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UCRL-2131

UNIVERSITY OF CALIFORNIA

Radiation Laboratory

Contract No. W-7405-eng-48

UCRL LECTURES ON NUMERICAL ANALYSIS AND APPLIED MATHEMATICS

Lecture VIII

Don Criley

November 11, 1952

Berkeley, California

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University of California
Radiation Laboratory
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CONVOLUTION INTEGRALS

1. Introduction

Let us set up a hypothetical experiment in which we are to observe a distribution spread. This observed spread will consist of two or more components--the "true" distribution which we wish to observe, and a "spread" due to other effects, notably the resolution of the instrument used in the observation.

It is often desirable to separate these components, if possible, and this is the problem under consideration in this paper.

2. The Faltung Integral

Let the "true" distribution be denoted by $f(x)$, the resolution "spread" by $g(x)$, and the observed distribution by $\phi(x)$.

If we examine these separately, they might appear as follows:

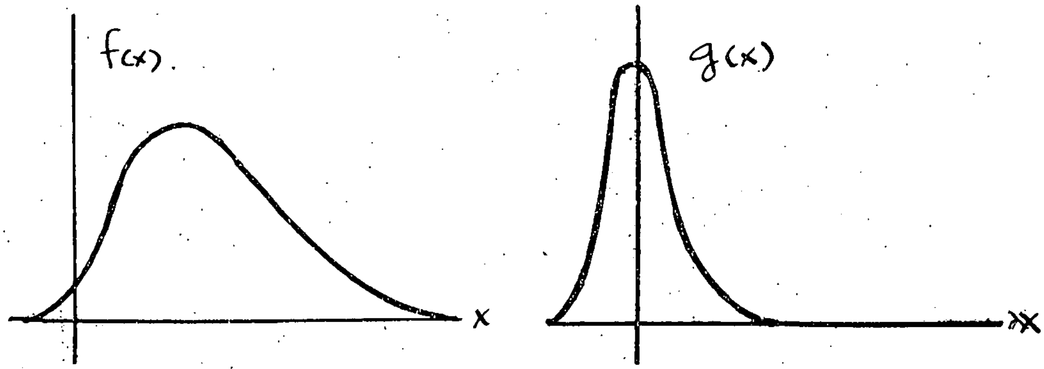

Fig. 2.1

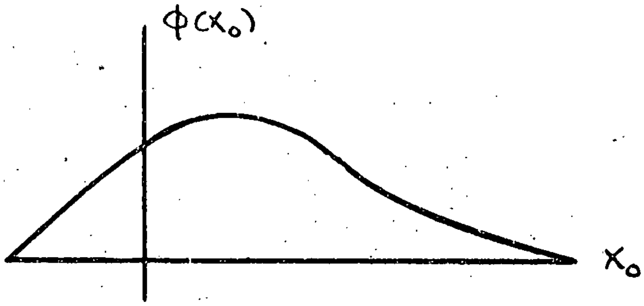

Fig. 2.2

Let us examine an arbitrary x, say x_o, and consider $f(x_o)$ and the spread $g(x_o - x)$ associated with $f(x_o)$.

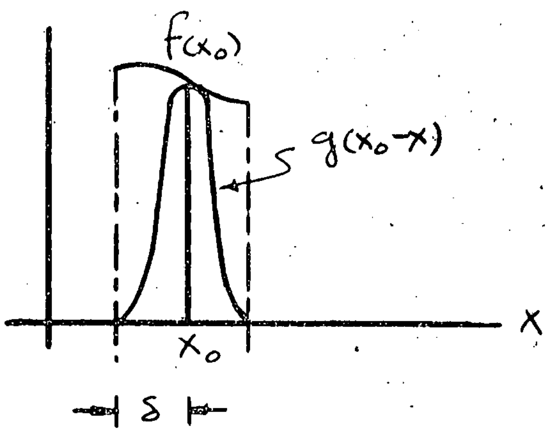

Fig. 2.3

Let $g(x)$ be a Gaussian of width 2δ. It is seen that

$\phi(x_o) = kf(x_o) +$ contributions of the form $f(x)g(x_o - x)$,

$x_o - \delta < x < x_o + \delta$ where k is an arbitrary normalizing constant.

The expression $\phi(x_o)$ may be expressed as

$$\phi(x_o) = \int_{x_o-\delta}^{x_o+\delta} f(x)\, g(x_o - x)dx \quad .$$

If the integration is carried out from $-\infty$ to $+\infty$,

$$\phi(x_o) = \int_{-\infty}^{\infty} f(x)\, g(x_o - x)dx \quad .$$

This is known as the <u>Faltung Integral</u> (Faltung = German "folding").

The process of evaluating ϕ may be graphically represented as follows:

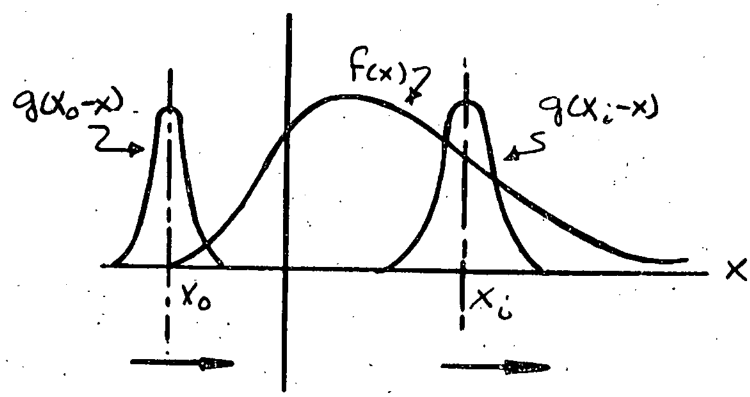

Fig. 2.4

Beginning at $x = x_o$ (at the left of the diagram) the integral is evaluated with the origin of $g(x)$ at x_o .

$$\phi(x_o) = \int_{-\infty}^{\infty} f(x)\, g(x_o - x)dx \quad .$$

Then the origin of $g(x)$ "slides" along x, assuming successively values $x_1, x_2, \ldots, x_n$. The integral evaluated at these points gives

$\phi(x_1)$, $\phi(x_2)$, ... , $\phi(x_n)$. Numerically, the integral is treated as a summation:

$$\phi(x_i) = \sum_j f(x_j)\, g(x_i - x_j)\Delta x \quad .$$

3. Commutativity of the "true" and "resolution" Functions.

As a matter of terminology, we will refer to $f(x)$ as the "stationary" function and $g(x)$ as the "sliding" function.

Given arbitrary $f(x)$ and $g(x)$, which is to be treated as "stationary" and which is to be treated as sliding? It is immaterial, as will be proved.

Theorem: $$\int_{-\infty}^{\infty} f(x)\, g(x_o - x)dx = \int_{-\infty}^{\infty} f(x_o - x)\, g(x)dx \quad .$$

Let

$$y = x_o - x \qquad x \rightarrow \infty, \quad y \rightarrow -\infty$$

$$dy = -dx \qquad x \rightarrow -\infty, \quad y \rightarrow \infty$$

$$x = x_o - y$$

Then

$$\int_{-\infty}^{\infty} f(x)\, g(x_o - x)dx = -\int_{\infty}^{-\infty} f(x_o - y)\, g(y)dy$$

$$= \int_{-\infty}^{\infty} f(x_o - y)\, g(y)dy \quad .$$

Let $g(x)$ = a delta function $\delta(x)$, and evaluate $\phi(x_i)$.

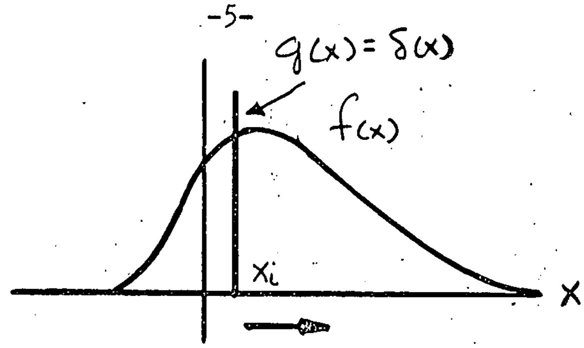

Fig. 3.1

It is readily seen that the resultant ϕ

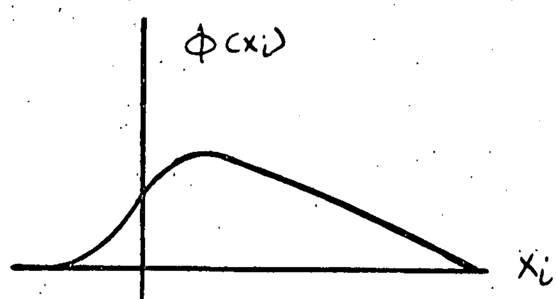

Fig. 3.2

is identical in form to f(x), except for a constant scale factor.

Now, let δ (x) be stationary, and let f(x) be the "sliding" function. A seemingly contradictory result is obtained, for, by the preceding proof, the resultant ϕ should be identical in both cases.

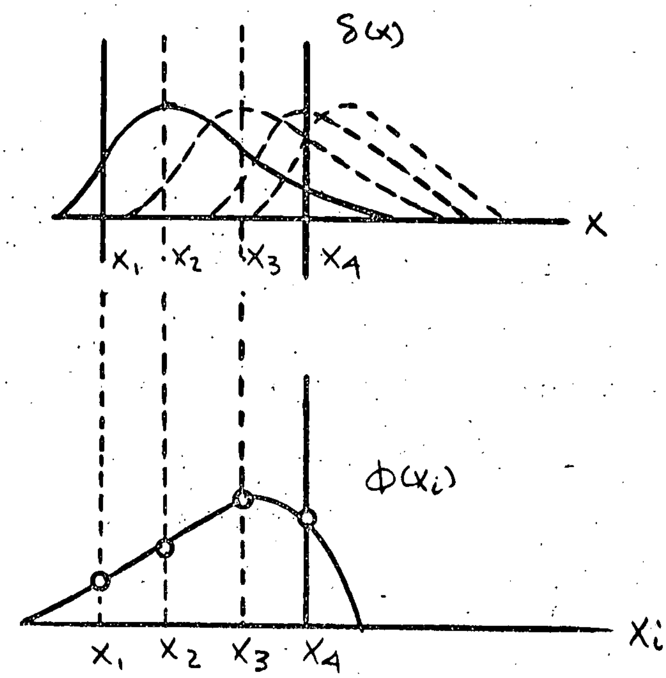

Fig. 3.3

The resultant is apparently the mirror image of the previous resultant!

However, this seeming contradiction is removed when we consider the argument of the sliding function. $f(x_o - x)$ is a reflection of $f(x)$ about the point x_o, and we are actually folding not the sliding function itself, but its mirror image about its origin. If the sliding function is symmetric about its origin, no modifications of form are necessary.

If, however, we are dealing with asymmetric functions, the sliding function must be inverted before the integral is computed.

4. Conditions for Existence of the Faltung Integral.

There are several restrictions upon the functions $f(x)$ and $g(x)$. In most cases,

$$\int_{-\infty}^{\infty} f(x)dx \quad \text{and} \quad \int_{-\infty}^{\infty} g(x)dx$$

must converge.

When the Faltung is numerically integrated, it is treated as a definite integral with the limits so chosen that

$$\int_{-\infty}^{a} f(x)dx \quad \text{and} \quad \int_{b}^{\infty} f(x)dx \sim 0 .$$

Then we have

$$\int_{a}^{b} f(x)dx \sim \int_{-\infty}^{\infty} f(x)dx .$$

If $f(x)$ has a singularity of degree < 1, it may be used, although its infinite integral does not converge.

If $f(x)$ has an integrable singularity at $x = x_1$, it may be treated as follows:

Write

$$\phi(x_o) = \int_a^b f(x)\, g(x_o - x)dx$$ as

$$\phi(x_o) = \int_a^{x_1-\epsilon} f(x)g(x_o - x)dx + \int_{x_1-\epsilon}^{x_1+\epsilon} f(x)g(x_o - x)dx + \int_{x_1+\epsilon}^{b} f(x)g(x_o - x)dx$$

If the singularity is integrable,

$$\int_{x_1-\epsilon}^{x_1+\epsilon} f(x)dx$$

is defined. (It is usually convenient to choose $\epsilon = \frac{\Delta}{2}$, where Δ is the tabular interval.)

We define $\bar{g}(x_o - x)$ as a mean value of $g(x_o - x)$ in the interval $x_1 - \epsilon \leq x \leq x_1 + \epsilon$, and approximate

$$\int_{x_1-\epsilon}^{x_1+\epsilon} f(x)g(x_o - x)dx$$

by

$$\bar{g}(x_o - x) \int_{x_1-\epsilon}^{x_1+\epsilon} f(x)dx \quad .$$

5. Applications of Transform Methods.

Let $T(f)$ = the Fourier or Laplace transform of f, and denote the faltung

$$\int_{-\infty}^{\infty} f(x)g(x_o - x)dx$$ by $f * g$.

A theorem due to Borel states:

$$T(f) \cdot T(g) = T(f * g) \qquad \text{or}$$

$$f * g = T^{-1}\left\{ T(f) \cdot T(g) \right\} .$$

This is a powerful method of finding an explicit expression for the faltung if the transforms of f and g are known, and the inverse transform of their product can be found.

6. Applications to Linear Integral Equations.

Consider a linear integral equation of the form

$$f(x_o) = h(x_o) + \int_a^b K(x, x_o)\, f(x)dx .$$

There are several methods of solution of this equation. In particular, let us examine the method of solution by successive approximation.

If the kernel $K(x, x_o)$ can be expressed as a function of the form $g(x_o - x)$, the definite integral takes the form of the faltung integral with limits a, b.

Choose any real function $f_1(x_o)$ which is continuous in $a \leq x \leq b$. Then define $f_2(x_o)$; $f_3(x_o)$; ... $f_n(x_o)$

$$f_2(x_o) = h(x_o) + \int_a^b g(x_o - x)\, f_1(x)dx$$

$$f_3(x_o) = h(x_o) + \int_a^b g(x_o - x)\, f_2(x)dx$$

$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$$

$$f_n(x_o) = h(x_o) + \int_a^b g(x_o - x)\, f_{n-1}(x)dx .$$

I will state without proof that

$$\lim_{n \to \infty} f_n(x_o) \equiv f(x_o)$$

and that this limit is independent of the choice of $f_1(x_o)$. The form of $f_n(x_o)$ is dependent upon $f_1(x_o)$, so the rapidity of convergence of the sequence is affected by this choice.

Example: Consider an infinitely long cylinder with an absorber arranged coaxially as follows:

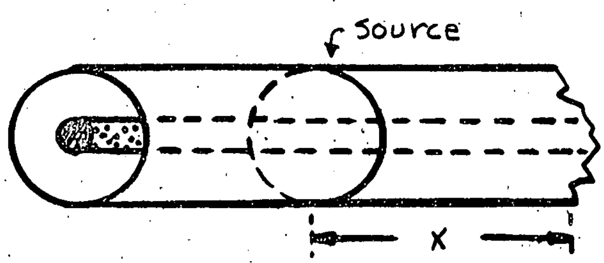

Fig. 6.1

Particles are injected into the cylinder with an isotropic angular distribution. If they strike

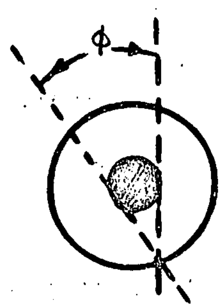

Fig. 6.2

the absorber they are lost. If they strike the walls they are absorbed, then reemitted with an isotropic angular distribution.

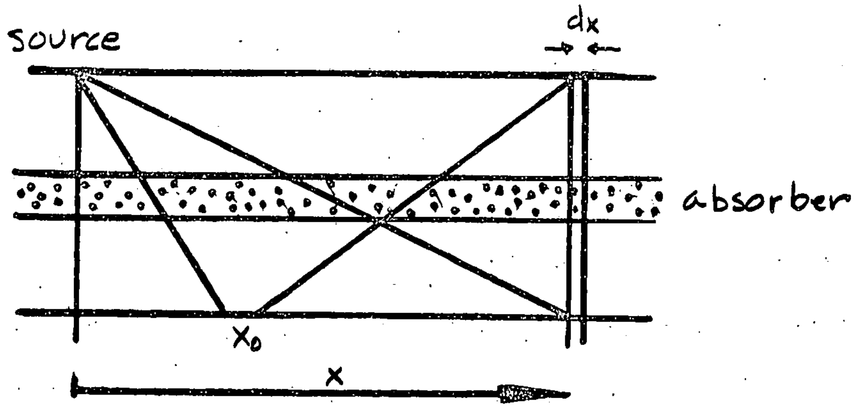

Fig. 6.3

Suppose that we are interested in the flux incident upon the area defined by a ring of width dx.

Let total flux = f(x) ; the number of particles arriving directly from source = S(x) . As a particle reemitted at x_0 has an isotropic angular distribution, it may be treated as a "random walk" problem. The flux incident upon the area under consideration, therefore, is

$$P(x_o - x)S(x_o)$$

where $P(x_o - x)$ is a probability function obtained from other considerations, notably the geometry of the experiment. The total flux is

$$f(x) = S(x) + \int_{-\infty}^{\infty} P(x_o - x)\, S(x_o) dx_o$$

which may be treated by the method of successive approximation

7. Bibliography.

7.1 Lovitt, "Linear Integral Equations".

7.2 Doetsch, G., "Theorie und anwendung der Laplacetransformation".