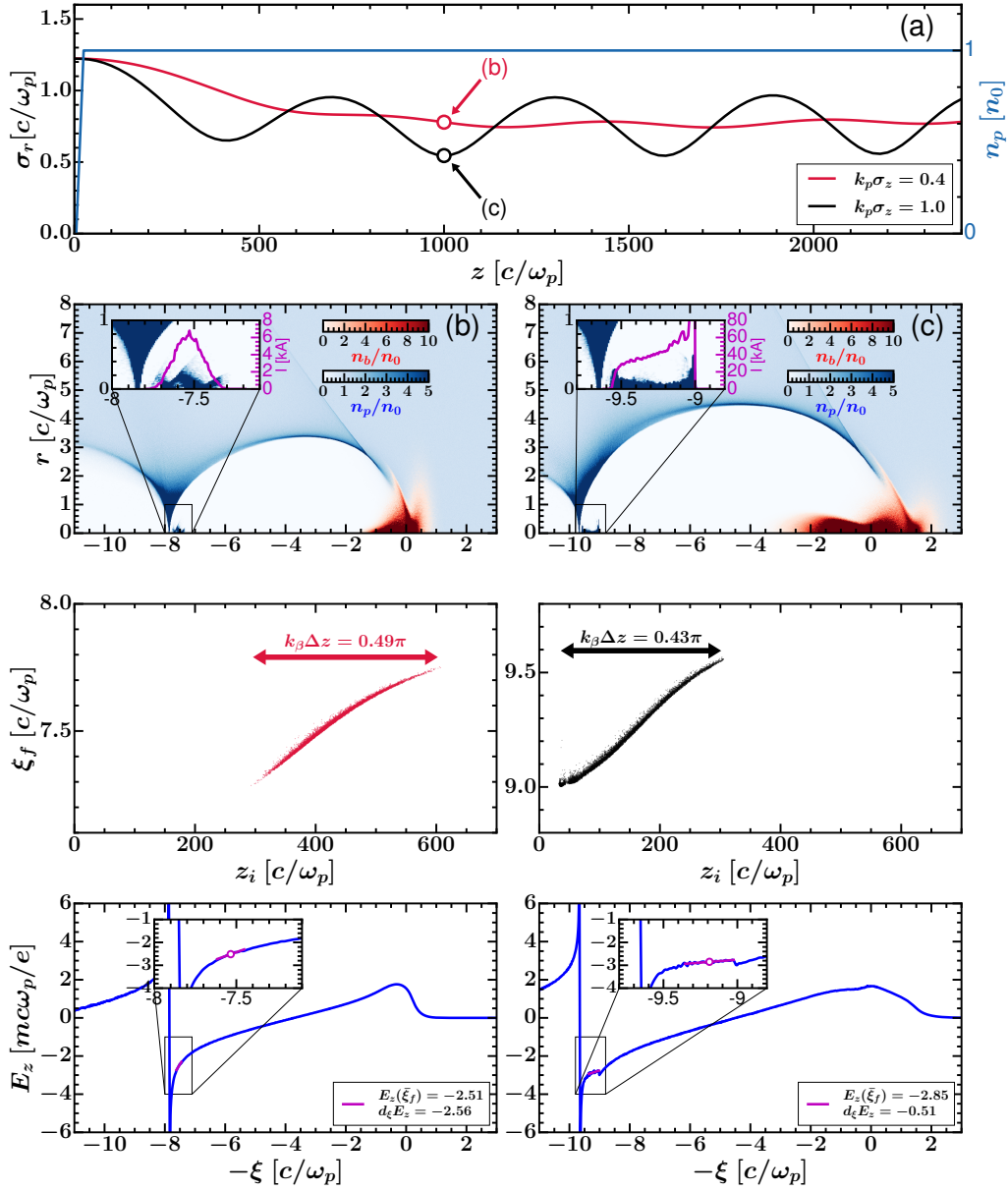


Figure 5.18: (a) $\sigma_r(z)$ of electron drivers $\{\gamma_b = 20000, \Lambda = 6, k_p\sigma_0 = 0.5\sqrt{\Lambda}, k_\beta\beta^* = 16\}$ with different durations $k_p\sigma_z = 0.4$ (red) and $k_p\sigma_z = 1$ (black) propagating in a constant plasma density (blue). (Top) The electron charge density, (middle) the phase space (ξ_f, z_i) of the injected charge and (bottom) $E_z(\xi)$ at $z = 1000 c/\omega_p$ for (b) $k_p\sigma_z = 0.4$ and (c) $k_p\sigma_z = 1$. The injected beam current profiles (purple) are shown in the insets. Linear fits of the electric fields within the beam cores are shown (purple).

while the spot size decreases. This is borne out by the injected electron phase space $\xi_f - z_i$ (red) plotted in the middle panel of Fig. 5.18(b), where it can be seen that trapping only occurs for electrons at $k_p z_i \gtrsim 300$. Despite this, injection still occurs over a distance of $L_{inj} \approx 0.49\pi/k_\beta$ since the spot size focusing is stretched. As a result, the length of the injected beam is $\Delta\xi_{tr} \simeq 0.49 c/\omega_p$.

In the case of $k_p\sigma_z = 1$, significantly higher slice currents ranging from ~ 30 kA to ~ 60 kA are produced within the core of the beam in Fig. 5.18(c). The increase in the slice currents $I = \gamma_\phi^2(k_p r_i)(k_p \Delta r_i)$ [Eq. (5.24)] is primarily driven by larger r_i and larger Δr_i . In addition, the trapping condition is satisfied as soon as the spot size begins to focus. This is indeed what is observed in the injected electron phase space $\xi_f - z_i$ (black) plotted in the middle panel of Fig. 5.18(c). Injection continues to occur over a distance of $L_{inj} \approx 0.43\pi/k_\beta$ until the wake length is fully expanded, resulting in an injected beam length of $\Delta\xi_{tr} \simeq 0.62 c/\omega_p$.

The bottom panels of Figs. 5.18(b)-(c) show the electric field of the plasma wakefield at $z = 1000 c/\omega_p$ for $k_p\sigma_z = 0.4$ and $k_p\sigma_z = 1$, respectively. Since the shorter driver $k_p\sigma_z = 0.4$ produces lower injected slice currents, the wakefield is not meaningfully loaded around the injected beam centroid where $\langle E_z \rangle_{acc} \approx -2.51$ and $\langle d_\xi E_z \rangle_{acc} \approx -2.56$. On the other hand, for the longer driver $k_p\sigma_z = 1$, the wakefield is significantly altered by the higher slice currents. Near the beam centroid, the simulation results indicate $\langle E_z \rangle_{acc} \approx -2.85$ and $\langle d_\xi E_z \rangle_{acc} \approx -0.51$. As the accelerating fields $\langle E_z \rangle_{acc}$ are comparable in these two cases, the acceleration distance needed to dechirp the beam L_{acc} [Eq. (5.20)] scales with the inverse of the electric field slope $\langle d_\xi E_z \rangle_{acc}^{-1}$. Thus, it follows that the injected beam produced by the longer driver $k_p\sigma_z = 1$ can be accelerated over a much longer distance before the energy distribution is flattened. However, in the case of $k_p\sigma_z = 1$, an interesting phenomenon occurs where the wakefield eventually becomes optimally loaded due to head erosion of the driver. As a result, the injected beam can be accelerated to multi-GeV energies while maintaining low energy spreads until the driver loses all its energy, i.e., pump depletes.

To understand how this process takes place, we plot the spot size evolution of the longer

driver ($k_p \sigma_z = 1$) from the plasma entrance at $z = 0$ to $z = 13500 \text{ } c/\omega_p$ in Fig. 5.19(a). Snapshots of the plasma wake (top panel), electric field $E_z(\xi)$ (middle panel), and energy-position phase space $\gamma_e - \xi$ of the driver and injected bunch (bottom panel) are shown at $z = 4000, 10000, 12000, 13200 \text{ } c/\omega_p$ in Figs. 5.19(b)-(e), respectively. To assist in the analysis, we use the multi-sheath model [100] to numerically calculate the bubble trajectory $r_b(\xi)$ and electric field $E_z(\xi)$ for the unloaded (dashed red) and loaded wakes (solid red). The integration parameters are provided in the caption of Fig. 5.19.

At $z = 4000 \text{ } c/\omega_p$ [Fig. 5.19(b)], head erosion begins to take effect since the driver has propagated over a distance longer than its diffraction length $\beta^* = 3200 \text{ } c/\omega_p$. This is illustrated by the increasing slice spot size $\sigma_r(\xi)$ (dashed black) at the head of the driver. As the driver head diffracts, the blowout radius shrinks and the wake front recedes. Due to these effects, the rear of the bubble begins to move forward. However, as shown by the numerical calculations of $r_b(\xi)$ using the multi-sheath model, the injected beam is still fully contained within the first bucket as the rear of the wake is extended by $\Delta\xi \approx 0.16c/\omega_p$ from $\xi(r_b = 0) \approx 9.5 \text{ } c/\omega_p$ (dashed red) to $\xi(r_b = 0) \approx 9.66 \text{ } c/\omega_p$ (solid red) due to beam loading effects. In addition, the multi-sheath model (solid red) now predicts $d_\xi E_z \approx -0.33$ within the center of the injected beam, which is in good agreement with the simulation results (black). As a result, the acceleration distance L_{acc} [Eq. (5.20)] needed to dechirp the injected beam is extended because the magnitude of $d_\xi E_z$ is now reduced.

As the driver propagates over longer distances, wake front recession and bubble contraction continue to occur due to head erosion. Since beam loading plays a larger role in modifying the bubble radius to keep the wake expanded, the electric field slope becomes less negative within the core of the injected beam. Eventually at $z = 10000 \text{ } c/\omega_p$ [Fig. 5.19(c)], both the multi-sheath model (solid red) and PIC simulation results (black) indicate that the loaded wakefield $E_z(\xi)$ is flattened with $d_\xi E_z \approx 0$ near the injected beam centroid. From the numerical calculations of $r_b(\xi)$ using the multi-sheath model, the optimal beam loading observed at this juncture can be attributed to the injected beam extending the wake

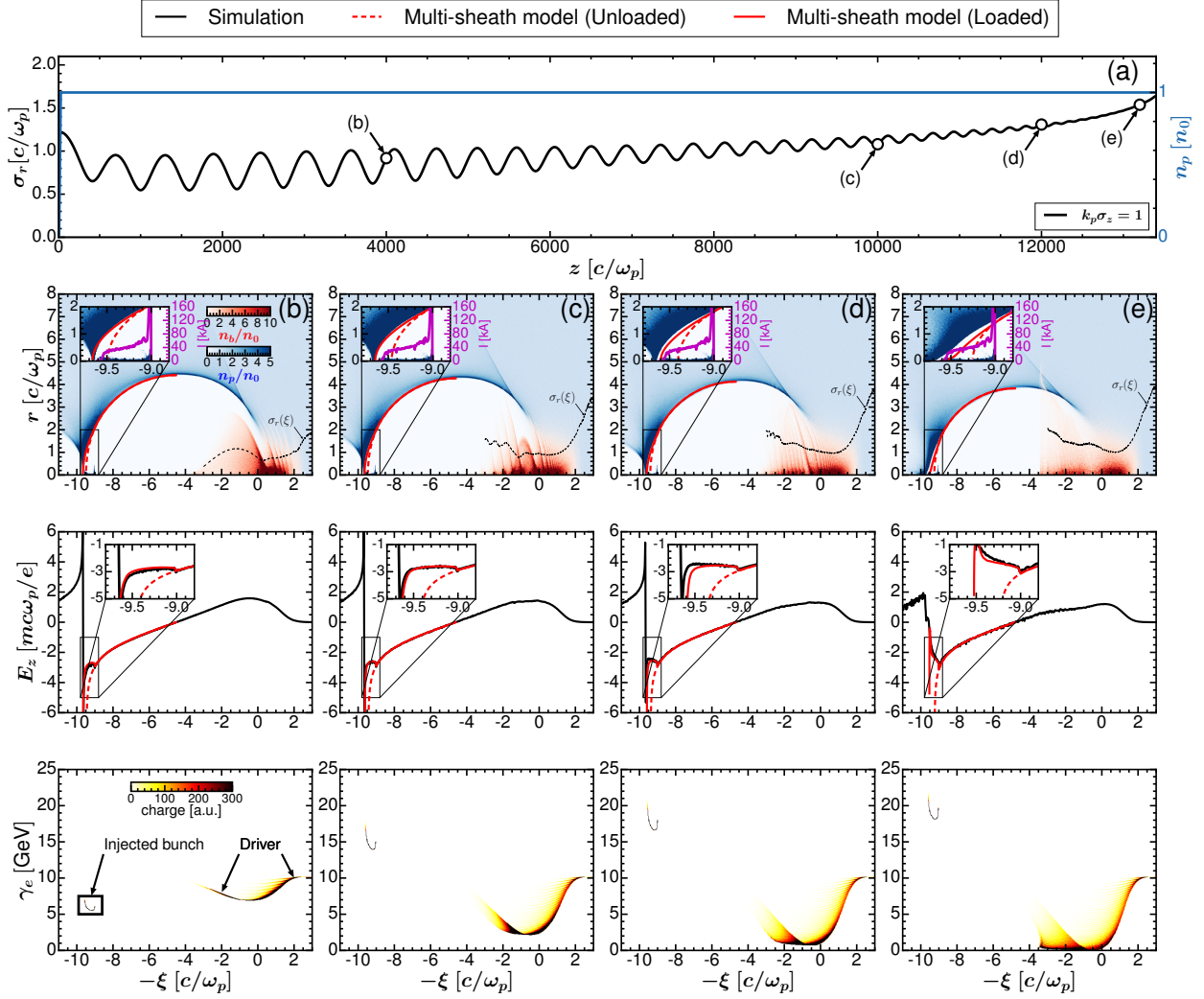


Figure 5.19: (a) $\sigma_r(z)$ of a driver $\{\gamma_b = 20000, \Lambda = 6, k_p\sigma_0 = 0.5\sqrt{\Lambda}, k_p\sigma_z = 1, k_\beta\beta^* = 16\}$ propagating in a constant density plasma (blue). (Top) The electron density distribution of the plasma and driver, (middle) longitudinal electric field, (bottom) energy-position phase space $\gamma_e - \xi$ of the driver and injected bunch at (b) $z = 4000 c/\omega_p$, (c) $z = 10000 c/\omega_p$, (d) $z = 12000 c/\omega_p$, (e) $z = 13200 c/\omega_p$. The slice spot size $\sigma_r(\xi)$ is plotted (dashed black). Numerical calculations of $r_b(\xi)$ and $E_z(\xi)$ using the multi-sheath model are shown with (solid red) and without (dashed red) the injected bunch. The integration parameters are $\Delta_1 = 1 + 0.05r_b$, $\Delta_2 = 3$, $\psi_{min} = -1$, and $n_2 = n_{20} \exp^{-sr_b^t/r_m^t}$, where $s = t = 3$.

length by a larger amount $\Delta\xi \approx 0.19 \text{ } c/\omega_p$ from $\xi(r_b = 0) \approx 9.46 \text{ } c/\omega_p$ (dashed red) to $\xi(r_b = 0) \approx 9.65 \text{ } c/\omega_p$ (solid red).

Over longer distances, the electric field slope within the injected bunch turns positive. This can be seen at $z = 12000 \text{ } c/\omega_p$ [Fig. 5.19(d)], where the multi-sheath model (solid red) predicts $d_\xi E_z \approx 0.15$ around the injected bunch centroid. However, as the electric field slope $d_\xi E_z$ is very small, the injected beam can still be accelerated without any meaningful growth in the projected energy spread. From the numerical calculations of $r_b(\xi)$, the multi-sheath model now predicts that the injected beam extends the wake length by nearly $\Delta\xi \approx 0.2 \text{ } c/\omega_p$ from $\xi(r_b = 0) \approx 9.4 \text{ } c/\omega_p$ (dashed red) to $\xi(r_b = 0) \approx 9.6 \text{ } c/\omega_p$ (solid red). Even at this position, the loaded trajectory and wakefield obtained using the multi-sheath model (solid red) are still in good agreement with the PIC simulation results (black). Finally, around $z = 13200 \text{ } c/\omega_p$ [Fig. 5.19(e)], the plasma wake begins to collapse due to pump depletion of the driver. From the numerical calculations using the multi-sheath model (dashed/solid red), it can be seen that the injected bunch barely remains within the first bucket due to beam loading effects.

The results in Fig. 5.19 show that the electric field slope $d_\xi E_z(\xi)$ changes from negative to zero to positive over distances greater than the diffraction length for the driver with $k_p \sigma_z = 1$. As a result, the average slope of the electric field $\langle d_\xi E_z(\xi) \rangle_{acc}$ over the entire acceleration process approaches 0 over sufficiently longer distances. Thus, the injected beam can be accelerated to multi-GeV energies with low projected energy spreads until the drive beam energy fully depletes, which occurs at $z = 13240 \text{ } c/\omega_p$ according to the PIC simulation results. However, for the shorter driver with $k_p \sigma_z = 0.4$, the slice currents of the injected beam are too small to sufficiently load the wake. Therefore, due to the large negative electric field slope $\langle d_\xi E_z(\xi) \rangle_{acc} \simeq -2.56$ [Fig. 5.18(b)], the injected beam needs to be extracted early on as the energy distribution is flattened around $z = 1140 \text{ } c/\omega_p$ based on the PIC simulation results.

In the top panels of Figs. 5.20, we show the energy distributions of the injected beams at

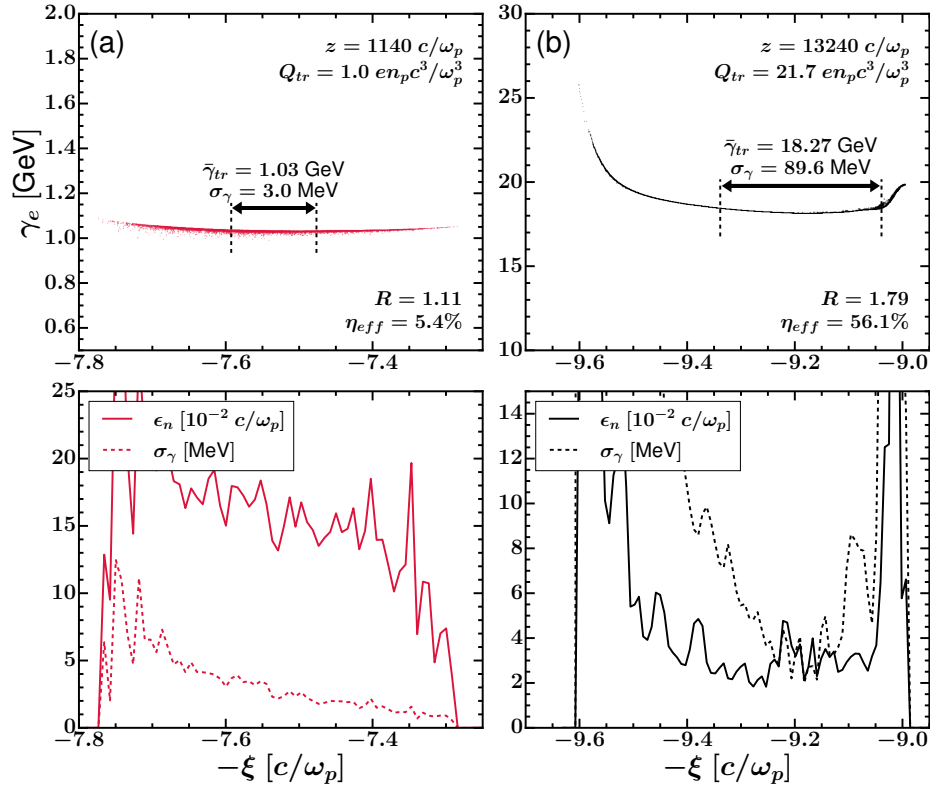


Figure 5.20: (Top) The injected electron energy distribution after optimal acceleration for drivers $\{\gamma_b = 20000, \Lambda = 6, k_p \sigma_0 = 0.5\sqrt{\Lambda}, k_\beta \beta^* = 16\}$ with different durations (a) $k_p \sigma_z = 0.4$ (red) and (b) $k_p \sigma_z = 1$ (black). The dashed black lines enclose 50% of total injected charge surrounding the centroid. (Bottom) The slice energy spread σ_γ in units of MeV (dashed) and normalized slice emittance ϵ_n in units of c/ω_p (solid).

extraction for $k_p \sigma_z = 0.4$ (red) and $k_p \sigma_z = 1$ (black), respectively. The injected beam cores are delimited by dashed black lines enclosing 50% of the total injected charge. Within these regions, we find average energies of 1.03 GeV and 18.27 GeV and projected energy spreads of 3 MeV (0.29%) and 89.6 MeV (0.49%) for $k_p \sigma_z = 0.4$ and $k_p \sigma_z = 1$, respectively. The total injected charge $Q_{tr}/en_p k_p^{-3} \simeq 21.7$ for $k_p \sigma_z = 1$ is over an order of magnitude higher than $Q_{tr}/en_p k_p^{-3} \simeq 1$ for $k_p \sigma_z = 0.4$, driven primarily by much higher slice currents. In addition, a higher transformer ratio $R = 1.79$ is achieved for $k_p \sigma_z = 1$ compared to $R = 1.11$ for $k_p \sigma_z = 0.4$ due to the larger accelerating field within the injected bunch. Thus, the transfer

efficiency $\eta_{eff} \simeq 56.1\%$ for $k_p\sigma_z = 1$ is over an order of magnitude higher than $\eta_{eff} \simeq 5.4\%$ for $k_p\sigma_z = 0.4$. The increase in η_{eff} is lower than that of Q_{tr} because a larger proportion of the driver charge loses energy to the wakefield in the case of $k_p\sigma_z = 1$.

In the bottom panels of Figs. 5.20, we plot the slice energy spreads σ_γ and normalized slice emittances ϵ_n for $k_p\sigma_z = 0.4$ (red) and $k_p\sigma_z = 1$ (black), respectively. In both cases, low slice energy spreads $\sigma_\gamma \lesssim 10$ MeV are observed within the middle of the beams. The normalized slice emittances observed within the beam core are as low as $\epsilon_n \sim 0.02$ c/ω_p for $k_p\sigma_z = 1$ and nearly a magnitude higher $\epsilon_n \sim 0.1 - 0.2$ c/ω_p for $k_p\sigma_z = 0.4$. As a result, the peak normalized brightness of the injected beam $B_n \sim 9$ [n_0 cm $^{-3}$] A/m 2 /rad 2 for $k_p\sigma_z = 1$ is more than two orders of magnitude higher than $B_n \sim 0.03$ [n_0 cm $^{-3}$] A/m 2 /rad 2 for $k_p\sigma_z = 0.4$.

These results are significant because they show that it is possible to simultaneously generate high quality beams through injection and optimally load the plasma wakefield. For $k_p\sigma_z = 1$, the injected beam can be accelerated to 18.27 GeV with a low projected energy spread of 0.49% and a high transfer efficiency of 56.1%. In addition, the results indicate that high peak normalized brightnesses of $B_n \sim 10^{20}$ A/m 2 /rad 2 can be achieved for plasma densities of 10^{19} cm $^{-3}$.

5.4 Summary

In this chapter, we discussed the advantages of self-injection triggered by operating the plasma self-focusing regime ($k_\beta\beta^* \gg 1$). We then developed a more robust model for the wake evolution $\frac{dL_b}{dz}$ and the wake velocity γ_ϕ in this regime based on the periodicity of the spot size oscillation. The model was then used to develop expressions for the optimal driver propagation distance z_d and output beam energy $\bar{\gamma}_{tr}$. The model was shown to be in good agreement with PIC simulations over a range of different drivers. Based on PIC simulations, drivers with sufficiently long diffraction lengths, i.e., $k_\beta\beta^* \gg 1$, can remain well-guided in the

plasma over distances much greater than the betatron wavelength. For drivers with different energies γ_b , the optimal output beam energy scales as $\bar{\gamma}_{tr} \sim \gamma_b$. For higher intensity drivers, more charge can be injected, resulting in higher energy transfer efficiencies. For focal spot sizes $k_p\sigma_0$ in the range of $0.4 - 0.6\sqrt{\Lambda}$, the injected beam length scales as $\Delta\xi \sim \sigma_0$. Injection and optimal beam loading were also demonstrated simultaneously using a longer duration driver with $k_p\sigma_z = 1$ instead of $k_p\sigma_z = 0.7$. The multi-sheath model was used to characterize the beam loading effects of the self-injected beam. We found that injected beams with energies of up to 18.27 GeV could be produced with high normalized peak brightnesses of $B_n \sim 10^{20}$ A/m²/rad² for plasma densities of 10^{19} cm⁻³.

CHAPTER 6

Summary and Final Thoughts

In this dissertation, several important areas of plasma wakefield acceleration in the nonlinear blowout regime are investigated. Here, the main results of each chapter are summarized and possible next steps in future work are suggested.

In chapter 2, the basic equations governing nonlinear plasma wakefield theory in the blowout regime and the single-sheath model for the wake potential proposed by Lu et al. [35, 36] are reviewed. In chapter 3, an improved multi-sheath phenomenological model for describing the source term profile $S \equiv -\frac{1}{en_p}(\rho - J_z/c)$ of plasma wakefields excited by relativistic electron drivers in the nonlinear blowout regime is proposed. The multi-sheath model is then incorporated into the formalism of Lu et al. [35, 36]. Previously, the wake potential was modeled as positive-definite due to the underlying assumptions of the source term in the single-sheath model [35, 36]. The multi-sheath model developed in this dissertation structurally addresses the negative wake potentials by introducing a second plasma sheath in which the source term is negative ($S < 0$) outside the bubble ($r > r_b$). This new description is also consistent with the empirical plasma source term obtained from PIC simulations of nonlinear plasma wakefields. Using the multi-sheath model, a new expression for the wake potential $\psi(r, \xi)$ is obtained and then used to solve for the trajectory of the innermost sheath electron r_b by integrating the equation of motion from the nonlinear blowout theory [35, 36]. In cases with and without trailing bunches, it is shown that the bubble radius r_b , wake potential ψ_0 , and electric field E_z predicted by the multi-sheath model demonstrate significantly improved agreement with simulations results at the rear of the wake when compared

with the results from the simple sheath model used by Lu et al. [35, 36]. In addition, the model demonstrates the capability to predict plasma wakefields in cases where electrons are injected at the rear of the bubble. It was shown how the multi-sheath model can be used to design beams that can load a constant wakefield and have discussed differences between the predictions for beam loading based on the multi-sheath model and single-sheath model in the ultrarelativistic limit used by Tzoufras et al. [38, 39]. This limit permits examining purely analytical solutions. In the cases examined, the multi-sheath model demonstrated improved accuracy over the analytic solutions [38, 39] in terms of flattening the wake. It also predicted significantly longer beam loads with 23 – 25% more charge. Two examples are presented in which the multi-sheath model is used to load longitudinally varying wakefields. The multi-sheath model is also applied to wakes driven by lasers and approaches to improving it were presented.

In chapter 4, a new method of controllable injection for high quality beam generation is proposed. The scheme relies on expanding the ion column by focusing the electron driver from spot sizes, σ_r , on the order of the blowout radius $\sim r_m$ to spot sizes much less than the blowout radius r_m . As the spot size focuses, the phase velocity of the wake is reduced and energetic sheath electrons can be trapped. Two regimes were identified and studied in proof-of-concept PIC simulations. In case A ($k_\beta \beta^* \ll 1$), the driver evolves according to vacuum diffraction. In case B ($k_\beta \beta^* \gg 1$), the driver is self-focused by the plasma. We used the multi-sheath model to characterize the effects of beam loading in case B (where the quasistatic approximation is valid). We then performed numerous quasistatic PIC simulations to quantify the dependence of the wake length and sheath electron energies on the driver spot size. The results were used to describe the trapping dynamics for each case. After optimal acceleration, injected beams with average energies of 169.7 MeV and 1.86 GeV and projected energy spreads of 1.1% and 0.7% are achieved in cases A and B, respectively. For a plasma density of $\sim 10^{19} \text{ cm}^{-3}$, peak normalized brightnesses of $B_n \sim 10^{20} \text{ A/m}^2/\text{rad}^2$ are also observed within the injected beams.

In chapter 5, the advantages of operating in the plasma focusing regime ($k_\beta \beta^* \gg 1$) was discussed. A robust model to characterize the wake evolution $\frac{dL_b}{dz}$ and wake velocity γ_ϕ in this regime was developed. Expressions were also derived for the optimal driver propagation distance z_d and the injected beam energy $\bar{\gamma}_{tr}$. The model predictions were shown to be in reasonable agreement with PIC simulations of different drivers. Based on the PIC simulation results, the driver can generally remain well-guided as long as the diffraction length is sufficiently long, i.e., $k_\beta \beta^* \gtrsim 10$. When varying the drive beam energy γ_b , the optimal injected beam energy was shown to scale as $\bar{\gamma}_{tr} \sim \sqrt{\gamma_b}$. For higher intensity drivers, significantly more charge is injected, resulting in higher transfer efficiencies. For focal spot sizes $k_p \sigma_0$ in the range of $0.4 - 0.6\sqrt{\Lambda}$, the length of the injected bunch was shown to scale linearly as $\Delta\xi_{tr} \sim \sigma_0$. Most importantly, injection and optimal beam loading was demonstrated simultaneously using a longer duration driver ($k_p \sigma_z \sim 1$). The multi-sheath model was used to explain the beam loading effects during acceleration. From the PIC simulation results, the injected beam was accelerated with a high transfer efficiency (56.1%) to 18.27 GeV while maintaining a low projected energy spread of 0.49% and a high peak normalized brightness of $B_n \sim 10^{20}$ A/m²/rad² for a plasma density of $\sim 10^{19}$ cm⁻³.

This work naturally leads to ideas for future work. Perhaps the most complicated and interesting problem is how to adapt the multi-sheath model to laser wakefield problems in which the electrons are not completely blown out of the bubble and the sheath is more complicated. It is worth recalling that at the end of chapter 3, a discussion was given on how the source term can be modified for the laser case. However, some assumptions will need to be made about the finite charge and current densities of the electrons inside the bubble. This means that the focusing force inside the channel also needs to necessarily be modified. In addition, the electrons inside the bubble can also be blown out by the presense of a trailing bunch. These issues may require a different approach in which the phenomenological profiles of electrons inside the bubble are functions of ξ rather than r_b .

Further optimization of injection using the evolving driver method is also an area for

future work. The results shown in chapter 5 can be used as a blueprint for the scaling of the injected beam parameters. Assuming the trapping condition is met, the driver should evolve in a similar manner if the betatron wavelength λ_β and diffraction length β^* were increased together so that $k_\beta\beta^*$ remains constant. Thus, the physics in the optimal 18.27 GeV beam loading case should be scalable to even higher energies. In addition, it should also be possible to achieve optimal beam loading with even higher efficiencies using a higher intensity, longer duration driver. The tolerances of the injected beam qualities to transverse asymmetries in the driver spot size and plasma density should also be studied. Understanding how the injection and acceleration depends on such parameters is critical to designing future experiments. Developing a theory for asymmetric drivers is very challenging, but simulations permit the ability to examine tolerances.

Last, another area for future work is to combine the self-focusing regime and a density down ramp [108]. If the spot size oscillates then injection will be triggered during each oscillation. If there is no density gradient, then electrons during each injection period will reside at the same location within the wake. However, if a downramp is also used then the electrons from each injection period will reside at progressively later locations in the bubble as it elongates. This could lead to a train of bunches whose spacing can be controlled through β^* and the density scale length.

Appendix A

Implementation of Tunneling Ionization into the Quasi-3D OSIRIS algorithm

A.1 Implementation

While much of this disserataion has focused on self-injection in beam-driven plasma wakefields (PWFA), we have also implemented a tunneling ionization module into the quasi-3D algorithm [67] in OSIRIS [84] to study ionization injection [42, 43, 44]. To model this process, the Ammonsov-Delone-Krainov (ADK) tunneling ionization model [109] was used. This model is valid for most wakefield problems as long as the Keldysh parameter $\gamma_K = \sqrt{I_p/2U_p} \ll 1$, where I_p is the ionization energy and U_p is the ponderomotive energy of the laser. In Ref. 110, it was shown that the ADK ionization probability rate W could be written in a more convenient form which depends only on the local electric field magnitude E (in units of GV/m) and the ionization energy I_p (in units of eV) of the neutral atom or ion,

$$W(s^{-1}) \approx 1.52 \times 10^{15} \frac{4^{n^*} I_p[eV]}{n^* \Gamma(2n^*)} \left(20.5 \frac{I_p^{3/2}[eV]}{E[GV/m]} \right)^{2n^*-1} \times \exp \left(-6.83 \frac{I_p[eV]}{E[GV/m]} \right), \quad (\text{A.1})$$

where $n^* \approx 3.69Z/I_p^{1/2}[eV]$ is the effective principal quantum number and Γ is the standard gamma function.

To model the tunneling ionization process in the quasi-3D geometry, we used a simple grid-based approach in which a neutral or pre-ionized gas is represented by an array of $r - z$ grids along ϕ . The number of grids is defined by the number of particles in the azimuthal direction, n_ϕ , so that the grids are equally space with $\Delta\phi = 2\pi/n_\phi$. Since the neutral or pre-ionized particles are much more massive than the plasma electrons, they are considered stationary and are not pushed. At each time step, the local electric field E is calculated by summing all the modes at every grid cell $(k\Delta r, l\Delta z, n\Delta\phi)$, where k , l , and n are integers. The ionization probability rate $W(I_p, E)$ is then calculated for each ionization level I_p of the gas. Recall that the quasi-3D approach [65, 66, 67] expands the electromagnetic fields and plasma charge and current densities into azimuthal fourier modes in ϕ ,

$$\begin{aligned} \mathbf{E}(r, z, \phi) = \Re \left\{ \sum_m \mathbf{E}^m(r, z) e^{im\phi} \right\} \\ \mathbf{E}^0(r, z) + \Re \{ \mathbf{E}^1 \} \cos(\phi) - \Im \{ \mathbf{E}^1 \} \sin \phi \\ + \Re \{ \mathbf{E}^2 \} \cos(2\phi) - \Im \{ \mathbf{E}^2 \} \sin(2\phi) \\ + \dots \end{aligned} \tag{A.2}$$

Once the local electric field is determined, the distribution of the ionization states for each gas within the cell are updated by integrating Eq. (A.1) over the time step Δt . Electrons can then be ionized (with zero momentum) if the average ionization level exceeds chosen thresholds, which are pre-defined based on the number of macroparticles used to model the bound electrons.

A.2 Comparisons of Quasi-3D and Full-3D Simulations

To validate the quasi-3D ionization algorithm, we revisit a result published in Ref. 111 in which a laser pulse with $\{a_0 = 2, W_0 = 14\mu\text{m}, \lambda = 0.8\mu\text{m}, \tau = 26\text{fs (FWHM)}\}$ propagates into a constant plasma density $n_p = 2 \times 10^{18} \text{ cm}^{-3}$. Pre-ionized nitrogen ions $n_{N^{5+}} = 10^{-3}n_p$

are distributed over a distance of $90\mu\text{m}$ from $z = 0.038$ to 0.128 mm. The ionization potentials of the sixth and seventh nitrogen electrons are $I_p \approx 552.1$ and $I_p \approx 667.0$ eV. Since the corresponding Keldysh parameters are $\gamma_K \approx 0.023, 0.021 \ll 1$, the ADK model can be used. To model this problem, we performed a quasi-3D simulation using only the first two modes $m \leq 1$ and grid dimensions of $26.75 \times 38.1 \mu\text{m}$ with 250×1500 cells in the r and z directions, respectively. The plasma electrons and nitrogen ions were both modeled using $1 \times 1 \times 8$ ppc in (r, z, ϕ) .

In Fig. A.1, snapshots of the laser field $E_x(m = 1)$ (top), plasma electron density ρ_e (middle), and ionized nitrogen electrons from the sixth and seventh levels $\rho_{N^{5+}}$ (bottom) are shown after the laser has propagated a distance of $z = 0.127$ mm into the plasma. Due to the high ionizing potential of the sixth and seventh levels of nitrogen, electron ionization only occurs where the laser field is most intense. Therefore, the ionized nitrogen electrons are born in regions where the wake potential is positive (where the laser exists) due to the formation of the ion channel, thereby easing the trapping threshold $\Delta\psi \approx -1$ [44]. As a result, ionized electrons are injected and accelerated at the rear of the wake where ψ is negative.

In Fig. A.2, we plot the current profile, $I(z)$, and bunching factor, $b(k) = \int dz n(z) \exp(ikz)$ of the injected electrons from the quasi-3D simulation (red) after the laser has propagated a distance of $z = 0.381$ mm into the plasma. For comparison, we have also plotted full-3D simulation results (black) using a window with dimensions of $63.5 \times 63.5 \times 38.1 \mu\text{m}$ with $500 \times 500 \times 1500$ cells in the x , y , and z directions, respectively. From the results, it can be seen that the amplitude and modulation of the full-3D current profile are qualitatively reproduced by the quasi-3D results. Integrating the currents profiles, we find that the modulation of the output beams peak at four times the laser wavelength, i.e., $k \approx 4k_0$, in both cases. It is worth noting that in this case the quasi-3D simulation is able to reproduce the full-3D results while using about $100\times$ fewer cpu-hours.

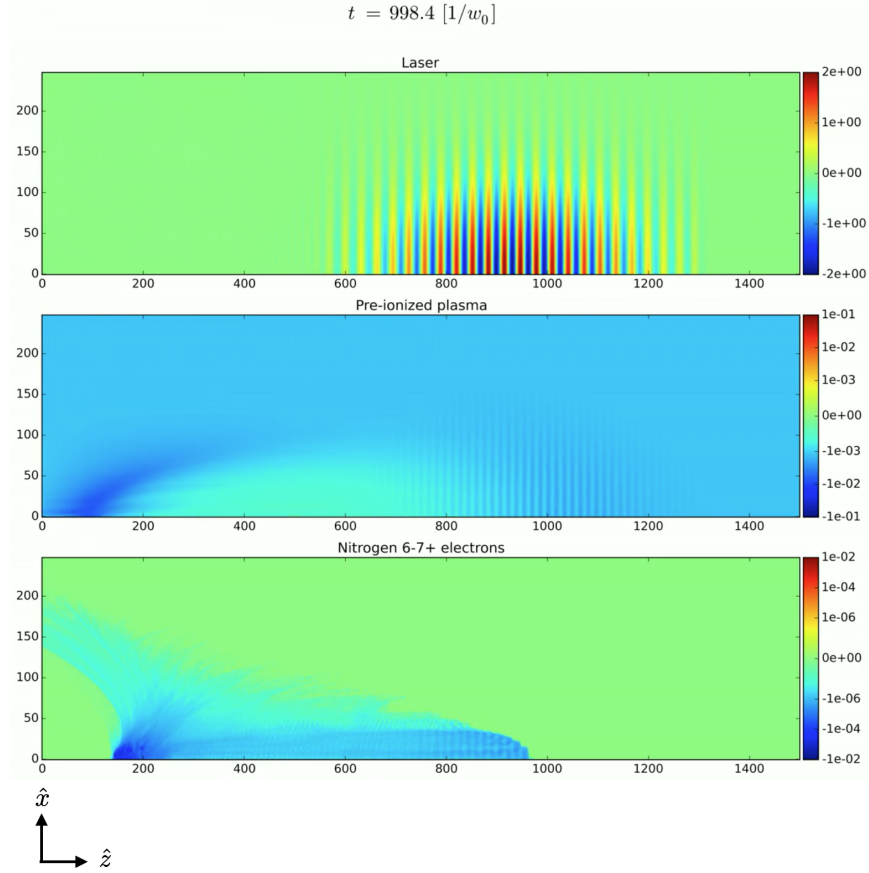


Figure A.1: A Quasi-3D PIC simulation of ionization injection using a laser pulse $\{a_0 = 2, W_0 = 14\mu\text{m}, \lambda = 0.8\mu\text{m}, \tau = 26\text{fs (FWHM)}\}$ propagating in a constant plasma density $n_p = 2 \times 10^{18} \text{ cm}^{-3}$. The laser pulse is focused at the plasma entrance ($z = 0 \text{ mm}$). Pre-ionized N^{5+} ions are distributed from $z = 0.038$ to 0.128 mm and $n_{N^{5+}} = 10^{-3}n_p$. Snapshots of the laser pulse (top), plasma electrons (middle), and ionized nitrogen (bottom) at $z = 0.127 \text{ mm}$ into the plasma.

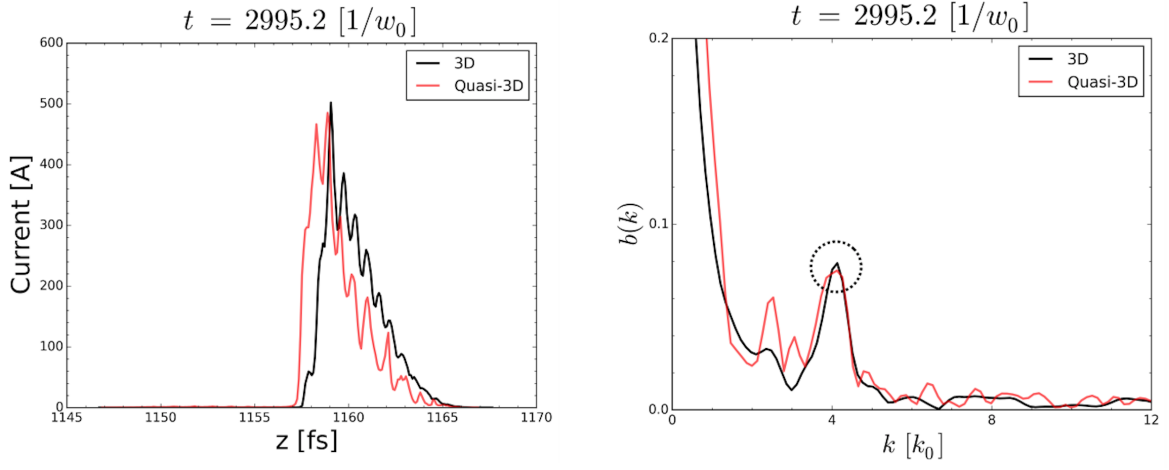


Figure A.2: (a) The current profile $I(z)$ and (b) bunching factor $b(k) = \int dz n(z) \exp(ikz)$ of the injected $n_{N^{5+}}$ electrons at $z = 0.381$ mm from quasi-3D (red) and full-3D (black). The laser and plasma parameters are provided in Fig. A.1.

Appendix B

Azimuthal Fourier Mode Decomposition Diagnostic

We have also implemented a post-processing diagnostic for transforming full-3D cartesian simulation results into azimuthal Fourier harmonics in the quasi-3D (r, z, ϕ) geometry. The diagnostic first interpolates full-3D cartesian data for electromagnetic fields and plasma charge and current densities onto $r - z$ grids equally spaced in ϕ . It then computes an FFT along the azimuthal direction ϕ to obtain the complex amplitudes of each fourier harmonic m . In Fig. B.1, we show how this post-processing diagnostic can be used to separate out azimuthal Fourier modes from the full-3D cartesian simulation performed in Appendix. A. From the full-3D simulation results, it can be seen that the longitudinal electric field E_1 contains both the wakefield and laser field. Using the post-processing diagnostic, the wakefield and laser field can be fully separated into $m = 0$ and $m = 1$ modes, respectively. This diagnostic can be used to study the effects of direct laser acceleration (DLA) [69, 70] in 3D cartesian simulations and to compare accuracy of quasi-3D and full-3D simulations [81].

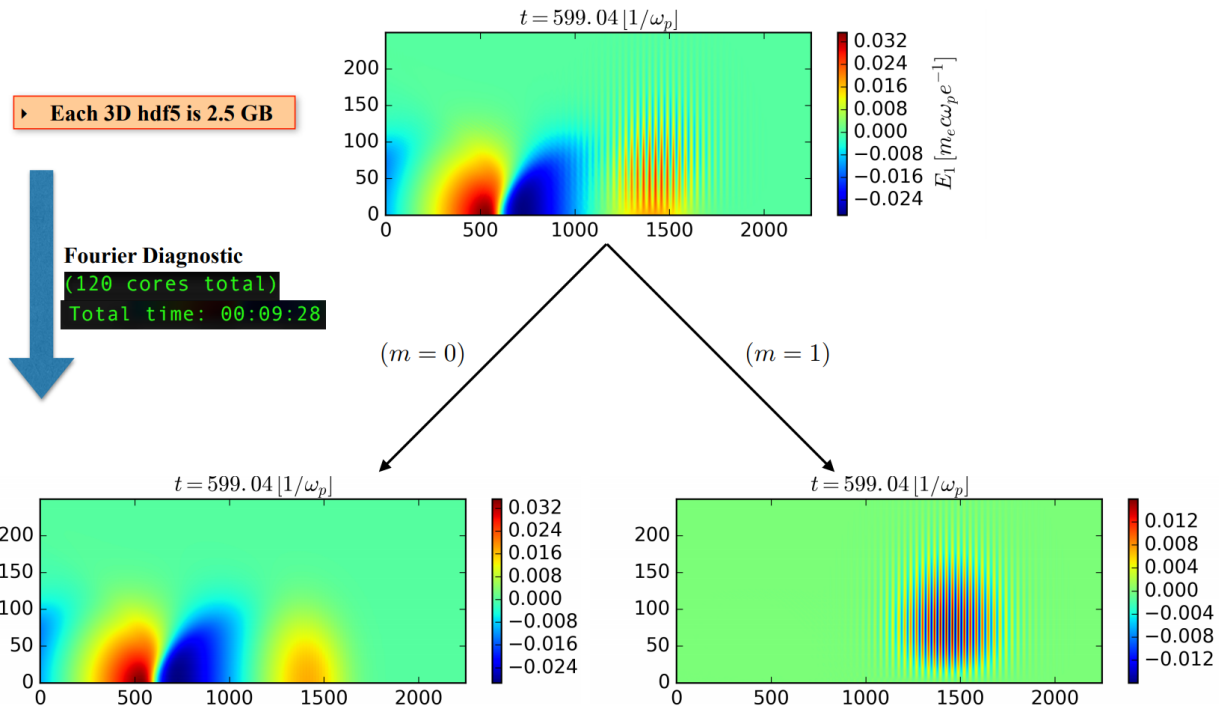


Figure B.1: Diagnostic results for decomposing full-3D simulation results into azimuthal Fourier modes.

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